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CHAIN RECURRENCE IN SURFACE FLOWS

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Abstract. We investigate the topological and dynamical structure of internally chain recurrent sets for surface flows having particularly simple limit sets, including planar flows with finitely many equilibria. We verify a conjecture of Thieme (1992) concerning the limit sets of planar asymptotically autonomous equations.

0. Introduction. A dynamical system in a compact metric space is *chain recurrent* if, crudely speaking, every orbit can be made periodic by allowing arbitrarily small errors at arbitrarily large times. An equivalent definition is that the system has no proper attractors or repellers. Examples include the restriction of a flow to a compact alpha or omega limit set. Bowen (1975) studied homeomorphisms of this type, showing that such a system can be identified with an omega limit set in a larger space. This was generalized to flows by Franke and Selgrade (1976). Conley (1978) studied the notion systematically, introducing many fruitful ideas, and coining the term "chain recurrent". The behavior of chain recurrence under bifurcations was studied by Hurley (1983).

Closely related to Bowen's fruitful technique of "shadowing", chain recurrence has become increasingly important in the analysis of dynamical systems, especially in the presence of hyperbolicity properties. Much of this work has been further developed in the recent systematic study of Akin (1993).

Several recent papers have connected chain recurrence and shadowing to divers dynamic, stochastic and geometric phenomena:

- Random Perturbations and Invariant Measures: Ruelle (1981); see also Kifer (1988);
- Asymptotically Autonomous Systems: Thieme (1992, 1993a, 1993b), Benaim and Hirsch (1993, 1994));
- Stochastic Approximations and Urn Processes: Benaim (1993), Benaim and Hirsch (1993, 1994));
- Algebraic Topology: Hirsch and Pugh (1988);
- Reaction Diffusion Systems: Hirsch (1993).

In this paper a modest beginning is made toward elucidating the topological and dynamical structure of internally chain recurrent sets for surface flows having particularly simple limit sets, including planar flows with finitely many equilibria. A key step in the proofs is application of a general result due to Akin, Nitecki and Shub, Theorem 3.1.

The results here enable us to develop elsewhere (Benaim and Hirsch, 1994) a Poincaré-Bendixson theory for nonautonomous systems, including certain kinds

of asymptotically autonomous equations, stochastic differential equations and stochastic approximation processes. In particular we verify a conjecture of Thieme (1992) concerning the limit sets of planar asymptotically autonomous equations (see Section 2), which in fact inspired this paper.

It turns out that we need only certain dynamical consequences of planarity; with these as hypotheses we obtain results for flows on arbitrary surfaces.

Notation and Definitions. Let $\Phi : \mathbf{R} \times X \to X$ denote a flow in a metric space X. We may write $\Phi(t,x) = \Phi_t x = x \cdot t$. The *trajectory* of $x \in X$ is the map $\Phi^x : \mathbf{R} \to X$,

$$t \mapsto \Phi_t x = \Phi(t,x) = x \cdot t.$$

For $Y \subset X$ and $I \subset \mathbf{R}$ we put $Y \cdot I = \Phi(Y \times I)$. The image of Φ^x is the *orbit* of x, denoted by γ_x or $\gamma(x)$; its closure is $\overline{\gamma_x} = \overline{\gamma(x)}$. The *forward orbit* of x is $\gamma_+(x) = x \cdot [0,\infty)$; the *backward orbit* is $\gamma_-(x) = x \cdot (-\infty,0]$.

If $I \subset \mathbf{R}$ is an interval and $\Phi^x|I$ is injective, then the set $\delta = \Phi^x I = x \cdot I$ is called an *orbit interval* (closed, open, etc., according to the nature of I). The *endpoints* of δ are the images under Φ^x of the endpoints of I. The *parameter length* of δ is the distance between the endpoints of I (possibly ∞).

Chain Recurrence. Let T and ϵ be positive numbers. A (T,ϵ)-*chain* of length $m \geq 1$ from p to q is a sequence of orbit intervals $\{\delta_i\}$, $i = 1, \ldots, m$ such that:

     (a): the parameter length of δ_i is $> T$;
     (b): the distance between the terminal endpoint of δ_i and the initial endpoint δ_{i+1} is $< \epsilon$;
     (c): $a_1 = p$, $b_m = q$, where a_1 denotes the initial endpoint of δ_1, and b_m the terminal endpoint of δ_m.

In this case we write $p \hookrightarrow_{T,\epsilon} q$. When this holds for all $T > 0, \epsilon > 0$ we write $p \hookrightarrow q$. Point p is *chain recurrent* if $p \hookrightarrow p$. If every point is chain recurrent then Φ is a *chain recurrent flow*. It was proved by (Conley, 1978) that a chain recurrent flow is also *chain transitive*, meaning that $p \hookrightarrow q$ for all $p, q \in X$, if and only if X is connected. A subset $L \subset X$ is *internally chain recurrent* provided L is a nonempty compact invariant set of which every point is chain recurrent for the restricted flow $\Phi|L$. The notion of an *internally chain transitive* set is analogously defined.

Orbit Chains. Next we define a different type of "chain". An *orbit chain* of *length* $m \geq 1$ is a finite sequence $\Gamma = \{\gamma_i, i = 1, \ldots, m\}$ of orbits such that: γ_i goes from an equilibrium (fixed point) e_{i-1} to an equilibrium e_i. That is, $e_{i-1} = \alpha(\gamma_i)$, the alpha limit set of γ_i, while $e_i = \omega(\gamma_i)$, the omega limit set of e_i. The equilibria e_i are the *nodes* of the orbit chain.

If all γ_i are in a subset L we say Γ is an *orbit chain in L*. We say the e_j ($j = 0, \ldots, m$) and the γ_i ($i = 1, \ldots, m$) *belong* to Γ. The *support* of Γ is the set $|\Gamma| = \bigcup_{i=1}^m \overline{\gamma_i}$. Each γ_i belonging to Γ has a natural linear order induced by its parameterization as a trajectory of the flow. We extend this to a closed transitive reflexive relation on $\overline{\gamma_i}$, denoted by $u \prec v$, in the obvious way. For a given orbit chain Γ we extend $\prec$ to $|\Gamma|$ by writing $\Gamma : u \rightsquigarrow v$ if either $u \prec v$ in one of the γ_i, or else $u \in \overline{\gamma_i}$ $v \in \overline{\gamma_j}$, and $i < j$.

To indicate that such an orbit chain Γ exists we write $u \rightsquigarrow v$. We may abuse notation slightly by writing $\gamma : u \rightsquigarrow v$ if $\Gamma = \{\gamma\}$. When $e_m = e_0$ we call Γ an *orbit cycle*. Note that the concatenation of two orbit cycles with a common node

or a common orbit is again an orbit cycle after a suitable reordering. An invariant set $\bar{R}$ is *strongly chain recurrent* if every point of R belongs to an orbit cycle or periodic orbit. It is *strongly chain transitive* provided every pair of points belong to a common orbit cycle or periodic orbit. If these orbit cycles and periodic orbits can be found in R we say R is *internally* strongly chain recurrent or chain transitive.

The main technical tools concern a *simple* flow in a compact metric space, by which we mean a flows having only a finite number of alpha and omega limit points. These results are presented in Section 3.

1. Statement of Results on Surface Flows.
In Section 2 we use the following result to verify a conjecture of H. Thieme.

Theorem 1.1. *Let M be an internally chain recurrent set for a planar flow, with finitely many equilibria in M. Then M is internally strongly chain recurrent.*

Most of our results are proved under the following Hypothesis 1.2:

Hypothesis 1.2. (STANDING ASSUMPTIONS*)*
Φ *denotes a flow in a connected surface S with empty boundary; $L \subset S$ is a nonempty compact, invariant set, having the following three properties:*

    H1: *L is internally chain recurrent.*
    H2: *Equilibria (fixed points) in S are isolated.*
    H3: *The alpha and omega limit sets of any nonperiodic orbit in L are equilibria.*

For planar flows H3 is a consequence of H1 and H2 in view of Lemma 5.6 below. For some results we further assume:

Hypothesis 1.3. (OCCASIONAL ASSUMPTION)

    H4: *Every nonconstant periodic orbit in L which preserves orientation, separates S.*

For planar flows ($S \subset \mathbf{R}^2$) this is a consequence of the Jordan separation theorem.[1] Thus only assumptions H1, H2 are significant for planar flows. Hypothesis 1.2 is assumed from now on.

Theorem 1.4. *Under Hypothesis 1.3, L is internally strongly chain recurrent.*

We denote by $P \subset L$ the union of all the nonstationary periodic orbits in L. Each connected component of P is chain transitive, but as points in P cannot be in an orbit chain, we have to handle P and $L \setminus P$ in different ways.

Theorem 1.5. *Each component of P which is not a single periodic orbit is homeomorphic to one of the following surfaces:*

    (i): *an annulus or Möbius band, with or without boundary,*
    (ii): *a torus,*
    (iii): *a Klein bottle.*

It follows that $L \setminus P$ is compact.

We denote the boundary in L of a subset $Z \subset L$ by $\partial_L Z$.

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[1]H4 also holds when S is a subset of the projective plane.

Theorem 1.6. (a): Every component of $\partial_L(L \setminus P)$ is internally strongly chain transitive. (b): Assume Hypothesis 1.3. Then every component of $L \setminus P$ is internally strongly chain transitive.

Theorem 1.5 is proved in Section 3 after topological preliminaries in Section 3. Theorems 1.4 and 1.6 are proved in Section 5. In the next section we present examples, counterexamples and related results illustrating the main theorems.

2. Application to Differential Equations; Discussion. Example: Asymptotically autonomous systems A nonautonomous differential equation

$$\frac{dx}{dt} = f(t,x); \ t \in \mathbf{R}, \ x \in \mathbf{R}^n \tag{1}$$

is called *asymptotically autonomous* with *limit equation*

$$\frac{dx}{dt} = g(x) \tag{2}$$

provided $\lim_{t \to \infty} f(t,x) = g(x)$ locally uniformly in x. We assume f and g are locally Lipschitz.

H. Thieme (1992) conjectured that *if $n = 2$ and system (2) has isolated equilibria, then the limit set L of any bounded solution to equation (1) is a connected union of equilibria, periodic orbits and supports of orbit chains of the flow of (2)*. In (Benaim and Hirsch 1994) we show that, for any n, L is a compact, connected internally chain recurrent set, whence Thieme's conjecture follows from Theorem 1.1.

Example: A flow on the torus The following example illustrates the necessity of Hypothesis 1.3 in Theorem 1.6(b). Consider the flow on the two-torus

$$T^2 = (\mathbf{R}/2\pi\mathbf{Z}) \times (\mathbf{R}/2\pi\mathbf{Z}) = S^1 \times S^1$$

induced by the differential equation:

$$\begin{aligned} \frac{d\theta}{dt} &= f(\theta), \\ \frac{d\psi}{dt} &= g(\theta) + h(\psi) \end{aligned}$$

where f, g, h are 2π-periodic smooth nonnegative functions such that

$$f^{-1}(0) \cap [-\pi, \ \pi] = [-\pi/2, \ \pi/2],$$

$$g^{-1}(0) \cap [-\pi, \ \pi] = [-\pi, \ -\pi/2] \cup [\pi/2, \ \pi],$$

and

$$h^{-1}(0) \cap [-\pi, \ \pi] = \{0\}.$$

See Figure 1. This flow admits two equilibria:

$$p_- = (-\pi/2, 0) \text{ and } p_+ = (\pi/2, 0).$$

The set of periodic points P is the set of points $p = (\theta, \psi)$ with $-\pi/2 < \theta < \pi/2$ and we have

$$\partial(T^2 \setminus P) = [\{-\pi/2\} \times S^1] \cup [\{\pi/2\} \times S^1] = \partial^+ \cup \partial^-.$$

If $p \in \text{Int}(T^2 \setminus P)$, $\alpha(p) = p_+$ and $\omega(p) = p_-$. If $p \in \partial^+$, $\alpha(p) = \omega(p) = p_-$ and if $p \in \partial^-$, $\alpha(p) = \omega(p) = p_+$. If $p \in \text{Int}(T^2 \setminus P)$ then $\alpha(p) = p_+$ and $\omega(p) = p_-$. If $p \in \partial^+$ then $\alpha(p) = \omega(p) = p_-$; and if $p \in \partial^-$, $\alpha(p) = \omega(p) = p_+$. From

these properties, it is easy to see that T^2 is internally chain recurrent and satisfies Hypothesis 1.2. But $T^2 \setminus P$ is not chain recurrent. Indeed, $T^2 \setminus P$ is a cylinder with ∂^+ as attractor and ∂^- as repeller.
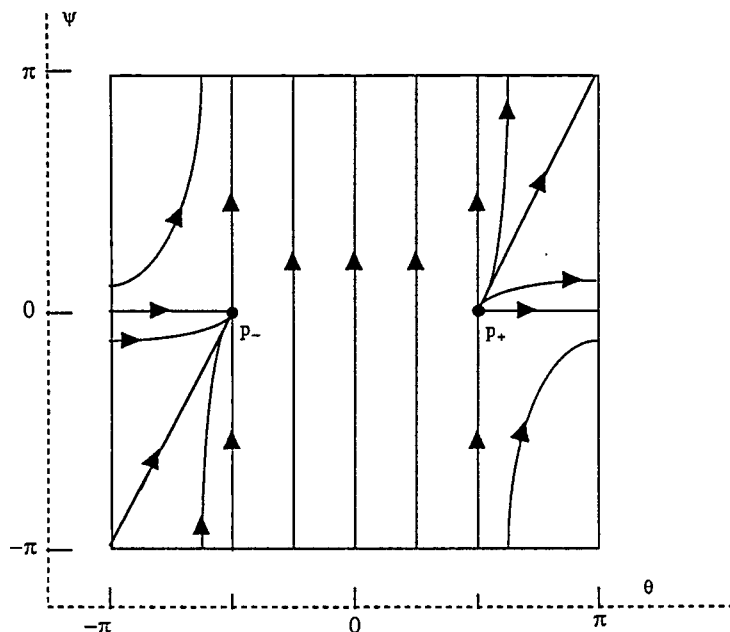
Example: Planar competitive systems. Consider the flow Ψ in the first quadrant $\mathbf{R}^2_+$ of a planar competitive system

$$\frac{dx_i}{dt} = x_i G_i(x_1, x_2); \ i = 1, 2 \tag{3}$$

where the C^1 growth rates $G_i : \mathbf{R}^2 \to \mathbf{R}^2$ satisfy Kolmogorov's competition condition:

$$\frac{\partial G_i}{\partial x_j} \leq 0 \text{ if } \neq j. \tag{4}$$

Assume the system is dissipative, that is, there is a global attractor.

Theorem 2.1. *If the equilibrium set is finite, then system* $(\Psi, \mathbf{R}^2_+)$ *admits a strict Liapunov function.*

Proof By a result of Conley (1978) (see also Akin 1993) it suffices to prove that the chain recurrent set is finite.

It is well known that the competition condition, together with dissipitavity, implies every bounded semitrajectory converges (Hadeler and Glas 1983; Hirsch 1985). Thus the flow is simple. Hence by Theorem 3.1, if x is chain recurrent then there is an orbit cycle from x to x. If x is not an equilibrium we may assume the support of such an orbit cycle is a Jordan curve C .

The reversed-time flow $\{\Theta_t = \Psi_{-t}\}$ also leaves C invariant, and preserves the vector order in the plane: $x \leq y \Leftrightarrow x_i \leq y_i$, $(i = 1, 2)$. But it is not hard to prove

that an order preserving flow in the plane cannot have an orbit cycle other than an equilibrium. $\square$

Discussion: For general surface flows, the structure of compact chain recurrent subsets is rather mysterious— even in the important special case of omega limit sets for planar flows.[2] The only general result known is the following (which is not needed for our proofs):

Theorem 2.2 (Hirsch and Pugh (1988)). *Let M be a compact invariant set for a flow Φ in a metric space X. Suppose there exists a nonstationary point $p \in M$ which is chain recurrent for $\Phi|M$, and at which M is 1-dimensional. Then:*

   (a): *There is an essential map from M to the circle.*

   (b): *The first Čech cohomology group of M is nontrivial.*

   (c): *If X is homeomorphic to an open set in the plane then M separates X.*

Thus nonstationary, 1-dimensional internally chain recurrent sets have nontrivial topology. Unfortunately one cannot conclude that such a set must have nontrivial fundamental group, as is shown by the solenoid attractor A for a Denjoy flow in the 2-torus T: While the 1-dimensional Čech cohomology of A is free on two generators (because $T \setminus A$ is an open 2-cell), its singular homology and fundamental group are trivial.

In contrast, Theorems 1.4 and 1.5 imply that under Hypothesis 1.2, a 1-dimensional, internally chain recurrent set L contains at least one compact invariant set homeomorphic to a circle: Either a periodic orbit, or an orbit cycle of minimal length.

Although even with these strong hypotheses we do not have a general structure theorem, from our results there emerges a clear picture of the dynamic connection between two points $x, y \in L \setminus P$. When the stronger Hypothesis 1.3 holds (in addition to Hypothesis 1.2) then $x \leadsto y$, i. e., there is an orbit chain from x to y. Hypothesis 1.2 by itself implies (by Theorems 1.4 and 1.5) that there is a finite sequence $x = x_1, y_1, \ldots, x_n, y_n = y$ where $x_i \leadsto y_i$, and x_i, y_{i+1} lie in the boundary of a component of $L \setminus P$.

3. Chain Recurrence in Simple Dynamical Systems.

Let Φ denote a flow in a metric space X. The following definitions are taken from Conley (1978).

The omega limit set (respectively alpha limit set) of Y, denoted by $\omega(Y)$ (respectively $\alpha(Y)$) is defined as the maximal invariant set in $\mathrm{clos}(Y \cdot [0, \infty))$ (respectively $\mathrm{clos}\, Y \cdot (-\infty, 0])$, where "clos" denotes closure.

A flow Φ in X is *simple* provided X is a compact metric space, and there is only a finite number of alpha and omega limit points, necessarily constituting the set $\mathcal{E}$ of equilibria.

In the remainder of this section Φ denotes a simple flow in X.

Theorem 3.1. $x \hookrightarrow y$ *if and only if* $x \leadsto y$.

Corollary 3.2. *Let $L \subset X$ be a compact invariant set. If L is internally chain recurrent, then L is internally strongly chain recurrent.*

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[2]Hartman (1964) shows that a compact omega limit set for a planar flow with finitely many equilibria is either a periodic orbit, or the union of equilibria and a countable family of nonperiodic orbits. To this we can add that it contains an orbit cycle through all the equilibria, by Theorem 1.6(b). We lack a reference to this folk theorem.

Corollary 3.3. *If $x \hookrightarrow y$ then x and y are in the same path component of X. Consequently, if X is chain transitive (or chain recurrent and connected), then X is pathwise connected.*

Proof If $x \hookrightarrow y$ then $x \rightsquigarrow y$, and the support of an orbit chain from x to Y contains a path from x to y. $\square$

Proof of Theorem 3.1[3] It suffices to prove that $x \rightsquigarrow y$ whenever $x \hookrightarrow y$, since the other implication is trivial. Asume $x \hookrightarrow y$. The set of z such that $x \hookrightarrow z$ is closed and invariant, and thus contains the alpha limit point q of y. It therefore suffices to prove $x \rightsquigarrow q$.

Consider the set

$$F = \{p \in \mathcal{E} : p \rightsquigarrow q\}.$$

Set $p = \omega(x)$ It suffices to show that $p \in F$. Suppose $p \notin F$. We then have the following situation:

(a): F is closed and open in $\mathcal{E}$,

(b): $x \hookrightarrow q \in F$,

(c): the omega limit set $(=\{p\})$ of x is disjoint from F.

This conditions constitute a special case of the hypothesis of Lemma 13(a) of Chapter 5 of (Akin 1993), which the author calls an extension of a lemma of Nitecki and Shub.[4] The conclusion of this Akin-Nitecki-Shub lemma, for this case, is that there exists $u \in X$ with the following properties:

(d): $x \hookrightarrow u$,

(e): $\omega(u) \in F$,

(f): $\alpha(u) \notin F$.

But (e) implies $u \rightsquigarrow q$, which implies $\alpha(u) \in F$, contradicting (f). Therefore $p \in F$. $\square$.

Remark 3.4. *For the special case of surface flows satisfying Hypotheses 1.2, 1.3, there is a proof of Theorem 3.1 which avoids the Akin-Nitecki-Shub Lemma. It is based on P. Hartman's theorem on the finiteness of hyperbolic sectors (Hartman, 1964).*

Define an equivalence relation $\mathcal{R} = \mathcal{R}(\Phi)$ in X as follows:

$$\mathcal{R} = \{(u, v) \in X \times X : u \rightsquigarrow v \text{ and } v \rightsquigarrow u.\}$$

Corollary 3.5. *For a simple flow in a compact metric space X, the relation $\mathcal{R}$ is closed in $X \times X$.*

Proof Follows from Theorem 3.1, because the relation $x \rightsquigarrow y$ is closed for any flow. $\square$

A nonempty compact invariant set $A \subset X$ is an *attractor* if A has an open neighborhood U in X such that $\omega(U) = A$; or a *repeller* if $\alpha(U) = A$. An attractor or repeller is *proper* provided it is not open in X.

The following proposition follows from sections 5 and 6 of (Conley 1978, Chapter 2).

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[3]The proof was kindly supplied to us by E. Akin.

[4]This result is stated for homeomorphisms, but as Akin points out, the proof is designed so that it applies equally to flows.

Proposition 3.6.   **(i):** *Let $x \in X$ be a chain recurrent point, $A \subset X$ an attractor. If $\omega(x) \subset A$ then $x \in A$.*

(ii): *Let $N \subset X$ be a compact set such that $\Phi_T(N) \subset \mathrm{Int}(N)$ for some $T > 0$. Then $\omega(N)$ is an attractor contained in $\mathrm{Int}(N)$.*

(iii): *Let $N \subset X$ be a compact set. Let $A \subset X$ be the maximal invariant set contained in N. If A is nonempty and A is not an attractor, there exists $p \in \partial N \subset A$ such that $\gamma_-(p) \subset N$ and $\alpha(p)$ is a nonempty subset of A.*

(iv): *A chain recurrent flow has no proper attractor or repeller.*

Corollary 3.7. *Let K be a nonempty compact invariant set for a simple flow in a compact metric space X. Then the set*

$$Q = \{y \in X : \exists x \in K,\ x \rightsquigarrow y\}$$

is an attractor.

Proof Q is invariant, and compact by Corollary 3.5. Let $N \subset X$ be a compact neighborhood of Q such that $N \cap \mathcal{E} \subset Q$. Suppose $\gamma_-(z) \subset N$ then, then there exists an equilibrium $w = \alpha(x) \in Q$. Since $x \rightsquigarrow w$ for some $x \in K$ it follows that $x \rightsquigarrow z$, so $z \in Q$ by definition of Q, whence $\gamma_-(z) \subset Q$ by invariance.

This shows, first, that Q is the maximal invariant set in N, and second, that Q is an attractor, by Proposition 3.6(iii). $\square$

4. Topology of the Set P of Nonstationary Periodic Points.
Throughout the remainder of the paper, Hypothesis 1.2 is in force. We first prove Theorem 1.5, restated here for convenience:

Theorem 4.1. *Each component of P which is not a single periodic orbit is homeomorphic to one of the following surfaces:*

(i): *an annulus or Möbius band, with or without boundary,*

(ii): *a torus,*

(iii): *a Klein bottle.*

Proof Let C be a component of P which is not a single periodic orbit. It follows from Theorem 4.3 that C is a surface embedded in S, perhaps with boundary. Consider first the case where the surface C is orientable. From Proposition 4.3 we see that the orbits in C constitute a foliation of C by topological circles. Moreover Remark 4.5 and orientability of C imply that this foliation is locally a product.

Denote by $\Pi : C \to B$ the identification map which collapses each orbit in C to a point. Give B the quotient space topology. It is not hard to prove from Proposition 4.3 and Remark 4.5 that B is a Hausdorff 1-dimensional manifold, and Π is a locally trivial fibration. As B is connected, it is homeomorphic to a circle, or an interval (open, closed or half-open). Since we are assuming C is orientable, it follows that the fibration is trivial. Thus C is homeomorphic to $B \times S^1$, so C is an annulus or torus.

Now suppose C is nonorientable. Then C has a connected two-fold covering space $\hat{C}$ which is orientable. The flow on C lifts to a flow on to which Proposition 4.3 applies. Therefore the previous argument shows that $\hat{C}$ is homeomorphic to an annulus or torus, whence C, being nonorientable, is a Möbius band or a Klein bottle. $\square$

Lemma 4.2. *Let Λ be an isolated nonstationary periodic orbit in L. Then Λ is a component of L.*

Proof Let $N \subset L$ be a compact neighborhood of Λ which contains no equilibrium. Then Λ is the maximal invariant set in N. To see this, let γ be any orbit in N. Then no limit point of γ can be an equilibrium, so γ is periodic by Hypothesis 1.2, whence $\gamma \subset \Lambda$. We claim Λ is an attractor for the flow in L. For otherwise Proposition 3.6(iii) leads to the impossible conclusion that Λ contains an equilibrium.

The same reasoning, applied to the time-reversed flow, shows that Λ is also a repeller in L. The only way this can happen is for Λ to be open in L, showing that Λ is a component of L. $\square$

Proposition 4.3. *Let $\Lambda \subset L$ be a nonstationary periodic orbit; assume Λ is not a connected component of L. Let $J \subset S$ be a compact arc which is a local section through a point $p \in \Lambda$. Then p belongs to a closed arc $I \subset J$ such that:*

    (i): *$I \cdot \mathbf{R}$ is a compact neighborhood of p in L;*
    (ii): *every point of I is periodic;*
    (iii): *p is an endpoint of I if and only if $p \in \partial L$.*

In particular, every neighborhood of Λ in L contains infinitely many periodic orbits.

Proof We identify J with a subinterval in $\mathbf{R}$ about 0, and p with 0. Let $J_1 \subset J$ be a smaller compact local section at p on which a Poincaré map $g : J_1 \to J$ is defined. By choosing J_1 sufficiently small we assume that the iterates $g^k : J_1 \to J$ are defined for $|k| \leq 2$.

Let $L_1 = (L \cap J_1)$. We claim *every point of L_1 is periodic*. More precisely, the second iterate $g^2 = g \circ g$ is the identity on L_1. For g^2 is preserves the natural linear order on J_1 (and h preserves that order if and only if S is orientable in a neighborhood of Λ). Suppose $x \in L_1$ is not fixed under g^2. Then (recalling that points in J are identified with real numbers) we see that either $|g^2(x)| < |x|$ or $|g^{-2}(x)| < |x|$; we may assume the first inequality. Then $g^2(x) \in J_1$, since J_1 is an interval containing x and 0. It follows that $g^{2k}(x) \in J_1$, $k = 1, 2, 3, \ldots$, and that this sequence converges monotonically to a point $y \in L_1$. It follows that $y \in \omega(x)$, and therefore y is an equilibrium for the flow. But as no point of the local section J is an equilibrium, no such x can exist. Thus $g^2|L_1 = \mathrm{Id}$.

Notice that p (identified with 0) is not the only point of L_1, because $\Lambda \neq L$. Suppose there exists $c > 0$ in L_1. We claim the entire interval $[0, c]$ is in L_1. For suppose not. Then there is an open subinterval $D \subset [0, c] \setminus L_1$ with endpoints in L_1. Since g^2 is continuous and injective, and has the endpoints of D as fixed points, it follows that g^2 maps D onto itself. From this one can see that $D \cdot \mathbf{R}$, the union of the orbits of D under the flow, is an invariant open annulus or Möbius band that separates p from c. But this contradicts the assumption that L is connected.

If all points of L_1 are ≥ 0, Proposition 4.3 is proved by setting $I = [0, c]$. If all points are ≤ 0, a similar construction give an interval $I = [-b, 0]$. If L_1 has points of both signs then an interval of the form $I = [-b, c]$ completes the proof. $\square$

Corollary 4.4. *If $C \subset P$ is a connected component containing more than one periodic orbit, then $\mathrm{Int}_S(C)$ is not empty.*

Remark 4.5.

(1) *It is easy to see from the proof of Proposition 4.3 that $h = $ Id if and only if the periodic orbit Λ preserves orientation; in this case the minimal period function* Per $: P \to \mathbf{R}_+$, *assigning to each point of P its minimal period, is a continuous at p.*

(2) *If Λ reverses orientation and $\Lambda \neq P$, then $\lim_{x \to p} \text{Per}(x) = 2\text{Per}(p)$.*

(3) *Only finitely many periodic orbits in a compact subset of S can reverse orientation. To see this, let $S_0 \subset S$ denote a compact surface (perhaps with boundary) containing L. Any set of orientation reversing simple closed curves in S represent linearly independent homology classes in the finite dimensional vector space $H_1(S_0; \mathbf{Z}_2)$. It follows that the number of such curves is bounded by the first Betti number of S_0 over $\mathbf{Z}_2$. (See e. g. Hirsch 1976, Chapter 9, Lemma 3.9.)*

Proposition 4.6. *(a): The flow in $\partial_L P$ is simple. (b): $\partial_L P$ has only finitely many components, and each component contains an equilibrium.*

Proof It follows from Proposition 4.3(i, ii) that no point of $\partial_L P$ can be non-stationary and periodic. Therefore the flow in $\partial_L P$ is simple by Hypotheses 1.2. As every component of $\partial_L P$ therefore contains an equilibrium, the number of such components is bounded by the number of equilibria. $\square$

Theorem 4.7. *Assume Hypothesis 1.3. If L is connected, then P has only finitely many components.*

Proof For simplicity we assume the surface S is orientable; the general case follows by lifting the flow to the orientable two-fold covering space. Since S is a connected surface, there cannot be more than one subset homeomorphic to a torus (and no such component if S is not compact); so we assume there are no such components.

Assume P has infinitely many components. Since L has only one component, the boundary in L of each component is nonempty, and therefore contains an equilibrium by Proposition 4.6. Since the equilibria are finite in number, there is an equilibrium $q \in L$ and a sequence C_i of distinct components of P, such that q is in the closure of each C_i. It follows that no C_i is a single orbit. Therefore by Theorem 4.1 and orientability, each C_i is an annulus.

Let $D \subset S$ be the interior of closed disk neighborhood $\overline{D}$ of q which contains no other equilibrium. We claim that *no orbit in $\bigcup C_i$ lies entirely within $\overline{D}$.* For suppose that $\Lambda \subset C_i \cap \overline{D}$ is an orbit. Replacing it with another orbit, we can assume $\Lambda \subset C_i \cap D$. Then Λ separates D into two disjoint connected open sets, an open disk B and an open annulus A. Now q must lie in B. For otherwise we could find another periodic orbit $\Lambda' \subset B \cap C_i$ which would surround an equilibrim in $D \setminus q$, contradicting the definition of q. Therefore $q \in B$. This implies $C_j \subset B$ for all $j \neq i$, because C_j contains q in its closure. Reasoning as before leads to the contradiction that any orbit in C_j surrounds an equilibrium in $D \setminus q$; this proves the claim.

It follows that for each i there are sequences $u_{ij}, v_{ij} \in C_i \cap \partial D$ converging respectively (as $j \to \infty$) to points $a_i, b_i \in \partial D$, and a positive sequence $t_{ij} \to \infty$ (as $j \to \infty$) such that $u_{ij} \cdot t_{ij} = v_{ij}$, and $u_{ij} \cdot t \in D$ for $0 < t < t_{ij}$. It follows from continuity of the flow that $a_i \cdot t, b_i \cdot (-t) \in \overline{D}$ for all $t \geq 0$. Therefore $a \cdot t \to q$

and $b \cdot (-t) \to q$ as $t \to \infty$. We replace a_i and b_i by other points on their orbits, if necessary, so as to have $a_i \cdot t$, $b_i \cdot (-t) \in D$ for all $t \geq 0$. By shrinking D we may assume $a_i \neq b_i$.

Set $H_i = \gamma_+(a_i) \cup p \cup \gamma_-(b_i)$. This is an arc in $\overline{D}$ meeting ∂D in $\{a, b\}$, and separating $\overline{D}$ into two components. The component of $\overline{D} \setminus H_i$ containing the arcs $J_{ij} = u_{ij} \cdot [0, t_{ij}]$, $1 \leq j < \infty$, is called U_i. The boundary of U_i is the Jordan curve $H_i \cup K_i$ where $K_i \subset \partial D$ is an arc joining a_i and b_i.

Because each arc J_{ij} separates U_i, it follows that no forward orbit starting in $H_i \setminus \{a, b_i\}$ can converge to q in U_i; each such forward orbit must exit $\overline{D}$. This makes U_i a *hyperbolic sector* at q (relative to the disk D) in the sense of (Hartman 1964). And it implies that U_i and U_l have disjoint interiors if $i \neq l$. Because Hartman proved that the number of sectors at q (relative to D) is finite, we have reached a contradiction. Therefore there cannot be infinitely many components of P. This completes the proof of 4.7. $\square$

The closure $\bar{P}$ of the set of nonstationary periodic orbits also has a finite number of components, each component Q being a torus, Klein bottle, or the closure of an open annulus or open Möbius band. The closures of the latter two kinds of sets we call the *pieces* of Q.

The combinatorics of how the pieces of Q fit together can be fairly complicated; but the topology of Q, and hence of $\bar{P}$, seem to be comparatively simple. We conjecture that Q is triangulable. We shall need the following results:

Proposition 4.8. *(a): Each component Q of $\bar{P}$ is path connected. (b): Let $A \subset P$ be a component which is an open annulus or Möbius band A. Then each component of $\partial_L A$ is path connected.*

Proof Clearly A is path connected, and it is easy to see that there is a path joining a point of A to a point of $\partial_L A$. Now $\partial_L A$ is a compact invariant set disjoint from P, and so $\Phi | \partial_L A$ is a simple flow by Proposition 4.6. Therefore Corollary 3.3 implies $\partial_L A$ is path connected. $\square$

Proposition 4.9. *Assume Hypothesis 1.3. Let $x, y \in \partial P$ be two points in the same component of $\bar{P}$, and also in the same path component of $S \setminus P$. Then x, y are in the same component of ∂P.*

The rest of this section is devoted to the proof. We can ignore components of P that are tori or Klein bottles, since there can only be one such component, equal to S. Henceforth we assume every component of P is either an annulus Möbius band.

Each component of P is foliated by periodic orbits Γ. In an annular component A these orbits all isotopic. and each one separates A; that is, $A \setminus \Gamma$ has two components. In a Möbius component M there is a unique *central orbit* which reverses orientation, and does not separate M; all others preserve orientation and separate.

Lemma 4.10. *Let C a component of P and S_0 a connected open set, $C \subset S_0 subset S$.*

(a): *Suppose C is an annulus containing a periodic orbit Γ. Then $S_0 \setminus C$ has exactly two components, each containing exactly one component of $S_0 \setminus \Gamma$.*

(b): *Suppose C is a Möbius band. Then $S_0 \setminus C$ is connected.*

Proof (a) By hypothesis Γ disconnects S, so it also disconnects S_0. From topology (e.g. Hirsch 1976, Chapter 4, Lemma 4.4) we learn that $S_0 \setminus \Gamma$ has exactly

two components, each having boundary C, and this property implies that each component of $S_0 \setminus C$ lies in exactly one component of $S \setminus \Gamma$.

Now let N be an open tubular neighborhood of Γ whose closure is compact in C. Then it is easy to see that there is a retraction of $S_0 \setminus \Gamma$ onto $S_0 \setminus N$. It follows that $S_0 \setminus N$ has just two components, each contained in one of the two components of $S_0 \setminus \Gamma$. Let $\{N_i\}$ be an increasing nested family of open tubuluar neighborhoods of Γ having compact closures in C, and whose union is C. Denote the two components of $S_0 \setminus N_i$ by L_i, R_i so that $\{L_i\}$ and $\{R_i\}$ are increasing nested families. Now

$$S_0 \setminus \Gamma = \bigcap_i (S_0 \setminus N_i)$$
$$= \bigcap_i L_i \cup \bigcap_i R_i.$$

Each of the sets $\bigcap_i L_i$, $\bigcap_i R$, being the decreasing intersection of compact connected sets, is connected; and they are disjoint because L_i, R_j are disjoint.

(b) The proof is similar, N_i now being a Möbius band with central orbit Γ. Then $N_i \setminus \Gamma$ is connected, so that $S_0 \setminus C$ is represented as the decreasing intersection of compact connected sets $\bigcap_i (S_0 \setminus N_i)$. $\square$

Let $\{C_j\}$ be a collection of components of P. Set $Q_0 = \bigcup C_j$.

Lemma 4.11. *Let $S_0 \subset S$ be an open neighborhood of Q. If two points are separated in S_0 by Q_0, they are separated by some C_j, which must be an annulus.*

Proof By induction on the number ν of C_j, the case $\nu = 1$ following from Lemma 4.11. Suppose $\nu > 1$. For each j let $\Gamma_j \subset C_j$ be an orbit, central if C_j is a Möbius band. Set $S_1 = S_0 \setminus \bigcup_{j=1}^{\nu-1} \Gamma_j$ Let x, y be in different components of $S_0 \setminus Q_0$. If x, y are in in different components of S_1, then by the induction hypothesis they are separated by some annulus C_j, $1 \le j \le \nu - 1$.

For the rest of this proof, assume x, y are in the same component S_2 of S_1. If $\Gamma_\nu \not\subset S_2$, then C_ν is disjoint from S_2, and thus S_2 is a connected subset of S_0 containing x, y.

On the other hand, suppose $\Gamma_\nu \subset S_2$. Then we apply Lemma 4.10 to $C_\nu \subset S_2$: (a) If C_ν is an annulus, there are just two components of $S_2 \setminus C_\nu$, each containing one one of the points x, y, and each contained in one of the two components of $S_0 \setminus C_\nu$. Thus x, y are in different components of $S_0 \setminus \Gamma_\nu$. (b) If C_ν is a Möbius band then x, y are in the same component of $S_2 \setminus C_\nu = S_0 \setminus Q_0$, contrary to hypothesis. $\square$

Lemma 4.12. *Let Q be a component of $\bar{P}$. If x, y are in distinct components of ∂Q, then there is an annular component A of $Q \cap P$ such that x, y are in distinct components of $S \setminus \Gamma$, for any orbit $\Gamma \subset A$.*

Proof Assume no such Γ separates x, y in S. Then by Lemma 4.11, no component A of $Q \cap P$ separates them.

Choose a decreasing family of compact subsets $K_i S$, each having connected interior S_i containing Q, and with $\bigcap K_i = P$. Set $Q_0 = Q \cap P$. By Lemma 4.11, x, y are in the same component of $S_i \setminus Q_0$, hence of $K_i \setminus Q_0$. Therefore they are in the same component of $\bigcap (K_i \setminus Q_0) = \partial P$. $\square$

Proof of Proposition 4.9 Let Q be the component of $\bar{P}$ which contains x, y. Set $Q_0 = Q \cap P$. Let $f : [0, \pi] \to Q$ be a path from $x = f(0)$ to $y = f(\pi)$. Let $g : [\pi, 2\pi] \to S \setminus P$ be a path from $y = g(\pi)$ to $x = g(2\pi)$. Fit f and g fit together

to determine a map $h : S^1 \to S$. Let Γ be an orbit in an annular component A of Q_0. *The mod 2 intersection number* $\#(h, \Gamma)$ *is* 0, because Γ separates S (compare Chapter 4, Section 4 of Hirsch 1976). Since the image of f is disjoint from Γ, it follows that $\#(g, \Gamma) = 0 (\text{mod } 2)$.

This means that g can be perturbed slightly (but not at the endpoints) so as to meet Γ in an even number of points (or be disjoint from Γ), crossing Γ at each point. From this it follows that the end points x, y of the arc g are in the same component of $S \setminus \Gamma$.

Lemma 4.12 now implies that x, y are in the same component of ∂P. $\square$

5. Internal Chain Recurrence in Surface Flows.

Proof of Theorem 1.6.

Lemma 5.1. $\partial_L(P)$ *is internally chain recurrent.*

Proof First remark that P being open in L (corollary 4.4) we have $\partial_L(P) = \text{clos}_L(\text{P}) \setminus \text{Int}_L(\text{P}) = \overline{\text{P}} \setminus \text{P}$ Let $p \in \partial_L(P)$, we have $p = \lim p_n$ for some sequence $p_n \in P$. Let γ_n be the periodic orbit which contains p_n. As γ_n is a compact subset of L, we may extract from the sequence $\{\gamma_n\}$ a subsequence which converge toward a compact set C for the Hausdorff metric in L. By reindexing, we assume this subsequence is again $\{\gamma_n\}$.

We claim that $C \subset \partial_L(P)$. As $\gamma_n \subset P$, $C \subset \overline{P}$. Suppose there exists $x \in C \cap P$. Then according to Proposition 4.3 there exists a closed neighborhood of x, $N_x = I_x \cdot \mathbf{R} \subset P$ consisting of periodic orbits. It follows that $\gamma_n \subset N_x$ for n large enough. Therefore $p = \lim p_n \in P$. But this contradicts the assumption that $p \in \overline{P} \setminus P$.

To conclude the proof of the lemma, we now show that for all $T > 0$, $\epsilon > 0$ there exists a (T, ϵ)-chain from p to p of orbit intervals *contained in* $\partial_L(P)$. By uniform continuity of the flow on L, there exists $0 < \alpha < \epsilon/2$ such that $d(x, y) \leq \alpha$ implies $d(\Phi_t(x), \Phi_t(y)) \leq \epsilon/2$ uniformly in $0 \leq t \leq 2T$. Choose n large enough such that $1/n \leq \epsilon/2$ and $d(\gamma_n, C) \leq \alpha$. Since $p_n \in \gamma_n$ and γ_n is periodic, we may always assume that $p_n = \Phi_{T_n}(p_n)$ for some $T_n > T$. We write $T_n = k_n t_n$ for some integer $k_n \geq 1$ and $T \leq t_n < 2T$.

Define
$$p_{i,n} = \Phi_{i \cdot t_n}(p_n), \ i = 0, \ldots, k_n.$$
Notice that $p_{0,n} = p_n = p_{k_n,n}$. Since $d(\gamma_n, C) \leq \alpha$ we can construct a finite periodic sequence
$$p_i^* \in C, \ i = 0, \ldots k_n$$
such that $p_0^* = p = p_{k_n}^*$ and $d(p_i^*, p_{i,n}) \leq \alpha$, $i = 0, \ldots, k_n$. We have
$$d(\Phi_{t_n}(p_i^*), p_{i+1}^*) \leq d(\Phi_{t_n}(p_i^*), \Phi_{t_n}(p_{i,n})) + d(\Phi_{t_n}(p_{i,n}), p_{i+1}^*)$$
$$\leq \epsilon/2 + d(p_{i+1,n}, p_{i+1}^*) \leq \epsilon/2 + \alpha \leq \epsilon.$$
As $p_i^* \in C \subset \partial_L(P)$ and $\partial_L(P)$ is invariant, this proves that there exists in $\partial_L(P)$ a (T, ϵ)-chain from p to p. $\square$

Corollary 5.2. $\partial_L P$ *is internally strongly chain recurrent.*

Proof The flow in $\partial_L(P)$ is simple by Proposition 4.6. The flow in $\partial_L(P)$, being chain recurrent by Lemma 5.1, is strongly chain recurrent by Corollary 3.3. $\square$.

Corollary 5.3. *Each component of $\partial_L P$ is path connected.*

Proof   Follows from Corollaries 5.2 and 3.3. □

Theorem 1.6(a) follows from Corollary 5.2.

Proposition 5.4. *Under Hypothesis 1.3, $L \setminus P$ is internally chain recurrent.*

We argue by contradiction, starting from the assumption that $\Phi|(L \setminus P)$ is not chain recurrent. We first prove:

Lemma 5.5. *If $\Phi|(L \setminus P)$ is not chain recurrent, then there is a component U of $L \setminus P$ which contains two disjoint components of $\partial_L P$ belonging to the same component of $\overline{P}$.*

Example 2 illustrates this lemma.

Proof   We may assume some component U of $\Phi|L \setminus P$ is not chain recurrent. Then U contains a dual attractor-repellor pair (Z, Z^*) for the flow in U, and $Z \subset U$ is a proper attractor. Let $B(Z) \subset L \setminus P$ be the basin of attraction of Z for the flow in $U \setminus P$. Then $Z^* = L \setminus (P \cup B(Z))$.

Since there are no proper attractors or repellors for the chain recurrent flow $\Phi|L$, it follows that the compact invariant sets $Y = Z \cap \partial_L P$ and $Y^* = Z^* \cap \partial_L P$ are nonempty, and form a dual attractor-repellor pair for $\Phi|\partial_L P$. But this flow too is chain recurrent (Lemma 5.1). Therefore the disjoint sets Y, Y^* are each unions of components of $\partial_L P$.

Fix any two points $y_0 \in Y, y_1 \in Y^*$. Then $y_0 \rightsquigarrow y_1$ in the internally chain recurrent set L. We assert that also $y_0 \rightsquigarrow y_1$ in $\overline{P}$.

To see this, observe that since Z is an attractor for the flow in $L \setminus P$, there is a small $\epsilon > 0$ and a large $T > 0$ such that no (T, ϵ)-chain in $L \setminus P$ leads from a point in Z to a point on $\partial_{L \setminus P} B(Z)$. From this it follows that $y_0 \rightsquigarrow y_1$ in $\overline{P}$; and this implies y_0, y_1 are in the same component of $\overline{P}$. □

To prove Proposition 5.4 we make the following simplifying assumptions, whose justifications are either obvious or similar to earlier arguments. We assume that L is connected. Therefore we may also assume, by Theorems 4.7 and 4.1, that P has only finitely many components. And we assume that no component of P is a torus or Klein bottle— for such a component would be all of S and there is nothing to prove.

Now observe that by Proposition 4.9, every component of $L \setminus P$ meets every component of $\overline{P}$ in a single component of $\partial_L P$. This contradicts Lemma 5.5; Proposition 5.4 follows. By Corollary 4.4(a) the invariant set $L \setminus P$ is compact. By Proposition 5.4, the flow $\Psi = \Phi|(L \setminus P)$ is chain recurrent. This flow is simple by Hypothesis 1.2. Therefore Ψ is strongly chain recurrent by Theorem 3.1. Theorem 1.6(b) follows. □

Proof of Theorem 1.4. It suffices to prove that each of the two invariant subsets P and $L \setminus P$ are internally strongly chain recurrent. Since every point of P is periodic, P is internally strongly chain recurrent. For $L \setminus P$ we use Proposition 5.4 and Corollary 3.2. □

Proof of Theorem 1.1. First we verify Hypothesis 1.2(H3) for a planar flow: .

Lemma 5.6. *Let M be an internally chain recurrent set for an arbitrary planar flow. Then every alpha or omega limit point of any nonperiodic orbit in M is an equilibrium.*

Proof Assume X is an open subset of the plane, $M \subset X$ is internally chain recurrent, and $p \in M$ is nonperiodic. Suppose there is a nonstationary point q in $\omega(p)$. A theorem of Whitney (1933) shows that there exists a *local section*[5] $J \subset S$ through q: this means J is a compact arc containing q in its interior, and for some $\epsilon > 0$, the map

$$h = \Phi|[-\epsilon, \epsilon] \times J : [-\epsilon, \epsilon] \times J \to S$$

is a homeomorphism onto a neighborhood of q. This map (and sometimes its image) is called a *flow box* at q.

Since q is an omega limit point of p the forward trajectory of p meets J infinitely often.

Let $t_0 > 0$ be such that $\Phi_{t_0}(p) \in J$ and define $t_1 = \inf\{t > t_0 : \Phi_t(p) \in J\}$. Let $(0, x_0) = h^{-1}(\Phi_{t_0}(p))$ and $(0, x_1) = h^{-1}(\Phi_{t_1}(p))$. The set

$$C = h(0 \times [x_0, x_1]) \bigcup \{\Phi_t(p) : t_0 < t < t_1\}$$

is a Jordan curve. Let K be the compact set whose interior is the unbounded component of $\mathbf{R}^2 \setminus C$, so that the boundary of K is C. It is clear from the definition of R that no trajectory can leave K. Moreover, if $T > t_1 - t_0$ then $\Phi_T(K \cap L)$ lies in the interior relative to L of $K \cap L$. Therefore by Proposition 3.6(ii) there is an attractor $A \subset \text{Int}(K)$. But implies that $p \in A$. As $p \notin K$, this attractor is proper, contradicting Proposition 3.6(iv). Therefore q is necessarily an equilibrium; and similarly for alpha limit points of p. The conclusion of Lemma 5.6 now follows since the equilibria are isolated and compact limit sets are connected. $\square$

Theorem 1.1 now follows from Lemma 5.6 and Theorem 1.4. $\square$.

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