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ABSTRACT

Certain asymptotic conditions for $l \to \infty$ are discussed, under which the analytic continuation of the S matrix in angular momentum is unique.

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Regge has proved from the Schrödinger equation with a particular class of potentials (a superposition of Yukawa potentials),

$$V(r) = \int_{m>0}^{\infty} \sigma(\mu)[\exp(-\mu r)/r]\, d\mu,$$

that it is possible to construct a function $a(E,\ell)$ of the energy E and the complex variable ℓ with the following properties;[1]

(a) It agrees with $a_n(E)$, the coefficients of the partial-wave expansion of the scattering amplitude, for $\ell = n$ (integer).

(b) It is meromorphic in the region $\mathrm{Re}\,\ell > -1/2$.

(c) It is represented asymptotically (for $\ell \to \infty$) by an exponential in every angle $(-\frac{\pi}{2}+\epsilon, \frac{\pi}{2}-\epsilon)$, where ϵ is an arbitrarily small positive number.

This analytic continuation of the scattering amplitude in angular momentum is important because its poles have a simple physical meaning (namely, they are connected with the bound states and the resonances) and are related to the problem of the subtractions in the Mandelstam representation.

Chew, Frautschi, and Mandelstam have conjectured that a similar continuation is also possible in the relativistic case[2] and have solved in this way the problem of the undertermined subtractions in the S-matrix theory. However, in the relativistic case there is no Schrödinger equation that provides the desired particular continuation, so it is important to study whether conditions like (a), (b), (c) are sufficient in themselves to characterize uniquely the function $a(E,\ell)$, if it is assumed that it exists.

We shall prove that if a function $a(\ell)$ exists which assumes given values a_n for ℓ integer, which is meromorphic to the right of a certain vertical line r in the complex plane, and which for $|\ell|$ sufficiently large satisfies the condition (uniformly in all directions)

$$|a(\ell)| < e^{-\tau \operatorname{Re} \ell} \quad \text{for} \quad \tau > 0, \tag{1}$$

then it is unique.

A proof of the same statement has been given by Squires on the grounds that the poles of $a(\ell)$ determine the asymptotic behavior in the momentum transfer of the scattering amplitude.[3] Our proof is, however, mathematically more direct, does not require any explicit reference to the possibility of Watson's transformation, and provides a more general prescription for the asymptotic behavior of $a(\ell)$ that makes the continuation unique. These enlarged prescriptions are not physically immediately significant, but may turn out to be useful for the problem in general and eventually for particular developments.

Our proof rests on a lemma that I shall give at present in a rather restrictive form:

A function $f(\ell)$, which is holomorphic to the right of a certain vertical line r and satisfies for sufficiently large $|\ell|$ the condition (uniformly in all directions)

$$|f(\ell)| < e^{-\sigma|\ell|} \quad \text{for} \quad \sigma > 0, \tag{2}$$

is zero everywhere.

Let us suppose for convenience that r is at the left of the imaginary axis. The function $f(i\zeta)$ of the real variable ζ can be represented by a Fourier integral

$$f(i\zeta) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ik\zeta}\, \phi(k) dk, \tag{3}$$

being $\phi(k)$ given by

$$\phi(k) = \frac{1}{i} \int_{-i\infty}^{+i\infty} e^{k\ell}\, f(\ell)\, d\ell \,.$$

The function $\phi(k)$ is analytic in the k plane in the strip $|\mathrm{Im}\, k| < \sigma$ which includes the whole k real axis. Since $f(\ell)$ is holomorphic everywhere, we have

$$\int_{-iR}^{iR} e^{k\ell}\, f(\ell)\, d\ell = \int_{C(R)} e^{k\ell}\, f(\ell)\, d\ell \,, \tag{4}$$

where $C(R)$ is the semicircle of radius R and center in the origin which lies at the right of the imaginary axis. The right-hand term in (4) goes to zero for $R \to \infty$, if the condition $|k| < \sigma$ is satisfied. Therefore $\phi(k)$ is zero in a circle of the k plane belonging to its analyticity domain, and then it is zero everywhere. Then $f(\ell)$ is zero on the imaginary axis and, since it is holomorphic, is zero everywhere.

The lemma now given is susceptible of the immediate generalizations: (a) we may admit for $f(\ell)$ a finite number of poles; in fact, in this case

$$f(\ell) = \frac{g(\ell)}{\prod_{k=1}^{n} (\ell - \ell_k)},$$

where $g(\ell)$ is an holomorphic function, which must satisfy a condition like (2).

(b) We may admit that the condition (2) is violated in certain neighborhoods of a succession of points off the imaginary axis; these neighborhoods are so small that it is possible to construct a succession of semicircles $C(R_n)$ with $R_n \to \infty$, over which the (2) holds without exceptions.

Let us suppose now that $a(\ell)$ is a solution of our problem and $a'(\ell)$ a second solution. Since $a'(\ell)$ and $a(\ell)$ must be equal for ℓ integer, we have

$$a'(\ell) = a(\ell) + f(\ell) \sin \pi\ell \,, \tag{5}$$

where $f(\ell)$ is a function meromorphic to the right of r (obviously without poles at the integers). Obviously, for sufficiently large $|\ell|$,

$$|f(\ell)| \times |\sin \pi\ell| \leq |a'(\ell)| + |a(\ell)| \leq e^{-\tau' \operatorname{Re} \ell} \,. \tag{6}$$

On the other side it is easy to see that it is possible to construct appropriate neighborhoods of the integers out of which we have, for instance,

$$|\sin \pi\ell| > \frac{1}{4} e^{\pi |\operatorname{Im} \ell|} \,.$$

It follows that, for sufficiently large ℓ,

$$|f(\ell)| \leq \frac{1}{4} e^{-\tau' |\ell| \, |\cos \phi| - \pi |\ell| \, |\sin \phi|} \leq e^{-\sigma |\ell|} \,,$$

$$\phi = \arg \ell \,. \tag{7}$$

If we observe now that the condition (1) prevents the possibility of an infinite number of poles for $a(\ell)$, we have $f(\ell) \equiv 0$, $a'(\ell) \equiv a(\ell)$.

If we relax the condition (1) and we ask that it be verified not uniformly in the entire half plane to the right of r but only inside an angle (centered on the real axis) of the form $(-\frac{\pi}{2} + \epsilon, \frac{\pi}{2} - \epsilon)$, then $a(\ell)$ may have an infinite number of poles

$$f(\ell)=\left[\ell^n \Big/ \prod_{k=1}^{n} (\ell-\ell_k)\right] g(\ell) ,$$

where $g(\ell)$ is holomorphic. We may see then that our considerations that follow (5) may be repeated if the condition

$$|a(\ell)| < |\ell|^q e^{\rho|\ell|\,|\sin\phi|}, \quad \text{for} \quad 0 < \rho < \pi, \tag{8}$$

is satisfied for large ℓ.

We may now observe that if we know that a meromorphic continuation from a_n's exists satisfying a required asymptotic condition, then any other condition for which it is possible to establish a very weak asymptotic boundary such as (8) must be equal to the correct one [vice versa, if a continuation exists satisfying (8) and violating, for instance (1), no continuation satisfying (1) can exist]. This observation may eventually be useful in problems of extension of the analyticity domain of a certain $a(\ell)$.

To conclude, we recall that a continuation for the amplitude of scattering in angular momentum has been proposed by Froissart,[4] also in the relativistic case. This continuation, apart from the inessential complication of the so-called j-parity, may be written with the usual notations:

$$a(\ell, s) = (1/2\pi p^2)\int_{m^2}^{\infty} Q_\ell(1 + t/2p^2)\, D(s,t)\, dt . \tag{9}$$

Equation (9) satisfies Eq. (1) and is completely holomorphic to the right of a certain line $\mathrm{Re}\,\ell = N$; the eventual proof (starting from the principles of the S-matrix theory) that the continuation of (9) is meromorphic in the whole physically interesting region is an actual task. The other continuation, outside every such angle. If this is the case and if a second vertical line r' exists, to the right of which only a finite number of poles falls and the condition (1) is again uniformly satisfied, then the continuation remains unique; the position of the line r is in fact irrelevant for our considerations, and these may be applied again to the right of r'. If instead the infinite poles of $a(\ell)$ do not remain to the left of a certain vertical line (as it happens if they lie, for instance, on a parabola which has for its axis the imaginary axis), then it does not seem possible to say anything in general.

Another consequence of the fact that the boundary of the region, in which our considerations may be applied, can be displaced arbitrarily toward the right is that the values $a_0 a_1 \cdots a_p$, assumed by $a(\ell)$ for ℓ equal $0, 1, \cdots p$, are completely determined from the values assumed for $\ell = p + 1, p + 2 \cdots$. It follows that only for particular assignments of the values a_n's we may expect that an analytic continuation that satisfies the required conditions exists, even if such values are in agreement with Eq. (1). This illustrates the dynamical content of the property of the analyticity of the S matrix in angular momentum.

From the mathematical point of view the condition (1) is clearly very particular; it corresponds to the case in which a finite ellipse exists for the convergence of the partial-wave expansion of the scattering amplitude. Physically, the most interesting case that violates condition (1) is represented by the Coulomb potential. To have a more general asymptotic condition (that makes the continuation unique) we observe that in our lemma the condition (2) may be replaced by

$$|f(\ell)| < \mu(|\ell|)\, e^{-\sigma|\ell|\;|\sin\phi|}\,, \qquad (2)$$

where $\mu(|\ell|)$ is an arbitrary function that goes to zero for $|\ell| \to \infty$. This is true for $f(\ell)$ either holomorphic or meromorphic; the latter is reduced to the former by

$$b(\ell, s) = \frac{1}{2} \int_{-1}^{1} A(s, \cos\theta)\, P_\ell(\cos\theta)\, d\cos\theta, \tag{10}$$

which could seem natural, violates in general condition (1) and obviously violates even condition (8).

Incidentally, the possibility of establishing, for $b(s, \ell)$, the boundary $|b(s,\ell)| < e^{\pi|\mathrm{Im}\,\ell|}$ (for large ℓ), which is the closest possible to (8), seems to indicate that (8) is in fact very general eral possible asymptotic condition that makes the continuation unique.

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FOOTNOTES AND REFERENCES

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