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Inverse Problems
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by

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ABSTRACT OF THE DISSERTATION

Inverse Problems in Mean Field Games

by

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Doctor of Philosophy in Mathematics
University of California, Los Angeles, 2023
Professor Wilfrid Dossou Gangbo, Chair

In this thesis, we propose a new class of inverse problems to recover Lagrangians in Mean Field Games from boundary data. We present strategies to address these problems when the Lagrangian we are searching for is assumed to be analytic. Our study can be viewed as an extension of inverse problems from Riemannian manifolds to infinite-dimensional metric spaces, such as the Wasserstein space, which possess differential structures. It can also be regarded as an infinite-dimensional version of the travel time tomography problem. The application of our inverse problem is to learn the rules governing people's migration when we have limited knowledge of their movements at the boundary.

The dissertation of Weiyi Liu is approved.

Christina Kim

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University of California, Los Angeles

2023

To my dearest

family and friends

for their consistent support

throughout years as

I chase my dream

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CHAPTER 1

Introduction

1.1 Some background of the Mean Field Games Theory

In this project, we propose a new class of inverse problems to recover Lagrangians in Mean Field Games from boundary data. We present strategies to address these problems when the Lagrangian we are searching for is assumed to be analytic.

Mean Field Games Theory is a theory of strategic decision-making in differential games played by large populations with small interactions. In this theory, each player acts based on his or her own optimization, like minimizing the cost or maximizing the benefits of the game, taking into account the decisions of other players. The term “mean field” was first introduced in statistical mechanics, where the number of particles tends to infinity to approximate the original model with a simpler one after averaging over degrees of freedom. Under this assumption, one have to consider many components interacting with each other. The mean field models we are interested in involve searching for Nash equilibria when there are infinitely many identical players. At a fixed time, the collection of players is represented by a probability measure μ with a finite second moment, denoted as $\mu \in \mathcal{P}_2(\mathbb{R}^d)$.

Mean Field Games Theory was first studied by Jean-Michel Lasry and Pierre-Louis Lions [19], and independently by P.E. Caines, M. Huang, and R.P. Malhamé [7], [9]. Over the past decades, it has become a popular field of research thanks to the contributions of many mathematicians. For an introduction to recent developments in this field, we refer to [2].

Over the past years, Mean Field Games Theory has found numerous applications in diverse subjects such as economics (see [5], [1], [8]), machine learning (see [5], [25]), robotics (see [22]), crowd motion (see [27]), large population dynamics (see [29], [20]), and public health (see [21], [17]).

There are three mechanisms at work in MFG models, one of which is governed by a Lagrangian function

$$\bar{L} : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}.$$

Another mechanism is induced by the so-called individual noise operator, which ensures that the measures representing the players are absolutely continuous with respect to the Lebesgue measure. The common noise operator induces a third mechanism, transforming the MFG system into a system of stochastic partial differential equations. In this dissertation, we focus solely on the first mechanism, which presents a greater challenge as we have to deal with potentially singular measures. Additionally, we make the simplification that our Lagrangian $\bar{L}$ is separable, meaning it can be expressed in the form

$$\bar{L}(q, \mu, v) = L(q, v) + F^0(q, \mu),$$

where F^0 is the Fréchet derivative of a prescribed function $F : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$.

A typical example is when we are given an interaction potential $\Phi \in C^2(\mathbb{R}^d \times \mathbb{R}^d)$ and

$$\bar{L}(q, \mu, v) = \frac{1}{2}|v|^2 + \int_\Omega \Phi(q, y)\mu(dy).$$

In the general theory of MFG, the players (also referred to as agents) move within a domain Ω contained in $\mathbb{R}^d$ over a prescribed time interval $[0, T]$. The cost of each player is determined not only by its own trajectory but also by the trajectories of all the other players. Once the Lagrangian is fixed, the Hamiltonian H in the game is defined such that $H(q, \cdot)$ is the Legendre transform of $L(q, \cdot)$ for each q. More precisely,

$$H(q, p) = \max_{\xi \in \mathbb{R}^d} \langle p, \xi \rangle - L(q, \xi).$$

The search for Nash equilibria involves studying the following classic system of partial differential equations (PDEs), known as MFG systems. This system is close to our inverse problem in the literature and it is stated as follows.

Given $G : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, we seek to find

$$u : [0,T] \times \mathbb{R}^d \to \mathbb{R} \quad \text{and} \quad \sigma : t \in [0,T] \mapsto \sigma_t \in \mathcal{P}_2(\mathbb{R}^d)$$

such that

$$\begin{cases} \partial_t u(t,q) + H\big(q, \nabla_q u(t,q)\big) = F^0(q,\sigma_t) & \text{in } (0,T) \times \mathbb{R}^d \\ \partial_t \sigma + \nabla_q \cdot \big(\sigma_t \nabla_p H\big(q, \nabla_q u(t,q)\big)\big) = 0 & \text{in } (0,T) \times \mathbb{R}^d \\ \sigma_T = \mu, \quad u(0,q) = G(q,\sigma_0) & \text{in } \quad \mathbb{R}^d \end{cases} \tag{1.1.1}$$

The first equation in (1.1.1), known as the Hamilton-Jacobi equation, is formulated in backward time, and u represents the value function. The second equation in (1.1.1) is a forward-time continuity equation that ensures the conservation of total mass for σ_t over time. In certain MFG models, $\mathbb{R}^d$ is replaced by a bounded domain, denoted as Ω, which is an open connected set, or by the torus $\mathbb{T}^d$ (see, for example, [18] and [6]).

1.2 Our inverse problem

In our case, we would like to study a type of inverse problem which involves making some measurements of a group of people passing through the boundary of a given region. From those measurements, we would like to predict people's behavior inside the region. In other words, we want to find the Lagrangian dictating the movement from boundary measurements.

Mathematically, we postulate that the Lagrangian of the system is $\overline{L}$, where

$$\overline{L}(q,\mu,v) = L(q,v) + F(q,\mu).$$

Our inverse problem is to determine L and F given partial knowledge on the boundary.

In this thesis, we consider a bounded open convex set $\Omega \subset \mathbb{R}^d$ that is of class $C^{1,1}$. We further focus on a class of Lagrangians induced by metrics. Let $0 < a < b < \infty$ be fixed real numbers, and denote by $\mathcal{G}(a, b)$ the set of $g \in C^2(\mathbb{R}^d, \mathbb{R}^{d\times d})$ such that $g_{ij} = g_{ji}$ and the eigenvalues of $g(q)$ are greater than or equal to a but less than or equal to b for all $q \in \overline{\Omega}$. For such g, we define the Lagrangian as

$$L_g(q, v) = \frac{1}{2}\sum_{i,j=1}^{d} g_{ij}(q)v^i v^j, \qquad \forall (q, v) \in \bar{\Omega} \times \mathbb{R}^d.$$

When $\Phi \in C^2(\mathbb{R}^d \times \mathbb{R}^d)$ is a symmetric function, we set

$$F_\Phi(\mu) := \frac{1}{2}\int_{\overline{\Omega}^2} \Phi(q_1, q_2)\mu(dq_1)\mu(dq_2), \qquad \forall \mu \in \mathcal{P}_2(\mathbb{R}^d).$$

The Fréchet derivative of F_Φ is the function F^0_Φ given by

$$F^0_\Phi(q_1, \mu) := \frac{1}{2}\int_{\overline{\Omega}} \Phi(q_1, q_2)\mu(dq_2), \qquad \forall \mu \in \mathcal{P}_2(\mathbb{R}^d).$$

The set $\mathcal{P}_2(\mathbb{R}^d)$ can be replaced by $\mathcal{P}(\overline{\Omega})$, the set of probability measures supported by $\overline{\Omega}$. A prescribed data on the boundary $\partial\Omega$ is a piecewise narrowly continuous path of measures $t \mapsto \psi_t$ such that

$$\psi_t(\partial\Omega) < 1, \quad \forall t \in (0, T) \qquad \text{and} \qquad \psi_0(\partial\Omega) = \psi_T(\partial\Omega) = 1.$$

We define the cost for transporting ψ_0 to ψ_T to be

$$C^F_L(\psi_0, \psi_T) := \inf_{(\sigma,v)} \left\{ \int_0^T \Big(F(\sigma_t) + \int_{\overline{\Omega}} L_g(q, v_t(q))\sigma_t(dq)\Big) dt \right\},$$

where the infimum is performed over the set of (σ, v) such that $\sigma : [0, T] \to \mathcal{P}(\bar{\Omega})$ satisfies some regularity properties which will be specified later. v is the velocity field driving σ. The initial condition is given by $\sigma_0 = \psi_0$, the terminal condition by $\sigma_T = \psi_T$, and the constraint $\sigma_t \geq \psi_t$ holds on $\partial\Omega$. The constraint $\sigma_t \geq \psi_t$ distinguishes our study from previous works in optimal transportation theory.

We have discovered that the dual problem to (3.0.1) involves maximizing a function $\mathcal{J}(\cdot, \cdot, \cdot|\psi)$ expressed in terms of dual functions (u, h, α) as given in (4.1.12). We require u

to be continuous on $[0,T]\times\bar{\Omega}$, but we can only impose the condition that α and h are Borel maps on $[0,T]\times\bar{\Omega}$, since we expect h to be non-negative and equal to zero outside $\partial\Omega$. Unlike the conventional conditions satisfied by dual functions in classical optimal transport theory, the functions (u,h,α) are connected by a more intricate inequality.

$$u(t,\gamma(t))-u(s,\gamma(s))\leq\int_s^t\Big(L_g(\gamma,\dot{\gamma})-h(\tau,\gamma(\tau))+\alpha(\tau,\gamma(\tau))\Big)d\tau$$

for all $0\leq s<t\leq T$ and all $\gamma\in W^{1,\infty}(s,t;\bar{\Omega})$. We demonstrate that $C_L^F(\psi_0,\psi_T)$ is equal to the supremum of $\mathcal{J}(u,h,\alpha|\psi)$ subject to the aforementioned constraints. However, unlike in standard optimal transport problems, it is not expected that this supremum of $\mathcal{J}(u,h,\alpha|\psi)$ is attained except in special cases. This reality significantly increases the challenges we face in our study and leads to a system of variational inequalities that is more complex than (1.1.1). When (u,h,α) is a maximizer, the expression $\mathcal{J}(u,h,\alpha|\psi)$ contains boundary information that is accessible to us and is suitable for setting up an inverse problem. We establish that, by appropriately selecting F and $P_0,P_T\in\partial\Omega$, we can choose $(\psi^\epsilon)_{\epsilon>0}$ such that

$$2\epsilon J\big(u_{\psi^\epsilon},h_{\psi^\epsilon}|\psi^\epsilon\big)=\frac{1}{2}\mathrm{dist}_g^2\big(P_0,P_T\big)+o(\epsilon). \tag{1.2.1}$$

This identity is utilized to recover the metric g, leaving only Φ to be determined. We have access to the values of the functional

$$I_g[\Phi](P_0,P_T,Q_0,Q_T)=\int_0^1\Phi(\gamma_P(t),\gamma_Q(t))dt,$$

where γ_P and γ_Q are constant-speed geodesics joining P_0 to P_T and Q_0 to Q_T, respectively. Unfortunately, we can only recover Φ from I_g when Φ is real analytic. We lack a stability result to extend the recovery to the case where Φ belongs only to the class C^k.

The remaining part of this thesis is organized and recapitulated as follows. Chapter 2 introduces the notations used throughout the project and states some preliminary results. We prove that $C_L^F(\psi_0,\psi_T)$ is produced by a unique minimizer under appropriate convexity conditions on F_Φ and L. For educational purpose and dual problem settings, Chapter 3 focuses on the case with zero potential $F_\Phi = 0$. In Chapter 4, we prove the identity in (1.2.1) and recover the metric g from it. Chapter 5 demonstrates that Φ can be recovered when it is assumed to be real analytic. An appendix containing useful information can be found in Chapter 6.

The first example appeared in the literature, an extensively studied finite dimensional version of the inverse problem, is the travel time tomography problem. The travel time tomography theory provides ways we can estimate the subsurface structure of the Earth from the boundary measurements of travel time of seismic waves.

Our study can be viewed as an extension of inverse problems from Riemannian manifolds to infinite-dimensional metric spaces, such as the Wasserstein space, which possess differential structures. It can also be regarded as an infinite dimensional version of the travel time tomography problem. The application of our inverse problem is to learn the rules governing people's migration when we have limited knowledge of their movements on the boundary. Other related inverse problems, although of a completely different nature, have been studied in [24], and some numerical methods were developed in [10].

1.3 Notations and settings of our MFG problem

In this section, we introduce some basic notations and settings.

Our domain Ω is a bounded open subset of $\mathbb{R}^d$ with $C^{1,1}$ boundary. It is also geodesic convex, which means that for any two points in Ω, there exists a unique minimizing geodesic within Ω that joins those two points.

We denote $\mathcal{M}^+(\Omega)$ as the set of finite Radon measures on $\mathbb{R}^d$ that are supported by Ω. The set $\mathcal{M}^+(\partial\Omega)$ is equipped with the narrow convergence topology.

For any subset $E \subset \mathbb{R}^d$, we use $\mathcal{M}_2(E)$ to denote the set of measures μ on E with finite second moment, i.e., $\int_E |x|^2 d\mu(x) < \infty$.

Given $\mu, \nu \in \mathcal{P}(\bar{\Omega})$, we denote $\Pi(\mu, \nu)$ as the set of measures on $\bar{\Omega}^2$ that have μ and ν as the first and the second marginal respectively. We define the Wasserstein distance $W_2(\mu, \nu)$ as

$$W_2(\mu, \nu) = \inf \left\{ \int_{\Omega\times\Omega} |x-y|^2 d\gamma : \gamma : \Pi(\mu, \nu) \right\}^{\frac{1}{2}}.$$

In the context of a metric space $(\mathbb{S}, \mathrm{dist})$ and a path $\sigma : [0, T] \to \mathbb{S}$, we use σ_t to denote $\sigma(t)$. If there exists $m \in L^2(0, T)$ such that

$$\mathrm{d}ist\big(\sigma_s, \sigma_t\big) \leq \int_s^t m(r)dr, \qquad \forall\, 0 \leq s < t \leq T,$$

we say that σ is 2-absolutely continuous and we write $\sigma \in AC^2(0, T; \mathbb{S})$.

For $\sigma \in AC^2(0, T; \mathbb{S})$, we also define the limit

$$|\sigma'|(t) := \lim_{s\to t} \frac{dist(\sigma(s), \sigma(t))}{|s-t|}.$$

The study of $AC^2(0, T; \mathbb{S})$ when $\mathbb{S} = \mathcal{P}_2(\mathbb{R}^d)$ is discussed in [3]. It is shown that $\sigma \in AC^2(0, T; \mathcal{P}_2(\mathbb{R}^d))$ if and only if there exists a Borel velocity field $v : (0, T) \times \mathbb{R}^d \to \mathbb{R}^d$ such that the continuity equation

$$\partial_t \sigma + \nabla \cdot (v\sigma) = 0 \quad \text{ on } \mathcal{D}'\big((0, T) \times \mathbb{R}^d\big).$$

is satisfied in the distribution sense. In this case, we say v is the velocity field driving σ.

We define the set of data $\mathcal{S}$ as follows:

$$\mathcal{S} := \Big\{ f \in C\big([0, T]; \mathcal{M}^+(\partial\Omega)\big),\ f_t \geq 0, \int_{\partial\Omega} f_t(dq) \leq 1,\ f_0(\partial\Omega) = f_T(\partial\Omega) = 1 \Big\}.$$

Given $\mu, \nu \in \mathcal{P}(\bar{\Omega})$, we denote $\Sigma_T(\mu, \nu)$ as the set of pairs (σ, v) such that $\sigma \in AC^2(0, T; \mathcal{P}_2(\mathbb{R}^d))$ and $v : (0, T) \times \overline{\Omega} \to \mathbb{R}^d$ is a Borel vector field satisfying

(i)

$$\sigma_0 = \mu,\ \sigma_T = \nu, \quad \text{supp } (\sigma_t) \subset \overline{\Omega}, \quad \forall t \in [0, T]. \tag{1.3.1}$$

(ii)

$$\partial_t \sigma + \nabla \cdot (v\sigma) = 0 \quad \text{on } \mathcal{D}'\big((0,T)\times\mathbb{R}^d\big). \tag{1.3.2}$$

Given $\psi \in \mathcal{S}$, we denote by $\Sigma_T(\mu,\nu|\psi)$ the set of $(\sigma, v) \in \Sigma_T(\mu,\nu)$ such that

$$\sigma_t|_{\partial\Omega} \geq \psi_t \quad \text{on } [0,T]\times\partial\Omega.$$

Notice that since ψ_0 and ψ_T are probability measures on $\partial\Omega$, $(\sigma, v) \in \Sigma_T(\mu,\nu|\psi)$ implies that

$$\sigma_0 = \mu = \psi_0, \quad \sigma_T = \nu = \psi_T.$$

Given a symmetric function $\Phi \in C^2(\mathbb{R}^d\times\mathbb{R}^d)$ we define

$$F_\Phi(\mu) := \frac{1}{2}\int_{\overline{\Omega}^2} \Phi(q_1,q_2)\mu(dq_1)\mu(dq_2).$$

We always assume that the functional F_Φ is strictly convex on $\mathcal{P}(\overline{\Omega})$.

For a given Lagrangian $L : \overline{\Omega}\times\mathbb{R}^d \to \mathbb{R}$ and a function $F : \mathcal{P}(\overline{\Omega}) \to \mathbb{R}$, we define

$$\mathcal{A}_L^F[\sigma, v] := \int_0^T \left(F(\sigma_t) + \int_{\overline{\Omega}} L(q, v_t(q))\sigma_t(dq)\right) dt,$$

and refers to $\mathcal{A}_L^F$ as an action functional. The minimizers (σ, v) of $\mathcal{A}_L^F$ over $\Sigma_T(\mu,\nu|\psi)$ are formally characterized by the following given system of partial differential equations, where $u, \alpha \in C([0,T]\times\bar{\Omega})$, $h : [0,T]\times\bar{\Omega} \to [0,\infty]$ are Borel maps.

$$\begin{cases} \partial_t u(t,q) + h(t,q) + H\big(q, \nabla u(t,q)\big) = \alpha, \text{ in } (0,T)\times\overline{\Omega} \\ \partial_t \sigma + \nabla\cdot\big(\sigma D_p H(q, \nabla u(t,q))\big) = 0 \\ \alpha_t = \partial_\sigma F(\sigma_t) \\ \sigma_t = \psi_t, \ h_t - \text{a.e.} \end{cases} \tag{1.3.3}$$

CHAPTER 2

Forward Problem

In this chapter, we study the forward problem by first working on the following optimization problem.

Given $\psi \in \mathcal{S}$, we define the cost for transporting ψ_0 to ψ_T to be

$$C_L^F(\psi_0, \psi_T) := \inf_{(\sigma,v)} \mathcal{A}_L^F[\sigma, v],$$

where the infimum is performed over the set of $(\sigma, v) \in \Sigma_T(\psi_0, \psi_T|\psi)$.

We prove the existence and the uniqueness of the minimizer to the above problem.

2.1 Preliminaries

Throughout this chapter, we make the following assumptions about general L. Notice that here L does not rely on any metric g. We would like to first establish some general lemmas for our future application.

Assume

$$H, L \in C^3(\mathbb{R}^d \times \mathbb{R}^d), \quad L \geq 0, \tag{2.1.1}$$

such that $L(q, \cdot)$ and $H(q, \cdot)$ are Legendre transforms of each other for any $q \in \mathbb{R}^d$. We assume there exist constants κ and κ_0 such that

$$D_{vv}^2 L \geq \kappa I_d, \quad D_{pp}^2 H > 0, \tag{2.1.2}$$

and

$$DH, \quad DL \quad \text{are} \quad \kappa_0\text{-Lipschitz.} \tag{2.1.3}$$

Thus, for all $(q_1, q_2), (q_1', q_2') \in \mathbb{R}^d \times \mathbb{R}^d$, we have

$$|DH(q_1, q_2) - DH(q_1', q_2')| \leq \kappa_0 \sqrt{(q_1 - q_1')^2 + (q_2 - q_2')^2}.$$

We further assume that there exist $\lambda_1 > 0$ and $\lambda_0 < 0$ such that

$$\lambda_1 |\xi|^2 + \lambda_0 \leq L(q, \xi) \leq \lambda_1^{-1} |\xi|^2 - \lambda_0. \tag{2.1.4}$$

By duality, we have

$$\frac{1}{4}\lambda_1 |p|^2 + \lambda_0 \leq H(q, p) \leq \frac{1}{4}\lambda_1^{-1} |p|^2 - \lambda_0. \tag{2.1.5}$$

In the case when L is given by a metric g as we mentioned in Chapter 1, the above settings of L are easily satisfied.

We still assume that F comes from a symmetric function $\Phi \in C^2(\mathbb{R}^d \times \mathbb{R}^d)$ and set

$$F = F_\Phi(\mu) := \frac{1}{2} \int_{\overline{\Omega}^2} \Phi(q_1, q_2) \mu(dq_1) \mu(dq_2), \qquad \forall \mu \in \mathcal{P}(\mathbb{R}^d).$$

Notice that Φ is bounded as $\overline{\Omega}$ is compact. We assume F is convex on $\mathcal{P}(\mathbb{R}^d)$.

2.2 Kinetic formulation

Recall in Chapter 1 that given $\psi \in \mathcal{S}$, we denote by $\Sigma_T(\sigma_0, \sigma_T | \psi)$ the set of (σ, v) satisfies

$$supp(\sigma_t) \subset \overline{\Omega}, \qquad \forall t \in [0, T]. \tag{2.2.1}$$

$$\sigma_0 = \psi_0, \quad \sigma_T = \psi_T \quad \text{on } \partial\Omega \tag{2.2.2}$$

$$\sigma_t|_{\partial\Omega} \geq \psi_t \quad \text{on } [0, T] \times \partial\Omega. \tag{2.2.3}$$

$$\partial_t \sigma + \nabla \cdot (v\sigma) = 0 \quad \text{on } \mathcal{D}'\big((0, T) \times \mathbb{R}^d\big). \tag{2.2.4}$$

We will enlarge $\Sigma_T(\sigma_0, \sigma_T | \psi)$ to a bigger set of f, which describes kinetic movement.

We denote that

$$C := [0,T] \times \overline{\Omega} \times \mathbb{R}^d.$$

Let $\mathcal{M}_2(C)$ be the set of signed Borel measures f on $\mathbb{R}^{2d+1}$ which are supported by C. Note that $\mathcal{M}_2(C)$ is a topological vector space when endowed with the narrow convergence topology. The set $\mathcal{M}_2^+(C)$ consists of non–negative elements f of $\mathcal{M}(C)$ such that

$$\int_C |\xi|^2 f(dt, dq, d\xi) < \infty.$$

The kinetic formulation provides us with a linearized problem for which it is easier for us to identify the dual problem.

In kinetic formulation, the transport equation of the measure is given by

$$\int_{\mathbb{R}^d} \Big(u_T(q)\sigma_T(dq) - u_0(q)\sigma_0(dq) \Big) = \int_C \big(\partial_t u(t,q) + \langle \xi, \nabla u(t,q) \rangle \big) f(dt, dq, d\xi) \tag{2.2.5}$$

for any $u \in C_c^\infty(\mathbb{R}^{d+1})$

We denote by $\overline{\mathcal{F}}(\sigma_0, \sigma_T)$ the set of measures f in $\mathcal{M}_2^+(C)$ satisfying (2.2.5).

A sufficient condition for (2.2.5) to hold is

$$\partial_t f + \nabla \cdot (vf) = 0 \qquad \text{on} \quad \mathcal{D}'(\mathbb{R}^{2d+1}).$$

Similarly, we define $\overline{\mathcal{F}}(\sigma_0, \sigma_T|\psi)$ by

$$\overline{\mathcal{F}}(\sigma_0, \sigma_T|\psi) = \{ f \in \overline{\mathcal{F}}(\sigma_0, \sigma_T) : \sigma_t^f \geq \psi_t, \forall t \in [0,T] \}.$$

Given $\Sigma_T(\sigma_0, \sigma_T|\psi)$, by the Riesz representation theorem, we can construct a kinetic measure $f^{\sigma,v}$ as

$$\int_{\mathbb{R}^{2d+1}} \varphi(t,q,\xi) f^{\sigma,v}(dt, dq, d\xi) = \int_0^T dt \int_{\mathbb{R}^d} \varphi\big(t, q, v_t(q)\big)\sigma_t(dq). \tag{2.2.6}$$

In other words,

$$f^{\sigma,v} = (id \times v)_\# \sigma \in \mathcal{M}_2^+(C).$$

We denote the measure at the left hand-side of (2.2.6) as $f^{\sigma,v}$. For any $u \in C_c^\infty(\mathbb{R}^{d+1})$, we have

$$\int_{\mathbb{R}^d} \Big(u_T(q)\sigma_T(dq) - u_0(q)\sigma_0(dq)\Big) = \int_C \big(\partial_t u(t,q) + \langle \xi, \nabla u(t,q)\rangle\big) f^{(\sigma,v)}(dt,dq,d\xi). \quad (2.2.7)$$

Given $f \in \overline{\mathcal{F}}(\sigma_0, \sigma_T)$, we use disintegration theory to build (σ, v).

Let $\overline{f}$ be marginal of f on $[0,T] \times \overline{\Omega} \subset \mathbb{R}^{d+1}$. The theory of disintegration of measures ensures the existence of Borel probability measures $(t,q) \mapsto f^{(t,q)}$ such that

$$\int_{\mathbb{R}^{2d+1}} \varphi f(dt,dq,d\xi) = \int_{\mathbb{R}^{d+1}} \overline{f}(dt,dq) \int_{\mathbb{R}^d} \varphi(t,q,\xi) f^{(t,q)}(d\xi), \quad \forall \varphi \in C_c^\infty(\mathbb{R}^{2d+1}).$$

Let η^f be the projection of $\overline{f}$ on $[0,T]$. Disintegrate further, we find a Borel probability measure σ_t^f such that

$$\int_{\mathbb{R}^{2d+1}} \varphi f(dt,dq,d\xi) = \int_0^T \eta(dt) \int_{\mathbb{R}^d} \sigma_t^f(dq) \int_{\mathbb{R}^d} \varphi(t,q,\xi) f^{(t,q)}(d\xi), \qquad \forall \varphi \in C_c^\infty(\mathbb{R}^{2d+1}).$$

Proposition 2.2.1. *Assume $\overline{f}$ is a finite Borel measure on $\mathbb{R}^{d+1}$ that is supported on $[0,T] \times \overline{\Omega}$. Assume $v : \mathbb{R}^d \to \mathbb{R}^d$ is a Borel vector field such that*

$$\partial_t \overline{f} + \nabla \cdot (v\overline{f}) = 0 \text{ on } \mathcal{D}'\big((0,T) \times \mathbb{R}^d\big).$$

That is, for any $u \in C_c^\infty(\mathbb{R}^{d+1})$, we have

$$\int_{\mathbb{R}^d} \Big(u_T(q)\sigma_T(dq) - u_0(q)\sigma_0(dq)\Big) = \int_{\mathbb{R}^{d+1}} \big(\partial_t u(t,q) + \langle v, \nabla u(t,q)\rangle\big) \overline{f}(dt,dq). \quad (2.2.8)$$

Let η^f be the projection of $\overline{f}$ on $\mathbb{R}$ so that we can disintegrate $\overline{f}$ to obtain

$$\int_{\mathbb{R}^{d+1}} \varphi(t,q)\overline{f}(dt,dq) = \int_{\mathbb{R}} \eta^f(dt) \int_{\mathbb{R}^d} \varphi(t,q)\overline{f}^t(dq), \quad \forall \varphi \in C_c^\infty(\mathbb{R}^{d+1}).$$

Then η^f is the Lebesgue meausre on $[0,T]$.

Proof. Indeed Choosing $u \equiv u(t)$ in (2.2.8) and using the fact that σ_0 and σ_T are probability measures, we obtain

$$\int_0^T \dot{u}(t)dt = u(T) - u(0) = \int_{\mathbb{R}} \dot{u}(t)\eta^f(dt), \quad \forall u \in C_c^\infty(\mathbb{R}).$$

Thus, η^f is the Lebesgue measure on $[0,T]$.

□

Define $\Pi^0 : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ such that $(t,q,\xi) \mapsto t$.

Taking $\varphi(t,q,\xi) = u(t)$, where $u \in C_c^\infty(\mathbb{R}^{d+1})$. For all $f \in \overline{\mathcal{F}}(\sigma_0, \sigma_T)$, by disintegration theory, we can write for any Borel $u(t)$

$$\int_C u(t) f(dt, dq, d\xi) = \int_0^T u(t)\eta(dt).$$

In case the conditions in Proposition 2.2.1 are satisfied, the η^f is Lebesgue measure restricted to $[0,T]$. That is

$$\eta^f := \Pi^0_\# f = L^1|[0,T].$$

In the sequel, we shall only consider f such that $\eta^f = \mathcal{L}^1_{[0,T]}$.

Then we can simplify our above disintegration

$$\int_C \varphi(t,q,\xi) f(dt,dq,d\xi) = \int_0^T dt \int_{\overline{\Omega}\times\mathbb{R}^d} \varphi(t,q,\xi) f^t(dq,d\xi),$$

for some continuous map $t \mapsto f^t \in \mathcal{P}(\overline{\Omega} \times \mathbb{R}^d)$.

Moreover,

$$\int_{\overline{\Omega}\times\mathbb{R}^d} \varphi(t,q,\xi) f^t(dq,d\xi) = \int_{\overline{\Omega}} \sigma_t(dq) \int_{\mathbb{R}^d} \varphi(t,q,\xi) f^{(t,q)}(d\xi).$$

In other words, $\sigma^f_t = \Pi^1_\# f^t$ a probability measure, where $\Pi^1 : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ such that $(q,\xi) \mapsto q$.

Set

$$v^f_t(q) := \int_{\mathbb{R}^d} \xi f^{(t,q)}(d\xi).$$

Assume

$$\int_{\overline{\Omega}\times\mathbb{R}^d} |\xi|^2 f^t(dq,d\xi) < \infty.$$

Since L is convex, by Jensen's inequality, we have

$$\int_{\mathbb{R}^d} L(q,\xi) f^{(t,q)}(d\xi) \geq L\Big(q, \int_{\mathbb{R}^d} \xi f^{(t,q)}(d\xi)\Big) = L\big(q, v_t^f(q)\big),$$

and the inequality is strict unless $f^{(t,q)} = \delta_{\xi(t,q)}$.

As L is bounded from 2.1.4, we thus have

$$\int_0^T dt \int_{\mathbb{R}^d} L\big(q, v_t^f(q)\big)\sigma_t^f(dq) \leq \int_C L(q,\xi) f(dt,dq,d\xi) < \infty.$$

Notice that the inequality above is strict unless $f^{(t,q)} = \delta_{v^f(t,q)}$ $\overline{f}$–almost everywhere.

From the construction, (σ^f, v^f) we constructed from f satisfies (2.2.4).

Lemma 2.2.2. *Assume $f_n \rightharpoonup f$ narrowly in $\overline{\mathcal{F}}(\sigma_0, \sigma_T)$. Then $\sigma_t^{f_n} \rightharpoonup \sigma_t^f$ narrowly in $\mathcal{P}(\overline{\Omega})$.*

Proof. We use proposition 3.3.1 (Arzela-Ascoli Theorem) in [4] to claim the lemma. In order to do so, we simplify our notation by denoting v^f as v and σ^f as σ and notice the following.

Let v be the velocity field driving σ. Notice we have

$$\int_0^T dt \int_{\mathbb{R}^d} |v_t|^2 \sigma_t(dq) < \infty.$$

By Theorem 8.3.1 in [4], we have that

$$|\sigma'|^2 \leq \int_{\mathbb{R}^d} |v_t|^2 \sigma_t(dq).$$

Also by Theorem 1.1.12 in [4], we have that

$$W_p(\sigma_t, \sigma_s) \leq \int_s^t |\sigma'| d\tau.$$

Thus

$$W_p(\sigma_t, \sigma_s) \leq \int_s^t |\sigma'| d\tau \leq (\int_s^t |\sigma'|^2 d\tau)^{\frac{1}{2}} (\int_s^t 1 d\tau)^{\frac{1}{2}}$$

$$\leq \Big(\int_s^t d\tau \int_{\mathbb{R}^d} |v_t|^2 \sigma_t(dq)\Big)^{\frac{1}{2}} (t-s)^{\frac{1}{2}} \leq \Big(\int_0^T d\tau \int_{\mathbb{R}^d} |v_t|^2 \sigma_t(dq)\Big)^{\frac{1}{2}} (t-s)^{\frac{1}{2}}$$

Therefore, $W_p(\sigma_t, \sigma_s)$ is bounded, where $0 \leq s < t \leq T$.

□

Based on our assumption, the following lemmas are standard (see [4]).

Lemma 2.2.3. *Assume $\mu_n \rightharpoonup \mu$ narrowly in $\mathcal{P}(\overline{\Omega})$. Then*

$$\lim_{n\to\infty} F(\mu_n) \to F(\mu_n).$$

Lemma 2.2.4. *If $f_n \rightharpoonup f$ narrowly in $\overline{\mathcal{F}}(\sigma_0, \sigma_T)$ and $\varphi : \overline{\Omega} \times \mathbb{R}^d \to [0, \infty]$ is continuous and bounded from below, then*

$$\liminf \int_C \varphi(q,\xi) f_n(dt, dq, d\xi) \geq \int_C \varphi(q,\xi) f(dt, dq, d\xi).$$

Recall the action functional we try to minimize over $\Sigma_T(\sigma_0, \sigma_T|\psi)$.

$$\mathcal{A}_L^F[\sigma, v] := \int_0^T \left(F(\sigma_t) + \int_{\overline{\Omega}} L(q, v_t(q))\sigma_t(dq) \right) dt.$$

Notice the action functional has exactly the same minimizers as the functional

$$\mathcal{A}_{L+\lambda}^F[\sigma, v] := \int_0^T \left(F(\sigma_t) + \int_{\overline{\Omega}} \big(L(q, v_t(q)) + \lambda\big)\sigma_t(dq) \right) dt = \mathcal{A}_L[\sigma, v] + \lambda T.$$

Therefore, without lost of generality, we may assume that $F \geq 0$.

We define

$$\overline{\mathcal{A}}_L^F[f] := \int_C L(q,\xi) f(dt, dq, d\xi) + \int_0^T F(\sigma_t^f) dt. \tag{2.2.9}$$

Then we have

$$\overline{\mathcal{A}}_L^F[f^{\sigma,v}] = \mathcal{A}_L^F[\sigma, v].$$

Proposition 2.2.5.

$$\inf_{f \in \overline{\mathcal{F}}(\sigma_0, \sigma_T|\psi)} \overline{\mathcal{A}}_L^F[f] \quad \textit{admits a minimizer.}$$

Proof. By Remark 5.1.5 in [4], we know that

$\{f_n\}_{n=1}^\infty$ is narrowly compact if and only if there exists G on $\mathbb{R}^d$ such that the set $\{G \leq c\}$ is compact for any real number c and

$$\sup_n \int_{\mathbb{R}^d} G f_n < \infty.$$

We take a minimizing sequence $\{f_n\}_{n=1}^{\infty}$ such that

$$\int_C L(q,\xi) f_n(dt,dq,d\xi) + \int_0^T F(\sigma_t^{f_n})dt$$

decreases to the infimum.

To use the narrowly compact property above, take $G = L$.

Notice that

$$\begin{aligned} &\int_C L(q,\xi) f_n(dt,dq,d\xi) + \int_0^T F(\sigma_t^{f_n})dt \\ \leq &\int_C L(q,\xi) f_1(dt,dq,d\xi) + \int_0^T F(\sigma_t^{f_1})dt := A. \end{aligned}$$

As F is bounded from below, we may write $F \leq M$ for some real number M.

Thus

$$\int_C L(q,\xi) f_n(dt,dq,d\xi) \leq A - MT, \quad \forall\, n \geq 1.$$

Then $\{f_n\}$ is narrowly compact and we can find a subsequence f_{n_k} which converges narrowly to f for some $f \in \overline{\mathcal{F}}(\sigma_0, \sigma_T|\psi)$. We can check easily that such f is a minimizer.

□

We then have that

$$\inf_{f \in \overline{\mathcal{F}}(\sigma_0,\sigma_T|\psi)} \overline{\mathcal{A}}_L^F[f] = \inf_{(\sigma,v)\in\mathcal{F}(\sigma_0,\sigma_T|\psi)} \mathcal{A}_L^F[\sigma, v].$$

Therefore, we can conclude the following proposition.

Proposition 2.2.6.

$$\inf_{(\sigma,v)\in\mathcal{F}(\sigma_0,\sigma_T|\psi)} \mathcal{A}_L^F[\sigma, v]$$

admits a minimizer. If $L(q,\cdot)$ is convex and F is strictly convex, then the minimizer (σ, v) is unique.

Proof. It is sufficient to check the uniqueness.

Let (σ_1, v^1) and (σ_2, v^2) be minimizers of $\mathcal{A}_L^F[\sigma, v]$. We have $f_1 = f^{(\sigma_1, v^1)}$ and $f_2 = f^{(\sigma_2, v^2)}$, which are minimizers of $\overline{\mathcal{A}}_L^F[f]$.

Let $f = \frac{1}{2}(f_1 + f_2)$. Then $\sigma_t^f = \frac{1}{2}\sigma_t^{f_1} + \frac{1}{2}\sigma_t^{f_2}$.

Then

$$\overline{\mathcal{A}}_L^F[f] \geq \frac{1}{2}\overline{\mathcal{A}}_L^F[f_1] + \frac{1}{2}\overline{\mathcal{A}}_L^F[f_1].$$

By linearity of $\overline{\mathcal{A}}$ in f, we get

$$\int_0^T F(\sigma_t^f)dt \geq \frac{1}{2}\int_0^T F(\sigma_t^{f_1})dt + \frac{1}{2}\int_0^T F(\sigma_t^{f_2})dt.$$

As F is strictly convex, we have $F(\sigma_t^f) < \frac{1}{2}F(\sigma_t^{f_1}) + \frac{1}{2}F(\sigma_t^{f_2})$.

Thus

$$\int_0^T F(\sigma_t^f)dt \leq \frac{1}{2}\int_0^T F(\sigma_t^{f_1})dt + \frac{1}{2}\int_0^T F(\sigma_t^{f_2})dt.$$

Therefore, we have $\sigma_t^{f_1} = \sigma_t^{f_2}$ and $\sigma_1 = \sigma_2$.

Assume $v_t^1(q) \neq v_t^2(q)$ for some $t \in [0, T], q \in \mathbb{R}^d$.

Then $\frac{1}{2}\delta_{v_t^1(q)} + \frac{1}{2}\delta_{v_t^2(q)}$ is not a Dirac mass.

By the convexity of $L(q, \cdot)$ and Jensen's inequality, we have

$$\int_{\mathbb{R}^d} L(q, \xi)(\frac{1}{2}\delta_{v_t^1(q)} + \frac{1}{2}\delta_{v_t^2(q)})d\xi > L(q, \xi \int_{\mathbb{R}^d} (\frac{1}{2}\delta_{v_t^1(q)} + \frac{1}{2}\delta_{v_t^2(q)})dq) = L(q, \frac{v_t^1(q) + v_t^2(q)}{2}).$$

Thus

$$\frac{1}{2}L(q, v_t^1(q)) + \frac{1}{2}L(q, v_t^2(q)) > L(q, \frac{v_t^1(q) + v_t^2(q)}{2}),$$

contradicting to the minimality of (σ_1, v^1) and (σ_2, v^2).

Therefore, $v^1 = v^2$ and the minimizer is unique.

□

2.3 Duality

Notice that $\sigma_t^f \geq \psi_t$ is equivalent to

$$\int_C h(t,q) f(dt,dq,d\xi) \geq \int_{[0,T]\times\overline{\Omega}} h(t,q)\psi(dt,dq),$$

for all non-negative $h \in C_0^1(\mathbb{R}^{d+1})$.

We define $\mathcal{U}_{*,T}^{\alpha}$ to be the set of pairs (u,h) such that $u \in C([0,T]\times\bar{\Omega})$, $h : [0,T]\times\bar{\Omega} \to [0,\infty]$ is Borel and non–negative and

$$u(t,\gamma(t)) - u(s,\gamma(s)) \leq \int_s^t \Big(L(\gamma,\dot{\gamma}) + \alpha(\tau,\gamma) - h(\tau,\gamma) \Big) d\tau,$$

for all $0 \leq s < t \leq T$.

On $\mathcal{U}_{*,T}^{\alpha}$, the following functional is well–defined.

We also define

$$J(u,h) := \int_{\overline{\Omega}} \big(u_T(q)\sigma_T(dq) - u_0(q)\sigma_0(dq) \big) + \int_0^T dt \int_{\overline{\Omega}} h_t(q)\psi_t(dq). \tag{2.3.1}$$

Set

$$\hat{\mathcal{L}}(f,u,h) := J(u,h) + \int_0^T F(\sigma_t^f) + \int_C \Big(L(q,\xi) - \partial_t u(t,q) - \nabla_q u(t,q)\cdot\xi - h(t,q) \Big) f(dt,dq,d\xi).$$

Proposition 2.3.1. *If we set*

$$\mathcal{U} := C_o^1\big(\mathbb{R}^{d+1}\big) \times C_o^1\big(\mathbb{R}^{d+1}\big)^+,$$

then

$$\sup_{(u,h)\in\mathcal{U}} \hat{\mathcal{L}}(f,u,h) = \begin{cases} \overline{\mathcal{A}}_L^F[f] & \text{if } f \in \tilde{\mathcal{F}}(\sigma_0,\sigma_T|\psi) \\ \infty & \text{if otherwise} \end{cases}$$

Notice that h is non-negative here.

Proof. Let's define

$$\tilde{L}(f,u,h) = \hat{\mathcal{L}}(f,u,h) - \overline{\mathcal{A}}_L^F[f]$$

That is,

$$\tilde{L}(f,u,h) = \int_{\overline{\Omega}} (u_T\sigma_T - u_0\sigma_0) - \int_C (\partial_t u + \nabla_q u \cdot \xi) f + \int_0^T \int_{\overline{\Omega}} h_t(q)\psi_t(dq) - \int_C hf(dt,dq,d\xi)$$

We will prove that

$$\sup_{(u,h)\in\mathcal{U}} \tilde{L}(f,u,h) = \begin{cases} 0 & \text{if } \ f \in \tilde{\mathcal{F}}(\sigma_0,\sigma_T|\psi) \\ \infty & \text{if } \textit{otherwise.} \end{cases}$$

Indeed, if $f \in \tilde{\mathcal{F}}(\sigma_0,\sigma_T|\psi)$, then $\tilde{L}(f,u,h) \leq 0$ by definition of $\tilde{\mathcal{F}}(\sigma_0,\sigma_T|\psi)$. Notice that here we have

$$\tilde{L}(f,u,h) \leq 0 = \tilde{L}(f,0,0).$$

Thus $L(f,0,0)$ is the maximum.

Moreover, if $f \notin \tilde{\mathcal{F}}(\sigma_0,\sigma_T|\psi)$, then either $f \notin \overline{\mathcal{F}}(\sigma_0,\sigma_T)$ or $\sigma_t^f < \psi_t$ on a set $A \subset [0,1]$ of positive measure.

Let $h_\lambda^A = \lambda\chi_A$, where $\lambda \geq 0$. Then

$$\int (\psi_t - \sigma_t^f)h_\lambda^A = \lambda \int_{Ax\overline{\Omega}} (\psi_t - \sigma_t^f)(dq)(dt) \geq 0$$

As $\lambda \to \infty$, $\tilde{L}(f,u,h_\lambda^A) \to \infty$.

Therefore $\sup_{(u,h)\in\mathcal{U}} \tilde{L}(f,u,h) = \infty$.

□

Proposition 2.3.2. *Show that*

$$\inf_{f\in\mathcal{P}^1(C)} \sup_{(u,h)\in\mathcal{U}} \hat{\mathcal{L}}(f,u,h) = \sup_{(u,h)\in\mathcal{U}} \inf_{f\in\mathcal{P}^1(C)} \hat{\mathcal{L}}(f,u,h).$$

and the infimum on the left hand side is in fact a minimum. Here,

$$\mathcal{P}^1(C) = \{f \in \mathcal{P}(C) \ : \ \eta^f = 1\}.$$

Proof. We will use Sion's Theorem 1.6 in [23].

Notice first that we can equip $C_o^1\big(\mathbb{R}^{d+1}\big)$ with the C^1 norm. i.e.

$$||u||_{C^1} = ||u||_{L^\infty} + ||\nabla u||_{L^\infty}, \forall u \in C_o^1\big(\mathbb{R}^{d+1}\big).$$

Then $(C_o^1\big(\mathbb{R}^{d+1}\big), ||\cdot||_{C^1})$ is a topological vector space and $\{u|\hat{\mathcal{L}}(f,u,h) \geq c\}$ is closed and convex for any constant c.

Indeed, take $\{u_n\} \subset \{u|\hat{\mathcal{L}} \geq c\}$ such that $u_n \to u$ in $(C_o^1\big(\mathbb{R}^{d+1}\big), ||\cdot||_{C^1})$. Let's show that $u \in \{u|\hat{\mathcal{L}} \geq c\}$.

It suffices to show that $|\hat{\mathcal{L}}(f,u_n,h) - \hat{\mathcal{L}}(f,u,h)| \to 0$.

$$\hat{\mathcal{L}}(f,u_n,h) - \hat{\mathcal{L}}(f,u,h)$$

Notice that

$$= \int_{\overline{\Omega}}((u_n)_T - u_T)(q)\sigma_T(dq) - \int_{\overline{\Omega}}((u_n)_0 - u_0)(q)\sigma_0(dq) - \int_C(\partial_t(u_n - u) + \nabla(u_n - u)\cdot\xi)f(dt,dq,d\xi).$$

By Cauchy-Schwarz inequality,

$$|\hat{\mathcal{L}}(f,u_n,h) - \hat{\mathcal{L}}(f,u,h)| \leq 2||u_n - u||_{L^\infty} + ||\nabla(u_n - u)||_{L^\infty} + ||\nabla(u_n - u)||_{L^\infty}\int_C|\xi|f(dt,dq,d\xi).$$

By Jensen's inequality,

$$(\int_C|\xi|f(dt,dq,d\xi))^2 \leq \int_C|\xi|^2f(dt,dq,d\xi) < \infty.$$

Therefore

$$|\hat{\mathcal{L}}(f,u_n,h) - \hat{\mathcal{L}}(f,u,h)| \to 0 \quad \text{as } n \to \infty.$$

Moreover, since $\hat{\mathcal{L}}(f,u_n,h)$ is linear in u, $\{u|\hat{\mathcal{L}}(f,u,h) \geq c\}$ is convex.

Notice next that we can equip $P^2(C)$ with narrow convergence topology and $P^2(C) \hookrightarrow \mathcal{M}(C)$, where $\mathcal{M}(C)$ is also equipped with the narrow convergence topology. Then $P^2(C)$ equipped with narrow convergence topology is compact.

We know that $P^2(C)$ with narrow convergence topology is a convex subset of $\mathcal{M}(C)$.

Indeed, it suffices to check that $[0,T] \times P^2(C) \times P^2(C) \to P^2(C)$ via $(\lambda, \mu, \nu) \longmapsto (1-\lambda)\mu + \lambda\nu$ is continuous. Thus if $\lambda_n \to \lambda, \mu_n \rightharpoonup \mu, \nu_n \rightharpoonup \nu$, and given any $\phi \in C_b(C)$, we want to show that

$$\int_C \phi\big((1-\lambda_n)\mu_n + \lambda_n\nu_n\big) \to \int_C \phi\big((1-\lambda)\mu + \lambda\nu\big).$$

This follows from

$$(1-\lambda_n)\int_C \phi\mu_n \to (1-\lambda)\int_C \phi\mu$$

and

$$\lambda_n \int_C \phi\nu_n \to \lambda \int_C \phi\nu.$$

Now let's check that $\{f : \hat{\mathcal{L}}(f, u_n, h) \leq c\}$ is convex and closed in $P^2(C)$ for any given $(u, h) \in \mathcal{U}$.

Since $\hat{\mathcal{L}}(f, u_n, h)$ is linear in f, $\{f : \hat{\mathcal{L}}(f, u_n, h) \leq c\}$ is convex.

For closeness, notice that if $f_n \rightharpoonup f$ narrowly, it suffices to show that

$$\liminf \hat{\mathcal{L}}(f_n, u, h) \geq \liminf \hat{\mathcal{L}}(f, u_n, h).$$

By Lemma 2.2.2 and Lemma 2.2.3, it suffices to prove that

$$\liminf \int_C [L(q,\xi) - (\partial_t u + \nabla_q u(t,q)\cdot\xi) - h] f_n \geq \int_C [L(q,\xi) - (\partial_t u + \nabla_q u(t,q)\cdot\xi) - h] f.$$

Then by Lemma 2.2.4, we just need to check that

$$L(q,\xi) - (\partial_t u + \nabla_q u(t,q)\cdot\xi) - h \quad \text{is bounded from below.}$$

Indeed, we may assume that $\partial_t u(t,q), \nabla_q u(t,q)$, and h are bounded by a, b, c from below respectively. Then

$$L(q,\xi) - (\partial_t u + \nabla_q u(t,q)\cdot\xi) - h \geq \lambda_1|\xi|^2 + \lambda_0 - a - b|\xi| - c.$$

$\lambda_1|\xi|^2 - b|\xi|$ has minimum value λ since $\lambda_1 \geq 0$. Then

$L(q,\xi) - (\partial_t u + \nabla_q u(t,q)\cdot\xi) - h \geq \lambda + \lambda_0 - a - c$ as wanted.

Finally since $P^2(C)$ is narrowly compact, the infimum on the left hand side is actually a minimum.

$\square$

We thus obtained the following proposition.

Proposition 2.3.3. *For any $(u,h)\in\mathcal{U}$, we have*

$$\inf_{f\in\mathcal{P}^1(C)} \hat{\mathcal{L}}(f,u,h)$$

$$= J(u,h) - \int_0^T F^*\Big(\partial_t u(t,q) + h(t,q) + H\big(q,\nabla u(t,q)\big)\Big)dt.$$

Here F^* is the Legendre transform of F, which is defined as

$$F^*(\alpha) = \sup_{\mu\in\mathcal{M}^+(\overline{\Omega})} \Big\{ \int_{\overline{\Omega}} \alpha(q)\mu(dq) - F(\mu)\Big\}.$$

If we denote

$$\hat{J}(u,h) := J(u,h) - \int_0^T F^*_\Phi\Big(\partial_t u(t,\cdot) + h(t,\cdot) + H\big(\cdot,\nabla u(t,\cdot)\big)\Big)dt,$$

as a corollary, we get

Corollary 2.3.4.

$$C_L^F(\psi_0,\psi_T) = \sup_{(u,h)\in\mathcal{U}} \hat{J}(u,h)$$

When $\alpha\in C\big([0,T]\times\bar{\Omega}\big)$ Borel, we define

$$\mathcal{A}_L^\alpha[\sigma,v] := \int_0^T dt \int_{\overline{\Omega}} \Big(L(q,v_t(q)) + \alpha(t,q)\Big)\sigma_t(dq),$$

where $(\sigma,v)\in\Sigma_T(\psi_0,\psi_T|\psi)$.

We fix $p > d$ and define

$$\mathcal{U}_T^\alpha \equiv \mathcal{U}_{T,p}^\alpha = \mathcal{U}_{*,T}^\alpha \cap \Big(W^{2,p} \times W^{1,p}\Big).$$

We set

$$\Gamma_T = C([0,T];\mathbb{R}^d)$$

We consider $(\sigma, v) \in \Sigma_T(\psi_0, \psi_T|\psi)$ such that

$$\mathcal{A}_L^\alpha[\sigma, v] < \infty.$$

Using η^σ as the probabilistic representation of (σ, v), for almost every $(q,\gamma) \in \bar{\Omega} \times \Gamma_T$ with respect to η^σ, we have that $\gamma \in AC_2(0,T;\bar{\Omega})$.

Now, let $u \in C^1([0,T] \times \bar{\Omega})$ and $h : ([0,T] \times \bar{\Omega}) \to [0,\infty)$ be a Borel function. Based on the fact that $\sigma_t \geq \psi_t$ almost everywhere, we can make the following observation:

$$\begin{aligned}
J(u,h) &\leq \int_0^T dt \int_{\bar{\Omega}} \Big(\partial_t u + \langle \nabla u, v\rangle + h\Big)\sigma_t(dq) \\
&= \int_{\bar{\Omega}\times\Gamma_T} \left(\int_0^T \Big(\partial_t u(t,\gamma) + \langle \nabla u(t,\gamma,\dot{\gamma}\rangle + h(t,\gamma)\Big)dt\right)\eta^\sigma(dq,d\gamma) \\
&= \int_{\bar{\Omega}\times\Gamma_T} \left(u(T,\gamma(T)) - u(0,\gamma(0)) + \int_0^T h(t,\gamma)dt\right)\eta^\sigma(dq,d\gamma).
\end{aligned}$$

If we further assume that for $\epsilon > 0$,

$$u(T,\gamma(T)) - u(0,\gamma(0)) \leq 2\epsilon + \int_0^T \Big(L(\gamma,\dot{\gamma}) + \alpha(\tau,\gamma) - h(\tau,\gamma)\Big)d\tau,$$

for all $\gamma \in AC_2(0,T;\bar{\Omega})$, then

$$J(u,h) \leq 2\epsilon + \int_{\bar{\Omega}\times\Gamma_T} \left(\int_0^T \big(L(\gamma,\dot{\gamma}) + \alpha(t,\gamma)\big)dt\right)\eta^\sigma(dq,d\gamma).$$

Therefore we have

$$J(u,h) \leq 2\epsilon + \mathcal{A}_L^\alpha[\sigma,v]. \tag{2.3.2}$$

Lemma 2.3.5. *If* $(u,h) \in \mathcal{U}^{\alpha}_{*,T}$ *and* $(\sigma, v) \in \Sigma_T(\psi_0, \psi_T|\psi)$, *then*

$$J(u,h) \leq \int_0^T \bigg(\int_{\bar{\Omega}} \big(L(q,v_t) + \alpha(t,q)\big)\sigma_t(dq)\bigg) dt.$$

Proof. Let $(u,h) \in \mathcal{U}^{\alpha}_{*,T}$ and $(\sigma, v) \in \Sigma_T(\mu, \nu|\psi)$. We can choose $(u_\epsilon)_\epsilon \subset C^1([0,T] \times \bar{\Omega})$ such that $|u - u_\epsilon| \leq \epsilon$. For any $\gamma \in AC_2(0,T;\bar{\Omega})$, we have

$$\begin{aligned} u^\epsilon(T,\gamma(T)) - u^\epsilon(0,\gamma(0)) &\leq 2\epsilon + u(T,\gamma(T)) - u(0,\gamma(0)) \\ &\leq 2\epsilon + \int_0^T \Big(L(\gamma,\dot{\gamma}) + \alpha(\tau,\gamma) - h(\tau,\gamma)\Big) d\tau. \end{aligned}$$

By (2.3.2), we have

$$J(u^\epsilon, h) \leq 2\epsilon + \int_0^T \bigg(\int_{\bar{\Omega}} \big(L(q,v_t) + \alpha(t,q)\big)\sigma_t(dq)\bigg) dt.$$

Let ϵ tend to 0. We conclude the proof of the Lemma.

□

Corollary 2.3.6. *For any set* $\mathcal{U}^\alpha$ *such that* $\mathcal{U}^\alpha_T \subset \mathcal{U}^\alpha \subset \mathcal{U}^\alpha_{*,T}$, *we have*

$$\sup_{(u,h)\in\mathcal{U}^\alpha_T} J(u,h) = \sup_{(u,h)\in\mathcal{U}^\alpha} J(u,h) = \sup_{(u,h)\in\mathcal{U}^\alpha_{*,T}} J(u,h) = \min_{(\sigma,v)\in\Sigma_T(\mu,\nu|\psi)} \mathcal{A}^\alpha_L[\sigma,v].$$

Proof. By Lemma 2.3.5, we easily have

$$\sup_{(u,h)\in\mathcal{U}^\alpha_T} J(u,h) \leq \sup_{(u,h)\in\mathcal{U}^\alpha} J(u,h) \leq \sup_{(u,h)\in\mathcal{U}^\alpha_{*,T}} J(u,h) \leq \min_{(\sigma,v)\in\Sigma_T(\mu,\nu|\psi)} \mathcal{A}^\alpha_L[\sigma,v].$$

By Proposition 2.3.1, Proposition 2.3.2, and Proposition 2.3.3, we have that

$$\sup_{(u,h)\in\mathcal{U}^\alpha_T} J(u,h) \geq \min_{(\sigma,v)\in\Sigma_T(\mu,\nu|\psi)} \mathcal{A}^\alpha_L[\sigma,v].$$

Hence the corollary holds.

2.4 Some useful lemmas

Lemma 2.4.1. *Let $\mu \in \mathcal{P}(\bar{\Omega})$ and $\alpha : \bar{\Omega} \to \mathbb{R}$ be a bounded Borel function such that*

$$\int_{\bar{\Omega}} \alpha(q) f(q) \mu(dq) = 0$$

for all $f \in C(\bar{\Omega})$ such that $\int_{\bar{\Omega}} f(q)\mu(dq) = 0$. Then there exists a constant c_α such that $\alpha = c_\alpha$ μ–a.e.

Proof. Let $f_0 \in C(\bar{\Omega})$. We set

$$f(q) := f_0(q) - \int_{\bar{\Omega}} f_0(q_2)\mu(dq_2).$$

Then we have

$$\int_{\bar{\Omega}} f(q)\mu(dq) = 0.$$

Thus

$$\begin{aligned} 0 &= \int_{\bar{\Omega}} \alpha(q) f_0(q)\mu(dq) - \int_{\bar{\Omega}} \alpha(q)\mu(dq) \int_{\bar{\Omega}} f_0(q_2)\mu(dq_2) \\ &= \int_{\bar{\Omega}} f_0(q_2) \Big(\alpha(q_2) - \int_{\bar{\Omega}} \alpha(q)\mu(dq) \Big) \mu(dq_2). \end{aligned}$$

Since $f_0 \in C(\bar{\Omega})$ is arbitrary, this implies

$$\alpha(q_2) = \int_{\bar{\Omega}} \alpha(q)\mu(dq) \qquad \mu - \text{a.e.}$$

□

Corollary 2.4.2. *Let $\alpha : \bar{\Omega} \to \mathbb{R}$ be a bounded Borel function. Assume μ maximizes*

$$\bar{\mu} \mapsto I(\bar{\mu}) := \int_{\bar{\Omega}} \alpha(q)\bar{\mu}(dq) - F_\Phi(\bar{\mu})$$

over $\mathcal{P}(\bar{\Omega})$.

Then

$$q \mapsto \alpha(q) - \int_{\bar{\Omega}} \Phi(q, q_2)\mu(dq_2)$$

is constant μ–a.e.

Proof. Take $f \in C(\bar{\Omega})$ such that

$$\int_{\bar{\Omega}} f(q)\mu(dq) = 0.$$

Set $\mu^\epsilon = \mu(1 + \epsilon f)$. We have that $\mu^\epsilon \in \mathcal{P}(\bar{\Omega})$ and

$$I(\mu^\epsilon) = I(\mu) + \epsilon \int_{\bar{\Omega}} \left(\alpha(q_1) - \int_{\bar{\Omega}} \Phi(q_1, q_2)\mu(dq_2) \right) f(q_1)\mu(dq_1) + o(\epsilon).$$

By the maximality property of $\mu^0 = \mu$, we have that

$$0 = \int_{\bar{\Omega}} \left(\alpha(q_1) - \int_{\bar{\Omega}} \Phi(q_1, q_2)\mu(dq_2) \right) f(q_1)\mu(dq_1).$$

Apply Lemma 2.4.1, we obtain the desired result.

□

For the rest of this section, let $\Phi \in C^2(\mathbb{R}^d \times \mathbb{R}^d)$. We define

$$\alpha(t, q) := \int_{\bar{\Omega}} \Phi(q, q_2)\sigma_t(dq_2)$$

and set

$$\bar{L}(t, q, v) = L(q, v) + \alpha(t, q), \qquad \bar{H}(t, q, p) = H(q, p) - \alpha(t, q).$$

Given $P_0 \in \mathbb{R}^d$, we define

$$u(t, x) := \inf_{\gamma} \left\{ \int_0^t \bar{L}(\tau, \gamma, \dot{\gamma})d\tau : \gamma \in W^{1,2}(s, t; \bar{\Omega}),\ \gamma(0) = P_0,\ \gamma(t) = x \right\}. \tag{2.4.1}$$

By the definition of u and the boundedness of $\bar{L}$, we have the following lemma.

Lemma 2.4.3. *For any $0 \leq s < t \leq T$,*

$$u(t, \gamma(t)) - u(s, \gamma(s)) \leq \int_s^t \bar{L}(\tau, \gamma, \dot{\gamma})d\tau, \quad \forall \gamma \in W^{1,2}(s, t; \bar{\Omega}). \tag{2.4.2}$$

The function u defined in (2.4.1) *is Lipschitz continuous.*

Lemma 2.4.4. *(i) For any $0 \le s < t \le T$, we have*

$$\int_{\bar{\Omega}} u(t,q)\sigma_t(dq) - \int_{\bar{\Omega}} u(s,q)\sigma_s(dq) \le \int_s^t d\tau \int_{\bar{\Omega}} L(q, v_\tau(q))\sigma_\tau(dq).$$

(ii) Therefore, the function

$$t \mapsto \omega(t) := \int_0^t d\tau \int_{\bar{\Omega}} L(q, v_\tau(q))\sigma_\tau(dq) - \int_{\bar{\Omega}} u(t,q)\sigma_t(dq)$$

is monotone non-decreasing.

Proof. Note that since $\alpha(\tau, \cdot)$ is of null σ_τ average, we have

$$\int_{\bar{\Omega}} \bar{L}(\tau, (q, v_\tau(q))\sigma_\tau(dq) = \int_{\bar{\Omega}} L(q, v_\tau(q))\sigma_\tau(dq).$$

Thus, if η^σ is the probabilistic representation of (σ, v), then

$$\int_{\bar{\Omega}} L(q, v_\tau(q))\sigma_\tau(dq) = \int_{\bar{\Omega}\times\Gamma_T} \bar{L}(\tau, \gamma(\tau), \dot{\gamma}(\tau))\eta^\sigma(dq, d\gamma). \tag{2.4.3}$$

Since

$$\int_{\bar{\Omega}} u(t,q)\sigma_t(dq) - \int_{\bar{\Omega}} u(s,q)\sigma_s(dq) = \int_{\bar{\Omega}\times\Gamma_T} \Big(u(t, \gamma(t)) - u(s, \gamma(s))\Big)\eta^\sigma(dq, d\gamma),$$

we use (2.4.2) to conclude that

$$\int_{\bar{\Omega}} u(t,q)\sigma_t(dq) - \int_{\bar{\Omega}} u(s,q)\sigma_s(dq) \le \int_{\bar{\Omega}\times\Gamma_T} \left(\int_s^t \bar{L}(\tau, \gamma(\tau), \dot{\gamma}(\tau))d\tau\right)\eta^\sigma(dq, d\gamma).$$

Since $\bar{L}$ is bounded from below, we can use Fubini's theorem to obtain

$$\int_{\bar{\Omega}} u(t,q)\sigma_t(dq) - \int_{\bar{\Omega}} u(s,q)\sigma_s(dq) \le \int_s^t \left(\int_{\bar{\Omega}\times\Gamma_T} \bar{L}(\tau, \gamma(\tau), \dot{\gamma}(\tau))\eta^\sigma(dq, d\gamma)\right)d\tau.$$

This, together with (2.4.3) implies that ω is monotone non–decreasing.

□

We define

$$\bar{h}(t,q) := \begin{cases} L(P_t,\dot{P}_t) + \alpha(t,P_t) - \frac{d^+}{dt}u(t,P_t) & \text{if } \; q = P_t \\ 0 & \text{if } \; q \neq P_t \end{cases}$$

Here $\frac{d^+}{dt}u(t,P_t)$ is the right derivative of u with respect to t.

Lemma 2.4.5. *The function $\bar{h}$ is non–negative and for $0 \le s < t \le T$, the u defined in* (2.4.1) *satisfies*

$$u(t,\gamma(t)) - u(s,\gamma(s)) + \int_s^t h(\tau,\gamma(\tau))d\tau \le \int_s^t \bar{L}(\tau,\gamma,\dot{\gamma})d\tau, \quad \forall \gamma \in W^{1,\infty}(s,t;\bar{\Omega}).$$

Proof. The fact that $\bar{h} \ge 0$ is a direct consequence of (2.4.2) when we use $\gamma(t) = P_t$.

Fix $0 \le s < t \le T$ and let $\bar{\gamma} : [s,t] \to \bar{\Omega}$ be a Lipschitz curve and set

$$S_0 = \{\tau \in (s,t) \,:\, P_\tau = \bar{\gamma}(\tau)\}, \quad S_1 = \Big\{\tau \in (s,t) \,:\, \dot{P}_\tau = \dot{\bar{\gamma}}(\tau),\; \frac{d}{d\tau}u(\tau,\bar{\gamma}(\tau)) = \frac{d}{d\tau}u(\tau,P_\tau)\Big\}.$$

Recall that the set $S_0 \setminus S_1$ is of null Lebesgue measure (see Lemma 6.3.1).

Let τ be a point of differentiability of $u(\cdot,\bar{\gamma})$. On the one hand if $\tau \in S_0 \cap S_1$ then

$$\frac{d^+}{d\tau}u(\tau,\bar{\gamma}_\tau) + h(\tau,\bar{\gamma}_\tau) - L(\bar{\gamma}_\tau,\dot{\bar{\gamma}}_\tau) = \frac{d^+}{d\tau}u(\tau,P_\tau) + h(\tau,P_\tau) - L(P_\tau,\dot{P}_\tau) = 0.$$

On the other hand, if $\bar{\gamma}_\tau \neq P_\tau$, then there exists $\delta > 0$ such that $\bar{\gamma}_l \neq P_l$ for all $l \in [\tau-\delta,\tau+\delta]$. We use (2.4.2) to obtain that

$$\frac{d^+}{d\tau}u(\tau,\bar{\gamma}_\tau) + h(\tau,\bar{\gamma}_\tau) - L(\bar{\gamma}_\tau,\dot{\bar{\gamma}}_\tau) = \frac{d^+}{d\tau}u(\tau,\bar{\gamma}_\tau) - L(\bar{\gamma}_\tau,\dot{\bar{\gamma}}_\tau) \le 0.$$

In conclusion,

$$\frac{d^+}{dt}u(t,\bar{\gamma}_t) + h(t,\bar{\gamma}_t) - L(\bar{\gamma}_t,\dot{\bar{\gamma}}_t) \le 0 \qquad \mathcal{L}^1 \text{ a.e.}$$

Integrating, we obtain the desired inequality.

□

Let ϵ_0 be a positive real number and let $\lambda : [0, T] \to [\epsilon_0, 1]$ be a piecewise continuous function which is continuous at 0 and 1, such that $\lambda(0) = \lambda(1) = 1$. We assume

$$\psi_t = \lambda_t \delta_{P_t}, \quad \forall t \in [0, T].$$

We define $h : \{(t, P_t) : \ t \in [0, T]\} \to [0, \infty]$ by

$$h(t, P_t) = \frac{1}{\lambda_t} \limsup_{\delta \to 0^+} \frac{H(t+\delta) - H(t)}{\delta}.$$

Note that

$$\int_0^T \lambda_t h(t, P_t) dt = \int_0^T H'(t) dt = H(T) - H(0).$$

In other words,

$$\int_{\bar{\Omega}} \Big(u_T(q)\sigma_T(dq) - u_0(q)\sigma_0(dq) \Big) + \int_0^T dt \int_{\partial\Omega} h(t, q)\psi_t(dq) = \int_0^T d\tau \int_{\bar{\Omega}} L(q, v_\tau(q))\sigma_\tau(dq).$$

Since $F^*_\Phi(\alpha_t) + F_\Phi(\sigma_t) = 0$, we conclude that

$$J(u, h) - \int_0^T F^*_\Phi(\alpha_t) dt = \mathcal{A}^F_L[\sigma, v]. \tag{2.4.4}$$

Recall that if γ is a minimizer in (2.4.1), then setting $p(s) = D_v \bar{L}(s, \gamma(s), \dot{\gamma}(s))$, we have

$$\dot{\gamma} = D_p \bar{H} L(s, \gamma, p), q \quad \dot{p} = -D_q \bar{H} L(s, \gamma, p).$$

Thus

$$\frac{d}{dt} \bar{H}(t, \gamma, p) = \partial_t \bar{H}(t, \gamma, p) + \langle D_q \bar{H}(t, \gamma, p), \dot{\gamma} \rangle + \langle D_q \bar{H}(t, \gamma, p), \dot{p} \rangle = \partial_t \bar{H}(t, \gamma, p).$$

Thus,

$$\bar{H}(s, \gamma(s), p(s)) = \bar{H}(t, x, p(t)) - \int_t^s \partial_t \alpha(\tau, \gamma) d\tau.$$

From this, we get a uniform bound on the L^∞-norm of p. Thus, there is a bound on the L^∞-norm of $\dot{\gamma}$.

CHAPTER 3

Zero Potential and Duality

In this chapter, we first derive a useful result by assuming the linear potential given by

$$F(\mu) = F_\Phi(\mu) := \int_{\overline{\Omega}} \Phi(q)\mu(dq),$$

for some strict convex $\Phi \in C^2(\mathbb{R}^d)$. Then we assume $\Phi = 0$ for the remaining part of the chapter. Although most results can be easily extended to linear or general potentials, we keep it zero in this chapter to make our argument clear.

Our goal is to study the following minimization problem via duality

$$\inf_{(\sigma,v)} \Big\{ \mathcal{A}_L^F[\sigma, v] \, : \, (\sigma, v) \in F(\sigma_0, \sigma_T|\psi) \Big\}, \tag{3.0.1}$$

where

$$\mathcal{A}_L^F[\sigma, v] = \int_0^T \int_{\overline{\Omega}} L(q, v_t(q))\sigma_t(dq)dt.$$

We assume $t \to P_t \in \partial\Omega$ is of class C^2.

We define $\mathcal{U}_0$ to be the set of pair (u, h) such that

$$u \in C([0,T] \times \bar{\Omega}), \quad h : [0,T] \times \partial\Omega \to [0, +\infty) \text{ Borel}$$

and

$$u(t, \gamma(t)) - u(s, \gamma(s)) \leq \int_s^t \Big(L(\gamma, \dot{\gamma}) - h(\tau, \gamma(\tau)) \Big) d\tau,$$

for all $0 \leq s < t \leq T$ and all $\gamma \in W^{1,\infty}(s, t; \bar{\Omega})$.

3.1 Minimizer of the cost function

For $s, t \in [0, T]$ and $\gamma \in W^{1,2}(s, t; \mathbb{R}^d)$, we define

$$A_s^t(\gamma) := \int_s^t L(\gamma, \dot{\gamma}) + \Phi(\gamma) d\tau.$$

Since L is convex and Φ is strict convex, we have

$$(q, \xi) \mapsto L(q, \xi) + \Phi(q) \qquad \text{is strictly convex.}$$

Therefore, for all $s, t \in [0, T]$ and $x, y \in \mathbb{R}^d$, we have

$$\gamma \mapsto A_s^t(\gamma) \quad \text{is strictly convex on} \quad \{\gamma \in W^{1,2}(s, t; \mathbb{R}^d),\ \gamma(s) = x, \gamma(t) = y\}. \tag{3.1.1}$$

For $s, t \in [0, T]$ and $x, y \in \mathbb{R}^d$ we define

$$c_s^t(x, y) := \min_\gamma \Big\{ A_s^t(\gamma) :\ \gamma \in W^{1,2}(s, t; \mathbb{R}^d),\ \gamma(s) = x, \gamma(t) = y \Big\}. \tag{3.1.2}$$

Since Φ is continuous on compact subset $\overline{\Omega}$, $\Phi|_{\overline{\Omega}}$ is bounded. Write $K_1 \le \Phi(q) \le K_2$ for any $q \in \overline{\Omega}$.

By (2.1.4)

$$\lambda_1 \frac{|y - x|^2}{t - s} + \lambda_0(t - s) + K_1 \le c_s^t(x, y) \le \lambda_1^{-1} \frac{|y - x|^2}{t - s} - \lambda_0(t - s) + K_2. \tag{3.1.3}$$

First let's prove that (3.1.2) indeed admits a minimizer, so it is well-defined.

Lemma 3.1.1. (3.1.2) *admits a minimizer. By* (3.1.1)*, this minimizer is unique and will be denoted by* $\gamma_{s,x}^{t,y}$.

Proof. For simplicity in notation, we denote $A_a^b[\gamma]$ as $A[\gamma]$ and work with $\gamma \in W^{1,2}(a, b, \mathbb{R}^d)$.

Fix x and y in $\overline{\Omega}$, let $\gamma \in W^{1,2}(a, b, \mathbb{R}^d)$ such that $\gamma(a) = x$ and $\gamma(b) = y$.

Set $\overline{\gamma}(t) = (1 - \frac{t-a}{b-a})x + \frac{t-a}{b-a}y$. Then $\dot{\overline{\gamma}}(t) = \frac{y-x}{b-a}$.

Let $\{\gamma_n\}$ be a minimizing sequence in $W^{1,2}$.

Since $A[\gamma_n] \leq A[\overline{\gamma}]$ for larger enough n, for simplicity, we assume $A[\gamma_n] \leq A[\overline{\gamma}]$.

Notice that

$$\int_a^b \lambda_0 + |\dot{\gamma}_n|^2\lambda_1 + \Phi(\gamma_n)dt \leq A[\gamma_n] \leq A[\overline{\gamma}] = \int_a^b L(\overline{\gamma}, \frac{y-x}{b-a}) + \Phi(\overline{\gamma})dt =: M.$$

Then

$$\int_a^b |\dot{\gamma}_n|^2 dt \leq \frac{M - (K_1 + \lambda_0)(b-a)}{\lambda_1} =: M_1.$$

Thus

$$|\gamma_n(t) - x| = |\gamma_n(t) - \gamma_n(a)| = |\int_a^t \dot{\gamma}_n(\tau)d\tau| \leq \sqrt{\int_a^t |\dot{\gamma}_n(\tau)^2|d\tau}\sqrt{t-a} \leq \sqrt{b-a}\sqrt{M_1}.$$

Then $(\gamma_n)_n$ is bounded in $W^{1,2} \cap L^\infty(a,b)$. Indeed,

$$||\gamma_n||_{L^\infty} \leq |x| + \sqrt{b-a}M_1 \leq \lambda_\Omega + \sqrt{b-a}M_1,$$

where $\lambda_\Omega := \sup_{v\in\Omega} |v|$.

By Banach-Alaoglu Theorem, there is a subsequence of $\{\gamma_n\}$, still denoted as $\{\gamma_n\}$ for simplicity, that converges weakly to some $\gamma \in W^{1,2}$. That is, $\gamma_n \to \gamma$ in L^2 and $\dot{\gamma}_n \rightharpoonup \dot{\gamma}$ in L^2.

Notice that

$$L(q,v) + \Phi(q) \geq \lambda_0 + \lambda_1|v|^2 + \inf_\Omega \Phi \geq \lambda_0 + K_1 =: \lambda_2.$$

Then

$$L(q,v) + \Phi(q) - \lambda_2 \geq 0.$$

Observe that

$$\int_a^b (L(\gamma,\dot{\gamma}) + \Phi(\gamma) + C)dt = A[\gamma] + (b-a)C,$$

where C is any fixed constant.

Without loss of generality, we may assume that $L + \Phi \geq 0$.

Set

$$T_r := \{\, t \in (a,b) : |\dot\gamma(t)| \le r \,\}.$$

Notice that $L(q,v)$ is convex and Φ is convex, we have

$$\int_a^b L(\gamma_n, \dot\gamma_n) + \Phi(\gamma_n) dt - \int_a^b L(\gamma, \dot\gamma) + \Phi(\gamma) dt$$

$$= \int_a^b \Phi(\gamma_n) - \Phi(\gamma) dt + \int_a^b L(\gamma_n, \dot\gamma_n) - L(\gamma_n, \dot\gamma) dt + \int_a^b L(\gamma_n, \dot\gamma) - L(\gamma, \dot\gamma) dt$$

$$\ge \int_a^b \langle\, \nabla\Phi(\gamma), \gamma_n - \gamma\rangle dt + \int_a^b \langle\, D_v L(\gamma, \dot\gamma), \dot\gamma_n - \dot\gamma\rangle dt + \int_a^b L(\gamma_n, \dot\gamma) - L(\gamma, \dot\gamma) dt.$$

Therefore, we get

$$\int_a^b L(\gamma_n, \dot\gamma_n) + \Phi(\gamma_n) dt \ge \int_a^b \chi_{T_r}(L(\gamma, \dot\gamma) + \Phi(\gamma)) dt + \int_a^b \chi_{T_r}\langle\, D_v L(\gamma, \dot\gamma), \dot\gamma_n - \dot\gamma\rangle dt$$

$$+ \int_a^b \chi_{T_r}\langle\, \nabla\Phi(\gamma), \gamma_n - \gamma\rangle dt + \int_a^b \chi_{T_r}(L(\gamma_n, \dot\gamma) - L(\gamma, \dot\gamma)) dt$$

Notice that

$$\chi_{T_r}|L(\gamma_n, \dot\gamma) - L(\gamma, \dot\gamma)| \le |\gamma_n(t) - \gamma(t)| e_\delta(r),$$

where

$$e_\delta(r) := \sup_{|v|\le r, |x|\le\delta} D_q L(q,v).$$

Hence

$$\int_a^b \chi_{T_r}|L(\gamma_n, \dot\gamma) - L(\gamma, \dot\gamma)| dt \le e_\delta(r) \int_a^b |\gamma_n(t) - \gamma(t)| dt \le e_\delta(r)\sqrt{b-a}||\gamma_n - \gamma||_{L^2(a,b)}.$$

Since $\chi_{T_r} D_v L(\gamma, \dot\gamma) \in L^2$, as $n \to \infty$, we get that

$$\int_a^b \chi_{T_r}\langle\, D_v L(\gamma, \dot\gamma), \dot\gamma_n - \dot\gamma\rangle dt + \int_a^b \chi_{T_r}\langle\, \nabla\Phi(\gamma), \gamma_n - \gamma\rangle dt + \int_a^b \chi_{T_r}(L(\gamma_n, \dot\gamma) - L(\gamma, \dot\gamma)) dt \to 0.$$

Let $r \to \infty$, we get

$$\liminf_{n\to\infty} \int_a^b L(\gamma_n, \dot\gamma_n) + \Phi(\gamma_n) dt = \liminf_{n\to\infty} A[\gamma_n] \ge A[\gamma].$$

Thus γ is the minimizer we want.

Let $\phi \in C_c^\infty(a,b)$. Set $\gamma_\epsilon = \gamma + \epsilon\phi$. Since

$$\lim_{\epsilon\to 0} \frac{A[\gamma_\epsilon] - A[\gamma]}{\epsilon} = 0,$$

we have the Euler-Lagrange equation

$$\frac{d}{dt}(D_v L(\gamma,\dot\gamma)) = D_q L(\gamma,\dot\gamma) \in L^\infty.$$

Also $D_v L(\gamma,\dot\gamma) \in W^{1,2}(a,b)$ since $D_v L(\gamma,\dot\gamma) \in L^2$. Therefore

$$p(t) := D_v L(\gamma,\dot\gamma) \in W^{1,\infty}.$$

Then $\dot\gamma = D_p H(\gamma, p) \in W^{1,\infty}$. Therefore $\gamma \in W^{2,\infty}$.

Now we show that $\dot\gamma \in L^\infty$. Indeed, let $A \in C_c^\infty(a,b)$ and define $S_\epsilon(t) = t + \epsilon A(t)$.

If $|\epsilon| \ll 1$, we may assume that $\dot S_\epsilon(t) \geq \frac{1}{2}$.

Then by the inverse function theorem, $S_\epsilon : [a,b] \to [a,b]$ is a bijection with a differentiable inverse T_ϵ.

Notice that $\gamma_\epsilon(a) = \gamma(S_\epsilon(a)) = \gamma(a) = x$ and $\gamma_\epsilon(b) = \gamma(S_\epsilon(b)) = \gamma(b) = y$.

Moreover,

$$\frac{d}{dt}\gamma(S_\epsilon) = \dot\gamma(S_\epsilon)\dot S_\epsilon \in L^\infty.$$

By the fact that γ is a minimizer, we have $A[\gamma(S_\epsilon)] \geq A[\gamma]$.

Denote $\overline{L} = L + \Phi$. Notice that

$$A[\gamma(S_\epsilon)] = \int_a^b \overline{L}(\gamma(S_\epsilon(t)), \dot\gamma(S_\epsilon(t))\dot S_\epsilon(t)))dt = \int_a^b \overline{L}(\gamma(S_\epsilon(t)), \dot\gamma(S_\epsilon(t))\frac{1}{\dot T_\epsilon(S_\epsilon(t))})dt.$$

Denote $\tau = S_\epsilon(t)$. Notice that $\dot S_\epsilon(t) = 1 + \epsilon \dot A(t)$. Then $\dot T_\epsilon(\tau) = 1 - \epsilon \dot A(\tau) + o(\epsilon)$.

Then

$$A[\gamma(S_\epsilon)] = \int_a^b \overline{L}(\gamma(\tau), \dot\gamma(\tau)\frac{1}{\dot T_\epsilon(\tau)})\dot T_\epsilon(\tau)d\tau$$

$$= \int_a^b \overline{L}(\gamma(\tau), \dot{\gamma}(\tau) + \epsilon \dot{A}\dot{\gamma}(\tau) + o(\epsilon))(1 - \epsilon \dot{A}(\tau) + o(\epsilon))d\tau$$

$$= \int_a^b \overline{L}(\gamma(\tau), \dot{\gamma}(\tau)) + \epsilon \dot{A}(\langle D_v \overline{L}(\gamma, \dot{\gamma}), \dot{\gamma}\rangle - \overline{L}(\gamma, \dot{\gamma})) + o(\epsilon)d\tau.$$

Recall that $D_v\overline{L}(\gamma, \dot{\gamma}) = D_v L(\gamma, \dot{\gamma}) = p$ and $\langle p, \dot{\gamma}\rangle = \overline{L} + H$.

We thus get

$$0 = \lim_{\epsilon \to 0} \frac{A[\gamma(S_\epsilon)] - A[\gamma]}{\epsilon(t)} = \int_a^b \dot{A}(\langle D_v \overline{L}(\gamma, \dot{\gamma}), \dot{\gamma}\rangle d\tau = \int_a^b \dot{A} H(\gamma, p) d\tau.$$

Hence we get the conservation of the Hamiltonian, i.e. $H(\gamma, p)$ is a constant.

By (2.1.5), there exist constants $\frac{1}{4}\lambda_1 > 0$ and $\lambda_0 < 0$, such that

$$\lambda_1 |p|^2 + \lambda_0 \leq H(\gamma(\tau), p(\tau)) = H(\gamma(0), p(0)).$$

Then $p \in L^\infty$ since

$$|p|^2 \leq \frac{H(\gamma(0), p(0)) - \lambda_0}{\frac{1}{4}\lambda_1}.$$

Since $\dot{\gamma} = D_p H(\gamma, p) \in W^{1,\infty}$, we conclude that $\dot{\gamma} \in L^\infty$ as wanted.

□

We assume $\Phi = 0$ for the rest part of this chapter.

We make appropriate additional assumptions on Ω so that

$$\gamma_{s,x}^{t,y}\Big((s,t)\Big) \subset \Omega \qquad \forall 0 \leq s < t < l \leq T. \tag{3.1.4}$$

This means

$$c_s^t(x, y) := \min_\gamma \Big\{ A_s^t(\gamma) : \gamma \in W^{1,2}(s, t; \bar{\Omega}),\ \gamma(s) = x, \gamma(t) = y \Big\}. \tag{3.1.5}$$

We set

$$p_{s,x}^{t,y}(\tau) = D_q L\Big(\gamma_{s,x}^{t,y}(\tau), \dot{\gamma}_{s,x}^{t,y}(\tau)\Big).$$

From the definition above, it is easy to see that we have standard property

$$c_s^t(x,y) + c_t^l(y,z) \geq c_s^l(x,z), \qquad \forall 0 \leq s < t < l \leq T. \tag{3.1.6}$$

If γ is a minimizer in $c_s^l(x,z)$, then we have the following so-called semi–group property.

$$c_s^t\big(x,\gamma(t)\big) + c_t^l\big(\gamma(t),z\big) = c_s^l(x,z), \qquad \forall 0 \leq s < t < l \leq T. \tag{3.1.7}$$

The following properties are standard ([12] [14]).

Proposition 3.1.2. *Assume* $0 \leq s < t \leq T$.

(i) $\gamma_{s,x}^{t,y} \in C^2([s,t])$ *and*

$$\dot{\gamma}_{s,x}^{t,y} = D_p H\Big(\gamma_{s,x}^{t,y},\, p_{s,x}^{t,y}\Big), \quad \dot{p}_{s,x}^{t,y} = -D_q H\Big(\gamma_{s,x}^{t,y},\, p_{s,x}^{t,y}\Big)$$

(ii) There exists a monotone function $\lambda_\Omega : [0,T] \to [0,\infty)$ *such that* $\lim_{\tau\to 0^+} \lambda_\Omega(\tau) = +\infty$ *and for all* $x,y \in \bar{\Omega}$, $c_s^t(\cdot,y)$ *and* $c_s^t(x,\cdot)$ *are* $\lambda_\Omega(t-s)$*–concave in a neighborhood of* $\bar{\Omega}$.

(iii) For all $x,y \in \bar{\Omega}$, $c_s^t(\cdot,y)$ *and* $c_s^t(x,\cdot)$ *are* λ_Ω*–concave in a neighborhood of* $\bar{\Omega}$ *and*

$$-p_{s,x}^{t,y}(s) \in \partial^{\cdot} c_s^t(\cdot,y)(x) \quad \textit{and} \quad p_{s,x}^{t,y}(t) \in \partial^{\cdot} c_s^t(x,\cdot)(y). \tag{3.1.8}$$

(iv) Increasing the value of λ_Ω *if necessary, we may assume that*

$$c_s^t(\cdot,y),\ c_s^t(x,\cdot) \quad \textit{are} \quad \lambda_\Omega(t-s)\text{-}\textit{Lipschitz} \qquad \forall x,y \in \bar{\Omega}.$$

Furthermore, for every $\tau \in (s,t)$ *and* $z := \gamma_{s,x}^{t,y}(\tau)$.

(v) There is a monotone function $\bar{\lambda}_\Omega : [0,T] \to [0,\infty)$ *such that* $\lim_{\tau\to 0^+} \bar{\lambda}_\Omega(\tau) = +\infty$ *and*

$$\Big|DL\Big(\gamma_{s,x}^{t,y}, \dot{\gamma}_{s,x}^{t,y}\Big)\Big| \leq \bar{\lambda}_\Omega \forall\, x,y \in \bar{\Omega}.$$

Remark 3.1.3. *Continuing with the notation of Proposition 3.1.2, for every* $\tau \in (s,t)$ *and* $z := \gamma_{s,x}^{t,y}(\tau)$

$$c_s^\tau(\cdot,z),\ c_\tau^t(z,\cdot) \quad \textit{are} \quad \lambda_\Omega(t-s)\text{-}\textit{Lipschitz} \qquad \forall x,y \in \bar{\Omega}.$$

which means the Lipschitz constant is better than $\lambda_\Omega(\tau - s)$ *or* $\lambda_\Omega(t-\tau)$.

We define $\mathcal{E}_s^t$ to be the set of pairs $(u, v) \in C(\bar{\Omega}) \times C(\bar{\Omega})$ such that

$$u(x) + v(y) \leq c_s^t(x, y), \qquad \forall x, y \in \bar{\Omega}$$

and we denote above inequality as $u \oplus v \leq c_s^t$.

Definition 3.1.4 (c-transform). *Let $u, v : \overline{\Omega} \to \mathbb{R} \cup \{-\infty\}$ and set $c := c_s^t$. The* first c-transform *of u, $u^c : \bar{\Omega} \to \mathbb{R} \cup \{-\infty\}$, and the* second c-transform *of v, $v_c : \bar{\Omega} \to \mathbb{R} \cup \{-\infty\}$, are defined by*

$$u^c(y) := \inf_{x \in \bar{\Omega}} \big\{c(x, y) - u(x)\big\}, \qquad v_c(x) := \inf_{y \in \bar{\Omega}} \big\{c(x, y) - v(y)\big\}. \tag{3.1.9}$$

Lemma 3.1.5. *Let λ_Ω be the function in Proposition 3.1.2 and set $c := c_s^t$. If $u \in C(\bar{\Omega})$, then*

(i) Then $(u_c)^c \geq u$, $(u^c)_c \geq u$, $\big((u_c)^c\big)_c = u_c$, and $\big((u^c)_c\big)^c = u^c$.

(ii) If $u = v^c$ for some $v : \bar{\Omega} \to \mathbb{R} \cup \{-\infty\}$ and $v \not\equiv -\infty$, then:

(a) u is $\lambda_\Omega(t-s)$-Lipschitz and $\lambda_\Omega(t-s)$-semiconcave.

(b) If $\bar{x} \in \bar{\Omega}$ is a point of differentiability of u, $\bar{y} \in \bar{\Omega}$, and $u(\bar{x}) + v(\bar{y}) = c(\bar{x}, \bar{y})$, then $\bar{x}$ is a point of differentiability of $c(\cdot, \bar{y})$ and $\nabla u(\bar{x}) = \nabla_x c(\bar{x}, \bar{y}) = D_v L\Big(\gamma_x^y(0), \dot{\gamma}_x^y(0)\Big)$. Furthermore $\bar{y}$ is uniquely determined.

(iii) If $v = u_c$, then the symmetric analogue of above holds.

(iv) As a consequence of (i-iii), if $K \subset \mathbb{R}$ is bounded, the set

$$\{v^c : v : \bar{\Omega} \to \mathbb{R} \text{ is bounded from above}, \;\; v^c(\bar{\Omega}) \cap K \neq \emptyset\}$$

is compact in $C(\bar{\Omega})$, and weak compact in $W^{1,\infty}(\Omega)$. The uniform norm as well as the $W^{1,\infty}$ norm of any element of $\{v^c : v \in C(\bar{\Omega}), v^c(\bar{\Omega}) \cap K \neq \emptyset\}$ depends only on $\lambda_\Omega(t-s)$ and K. In particular, we can take $K = \{0\}$.*

Proof. (i–iii) of the Lemma 3.1.5 can be found in [4]. We prove (iv) here.

Let $x_1, x_2 \in \bar{\Omega}$ and assume without loss of generality that $v^c(x_2) - v^c(x_1) \geq 0$. For $\epsilon > 0$ arbitrary, choose $y \in \bar{\Omega}$ such that $v^c(x_1) \geq c(x_1, y) - v(y) - \epsilon$.

We conclude that

$$|v^c(x_2) - v^c(x_1)| = v^c(x_2) - v^c(x_1) \leq c(x_2, y) - v(y) - c(x_1, y) + v(y) + \epsilon = c(x_2, y) - c(x_1, y) + \epsilon.$$

Thus

$$|v^c(x_2) - v^c(x_1)| \leq \mathrm{L}ip(c)|x_2 - x_1| + \epsilon.$$

Since ϵ is arbitrary, we conclude that $\mathrm{L}ip(v^c) \leq \mathrm{L}ip(c)$. Let B be a ball centered at the origin and containing K. Choose $x_0 \in \Omega$ such that $v^c(x_0) \in K$. We have $|v^c(x_0)| \leq R$.

Thus

$$|v^c(x) - v^c(x_0)| \leq R + \mathrm{L}ip(c)\mathrm{d}iam(\Omega).$$

This proves (iv).

□

For $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$, we denote by $\Gamma(\mu, \nu)$ the set of Borel measures on $\mathbb{R}^{2d}$ which have μ_0 as the first marginal and μ_1 as the second marginal. For $0 \leq s < t \leq T$, we define

$$C_s^t(\mu, \nu) := \min_{\pi \in \Gamma(\mu,\nu)} \int_{\mathbb{R}^{2d}} c_s^t(x, y)\pi(dx, dy).$$

Remark 3.1.6. *By the standard theory of optimal transportation (see [15]) and by the compactness property provided in Lemma 3.1.5, we have that*

$$C_{t_0}^{t_1}(\delta_{P_{t_0}}, \delta_{P_{t_1}}) = \max_u \big\{u(t_1, P_{t_1}) - u(t_0, P_{t_0})\big\}. \tag{3.1.10}$$

Here B is any bounded set containing $\bar{\Omega}$. The maximum is performed over the set of $u \in C\big([t_0, t_1] \times B\big)$ such that

$$u(t, y) - u(s, x) \leq c_s^t(x, y), \tag{3.1.11}$$

for all $(x, y) \in B^2$ and $(s, t) \subset [t_0, t_1]$. The maximizer exists and is denoted as u^*.

Given a path $\gamma : [0,T] \to \bar{\Omega}$ and a function $u : [0,T] \times \bar{\Omega} \to \mathbb{R}^d$, the following function is pointwise well–defined although it may take the values $\pm\infty$:

$$t \mapsto \frac{d^+}{dt} u(t, \gamma(t)) =: \limsup_{h \downarrow 0} \frac{u(t+h, \gamma(t+h)) - u(t, \gamma(t))}{h}.$$

When γ and u are Lipschitz then so is $t \mapsto u(t, \gamma(t))$. Thus, the latter function is differentiable almost everywhere, which coincides almost everywhere with the bounded Borel function $t \mapsto \frac{d^+}{dt} u(t, \gamma(t))$.

Lemma 3.1.7. *Let $u : [0,T] \times \bar{\Omega} \to \mathbb{R}$ be a Lipschitz function. The following conditions are equivalent:*

(i) For every $0 \le s < t \le T$ and every $x, y \in \bar{\Omega}$,

$$u(t, y) - u(s, x) \le c_s^t(x, y).$$

(ii) For every Lipschitz curve $\gamma : [0,T] \to \bar{\Omega}$

$$\frac{d^+}{dt} u(t, \gamma(t)) \le L(\gamma, \dot{\gamma}) \quad a.e. \ on \quad (0,T).$$

Proof. (i) implies (ii):

For any Lipschitz curve $\gamma : [0,T] \to \overline{\Omega}$

$$\frac{d^+}{dt} u(t, \gamma(t)) = \limsup_{h \downarrow 0} \frac{u(t+h, \gamma(t+h)) - u(t, \gamma(t))}{h} \le \limsup_{h \downarrow 0} \frac{c_t^{t+h}(\gamma(t), \gamma(t+h))}{h}.$$

By the definition of $c_t^{t+h}(\gamma(t), \gamma(t+h))$, we get

$$\frac{d^+}{dt} u(t, \gamma(t)) \le \limsup_{h \downarrow 0} \frac{1}{h} \int_t^{t+h} L(\gamma(\tau), \dot{\gamma}(\tau)) + \Phi(\gamma(\tau)) d\tau = L(\gamma(t), \dot{\gamma}(t)) + \Phi(\gamma(t)).$$

Notice that $L(\gamma(\tau), \dot{\gamma}(\tau)) + \Phi(\gamma(\tau))$ is locally L^1 integrable.

The last equality thus follows from the Lebesgue differentiation theorem.

(ii) implies (i):

For every $0 \le s < t \le t$ and every $x, y \in \overline{\Omega}$, take any Lipschitz curve γ such that $\gamma(s) = x$ and $\gamma(t) = y$.

Let $\Gamma : [0, T] \to \mathbb{R}$ such that $\Gamma(t) = u(t, \gamma(t))$. Then since u and γ are Lipschitz, let a and b be their Lipschitz constant respectively. It is easy to see that Γ is Lipschitz with Lipschitz constant $a(1 + b)$.

Indeed, we have that

$$|\Gamma(t+h) - \Gamma(t)| = |u(t+h, \gamma(t+h)) - u(t, \gamma(t))|$$

$$\le a(|h| + |\gamma(t+h) - \gamma(t)| \le (a + ab)|h|.$$

Thus, Γ is differentiable almost everywhere and

$$u(t, y) - u(s, x) = \Gamma(t) - \Gamma(s) = \int_s^t \dot{\Gamma}(\tau) d\tau = \int_s^t \lim_{h \to 0} \frac{\Gamma(\tau + h) - \Gamma(\tau)}{h} d(\tau)$$

$$= \int_s^t \frac{d^+}{dt} u(\tau, \gamma(\tau)) \le \int_s^t L(\gamma(\tau), \dot{\gamma}(\tau)) + \Phi(\gamma(\tau)) d\tau.$$

Take inf over all such γ's, we get $[u(t, y) - u(s, x) \le c_s^t(x, y)$ as wanted.

□

3.2 Maximizers in $[t_0, t_1]$

Given $t_0 < t_1$ in $[0, T]$ and B any bounded set containing $\bar{\Omega}$, let $u^* \in C\big([t_0, t_1], B\big)$ be a maximizer in the problem at the right hand side of (3.1.10) and set

$$\gamma^* := \gamma_{P_{t_0}}^{P_{t_1}}$$

so that

$$\mathcal{A}_{t_0}^{t_1}(\gamma^*) = C_{t_0}^{t_1}(\delta_{P_{t_0}}, \delta_{P_{t_1}}) = u^*(t_1, P_{t_1}) - u^*(t_0, P_{t_0}). \tag{3.2.1}$$

Remark 3.2.1. *Let* $t \in (t_0, t_1)$.

(i) We have

$$c_t^{t_1}(\gamma_t^*, P_{t_1}) = u^*(t_1, P_{t_1}) - u^*(t, \gamma_t^*) \quad \textit{and} \quad u^*(t, \gamma_t^*) = u^*(t_0, P_{t_0}) + c_{t_0}^t(P_{t_0}, \gamma_t^*). \tag{3.2.2}$$

In fact, by a direct application of Lemma 3.1.5 (i), we can choose $u^* \in C\big([t_0, t_1], \bar{\Omega}\big)$ *such that*

$$u^*(t, y) = \min_{\bar{x} \in \bar{\Omega}} \big\{ c_{t_0}^t(\bar{x}, y) + u^*(t_0, \bar{x}) \big\}, \qquad \forall y \in \bar{\Omega}. \tag{3.2.3}$$

Then

$$u^*(t_0, x) = \max_{\bar{y} \in \bar{\Omega}} \big\{ - c_{t_0}^{t_1}(x, \bar{y}) + u^*(t_1, \bar{y}) \big\}, \qquad \forall x \in \bar{\Omega} \tag{3.2.4}$$

and

$$u^*(t_0, P_{t_0}) = 0. \tag{3.2.5}$$

(ii) By Lemma 3.1.5 (iv), (3.2.4) *implies that* $u^*(t_0, \cdot)$ *is* $\lambda_\Omega(t_1 - t_0)$*-Lipschitz and* $\lambda_\Omega(t_1 - t_0)$*-semiconvex.*

We shall prove in the next lemma that $\nabla u^*(t, \gamma^*(t))$ exists for any $t \in (t_0, t_1)$. Since

$$u^*(t + h, \gamma^*(t + h)) - u^*(t, \gamma^*(t)) = \int_t^{t+h} L(\gamma^*, \dot{\gamma}^*) d\tau,$$

it is obvious that $u^*(\cdot, \gamma^*)$ is differentiable at t. This means, we should expect from Lemma 3.2.2 that $\partial_t u^*(t, \gamma^*(t))$ exists, which means that u^* is differentiable at $(t, \gamma^*(t))$.

Lemma 3.2.2. *Let* (u^*, γ^*) *be as in Remark 3.2.1. Then for any* $t \in (t_0, t_1)$*, we have*

$$u^*(t, \gamma_t^*) = \min_{x \in \bar{\Omega}} \big\{ c_{t_0}^t(x, \gamma_t^*) + u^*(t_0, x) \big\} = \max_{y \in \bar{\Omega}} \big\{ - c_t^{t_1}(\gamma_t^*, y) + u^*(t_1, y) \big\}.$$

Thus, $u^*(t, \cdot)$ *is continuously differentiable at* γ_t^*.

Proof. Set

$$\alpha := \min_{x \in \bar{\Omega}} \big\{ c_{t_0}^t(x, \gamma_t^*) + u^*(t_0, x) \big\}, \qquad \beta := \max_{y \in \bar{\Omega}} \big\{ - c_t^{t_1}(\gamma_t^*, y) + u^*(t_1, y) \big\}.$$

We use (3.1.11) to conclude that $u^*(t,\gamma_t^*) \le \alpha$. Since α is an infimum, by the second identity in (3.2.2), we have

$$\alpha \le c_{t_0}^t(P_{t_0},\gamma_t^*) + u^*(t_0,P_{t_0}) = u^*(t,\gamma_t^*).$$

This proves that $u^*(t,\gamma_t^*) = \alpha$.

The identity $u^*(t,\gamma_t^*) = \beta$ is obtained similarly by first using (3.1.11) and second using the first identity in (3.2.2).

Let $z \in \bar{\Omega}$. By (3.1.11) and the second identity in (3.2.2) we have

$$u^*(t,z) \le u^*(t_0,P_{t_0}) + c_{t_0}^t(P_{t_0},z), \quad \text{and} \quad u^*(t_0,P_{t_0}) = u(t,\gamma_t^*) - c_{t_0}^t(P_{t_0},\gamma_t^*).$$

Thus,

$$u^*(t,z) \le u(t,\gamma_t^*) - c_{t_0}^t(P_{t_0},\gamma_t^*) + c_{t_0}^t(P_{t_0},z).$$

We apply Lemma 3.1.5 (ii) to find $\delta_t \in \mathbb{R}^d$ and a constant A depending on t_0 such that

$$u^*(t,z) \le u(t,\gamma_t^*) + \langle \delta_t, z-\gamma_t^*\rangle + A|z-\gamma_t|^2. \tag{3.2.6}$$

Similarly, using (3.1.11) and the first identity in (3.2.2), increasing the value of A if necessary, we find $\bar{\delta}_t \in \mathbb{R}^d$ such that

$$u^*(t,z) \ge u(t,\gamma_t^*) + \langle \bar{\delta}_t, z-\gamma_t^*\rangle - A|z-\gamma_t|^2 \tag{3.2.7}$$

This, together with (3.2.6), means that the sub differential and the super differential of $u^*(t,\cdot)$ at γ_t^* is not empty. Thus, $u^*(t,\cdot)$ is differentiable at γ_t^*. Since $u^*(t,\cdot)$ is a semi–concave function, then it is continuously differentiable at γ_t^*.

□

3.3 Maximizers on the whole interval $[0,T]$

Assume (u^*,γ^*) is as in Remark 3.2.1 and satisfies (3.2.3).

Set

$$u(t,a) := \max_{x\in\bar{\Omega}}\{-c_t^{t_0}(a,x) + u^*(t_0,x)\} \quad \text{if} \quad t \in [0,t_0), \tag{3.3.1}$$

$$u(t,a) := u^*(t,a) \quad \text{if} \quad t \in [t_0,t_1], \tag{3.3.2}$$

and

$$u(t,a) := \min_{y\in\bar{\Omega}}\{c_{t_1}^t(y,a) + u^*(t_1,y)\} \quad \text{if} \quad t \in (t_1,T]. \tag{3.3.3}$$

For $t \in (t_1, T]$, we use (3.2.3) to conclude that

$$u(t,a) = \min_{x,y\in\bar{\Omega}}\{c_{t_1}^t(y,a) + c_{t_0}^{t_1}(x,y) + u^*(t_0,x)\}.$$

We combine (3.1.6) and (3.1.7) to conclude that

$$u(t,a) = \min_{x\in\bar{\Omega}}\{c_{t_0}^t(x,a) + u^*(t_0,x)\} \qquad \forall t \in (t_1,T].$$

By (3.2.3), this means

$$u(t,a) = \min_{x\in\bar{\Omega}}\{c_{t_0}^t(x,a) + u^*(t_0,x)\} \qquad \forall t \in [t_0,T]. \tag{3.3.4}$$

Proposition 3.3.1. *We have*

$$u : [0,t_0] \times \bar{\Omega} \to \mathbb{R} \quad \textit{and} \quad u : [t_0,T] \times \bar{\Omega} \to \mathbb{R} \quad \textit{are Lipschitz}$$

Since $u(t_0,\cdot)$ is Lipschitz, we conclude that u is Lipschitz on $[0,T] \times \bar{\Omega}$.

Proof. By Remark 3.2.1 (ii), $u(t_0,\cdot)$ is Lipschitz. In light of Lemma 3.1.5 (iv), the formulation used in (3.3.4) shows that for each $t \in (t_0,T]$, $u(t,\cdot)$ is $\lambda_\Omega(t-t_0)$-Lipschitz. Similarly, by (3.3.1), we conclude that $u(t,\cdot)$ is $\lambda_\Omega(t_0-t)$-Lipschitz for $t \in [0,t_0)$.

Our goal is to improve these statements so that the Lipschitz constant of $u(t,\cdot)$ is independent of t. To achieve this goal, we first show that u is Lipschitz in $[0,T]\times\Omega$. Since for any $t \in [0,T]$, $u(t,\cdot)$ is continuous on $\bar{\Omega}$, by a simple approximation argument, u is globally Lipschitz.

Part 1. Lipschitz in space. Let $t \in (t_0, T]$. Recall that by Remark 3.2.1, $u(t_0, \cdot)$ is κ–Lipschitz for some $\kappa > 0$. Let $y_1, y_2 \in \Omega$. Interchanging the role of y_1 and y_2 if necessary, we may assume that $u(t, y_2) - u(t, y_1) \geq 0$. Take $\gamma \in W^{1,2}(t_0, t; \bar{\Omega})$ such that

$$u(t, y_1) = u(t_0, \gamma(t_0)) + \int_{t_0}^{t} L(\gamma, \dot{\gamma}) d\tau, \qquad \gamma(t) = y_1.$$

We first establish a control on $\dot{\gamma}$. Indeed,

$$u(t_0, \gamma(t_0)) + \int_{t_0}^{t} L(\gamma, \dot{\gamma}) d\tau \leq u(t_0, y_1) + \int_{t_0}^{t} L(y, 0) d\tau \leq \|u(t_0, \cdot)\|_{C(\bar{\Omega})} + T\|L(\cdot, 0)\|_{C(\bar{\Omega})}.$$

This implies

$$\int_{t_0}^{t} L(\gamma, \dot{\gamma}) d\tau \leq 2\|u(t_0, \cdot)\|_{C(\bar{\Omega})} + T\|L(\cdot, 0)\|_{C(\bar{\Omega})}.$$

We use (2.1.4) to conclude that

$$\int_{t_0}^{t} |\dot{\gamma}|^2 d\tau \leq 2\lambda_1^{-1}\|u(t_0, \cdot)\|_{C(\bar{\Omega})} + T\lambda_1^{-1}\|L(\cdot, 0)\|_{C(\bar{\Omega})} - \lambda_0. \tag{3.3.5}$$

By Cauchy-Schwarz inequality,

$$\int_{t_0}^{t} |\dot{\gamma}| d\tau \leq \sqrt{T}\sqrt{2\lambda_1^{-1}\|u(t_0, \cdot)\|_{C(\bar{\Omega})} + T\lambda_1^{-1}\|L(\cdot, 0)\|_{C(\bar{\Omega})} - \lambda_0}. \tag{3.3.6}$$

Set

$$R(\tau) := \gamma(\tau) + y_2 - y_1, \qquad \forall \tau \in [0, t].$$

Note that $R \in W^{1,2}(t_0, t; \mathbb{R}^d)$ and $R(t) = y_2$. Without loss of generality, assume that y_1 and y_2 are chosen so that the range of R is contained in $\bar{\Omega}$. We have

$$\begin{aligned} u(t, y_2) \leq & u(t_0, R(t_0)) + \int_{t_0}^{t} L(R, \dot{R}) d\tau \\ = & u(t, y_1) + u(t_0, R(t_0)) - u(t_0, \gamma(t_0)) + \int_{t_0}^{t} \Big(L(R, \dot{R}) - L(\gamma, \dot{\gamma}) \Big) d\tau. \end{aligned}$$

Thus

$$u(t, y_2) - u(t, y_1) \leq \mathrm{Lip}\big(u(t_0, \cdot)\big)|y_1 - y_2| + \int_{t_0}^{t} \int_0^1 \langle D_q L\big(\gamma + s(y_2 - y_1), \dot{\gamma}\big), y_2 - y_1 \rangle ds d\tau.$$

Since by (2.1.3), DL is κ_0–Lipschitz, we have

$$u(t,y_2)-u(t,y_1)\le \mathrm{L}ip\big(u(t_0,\cdot)\big)|y_1-y_2|+\kappa_0\int_{t_0}^{t}\int_0^1\Big(|\gamma|+|\dot\gamma|+|y_2-y_1|\Big)|y_2-y_1|dsd\tau.$$

Using (3.3.6) and the fact that the range of γ is contained in the bounded set $\bar\Omega$, there is a constant K independent of t, y_1 and y_2 such that

$$|u(t,y_2)-u(t,y_1)|\le K|y_1-y_2|.$$

Thus the Lipschitz constant of $u(t,\cdot)$ on $\bar\Omega$ is independent of $t\in[t_0,T]$. We apply the same reasoning to conclude that the Lipschitz constant of $u(t,\cdot)$ on $\bar\Omega$ is independent of $t\in[0,t_0]$.

Part 2. Lipschitz in time. Let $y\in\Omega$ and let $t_1\le s_1<s_2\le T$. By the semi–group property in 3.1.7, choosing the trajectory $\gamma(\tau)\equiv y$, we have

$$u(s_2,y)-u(s_1,y)\le\int_{s_1}^{s_2}L(y,0)d\tau\le(s_2-s_1)\max_{\bar\Omega}L(\cdot,0). \tag{3.3.7}$$

Next, choose $\gamma\in W^{1,2}(0,s_2;\bar\Omega)$ such that

$$u(s_2,y)=u(s_1,\gamma(s_1))+\int_{s_1}^{s_2}L(\gamma,\dot\gamma)d\tau,\qquad \gamma(s_2)=y.$$

Since $u(s_1,\cdot)$ is K–Lipschitz,

$$u(s_1,\gamma(s_1))\ge u(s_1,y)-K|y-\gamma(s_1)|=u(s_1,y)-K|\gamma(s_2)-\gamma(s_1)|.$$

Thus

$$u(s_1,\gamma(s_1))\ge u(s_1,y)-K\int_{s_1}^{s_2}|\dot\gamma|d\tau.$$

Using the definition of γ, we conclude that

$$u(s_2,y)\ge u(s_1,y)+\int_{s_1}^{s_2}\big(L(\gamma,\dot\gamma)-K|\dot\gamma|\big)d\tau.$$

Thanks to (2.1.4), there is a constant K_0 bigger than $\max_{\bar\Omega}L(\cdot,0)$ such that

$$u(s_2,y)-u(s_1,y)\ge -K_0(s_2-s_1).$$

This, together with (3.3.7), implies $u(\cdot, y)$ is K_0–Lipschitz on $[t_0, T]$ for every $y \in \Omega$. Increasing the value of K_0 is necessary and using an analogous argument, we obtain that $u(\cdot, y)$ is K_0–Lipschitz on $[0, t_0]$ for every $y \in \Omega$.

$\square$

Lemma 3.3.2. *We have*

$$u(t, b) - u(s, a) \leq c_s^t(a, b), \qquad \forall 0 \leq s < t \leq T.$$

Proof. By (3.3.4), u satisfied the semi–group property on $[t_0, T]$. It suffices to prove the lemma only for $s \in [0, t_0)$. If $s < t \leq t_0$, we need to check that

$$u^*(t_0, x) \leq u(s, a) + c_s^t(a, b) + c_t^{t_0}(b, x).$$

This inequality holds since

$$u(s, a) \geq -c_s^{t_0}(a, x) + u^*(t_0, x) \quad \text{and} \quad c_s^{t_0}(a, x) \leq c_s^t(a, b) + c_t^{t_0}(b, x).$$

Thus it remains to study the case when $s \in [0, t_0)$ and $t \in [t_0, T]$, which are the conditions we impose in the sequel that $s \in [0, t_0)$ and $t \in [t_0, T]$. Let $a, b \in \bar{\Omega}$. By (3.1.7), we can choose $\gamma : [s, t] \to \bar{\Omega}$ such that

$$c_s^t(a, b) = c_s^{t_0}\big(a, \gamma(t_0)\big) + c_{t_0}^t\big(\gamma(t_0), b\big),$$

and

$$u(t, b) \leq c_{t_0}^t\big(\gamma(t_0), b\big) + u(t_0, \gamma(t_0)) = c_s^t(a, b) - c_s^{t_0}\big(a, \gamma(t_0)\big) + u(t_0, \gamma(t_0)).$$

We use (3.3.4) and the fact that $s \leq t_0$ to conclude that

$$u(t, b) \leq c_s^t(a, b) + u(s, a).$$

This concludes the proof.

$\square$

3.4 Duality property in our particular case

We are halfway towards establishing the duality property in our particular case.

We endow $\Gamma_T = C([0,T];\mathbb{R}^d)$ with the supremum norm. Let $(\sigma, v) \in \Sigma_T(\sigma_0, \sigma_T|\psi)$ and let

$$\mathcal{E} := \Big\{(q,\gamma) : \ q \in \mathbb{R}^d, \quad \gamma \in AC_2(0,T,\mathbb{R}^d), \quad \gamma(0) = q, \quad \dot{\gamma} = v(\cdot, \gamma) \quad \mathcal{L}^1\text{-a.e.}\Big\}$$

By Theorem 8.2.1 [4], there exists a Borel probability measure η on $\mathbb{R}^d \times \Gamma_T$ which is concentrated on the set $\mathcal{E}$ and such that

$$\int_{\mathbb{R}^d} \varphi(q)\sigma_t(dq) = \int_{\mathbb{R}^d\times\Gamma_T} \varphi(\gamma(t))\eta(dq, d\gamma), \qquad \forall \varphi \in C_b(\mathbb{R}^d), \ t \in [0,T]. \tag{3.4.1}$$

Following [4], we shall use the notation

$$\sigma \equiv \sigma^\eta. \tag{3.4.2}$$

Lemma 3.4.1. *The identity* (3.4.1) *holds for every bounded Borel function* $\varphi : \mathbb{R}^d \to \mathbb{R}$. *This implies the identity holds for every non-negative Borel function* $\varphi : \mathbb{R}^d \to \mathbb{R}$.

Proof. It suffices to prove the lemma when $\varphi = \chi_A$ and $A \subset \mathbb{R}^d$ is a Borel set. Fix $t \in [0,T]$ and fix a Borel set $A \subset \mathbb{R}^d$. For each natural number $n > 1$, we can find a compact set $K_n^0 \subset A$ such that

$$\sigma_t(A \setminus K_n) < \frac{1}{n}.$$

Since $A \times \Gamma_T$ is a Borel subset and $\mathbb{R}^d \times \Gamma_T$ is a separable metric space, we can find a compact set $\tilde{K} \subset A \times \Gamma_T$ (see section 5.1 [4]) such that

$$\eta\Big(A \times \Gamma_T \setminus \tilde{K}\Big) < \frac{1}{n}.$$

The projection $P^d_{\mathbb{R}} : \mathbb{R}^d \times \Gamma_T \to \mathbb{R}^d$ be a continuous map, we conclude that $K_n^1 := P^d_{\mathbb{R}}(\tilde{K})$ is a compact subset of $\mathbb{R}^d$ contained in A. Note that the compact set

$$K_n := \cup_{j=1}^n \big(K_n^0 \cup K_n^1\big)$$

is contained in A and satisfies

$$K_{n-1} \subset K_n, \quad \sigma_t(A \setminus K_n) < \frac{1}{n}, \qquad \eta\Big(A \times \Gamma_T \setminus K_n \times \Gamma_T\Big) < \frac{1}{n}.$$

As done above, we can find O_n^0, O_n^1 open sets containing A and such that

$$\sigma_t(O_n^0 \setminus A) < \frac{1}{n}, \quad \eta\Big(O_n^1 \times \Gamma_T \setminus A \times \Gamma_T\Big) < \frac{1}{n}.$$

Set

$$O_n = \cap_{j=1}^n \big(O_n^0 \cap O_n^1\big).$$

We have

$$O_n \subset O_{n-1}, \quad \sigma_t(O_n \setminus A) < \frac{1}{n}, \quad \eta\Big(O_n \times \Gamma_T \setminus A \times \Gamma_T\Big) < \frac{1}{n}.$$

By Urysohn's lemma there exists a continuous function $g_n \in C(\mathbb{R}^d, [0,1])$ such that

$$\bar{g}_n|_{K_n} \equiv 1, \quad \bar{g}_n|_{O_n} \equiv 0.$$

Set

$$g_n = \max\{\bar{g}_1, \cdots, \bar{g}_n\}.$$

Note that

$$g_{n-1} \leq g_n, \quad g_n|_{K_n} \equiv 1, \quad g_n|_{O_n} \equiv 0$$

Therefore

$$g(x) := \lim_{n\to\infty} g_n(x) \in [0,1]$$

exists for every $x \in \mathbb{R}^d$. The Borel sets

$$K := \cup_{n=1}^\infty K_n, \quad O := \cap_{n=1}^\infty O_n$$

satisfy

$$g(x) = 1, \quad \forall x \in K, \quad \text{and} \quad g(x) = 0, \quad \forall x \in O.$$

Moreover,

$$K \subset A \subset O, \qquad \sigma_t(K) = \sigma_t(A), \qquad \eta(K \times \Gamma_T) = \eta(A \times \Gamma_T).$$

Since

$$\int_{\mathbb{R}^d \times \Gamma_T} \chi_{K_n \times \Gamma_T}(\gamma(t))\eta(dq, d\gamma) \leq \int_{\mathbb{R}^d \times \Gamma_T} g_n(\gamma(t))\eta(dq, d\gamma) \leq \int_{\mathbb{R}^d \times \Gamma_T} \chi_{O_n \times \Gamma_T}(\gamma(t))\eta(dq, d\gamma),$$

we conclude that

$$\eta\Big(O \times \Gamma_T\Big) - \frac{2}{n} \leq \int_{\mathbb{R}^d \times \Gamma_T} g_n(\gamma(t))\eta(dq, d\gamma) \leq \eta\Big(K \times \Gamma_T\Big) + \frac{2}{n}.$$

Let n tend to ∞ and apply the dominated convergence theorem. We obtain

$$\int_{\mathbb{R}^d \times \Gamma_T} \chi_{O \times \Gamma_T}(\gamma(t))\eta(dq, d\gamma) \leq \int_{\mathbb{R}^d \times \Gamma_T} g(\gamma(t))\eta(dq, d\gamma) \leq \int_{\mathbb{R}^d \times \Gamma_T} \chi_{K \times \Gamma_T}(\gamma(t))\eta(dq, d\gamma).$$

In light of the inclusions $K \subset A \subset O$, we conclude that

$$g(\gamma(t)) = \chi_A(\gamma(t)), \text{ for } (q, \gamma)\ \eta\text{-a.e.}. \tag{3.4.3}$$

It is straightforward to obtain the identity

$$g = \chi_A, \qquad \sigma_t\text{-a.e.} \tag{3.4.4}$$

By (3.4.1),

$$\int_{\mathbb{R}^d} g_n(q)\sigma_t(dq) = \int_{\mathbb{R}^d \times \Gamma_T} g_n(\gamma(t))\eta(dq, d\gamma).$$

We use the dominated convergence theorem to conclude that

$$\int_{\mathbb{R}^d} g(q)\sigma_t(dq) = \int_{\mathbb{R}^d \times \Gamma_T} g(\gamma(t))\eta(dq, d\gamma).$$

This, combined with (3.4.3) and (3.4.4), yields the desired result.

□

Remark 3.4.2. *Set $A := supp(\sigma_t)$. Using the definition of $\mathcal{E}$ and Lemma 3.4.1, we have*

$$1 = \eta\Big\{(q,\gamma) \in \mathbb{R}^d \times \Gamma_T : \ \gamma(0) = q, \ \gamma(t) \in \bar{\Omega}\Big\}.$$

Since $\sigma_0 = \delta_{P_0}$ and $\sigma_T = \delta_{P_T}$, we conclude that

$$\gamma(0) = P_0, \quad \gamma(T) = P_T, \quad \text{and} \quad v(t,\gamma(t)) = \dot{\gamma}(t) \quad \text{for} \quad \mathcal{L}^1 a.e.(q,\gamma) \in supp(\eta).$$

Recall $\mathcal{U}_0$ given at the beginning of this chapter.

Proposition 3.4.3. *For any $(\bar{u},\bar{h}) \in \mathcal{U}_0$, we have*

$$J(\bar{u},\bar{h}) \leq \mathcal{A}_L^F[\sigma,v] := J$$

Proof. For each $M > 0$, set

$$h_M(t,q) = \min\{M, \bar{h}(t,q)\}$$

so that h_M is bounded. Since $\bar{h} \geq h_M$, we have that $(\bar{u}, h_M) \in \mathcal{U}_0$.

We use Lemma 3.4.1 and the fact that L is bounded from below (in fact non-negative) and the definition of $\mathcal{E}$ to obtain that

$$\begin{aligned}\mathcal{A}_L^F[\sigma,v] &= \int_0^T \left(\int_{\mathbb{R}^d\times\Gamma_T} L\Big(\gamma(t), v(t,\gamma(t))\Big)\eta(dq,d\gamma)\right)dt \\ &= \int_0^T \left(\int_{\mathbb{R}^d\times\Gamma_T} L\Big(\gamma(t), \dot{\gamma}(t)\Big)\eta(dq,d\gamma)\right)dt.\end{aligned}$$

We apply Fubini's theorem and then use the fact that $(\bar{u},\bar{h}) \in \mathcal{U}_0$ to conclude that

$$J \geq \int_{\mathbb{R}^d\times\Gamma_T} \left(\bar{u}(T,\gamma(T)) - \bar{u}(0,\gamma(0)) + \int_0^T h_M(\tau,\gamma(\tau)d\tau\right)\eta(dq,d\gamma).$$

We use Remark 3.4.2 and apply Fubini's theorem to conclude that

$$\begin{aligned}J \geq& \bar{u}(T,P_T) \ - \bar{u}(0,P_0) + \int_0^T \left(\int_{\mathbb{R}^d\times\Gamma_T} h_M(\tau,\gamma(\tau))\eta(dq,d\gamma)\right)d\tau \\ =& \bar{u}(T,P_T) \ - \bar{u}(0,P_0) + \int_0^T \left(\int_{\mathbb{R}^d} h_M(\tau,q)\sigma_\tau(dq)\right)d\tau.\end{aligned}$$

As $h_M \geq 0$, we can apply Fatou's lemma to conclude that

$$J \geq \bar{u}(T, P_T) - \bar{u}(0, P_0) + \int_0^T \left(\int_{\mathbb{R}^d} h(\tau, q)\sigma_\tau(dq) \right) d\tau,$$

which is the desired result.

□

3.5 Duality and minimizers in a particular case

In this section, we show that our duality holds in a case with a special boundary measure ψ given below.

Let's consider the following special boundary data

$$\psi_t := \begin{cases} \delta_{P_t} & \text{if } t \in [0, T] \setminus [t_0, t_1] \\ \frac{1}{2}\delta_{P_t} & \text{if } t \in [t_0, t_1] \end{cases}$$

and define

$$\sigma_t := \begin{cases} \delta_{P_t} & \text{if } t \in [0, T] \setminus [t_0, t_1] \\ \frac{1}{2}\delta_{P_t} + \frac{1}{2}\delta_{\gamma_t^*} & \text{if } t \in [t_0, t_1] \end{cases}$$

We define

$$h(t, q) := \begin{cases} L(P_t, \dot{P}_t) - \frac{d^+}{dt} u(t, P_t) & \text{if } q = P_t \\ 0 & \text{if } q \neq P_t \end{cases}$$

We combine Lemma 3.1.7 and Lemma 3.3.2 to conclude that

$$h \geq 0.$$

By Proposition 3.3.1

$$\sup_{[0,T] \times \bar{\Omega}} h < +\infty.$$

Lemma 3.5.1. *Let u be as in (3.3.1 - 3.3.3) so that in particular Lemma 3.3.2 holds. Let h be defined as before Lemma 2.4.3. We have*

$$J(u,h) = \mathcal{A}_L^F[\sigma, v].$$

Here σ is defined as the σ_t above and $(\sigma, v) \in \Sigma_T(\sigma_0, \sigma_T|\psi)$.

Proof. Since $u_{t_1}(P_{t_1}) = u_{t_1}(\gamma_{t_1})$ and $u_{t_0}(P_{t_0}) = u_{t_0}(\gamma_{t_0})$, we conclude that

$$\begin{aligned} u_T(P_T) - u_0(P_0) =& u_T(P_T) - u_{t_1}(P_{t_1}) + \frac{1}{2}\Big(u_{t_1}(P_{t_1}) - u_{t_0}(P_{t_0})\Big) + \frac{1}{2}\Big(u_{t_1}(\gamma_{t_1}) - u_{t_0}(\gamma_{t_0})\Big) \\ &+ u_{t_0}(P_{t_0}) - u_0(P_0). \end{aligned}$$

We have

$$\begin{aligned} u_T(P_T) - u_0(P_0) =& \int_{t_1}^T \Big(-h(\tau, P_\tau) + L(P_\tau, \dot{P}_\tau)\Big) d\tau \\ &+ \frac{1}{2}\int_{t_0}^{t_1} \Big(-h(\tau, P_\tau) + L(P_\tau, \dot{P}_\tau)\Big) d\tau \\ &+ \frac{1}{2}\int_{t_0}^{t_1} \Big(-h(\tau, \gamma_\tau) + L(\gamma_\tau, \dot{\gamma}_\tau)\Big) d\tau \\ &+ \frac{1}{2}\int_{0}^{t_0} \Big(-h(\tau, P_\tau) + L(P_\tau, \dot{P}_\tau)\Big) d\tau. \end{aligned}$$

Then

$$u_T(P_T) - u_0(P_0) = \mathcal{A}_L^F[\sigma, v] - \int_{(0,t_0)\cup(t_1,T)} h(\tau, P_\tau) d\tau - \frac{1}{2}\int_{t_0}^{t_1} h(\tau, P_\tau) d\tau.$$

Thus

$$u_T(P_T) - u_0(P_0) = \mathcal{A}_L^F[\sigma, v] - \int_0^T h_t(q)\psi_t(dq).$$

This concludes the proof.

□

Lemma 3.5.2. *Let u be as in (3.3.1 - 3.3.3) so that in particular Lemma 3.3.2 holds. Let h be defined as before Lemma 2.4.3. Then, we have*

$$J(u,h) = \sup_{\mathcal{U}_0} J = \mathcal{A}_L^F[\sigma, v].$$

Proof. By Proposition 3.4.3 we have

$$\sup_{\mathcal{U}_0} J \leq \mathcal{A}_L^F[\sigma, v].$$

In light of Lemma 3.5.1, it suffices to show that $(u, h) \in \mathcal{U}_0$.

By Proposition 3.3.1, u is Lipschitz and h is bounded from above by a constant. We combine Lemma 3.1.7 and Lemma 3.3.2 to conclude that $h \geq 0$. As a limit of Borel functions, $\frac{d^+}{dt}u$ is also a Borel function. Thus, h is a Borel function.

It remains to show that for any Lipschitz curve $\bar{\gamma} : [0, T] \to \bar{\Omega}$, we have

$$u(t, \bar{\gamma}_t) - u(s, \bar{\gamma}_s) \leq \int_s^t \Big(L(\bar{\gamma}_\tau, \dot{\bar{\gamma}}_\tau) - h(\tau, \bar{\gamma}_\tau)\Big) d\tau, \qquad \forall 0 \leq s < t \leq T. \tag{3.5.1}$$

Let $\bar{\gamma} : [0, T] \to \bar{\Omega}$ be a Lipschitz curve and set

$$S_0 = \big\{t \in (0, t_0) \,:\, P_t = \bar{\gamma}(t)\big\}, \quad S_1 = \Big\{t \in (0, t_0) \,:\, \dot{P}_t = \dot{\bar{\gamma}}(t),\ \frac{d}{dt}u(t, \bar{\gamma}(t)) = \frac{d}{dt}u(t, P_t)\Big\}.$$

Recall that the set $S_0 \setminus S_1$ is of null Lebesgue measure (see Lemma 6.3.1).

Let t be a point such that $u(\cdot, \bar{\gamma})$ is differentiable at t.

On one hand, if $t \in S_0 \cap S_1$, then

$$\frac{d^+}{dt}u(t, \bar{\gamma}_t) + h(t, \bar{\gamma}_t) - L(\bar{\gamma}_t, \dot{\bar{\gamma}}_t) = \frac{d^+}{dt}u(t, P_t) + h(t, P_t) - L(P_t, \dot{P}_t) = 0.$$

On the other hand, if $\bar{\gamma}_t \neq P_t$, we combine Lemma 3.1.7 and Lemma 3.3.2 to conclude that

$$\frac{d^+}{dt}u(t, \bar{\gamma}_t) + h(t, \bar{\gamma}_t) - L(\bar{\gamma}_t, \dot{\bar{\gamma}}_t) = \frac{d^+}{dt}u(t, \bar{\gamma}_t) - L(\bar{\gamma}_t, \dot{\bar{\gamma}}_t) \leq 0.$$

In conclusion,

$$\frac{d^+}{dt}u(t, \bar{\gamma}_t) + h(t, \bar{\gamma}_t) - L(\bar{\gamma}_t, \dot{\bar{\gamma}}_t) \leq 0 \qquad \mathcal{L}^1 \text{ a.e.}$$

Integrating, we have (3.5.1), which concludes the proof of the lemma.

□

3.6 Properties of optimal curves

In this section, we discuss optimal curves properties in probabilistic representation of measures.

We set

$$\Gamma_{T,2} := AC_2(0,T;\bar{\Omega})$$

Recall that $\Gamma_{T,2}$ is a subset of the Hilbert space $W^{1,2}(0,T;\mathbb{R}^d)$. For $\gamma^0 \in \Gamma_{T,2}$ and $\delta > 0$, we denote by $\mathbb{B}_\delta(\gamma_0)$ the ball in $\Gamma_{T,2}$, of radius δ, centered at γ_0.

Given $\gamma \in \Gamma_{T,2}$, if there exists $s_0 \in (0,T)$ such that $\gamma(s_0) =: x_0 \in \Omega$, we define

$$t^-(\gamma, x_0) := \inf_{s\in[0,T]} \big\{s :\ s < s_0, \gamma\big((s,s_0)\big) \subset \Omega\big\}$$

and

$$t^+(\gamma, x_0) := \inf_{s\in[0,T]} \big\{s :\ s > s_0, \gamma\big((s_0,s)\big) \subset \Omega\big\}.$$

For $\gamma \in AC_2(s,t;\bar{\Omega})$ such that $0 \le s < t \le T$, we define

$$E_s^t(\gamma) := c_s^t\big(\gamma(s),\gamma(t)\big) - \int_s^t L(\gamma,\dot{\gamma})d\tau.$$

When $\gamma \in AC_2(0,T;\bar{\Omega})$, $E_s^t(\gamma)$ means $E_s^t(\gamma|_{[s,t]})$.

Let $s,t \in [0,T]$ be such that $s < t$. Since $\gamma_{s,x}^{t,y}$ satisfies the system of differential equations

$$\frac{d}{dt} D_v L(\gamma_{s,x}^{t,y}, \dot{\gamma}_{s,x}^{t,y}) = D_q L(\gamma_{s,x}^{t,y}, \dot{\gamma}_{s,x}^{t,y}),$$

the uniform bound $\|\gamma_{s,x}^{t,y}\|_{W^{1,2}(s,t)}$ obtained in Proposition 3.1.2 implies that

$$S_{st} := \sup_{x,y\in\bar{\Omega}} \|\gamma_{s,x}^{t,y}\|_{W^{2,2}(s,t)} < \infty. \tag{3.6.1}$$

By the Sobolev embedding theorem, increasing the value of S_{st} if necessary, we have

$$\|\gamma_{s,x}^{t,y}\|_{C([s,t])} \le S_{st}\|\gamma_{s,x}^{t,y}\|_{W^{1,2}(s,t)} \quad \text{and} \quad \|\gamma_{s,x}^{t,y}\|_{C^1([s,t])} \le S_{st}\|\gamma_{s,x}^{t,y}\|_{W^{2,2}(s,t)} \tag{3.6.2}$$

and both injections from $C([0,T];\bar{\Omega})$ into $W^{1,2}(0,T;\bar{\Omega})$ and from $C^1([0,T];\bar{\Omega})$ into $W^{2,2}(0,T;\bar{\Omega})$ are compact.

Lemma 3.6.1. *Take $\gamma \in \Gamma_{T,2}$ such that there exists $s_0 \in (0, T)$ such that $\gamma(s_0) =: x_0 \in \Omega$ and set $t^{\pm} = t^{\pm}(\gamma, x_0)$. Assume $\gamma(0), \gamma(T) \in \partial\Omega$.*

(i) We have $0 \le t^- < s_0 < t^+ \le T$.

(ii) We have $\gamma\big((t^-, t^+)\big) \subset \Omega$ and $\gamma(t^-), \gamma(t^+) \in \partial\Omega$.

Proof. (i) Take $\epsilon > 0$ such that the open ball B in $\mathbb{R}^d$ centered at $\gamma(s_0)$ and of radius ϵ is contained in Ω. Since γ is continuous at t_0, there exists $\delta > 0$ such that when $|s - s_0| \le \delta$ then $\gamma(s)$ belongs to B. We have $t^- \le s_0 - \delta$ and $s_0 + \delta \le t^+$.

(ii) Let $(s_n)_n$ be a monotone sequence in $[0, s_0)$ decreasing to t^- such that $\gamma\big((s_n, s_0)\big) \subset \Omega$. Since

$$\gamma\big((t^-, s_0)\big) = \bigcup_{n=1}^{\infty} \gamma\big((s_n, s_0)\big),$$

we conclude that $\gamma\big((t^-, s_0)\big) \subset \Omega$. If $t^- = 0$ then $\gamma(t^-) \in \partial\Omega$. If $t^- > 0$ and $\gamma(t^-) \notin \partial\Omega$, then by (i), $t^-(\gamma, \gamma(t^-)) < t^-$, which yields a contradiction. Using similar when t^- is replaced by t^+, we conclude the proof of (ii).

□

Lemma 3.6.2. *For any $s, t \in [0, T]$ such that $s < t$, the function A_s^t is Lipschitz on bounded subsets of $AC_2(s, t; \bar{\Omega})$.*

Proof. Let $\gamma, \bar{\gamma} \in AC_2(s, t; \bar{\Omega})$.

Notice

$$L(\gamma, \dot{\gamma}) - L(\bar{\gamma}, \dot{\bar{\gamma}}) = \int_0^1 \left\langle DL \begin{pmatrix} (1-\lambda)\bar{\gamma} + \lambda\gamma \\ (1-\lambda)\dot{\bar{\gamma}} + \lambda\dot{\gamma} \end{pmatrix}, \begin{pmatrix} \gamma - \bar{\gamma} \\ \dot{\gamma} - \dot{\bar{\gamma}} \end{pmatrix} \right\rangle d\lambda.$$

By the fact that DL is κ_0–Lipschitz continuous, we conclude that there exists a constant C, independent of s, t, γ or $\bar{\gamma}$ such that

$$\Big| L(\gamma, \dot{\gamma}) - L(\bar{\gamma}, \dot{\bar{\gamma}}) \Big| \le C\Big(\kappa_0(|\gamma| + |\dot{\gamma}| + |\bar{\gamma}| + |\dot{\bar{\gamma}}|) + |DL(0)|\Big)\Big(|\bar{\gamma} - \gamma| + |\dot{\bar{\gamma}} - \dot{\gamma}|\Big).$$

Integrating over $[s,t]$ and increasing the value of C if necessary, we conclude that

$$\left|A_s^t(\gamma) - A_s^t(\bar\gamma)\right| \leq C\Big(\|\gamma\|_{W^{1,2}(s,t)} + \|\bar\gamma\|_{W^{1,2}(s,t)} + 1\Big)\|\bar\gamma - \gamma\|_{W^{1,2}(s,t)}.$$

□

Remark 3.6.3. *Let $s,t \in [0,T]$ such that $s < t$.*

(i) Since by Proposition 3.1.2, c_s^t is continuous on $\bar\Omega^2$ and $C([s,t];\bar\Omega)$ compactly embeds in $W^{1,2}(s,t;\bar\Omega)$, Lemma 3.6.2 implies that E_s^t is a continuous function on $W^{1,2}(s,t;\bar\Omega)$.

(ii) Let $(x^n)_n$ and $(y^n)_n$ be two sequences in $\bar\Omega$ converging respectively to x and y in $\bar\Omega$. We use the $W^{2,2}$ uniform bound in (3.6.1)*, the compactness property in* (3.6.2)*,(i) above, and the fact that $\gamma_{s,x}^{t,y}$ if the unique minimizer connecting x to y to conclude that $\left(\gamma_{s,x^n}^{t,y^n}\right)_n$ converges to $\gamma_{s,x}^{t,y}$ in $C^1([s,t];\Omega)$.*

Take $\gamma^0 \in \Gamma_{T,2}$ such that the range of γ^0 intersects Ω at a point x^0. Take s,t such that

$$t^-(\gamma, x_0) < s < t < t^+(\gamma, x_0). \tag{3.6.3}$$

Fix $\delta > 0$ and define the map $M^\delta \equiv M^\delta_{\gamma^0} : \Gamma_{T,2} \to \Gamma_{T,2}$ by setting,

$$M^\delta(\gamma) = \gamma$$

for $\gamma \in \Gamma_{T,2} \setminus \mathbb{B}_\delta(\gamma^0)$.

For $\gamma \in \mathbb{B}_\delta(\gamma^0)$, we define $M^\delta(\gamma)$ by

$$M^\delta(\gamma)(\tau) := \begin{cases} \gamma(\tau) & \text{if } \tau \in [0,s] \cup [t,T] \\ \\ c_{s,\gamma(s)}^{t,\gamma(t)}(\tau) & \text{if } \tau \in (s,t). \end{cases} \tag{3.6.4}$$

Lemma 3.6.4. *Take $\gamma^0 \in \Gamma_{T,2}$ and s,t such that* (3.6.3) *holds and let $\delta > 0$.*

(i) The map M^δ is continuous on $\mathbb{B}_\delta(\gamma^0)$ and on $\{\gamma \in \Gamma_{T,2} : \|\gamma - \gamma^0\|_{W^{(1,2)}(0,T)} > \delta\}$.

(ii) The map M^δ is a Borel map.

Proof. (i) Since M^δ coincides with the identity map on the open set $\{\gamma \in \Gamma_{T,2} : \|\gamma - \gamma^0\|_{W^{(1,2)}(0,T)} > \delta\}$, it is continuous there. To complete the proof of (i), we fix $\gamma \in \mathbb{B}_\delta(\gamma^0)$ and take any arbitrary sequence $(\gamma^n)_n \subset \Gamma_{T,2}$ converging to γ. We need to show that $\big(M^\delta(\gamma^n)\big)_n$ converges to $M^\delta(\gamma)$.

Since $\mathbb{B}_\delta(\gamma^0)$ is an open set, there exists N_0 such that $\gamma^n \in \mathbb{B}_\delta(\gamma^0)$ for all $n \geq N_0$. Without loss of generality, we assume that $N_0 = 1$. Applying the Sobolev embedding theorem $C([0,T];\bar{\Omega}) \subset W^{1,2}(0,T;\bar{\Omega})$, we obtain existence of the limits

$$x := \lim_{n\to+\infty} x^n, \quad y := \lim_{n\to+\infty} y^n, \quad \text{where} \quad x^n := \gamma^n(s), \quad y^n := \gamma^n(t).$$

Remark 3.6.3 ensures that $\left(\gamma^{t,y^n}_{s,x^n}\right)_n$ converges to $\gamma^{t,y}_{s,x}$ in $C^1([s,t];\Omega)$. Thus,

$$\limsup_{n\to+\infty} \left\|M^\delta(\gamma^n) - M^\delta(\gamma)\right\|_{W^{1,2}(s,t)} = \limsup_{n\to+\infty} \left\|\gamma^{t,y^n}_{s,x^n} - \gamma^{t,y}_{s,x}\right\|_{W^{1,2}(s,t)} = 0.$$

Since

$$\left\|M^\delta(\gamma^n) - M^\delta(\gamma)\right\|_{W^{1,2}\left((0,s)\cup(t,T)\right)} = \|\gamma^n - \gamma\|_{W^{1,2}\left((0,s)\cup(t,T)\right)},$$

we conclude that

$$\limsup_{n\to+\infty} \left\|M^\delta(\gamma^n) - M^\delta(\gamma)\right\|_{W^{1,2}\left((0,s)\cup(t,T)\right)} \leq \limsup_{n\to+\infty} \|\gamma^n - \gamma\|_{W^{1,2}(0,T)} = 0.$$

This shows that M^δ is continuous at γ.

(ii) Let $\mathcal{O} \subset \Gamma_{T,2}$ be an open set and denote by $\mathcal{O}_\delta$, the inverse of $\mathcal{O}$ under M^δ. Then $\mathcal{O}_\delta$ is the union of

$$\mathcal{O}^1_\delta := \{\gamma \in \mathbb{B}_\delta : M^\delta(\gamma) \in \mathcal{O}\}$$

and

$$\mathcal{O}^2_\delta := \{\gamma \in \Gamma_{T,2} \setminus \mathbb{B}_\delta : M^\delta(\gamma) \in \mathcal{O}\} = (\Gamma_{T,2} \setminus \mathbb{B}_\delta) \cap \mathcal{O}.$$

By (i), $\mathcal{O}^1_\delta$ is an open set. Thus it is a Borel set. Since, $\mathcal{O}^2_\delta$ is the intersection of a closed set and an open set, it is also a Borel set. As a consequence, $\mathcal{O}_\delta$ is a Borel set. This proves that M^δ is a Borel map.

Theorem 3.6.5. *Let $(\sigma, v) \in \Sigma_T(\sigma_0, \sigma_T|\psi)$ and let η be a Borel probability measure on $\mathbb{R}^d \times \Gamma_T$ be such that $\sigma = \sigma^\eta$, according to (3.4.2). Assume $\gamma^0 \in \Gamma_{T,2} \cap supp(\eta)$ has the range of γ^0 intersecting Ω at $x_0 = \gamma(s_0)$ and $s, t, s_0 \in \Big(t^-(\gamma^0, x_0), t^+(\gamma^0, x_0)\Big)$, where $s < s_0 < t$. If $E_s^t(\gamma^0) = -2\epsilon < 0$ then for $\delta > 0$ small enough, there exists $(\sigma^\delta, v^\delta) \in \Sigma_T(\sigma_0, \sigma_T|\psi)$ such that*

$$\mathcal{A}_L^F[\sigma^\delta, v^\delta] < \mathcal{A}_L^F[\sigma, v].$$

Proof. By Remark 3.6.3 (i), E_s^t is continuous. Thus there exists $\delta > 0$ such that

$$E_s^t(\gamma) < -\epsilon, \qquad \forall \gamma \in \mathbb{B}_\delta(\gamma^0). \tag{3.6.5}$$

Increase the value of S_{st} if necessary. The Sobolev embedding theorem gives

$$\big\|\gamma\big\|_{C([s,t])} \leq S_{st}\big\|\gamma\big\|_{W^{1,2}(s,t)}, \qquad \forall \gamma \in W^{1,2}(s,t). \tag{3.6.6}$$

Since $\gamma^0([s,t]) \subset \Omega$ and both $\gamma^0([s,t])$ and $\partial\Omega$ are compact sets,

$$\delta_0 := \mathrm{d}ist\big(\gamma^0[s,t], \partial\Omega\big) > 0. \tag{3.6.7}$$

Choose $\delta > 0$ small enough so that we further have $\delta S_{st} < \delta_0/2$.

Let M^δ be the map defined in (3.6.4) and set

$$\eta^\delta := (\mathrm{i}d \times M^\delta)_\# \eta.$$

By Lemma 3.6.4, η^δ is also a Borel probability measure. We define σ_τ^δ by

$$\int_{\mathbb{R}^d} \varphi(y)\sigma_\tau^\delta(dy) := \int_{\mathbb{R}^d \times \Gamma_{T,2}} \varphi(\gamma(\tau))\eta^\delta(dq, d\gamma).$$

In other words,

$$\int_{\mathbb{R}^d} \varphi(y)\sigma_\tau^\delta(dy) := \int_{\mathbb{R}^d \times \Gamma_{T,2}} \varphi\Big(M^\delta(\gamma)(\tau)\Big)\eta(dq, d\gamma).$$

If $\gamma \in \mathbb{B}_\delta(\gamma^0)$, then

$$\int_0^T L(\gamma, \dot\gamma)d\tau = \int_{(0,s)\cup(t,T)} L(\gamma, \dot\gamma)d\tau + \int_s^t L(\gamma, \dot\gamma)d\tau.$$

Since
$$\int_0^T L(\gamma,\dot\gamma)d\tau = \int_0^T L\Big(M^\delta(\gamma), \dot{M^\delta}(\gamma)\Big)d\tau - E_s^t(\gamma),$$
we use (3.6.5) to conclude that
$$\int_0^T L(\gamma,\dot\gamma)d\tau \geq \epsilon + \int_0^T L\Big(M^\delta(\gamma), \dot{M^\delta}(\gamma)\Big)d\tau.$$

Thus
$$\int_{\mathbb{R}^d\times\mathbb{B}_\delta(\gamma^0)} \Big(\int_0^T L(\gamma,\dot\gamma)d\tau\Big)\eta(dq,d\gamma) \geq \int_{\mathbb{R}^d\times\mathbb{B}_\delta(\gamma^0)} \Big(\int_0^T L\Big(M^\delta(\gamma), \dot{M^\delta}(\gamma)\Big)d\tau\Big)\eta(dq,d\gamma) \tag{3.6.8}$$
$$+\epsilon\eta\Big(\mathbb{R}^d\times\mathbb{B}_\delta(\gamma^0)\Big). \tag{3.6.9}$$

Since $M^\delta(\gamma)\equiv\gamma$ on the complement of $\mathbb{B}_\delta(\gamma^0)$, we conclude that
$$\int_{\mathbb{R}^d\times\big(\Gamma_T\setminus\mathbb{B}_\delta(\gamma^0)\big)} \Big(\int_0^T L(\gamma,\dot\gamma)d\tau\Big)\eta(dq,d\gamma)$$
$$= \int_{\mathbb{R}^d\times\big(\Gamma_{T,2}\setminus\mathbb{B}_\delta(\gamma^0)\big)} \Big(\int_0^T L\Big(M^\delta(\gamma), \dot{M^\delta}(\gamma)\Big)d\tau\Big)\eta(dq,d\gamma).$$

This, together with (3.6.8) implies
$$\int_{\mathbb{R}^d\times\Gamma_{T,2}} \Big(\int_0^T L(\gamma,\dot\gamma)d\tau\Big)\eta(dq,d\gamma)$$
$$\geq \int_{\mathbb{R}^d\times\Gamma_{T,2}} \Big(\int_0^T L(\gamma,\dot\gamma)d\tau\Big)\eta^\delta(dq,d\gamma) + \epsilon\eta\Big(\mathbb{R}^d\times\mathbb{B}_\delta(\gamma^0)\Big). \tag{3.6.10}$$

We use Proposition 6.1.1 with η replaced by η^δ to conclude that the path $t\mapsto\sigma^\delta$ belongs to $\Gamma_{T,2}$ and there is a velocity v^δ for σ^δ such that
$$\int_0^T\int_{\mathbb{R}^d} L(x, v_\tau^\delta(x))\sigma_\tau^\delta(dx) \leq \int_{\mathbb{R}^d\times\Gamma_{T,2}} \Big(\int_0^T L\big(\gamma(\tau),\dot\gamma(\tau)\big)d\tau\Big)\eta^\delta(dq,d\gamma).$$

This, together with (3.6.10) and the fact that $\sigma=\sigma^\eta$, implies
$$\int_0^T\int_{\mathbb{R}^d} L(x, v_\tau^\delta(x))\sigma_\tau^\delta(dx) + \epsilon\eta\Big(\mathbb{R}^d\times\mathbb{B}_\delta(\gamma^0)\Big) \leq \int_0^T\int_{\mathbb{R}^d} L(x, v_\tau(x))\sigma_\tau(dx).$$

Since γ^0 belongs to the support of η, we have $\eta\Big(\mathbb{R}^d \times \mathbb{B}_\delta(\gamma^0)\Big) > 0$.

Thus

$$\int_0^T \int_{\mathbb{R}^d} L(x, v_\tau^\delta(x))\sigma_\tau^\delta(dx) < \int_0^T \int_{\mathbb{R}^d} L(x, v_\tau(x))\sigma_\tau(dx). \tag{3.6.11}$$

Note that if $\gamma \in \mathbb{B}_\delta(\gamma^0)$, then for any $\tau \in [s,t]$, we have

$$|\gamma(\tau) - \gamma^0(\tau)| \leq S_{st}\|\gamma - \gamma^0\|_{W^{1,2}(s,t)} \leq S_{st}\delta < \frac{\delta_0}{2}.$$

Thus

$$\gamma(\tau) \notin \partial\Omega \qquad \forall \tau \in [s,t]. \tag{3.6.12}$$

1. Claim. If $A \subset \partial\Omega$ is a Borel set, then $\chi_A\Big(M^\delta(\gamma)(\tau)\Big) \geq \chi(\gamma(\tau))$ for all $\tau \in [0,T]$.

*P*roof of the Claim. We need to prove the claim only for $\gamma \in \mathbb{B}_\delta(\gamma^0)$ and for $\tau \in [s,t]$. Under these additional assumptions,(3.6.12) implies that $\chi(\gamma(\tau)) = 0$, which concludes the proof.

2. Let $A \subset \partial\Omega$ be a Borel set. We have

$$\sigma_\tau^\delta(A) = \int_{\mathbb{R}^d \times \Gamma_{T,2}} \chi_A(\gamma(\tau))\eta^\delta(dq, d\gamma) = \int_{\mathbb{R}^d \times \Gamma_{T,2}} \chi_A\Big(M^\delta(\gamma)(\tau)\Big)\eta(dq, d\gamma).$$

We use Claim 1 to conclude that

$$\sigma_\tau^\delta(A) \geq \int_{\mathbb{R}^d \times \Gamma_{T,2}} \chi_A(\gamma(\tau))\eta(dq, d\gamma) = \sigma_\tau(A) \geq \psi_\tau(A).$$

This proves that

$$\sigma_\tau^\delta \geq \psi_\tau \qquad \forall \tau \in [0,T]. \tag{3.6.13}$$

By Remark 3.4.2 $M^\delta(\gamma(\tau)) = \gamma(\tau)) = P_\tau$ for $\tau \in \{0,T\}$. Hence,

$$\sigma_\tau^\delta = \sigma_\tau \qquad \forall \tau \in \{0,T\}. \tag{3.6.14}$$

We combine (3.6.13) and (3.6.14) to conclude that $(\sigma^\delta, v^\delta) \in \Sigma_T(\sigma_0, \sigma_T|\psi)$.

□

Corollary 3.6.6. *Let* $(\sigma, v) \in \Sigma_T(\sigma_0, \sigma_T|\psi)$ *be the minimizer of* $\mathcal{A}_L^F$ *over* $\Sigma_T(\sigma_0, \sigma_T|\psi)$ *and let* η *be a Borel probability measure on* $\mathbb{R}^d \times \Gamma_T$ *such that* $\sigma = \sigma^\eta$. *If* $\gamma^0 \in \Gamma_{T,2} \cap supp(\eta)$ *such that the range of* γ^0 *intersection* Ω *at* $x_0 = \gamma(s_0)$, *and if* $s, t, s_0 \in \Big(t^-(\gamma^0, x_0), t^+(\gamma^0, x_0)\Big)$ *such that* $s < s_0 < t$, *then*

$$\gamma^0|_{[t^-,t^+]} = \gamma_{t^-,x}^{t^+,y},$$

where

$$t^- := (t^-(\gamma^0, x_0), \quad t^+ := (t^-(\gamma^0, x_0), \quad x = \gamma^0(t^-), \quad y = \gamma^0(t^+).$$

Proof. By Theorem 3.6.5,

$$\gamma^0|_{[s,t]} = \gamma_{s,\gamma^0(s)}^{t,\gamma^0(t)}, \qquad \forall t^- < s < t < t^+.$$

Letting s tend to t^- and t tend to t^+, we conclude the proof.

□

CHAPTER 4

Action Involving a Metric and a Potential

Recall that $\mathcal{M}^+(\partial\Omega)$ denotes the set of Borel measures on $\mathbb{R}^d$ with their supports in $\partial\Omega$. Let $\mathcal{S}$ be the set of Borel paths ψ from $[0,T]$ to $\mathcal{M}^+(\partial\Omega)$ that are piecewise continuous on $[0,T]$ with respect to narrow convergence topology and are also continuous at 0 and T.

We first review some settings for this chapter.

4.1 Preliminaries

We define

$$\mathcal{S} := \left\{\psi \in \mathcal{S}_0 : \psi_0(\bar{\Omega}) = \psi_T(\bar{\Omega}) = 1, \psi_t(\partial\Omega) \le 1\right\}.$$

We denote by $\mathcal{G}(a,b)$, the set of $g \in C^2(\mathbb{R}^d, \mathbb{R}^{d\times d})$ such that $g_{ij} = g_{ji}$ and there exist $0 < a < b < \infty$ such that the eigenvalues of $g(x)$ are between a and b for all $x \in \bar{\Omega}$. For such a g, we define the Lagrangian

$$L_g(x,v) = \frac{1}{2}\sum_{i,j=1}^{d} g_{ij}(x)v^i v^j, \qquad \forall (x,v) \in \bar{\Omega}\times\mathbb{R}^d.$$

We denote by g^{-1} the inverse of g and the associated Hamiltonian is

$$H_g(x,p) = \frac{1}{2}\sum_{i,j=1}^{d} g_{ij}^{-1}(x)p^i p^j, \qquad \forall (x,p) \in \bar{\Omega}\times\mathbb{R}^d.$$

At $x \in \bar{\Omega}$, we define the inner product and the norm

$$\langle v, w\rangle_{g(x)} := \langle g(x)v, w\rangle, \quad |v|_{g(x)} = \sqrt{\langle v, v\rangle_{g(x)}}$$

Our Lagrangian can then be written as

$$L_g(x,v) = \frac{1}{2}|v|^2_{g(x)}, \qquad \forall (x,v) \in \bar{\Omega} \times \mathbb{R}^d.$$

The distance dist_g between $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$ is

$$\mathrm{dist}_g(x,y) = \min_{\gamma} \Big\{ \int_0^1 |\dot{\gamma}|_{g(\gamma)} dt : \ \gamma(0) = x, \gamma(1) = y, \ \gamma \in AC_2\big(0,1;\mathbb{R}^d\big) \Big\}. \tag{4.1.1}$$

We assume that if γ is a minimizer in (4.1.1) and $x, y \in \bar{\Omega}$, then

$$\gamma(0,1) \subset \Omega, \tag{4.1.2}$$

which means that the range of γ minus $\{\gamma(0), \gamma(1)\}$ is entirely contained in Ω.

We set $g^0_{ij}(x) \equiv \delta_{ij}$ and

$$\mathrm{dist}_\Omega \equiv \mathrm{dist}_{g^0}. \tag{4.1.3}$$

Given $\Phi \in C^2(\bar{\Omega}^2)$, we define

$$F_\Phi(\mu) := \frac{1}{2} \int_{\bar{\Omega}^2} \Phi(x_1,x_2)\mu(dx_1)\mu(dx_2), \qquad \forall \mu \in \mathcal{P}(\bar{\Omega}).$$

We define $\mathcal{X}$ to be the set of pairs (g, Φ) such that $g \in \mathcal{G}$, $\Phi \in C^2(\bar{\Omega}^2)$, Φ is symmetric and F_Φ is strictly convex.

Given $\mu, \nu \in \mathcal{P}(\overline{\Omega})$, we recall that $\Sigma_T(\mu,\nu)$ is the set of (σ, v) satisfies

$$supp(\sigma_t) \subset \overline{\Omega}, \qquad \forall t \in [0,T]. \tag{4.1.4}$$

$$\sigma_0 = \mu, \quad \sigma_T = \nu \quad \text{ on } \ \partial\Omega \tag{4.1.5}$$

$$\partial_t \sigma + \nabla \cdot (v\sigma) = 0 \quad \text{ on } \ \mathcal{D}'\big((0,T) \times \mathbb{R}^d\big). \tag{4.1.6}$$

We also recall that if $\psi \in \mathcal{S}$, then $\Sigma_T(\mu,\nu|\psi)$ is the set of $(\sigma, v) \in \Sigma_T(\mu,\nu)$ satisfies

$$\sigma_t|_{\partial\Omega} \geq \psi_t \quad \text{ on } \quad [0,T] \times \partial\Omega. \tag{4.1.7}$$

For $(g,\Phi)\in\mathcal{X}$ and $(\sigma,v)\in\Sigma_T(\mu_0,\mu_T)$, our action functional becomes

$$\mathcal{A}_g^\Phi[\sigma,v]:=\int_0^T\bigg(\int_{\bar\Omega}\frac{1}{2}|v_t(x)|^2_{g(x)}\sigma_t(dx)+F_\Phi(\sigma_t)\bigg)dt \tag{4.1.8}$$

Similarly, for $\alpha\in C([0,T]\times\bar\Omega)$, we define the action

$$\mathcal{A}_g^\alpha[\eta,w]:=\int_0^T\int_{\bar\Omega}\bigg(\frac{1}{2}\langle g(q)w_t(q),w_t(q)\rangle+\alpha(t,q)\bigg)\eta_t(dq)$$

and study the variational problem

$$\inf_{(\eta,w)}\Big\{\mathcal{A}_g^\alpha[\eta,w]:\,(\eta,w)\in\Sigma_T(\sigma_0,\sigma_T,|\psi)\Big\}. \tag{4.1.9}$$

Let $\mathcal{U}$ be the set of tuples (u,h,α) such that $u,\alpha\in C([0,T]\times\bar\Omega)$, $h:[0,T]\times\bar\Omega\to\mathbb{R}$ are Borel maps such that $h\geq 0$. Note that h is defined pointwise and we are not assuming there is an underlying measure for which h is define up to a set of zero measure. It makes sense to consider those tuples (u,h,α) satisfying the condition

$$u(t,\gamma(t))-u(s,\gamma(s))\leq\int_s^t\Big(L_g(\gamma,\dot\gamma)-h(\tau,\gamma(\tau))+\alpha(\tau,\gamma(\tau))\Big)d\tau \tag{4.1.10}$$

for all $0\leq s<t\leq T$ and all $\gamma\in W^{1,\infty}(s,t;\bar\Omega)$.

We define

$$\mathcal{U}_g:=\Big\{(u,h,\alpha)\in\mathcal{U}:(4.1.10)\ \text{ holds}\Big\}. \tag{4.1.11}$$

For $(u,h,\alpha)\in\mathcal{U}_g$ and $\psi\in\mathcal{S}$, we set

$$\mathcal{J}(u,h,\alpha|\psi):=J(u,h|\psi)-\int_0^T F^*_\Phi(\alpha_t)dt, \tag{4.1.12}$$

where

$$J(u,h|\psi):=\int_{\bar\Omega}u(T,q)\psi_T(dq)-\int_{\bar\Omega}u(0,q)\psi_0(dq)+\int_0^T dt\int_{\partial\Omega}h(t,q)\psi_t(dq).$$

Here F^*_Φ is the Legendre transform of F_Φ, which is defined as

$$F^*_\Phi(\alpha)=\sup_{\mu\in\mathcal{M}^+(\overline{\Omega})}\Big\{\int_{\overline{\Omega}}\alpha(q)\mu(dq)-F_\Phi(\mu)\Big\}.$$

Since we can choose the extension of F to be $+\infty$ outside $\mathcal{P}(\overline{\Omega})$, then

$$F_\Phi^*(\alpha) = \sup_{\mu \in \mathcal{P}(\overline{\Omega})} \Big\{ \int_{\overline{\Omega}} \alpha(q)\mu(dq) - F_\Phi(\mu) \Big\}.$$

The first main result of this chapter is the following theorem whose proof will be divided into several steps.

Theorem 4.1.1. *Let $\psi \in \mathcal{S}$ and let (σ^ψ, v^ψ) be the unique minimizer of $\mathcal{A}_g^\Phi$ over the set $\Sigma_T(\psi_0, \psi_T|\psi)$. Recall that our duality asserts that*

$$\mathcal{A}_g^\Phi[\sigma^\psi, v^\psi] = \sup_{(u,h,\alpha)\in \mathcal{U}_g} \mathcal{J}(u, h, \alpha|\psi).$$

The optimal α^ψ can be chosen to satisfy

$$\alpha_t^\psi(q) = \int_{\overline{\Omega}} \Phi(q, q_2)\sigma_t^\psi(dq_2) - 2F_\Phi(\alpha_t^\psi). \tag{4.1.13}$$

We only need to show (4.1.13)

Whether or not $\mathcal{J}$ admits a maximizer over $\mathcal{U}_g$, thanks to Theorem 4.1.1, we will continue to assume that we can measure the supremum of $\mathcal{J}$ which is uniquely determined.

4.2 The relevant maps for our inverse problem

We endow the set $\mathcal{S}$ with the following topology: a sequence $(\psi^n)_n \subset \mathcal{S}$ converges to ψ in $\mathcal{S}$ if, for every $\varphi \in C_c(\mathbb{R}^{d+1})$, we have

$$\lim_{n\to\infty} \int_0^T \Big(\int_{\bar{\Omega}} \varphi(t,q)\psi_t^n(dq)\Big) dt = \int_0^T \Big(\int_{\bar{\Omega}} \varphi(t,q)\psi_t(dq)\Big) dt,$$

and for every $f \in C_c(\mathbb{R})$ we have

$$\lim_{n\to\infty} \int_{\bar{\Omega}} f(q)\psi_T^n(dq) = \int_{\bar{\Omega}} f(q)\psi_T(dq) \quad \text{and} \quad \lim_{n\to\infty} \int_{\bar{\Omega}} f(q)\psi_0^n(dq) = \int_{\bar{\Omega}} f(q)\psi_0(dq).$$

Remark 4.2.1. *Suppose that $(\psi^n)_n \subset \mathcal{S}$ converges to ψ in $\mathcal{S}$ and let (σ^ψ, v^ψ) is the unique minimizer of $\mathcal{A}_g^\Phi$ on the set $\Sigma_T(\psi_0, \psi_T|\psi)$. For every $\epsilon > 0$, we can find $(u, h, \alpha) \in \mathcal{U}_g$ such that h is continuous and*

$$\mathcal{A}_g^\Phi[\sigma^\psi, v^\psi] \leq \epsilon + \mathcal{J}(u, h, \alpha|\psi) = \epsilon + \liminf_{n\to\infty} \mathcal{J}(u, h, \alpha|\psi^n) \leq \epsilon + \liminf_{n\to\infty} \mathcal{A}_g^\Phi[\sigma^n, v^n].$$

Hence $\psi \mapsto \mathcal{A}_g^\Phi[\sigma^\psi, v^\psi]$ is lower semicontinuous from $\mathcal{S}$ to $\mathbb{R}$.

We denote by $\mathcal{F}(\mathcal{S};\mathbb{R})$ the set of real valued function on $\mathcal{S}$. We define

$$\mathcal{I} : \mathcal{G} \times C^2(\bar{\Omega}^2) \to \mathcal{F}(\mathcal{S};\mathbb{R})$$

by

$$\mathcal{I}(g, \Phi)(\psi) = \mathcal{A}_g^\Phi[\sigma^\psi, v^\psi] + \int_0^T F_\Phi^*(\alpha_t^\psi)dt, \tag{4.2.1}$$

where (σ^ψ, v^ψ) is the unique minimizer of $\mathcal{A}_g^\Phi$ on the set $\Sigma_T(\psi_0, \psi_T|\psi)$ and α_t^ψ is given by (4.1.13).

We shall have access to the boundary measurement $\mathcal{I}(g, \Phi)$ on $\mathcal{S}$. The inverse problem we are interested in is to find (g, Φ) knowing $\mathcal{I}(g, \Phi)$.

4.3 Some results when $T = 1$

In this section, we take $T = 1$ for simplicity. This does not make our results special as it is just reparameterization.

It is well known that if $g \in \mathcal{G}$ and γ is a minimizer in (4.1.1), then $|\dot{\gamma}(t)|_{g(\gamma(t))}$ is time independent and

$$\mathrm{dist}_g^2(x, y) = \int_0^1 |\dot{\gamma}|^2_{g(\gamma)}dt = \min_{\tilde{\gamma}} \Big\{ \int_0^1 |\dot{\tilde{\gamma}}|^2_{g(\tilde{\gamma})}dt : \ \tilde{\gamma}(0) = x, \tilde{\gamma}(1) = y \Big\}. \tag{4.3.1}$$

Let $\alpha : \bar{\Omega} \to \mathbb{R}$ be a Borel function and let $\Phi \in C^2(\bar{\Omega}^2)$ be a symmetric function such that F_Φ is strictly convex.

Define

$$I_\alpha(\mu) := \int_{\bar{\Omega}} \alpha(q)\mu(dq) - F_\Phi(\mu), \qquad \forall \mu \in \mathcal{P}(\bar{\Omega}).$$

Given $\mu_0, \mu_1 \in \mathcal{P}(\bar{\Omega})$, we define $\mu_t := (1-t)\mu_0 + t\mu_1$. We have

$$\begin{aligned} I_\alpha(\mu_t) =& I_\alpha(\mu_0) + t\int_{\bar{\Omega}} \left(\alpha(q_1) - \int_{\bar{\Omega}} \Phi(q_1,q_2)\mu_0(dq_2)\right)(\mu_1-\mu_0)(dq_1) \\ &- \frac{t^2}{2}\int_{\bar{\Omega}^2} \Phi(q_1,q_2)(\mu_1-\mu_0)(dq_1)(\mu_1-\mu_0)(dq_2). \end{aligned} \tag{4.3.2}$$

Similarly,

$$\begin{aligned} I_\alpha(\mu_t) =& I_\alpha(\mu_1) + (1-t)\int_{\bar{\Omega}} \left(\alpha(q_1) - \int_{\bar{\Omega}} \Phi(q_1,q_2)\mu_1(dq_2)\right)(\mu_0-\mu_1)(dq_1) \\ &- \frac{(1-t)^2}{2}\int_{\bar{\Omega}^2} \Phi(q_1,q_2)(\mu_1-\mu_0)(dq_1)(\mu_1-\mu_0)(dq_2). \end{aligned}$$

Write $I_\alpha(\mu_t) = (1-t)I_\alpha(\mu_t) + tI_\alpha(\mu_t)$. We conclude that

$$\begin{aligned} I_\alpha(\mu_t) =& (1-t)I_\alpha(\mu_0) + tI_\alpha(\mu_1) \\ &+ \frac{1}{2}(1-t)t\int_{\bar{\Omega}^2} \Phi(q_1,q_2)\mu_0(dq_2)(\mu_1-\mu_0)(dq_1)(\mu_1-\mu_0)(dq_2). \end{aligned}$$

The strict convexity of F_Φ is equivalent to the strict concavity of I_α, which means

$$\int_{\bar{\Omega}^2} \Phi(q_1,q_2)\mu_0(dq_2)(\mu_1-\mu_0)(dq_1)(\mu_1-\mu_0)(dq_2) > 0, \qquad \forall \mu_0 \neq \mu_1. \tag{4.3.3}$$

Proposition 4.3.1. *Assume $\mu \in \mathcal{P}(\overline{\Omega})$ and set*

$$\alpha_\mu(q) := \int_{\bar{\Omega}} \Phi(q,q_2)\mu(dq_2).$$

Then

$$F_\Phi^*(\alpha_\mu) = I_{\alpha_\mu}(\mu).$$

Thus $\mu \in \partial.F_\Phi^*(\alpha_\mu)$ and $\alpha_\mu \in \partial.F_\Phi(\mu)$. Furthermore, for any $\lambda \in \mathbb{R}$, we have

$$F_\Phi^*(\alpha+\lambda) = F_\Phi^*(\alpha) + \lambda.$$

Proof. Let $\mu_t \in \mathcal{P}(\bar{\Omega})$. Take $\mu_0 = \mu$ in (4.3.2) and (4.3.3). Notice that

$$I_{\alpha_\mu}(\mu_t) = I_{\alpha_\mu}(\mu) - \frac{t^2}{2}\int_{\bar{\Omega}^2} \Phi(q_1, q_2)(\mu_1 - \mu_0)(dq_1)(\mu_1 - \mu_0)(dq_2) \leq I_{\alpha_\mu}(\mu),\ \forall t \in [0,1].$$

Thus $F^*_\Phi(\alpha_\mu) = I_{\alpha_\mu}(\mu)$. Therefore, $\mu \in \partial_\cdot F^*_\Phi(\alpha_\mu)$ and $\alpha_\mu \in \partial_\cdot F_\Phi(\mu)$.

Since $I_{\alpha_\mu+\lambda}(\mu) = I_{\alpha_\mu}(\mu) + \lambda$, we conclude that $F^*_\Phi(\alpha_\mu + \lambda) = F^*_\Phi(\alpha_\mu) + \lambda$.

□

Remark 4.3.2. *Note that by Proposition 4.3.1, we have*

$$\overline{\alpha}_\mu \in \partial_\cdot F_\Phi(\mu),$$

where

$$\overline{\alpha}_\mu(q) := \int_{\bar{\Omega}} \Phi(q, q_2)\mu(dq_2) - 2F_\Phi(\mu).$$

Then

$$F^*_\Phi(\overline{\alpha}_\mu) + F_\Phi(\mu) = 0, \quad \textit{and} \quad \int_{\bar{\Omega}} \overline{\alpha}_\mu(q)\mu(dq) = 0.$$

Lemma 4.3.3. *If $P_0, P_1 \in \partial\Omega$, then*

$$\min_{(\sigma,v)} \left\{ \int_0^1 \int_{\bar{\Omega}} \left|v_t(x)\right|^2_{g(x)} \sigma_t(dx)dt :\ (\sigma, v) \in \Sigma_1\big(\delta_{P_0}, \delta_{P_1}\big) \right\} = dist^2_g\big(P_0, P_1\big)$$

and the minimum is attained by any (σ^*, v^*) such that $\sigma^* = \delta_\gamma$ where γ is any minimizer in (4.3.1) *and $v^*_t(\gamma_t) = \dot{\gamma}_t$.*

Proof. Observe first that if γ is a minimizer in (4.3.1), then

$$\{(s,t) \in (0,1)^2 :\ \gamma_t = \gamma_s, \dot{\gamma}_t \neq \dot{\gamma}_s\} \quad \text{has null } \mathcal{L}^2_{(0,1)^2}\text{–measure.}$$

Therefore, there exists a Borel vector field $v^* : (0,1) \times \bar{\Omega} \to \mathbb{R}^d$ such that $v^*_t(\gamma_t) = \dot{\gamma}_t$.

We have

$$\int_0^1 \int_{\bar{\Omega}} \left|v^*_t(x)\right|^2_{g(x)} \sigma^*_t(dx)dt = \int_0^1 |\dot{\gamma}_t|^2_{g(\gamma_t)} dt = \text{dist}^2_g\big(P_0, P_1\big). \tag{4.3.4}$$

Let $(\sigma, v) \in \Sigma_1\big(\delta_{P_0}, \delta_{P_1}\big)$ and let η^σ be its probabilistic representation given by Theorem 8.2.1 [4] such that

$$v_t(\gamma_t) = \dot\gamma \quad \text{for a.e. } t, \quad \eta^\sigma \text{ a.e. } (q,\gamma). \tag{4.3.5}$$

Denote $\Gamma_{1,2} := AC_2(0,1;\bar\Omega)$. We use (6.1.3), (4.3.5), and apply Fubini's theorem to obtain

$$\int_0^1 \int_{\bar\Omega} \big|v_t(x)\big|^2_{g(x)} \sigma_t(dx)dt = \int_0^1 \int_{\mathbb{R}^d \times \Gamma_{1,2}} \big|\dot\gamma_t\big|^2_{g(\gamma_t)} \eta^\sigma(dq, d\gamma)dt. \tag{4.3.6}$$

Since (6.1.3) also ensures that the set of (q,γ), with either $\gamma(0) \neq P_0$ or $\gamma(1) \neq P_1$, is of null η^σ–measure, we have

$$\int_0^1 \int_{\mathbb{R}^d \times \Gamma_{1,2}} \big|\dot\gamma_t\big|^2_{g(\gamma_t)} \eta^\sigma(dq, d\gamma)dt \geq \int_{\mathbb{R}^d \times \Gamma_{1,2}} \mathrm{dist}_g^2\big(\gamma(0), \gamma(1)\big)\eta^\sigma(dq, d\gamma)dt = \mathrm{dist}_g^2\big(P_0, P_1\big).$$

This, together with (4.3.4) and (4.3.6), yields the desired result.

□

4.4 Minimizers of $\mathcal{A}_g^\Phi$

In this section, we will see that minimizers of $\mathcal{A}_g^\Phi$ are also minimizers of an auxiliary linearized problem.

Our goal is to prove a fixed point property, which means that we can choose α such that there exists $(\bar\eta, \bar w)$ which is a minimizer in (4.1.9) and such that

$$\alpha(t,q) := \int_{\bar\Omega} \Phi(q, q_2)\bar\eta_t(dq_2).$$

We first explain the idea that guided our intuition in this section. Assume we have two functions A and B and we know that $A+B$ is minimized at x_0 and A is convex. Consider the function $A + \tilde B_{x_0}$, where $\tilde B_{x_0}(x) = B(x_0) + \langle B(x_0), x - x_0\rangle$. The function $A + \tilde B_{x_0}$ is convex and its gradient at x_0 is $\nabla A(x_0) + \nabla B(x_0)$. Since x_0 is a critical point for $A+B$, it is also a critical point for $A + \tilde B_{x_0}$. For convex functions, critical points are minimizers. so,

x_0 is a minimizer for $A + \tilde{B}_{x_0}$. In conclusion, every minimizer of $A + B$ is also a minimizer of $A + \tilde{B}_{x_0}$. Furthermore, let us assume we have a duality result

$$\sup_{\beta} J(\beta) = \min_{x} A + \tilde{B}_{x_0}.$$

This implies

$$\sup_{\beta} J(\beta) = \min_{x} A + B.$$

We will show that the argument above, which is applied in a Hilbert setting, works in our case as well. To achieve this goal, we need to convert $\mathcal{A}_g^{\Phi}$ into a convex functional.

Let's set

$$G(q, a, m) := \begin{cases} \frac{1}{2a} \langle g(q)m, m \rangle & \text{if } \ a > 0 \\ 0 & \text{if } \ a = 0, m = 0 \\ +\infty & \text{if } \ (a = 0, m \neq 0) \text{ or } a < 0 \end{cases}$$

The Legendre transform of $G(q, \cdot, \cdot)$ is

$$G^*(q, b, n) = \sup_{a>0} \left\{ ab + \frac{a}{2} \langle g^{-1}(q)n, n \rangle \right\} = \begin{cases} +\infty & \text{if } \ b + \frac{1}{2} \langle g^{-1}(q)n, n \rangle > 0 \\ \\ 0 & \text{if } \ b + \frac{1}{2} \langle g^{-1}(q)n, n \rangle \leq 0 \end{cases}$$

Thus

$$G^{**}(q, a, m) = \sup_{b,n} \left\{ ab + \langle m, n \rangle \, : \, b + \frac{1}{2} \langle g^{-1}(q)n, n \rangle \leq 0 \right\}.$$

We observe that $G^{**}(q, 0, 0) = 0$ and $G^{**}(q, a, m) = +\infty$ if either $a < 0$ or $a = 0$ and $m \neq 0$. When $a > 0$, we have

$$G^{**}(q, a, m) = \sup_{n} \left\{ \langle m, n \rangle - \frac{a}{2} \langle g^{-1}(q)n, n \rangle \right\} = \frac{1}{2a} \langle g(q)m, m \rangle.$$

This proves that $G(q, \cdot, \cdot) = G^{**}(q, \cdot, \cdot)$.

Thus, $G(q, \cdot, \cdot)$ is convex and lower semicontinuous.

We identify $\sigma \in AC_2(0,T;\mathcal{P}(\bar\Omega))$ with the measure on $[0,T]\times\bar\Omega$, which we continue to denote by σ for simplicity.

We define

$$\int_{[0,T]\times\bar\Omega} \varphi(t,q)\sigma(dt,dq) := \int_0^T \int_{\bar\Omega} \varphi(t,q)\sigma_t(q), \qquad \forall\varphi \in C([0,T]\times\bar\Omega).$$

If v is a velocity field driving σ, we define the vector field $m=\sigma v$, whose components are signed measures, by

$$\int_{[0,T]\times\bar\Omega} \langle\psi(t,q), m(dt,dq)\rangle := \int_0^T dt \int_{\bar\Omega} \langle\psi(t,q), v_t(q)\rangle\sigma_t(q), \qquad \forall\psi \in C([0,T]\times\bar\Omega,\mathbb{R}^d). \tag{4.4.1}$$

We write $m \in \mathcal{M}((0,T)\times\bar\Omega)^d$ to express the fact that each one of the component of m is a signed Borel measure. Note that $|m| << \sigma$.

We now consider the function

$$\tilde A_g(\sigma,m) := \begin{cases} \int_{[0,T]\times\bar\Omega} G(q,f,dm/df)\tilde\sigma(dt,dq) & \text{if } \ |m| << \sigma, dm/d\sigma = v \\ \\ +\infty & \text{if } \ |m| \not<< \sigma \end{cases}$$

Here, $\tilde\sigma$ is a probability measure and f is a non–negative function such that $\sigma = f\tilde\sigma$. Since $G(,q,\cdot,\cdot)$ is convex and 1–homogeneous, the definition is independent of f. In particular, we can take $f\equiv 1$.

If (σ,m,v) are as in (4.4.1). then

$$\tilde A_g(\sigma,m) = \frac{1}{2}\int_0^T \int_{\bar\Omega} \langle g(q)v_t(q), v_t(q)\rangle\sigma_t(dq). \tag{4.4.2}$$

One advantage of the new formulation is that $\tilde A_g$ is convex on the set

$$\tilde{\mathcal{P}} := \Big\{(\sigma,m):\ \sigma\in\mathcal{P}((0,T)\times\bar\Omega),\ m\in\mathcal{M}((0,T)\times\bar\Omega)^d, \partial_t\sigma + \nabla\cdot m = 0,\ \mathcal{D}'\big((0,T)\times\mathbb{R}^d\big)\Big\}.$$

Therefore, $\tilde A_g$ is convex on the set

$$\tilde{\mathcal{P}}_0 := \Big\{(\sigma,m)\in\tilde{\mathcal{P}}:\ \sigma_t\ge\psi_t,\ \sigma_0=\delta_{P_0}, \sigma_T=\delta_{P_T}\Big\}.$$

Theorem 4.4.1. *Suppose (σ, v) minimizes $\mathcal{A}_g^\Phi$ over $\Sigma_T(\sigma_0, \sigma_T|\psi)$. Then (σ, v) minimizes $\mathcal{A}_g^\alpha$ over $\Sigma_T(\sigma_0, \sigma_T|\psi)$, where*

$$\alpha(t,q) := \int_{\bar{\Omega}} \Phi(q, q_2)\sigma_t(dq_2).$$

Proof. We first observe that

$$F_\Phi(\eta) = F_\Phi(\sigma) + \int_{\bar{\Omega}} \alpha(t,q_1)(\eta - \sigma)(dq_1) + \frac{1}{2}\int_{\bar{\Omega}^2} \Phi(q_1,q_2)(\eta-\sigma)(dq_1)(\eta-\sigma)(dq_2). \quad (4.4.3)$$

Thus

$$\mathcal{A}_g^\alpha[\eta,w] = \int_0^T \int_{\bar{\Omega}} \frac{1}{2}\langle g(q)w_t(q), w_t(q)\rangle \eta_t(dq) + \int_0^T \Big(F_\Phi(\eta) - F_\Phi(\sigma) + \int_{\bar{\Omega}} \alpha(t,q_1)\sigma(dq_1)\Big)dt$$
$$-\frac{1}{2}\int_{\bar{\Omega}^2} \Phi(q_1,q_2)(\eta-\sigma)(dq_1)(\eta-\sigma)(dq_2).$$

We conclude that

$$\begin{aligned}\mathcal{A}_g^\alpha[\eta,w] =& \mathcal{A}_g^\Phi[\eta,w] + \int_0^T \Big(- F_\Phi(\sigma) + \int_{\bar{\Omega}} \alpha(t,q_1)\sigma(dq_1)\Big)dt \\ &-\frac{1}{2}\int_{\bar{\Omega}^2} \Phi(q_1,q_2)(\eta-\sigma)(dq_1)(\eta-\sigma)(dq_2).\end{aligned}$$

Using the minimality property of (σ, v), we deduce that

$$\begin{aligned}\mathcal{A}_g^\alpha[\eta,w] \geq& \mathcal{A}_g^\Phi[\sigma,v] + \int_0^T \Big(- F_\Phi(\sigma) + \int_{\bar{\Omega}} \alpha(t,q_1)\sigma(dq_1)\Big)dt \\ &-\frac{1}{2}\int_{\bar{\Omega}^2} \Phi(q_1,q_2)(\eta-\sigma)(dq_1)(\eta-\sigma)(dq_2) \\ =& \mathcal{A}_g^\alpha[\sigma,v] - \frac{1}{2}\int_{\bar{\Omega}^2} \Phi(q_1,q_2)(\eta-\sigma)(dq_1)(\eta-\sigma)(dq_2).\end{aligned}$$

Replacing $[\eta, w]$ by $[\eta^\lambda, w^\lambda] := (1-\lambda)[\eta, w] + \lambda[\sigma, v]$, the previous identity provides

$$\mathcal{A}_g^\alpha[\eta^\lambda, w^\lambda] \geq \mathcal{A}_g^\alpha[\sigma,v] - \frac{(1-\lambda)^2}{2}\int_{\bar{\Omega}^2} \Phi(q_1,q_2)(\eta-\sigma)(dq_1)(\eta-\sigma)(dq_2). \quad (4.4.4)$$

We set

$$m^\lambda := (1-\lambda)\eta w + \lambda v\sigma,$$

and use the convexity of $\tilde{A}_g$ to conclude that

$$
\begin{aligned}
&(1-\lambda)\Big(\tilde{A}_g[\eta,\eta w] + \int_0^T dt \int_{\bar{\Omega}} \alpha(t,q_1)\eta(dq_1)\Big) + \lambda\Big(\tilde{A}_g[\sigma,\sigma v] + \int_0^T dt \int_{\bar{\Omega}} \alpha(t,q_1)\sigma(dq_1)\Big) \\
\geq &\tilde{A}_g[\eta^\lambda, m^\lambda] + \int_0^T dt \int_{\bar{\Omega}} \alpha(t,q_1)\eta^\lambda(dq_1).
\end{aligned}
$$

This means

$$
(1-\lambda)\mathcal{A}_g^\alpha[\eta,w] + \lambda\mathcal{A}_g^\alpha[\sigma,v] \geq \mathcal{A}_g^\alpha[\eta^\lambda, w^\lambda].
$$

This, together with (4.4.4), implies

$$
(1-\lambda)\mathcal{A}_g^\alpha[\eta,w] + \lambda\mathcal{A}_g^\alpha[\sigma,v] \geq \mathcal{A}_g^\alpha[\sigma,v] - \frac{(1-\lambda)^2}{2}\int_{\bar{\Omega}^2} \Phi(q_1,q_2)(\eta-\sigma)(dq_1)(\eta-\sigma)(dq_2).
$$

Rearranging and then simplifying the subsequent inequality by $(1-\lambda)$, we get

$$
\mathcal{A}_g^\alpha[\eta,w] \geq \mathcal{A}_g^\alpha[\sigma,v] - \frac{(1-\lambda)}{2}\int_{\bar{\Omega}^2} \Phi(q_1,q_2)(\eta-\sigma)(dq_1)(\eta-\sigma)(dq_2).
$$

Let λ tend to 1. We conclude that $\mathcal{A}_g^\alpha[\eta,w] \geq \mathcal{A}_g^\alpha[\sigma,v]$.

□

Remark 4.4.2. *Note that if (σ,v) is the unique minimizer $\mathcal{A}_g^\Phi$ over $\Sigma_T(\sigma_0,\sigma_T|\psi)$ and set $\bar{\alpha}(t,\cdot) = \alpha_{\sigma_t}$, which means*

$$
\bar{\alpha}(t,q) := \int_{\bar{\Omega}} \Phi(q,q_2)\sigma_t(dq_2).
$$

Then (see also Remark 4.3.2)

$$
F_\Phi(\sigma_t) - \int_{\bar{\Omega}} \bar{\alpha}(t,q)\sigma_t(dq) = -F_\Phi^*(\bar{\alpha}(t,\cdot)).
$$

Corollary 4.4.3. *If (σ,v) and $\bar{\alpha}$ are as in Remark 4.4.2, then the following hold.*

(a) For any $(\eta,w) \in \Sigma_T(\mu,\nu)$, we have

$$
\mathcal{A}_g^\Phi[\eta,w] \geq \mathcal{A}_g^{\bar{\alpha}}[\eta,w] + \int_0^T \Big(F_\Phi(\sigma_t) - \int_{\bar{\Omega}} \alpha(t,q)\sigma_t(dq)\Big)dt.
$$

(b) Hence,

$$\min_{(\eta,w)} \Big\{ \mathcal{A}_g^{\Phi}[\eta,w] : \ (\eta,w) \in \Sigma_T(\sigma_0,\sigma_T|\psi) \Big\} = \mathcal{A}_g^{\Phi}[\sigma,v]$$

$$= \min_{(\eta,w)} \Big\{ \mathcal{A}_g^{\bar\alpha}[\eta,w] : \ (\eta,w) \in \Sigma_T(\sigma_0,\sigma_T|\psi) \Big\} + \int_0^T F_\Phi(\sigma_t)dt = \mathcal{A}_g^{\bar\alpha}[\sigma,v] - \int_0^T F_\Phi^*(\bar\alpha_t)dt.$$

Proof. By Proposition 4.3.1,

$$F_\Phi(\eta_t) \geq F_\Phi(\sigma_t) + \int_{\bar\Omega} \bar\alpha(t,q)(\eta_t - \sigma_t)(dq).$$

Thus

$$\mathcal{A}_g^F[\sigma,v] \geq \int_0^T dt \int_{\bar\Omega} \left(\frac{1}{2}|w_t(q)|^2_{g(q)} + \int_{\bar\Omega} \bar\alpha(t,q)\eta_t(dq) \right) + \int_0^T \left(F_\Phi(\sigma_t) - \int_{\bar\Omega} \bar\alpha(t,q)\sigma_t(dq) \right) dt,$$

which proves (a).

Use (a), Remark 4.4.2, and the fact that when $\eta = \sigma$, we have

$$\int_0^T F_\Phi(\eta_t)dt = \int_0^T dt \int_{\bar\Omega} \alpha(t,q)\eta_t(dq) - \int_0^T F_\Phi^*(\bar\alpha_t)dt,$$

which concludes that (b) holds.

□

We now finish the proof of Theorem 4.1.1.

By the duality relation

$$\min_{(\eta,w)} \Big\{ \mathcal{A}_g^{\Phi}[\eta,w] : \ (\eta,w) \in \Sigma_T(\sigma_0,\sigma_T|\psi) \Big\} = \sup_{(u,h)} \Big\{ J(u,h|\psi) : \ (u,h,\bar\alpha) \in \mathcal{U}_g \Big\}$$

and the fact that, by Theorem 4.4.1, (σ,v) minimizes $\mathcal{A}_g^{\bar\alpha}$ over $\Sigma_T(\sigma_0,\sigma_T|\psi)$, we have

$$\mathcal{A}_g^{\bar\alpha}[\sigma,v] = \sup_{(u,h)} \Big\{ J(u,h|\psi) : \ (u,h,\bar\alpha) \in \mathcal{U}_g \Big\}.$$

We use Corollary 4.4.3 and the fact that $F_\Phi(\sigma_t) + F^*(\bar\alpha_t) = 0$ to conclude that

$$\mathcal{A}_g^{\Phi}[\sigma,v] = \sup_{(u,h,\bar\alpha)} \Big\{ \mathcal{J}(u,h,\bar\alpha|\psi) : \ (u,h,\bar\alpha) \in \mathcal{U}_g \Big\}.$$

This concludes the proof of the theorem.

□

4.5 Special family $(\psi^\epsilon)_\epsilon \subset \mathcal{D}$

In this section, we study minimizers of $\mathcal{A}_g^\Phi$ over $\Sigma_T(\delta_{P_0}, \delta_{P_T}|\psi^\epsilon)$ for a special class of $(\psi^\epsilon)_\epsilon \subset \mathcal{D}$.

For $\epsilon \in (0, T/4)$, we set

$$t_\epsilon := \frac{T}{2} - \epsilon,\ t^\epsilon := \frac{T}{2} + \epsilon.$$

Fix P_0, P_T in $\partial\Omega$ and assume $t \mapsto P_t^\epsilon \in \partial\Omega$ is a differentiable function such that

$$P_t^\epsilon = P_0 \quad \forall t \in [0, t_\epsilon], \qquad P_s^\epsilon = P_T \quad \forall s \in [t^\epsilon, T], \qquad P_l^\epsilon \notin \{P_0, P_T\} \quad \forall l \in (t_\epsilon, t^\epsilon).$$

Let $\lambda^\epsilon : [0, T] \to \mathbb{R}$ and ψ^ϵ be defined by

$$\lambda_t^\epsilon := \chi_{[0,t_\epsilon]\cup[t^\epsilon,T]}, \qquad \psi_t^\epsilon = \lambda_t^\epsilon \delta_{P_t^\epsilon}. \tag{4.5.1}$$

Note that $\psi^\epsilon \in \mathcal{D}$.

Let $(\sigma^\epsilon, v^\epsilon)$ be the unique minimizer of $\mathcal{A}_g^\Phi$ over $\Sigma_T(\delta_{P_0}, \delta_{P_T}|\psi^\epsilon)$ and set

$$\alpha^\epsilon(t, q) := \int_{\bar{\Omega}} \Phi(q, q_2)\sigma_t^\epsilon(dq_2) - F_\Phi(\sigma_t^\epsilon), \quad \forall (t, q) \in [0, T] \times \bar{\Omega}.$$

We have that

$$\sigma_t^\epsilon = \begin{cases} P_0 & \text{if } \ t \in [0, t_\epsilon] \\ P_T & \text{if } \ t \in [t^\epsilon, T] \end{cases}$$

and the restriction of $(\sigma^\epsilon, v^\epsilon)$ to $[t_\epsilon, t^\epsilon]$ is the unique minimizer of

$$\int_{t_\epsilon}^{t^\epsilon} \int_{\bar{\Omega}} \Big(L_g(x, v_t(x))\sigma_t(dx) + F_\Phi(\sigma_t) \Big) dt$$

over the set of (σ, v) such that v is a velocity driving σ and

$$\sigma \in AC_2(t_\epsilon, t^\epsilon; \bar{\Omega}), \qquad \sigma_{t_\epsilon} = \delta_{P_0}, \quad \sigma_{t^\epsilon} = \delta_{P_T}.$$

To reparameterize the time into $T = 1$, we define

$$\hat{\sigma}_s^\epsilon := \sigma_{2\epsilon s + t_\epsilon}, \quad \hat{v}_s^\epsilon := 2\epsilon v_{2\epsilon s + t_\epsilon}^\epsilon, \qquad \forall s \in [0, 1].$$

Then $(\hat{\sigma}^\epsilon, \hat{v}^\epsilon)$ is the unique minimizer of

$$\hat{\mathcal{A}}^\epsilon[\sigma, v] := \int_0^1 \Big(\int_{\bar{\Omega}} \frac{1}{2} \langle g(x) v_s(x), v_s(x) \rangle \sigma_s(dx) + 4\epsilon^2 F_\Phi(\sigma_s) \Big) ds$$

over the set $\Sigma_T\big(\delta_{P_0}, \delta_{P_T}\big)$ and we have the identity

$$\frac{1}{2\epsilon} \hat{\mathcal{A}}^\epsilon[\hat{\sigma}^\epsilon, \hat{v}^\epsilon] = \int_{t_\epsilon}^{t^\epsilon} \int_{\bar{\Omega}} \Big(L_g(x, v_t^\epsilon(x)) \sigma_t(dx) + F_\Phi(\sigma_t^\epsilon) \Big) dt.$$

Observe that

$$\mathcal{A}_g^\Phi[\sigma^\epsilon, v^\epsilon] = t_\epsilon \Big(F_\Phi\big(\delta_{P_0}\big) + F_\Phi\big(\delta_{P_T}\big) \Big) + \int_{t_\epsilon}^{t^\epsilon} \Big(\int_{\bar{\Omega}} \frac{1}{2} \big| v_t^\epsilon(x) \big|^2_{g(x)} \sigma_t^\epsilon(dx) + F_\Phi(\sigma_t^\epsilon) \Big) dt.$$

Thus

$$\mathcal{A}_g^\Phi[\sigma^\epsilon, v^\epsilon] = t_\epsilon \Big(F_\Phi\big(\delta_{P_0}\big) + F_\Phi\big(\delta_{P_T}\big) \Big) + \frac{1}{2\epsilon} \hat{\mathcal{A}}^\epsilon[\hat{\sigma}^\epsilon, \hat{v}^\epsilon]. \qquad (4.5.2)$$

We now study supports of the minimizers of $\hat{\mathcal{A}}^\epsilon$.

Using the notation of section 4.5, we recall $(\hat{\sigma}^\epsilon, \hat{v}^\epsilon)$ is the unique minimizer of $\hat{\mathcal{A}}^\epsilon[\sigma, v]$ over the set $\Sigma_T(\delta_{P_0}, \delta_{P_T} | \psi^\epsilon)$.

We define

$$\hat{\alpha}_t^\epsilon(q) = \int_{\bar{\Omega}} \Phi(q, q_2) \hat{\sigma}_t^\epsilon(dq_2) - 2F_\Phi\big(\hat{\sigma}_t^\epsilon\big), \qquad \hat{L}^{g,\epsilon}(t, x, v) := \frac{1}{2} \langle g(x) v, v \rangle + 4\epsilon^2 \hat{\alpha}_t^\epsilon(q).$$

We denote

$$\hat{B}^{g,\epsilon}[\gamma] := \int_0^1 \hat{L}^{g,\epsilon}\big(t, \gamma_t, \dot{\gamma}_t\big) dt.$$

We linearize F_Φ around $\hat{\sigma}_t^\epsilon$ and consider the functional

$$\begin{aligned} \hat{\mathcal{B}}_0^{g,\epsilon}[\sigma, v] :=& \frac{1}{2} \int_0^1 dt \int_{\bar{\Omega}} \langle g(x) v_t(x), v_t(x) \rangle \sigma_t(dx) \\ &+ 4\epsilon^2 \int_0^1 dt \int_{\bar{\Omega}} \hat{\alpha}_t^\epsilon(x) (\sigma_t - \hat{\sigma}_t^\epsilon)(dx) + 4\epsilon^2 \int_0^1 F_\Phi\big(\hat{\sigma}_t^\epsilon\big) dt. \end{aligned}$$

We set

$$\hat{\mathcal{B}}^{g,\epsilon}[\sigma, v] := \int_0^1 dt \int_{\bar{\Omega}} \hat{L}^{g,\epsilon}(t, x, v_t(x)) \sigma_t(dx).$$

In light of the last identity in Remark 4.3.2, the previous expression can be written as

$$\hat{\mathcal{B}}_0^{g,\epsilon}[\sigma, v] = \hat{\mathcal{B}}^{g,\epsilon}[\sigma, v] + 4\epsilon^2 \int_0^1 F_\Phi\big(\hat{\sigma}_t^\epsilon\big) dt.$$

Let $\hat{\gamma}^\epsilon$ be a minimizer of $\hat{B}^{g,\epsilon}$ over the set of $\gamma \in W^{1,2}(0,1;\bar{\Omega})$ such that $\gamma_0 = P_0$ and $\gamma_1 = P_T$. We assume that the minimizer in (4.3.1) is unique, which means $\hat{\gamma}^0$ is unique. In Theorem 4.9.1 of section 4.9, we will give a sufficient condition to conclude the uniqueness of the minimizer.

The relation

$$c_0|v|^2 - 4\epsilon^2 \|\Phi\|_{L^\infty(\bar{\Omega}^2} \leq \hat{L}^\epsilon(t,x,v) \leq c_0|v|^2 - 4\epsilon^2 \|\Phi\|_{L^\infty(\bar{\Omega}^2)}$$

implies that $\{\hat{\gamma}^\epsilon\}_{\epsilon \geq 0}$ is bounded in $W^{1,2}(0,1;\bar{\Omega})$. If $(\hat{\gamma}^{\epsilon_j})_j$ is a subsequence converging weakly to some γ^* in $W^{1,2}(0,1;\bar{\Omega})$, in light of the Sobolev embedding theorem, we may assume without loss of generality that $(\hat{\gamma}^{\epsilon_j})_j$ converges to γ^* in $L^\infty(0,1;\bar{\Omega})$. We have

$$\hat{B}^{g,0}[\gamma^*] \leq \liminf_{j\to\infty} \hat{B}^{g,\epsilon_j}[\gamma^{\epsilon_j}] \leq \liminf_{j\to\infty} \hat{B}^{g,\epsilon_j}[\gamma] = \hat{B}^{g,0}[\gamma],$$

for any arbitrary $\gamma \in W^{1,2}(0,1;\bar{\Omega})$ such that $\gamma_0 = P_0$ and $\gamma_1 = P_T$. This means that $\gamma^* = \hat{\gamma}^0$. Thus, $\{\hat{\gamma}^\epsilon\}_{\epsilon \geq 0}$ has a unique accumulation point. We obtain the following lemma.

Lemma 4.5.1. *Suppose $\hat{\gamma}^0$ is unique and $\hat{\gamma}^0(0,1) \subset \Omega$. Then any minimizers $(\hat{\gamma}^\epsilon)_{\epsilon\geq 0}$ converges weakly to $\hat{\gamma}^0$ in $W^{1,2}(0,1;\bar{\Omega})$ and strongly in $L^\infty(0,1;\bar{\Omega})$.*

Lemma 4.5.2. *Suppose $\hat{\gamma}^0$ is unique and $\hat{\gamma}^0(0,1) \subset \Omega$. Let η^ϵ be the probabilistic representation of $(\hat{\sigma}^\epsilon, \hat{v}^\epsilon)$ given by (3.4.2). Then except on a set of zero–η^ϵ measure, every γ in the support of η^ϵ, minimizes $\hat{B}^{g,\epsilon}$ over the set of $\gamma \in W^{1,2}(0,1;\bar{\Omega})$ such that $\gamma_0 = P_0$ and $\gamma_1 = P_T$.*

Proof. Set $\hat{\sigma}^{*\epsilon} = \delta_{\hat{\gamma}^\epsilon}$ and let $\hat{v}^{*\epsilon}$ its unique velocity field. We have $(\hat{\sigma}^{*\epsilon}, \hat{v}^{*\epsilon}) \in \Sigma_1(\delta_{P_0}, \delta_{P_T}|0)$. Let $(\sigma, v) \in \Sigma_1(\delta_{P_0}, \delta_{P_T}|0)$.

Let η be the probabilistic representation of an arbitrary (σ, v). By Remark 3.4.2, except on a set of zero η measure, every γ in the support of η, belongs to $W^{1,2}(0,1;\bar{\Omega})$ and satisfies $\gamma_0 = P_0$ and $\gamma_1 = P_T$.

Since η is a probability measure, we have

$$\mathcal{B}^{g,\epsilon}[\sigma, v] = \int_{\bar{\Omega}\times W^{1,2}(0,1;\bar{\Omega})} \left(\int_0^1 \hat{L}^\epsilon(t, a_t, \dot{a})dt \right) \eta(dx, da) \geq \hat{B}^{g,\epsilon}[\gamma^\epsilon] = \mathcal{B}^{g,\epsilon}[\hat{\sigma}^{*\epsilon}, \hat{v}^{*\epsilon}]. \quad (4.5.3)$$

This, together with Theorem 4.4.1, implies

$$\min_{(\sigma,v)} \left\{ \mathcal{B}^{g,\epsilon}[\sigma, v] :\ (\sigma, v) \in \Sigma_1(\delta_{P_0}, \delta_{P_T}|0) \right\} = \mathcal{B}^{g,\epsilon}[\hat{\sigma}^{*\epsilon}, \hat{v}^{*\epsilon}] = \hat{B}^{g,\epsilon}[\gamma^\epsilon] = \mathcal{B}^{g,\epsilon}[\hat{\sigma}^\epsilon, \hat{v}^\epsilon]. \quad (4.5.4)$$

We combine (4.5.3) and (4.5.4) to conclude the proof of the Lemma.

□

Corollary 4.5.3. *For each $\delta \in (0, 1/2)$ there exists $\epsilon_0 > 0$ such that the support of the $\hat{\sigma}_t^\epsilon$ denoted by supp$(\hat{\sigma}_t^\epsilon)$, is contained in compact set in Ω for all $t \in [\delta, 1-\delta]$ and all $\epsilon \in [0, \epsilon_0]$.*

Proof. Let $\delta \in (0, 1/2)$. Since $\hat{\gamma}^0$ is a continuous function, $\hat{\gamma}^0([\delta, 1-\delta])$ is a compact set which by assumption, is contained in Ω. Let $d_0 > 0$ be the distance between $\partial\Omega$ and $\hat{\gamma}^0([\delta, 1-\delta])$.

By Lemma 4.5.1, there exists $\epsilon_0 > 0$ such that if $\epsilon \in [0, \epsilon_0]$, and $\hat{\gamma}$ minimizes $\hat{B}^{g,\epsilon}$ over the set of paths in $W^{1,2}(0,1;\bar{\Omega})$ which start at $\gamma_0 = P_0$ and end at $\gamma_1 = P_T$, then $\hat{\gamma}([\delta, 1-\delta])$ is contained in the $d_0/2$–open neighborhood of $\hat{\gamma}^0([\delta, 1-\delta])$, which we denote by $\mathcal{O}$. Thanks to Lemma 4.5.2, (3.4.1) implies

$$supp(\hat{\sigma}_t^\epsilon) \subset \left\{ \gamma(t) :\ \gamma \in W^{1,2}(0,1;\bar{\Omega}),\ \gamma_0 = P_0,\ \gamma_1 = P_T,\ \hat{B}^{g,\epsilon}[\gamma] = \hat{B}^{g,\epsilon}[\hat{\gamma}^\epsilon] \right\} \subset \mathcal{O},$$

which yields the desired result.

□

4.6 Existence of a dual maximizer when $\psi = \psi^\epsilon$

Throughout this section, we choose ψ^ϵ as in section 4.5. Set

$$\alpha^\epsilon(t,q) := \int_{\bar\Omega} \Phi(q,q_2)\sigma_t^\epsilon(dq_2) - F_\Phi(\sigma_t^\epsilon), \quad \forall (t,q) \in [0,T]\times\bar\Omega.$$

Note that

$$\alpha^\epsilon(t,P_t) = 0, \qquad \forall t \in [0,t_\epsilon]\cup[t^\epsilon,T]. \tag{4.6.1}$$

On $[t_\epsilon,T]\times\bar\Omega$, we define

$$u^\epsilon(t,z) := \min_\gamma \left\{ \int_{t_\epsilon}^t \Big(L_g(\gamma,\dot\gamma) + \alpha^\epsilon(t,\gamma)\Big)d\tau : \ \gamma(t_\epsilon) = P_0, \gamma(t) = z, \ \ \gamma([t_\epsilon,t]) \subset \bar\Omega \right\}.$$

On $[0,t_\epsilon]\times\bar\Omega$, we define

$$u^\epsilon(t,z) := \max_\gamma \left\{ -\int_t^{t_\epsilon} \Big(L_g(\gamma,\dot\gamma) + \alpha^\epsilon(\tau,\gamma)\Big)d\tau : \ \gamma(t_\epsilon) = P_0, \gamma(t) = z, \ \ \gamma([t_\epsilon,t]) \subset \bar\Omega \right\}.$$

Let γ^ϵ be a minimizer of

$$\gamma \mapsto \int_{t_\epsilon}^{t^\epsilon} \Big(L_g(\gamma,\dot\gamma) + \alpha^\epsilon(\tau,\gamma)\Big)d\tau$$

over the set of $\gamma \in W^{1,2}(t_\epsilon,t^\epsilon;\bar\Omega)$ such that $\gamma(t_\epsilon) = P_0$ and $\gamma(t^\epsilon) = P_T$.

We have

$$u^\epsilon(s,\gamma^\epsilon(s)) - u^\epsilon(t,\gamma^\epsilon(t)) = \int_t^s \Big(L_g(\gamma,\dot\gamma) + \alpha^\epsilon(t,\gamma)\Big)d\tau, \qquad \forall t_\epsilon \le t < s \le t^\epsilon. \tag{4.6.2}$$

Remark 4.6.1. *We know that the restriction of u^ϵ is Lipschitz continuous on $[t_\epsilon,T]\times\bar\Omega$. By the representation formula above, the restriction of u^ϵ is also Lipschitz continuous on $[0,t_\epsilon]\times\bar\Omega$. Since u^ϵ is continuous on $\{t_\epsilon\}\times\bar\Omega$, we conclude that u^ϵ is Lipschitz continuous on $[0,T]\times\bar\Omega$.*

Lemma 4.6.2. *The function $t \mapsto u^\epsilon(t,P_t^\epsilon)$ is monotone non-increasing on respectively $[0,t_\epsilon]$ and on $[t^\epsilon,T]$. Similarly, the function*

$$t \mapsto m(t) := \int_{t_\epsilon}^t \Big(L_g(P^\epsilon,\dot P^\epsilon) + \alpha^\epsilon(\tau,P^\epsilon)\Big)d\tau - u^\epsilon(t,P_t^\epsilon)$$

is monotone non-decreasing on $[t_\epsilon,t^\epsilon]$.

Proof. (i) Assume $0 \le t < s \le t_\epsilon$. Choose $\gamma \in W^{1,2}(s, t_\epsilon; \bar\Omega)$ such that $\gamma(s) = P_0$ and

$$u^\epsilon(s, P_0) = -\int_s^{t_\epsilon} \Big(L_g(\gamma, \dot\gamma) + \alpha^\epsilon(t, \gamma)\Big) d\tau,$$

Let $\tilde\gamma$ be the extension of γ to $[t, s)$ obtain by setting $\tilde\gamma(\tau) = P_0$ on this interval. Since $\tilde\gamma \in W^{1,2}(t, t_\epsilon; \bar\Omega)$ and $\gamma(t) = P_0$, using (4.6.1), we have

$$\begin{aligned} u^\epsilon(t, P_0) &\ge -\int_t^{t_\epsilon} \Big(L_g(\tilde\gamma, \dot{\tilde\gamma}) + \alpha^\epsilon(t, \tilde\gamma)\Big) d\tau \\ &= -\int_t^s 0 d\tau - \int_s^{t_\epsilon} \Big(L_g(\gamma, \dot\gamma) + \alpha^\epsilon(t, \gamma)\Big) d\tau = u^\epsilon(s, P_0). \end{aligned}$$

Thus, $t \mapsto u^\epsilon(t, P_0)$ is monotone non-increasing on $[0, t_\epsilon]$ A similar argument allows to conclude that $t \mapsto u^\epsilon(t, P_0)$ is monotone non-increasing on $[t^\epsilon, T]$.

(ii) Assume $t_\epsilon \le t < s \le t^\epsilon$. Since u^ϵ satisfies the semi–group property, we have

$$u^\epsilon(s, P^\epsilon_s) \le u^\epsilon(t, P^\epsilon_t) + \int_t^s \Big(L_g(\gamma, \dot\gamma) + \alpha^\epsilon(\tau, \gamma)\Big) d\tau$$

whenever $\gamma(s) = P^\epsilon_s$ and $\gamma(t) = P^\epsilon_t$. In particular,

$$\begin{aligned} u^\epsilon(s, P^\epsilon_s) \le& u^\epsilon(t, P^\epsilon_t) + \int_t^s \Big(L_g(P^\epsilon, \dot P^\epsilon) + \alpha^\epsilon(\tau, P^\epsilon)\Big) d\tau \\ =& u^\epsilon(t, P^\epsilon_t) - \int_{t_\epsilon}^t \Big(L_g(P^\epsilon, \dot P^\epsilon) + \alpha^\epsilon(\tau, P^\epsilon)\Big) d\tau + \int_{t_\epsilon}^s \Big(L_g(P^\epsilon, \dot P^\epsilon) + \alpha^\epsilon(\tau, P^\epsilon)\Big) d\tau. \end{aligned}$$

Rearranging, we obtain the last part of the proof.

□

Remark 4.6.3. *(i) In light of Lemma 4.6.2, we can define the non–negative function*

$$h^\epsilon(t, P_0) := -\lim_{h \to 0^+} \frac{u^\epsilon(t + h, P_0) - u^\epsilon(t, P_0)}{h}, \quad \forall t \in [0, t_\epsilon) \cap (t^\epsilon, T].$$

We extend h^ϵ to $([0, t_\epsilon) \cap (t^\epsilon, T]) \times (\bar\Omega \setminus \{P_0\})$ by setting its value to be 0 on this set. We have

$$u^\epsilon(s, P_0) - u^\epsilon(t, P_0) = -\int_t^s h^\epsilon(\tau, P_0) d\tau, \quad \forall s, t \in [0, t_\epsilon]. \tag{4.6.3}$$

Since $L_g(P^\epsilon, \dot P^\epsilon) + \alpha^\epsilon(\tau, P^\epsilon) \equiv 0$ on $[0, t_\epsilon]$, this is equivalent to

$$u^\epsilon(s, P^\epsilon_s) = u^\epsilon(t, P^\epsilon_t) + \int_t^s \Big(L_g(P^\epsilon, \dot P^\epsilon) + \alpha^\epsilon(\tau, P^\epsilon) - h^\epsilon(\tau, P^\epsilon)\Big) d\tau, \quad \forall s, t \in [0, t_\epsilon].$$

The same identity holds for $s, t \in [t^\epsilon, T]$.

(ii) Lemma 4.6.2 also implies that for almost every $t \in (t_\epsilon, t^\epsilon)$, the following limit exists and is non–negative:

$$h^\epsilon(t, P^\epsilon_t) := \lim_{h \to 0^+} \frac{m^\epsilon(t+h) - m^\epsilon(t)}{h}.$$

The identity

$$m^\epsilon(s) - m^\epsilon(t) = \int_t^s h^\epsilon(\tau, P^\epsilon) d\tau, \quad \forall s, t \in [t_\epsilon, t^\epsilon]$$

reads off

$$u^\epsilon(s, P^\epsilon_s) = u^\epsilon(t, P^\epsilon_t) + \int_t^s \Big(L_g(P^\epsilon, \dot P^\epsilon) + \alpha^\epsilon(\tau, P^\epsilon) - h^\epsilon(\tau, P^\epsilon)\Big) d\tau, \quad \forall s, t \in [t_\epsilon, t^\epsilon].$$

We extend h^ϵ by setting $h^\epsilon(t, x) = 0$ if $t \in [t_\epsilon, t^\epsilon]$ but $x \neq P^\epsilon_t$.

Lemma 4.6.4. *If $\mathcal{U}_g$ is defined as in* (4.1.11), *then $(u^\epsilon, h^\epsilon, \alpha^\epsilon) \in \mathcal{U}_g$.*

Proof. Recall that u^ϵ and α^ϵ are Lipschitz and h^ϵ is a non–negative Borel function. It remains to show that (4.1.10) holds. For this, we fix $0 \leq s < t \leq T$ and a Lipschitz continuous path $\gamma : [t, s] \to \bar\Omega$. One readily check that

$$u^\epsilon(s, \gamma(s)) - u^\epsilon(t, \gamma(t)) \leq \int_t^s \Big(L_g(\gamma, \dot\gamma) + \alpha^\epsilon(\tau, \gamma)\Big) d\tau. \tag{4.6.4}$$

Set

$$S_0 = \big\{ l \in (t, s) \,:\, P^\epsilon_l = \gamma(l) \big\}$$

and

$$S_1 = \Big\{ l \in (t, s) \,:\, \dot P^\epsilon_l = \dot\gamma(l),\ \frac{d}{dl} u^\epsilon(l, \gamma(l)) = \frac{d}{dl} u(l, P^\epsilon_l) \Big\}.$$

Recall that the set $S_0 \setminus S_1$ is of null Lebesgue measure in Lemma 6.3.1. By Remark 4.6.3

$$\frac{d}{dl}\big(u^\epsilon(\cdot, \gamma)\big) = L_g(\gamma, \dot\gamma) + \alpha^\epsilon(\cdot, \gamma) - h^\epsilon(\cdot, \gamma), \quad \text{a.e. on} \quad S_0 \cap S_1.$$

If $l_0 \in (t,s)$ such that $\gamma(l_0) \neq P^\epsilon_{l_0}$, then there exists $\delta > 0$ such that $\gamma(l) \neq P^\epsilon_l$ for $l \in (l_0-\delta, l_0+\delta)$. Thus, $l \to -h^\epsilon(l,\gamma(l))$ is identically null on $(l_0-\delta, l_0+\delta)$. We use (4.6.4) to conclude that if we further assume that l_0 is a point of differentiability for $u^\epsilon(\cdot,\gamma)$, then

$$\frac{d}{dl}\big(u^\epsilon(\cdot,\gamma)\big)\big|_{l=l_0} \leq \Big(L_g(\gamma,\dot\gamma)+\alpha^\epsilon(\tau,\gamma)\Big)\Big|_{l=l_0} = L_g\big(\gamma(l_0),\dot\gamma(l_0)\big)+\alpha^\epsilon\big(l_0,\gamma(l_0)\big)-h^\epsilon\big(l_0,\gamma(l_0)\big).$$

In conclusion, we have

$$\frac{d}{dl}\big(u^\epsilon(\cdot,\gamma)\big) \leq L_g(\gamma,\dot\gamma)+\alpha^\epsilon(\cdot,\gamma)-h^\epsilon(\cdot,\gamma), \quad \text{a.e. on} \quad (0,T).$$

Integrating over $[t,s]$, we conclude the proof of the lemma.

□

Recall that $\mathcal{J}$ is given in (4.1.12), $\mathcal{A}^\Phi_g$ is given in (4.1.8), and ψ^ϵ is given in (4.5.1).

Theorem 4.6.5. *We have*

$$\mathcal{A}^\Phi_g[\sigma^\epsilon,v^\epsilon] = \mathcal{J}(u^\epsilon,h^\epsilon,\alpha^\epsilon|\psi^\epsilon). \tag{4.6.5}$$

Thus $(\sigma^\epsilon,v^\epsilon)$ *minimizes* $\mathcal{A}^\Phi_g$ *over* $\Sigma^T_0\Big(\delta_{P_0},\delta_{P_T}|\psi^\epsilon\Big)$ *and* $(u^\epsilon,h^\epsilon,\alpha^\epsilon)$ *maximizes* $\mathcal{J}$ *over* $\mathcal{U}_g$.

Proof. We know that $\mathcal{A}^\Phi_g(\sigma,v) \geq \mathcal{J}(u,h,\alpha)$ for any $(\sigma,v) \in \Sigma^T_0\Big(\delta_{P_0},\delta_{P_T}|\psi^\epsilon\Big)$ and any $(u,h,\alpha) \in \mathcal{U}_g$. Since Lemma 4.6.4 asserts that $(u^\epsilon,h^\epsilon,\alpha^\epsilon) \in \mathcal{U}_g$, it suffices to show (4.6.5).

By (4.6.3),

$$u^\epsilon(T,P^\epsilon_T)-u^\epsilon(t^\epsilon,P^\epsilon_{t^\epsilon})+\int_{t^\epsilon}^T dt\int_{\partial\Omega} h^\epsilon(t,q)\psi^\epsilon_t(dq)=0.$$

We use Remark 4.3.2 and the fact that

$$\int_{\bar\Omega} L_g(q,v^\epsilon_t)\sigma^\epsilon_t(dq) \equiv 0$$

on $[t^\epsilon,T]$ to conclude that

$$\begin{aligned} &u^\epsilon(T,P^\epsilon_T)-u^\epsilon(t^\epsilon,P^\epsilon_{t^\epsilon})+\int_{t^\epsilon}^T dt\int_{\partial\Omega} h^\epsilon(t,q)\psi^\epsilon_t(dq)-\int_{t^\epsilon}^T F^*_\Phi(\alpha^\epsilon_t)dt \\ &= \int_{t^\epsilon}^T\bigg(\int_{\bar\Omega} L_g(q,v^\epsilon_t)\sigma^\epsilon_t(dq)+F_\Phi(\sigma^\epsilon_t)\bigg)dt. \end{aligned} \tag{4.6.6}$$

Similarly,

$$
\begin{aligned}
&u^\epsilon(t_\epsilon, P^\epsilon_{t_\epsilon}) - u^\epsilon(0, P^\epsilon_0) + \int_0^{t_\epsilon} dt \int_{\partial\Omega} h^\epsilon(t,q)\psi^\epsilon_t(dq) - \int_0^{t_\epsilon} F^*_\Phi(\alpha^\epsilon_t)dt \\
&= \int_0^{t_\epsilon} \Big(\int_{\bar\Omega} L_g(q, v^\epsilon_t)\sigma^\epsilon_t(dq) + F_\Phi(\sigma^\epsilon_t)\Big) dt. \qquad (4.6.7)
\end{aligned}
$$

We use first use Lemma 4.6.4, then we use the fact that $h^\epsilon \geq 0$ and finally use (4.6.2) to obtain that

$$
\begin{aligned}
u^\epsilon(t^\epsilon, \gamma^\epsilon(t^\epsilon)) - u^\epsilon(t_\epsilon, \gamma^\epsilon(t_\epsilon)) &\leq \int_{t_\epsilon}^{t^\epsilon} \Big(L_g(\gamma^\epsilon, \dot\gamma^\epsilon) + \alpha^\epsilon(\tau, \gamma^\epsilon) - h^\epsilon(\tau, \gamma^\epsilon)\Big) d\tau \\
&\leq \int_{t_\epsilon}^{t^\epsilon} \Big(L_g(\gamma^\epsilon, \dot\gamma^\epsilon) + \alpha^\epsilon(\tau, \gamma^\epsilon)\Big) d\tau \\
= u^\epsilon(t^\epsilon, \gamma^\epsilon(t^\epsilon)) - u^\epsilon(t_\epsilon, \gamma^\epsilon(t_\epsilon)).
\end{aligned}
$$

Thus

$$
u^\epsilon(t^\epsilon, \gamma^\epsilon(t^\epsilon)) - u^\epsilon(t_\epsilon, \gamma^\epsilon(t_\epsilon)) = \int_{t_\epsilon}^{t^\epsilon} \Big(L_g(\gamma^\epsilon, \dot\gamma^\epsilon) + \alpha^\epsilon(\tau, \gamma^\epsilon) - h^\epsilon(\tau, \gamma^\epsilon)\Big) d\tau \qquad (4.6.8)
$$

and

$$
h^\epsilon(\cdot, \gamma^\epsilon) \equiv 0 \quad \text{on} \quad [t_\epsilon, t^\epsilon]. \qquad (4.6.9)
$$

In light of Lemma 4.5.2, by the minimality property of γ^ϵ, we have

$$
\begin{aligned}
&\int_{t_\epsilon}^{t^\epsilon} \Big(L_g(\gamma^\epsilon, \dot\gamma^\epsilon) + \alpha^\epsilon(\tau, \gamma^\epsilon)\Big) d\tau \\
&= \min_{(\sigma, v)} \Big\{ \int_{t_\epsilon}^{t^\epsilon} dt \int_{\bar\Omega} \Big(L_g(q, v_t(q)) + \alpha^\epsilon(\tau, q)\Big) \sigma_t(dq) : \ (\sigma, v) \in \Sigma_{t_\epsilon}^{t^\epsilon}(\delta_{P_0}, \delta_{P_T}, 0)\Big\}.
\end{aligned}
$$

We use Theorem 4.4.1 and Remark 4.3.2 to conclude that

$$
\begin{aligned}
\int_{t_\epsilon}^{t^\epsilon} \Big(L_g(\gamma^\epsilon, \dot\gamma^\epsilon) + \alpha^\epsilon(\tau, \gamma^\epsilon)\Big) d\tau &= \int_{t_\epsilon}^{t^\epsilon} dt \int_{\bar\Omega} \Big(L_g(q, v^\epsilon_t(q)) + \alpha^\epsilon(\tau, q)\Big) \sigma^\epsilon_t(dq) \\
&= \int_{t_\epsilon}^{t^\epsilon} \Big(\int_{\bar\Omega} L_g(q, v^\epsilon_t(q)\sigma^\epsilon_t(dq) + F_\Phi(\sigma^\epsilon_t)\Big) dt + \int_{t_\epsilon}^{t^\epsilon} F^*_\Phi(\alpha^\epsilon_t).
\end{aligned}
$$

This, together with (4.6.8)-(4.6.9), implies

$$u^\epsilon(t^\epsilon,\gamma(t^\epsilon)) - u^\epsilon(t_\epsilon,\gamma(t_\epsilon)) + \int_{t_\epsilon}^{t^\epsilon} dt \int_{\partial\Omega} h^\epsilon(t,q)\psi_t^\epsilon(dq) - \int_{t_\epsilon}^{t^\epsilon} F_\Phi^*(\alpha_t^\epsilon)dt$$
$$= \int_{t_\epsilon}^{t^\epsilon} \bigg(\int_{\bar\Omega} L_g(q,v_t^\epsilon(q)\sigma_t^\epsilon(dq) + F_\Phi(\sigma_t^\epsilon)\bigg)dt. \quad (4.6.10)$$

We combine (4.6.6), (4.6.7) and (4.6.10) to conclude that

$$\int_{\bar\Omega} u^\epsilon(T,q)\sigma_T^\epsilon(dq) - \int_{\bar\Omega} u^\epsilon(0,q)\sigma_0^\epsilon(dq) + \int_0^T dt \int_{\partial\Omega} h^\epsilon(t,q)\psi_t^\epsilon(dq) - \int_0^T F_\Phi^*(\alpha_t^\epsilon)dt$$
$$= \int_0^T \bigg(\int_{\bar\Omega} L_g(q,v_t^\epsilon(q)\sigma_t^\epsilon(dq) + F_\Phi(\sigma_t^\epsilon)\bigg)dt.$$

Thus (4.6.5) holds.

□

4.7 Recovery of g

With all the settings in the previous sections, we now recover g in our inverse problem depicted by (4.2.1).

By Theorem 4.6.5, $\mathcal{J}(\cdot,\cdot,\cdot|\psi^\epsilon)$ admits a maximizer over $\mathcal{U}_g$. Assume $(u^\epsilon, h^\epsilon, \alpha^\epsilon)$ maximizes $\mathcal{J}(\cdot,\cdot,\cdot|\psi^\epsilon)$ over $\mathcal{U}_g$, so that

$$\mathcal{J}\big(u^\epsilon, h^\epsilon, \alpha^\epsilon|\psi^\epsilon\big) = \mathcal{A}_g^\Phi[\sigma^\epsilon, v^\epsilon].$$

By Proposition 4.3.1, $\alpha_t^\epsilon \in \partial.F_\Phi\big(\sigma_t^\epsilon\big)$ for a.e. t. Without loss of generality, assume that the average of α_t^ϵ with respect to σ_t^ϵ is null. We have

$$\alpha_t^\epsilon(q) = \int_{\bar\Omega} \Phi(q,q_2)\sigma_t^\epsilon(dq_2) - 2F_\Phi\big(\sigma_t^\epsilon\big). \quad (4.7.1)$$

The information which can be gathered from the boundary by direct measurements is

$$J\big(u^\epsilon, h^\epsilon|\psi^\epsilon\big) = \mathcal{A}_g^\Phi[\sigma^\epsilon, v^\epsilon] + \int_0^T F_\Phi^*(\alpha_t^\epsilon)dt.$$

We use Remark 4.3.2 and the definition of $\mathcal{A}_g^\Phi$ from (4.1.8) to obtain

$$J\big(u^\epsilon, h^\epsilon|\psi^\epsilon\big) = \mathcal{A}_g^\Phi[\sigma^\epsilon, v^\epsilon] - \int_0^T F_\Phi(\sigma_t^\epsilon)dt = \frac{1}{2}\int_0^T \int_{\bar\Omega} |v_t^\epsilon(x)|^2_{g(x)}\sigma_t^\epsilon(dx)dt.$$

We use (4.5.2) to conclude that

$$2\epsilon J\big(u^\epsilon, h^\epsilon|\psi^\epsilon\big) = \frac{1}{2}\int_0^1 \int_{\bar\Omega} |\hat v_s^\epsilon(x)|^2_{g(x)}\sigma_s^\epsilon(dx)ds = \hat{\mathcal{A}}^\epsilon[\hat\sigma^\epsilon, \hat v^\epsilon] - 4\epsilon^2 \int_0^1 F_\Phi(\hat\sigma_s^\epsilon)ds. \tag{4.7.2}$$

Proposition 4.7.1. *(i) We have*

$$2\epsilon J\big(u^\epsilon, h^\epsilon|\psi^\epsilon\big) = \frac{1}{2}dist_g^2\big(P_0, P_T\big) + O(\epsilon^2)$$

(ii) If we further assume that there is a unique path minimizing $dist_g^2\big(P_0, P_T\big)$, then

$$2\epsilon J\big(u^\epsilon, h^\epsilon|\psi^\epsilon\big) = \frac{1}{2}dist_g^2\big(P_0, P_T\big) + o(\epsilon^2).$$

Proof. We use Lemma 4.3.3 and the first identity in (4.7.2) to obtain that

$$\frac{1}{2}\mathrm{dist}_g^2\big(P_0, P_T\big) \le 2\epsilon J\big(u^\epsilon, h^\epsilon|\psi^\epsilon\big). \tag{4.7.3}$$

We use the minimality property of $(\hat\sigma^\epsilon, \hat v^\epsilon)$ to obtain that

$$\hat{\mathcal{A}}^\epsilon[\hat\sigma^\epsilon, \hat v^\epsilon] \le \hat{\mathcal{A}}^\epsilon[\hat\sigma^0, \hat v^0] = \hat{\mathcal{A}}^0[\hat\sigma^0, \hat v^0] + 4\epsilon^2 \int_0^T F_\Phi(\hat\sigma_t^0)dt.$$

This means

$$2\epsilon J\big(u^\epsilon, h^\epsilon|\psi^\epsilon\big) + 4\epsilon^2 \int_0^T F_\Phi(\hat\sigma_t^\epsilon)dt \le \frac{1}{2}\mathrm{dist}_g^2\big(P_0, P_T\big) + 4\epsilon^2 \int_0^T F_\Phi(\hat\sigma_t^0)dt. \tag{4.7.4}$$

We combine (4.7.3) and (4.7.4) to obtain

$$0 \le 2\epsilon J\big(u^\epsilon, h^\epsilon|\psi^\epsilon\big) - \frac{1}{2}\mathrm{dist}_g^2\big(P_0, P_T\big) \le 4\epsilon^2 \int_0^T \Big(F_\Phi(\hat\sigma_t^0) - F_\Phi(\hat\sigma_t^\epsilon)\Big)dt. \tag{4.7.5}$$

We use the fact that F_Φ is bounded to conclude the proof of (i).

(ii) Assume next that there is a unique path minimizing $\mathrm{dist}_g^2\big(P_0, P_T\big)$. Then

$$\lim_{\epsilon\to 0^+} F_\Phi(\hat\sigma_t^\epsilon) = F_\Phi(\hat\sigma_t^\epsilon).$$

Thus since $F_\Phi(\hat{\sigma}_t^0) - F_\Phi(\hat{\sigma}_t^\epsilon)$ is bounded independently of ϵ and t, we have

$$\lim_{\epsilon\to 0^+} \int_0^T \Big(F_\Phi(\hat{\sigma}_t^0) - F_\Phi(\hat{\sigma}_t^\epsilon)\Big) dt = 0.$$

This, together with (4.7.5), completes the proof of (ii).

□

By the travel time tomography theory [28], g can be recovered once we know the distance between any two points on the boundary.

4.8 Partial recovery of Φ

Note that by Proposition 4.7.1, g has been recovered. It only remains to recover Φ under the assumption that we have access to boundary information and g is recovered. We further assume we have chosen P^ϵ in such a way that $t \mapsto \hat{P}_t^\epsilon$ converges in $C^1([0,1])$ to a path $t \mapsto \hat{P}_t^0$. Set

$$\hat{h}^\epsilon(s, \hat{P}_s^\epsilon) := h^\epsilon\Big(2\epsilon s + \frac{T}{2} - \epsilon, \hat{P}_s^\epsilon\Big), \qquad \forall s \in [0,1].$$

By Remark 4.6.3,

$$2\epsilon\Big(\hat{u}^\epsilon(s_2, \hat{P}_{s_2}^\epsilon) - \hat{u}^\epsilon(s_1, \hat{P}_{s_1}^\epsilon)\Big) = \int_{s_1}^{s_2} \Big(L_g(\hat{P}^\epsilon, \dot{\hat{P}}^\epsilon) + 4\epsilon^2\Big(\hat{\alpha}^\epsilon(l, \hat{P}_l^\epsilon) - \hat{h}^\epsilon(l, \hat{P}_l^\epsilon)\Big)\Big) dl.$$

Thus

$$\begin{aligned} &2\epsilon\Big(\hat{u}^\epsilon(s_2, \hat{P}_{s_2}^\epsilon) - \hat{u}^\epsilon(s_1, \hat{P}_{s_1}^\epsilon)\Big) + 4\epsilon^2 \int_{s_1}^{s_2} \hat{h}^\epsilon(l, \hat{P}_l^\epsilon) dl - \int_{s_1}^{s_2} L_g(\hat{P}^\epsilon, \dot{\hat{P}}^\epsilon) dl \\ =&4\epsilon^2 \int_{s_1}^{s_2} \Big(\hat{\alpha}^\epsilon(l, \hat{P}_l^\epsilon) - \hat{h}^\epsilon(l, \hat{P}_l^\epsilon)\Big) dl. \end{aligned} \tag{4.8.1}$$

Note that the expression in (4.8.1) contains only informations on $\partial\Omega$, except for the action involving L_g. Since g has been recovered in Proposition 4.7.1 and P^ϵ is our choice, the expression in (4.8.1) is part of our knowledge. Therefore, if $\{P^\epsilon\}_\epsilon$ converges to $\hat{P}^0$ in C^1, we have knowledge of

$$\hat{I}(s|\hat{P}^0) := \lim_{\epsilon\to 0^+} \frac{2\epsilon\Big(\hat{u}^\epsilon(s_2,\hat{P}^\epsilon_{s_2}) - \hat{u}^\epsilon(s_1,\hat{P}^\epsilon_{s_1})\Big) + 4\epsilon^2\int_{s_1}^{s_2}\hat{h}^\epsilon(l,\hat{P}^\epsilon_l)dl - \int_{s_1}^{s_2} L_g(\hat{P}^\epsilon,\dot{\hat{P}}^\epsilon)dl}{4\epsilon^2}.$$

In light of (4.8.1),

$$\hat{I}(s|\hat{P}^0) = \int_0^s \hat{\alpha}^0(l,\hat{P}^0_l)dl.$$

Thus we have

$$\dot{\hat{I}}(s|\hat{P}^0) = \hat{\alpha}^0(s,\hat{P}^0_s) = \Phi(\hat{P}^0_s,\hat{\gamma}^0_s) - \Phi(\hat{\gamma}^0_s,\hat{\gamma}^0_s).$$

Notice that if we assume that $\Phi(q_1,q_2)$ is of the form $\psi(q_1-q_2)$, where ψ is even and real analytic, then we have

$$\dot{\hat{I}}(s|\hat{P}^0) = \hat{\alpha}^0(s,\hat{P}^0_s) = \psi(\hat{P}^0_s - \hat{\gamma}^0_s) - \psi(0), \qquad \forall s\in[0,1].$$

For this special case, we prove uniqueness of Φ from analytic continuation.

Denote the set of even and analytic ψ as

$$S := \{\psi:\mathbb{R}^d\to\mathbb{R}|\ \psi(x)=\psi(-x), \psi\in C^\omega(\mathbb{R}^d)\}.$$

Consider the map $D: S\to C(\mathbb{R}^d\times\mathbb{R}^d)$ such that $D_\Phi(x,y) := \Phi(x,y)-\Phi(x,x)$, where $x\in\bar{\Omega}, y\in\partial\Omega$.

Lemma 4.8.1. *There exists an open set $U\subset\{x-y : x\in\bar{\Omega}, y\in\partial\Omega\}$.*

Proof. Take any open subset $W\subset\Omega$. Then fix any point $p\in\partial\Omega$. Notice that

$$U := W-\{p\}\subset\{x-y : x\in\bar{\Omega}, y\in\partial\Omega\}$$

is an open subset.

Indeed, for any point $x-p$, where $x\in W$, we can find $\delta>0$ such that $B_\delta(x)\subset W$. Then for any point $q\in B_\delta(x-p)$, $q+p\in B_\delta(x)\subset W$. Thus $q=w-p$ for some $w\in W$. Hence $B_\delta(x-p)\subset U$ and U is open.

□

Lemma 4.8.2. *Assume that* $\Phi_i(x,y) = \psi_i(x-y)$ *and* $\psi_i \in S$*, for* $i = 1,2$*. If* $D_{\Phi 1} = D_{\Phi 2}$*, then* $\exists c$ *such that* $\psi_1 = \psi_2 + c$*.*

Proof. Since there exists an open set $U \subset \{x - y : x \in \bar{\Omega}, y \in \partial\Omega\}$ and $D_\Phi(x,y) = \Phi(x,y) - \Phi(x,x)$, where $x \in \bar{\Omega}, y \in \partial\Omega$, then $(\psi_1 - \psi_2)|_U = \psi_1(0) - \psi_2(0)$. Take $c = \psi_1(0) - \psi_2(0)$. We get $\psi_1 = \psi_2 + c$ on U. By analytic continuation, $\psi_1 = \psi_2 + c$.

□

Lemma 4.8.3. *If* $D_\Phi = 0$*, then* $\Phi(x,y) = c$ *for some constant* c*.*

Proof. There exists an open set $U \subset \{x - y : x \in \bar{\Omega}, y \in \partial\Omega\}$. If $D_\Phi = 0$, then $\psi|_u = \psi(0)$. Thus by analytic continuation, $\psi = \psi(0)$. Take $c = \psi(0)$. We get $\Phi(x,y) = c$.

□

Based on the lemmas above, we impose the assumption that $\Phi(x,x) = 0$ when we try to recover Φ. This is the case in Chapter 5, in which we recover Φ when g and Φ are real analytic.

4.9 $\hat{\sigma}^\epsilon$ concentrated on a curve

In this section, we give sufficient conditions for $\hat{\sigma}^\epsilon$ to concentrate on a curve. The main Theorem 4.9.1 in this section provides the uniqueness of the minimizer $\hat{\gamma}^0$ in Lemma 4.5.1.

Set

$$\hat{c}_s^{t,\epsilon}(x,y) := \min_\gamma \Big\{ \int_s^t \hat{L}^{g,\epsilon}\big(\tau, \gamma_\tau, \dot{\gamma}_\tau\big) d\tau : \ \gamma \in W^{1,2}(s,t;\bar{\Omega}), \ \gamma(s) = x, \gamma(t) = y \Big\}.$$

The inequalities

$$-4\epsilon^2 \|\Phi\|_{L^\infty} + \hat{L}^{g,0} \le \hat{L}^{g,\epsilon}(t,\cdot,\cdot) \le \hat{L}^{g,0}(x,v) + 4\epsilon^2 \|\Phi\|_{L^\infty}$$

imply

$$\left|\hat{c}_s^{t,\epsilon}(x,y) - \frac{1}{2(t-s)}\mathrm{dist}_g^2(x,y)\right| \leq 4\epsilon^2(t-s)\|\Phi\|_{L^\infty}. \tag{4.9.1}$$

Adapting the ideas of the proof of Proposition 3.11 [16] to the actions $\hat{B}^{g,\epsilon}$, we obtain

$$\frac{d}{dt}\left(\frac{1}{2}\big\langle g\big(\hat{\gamma}_t^\epsilon\big)\dot{\hat{\gamma}}_t^{\,\epsilon}, \dot{\hat{\gamma}}_t^\epsilon\big\rangle - 4\epsilon^2\hat{\alpha}_t^\epsilon\big(\hat{\gamma}_t^\epsilon\big)\right) = -\partial_t\hat{L}^{g,\epsilon}\big(t, \hat{\gamma}_t^\epsilon, \dot{\hat{\gamma}}_t^\epsilon\big) = -4\epsilon^2\partial_t\hat{\alpha}_t^\epsilon(\hat{\gamma}_t^\epsilon). \tag{4.9.2}$$

This proves that $t \mapsto \frac{1}{2}\big\langle g\big(\hat{\gamma}_t^\epsilon\big)\dot{\hat{\gamma}}_t^\epsilon, \dot{\hat{\gamma}}_t^\epsilon\big\rangle - 4\epsilon^2\hat{\alpha}_t^\epsilon\big(\hat{\gamma}_t^\epsilon\big)$ is $W^{1,\infty}$. Thus it is continuous.

Since $t \mapsto \hat{\alpha}_t^\epsilon\big(\hat{\gamma}_t^\epsilon\big)$ is continuous, we conclude that

$$t \mapsto \frac{1}{2}\big\langle g\big(\hat{\gamma}_t^\epsilon\big)\dot{\hat{\gamma}}_t^\epsilon, \dot{\hat{\gamma}}_t^\epsilon\big\rangle + 4\epsilon^2\hat{\alpha}_t^\epsilon\big(\hat{\gamma}_t^\epsilon\big) = \hat{L}^{g,\epsilon}\big(t, \hat{\gamma}_t^\epsilon, \dot{\hat{\gamma}}_t^\epsilon\big) \quad \text{is continuous.} \tag{4.9.3}$$

Thus there exists $t_\epsilon \in [0,1]$ such that

$$\hat{c}_0^{1,\epsilon}(P_0, P_T) = \hat{L}^{g,\epsilon}\big(t_\epsilon, \hat{\gamma}_{t_\epsilon}^\epsilon, \dot{\hat{\gamma}}_{t_\epsilon}^\epsilon\big). \tag{4.9.4}$$

Integrating both sides of (4.9.2) and using (4.9.4), we obtain

$$\begin{aligned}\frac{1}{2}\big\langle g\big(\hat{\gamma}_t^\epsilon\big)\dot{\hat{\gamma}}_t^\epsilon, \dot{\hat{\gamma}}_t^\epsilon\big\rangle - 4\epsilon^2\hat{\alpha}_t^\epsilon\big(\hat{\gamma}_t^\epsilon\big) =& \frac{1}{2}\big\langle g\big(\hat{\gamma}_{t_\epsilon}^\epsilon\big)\dot{\hat{\gamma}}_{t_\epsilon}^\epsilon, \dot{\hat{\gamma}}_{t_\epsilon}^\epsilon\big\rangle - 4\epsilon^2\hat{\alpha}_{t_\epsilon}^\epsilon\big(\hat{\gamma}_{t_\epsilon}^\epsilon\big) - \int_{t_\epsilon}^t \partial_t\hat{L}^{g,\epsilon}\big(\tau, \hat{\gamma}_\tau^\epsilon, \dot{\hat{\gamma}}_\tau^\epsilon\big) \\ =& \hat{c}_0^{1,\epsilon}(P_0, P_T) - 8\epsilon^2\hat{\alpha}_{t_\epsilon}^\epsilon\big(\hat{\gamma}_{t_\epsilon}^\epsilon\big) - \int_{t_\epsilon}^t \partial_t\hat{L}^{g,\epsilon}\big(\tau, \hat{\gamma}_\tau^\epsilon, \dot{\hat{\gamma}}_\tau^\epsilon\big).\end{aligned}$$

Rearranging and using (6.5.3) and (4.9.1), we conclude that

$$\begin{aligned}\frac{1}{2}\left|\big\langle g\big(\hat{\gamma}_t^\epsilon\big)\dot{\hat{\gamma}}_t^\epsilon, \dot{\hat{\gamma}}_t^\epsilon\big\rangle - \mathrm{dist}_g^2(P_0, P_T)\right| \leq& 10\epsilon^2\|\Phi\|_{L^\infty} + \left|\int_{t_\epsilon}^t \partial_t\hat{L}^{g,\epsilon}\big(\tau, \hat{\gamma}_\tau^\epsilon, \dot{\hat{\gamma}}_\tau^\epsilon\big)\right| \\ \leq& 2\epsilon^2\left(5\|\Phi\|_{L^\infty} + \|\nabla\Phi\|_{L^\infty}\Big(\mathrm{dist}_g^2\big(P_0, P_T\big) + 8\epsilon^2\|\Phi\|_{L^\infty}\Big)^{\frac{1}{2}}\right).\end{aligned} \tag{4.9.5}$$

We define some notations.

Let $\partial_{q_i}g(q)$ be the matrix obtained by differentiating the entries of g with respect to q_i. Since $\partial_{q_i}g(q)$ is symmetric, it is diagonalizable.

Let $\Lambda_{\nabla g}$ be the largest absolute value of the eigenvalues of all the $\partial_{q_i} g(q)$ when q varies in $\bar{\Omega}$. Similarly, each partial second derivative $\partial_{q_i q_j} g$ is diagonalizable. We denote $\lambda_{\nabla^2 g}$ as the smallest eigenvalue of all the $\partial_{q_i q_j} g(q)$'s. We also denote λ_g as the smallest eigenvalue of all the $g(q)$'s.

By (4.9.5), we have

$$\sqrt{\lambda_g}\big|\dot{\hat{\gamma}}_t^\epsilon\big| \leq \left[\text{dist}_g^2\big(P_0, P_T\big) + 2\epsilon^2\left(5\|\Phi\|_{L^\infty} + \|\nabla\Phi\|_{L^\infty}\Big(\text{dist}_g^2\big(P_0, P_T\big) + 8\epsilon^2\|\Phi\|_{L^\infty}\Big)^{\frac{1}{2}}\right)\right]^{\frac{1}{2}}.$$

This means there exists a constant $\kappa > 0$, which depends only on $\bar{\Omega}$, $\|\Phi\|_{L^\infty}$ and $\|\nabla\Phi\|_{L^\infty}$ such that

$$\sqrt{\lambda_g}\big|\dot{\hat{\gamma}}_t^\epsilon\big| \leq \text{dist}_g\big(P_0, P_T\big) + \epsilon\kappa. \tag{4.9.6}$$

Let $\gamma^0, \gamma^1 \in W^{1,2}(0,1;\bar{\Omega})$ be minimizers in $\hat{c}_0^{1,\epsilon}(P_0, P_T)$ and set

$$\gamma^\lambda = (1-\lambda)\gamma^0 + \lambda\gamma^1, \qquad \delta\gamma = \gamma^1 - \gamma^0, \qquad \forall\lambda \in [0,1].$$

Then

$$\frac{d^2}{d\lambda^2}\int_0^1 L_g\big(\gamma^\lambda, \dot{\gamma}^\lambda\big)dt = \int_0^1 \Big\langle D_{vv}L_g\big(\gamma^\lambda, \dot{\gamma}^\lambda\big)\delta\dot{\gamma}, \delta\dot{\gamma}\Big\rangle + \Big\langle D_{qq}L_g\big(\gamma^\lambda, \dot{\gamma}^\lambda\big)\delta\gamma, \delta\gamma\Big\rangle$$
$$+2\int_0^1 \Big\langle D_{qv}L_g\big(\gamma^\lambda, \dot{\gamma}^\lambda\big)\delta\dot{\gamma}, \delta\gamma\Big\rangle dt.$$

Since

$$\Big\langle D_{qv}L_g\big(\gamma^\lambda, \dot{\gamma}^\lambda\big)\delta\dot{\gamma}, \delta\gamma\Big\rangle = \sum_i \delta\gamma_i \langle\partial_i g(\gamma^\lambda, \dot{\gamma}^\lambda)\delta\dot{\gamma}, \dot{\gamma}^\lambda\rangle,$$

we have

$$\Big|\Big\langle D_{qv}L_g\big(\gamma^\lambda, \dot{\gamma}^\lambda\big)\delta\dot{\gamma}, \delta\gamma\Big\rangle\Big| \leq |\delta\gamma|\;|\delta\dot{\gamma}|\;|\dot{\gamma}^\lambda|\;\Lambda_{\nabla g}\sqrt{d}.$$

We use (4.9.6) to conclude that

$$\Big|\Big\langle D_{qv}L_g\big(\gamma^\lambda, \dot{\gamma}^\lambda\big)\delta\dot{\gamma}, \delta\gamma\Big\rangle\Big| \leq |\delta\gamma|\;|\delta\dot{\gamma}|\;\Lambda_{\nabla g}\sqrt{d}\Big(\text{dist}_g\big(P_0, P_T\big) + \epsilon\kappa\Big)\lambda_g^{\frac{-1}{2}}. \tag{4.9.7}$$

Similarly, since

$$\Big\langle D_{qq}L_g\big(\gamma^\lambda,\dot\gamma^\lambda\big)\delta\gamma,\delta\gamma\Big\rangle=\sum_{i,jk,l}\partial^2_{q_iq_j}g_{kl}\big(\gamma^\lambda\big)\dot\gamma^\lambda_k\dot\gamma^\lambda_l,$$

we conclude that

$$\Big\langle D_{qq}L_g\big(\gamma^\lambda,\dot\gamma^\lambda\big)\delta\gamma,\delta\gamma\Big\rangle\geq d^2\lambda_{\nabla^2 g}|\delta\gamma|^2. \tag{4.9.8}$$

We combine (4.9.7) and (4.9.8) to conclude that for every $e>0$ we have

$$\begin{aligned}\frac{d^2}{d\lambda^2}\int_0^1 L_g\big(\gamma^\lambda,\dot\gamma^\lambda\big)dt\geq&\int_0^1\Big(\lambda_g|\delta\dot\gamma|^2+d^2\lambda_{\nabla^2 g}|\delta\gamma|^2\Big)dt\\&-\int_0^1\Lambda_{\nabla g}\sqrt{d})\Big(e|\delta\dot\gamma|^2+|\delta\gamma|^2e^{-1}\Big)\Big(\text{dist}_g\big(P_0,P_T\big)+\epsilon\kappa\Big)\lambda_g^{\frac{-1}{2}}\,dt.\end{aligned}$$

Thus

$$\begin{aligned}\frac{d^2}{d\lambda^2}\int_0^1 L_g\big(\gamma^\lambda,\dot\gamma^\lambda\big)dt\geq&\int_0^1|\delta\dot\gamma|^2\Big[\lambda_g-e\Lambda_{\nabla g}\sqrt{d}\Big(\text{dist}_g\big(P_0,P_T\big)+\epsilon\kappa\Big)\lambda_g^{\frac{-1}{2}}\Big]dt\\&+\int_0^1|\delta\gamma|^2\Big[d^2\lambda_{\nabla^2 g}-\Lambda_{\nabla g}\sqrt{d}e^{-1}\Big(\text{dist}_g\big(P_0,P_T\big)+\epsilon\kappa\Big)\lambda_g^{\frac{-1}{2}}\Big]dt.\end{aligned} \tag{4.9.9}$$

We have

$$\frac{d^2}{d\lambda^2}\hat\alpha^\epsilon_t\big(\gamma^\lambda\big)=\int_{\bar\Omega}\Big\langle\nabla^2_{q_1q_1}\Phi\big(\gamma^\lambda,q_2\big)\delta\gamma,\delta\gamma\Big\rangle\hat\sigma^\epsilon_t(dq_2).$$

Thus, if $\lambda_{\nabla^2\Phi}$ is the smallest eigenvalue of all the $\lambda_{\nabla^2\Phi}(q_1,q_2)$'s, then

$$\frac{d^2}{d\lambda^2}\int_0^1\hat\alpha^\epsilon_t\big(\gamma^\lambda\big)\geq\lambda_{\nabla^2\Phi}\int_0^1|\delta\gamma|^2dt. \tag{4.9.10}$$

We combine (4.9.9) and (4.9.10) to deduce that

$$\begin{aligned}\frac{d^2}{d\lambda^2}\int_0^1\hat L^{g,\epsilon}\big(\gamma^\lambda,\dot\gamma^\lambda\big)dt\geq&\int_0^1|\delta\dot\gamma|^2\Big[\lambda_g-e\Lambda_{\nabla g}\sqrt{d}\Big(\text{dist}_g\big(P_0,P_T\big)+\epsilon\kappa\Big)\lambda_g^{\frac{-1}{2}}\Big]dt\\&+\int_0^1|\delta\gamma|^2\Big[d^2\lambda_{\nabla^2 g}+4\epsilon^2\lambda_{\nabla^2\Phi}-\Lambda_{\nabla g}\sqrt{d}e^{-1}\Big(\text{dist}_g\big(P_0,P_T\big)+\epsilon\kappa\Big)\lambda_g^{\frac{-1}{2}}\Big]dt.\end{aligned} \tag{4.9.11}$$

Theorem 4.9.1. *Assume*

$$\pi\lambda_g > \Lambda_{\nabla g}\sqrt{d}dist_g\big(P_0,P_T\big)\lambda_g^{\frac{-1}{2}} \quad and \quad \pi^2\lambda_g + d^2\lambda_{\nabla^2 g} > 2\pi\Lambda_{\nabla g}\sqrt{d}dist_g\big(P_0,P_T\big)\lambda_g^{\frac{-1}{2}}. \tag{4.9.12}$$

Then for small enough $\epsilon \geq 0$*, there exists a unique* $\hat{\gamma}^\epsilon \in W^{1,2}(0,1;\bar{\Omega})$ *minimizer in* $\hat{c}_0^{1,\epsilon}(P_0,P_T)$.

Proof. It suffices to show that we can find $e > 0$ such that the expression at the right handside of (4.9.11) is positive for $\epsilon = 0$. By continuity, the conclusion would hold for small enough ϵ, unless $\delta\gamma \equiv 0$.

By Poincare's inequality, we have

$$\int_0^1 |\delta\dot{\gamma}|^2 \geq \pi^2 \int_0^1 |\delta\gamma|^2 dt.$$

Thus, if $e \leq 1/\pi$, then $\lambda_g > e\Lambda_{\nabla g}\sqrt{d}\Big(\text{dist}_g\big(P_0,P_T\big) + \epsilon\kappa\Big)\lambda_g^{\frac{-1}{2}}$ for $\epsilon \geq 0$ sufficiently small.

Apply Poincare's inequality again, we obtain

$$\begin{aligned}&\int_0^1 |\delta\dot{\gamma}|^2\left[\lambda_g - e\Lambda_{\nabla g}\sqrt{d}\Big(\text{dist}_g\big(P_0,P_T\big)+\epsilon\kappa\Big)\lambda_g^{\frac{-1}{2}}\right]dt \\ \geq&\pi^2\left[\lambda_g - e\Lambda_{\nabla g}\sqrt{d}\Big(\text{dist}_g\big(P_0,P_T\big)+\epsilon\kappa\Big)\lambda_g^{\frac{-1}{2}}\right]\int_0^1 |\delta\gamma|^2 dt.\end{aligned}$$

We use (4.9.11) to conclude that

$$\frac{d^2}{d\lambda^2}\int_0^1 \hat{L}^{g,\epsilon}\big(\gamma^\lambda,\dot{\gamma}^\lambda\big)dt > \varphi_\epsilon(e)\int_0^1 |\delta\gamma|^2 dt,$$

where we have set

$$\begin{aligned}\varphi_\epsilon(e) :=&\pi^2\left[\lambda_g - e\Lambda_{\nabla g}\sqrt{d}\Big(\text{dist}_g\big(P_0,P_T\big)+\epsilon\kappa\Big)\lambda_g^{\frac{-1}{2}}\right] \\ &+\left[d^2\lambda_{\nabla^2 g} + 4\epsilon^2\lambda_{\nabla^2\Phi} - \Lambda_{\nabla g}\sqrt{d}e^{-1}\Big(\text{dist}_g\big(P_0,P_T\big)+\epsilon\kappa\Big)\lambda_g^{\frac{-1}{2}}\right].\end{aligned}$$

The study of φ_0 lead us to compute $\varphi_\epsilon(\pi^{-1})$ to discover that

$$\varphi_\epsilon(\pi^{-1}) = \pi^2\lambda_g + d^2\lambda_{\nabla^2 g} + 4\epsilon^2\lambda_{\nabla^2\Phi} - 2\pi^{-1}\Lambda_{\nabla g}\sqrt{d}\Big(\text{dist}_g\big(P_0,P_T\big)+\epsilon\kappa\Big)\lambda_g^{\frac{-1}{2}} > 0,$$

for $\epsilon \geq 0$ small enough.

□

4.10 Optimal paths in terms of Christoffel symbols

In this section, we assume that (4.9.12) holds. Set

$$\bar{\Phi}(q) = \Phi(q,q), \quad \bar{L}(q,v) := \frac{1}{2}|v|_g^2 + \eta\bar{\Phi}(q).$$

Note that in the above section $\eta = 4\epsilon^2 \geq 0$, but some of the comments we shall make in this section still make sense for $\eta < 0$ provided that $|\eta| << 1$.

There exists $\eta_0 > 0$ and $\delta_0 > 0$ such that if $|x-y| \leq \delta_0$ and $\eta \in [-\eta_0, \eta_0]$, then

$$Q \mapsto \int_0^1 \bar{L}(Q,\dot{Q})dt$$

admits a minimizer $Q(\cdot,\eta)$, over the set of Q such that $Q(0)=x$ and $Q(1)=y$. Furthermore, the minimizer is unique for $\eta \in [0,\eta_0]$.

Note

$$\frac{1}{2}\mathrm{dist}_g^2(x,y) - |\eta|\|\Phi\|_{L^\infty(\overline{\Omega}^2)} \leq \int_0^1 \bar{L}\big(Q(t,\eta),\dot{Q}(t,\eta)\big)dt \leq \frac{1}{2}\mathrm{dist}_g^2(x,y) + |\eta|\|\Phi\|_{L^\infty(\overline{\Omega}^2)}.$$

Thus

$$-3|\eta|\|\Phi\|_{L^\infty(\overline{\Omega}^2)} \leq \int_0^1 \left(\frac{1}{2}|\dot{Q}(t,\eta)|^2_{g(Q(t,\eta))} + \eta\bar{\Psi}(Q(t,\eta))\right)dt - \frac{1}{2}\mathrm{dist}_g^2(x,y) \leq 3|\eta|\|\Phi\|_{L^\infty(\overline{\Omega}^2)}.$$

By the conservation of the Hamiltonian, we have

$$-3|\eta|\|\Phi\|_{L^\infty(\overline{\Omega}^2)} \leq \frac{1}{2}|\dot{Q}(t,\eta)|^2_{g(Q(t,\eta))} + \eta\bar{\Psi}(Q(t,\eta)) - \frac{1}{2}\mathrm{dist}_g^2(x,y) \leq 3|\eta|\|\Phi\|_{L^\infty(\overline{\Omega}^2)}, \quad (4.10.1)$$

for all $t \in [0,1]$.

The Euler–Lagrange equation satisfies by $Q \equiv Q(\cdot,\eta)$ are

$$\frac{d}{dt}D_vL_0(Q,\dot{Q}) = D_qL_0(Q,\dot{Q}) + \eta\nabla\bar{\Phi}(Q), \quad Q(0,\eta)=x, Q(1,\eta)=y.$$

The system of ODEs is equivalent to

$$\frac{d}{dt}g(Q)\dot{Q} = D_qL_0(Q,\dot{Q}) + \eta\nabla\bar{\Phi}(Q). \quad (4.10.2)$$

This, together with (4.10.1), implies there exists a constant $c_0(\eta_0, \delta_0, \Phi, g)$ which depends only on η_0, δ_0, Φ, g such that

$$\|\ddot{Q}(t,\eta)\|_{L^\infty([0,1]\times[-\eta_0,\eta_0])} \leq c_0(\eta_0, \delta_0, \Phi, g). \tag{4.10.3}$$

Let $(g^{ij})_{ij}$ be the inverse of g. Then the Christoffel symbols for g are

$$\Gamma^\alpha_{jk}(q) = \sum_{i=1}^d g^{\alpha i}\Big(\frac{\partial g_{ij}(q)}{\partial q_k} + \frac{\partial g_{ik}(q)}{\partial q_j} - \frac{\partial g_{jk}(q)}{\partial q_i}\Big).$$

We define the bilinear form Γ by

$$\Gamma^\alpha(q)(v,v) = \sum_{jk} \Gamma^\alpha_{jk}(q) v_j v_k.$$

Remark 4.10.1. *By* (4.10.2),

$$\ddot{Q} = -\Gamma(Q)(\dot{Q},\dot{Q}) + \eta g^{-1}(Q)\nabla\bar{\Phi}(Q). \tag{4.10.4}$$

Thus

$$\begin{aligned}\partial_\eta \ddot{Q}^\alpha = &-\sum_{j,k,l} \frac{\partial \Gamma^\alpha_{jk}}{\partial q_l}(Q)\, \partial_\eta Q^l\, \dot{Q}^j\, \dot{Q}^k - \sum_{j,k} \Gamma^\alpha_{jk}(Q)\, \partial_\eta \dot{Q}^j\, \dot{Q}^k - \sum_{j,k} \Gamma^\alpha_{jk}(Q)\, \dot{Q}^j\, \partial_\eta \dot{Q}^k \\ &+ \sum_j g^{\alpha j}(Q) \frac{\partial \bar{\Phi}}{\partial q^j}(Q) + \eta \sum_{j,l} \frac{\partial g^{\alpha j}}{\partial q_l}(Q)\, \partial_\eta Q^l\, \frac{\partial \bar{\Phi}}{\partial q^j}(Q) + \eta \sum_{j,l} g^{\alpha j}(Q) \frac{\partial^2 \bar{\Phi}}{\partial q^j \partial q^l}(Q)\, \partial_\eta Q^l\end{aligned}$$

Lemma 4.10.2. *We have* $Q \in C\Big([0,\eta_0]; C^2([0,1];\mathbb{R}^d)\Big)$.

*Proof. P*art 1. We claim that $Q \in C\Big([0,\eta_0]; C^1([0,1];\mathbb{R}^d)\Big)$. Indeed, let $\bar{\eta} \in [0,\eta_0]$, We want to show that $\lim_{\eta\to\bar{\eta}} \|Q(\cdot,\eta) - Q(\cdot,\bar{\eta})\|_{C^1} = 0$. It suffices to show that if $(\eta_n)_n \subset [-\eta_0,\eta_0]$ is a sequence converging to $\bar{\eta}$, then up to a subsequence, we have

$$\lim_{n\to\infty} \|Q(\cdot,\eta_n) - Q(\cdot,\bar{\eta})\|_{C^1} = 0.$$

Note that if $(\eta_n)_n \subset [0,\eta_0]$ is a sequence converging to $\bar{\eta}$, then by (4.10.1) and (4.10.3), Ascoli–Arzela Lemma implies that $(\eta_n)_n$ is precompact in C^1. Thus, it admits an accumulation point $\bar{Q}$ in this topology. For any $Q \in C^1\big([0,1];\mathbb{R}^d\big)$ such that $Q(0) = x$ and $Q(1) = y$, we have

$$\int_0^1 \left(\frac{1}{2}|\dot{Q}(t,\eta_n)|^2_{g(Q(t,\eta_n))} + \eta_n \bar{\Phi}(Q(t,\eta_n))\right) dt \le \int_0^1 \left(\frac{1}{2}|\dot{Q}(t)|^2_{g(Q(t,\eta))} + \eta_n \bar{\Phi}(Q(t))\right) dt.$$

Using a converging subsequence of $(\eta_n)_n$, we conclude that

$$\int_0^1 \left(\frac{1}{2}|\dot{\bar{Q}}(t)|^2_{g(\bar{Q})} + \bar{\eta}\bar{\Phi}(\bar{Q}(t))\right) dt \le \int_0^1 \left(\frac{1}{2}|\dot{Q}(t)|^2_{g(Q(t,\eta))} + \bar{\eta}\bar{\Phi}(Q(t))\right) dt.$$

This proves that $\bar{Q} = Q(\cdot, \bar{\eta})$ is uniquely determined and, up to a subsequence,

$$\lim_{n\to\infty} \|Q(\cdot,\eta_n) - Q(\cdot,\bar{\eta})\|_{C^1} = 0.$$

Part 2. Notice that $Q \in C\Big([0,\eta_0]; C^1([0,1];\mathbb{R}^d)\Big)$. Since g^{-1}, $\nabla\bar{\Phi}$, and the Christoffel symbols Γ^i are all continuous, (4.10.4) implies that $Q \in C\Big([0,\eta_0]; C^2([0,1];\mathbb{R}^d)\Big)$.

□

Integrating (4.10.4), we obtain

$$\dot{Q}(t,\eta) = \dot{Q}(0,\eta) + \int_0^t \Big(-\Gamma\big(Q(\tau,\eta)\big)\big(\dot{Q}(\tau,\eta),\dot{Q}(\tau,\eta)\big) + \eta g^{-1}\big(Q(\tau,\eta)\big)\nabla\bar{\Phi}\big(Q(\tau,\eta)\big)\Big) d\tau$$

and

$$\begin{aligned} &Q(t,\eta) \\ =&x + t\dot{Q}(0,\eta) + \int_0^t ds \int_0^s \Big(-\Gamma\big(Q(\tau,\eta)\big)\big(\dot{Q}(\tau,\eta),\dot{Q}(\tau,\eta)\big) + \eta g^{-1}\big(Q(\tau,\eta)\big)\nabla\bar{\Phi}\big(Q(\tau,\eta)\big)\Big) d\tau. \end{aligned}$$

Thus

$$\dot{Q}(0,\eta) = y - x - \int_0^1 ds \int_0^s \Big(-\Gamma\big(Q(\tau,\eta)\big)\big(\dot{Q}(\tau,\eta),\dot{Q}(\tau,\eta)\big) + \eta g^{-1}\big(Q(\tau,\eta)\big)\nabla\bar{\Phi}\big(Q(\tau,\eta)\big)\Big) d\tau.$$

We then use this expression of $\dot{Q}(0,\eta)$ to conclude that

$$\begin{aligned} Q(t,\eta) =& x + t(y-x) \qquad (4.10.5) \\ &-t\int_0^1 ds \int_0^s \Big(-\Gamma\big(Q(\tau,\eta)\big)\big(\dot{Q}(\tau,\eta),\dot{Q}(\tau,\eta)\big) + \eta g^{-1}\big(Q(\tau,\eta)\big)\nabla\bar{\Phi}\big(Q(\tau,\eta)\big)\Big) d\tau \\ &+\int_0^t ds \int_0^s \Big(-\Gamma\big(Q(\tau,\eta)\big)\big(\dot{Q}(\tau,\eta),\dot{Q}(\tau,\eta)\big) + \eta g^{-1}\big(Q(\tau,\eta)\big)\nabla\bar{\Phi}\big(Q(\tau,\eta)\big)\Big) d\tau \end{aligned}$$

Hence

$$\dot Q(t,\eta) = y - x - \int_0^1 ds \int_0^s \Big(-\Gamma\big(Q(\tau,\eta)\big)\big(\dot Q(\tau,\eta), \dot Q(\tau,\eta)\big) + \eta g^{-1}\big(Q(\tau,\eta)\big)\nabla\bar\Phi\big(Q(\tau,\eta)\big)\Big) d\tau$$
$$+ \int_0^t \Big(-\Gamma\big(Q(\tau,\eta)\big)\big(\dot Q(\tau,\eta), \dot Q(\tau,\eta)\big) + \eta g^{-1}\big(Q(\tau,\eta)\big)\nabla\bar\Phi\big(Q(\tau,\eta)\big)\Big) d\tau. \qquad (4.10.6)$$

There exist $\eta_0 > 0$ and $\delta_0 > 0$ such that if $|x-y| \le \delta_0$, then there exists a map

$$\mathbb{M} : C\Big([0,1]\times[0,\eta_0]\Big)^d \to C\Big([0,1]\times[-\eta_0,\eta_0]\Big)^d,$$

which verifies the relation

$$\mathbb{M}(Q)(t,\eta) = x + t(y-x)$$
$$-t \int_0^1 ds \int_0^s \Big(-\Gamma\big(Q(\tau,\eta)\big)\big(\dot Q(\tau,\eta), \dot Q(\tau,\eta)\big) + \eta g^{-1}\big(Q(\tau,\eta)\big)\nabla\bar\Phi\big(Q(\tau,\eta)\big)\Big) d\tau$$
$$+ \int_0^t ds \int_0^s \Big(-\Gamma\big(Q(\tau,\eta)\big)\big(\dot Q(\tau,\eta), \dot Q(\tau,\eta)\big) + \eta g^{-1}\big(Q(\tau,\eta)\big)\nabla\bar\Phi\big(Q(\tau,\eta)\big)\Big) d\tau.$$

Furthermore, $\mathbb{M}(\cdot,\eta) \in C^2\big([0,1]\big)^d$.

Remark 4.10.3. *We have*

$$\frac{\partial}{\partial_\eta}\Big(\|\dot Q(s,\eta)\|^2_{g(Q(s,\eta))} \Big) = \frac{\partial}{\partial_\eta}\Big(\sum_{ij} g_{ij}(Q)\dot Q^i \dot Q^j \Big)$$
$$= \sum_{i,j,k} \frac{\partial g_{ij}(Q)}{\partial q^k}\frac{\partial Q^k}{\partial \eta}\dot Q^i \dot Q^j + \sum_{i,j} g_{ij}(Q)\Big(\frac{\partial \dot Q^i}{\partial \eta}\dot Q^j + g_{ij}(Q)\dot Q^i \frac{\partial \dot Q^j}{\partial \eta}\Big)$$

We now assume that (4.9.12) holds and study optimal paths depending on η.

Lemma 4.10.4. *We have*

$$\frac{1}{2}|\dot Q(t,\eta)|^2_{g(Q(t,\eta))} = \frac{1}{2} dist_g^2(x,y) + \eta\Big(\bar\Phi(Q(t,0)) - \int_0^1 \bar\Phi(Q(t,0))dt \Big) + o(\eta). \qquad (4.10.7)$$

Proof. As observed in Proposition 4.7.1

$$\lim_{\eta\to 0} \frac{\int_0^1 \frac{1}{2}|\dot Q(t,\eta)|^2_{g(Q(t,\eta))}dt - \frac{1}{2}\mathrm{dist}_g^2(x,y)}{\eta} = 0. \qquad (4.10.8)$$

By the conservation of the Hamiltonian,

$$\int_0^1 \left(\frac{1}{2}|\dot{Q}(t,\eta)|^2_{g(Q(t,\eta))} - \eta\bar{\Phi}(Q(t,\eta))\right)dt = \frac{1}{2}|\dot{Q}(t,\eta)|^2_{g(Q(t,\eta))} - \eta\bar{\Phi}(Q(t,\eta))$$

Since $(t,\eta) \mapsto \bar{\Phi}(Q(t,\eta))$ is uniformly continuous, using (4.10.8), we have

$$\frac{1}{2}\text{dist}_g^2(x,y) - \eta\int_0^1 \bar{\Phi}(Q(t,0))dt + o(\eta) = \frac{1}{2}|\dot{Q}(t,\eta)|^2_{g(Q(t,\eta))} - \eta\bar{\Phi}(Q(t,0)) + o(\eta).$$

□

4.11 Crossing curves

In this section, we consider a case with a special ψ^ϵ and four points $P_0, P_T, Q_0, Q_T \in \partial\Omega$. When $\delta = \epsilon$, we have the special case with two curves that starts at the same time $t = 0$ at points $P_0, P_T \in \partial\Omega$ and ends at the same time $t = T$ at points $Q_0, Q_T \in \partial\Omega$ respectively. This helps us separate the influences of the two curves and establish the map I_g in the next Chapter 5.

We set

$$\psi_t^\epsilon = \begin{cases} (1-\epsilon)\delta_{P_0} + \epsilon\delta_{Q_0} & \text{if } t \in \left[0, \frac{T}{2} - \epsilon\right] \\ \epsilon\delta_{Q_0} & \text{if } t \in \left(\frac{T}{2} - \epsilon, \frac{T}{2} - \delta\right] \\ 0 & \text{if } t \in \left(\frac{T}{2} - \delta, \frac{T}{2} + \delta\right) \\ \epsilon\delta_{Q_T} & \text{if } t \in \left[\frac{T}{2} + \delta, \frac{T}{2} + \epsilon\right) \\ (1-\epsilon)\delta_{P_T} + \epsilon\delta_{Q_T} & \text{if } t \in \left[\frac{T}{2} + \epsilon, T\right] \end{cases}$$

For any admissible (σ, v), we have

$$\inf_{(\sigma,v)} \mathcal{A}_g^\Phi[\sigma, v] = \frac{1}{2}\Big(\frac{T}{2} - \epsilon\Big)\Big((1-\epsilon)^2\Phi(P_0, P_0) + 2\epsilon(1-\epsilon)\Phi(P_0, Q_0) + \epsilon^2\Phi(Q_0, Q_0)\Big)$$
$$+\frac{1}{2}\Big(\frac{T}{2} - \epsilon\Big)\Big((1-\epsilon)^2\Phi(P_T, P_T) + 2\epsilon(1-\epsilon)\Phi(P_T, Q_T) + \epsilon^2\Phi(Q_T, Q_T)\Big)$$
$$+ \inf_{\rho^\epsilon_{T/2-\delta},\, \rho^\epsilon_{T/2+\delta}} \Big\{ I^1_\epsilon\Big(\psi^\epsilon_{\frac{T}{2}-\epsilon}, \rho^\epsilon_{\frac{T}{2}-\delta}\Big) + I^2_\epsilon\Big(\rho^\epsilon_{\frac{T}{2}-\delta}, \rho^\epsilon_{\frac{T}{2}+\delta}\Big) + I^3_\epsilon\Big(\rho^\epsilon_{\frac{T}{2}+\delta}, \psi^\epsilon_{\frac{T}{2}+\epsilon}\Big)\Big\} \tag{4.11.1}$$

$$= \frac{1}{2}\Big(\frac{T}{2} - \epsilon\Big)\Big((1-\epsilon)^2\Phi(P_0, P_0) + 2\epsilon(1-\epsilon)\Phi(P_0, Q_0) + \epsilon^2\Phi(Q_0, Q_0)\Big)$$
$$+\frac{1}{2}\Big(\frac{T}{2} - \epsilon\Big)\Big((1-\epsilon)^2\Phi(P_T, P_T) + 2\epsilon(1-\epsilon)\Phi(P_T, Q_T) + \epsilon^2\Phi(Q_T, Q_T)\Big)$$
$$+\Big\{ I^1_\epsilon\Big(\psi^\epsilon_{\frac{T}{2}-\epsilon}, \sigma^{\epsilon,*}_{\frac{T}{2}-\delta}\Big) + I^2_\epsilon\Big(\sigma^{\epsilon,*}_{\frac{T}{2}-\delta}, \sigma^{\epsilon,*}_{\frac{T}{2}+\delta}\Big) + I^3_\epsilon\Big(\sigma^{\epsilon,*}_{\frac{T}{2}+\delta}, \psi^\epsilon_{\frac{T}{2}+\epsilon}\Big)\Big\} \tag{4.11.2}$$

Here, in (4.11.1), the inf is taken over all $\rho^\epsilon_{T/2-\delta} \geq \epsilon\delta_{Q_0}$ and $\rho^\epsilon_{T/2+\delta} \geq \epsilon\delta_{Q_T}$. Also, we define

$$I^1_\epsilon\Big(\psi^\epsilon_{\frac{T}{2}-\epsilon}, \rho^\epsilon_{\frac{T}{2}-\delta}\Big) = \inf_{(\sigma,v)} \Big\{ \int_{\frac{T}{2}-\epsilon}^{\frac{T}{2}-\delta} \Big(\int_{\bar\Omega} \frac{1}{2}|v_t(x)|^2_{g(x)}\sigma_t(dx) + F_\Phi(\sigma_t)\Big) dt : \sigma_t \geq \psi^\epsilon_t, \quad \sigma_{\frac{T}{2}-\epsilon} = \psi^\epsilon_{\frac{T}{2}-\epsilon}, \quad \sigma_{\frac{T}{2}-\delta} = \rho^\epsilon_{\frac{T}{2}-\delta}\Big\}$$

and we define I^2_ϵ and I^3_ϵ by minimizing the actions on the appropriate intervals similarly. Moreover, in (4.11.2) we denote $(\sigma^{\epsilon,*}_{\frac{T}{2}-\delta}, \sigma^{\epsilon,*}_{\frac{T}{2}+\delta})$ as the minimizer of the problem (4.11.1), which is also σ^* at the time incident $t = \frac{T}{2} \pm \delta$ of the optimizer (σ^*, v^*) in the original problem $\inf_{(\sigma,v)} \mathcal{A}_g^\Phi[\sigma, v]$.

Reparametrize the time, we have

$$I^1_\epsilon\Big(\psi^\epsilon_{\frac{T}{2}-\epsilon}, \rho^\epsilon_{\frac{T}{2}-\delta}\Big) = \inf_{(\sigma,v)} \Big\{ \int_0^1 \Big(\int_{\bar\Omega} \frac{1}{2(\epsilon-\delta)}|\tilde v_s(x)|^2_{g(x)}\tilde\sigma_s(dx) + \frac{(\epsilon-\delta)}{2}\int_{\bar\Omega^2} \Phi(x,y)\tilde\sigma_s(dx)\tilde\sigma_s(dy)\Big) ds : \tilde\sigma_s \geq \psi^\epsilon_{(\epsilon-\delta)s+\frac{T}{2}-\epsilon}, \quad \tilde\sigma_0 = \psi^\epsilon_{\frac{T}{2}-\epsilon}, \quad \tilde\sigma_1 = \rho^\epsilon_{\frac{T}{2}-\delta}\Big\}. \tag{4.11.3}$$

Similarly, we have

$$
\begin{aligned}
&I_\epsilon^2\Big(\rho^\epsilon_{\frac{T}{2}-\delta},\rho^\epsilon_{\frac{T}{2}+\delta}\Big)\\
&=\inf_{(\sigma,v)}\Bigg\{\int_0^1\Bigg(\frac{1}{4\delta}\int_{\bar\Omega}|\tilde v_s(x)|^2_{g(x)}\tilde\sigma_s(dx)+\delta\int_{\bar\Omega^2}\Phi(x,y)\tilde\sigma_s(dx)\tilde\sigma_s(dy)\Bigg)ds:\\
&\qquad\qquad\tilde\sigma_s\ge\psi^\epsilon_{2\delta s+\frac{T}{2}-\delta},\quad\tilde\sigma_0=\rho^\epsilon_{\frac{T}{2}+\delta},\quad\tilde\sigma_1=\rho^\epsilon_{\frac{T}{2}+\delta}\Bigg\}
\end{aligned}
\tag{4.11.4}
$$

and

$$
\begin{aligned}
&I_\epsilon^3\Big(\rho^\epsilon_{\frac{T}{2}+\delta},\psi^\epsilon_{\frac{T}{2}+\epsilon}\Big)\\
&=\inf_{(\sigma,v)}\Bigg\{\int_0^1\Bigg(\int_{\bar\Omega}\frac{1}{2(\epsilon-\delta)}|\tilde v_s(x)|^2_{g(x)}\tilde\sigma_s(dx)+\frac{(\epsilon-\delta)}{2}\int_{\bar\Omega^2}\Phi(x,y)\tilde\sigma_s(dx)\tilde\sigma_s(dy)\Bigg)ds:\\
&\qquad\qquad\tilde\sigma_s\ge\psi^\epsilon_{(\epsilon-\delta)s+\frac{T}{2}+\delta},\quad\tilde\sigma_0=\rho^\epsilon_{\frac{T}{2}+\delta},\quad\tilde\sigma_1=\psi^\epsilon_{\frac{T}{2}+\epsilon}\Bigg\}.
\end{aligned}
\tag{4.11.5}
$$

For a Riemannian manifold Ω with metric g, we say g is simple if Ω is geodesic convex with respect to g.

Lemma 4.11.1. *Assume g is a simple metric in $\overline{\Omega}$ and consider the optimal transport problem*

$$
W_g^2\Big(\delta_{P_0},\nu\Big):=\inf_{(\sigma,v)}\Bigg\{\int_0^1\int_{\overline{\Omega}}|v(t,x)|^2_{g(x)}\rho(dq)dt:\ \partial_t\rho+\nabla(\rho v)=0,\ \rho_0=\delta_{P_0},\ \rho_1=\nu\Bigg\}.
\tag{4.11.6}
$$

Recall that $Q_0\in\partial\Omega$. If $P_0\neq Q_0$, then there exists an optimizer (σ,v) satisfies that

$$Q_0\notin supp(\sigma_t)\ \textit{for all}\ t<1\,,$$

and therefore, we can choose an optimizer (σ,v) satisfying,

$$v(t,Q_0)=0\ \textit{for all}\ t<1\,.$$

Proof. The set $\Gamma(\gamma_{P_0},\nu)$ of measures on $\overline{\Omega}^2$ which admit δ_{P_0} and ν as marginals is $\{\delta_{P_0}\times\nu\}$.

Hence

$$W_g^2(\delta_{P_0}, \nu) = \int_{\overline{\Omega}} \mathrm{dist}_g^2(P_0, y)\nu(dy).$$

Let Υ be the map defined in section 6.4. For any $y \in \overline{\Omega}$, we set

$$\gamma_y := \Upsilon(\cdot, y) = \gamma_{0,P_0}^{1,y}(t) : [0,1] \to \overline{\Omega}.$$

Then

$$\gamma_y(0) = P_0,\ \gamma_y(1) = y,\ |\dot{\gamma_y}(t)|_{g(\gamma_y(t))} = \mathrm{dist}_g(P_0, y) \text{ for all } t \in [0,1].$$

Therefore for $t \in [0,1]$, we have

$$\mathrm{dist}_g^2(P_0, \gamma_y(t)) = t^2 \mathrm{dist}_g^2(P_0, y).$$

We use (6.4.1) to show that the velocity of σ at $\gamma(t)$ is $\dot{\gamma}(t)$, σ is 2–absolutely continuous and $\sigma_0 = \delta_{P_0}$ and $\sigma_1 = \nu$. We have

$$W_g^2\left(\delta_{P_0}, \sigma_t\right) = t^2 W_g^2\left(\delta_{P_0}, \sigma_1\right)$$

Therefore σ is a constant speed W_g-geodesic and (σ, w) is an optimizer in (4.11.6).

Since

$$\Upsilon(t,\cdot)\big(\overline{\Omega}\big) \subset \Omega,\ \forall t \in (0,1), \quad \Upsilon(0,\cdot) \equiv P_0, \quad P_0 \neq Q_0,$$

we conclude that

$$Q_0 \notin \Upsilon\big([0,1) \times \overline{\Omega}\big).$$

Hence, if $t \in [0,1)$, since supp $(\sigma_t) \subset \Upsilon\big([0,1) \times \overline{\Omega}\big)$, we conclude that $Q_0 \notin$ supp (σ_t).

□

Lemma 4.11.2. *Assume $Q_0 \neq P_0$ and g is a simple metric in $\overline{\Omega}$, $\rho^\epsilon_{\frac{T}{2}-\delta} \geq \epsilon\delta_{Q_0}$ and $\rho^\epsilon_{\frac{T}{2}-\delta} \neq \epsilon\delta_{Q_0}$. Then*

$$I^1_\epsilon\Big(\psi^\epsilon_{\frac{T}{2}-\epsilon}, \rho^\epsilon_{\frac{T}{2}-\delta}\Big) \leq \frac{(\epsilon-\delta)}{2}\|\Phi\|_{L^\infty} + \frac{(1-\epsilon)}{2(\epsilon-\delta)} W_g^2\Big(\delta_{P_0}, (1-\epsilon)^{-1}\left(\rho^\epsilon_{\frac{T}{2}-\delta} - \epsilon\delta_{Q_0}\right)\Big).$$

Proof. From (4.11.4), we have

$$
\begin{aligned}
& I^1_\epsilon\left(\psi^\epsilon_{\frac{T}{2}-\epsilon}, \rho^\epsilon_{\frac{T}{2}-\delta}\right) \\
\leq & \frac{(\epsilon-\delta)}{2}\|\Phi\|_{L^\infty} + \inf_{(\sigma,v)}\left\{\int_0^1\int_\Omega \frac{1}{2(\epsilon-\delta)}|v(t,x)|^2_{g(x)}\rho(dq)dt : \delta_t\rho + \nabla(\rho v) = 0,\right. \\
& \left.\rho_s \geq \epsilon\delta_{Q_0}, \rho_0 = (1-\epsilon)\delta_{P_0} + \epsilon\delta_{Q_0}, \quad \rho_1 = \rho^\epsilon_{\frac{T}{2}-\delta}\right\}.
\end{aligned} \tag{4.11.7}
$$

From Lemma 4.11.1, we can choose an optimizer (σ, v) in (4.11.6) such that $Q_0 \notin \operatorname{supp}(\sigma_t)$ and $v(t, Q_0) = 0$ for $t < 1$.

We therefore have that

$$(\tilde{\sigma}_t, v) = ((1-\epsilon)\sigma_t + \epsilon\delta_{Q_0}, v)$$

is a competitor for the problem (4.11.4).

Note that

$$\tilde{\sigma}_0 = \psi^\epsilon_{\frac{T}{2}-\epsilon}, \quad \tilde{\sigma}_1 = \rho^\epsilon_{\frac{T}{2}-\delta}, \quad \text{and} \quad \tilde{\sigma}_t \geq \epsilon\delta_{Q_0}.$$

Moreover, since $v\delta_{Q_0} = 0$ for $t < 1$, we have

$$\partial_t\tilde{\sigma}_t + \nabla\cdot(v\tilde{\sigma}_t) = (1-\epsilon)\left(\partial_t\sigma_t + \nabla\cdot(v\sigma_t)\right) + \epsilon\left(\partial_t\delta_{Q_0} + \nabla\cdot(v\delta_{Q_0})\right) = 0.$$

Hence

$$
\begin{aligned}
& I^1_\epsilon\left(\psi^\epsilon_{\frac{T}{2}-\epsilon}, \rho^\epsilon_{\frac{T}{2}-\delta}\right) \\
\leq & \frac{(\epsilon-\delta)}{2}\|\Phi\|_{L^\infty} + \int_0^1\int_\Omega \frac{1}{2(\epsilon-\delta)}|v(t,x)|^2_{g(x)}\tilde{\sigma}_t(dq)dt \\
= & \frac{(\epsilon-\delta)}{2}\|\Phi\|_{L^\infty} + \frac{(1-\epsilon)}{(\epsilon-\delta)}\int_0^1\int_\Omega \frac{1}{2}|v(t,x)|^2_{g(x)}\sigma_t(dq)dt \\
+ & \frac{\epsilon}{(\epsilon-\delta)}\int_0^1\int_\Omega \frac{1}{2}|v(t,x)|^2_{g(x)}\delta_{Q_0}(dq)dt \\
= & \frac{(\epsilon-\delta)}{2}\|\Phi\|_{L^\infty} + \frac{(1-\epsilon)}{2(\epsilon-\delta)}W^2_g\left(\delta_{P_0}, (1-\epsilon)^{-1}\left(\rho^\epsilon_{\frac{T}{2}-\delta} - \epsilon\delta_{Q_0}\right)\right),
\end{aligned}
$$

where we used again that $v(t, Q_0) = 0$ for $t < 1$.

□

Lemma 4.11.3. *Let $Q_0 \neq P_0$, $Q_T \neq P_Q$ and assume g is a simple metric in $\overline{\Omega}$. For any $P^* \in \Omega \setminus \{Q_0, Q_T\}$, we have*

$$\frac{\delta}{2(\epsilon-\delta)}\mathcal{W}_g^2\Big(\psi^{\epsilon}_{\frac{T}{2}-\epsilon}, \sigma^{\epsilon,*}_{\frac{T}{2}-\delta}\Big) + \frac{1}{4}\mathcal{W}_g^2\Big(\sigma^{\epsilon,*}_{\frac{T}{2}-\delta}, \sigma^{\epsilon,*}_{\frac{T}{2}+\delta}\Big) + \frac{\delta}{2(\epsilon-\delta)}\mathcal{W}_g^2\Big(\rho^{\epsilon}_{\frac{T}{2}+\delta}, \psi^{\epsilon}_{\frac{T}{2}+\epsilon}\Big) - \epsilon\|\Phi\|_{L^\infty}$$
$$\leq \frac{\delta(1-\epsilon)}{2(\epsilon-\delta)} dist_g^2\Big(P_0, P^*\Big) + \frac{\epsilon}{4} dist_g^2\Big(Q_0, Q_T\Big) + \frac{\delta(1-\epsilon)}{2(\epsilon-\delta)} dist_g^2\Big(P^*, P_T\Big) + \epsilon\|\Phi\|_{L^\infty}.$$

Proof. By (4.11.3)

$$\frac{1}{2(\epsilon-\delta)}\mathcal{W}_g^2\Big(\psi^{\epsilon}_{\frac{T}{2}-\epsilon}, \rho^{\epsilon}_{\frac{T}{2}-\delta}\Big) - \frac{(\epsilon-\delta)}{2}\|\Phi\|_{L^\infty} \leq I^1_\epsilon\Big(\psi^{\epsilon}_{\frac{T}{2}-\epsilon}, \rho^{\epsilon}_{\frac{T}{2}-\delta}\Big). \tag{4.11.8}$$

Combining this with Lemma 4.11.2, we deduce that

$$\begin{aligned}
&\frac{1}{2(\epsilon-\delta)}\mathcal{W}_g^2\Big(\psi^{\epsilon}_{\frac{T}{2}-\epsilon}, \rho^{\epsilon}_{\frac{T}{2}-\delta}\Big) - \frac{(\epsilon-\delta)}{2}\|\Phi\|_{L^\infty} \\
\leq& I^1_\epsilon\Big(\psi^{\epsilon}_{\frac{T}{2}-\epsilon}, \rho^{\epsilon}_{\frac{T}{2}-\delta}\Big) \\
\leq& \frac{(1-\epsilon)}{2(\epsilon-\delta)} W_g^2\Big(\delta_{P_0}, (1-\epsilon)^{-1}\left(\rho^{\epsilon}_{\frac{T}{2}-\delta} - \epsilon\delta_{Q_0}\right)\Big) + \frac{(\epsilon-\delta)}{2}\|\Phi\|_{L^\infty}.
\end{aligned} \tag{4.11.9}$$

By (4.11.4),

$$\frac{1}{4\delta}\mathcal{W}_g^2\Big(\rho^{\epsilon}_{\frac{T}{2}-\delta}, \rho^{\epsilon}_{\frac{T}{2}+\delta}\Big) - \delta\|\Phi\|_{L^\infty} \leq I^2_\epsilon\Big(\rho^{\epsilon}_{\frac{T}{2}-\delta}, \rho^{\epsilon}_{\frac{T}{2}+\delta}\Big) \leq \frac{1}{4\delta}\mathcal{W}_g^2\Big(\rho^{\epsilon}_{\frac{T}{2}-\delta}, \rho^{\epsilon}_{\frac{T}{2}+\delta}\Big) + \delta\|\Phi\|_{L^\infty}. \tag{4.11.10}$$

Using P_T in place of P_0 and Q_T in place of Q_0, the analogue of (4.11.9) is

$$\begin{aligned}
&\frac{1}{2(\epsilon-\delta)}\mathcal{W}_g^2\Big(\rho^{\epsilon}_{\frac{T}{2}+\delta}, \psi^{\epsilon}_{\frac{T}{2}+\epsilon}\Big) - \frac{(\epsilon-\delta)}{2}\|\Phi\|_{L^\infty} \\
\leq& I^3_\epsilon\Big(\rho_{\frac{T}{2}+\delta}, \psi^{\epsilon}_{\frac{T}{2}+\epsilon}\Big) \\
\leq& \frac{(1-\epsilon)}{2(\epsilon-\delta)} W_g^2\Big((1-\epsilon)^{-1}\left(\rho^{\epsilon}_{\frac{T}{2}+\delta} - \epsilon\delta_{Q_T}\right), \delta_{P_T}\Big) + \frac{(\epsilon-\delta)}{2}\|\Phi\|_{L^\infty}.
\end{aligned} \tag{4.11.11}$$

We add up the expressions in the first identities in (4.11.9), (4.11.10), and (4.11.11) to obtain that

$$\frac{1}{4\delta}\mathcal{W}_g^2\Big(\psi^{\epsilon}_{\frac{T}{2}-\epsilon}, \sigma^{\epsilon,*}_{\frac{T}{2}-\delta}\Big) + \frac{1}{2(\epsilon-\delta)}\mathcal{W}_g^2\Big(\sigma^{\epsilon,*}_{\frac{T}{2}-\delta}, \sigma^{\epsilon,*}_{\frac{T}{2}+\delta}\Big) + \frac{1}{2(\epsilon-\delta)}\mathcal{W}_g^2\Big(\rho^{\epsilon}_{\frac{T}{2}+\delta}, \psi^{\epsilon}_{\frac{T}{2}+\epsilon}\Big) - \delta\|\Phi\|_{L^\infty} \tag{4.11.12}$$

$$\leq I^2_\epsilon\Big(\sigma^{\epsilon,*}_{\frac{T}{2}-\delta}, \sigma^{\epsilon,*}_{\frac{T}{2}+\delta}\Big) + I^1_\epsilon\Big(\psi^{\epsilon}_{\frac{T}{2}-\epsilon}, \sigma^{\epsilon,*}_{\frac{T}{2}-\delta}\Big) + I^3_\epsilon\Big(\rho^{\epsilon}_{\frac{T}{2}+\delta}, \psi^{\epsilon}_{\frac{T}{2}+\epsilon}\Big).$$

Below, we minimize over the set of pairs $\big(\eta_0, \eta_1\big)$ such that $\eta_0 \geq \epsilon\delta_{Q_0}$ and $\eta_1 \geq \epsilon\delta_{Q_T}$. By an approximation argument, we can assume without loss of generality that $\eta_0 \neq \epsilon\delta_{Q_0}$ and $\eta_1 \neq \epsilon\delta_{Q_T}$.

We combine (4.11.9), (4.11.10), and (4.11.11) to deduce that

$$\begin{aligned}
&\frac{1}{4\delta}\mathcal{W}_g^2\Big(\psi^{\epsilon}_{\frac{T}{2}-\epsilon}, \sigma^{\epsilon,*}_{\frac{T}{2}-\delta}\Big) + \frac{1}{4\delta}\mathcal{W}_g^2\Big(\sigma^{\epsilon,*}_{\frac{T}{2}-\delta}, \sigma^{\epsilon,*}_{\frac{T}{2}+\delta}\Big) + \frac{1}{4\delta}\mathcal{W}_g^2\Big(\rho^{\epsilon}_{\frac{T}{2}+\delta}, \psi^{\epsilon}_{\frac{T}{2}+\epsilon}\Big) - \epsilon\|\Phi\|_{L^\infty} \\
\leq &\inf_{\eta_0,\eta_1,\eta_0\neq\epsilon\delta_{Q_0},\epsilon\eta_1\neq\delta_{Q_T}} \Big\{ I^1_\epsilon\Big(\psi^{\epsilon}_{\frac{T}{2}-\epsilon}, \eta_0\Big) + I^2_\epsilon\Big(\eta_0,\eta_1\Big) + I^3_\epsilon\Big(\eta_1, \psi^{\epsilon}_{\frac{T}{2}+\epsilon}\Big) : \eta_0 \geq \epsilon\delta_{Q_0},\ \epsilon\eta_1 \geq \delta_{Q_T} \Big\} \\
\leq &\inf_{\eta_0,\eta_1} \Big\{ \frac{(1-\epsilon)}{2(\epsilon-\delta)} W_g^2\Big(\delta_{P_0}, (1-\epsilon)^{-1}\left(\eta_0 - \epsilon\delta_{Q_0}\right)\Big) + \frac{(1-\epsilon)}{2(\epsilon-\delta)} W_g^2\Big((1-\epsilon)^{-1}\left(\eta_1 - \epsilon\delta_{Q_T}\right), \delta_{P_T}\Big) \\
&\qquad + \frac{1}{4\delta}\mathcal{W}_g^2\Big(\eta_0,\eta_1\Big) + \epsilon\|\Phi\|_{L^\infty} : \eta_0 \geq \epsilon\delta_{Q_0}, \eta_0 \neq \epsilon\delta_{Q_0},\ \epsilon\eta_1 \geq \delta_{Q_T},\ \epsilon\eta_1 \neq \delta_{Q_T} \Big\}.
\end{aligned} \tag{4.11.13}$$

We choose an arbitrary $P^* \in \Omega \setminus \{Q_0, Q_T\}$ and in the optimization problem, use

$$\eta_0 := (1-\epsilon)\delta_{P*} + \epsilon\delta_{Q_0}, \quad \eta_1 := (1-\epsilon)\delta_{P*} + \epsilon\delta_{Q_T}$$

to conclude that

$$\begin{aligned}
&\frac{1}{2(\epsilon-\delta)}\mathcal{W}_g^2\Big(\psi^{\epsilon}_{\frac{T}{2}-\epsilon}, \sigma^{\epsilon,*}_{\frac{T}{2}-\delta}\Big) + \frac{1}{4\delta}\mathcal{W}_g^2\Big(\sigma^{\epsilon,*}_{\frac{T}{2}-\delta}, \sigma^{\epsilon,*}_{\frac{T}{2}+\delta}\Big) + \frac{1}{2(\epsilon-\delta)}\mathcal{W}_g^2\Big(\rho^{\epsilon}_{\frac{T}{2}+\delta}, \psi^{\epsilon}_{\frac{T}{2}+\epsilon}\Big) - \epsilon\|\Phi\|_{L^\infty} \\
\leq &\frac{(1-\epsilon)}{2(\epsilon-\delta)}\bigg(W_g^2\big(\delta_{P_0}, \delta_{P*}\big) + W_g^2\big(\delta_{P*}, \delta_{P_T}\big)\bigg) + \frac{1}{4\delta}\mathcal{W}_g^2\Big((1-\epsilon)\delta_{P*} + \epsilon\delta_{Q_0}, (1-\epsilon)\delta_{P*} + \epsilon\delta_{Q_T}\Big) \\
&+ \epsilon\|\Phi\|_{L^\infty} \\
= &\frac{(1-\epsilon)}{2(\epsilon-\delta)}\Big(\mathrm{dist}_g^2(P_0, P^*) + \mathrm{dist}_g^2(P^*, P_T)\Big) + \frac{1}{4\delta}W_g^2\Big((1-\epsilon)\delta_{P*} + \epsilon\delta_{Q_0}, (1-\epsilon)\delta_{P*} + \epsilon\delta_{Q_T}\Big) \\
&+ \epsilon\|\Phi\|_{L^\infty}
\end{aligned}$$

Since

$$W_g^2\Big((1-\epsilon)\delta_{P*} + \epsilon\delta_{Q_0}, (1-\epsilon)\delta_{P*} + \epsilon\delta_{Q_T}\Big) = \epsilon\mathrm{dist}_g^2(Q_0, Q_T),$$

We conclude the proof of the lemma.

□

Corollary 4.11.4. *Assume $Q_0 \neq P_0$, $Q_T \neq P_Q$ and g is a simple metric on $\overline{\Omega}$.*

(i) From every sequence in $(0,\infty)$ one can extract a subsequence $(\epsilon_n)_n$ and find $\rho^*_{\frac{1}{2}}$ such that

$$\limsup_{n\to+\infty,\delta_n=\epsilon_n^2} \Big\{\mathcal{W}_g^2\Big(\sigma^{\epsilon_n,*}_{\frac{T}{2}-\delta_n}, \rho^*_{\frac{1}{2}}\Big) + \mathcal{W}_g^2\Big(\rho^*_{\frac{1}{2}}, \sigma^{\epsilon_n,*}_{\frac{T}{2}+\delta_n}\Big)\Big\} = 0.$$

(ii) We have the following sharper inequality:

$$\limsup_{\epsilon\to 0,\delta=\epsilon^2} \epsilon^{-1}\mathcal{W}_g^2\Big(\sigma^{\epsilon,*}_{\frac{T}{2}-\delta}, \sigma^{\epsilon,*}_{\frac{T}{2}+\delta}\Big) \leq dist_g^2\Big(Q_0, Q_T\Big).$$

Proof. Since Ω is a bounded set, by Prokhorov's theorem, $\mathcal{P}_2(\overline{\Omega})$ is compact for the weak topology. Thus, it is compact for the $\mathcal{W}_g$ topology.

By Lemma 4.11.3, we obtain that $\{\sigma^{\epsilon,*}_{\frac{T}{2}-\delta}\}_{\epsilon>0}$ and $\{\sigma^{\epsilon,*}_{\frac{T}{2}+\delta}\}_{\epsilon>0}$ share the same points of accumulation, which proves (i).

By Lemma 4.11.3,

$$\begin{aligned}
&\frac{1}{2(\epsilon-\delta)}\mathcal{W}_g^2\Big(\psi^{\epsilon}_{\frac{T}{2}-\epsilon}, \rho^{\epsilon,*}_{\frac{T}{2}-\delta}\Big) + \frac{1}{4\delta}\mathcal{W}_g^2\Big(\sigma^{\epsilon,*}_{\frac{T}{2}-\delta}, \sigma^{\epsilon,*}_{\frac{T}{2}+\delta}\Big) + \frac{1}{2(\epsilon-\delta)}\mathcal{W}_g^2\Big(\rho^{\epsilon,*}_{\frac{T}{2}+\delta}, \psi^{\epsilon}_{\frac{T}{2}+\epsilon}\Big)\\
\leq&\frac{(1-\epsilon)}{2(\epsilon-\delta)}\mathrm{dist}_g^2\Big(P_0, P^*\Big) + \frac{\epsilon}{4\delta}\mathrm{dist}_g^2\Big(Q_0, Q_T\Big) + \frac{(1-\epsilon)}{2(\epsilon-\delta)}\mathrm{dist}_g^2\Big(P^*, P_T\Big) + 2\epsilon\|\Phi\|_{L^\infty}.
\end{aligned} \tag{4.11.14}$$

Observe that as $\epsilon \to 0$ and $\delta = \epsilon^2$, we have

$$\psi^{\epsilon}_{\frac{T}{2}-\epsilon} \rightharpoonup \delta_{P_0} \text{ and } \psi^{\epsilon}_{\frac{T}{2}+\epsilon} \rightharpoonup \delta_{P_T}.$$

Therefore, multiply inequality (4.11.14) by ϵ and let ϵ to zero. We use (i) to conclude that

$$\begin{aligned}
&\frac{1}{2}\mathcal{W}_g^2\Big(\delta_{P_0}, \rho^*_{\frac{1}{2}}\Big) + \frac{1}{2}\mathcal{W}_g^2\Big(\rho^*_{\frac{1}{2}}, \delta_{P_T}\Big) + \frac{1}{4}\limsup_{\epsilon\to 0,\delta=\epsilon^2} \epsilon^{-1}\mathcal{W}_g^2\Big(\sigma^{\epsilon,*}_{\frac{T}{2}-\delta}, \sigma^{\epsilon,*}_{\frac{T}{2}+\delta}\Big)\\
\leq&\frac{1}{2}\mathrm{dist}_g^2\Big(P_0, P^*\Big) + \frac{1}{4}\mathrm{dist}_g^2\Big(Q_0, Q_T\Big) + \frac{1}{2}\mathrm{dist}_g^2\Big(P^*, P_T\Big).
\end{aligned} \tag{4.11.15}$$

Since

$$\frac{1}{2}\mathcal{W}_g^2\big(\delta_{P_0},\delta_{P_T}\big)=\min_{\nu\in\mathcal{P}(\overline{\Omega})}\Big\{\mathcal{W}_g^2\big(\delta_{P_0},\nu\big)+\mathcal{W}_g^2\big(\nu,\delta_{P_T}\big)\Big\}\le\mathcal{W}_g^2\big(\delta_{P_0},\rho^*_{\frac{1}{2}}\big)+\mathcal{W}_g^2\big(\rho^*_{\frac{1}{2}},\delta_{P_T}\big),$$

(4.11.15) implies

$$\frac{1}{4}\mathcal{W}_g^2\big(\delta_{P_0},\delta_{P_T}\big)+\frac{1}{4}\limsup_{\epsilon\to0,\delta=\epsilon^2}\epsilon^{-1}\mathcal{W}_g^2\Big(\sigma^{\epsilon,*}_{\frac{T}{2}-\delta},\sigma^{\epsilon,*}_{\frac{T}{2}+\delta}\Big)\le\frac{1}{4}\mathcal{W}_g^2\big(\delta_{P_0},\delta_{P_T}\big)+\frac{1}{4}\mathrm{dist}_g^2\Big(\mathcal{Q}_0,\mathcal{Q}_T\Big).$$

This concludes the proof of the corollary.

□

CHAPTER 5

Recovery of Real Analytic Potential

Throughout this chapter, we assume that $g \in \mathcal{G}$ has already been recovered and is real analytic. We recover Φ when it is real analytic.

5.1 Euclidean geodesic

Before studying the general case, we first study a simple case when $g = Id$, the Euclidean metric. Since Ω is convex, the geodesics curves are straight line segments.

We set

$$D_{\overline{\Omega}} := \{\Phi \in C^\omega(\bar{\Omega} \times \bar{\Omega}) : \Phi(x,x) = 0, \Phi(x,y) = \Phi(y,x)\}$$

and

$$D_{\partial\Omega} := \{(P_0, P_T, Q_0, Q_T) \in (\partial\Omega)^4 : d^2(P_0, P_T) + d^2(Q_0, Q_T) \leq d^2(P_0, Q_T) + d^2(P_0, Q_T)\}.$$

Define $I_g : D_{\overline{\Omega}} \to C(D_{\partial\Omega}, \mathbb{R})$ by

$$I_g[\Phi](P_0, P_T, Q_0, Q_T) = \int_0^1 \Phi(\gamma_P(t), \gamma_Q(t))dt,$$

where γ_P and γ_Q are constant-speed geodesics joining P_0 to P_T and Q_0 to Q_T respectively.

We have access to the values of $I_g[\Phi](P_0, P_T, Q_0, Q_T)$ and would like to know if we can recover Φ uniquely. Since I_g is a well-defined linear functional from $D_{\overline{\Omega}}$ to $C(D_{\partial\Omega}, \mathbb{R})$, our problem is equivalent to showing that the kernel of I_g is trivial.

Lemma 5.1.1. *If* $I_g[\Phi] \equiv 0$*, then* $\Phi(x,y) = 0$*,* $\forall (x,y) \in (\partial\Omega)^2$*.*

Proof. Suppose otherwise. Without loss of generality, $\exists(x_0,y_0) \in (\partial\Omega)^2$ such that $x_0 \neq y_0$ and $\Phi(x_0,y_0) > 0$. By continuity, there exists $r > 0$ such that

$$\Phi(x,y) > 0,\ \forall(x,y) \in B_r(x_0) \times B_r(y_0).$$

Take $r' > 0$ small such that for all $P_0, P_T \in B'_r(x_0) \cap \partial\Omega$ and all $Q_0, Q_T \in B'_r(y_0) \cap \partial\Omega$, we have

$$(P_0, P_T, Q_0, Q_T) \in D_{\partial\Omega} \quad \text{and} \quad (\gamma_P(t), \gamma_Q(t)) \in (B'_r(x_0) \cap \Omega) \times (B'_r(y_0) \cap \Omega).$$

Then $I_g[\Phi](P_0, P_T, Q_0, Q_T) > 0$, thus a contradiction.

□

Lemma 5.1.2. *If $g = Id$ and $I_g[\Phi] \equiv 0$, then there exists $r > 0$, such that $\Phi(x,y) = 0, \forall x, y \in \Omega_r$, where $\Omega_r := \{x \in \Omega : d(x, \partial\Omega) < r\}$.*

Corollary 5.1.3. *As Φ is real analytic and Ω_r is open, Lemma 5.1.2 implies that $\Phi(x,y) = 0, \forall(x,y) \in \Omega^2$.*

Note that if g was complex analytic, Lemma 5.1.2 is a direct consequence of Lemma 5.1.1. We now prove lemma 5.1.2. We take $\tilde{\Omega}$ open such that $\bar{\Omega} \subset \tilde{\Omega}$ and suppose that $\Phi \in C^\omega(\tilde{\Omega} \times \tilde{\Omega})$.

Proof. Since $\Phi \in C^\omega(\tilde{\Omega} \times \tilde{\Omega})$ and $\Phi|_{\partial\Omega\times\partial\Omega} \equiv 0$, it is enough to show the following claim:

$$\partial_x^\alpha \partial_y^\beta \Phi(x,y) = 0,\ \forall \alpha, \beta \in \mathbb{N}^d,\ \forall x \neq y \in \partial\Omega.$$

We show the claim by induction.

Clearly, if $|\alpha| + |\beta| = 0$, by Lemma 5.1.1, we have $\Phi(x,y) = 0$, $\forall x, y \in \partial\Omega$. Suppose that the claim holds $\forall \alpha, \beta$ such that $|\alpha| + |\beta| < k$ for some $k \in \mathbb{N}$. Fix $x_0 \neq y_0 \in \partial\Omega$. Choose small enough neighborhoods $B_r(x_0)$ and $B_r(y_0)$ around x_0, y_0 respectively. Take $P_0, P_T \in \partial\Omega \cap \overline{Br(x_0)}$ and $Q_0, Q_T \in \partial\Omega \cap \overline{Br(y_0)}$.

Moreover, choose orthonormal coordinate frames $\{e_i^{x_0}\}_{i=1}^d$ and $\{e_i^{y_0}\}_{i=1}^d$ based at x_0, y_0 respectively such that $e_1^{x_0} = \vec{n}_{x_0}$ and $e_1^{y_0} = \vec{n}_{y_0}$, normal vectors pointing inward. In the flat Euclidean setting, we have $\gamma_P(t) = (1-t)P_0 + tP_T$ and $\gamma_Q(t) = (1-t)Q_0 + tQ_T$. We compute derivatives in the directions of the new coordinates. For $\alpha_i \in N$, we denote $\alpha = (\alpha_1, \cdots, \alpha_d)$ and set $\alpha! = \alpha_1!\alpha_2! \cdots \alpha_d!$.

By the Taylor expansion, we have

$$
\begin{aligned}
&I[\Phi](P_0, P_T, Q_0, Q_T) \\
=&\sum_{\alpha,\beta} \prod_{i,j=1}^d \frac{\partial^{\alpha_i}_{e_i^{x_0}} \partial^{\beta_j}_{e_j^{y_0}} \Phi(x_0, y_0)}{\alpha_i!\beta_j!} \int_0^1 \prod_{i,j=1}^d [(\gamma_P(t) - x_0)] \cdot e_i^{x_0}]^{\alpha_i} [(\gamma_Q(t) - y_0)] \cdot e_j^{y_0}]^{\beta_j} dt,
\end{aligned}
$$

where

$$\partial^{\alpha_i}_{e_i^{x_0}} \partial^{\beta_j}_{e_j^{y_0}} \Phi(x_0, y_0) = \frac{d^{\alpha_i}}{dt^{\alpha_i}}|_{t=0} \frac{d^{\beta_j}}{ds^{\beta_j}}|_{s=0} \Phi(x_0 + te_i^{x_0}, y_0 + se_j^{y_0}).$$

Notice we have the inductive hypothesis that

$$\partial^{\alpha_1}_{e_1^{x_0}} \cdots \partial^{\alpha_d}_{e_d^{x_0}} \partial^{\beta_1}_{e_1^{y_0}} \cdots \partial^{\beta_d}_{e_d^{y_0}} \Phi)(x, y) \equiv 0 \quad \text{on} \quad \partial\Omega \times \partial\Omega$$

such that $|\alpha| + |\beta| \leq k$.

Thus in the above expansion, we have that

$$
\begin{aligned}
&I[\Phi](P_0, P_T, Q_0, Q_T) \\
=&\sum_{|\alpha|+|\beta|>k} \prod_{i,j=1}^d \frac{\partial^{\alpha_i}_{e_i^{x_0}} \partial^{\beta_j}_{e_j^{y_0}} \Phi(x_0, y_0)}{\alpha_i!\beta_j!} \int_0^1 \prod_{i,j=1}^d [(\gamma_P(t) - x_0)] \cdot e_i^{x_0}]^{\alpha_i} [(\gamma_Q(t) - y_0)] \cdot e_j^{y_0}]^{\beta_j} dt.
\end{aligned}
$$

We need to show that if derivatives of order $k+1$ do not vanish, we have $I_\Phi(P_0, P_T, Q_0, Q_T) \neq 0$, hence a contradiction. First we notice the following. By inductive hypothesis, we have $\forall i, j \in \{2, 3, \cdots, d\}$

$$\partial_{e_i^{x_0}} (\partial^{\alpha_1}_{e_1^{x_0}} \cdots \partial^{\alpha_d}_{e_d^{x_0}} \partial^{\beta_1}_{e_1^{y_0}} \cdots \partial^{\beta_d}_{e_d^{y_0}} \Phi)(x, y) = 0, \tag{5.1.1}$$

and, by symmetry,

$$\partial_{e_j^{y_0}} (\partial^{\alpha_1}_{e_1^{x_0}} \cdots \partial^{\alpha_d}_{e_d^{x_0}} \partial^{\beta_1}_{e_1^{y_0}} \cdots \partial^{\beta_d}_{e_d^{y_0}} \Phi)(x, y) = 0. \tag{5.1.2}$$

Indeed whenever $\{e_j^{y_0}\}$ is not a normal direction, we need to compute, for instance,

$$\partial_{e_1^{x_0}}\partial_{e_2^{y_0}}\Phi(x_0,y_0) = \partial_{e_2^{y_0}}(\partial_{e_1^{x_0}}\Phi(x_0,y_0)) = \frac{d}{dt}|_{t=0}[\partial_x\Phi(x_0,\gamma(t))\cdot e_1^{x_0}], \tag{5.1.3}$$

where $\gamma(0)=y_0, \gamma'(0)=e_2^{y_0}, \gamma(t)\in\partial\Omega$.

Therefore, from the inductive step, one cannot have that $\partial^i_{e_1^{x_0}}\partial^j_{e_1^{y_0}}\Phi(x_0,y_0)=0$ if $i+j=k+1$ as all other derivatives of order $k+1$ will vanish.

Then the Taylor expansion yields that

$$\begin{aligned}
&I[\Phi](P_0,P_T,Q_0,Q_T)\\
&=\sum_{|\alpha|+|\beta|>k+1}\prod_{i,j=1}^d \frac{\partial^{\alpha_i}_{e_i^{x_0}}\partial^{\beta_j}_{e_j^{y_0}}\Phi(x_0,y_0)}{\alpha_i!\beta_j!}\int_0^1\prod_{i,j=1}^d[(\gamma_P(t)-x_0)]\cdot e_i^{x_0}]^{\alpha_i}[(\gamma_Q(t)-y_0)]\cdot e_j^{y_0}]^{\beta_j}dt\\
&+\sum_{i+j=k+1}\frac{\partial^i_{e_1^{x_0}}\partial^j_{e_1^{y_0}}\Phi(x_0,y_0)}{i!j!}\int_0^1[(\gamma_P(t)-x_0)]\cdot e_i^{x_0}]^i[(\gamma_Q(t)-y_0)]\cdot e_j^{y_0}]^j dt.
\end{aligned}$$

Set $P_0=x_0$ and $Q_0=y_0$, we get $\gamma_P(t)-x_0=(1-t)x_0+tP_T=t(P_T-x_0)$ and $\gamma_Q(t)-x_0=(1-t)y_0+tQ_T=t(Q_T-y_0)$

Suppose that $\partial^i_{e_1^{x_0}}\partial^j_{e_1^{y_0}}\Phi(x_0,y_0)\neq 0$ for some pair $(x_0,y_0)\in(\partial\Omega)^2$ and we will show that we can get a contradiction. i.e. $I[\Phi](x_0,P_T,y_0,Q_T)\neq 0$ for some $P_T,Q_T\in\partial\Omega$.

We have

$$\begin{aligned}
&\int_0^1[(\gamma_P(t)-x_0)]\cdot e_i^{x_0}]^i[(\gamma_Q(t)-y_0)]\cdot e_j^{y_0}]^j dt\\
&=\int_0^1 t^i[(P_T-x_0)]\cdot e_1^{x_0}]^i t^j[(Q_T-y_0)]\cdot e_j^{y_0}]^j dt = [(P_T-x_0)]\cdot e_1^{x_0}]^i[(Q_T-y_0)]\cdot e_j^{y_0}]^j\frac{1}{k+2}.
\end{aligned}$$

Set

$$i_0 := \text{a}rgmin\{i=\partial^i_{e_1^{x_0}}\partial^{k+1-i}_{e_1^{y_0}}\Phi(x_0,y_0)\neq 0\}.$$

Without loss of generality, assume that $i_0\leq\lfloor\frac{k+1}{2}\rfloor$. Otherwise, we can swap x_0 and y_0. We also assume that $\partial^{i_0}_{e_1^{x_0}}\partial^{k+1-i_0}_{e_1^{y_0}}\Phi(x_0,y_0)>0$.

Let

$$\epsilon := (Q_T-y_0)\cdot e_1^{y_0} \quad\text{and}\quad \delta(\epsilon):=(P_T-x_0)\cdot e_1^{x_0},$$

where δ is as a function of ϵ. We can take $\delta(\epsilon) = R\epsilon$.

We then have

$$\sum_{i+j=k+1} \frac{\partial^i_{e_1^{x_0}} \partial^j_{e_1^{y_0}} \Phi(x_0, y_0)}{i!j!} \int_0^1 \prod_{i,j=1}^d [(\gamma_P(t) - x_0)] \cdot e_i^{x_0}]^i [(\gamma_Q(t) - y_0)] \cdot e_j^{y_0}]^j dt$$

$$= \frac{\partial^{i_0}_{e_1^{x_0}} \partial^{k+1-i_0}_{e_1^{y_0}} \Phi(x_0, y_0)}{i_0!(k+1-i_0)!} \int_0^1 [(\gamma_P(t) - x_0)] \frac{1}{k+2} \delta(\epsilon)^{i_0} \epsilon^{k+1-i_0)}$$

$$+ \sum_{i=i_0+1}^{k+1} \frac{\partial^{i}_{e_1^{x_0}} \partial^{k+1-i_0}_{e_1^{y_0}} \Phi(x_0, y_0)}{i!(k+1-i)!} \frac{1}{k+2} R^i \epsilon^{k+1)},$$

where $i \in \{i_0 + 1, \cdots, k+1\}$.

Now it only remains to show that

$$\sum_{|\alpha|+|\beta|>k+1} \prod_{i,j=1}^d \frac{\partial^{\alpha_i}_{e_i^{x_0}} \partial^{\beta_j}_{e_j^{y_0}} \Phi(x_0, y_0)}{\alpha_i! \beta_j!} \int_0^1 \prod_{i,j=1}^d [(\gamma_P(t) - x_0)] \cdot e_i^{x_0}]^{\alpha_i} [(\gamma_Q(t) - y_0)] \cdot e_j^{y_0}]^{\beta_j} dt = o(\epsilon^{i_0+k+1}).$$

In particular,

$$\int_0^1 \prod_{i,j=1}^d t^{\alpha_i} [(\gamma_P(t) - x_0)] \cdot e_i^{x_0}]^{\alpha_i} t^{\beta_j} [(\gamma_Q(t) - y_0)] \cdot e_j^{y_0}]^{\beta_j} dt = \frac{1}{|\alpha| + |\beta|} \prod_{i,j=1}^d \delta(\epsilon)^{\alpha_i} \epsilon^{\beta_j}$$

$$= \frac{1}{|\alpha| + |\beta|} \prod_{i,j=1}^d R^{\alpha_i} \epsilon^{\alpha_i} \epsilon^{\beta_j}$$

Taking summations over i, j, we get $\frac{1}{|\alpha|+|\beta|} R^{|\alpha|} \epsilon^{|\alpha|+|\beta|}$. Notice that $|\alpha| + |\beta| \geq k+2$.

Choose ϵ small enough such that $\epsilon <$ [radius of convergence at $(x_0, y_0)]^2$. When $R < 1$,

$$\sum_{|\alpha|+|\beta| \geq k+2} C_{\alpha,\beta} R^{|\alpha|} \epsilon^{|\alpha|+|\beta|-(k+2)}$$

is bounded. Thus

$$\sum_{|\alpha|+|\beta| \geq k+2} \frac{1}{|\alpha|! + |\beta|!} C_{\alpha,\beta} R^{|\alpha|} \epsilon^{|\alpha|+|\beta|} = \epsilon^{k+2} \sum_{|\alpha|+|\beta| \geq k+2} C_{\alpha,\beta} R^{|\alpha|} \epsilon^{|\alpha|+|\beta|-(k+2)} = o(\epsilon^{k+2}),$$

as desired.

□

5.2 General geodesic

We now consider general $g \in C^{\omega}(\tilde{\Omega})$, where $\tilde{\Omega}$ is open such that $\bar{\Omega} \subset \tilde{\Omega}$. We prove the lemma 5.1.2 in this general case.

Since the geodesics are not straight line segments here, new local coordinates are needed. We use the exponential map exp defined locally over the smooth manifold. One can find more properties about the exponential map and its derivative in Chapter 4 section 6 of [30].

Given $v \in T_{x_0}\bar{\Omega}$, a geodesic $\gamma : [0,1] \to \bar{\Omega}$, with $\gamma(0) = x_0, \gamma'(0) = v$ is given by $\gamma(t) = \exp_{x_0}(tv)$. Also $d(\exp_{x_0})_0(v) = \frac{d}{dt}|_{t=0}\exp_{x_0}(tv) = v$.

Let $x_0 \in \partial\Omega$. Then $\exists \mathcal{U} \subset \overline{\Omega}$ open such that $x_0 \in \mathcal{U}$ and $\exp_{x_0} : \exp_{x_0}^{-1}(\mathcal{U}) \to \mathcal{U}$ is a smooth diffeomorphism.

For $x_0, P_T \in \mathcal{U}$, the geodesic connecting x_0 to P_T on $\partial\Omega$ is given by

$$\gamma_P(t) = \exp_{x_0}(t \cdot \exp_{x_0}^{-1}(P_T)).$$

We then have

$$\begin{aligned} I_g[\Phi](x_0, P_T, y_0, Q_T) &= \int_0^1 \Phi(\gamma_P(t), \gamma_Q(t))dt \\ &= \int_0^1 \Phi(\exp_{x_0}(t(\exp_{x_0})^{-1}(P_T)), \exp_{y_0}(t(\exp_{y_0})^{-1}(Q_T)))dt \\ &= \int_0^1 \tilde{\Phi}_{x_0,y_0}(t(\exp_{x_0})^{-1}(P_T), t(\exp_{y_0})^{-1}(Q_T))dt, \end{aligned}$$

where

$$\tilde{\Phi}_{x_0,y_0}(v,w) := \Phi(\exp_{x_0}(v), \exp_{y_0}(w)) = \Phi(\exp_{x_0}(\sum_{i=1}^{d} v_i e_i^{x_0}), \exp_{y_0}(\sum_{i=1}^{d} w_i e_i^{y_0})).$$

Here $v, w \in (T_{x_0}\tilde{\Omega}) \times (T_{y_0}\tilde{\Omega})$ and $(v_i)_{i=1}^d, (w_i)_{i=1}^d \in \mathbb{R}^d$ are the coordinates of v, w in the bases $\{e_i^{x_0}\}_{i=1}^d, \{e_i^{y_0}\}_{i=1}^d$ respectively.

Notice that Ω is geodesic convex with respect to g. We may choose x_0 close enough to P_T and $e_1^{x_0}$ in the direction of $P_T - x_0$ such that $Proj_{e_1^{x_0}}(\exp_{x_0}^{-1}(P_T)) = e_1^{x_0} \cdot \exp_{x_0}^{-1}(P_T) > 0$.

Likewise, we may assume that $Proj_{e_1^{y_0}}(\exp_{y_0}^{-1}(Q_T)) = e_1^{y_0} \cdot \exp_{y_0}^{-1}(Q_T) > 0$.

Set $v_{P_T} := \exp_{x_0}^{-1}(P_T), w_{Q_T} := \exp_{y_0}^{-1}(Q_T)$. Then

$$\int_0^1 \tilde{\Phi}_{x_0,y_0}(t(\exp_{x_0})^{-1}(P_T), t(\exp_{y_0})^{-1}(Q_T))dt$$

$$= \int_0^1 \sum_{\alpha,\beta}^{1} \frac{\partial_{v_1}^{\alpha_1} \cdots \partial_{v_d}^{\alpha_d} \partial_{w_1}^{\beta_1} \cdots \partial_{w_d}^{\beta_d} \tilde{\Phi}_{x_0,y_0}(0,0)}{(|\alpha|)!(|\beta|)!} \prod_{i,j=1}^{d} (t(v_{P_T})_i)^{\alpha_i} (t(w_{Q_T})_j)^{\beta_j} dt,$$

where $\alpha = (\alpha_1, \cdots, \alpha_d), \beta = (\beta_1, \cdots, \beta_d)$.

To compute the cross derivatives, let's prove the following lemma.

Lemma 5.2.1. *If $\partial_x^\alpha \partial_y^\beta \Phi(x,y) = 0$, where $x, y \in \partial\Omega \subset \tilde{\Omega}$, and $|\alpha| + |\beta| \leq k$, then*

$$\partial_x^\alpha \partial_y^\beta \Phi(x_0, y_0) = \partial_v^\alpha \partial_w^\beta \tilde{\Phi}(0,0) = 0,$$

for $|\alpha| + |\beta| = k+1$.

Proof. Define

$$\tilde{\Phi}_{x_0,y_0}(v,w) := \Phi(\exp_{x_0}(v), \exp_{y_0}(w)).$$

Then

$$\partial_v^\alpha \partial_w^\beta \tilde{\Phi}_{x_0,y_0}(v,w) = \partial_v^\alpha \partial_w^\beta \Phi(\exp_{x_0}(\sum_{i=1}^d v_i e_i^{x_0}), \exp_{y_0}(\sum_{j=1}^d w_j e_j^{y_0}))$$

To develop our intuition, we first consider the first-order derivatives, for instance,

$$\partial_{v_1}(\Phi(\exp_{x_0}(\sum_{i=1}^d v_i e_i^{x_0}), \exp_{y_0}(\sum_{j=1}^d w_j e_j^{y_0})))$$

$$= \sum_{j=1}^d \partial_{x_j} \Phi(\exp_{x_0}(\sum_{i=1}^d v_i e_i^{x_0}), \exp_{y_0}(\sum_{j=1}^d w_j e_j^{y_0}))(d(\exp_{x_0})|_{\sum_{i=1}^d v_i e_i^{x_0}} e_1^{x_0} \cdot e_j^{x_0}).$$

Then the second-order derivatives, for instance,

$$\partial_{v_1}^2(\Phi(\exp_{x_0}(\sum_{i=1}^d v_i e_i^{x_0}), \exp_{y_0}(\sum_{j=1}^d w_j e_j^{y_0})))$$

$$= \sum_{j,k}^{d} \partial^2_{x_j,x_k} \Phi(\exp_{x_0}(\sum_{i=1}^{d} v_i e_i^{x_0}), \exp_{y_0}(\sum_{j=1}^{d} w_j e_j^{y_0}))$$

$$\cdot (d(\exp_{x_0})|_{\sum_{i=1}^d v_i e_i^{x_0}} e_1^{x_0}) \cdot e_j^{x_0}) \cdot (d(\exp_{x_0})|_{\sum_{i=1}^d v_i e_i^{x_0}} e_1^{x_0}) \cdot e_k^{x_0})$$

$$+ \sum_{j=1}^{d} \partial_{x_j} \Phi(\exp_{x_0}(\sum_{i=1}^{d} v_i e_i^{x_0}), \exp_{y_0}(\sum_{j=1}^{d} w_j e_j^{y_0})) \partial_{v_1} (d(\exp_{x_0})|_{\sum_{i=1}^d v_i e_i^{x_0}} e_1^{x_0}) \cdot e_j^{x_0})$$

Restrict the last equality to $(v, w) = (0, 0)$, we get $\partial^2_{v_1} \tilde{\Phi}_{x_0,y_0}(0, 0) = \partial^2_{x_1,x_1} \Phi(x_0, y_0)$ as $d(\exp_{x_0})_0 = Id$ and $\partial_{x_j} \Phi(x_0, y_0) = 0$ by the assumption which says that all lower order derivatives of Φ at (x_0, y_0) vanish. Under these restrictions, we have

$$d(\exp_{x_0})|_0 e_i^{x_0} \cdot e_{\tilde{i}}^{x_0} = \delta_{i\tilde{i}},$$

and

$$d(\exp_{y_0})|_0 e_j^{y_0} \cdot e_{\tilde{j}}^{y_0} = \delta_{j\tilde{j}}.$$

Here $\delta_{i\tilde{i}}$ or $\delta_{j\tilde{j}}$ is the Kronecker-delta function.

Denote $\alpha = (\alpha_i)_{i=1}^d, \tilde{\alpha} = (\tilde{\alpha}_{\tilde{i}})_{\tilde{i}=1}^d$.

As all lower order derivatives of Φ at (x_0, y_0) vanish, we then have

$$\partial_v^\alpha \partial_w^\beta \tilde{\Phi}(v, w) = \partial_v^\alpha \partial_w^\beta \Phi(\exp_{x_0}(\sum_{i=1}^{d} v_i e_i^{x_0}), \exp_{y_0}(\sum_{j=1}^{d} w_j e_j^{y_0}))$$

$$= \sum_{|\tilde{\alpha}|+|\tilde{\beta}|=k+1} \partial_x^{\tilde{\alpha}} \partial_y^{\tilde{\beta}} \Phi(\exp_{x_0}(\sum_{i=1}^{d} v_i e_i^{x_0}), \exp_{y_0}(\sum_{j=1}^{d} w_j e_j^{y_0})) \cdot$$

$$\prod_{i=1,\tilde{i}=1}^{d,\tilde{d}} (d(\exp_{x_0})|_{\sum_{i=1}^d v_i e_i^{x_0}} e_1^{x_0} \cdot e_{\tilde{i}}^{x_0})^{\tilde{\alpha}_{\tilde{i}}} \prod_{j=1,\tilde{j}=1}^{d,\tilde{d}} (d(\exp_{y_0})|_{\sum_{j=1}^d w_j e_j^{x_0}} e_1^{y_0} \cdot e_{\tilde{j}}^{y_0})^{\tilde{\beta}_{\tilde{j}}}$$

+ (lower order terms for Φ) $\times$ (higher order terms for exp) $= 0$.

$\square$

Now by the Taylor expansion,

$$0 = I_g[\Phi](x_0, P_T, y_0, Q_T)$$

$$= \int_0^1 \tilde{\Phi}_{x_0,y_0}(t(\exp_{x_0})^{-1}(P_T), t(\exp_{y_0})^{-1}(Q_T))dt$$

$$= \int_0^1 \sum_{\alpha,\beta}^{1} \frac{\partial_{v_1}^{\alpha_1} \cdots \partial_{v_d}^{\alpha_d} \partial_{w_1}^{\beta_1} \cdots \partial_{w_d}^{\beta_d} \tilde{\Phi}_{x_0,y_0}(0,0)}{(|\alpha|)!(|\beta|)!} \prod_{i,j=1}^{d} (t(v_{P_T})_i)^{\alpha_i} (t(w_{Q_T})_j)^{\beta_j} dt.$$

We can use the same technique in the proof of the lemma 5.1.2 to check the derivatives vanishing by change of coordinates and induction. We thus get the following theorem.

Theorem 5.2.2. *If* $I_g[\Phi] \equiv 0$*, then* $\exists r > 0$*, such that* $\Phi(x,y) = 0, \forall x, y \in \Omega_r$*, where* $\Omega_r := \{x \in \Omega : d(x, \partial\Omega) < r\}$*. Here* $g \in C^\omega(\tilde{\Omega})$ *and* $\tilde{\Omega}$ *is open such that* $\bar{\Omega} \subset \tilde{\Omega}$*.*

CHAPTER 6

Appendix

6.1 Probabilistic representation of measures

We review some general properties of probabilistic representation of measures in this section. We use disintegration theory as in Chapter 2.

Assume η is a Borel probability measure on $\mathbb{R}^d \times \Gamma_T$ which satisfies

$$\int_{\mathbb{R}^d \times \Gamma_T} \left(\int_0^T |\dot{\gamma}(\tau)|^2 d\tau \right) \eta(dq, d\gamma) < \infty. \tag{6.1.1}$$

Define the probability Borel measure $\mathbf{m}$ on $[0,T] \times \mathbb{R}^{2d}$ by

$$\int_{[0,T]\times\mathbb{R}^{2d}} \varphi(t,x,\xi)\mathbf{m}(dt,dx,d\xi) := \int_{\mathbb{R}^d \times \Gamma_T} \left(\int_0^T \varphi(\tau,\gamma(\tau),\dot{\gamma}(\tau))d\tau \right) \eta(dq,d\gamma),$$

for all $\varphi \in C_c\big([0,T] \times \mathbb{R}^{2d}\big)$.

Taking $\varphi \equiv \varphi(t)$, we see that the projection of $\mathbf{m}$ onto $[0,T]$ is the Lebesgue measure. Therefore, by the theory of disintegration of measures (see [4] section 5.3), there exists a path $t \mapsto \mathbf{m}^t$ of Borel probability measures on $\mathbb{R}^{2d}$ such that

$$\int_{[0,T]\times\mathbb{R}^{2d}} \varphi(t,x,\xi)\mathbf{m}(dt,dx,d\xi) = \int_0^T \left(\int_{\mathbb{R}^{2d}} \varphi(\tau,x,\xi)\mathbf{m}^t(dx,d\xi) \right) dt$$

for all $\varphi \in C_c\big([0,T] \times \mathbb{R}^{2d}\big)$. The theory of disintegration of measures gives all the measurability properties we will rely on.

Let σ_t be the first marginal of $\mathbf{m}^t$. We apply the disintegration theory again to find a Borel map $(t,x) \mapsto \mathbf{m}^{t,x}$ on $\mathbb{R}^d$ such that

$$\int_{\mathbb{R}^{2d}} \varphi(x,\xi)\mathbf{m}^t(dx,d\xi) = \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \varphi(x,\xi)\mathbf{m}^t(d\xi) \right) \sigma_t(dx), \qquad \text{a.e. } t \in (0,T), \tag{6.1.2}$$

for all $\varphi \in C_c\big(\mathbb{R}^{2d}\big)$.

Note that

$$\int_{\mathbb{R}^d} f(x)\sigma_t(dx) = \int_{\mathbb{R}^d\times\Gamma_T} f(\gamma(t))\eta(dq,d\gamma), \qquad \text{a.e. } t\in(0,T), \tag{6.1.3}$$

for all $f \in C_c(\mathbb{R}^d)$.

We set

$$v_t(x) := \int_{\mathbb{R}^d} \xi \mathbf{m}^t(d\xi) \tag{6.1.4}$$

and set

$$\beta(\tau) := \bigg(\int_{\mathbb{R}^d\times\Gamma_T} |\dot\gamma(\tau)|^2\eta(dq,d\gamma)\bigg)^{\frac{1}{2}}.$$

By (6.1.1), $\beta \in L^2(0,T)$.

By Jensen's inequality

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} |\xi|^2\mathbf{m}^\tau(dx,d\xi)d\tau = \int_{\mathbb{R}^d}\bigg(\int_{\mathbb{R}^d}|\xi|^2\mathbf{m}^t(d\xi)\bigg)\sigma_\tau(dx) \geq \int_{\mathbb{R}^d}|v_\tau(x)|^2\sigma_\tau(dx).$$

Integrating both sides of the previous inequality over $[0,T]$, we obtain

$$\infty > \int_{\mathbb{R}^d\times\Gamma_T}\bigg(\int_0^T L(\gamma(\tau),\dot\gamma(\tau))d\tau\bigg)\eta(dq,d\gamma) \geq \int_0^T\bigg(\int_{\mathbb{R}^d}|v_\tau(x)|^2\sigma_\tau(dx)\bigg)d\tau. \tag{6.1.5}$$

Proposition 6.1.1. *Suppose a Borel probability measure on $\bar\Omega\times\Gamma_T$ and satisfies* (6.1.1)*. Then the following hold.*

(i) The path $t\mapsto\sigma$ belongs to $\Gamma_{T,2}$ and (6.1.3) *holds for every $t\in[0,T]$.*

(ii) The map $(t,x)\mapsto v_t(x)$ defined in (6.1.4) *is a velocity for σ.*

(iii) We have

$$\int_0^T\int_{\mathbb{R}^d} L(x,v_\tau(x))\sigma_\tau(dx) \leq \int_{\mathbb{R}^d\times\Gamma_T}\bigg(\int_0^T L\big(\gamma(\tau),\dot\gamma(\tau)\big)d\tau\bigg)\eta(dq,d\gamma).$$

Proof. For any $\gamma\in\Gamma_{T,2}$, we have

$$\chi_{\mathbb{R}^d\setminus\bar\Omega}(\gamma(\tau)) = 0, \qquad \forall t\in[0,T].$$

Thus

$$\int_0^T \sigma_t\big(\mathbb{R}^d \setminus \bar{\Omega}\big)dt = \int_{\mathbb{R}^d\times\Gamma_T} \bigg(\int_0^T \chi_{\bar{\Omega}}(\gamma(\tau))d\tau\bigg)\eta(dq,d\gamma) = 0.$$

This proves that for almost every t, σ_t is supported by $\bar{\Omega}$.

(i) For each $0 \le s < t \le T$, define the measures π^{st} by

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} f(x,y)\pi^{st}(dx,dy) = \int_{\mathbb{R}^d\times\Gamma_T} f\big(\gamma(s),\gamma(t)\big)\eta(dq,d\gamma),$$

for $f \in C_c(\mathbb{R}^d \times \mathbb{R}^d)$.

We use (6.1.3) to conclude that the first marginal of π^{st} is σ_s and the second marginal of π^{st} is σ_t. Thus

$$\begin{aligned} W_2^2(\sigma_s,\sigma_t) &\le \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2\pi^{st}(dx,dy) \\ &= \int_{\mathbb{R}^d\times\Gamma_T} \bigg|\int_s^t \dot{\gamma}(\tau)d\tau\bigg|^2 \eta(dq,d\gamma) \\ &\le \int_{\mathbb{R}^d\times\Gamma_T} \bigg(\int_s^t |\dot{\gamma}(\tau)|d\tau\bigg)^2 \eta(dq,d\gamma). \end{aligned}$$

We use Minkowski inequality to conclude that

$$W_2^2(\sigma_s,\sigma_t) \le \bigg(\int_s^t \beta(\tau)d\tau\bigg)^2.$$

This proves that $t \mapsto \sigma$ belongs to $\Gamma_{T,2}$. Thus (6.1.3) holds for every $t \in [0,T]$.

(ii) Let $\varphi \in C_c^\infty((0,T)\times\mathbb{R}^d)$. We use (6.1.2) and (6.1.3) to obtain that

$$\begin{aligned} &\int_0^T \Big(\int_{\mathbb{R}^d} \Big(\partial_t\varphi(t,q) + \langle\nabla\varphi(t,q)\rangle\Big)\sigma_t(dq) \\ =& \int_0^T \bigg(\int_{\mathbb{R}^d\times\Gamma_T} \partial_\tau\varphi(\tau,\gamma(\tau))\eta(dq,d\gamma)\bigg)d\tau + \int_0^T \bigg(\int_{\mathbb{R}^d\times\mathbb{R}^d} \langle\xi,\nabla\varphi(\tau,x)\rangle\mathbf{m}^\tau(dq,d\xi)\bigg)d\tau \\ =& \int_0^T \bigg(\int_{\mathbb{R}^d\times\Gamma_T} \Big(\partial_\tau\varphi(\tau,\gamma(\tau)) + \Big\langle\dot{\gamma}(\tau),\nabla\varphi\big(\tau,\gamma(\tau)\big)\Big\rangle\Big)\eta(dq,d\gamma)\bigg)d\tau. \end{aligned}$$

We use Fubini's theorem to conclude that

$$\begin{aligned}\int_0^T \Big(\int_{\mathbb{R}^d} \Big(\partial_t\varphi(t,q) + \langle \nabla\varphi(t,q)\rangle\Big)\sigma_t(dq) &= \int_{\mathbb{R}^d\times\Gamma_T} \Bigg(\int_0^T \partial_\tau\Big(\varphi\big(\tau,\gamma(\tau)\big)\Big)d\tau\Bigg)\eta(dq,d\gamma)\\ &= \int_{\mathbb{R}^d\times\Gamma_T} \Bigg(\varphi\big(T,\gamma(T)\big) - \varphi\big(0,\gamma(0)\big)\Bigg)\eta(dq,d\gamma)\\ &=0.\end{aligned}$$

This, together with (6.1.5), proves (ii).

Since $L(x,\cdot)$ is a convex function for all $x \in \bar{\Omega}$, we use Jensen's inequality to obtain that

$$L(x, v_\tau(x)) \leq \int_{\mathbb{R}^d} L(x,\xi)\mathbf{m}^{\tau,x}(d\xi).$$

Thus

$$\int_{\mathbb{R}^d} L(x, v_\tau(x))\sigma_\tau(dx) \leq \int_{\mathbb{R}^{2d}} L(x,\xi)\mathbf{m}^{\tau}(dx,d\xi).$$

Integrating, we conclude that

$$\int_0^T \Bigg(\int_{\mathbb{R}^d\times\mathbb{R}^d} L(x, v_\tau(x))\mathbf{m}^{\tau}(dx,d\xi)\Bigg)d\tau \leq \int_0^T \Bigg(\int_{\mathbb{R}^d\times\mathbb{R}^d} L(x,\xi)\mathbf{m}^{\tau}(dx,d\xi)\Bigg)d\tau.$$

This reads off the desired identity in (iii).

$\square$

6.2 Properties of cost functions

Assume

$$L(x,\xi) := L_0(v), \qquad \Phi \equiv 0.$$

Then the optimal path connecting $x := P_{t_0}$ to $y := P_{t_1}$ is given by

$$\gamma(\tau) = P_{t_0} + \frac{\tau - t_0}{t_1 - t_0}\Big(P_{t_1} - P_{t_0}\Big), \qquad \forall \tau \in [t_0, t_1].$$

Thus for $s < t$, we have

$$c_s^t\big(\gamma(s),\gamma(t)\big) = (t-s)L_0\bigg(\frac{P_t - P_t}{t-s}\bigg) = (t-s)L_0\bigg(\frac{P_{t_1} - P_{t_0}}{t_1 - t_0}\bigg).$$

We see that

$$\partial_s c_s^t\big(\gamma(s),\gamma(t)\big) = -L_0\bigg(\frac{P_{t_1} - P_{t_0}}{t_1 - t_0}\bigg), \quad \partial_t c_s^t\big(\gamma(s),\gamma(t)\big) = L_0\bigg(\frac{P_{t_1} - P_{t_0}}{t_1 - t_0}\bigg).$$

For general L, we have

$$c_s^t\big(\gamma(s),\gamma(t)\big) := \int_s^t \Big(L(\gamma,\dot\gamma) + \Phi(\gamma)\Big)d\tau, \tag{6.2.1}$$

Then

$$\partial_t c_s^t\big(\gamma(s),\gamma(t)\big) = L(\gamma(t),\dot\gamma(t)) + \Phi(\gamma(t)), \qquad \forall s,t \in (t_0,t_1). \tag{6.2.2}$$

and

$$\partial_s c_s^t\big(\gamma(s),\gamma(t)\big) = -L(\gamma(s),\dot\gamma(s)) - \Phi(\gamma(s)), \qquad \forall s,t \in (t_0,t_1). \tag{6.2.3}$$

We differentiable once more to conclude that

$$(s,t) \mapsto c_s^t\big(\gamma(s),\gamma(t)\big) \quad \text{is a map in} \quad C^2\big([t_0,t_1]\big)$$

6.3 An useful variant of Theorem 3.3 in [11]

Let $\alpha < \beta$ be real numbers and assume that $f \in W^{1,1}(\alpha,\beta)$. Let $g = f'$, where f' is the distributional derivative. We have $g \in L^1(\alpha,\beta)$ and

$$f(x) = f(\alpha) + \int_\alpha^x g(t)dt.$$

If $x_0 \in (\alpha,\beta)$ is a Lebesgue point for g then (see [11] Theorem 1.34)

$$\lim_{B\to x_0} \frac{1}{\mathcal{L}^1(B)} \int_B |g - g(x_0)|dx = 0,$$

where the limit is taken over all closed balls B containing x_0 and $\mathrm{d}iam(B) \to 0$. In particular, for $h > 0$, $B_h = [x_0, x_0 + h]$, we have

$$\lim_{h\to 0^+} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h\to 0^+} \frac{\int_{B_h} g(x)dx}{\mathcal{L}^1(B_h)} = g(x_0).$$

Similarly, for $h < 0$, $D_h = [x_0 + h, x_0]$, we have

$$\lim_{h\to 0^-} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h\to 0^+} \frac{\int_{D_h} g(x)dx}{\mathcal{L}^1(D_h)} = g(x_0).$$

This proves that f is differentiable at x_0.

Let

$$Z := \{x \in (\alpha, \beta) : \ f(x) = 0, \ x \in \ \mathrm{d}om(f')\}.$$

Lemma 6.3.1. *For almost every $x \in Z$, we have $f'(x) = 0$.*

Proof. Set

$$N := \{x \in Z : \ x \text{ is not a point of density one in } Z\}.$$

If $x_0 \in Z \setminus N$ such that $f'(x_0) > 0$, since

$$f(x_0 + h) = h\left(f'(x_0) + \frac{o(h)}{h}\right),$$

we conclude that there exists $\delta > 0$ such that

$$f > 0 \quad \text{on} \quad (x_0, x_0 + \delta].$$

Thus $[x_0, x_0 + \delta] \cap Z = \{x_0\}$. This contradicts the fact that $x_0 \in Z \setminus N$. Thus $f'(x_0) = 0$. Similarly, we show that $f'(x_0) < 0$.

In conclusion,

$$Z \setminus N \subset \mathrm{d}om(f') \cap \{f' = 0\}.$$

Since $\mathcal{L}^1(N) = 0$, this proves the Lemma.

□

6.4 Special properties of $\mathcal{W}_g(\delta_{P_0}, \nu)$

We use the notation $\gamma_{s,x}^{t,y}$ as in Remark 3.6.3 (ii) and define the map $(t, y) \in [0,1] \times \overline{\Omega} \mapsto \mathbb{R}^d$ by

$$\Upsilon(t, y) := \gamma_{0,P_0}^{1,y}(t).$$

Given $\nu \in \mathcal{P}(\overline{\Omega})$, we define the probability measures

$$\sigma_t = \Upsilon(t, \cdot)_{\#}\nu.$$

Note that

$$\int_{\overline{\Omega}} \varphi(y)\sigma_t(dy) = \int_{\overline{\Omega}} \varphi\big(\gamma_y(t)\big)\nu(dy), \qquad \forall \varphi \in C_c(\mathbb{R}^d) \tag{6.4.1}$$

For a compact set $K :=$ supp (ν), we set

$$\Omega_{P_0} := \Big(\cup_{y \in K}\, \gamma_{0,P_0}^{1,y}[0,1] \Big).$$

Lemma 6.4.1. *The map Υ is continuous. Thus Ω_{P_0} is a compact set.*

Proof. In order to show the lemma, assume that $(t^n, y^n) \subset [0,1] \times \overline{\Omega}$ converges to (t, y).

By Remark 3.6.3 (ii), $(\gamma_{0,P_0}^{1,y^n})_n$ converges to $\gamma_{0,P_0}^{1,y}$ in $C^1\big([0,1], \overline{\Omega}\big)$. Thus

$$\lim_{n\to\infty} \Upsilon(t_n, y^n) = \lim_{n\to\infty} \gamma_{0,P_0}^{1,y^n}(t_n) = \gamma_{0,P_0}^{1,y}(t) = \Upsilon(t, y).$$

This proves that Υ is continuous. Thus $\Omega_{P_0} = \Upsilon([0,1] \times K)$ is a compact set.

□

Remark 6.4.2. *The goal of this remark is to lay down detailed arguments supporting the fact that the identity* supp $(\sigma_t) = \Upsilon(t, \cdot)[0,1]$ *is a consequence of the continuity property of the map* $\Upsilon(t, \cdot)$.

(i) If $y \in \text{supp }(\nu)$*,* $\delta > 0$ *and we denote by* $B_\delta(a)$*, the open ball of center* a *and radius* δ*, then*

$$\sigma_t\big(B_\delta(\Upsilon(t,y)\big) = \nu\Big\{x:\, |\Upsilon(t,x) - \Upsilon(t,y) < \delta\Big\}.$$

Since $\Upsilon(t,\cdot)$ *is a continuous function and* $\Big\{x:\, |\Upsilon(t,x) - \Upsilon(t,y) < \delta\Big\}$ *is an open set containing* y*, we conclude that we can find* $r > 0$ *such that*

$$B_r(y) \subset \Big\{x:\, |\Upsilon(t,x) - \Upsilon(t,y) < \delta\Big\}.$$

Thus

$$\sigma_t\big(B_\delta(\Upsilon(t,y)\big) \geq \nu(B_r(y)) > 0, \qquad \forall \delta > 0,\ \forall y \in \text{supp }(\nu).$$

In other words, $\Upsilon(t,y) \in \text{supp }(\sigma_t)$*. Hence we have proven that*

$$\Upsilon(t,\cdot)(\text{supp }(\nu)) \subset \text{supp }(\sigma_t).$$

(ii) To show the reverse inclusion, we observe that since $\Upsilon(t,\cdot)$ *is continuous,* $\Upsilon(t,\cdot)(\text{supp }(\nu))$ *is a closed set. Hence if* $z \notin \Upsilon(t,\cdot)(\text{supp }(\nu))$*,* $2\delta_* := dist\big(z, \Upsilon(t,\cdot)(\text{supp }(\nu))\big) > 0$*. Let* $\phi \in C_c(B_{\delta_*}(z))$*. We have*

$$\int_{\mathbb{R}^d} \phi(\omega)\sigma_t(d\omega) = \int_{\mathbb{R}^d} \phi\big(\Upsilon(t,y)\big)\nu(dy) = 0, \qquad \forall \phi \in C_c(B_{\delta_*}(z)).$$

Then $\sigma_t\big(B_{\delta_*}(z)\big) = 0$ *and* $z \notin \text{supp }(\sigma_t)$*. Hence we have proven that*

$$\Big(\Upsilon(t,\cdot)(\text{supp }(\nu))\Big)^c \subset \big(\text{supp }(\sigma_t)\big)^c,$$

which means that $\text{supp }(\sigma_t) \ \subset \Upsilon(t,\cdot)(\text{supp }(\nu))$*.*

6.5 Uniform bounds on $\hat{v}^\epsilon$ and differentials of $\hat{L}^{g,\epsilon}$

We have

$$4\epsilon^2 \int_0^1 F_\Phi(\hat{\sigma}_s^\epsilon)ds \leq \hat{\mathcal{A}}^\epsilon[\hat{\sigma}^\epsilon, \hat{v}^\epsilon] - \frac{1}{2}\text{dist}_g^2\big(P_0, P_T\big) \leq 4\epsilon^2 \int_0^1 F_\Phi(\hat{\sigma}_s^0)ds \qquad (6.5.1)$$

Adapting the ideas of the proof of Proposition 3.11 in [16] to the actions $\hat{\mathcal{A}}^\epsilon$, we conclude conservation of the Hamiltonian in the sense that if we set $\hat{p}^\epsilon = g(x)\hat{v}^\epsilon$, then

$$s \mapsto h(s) := \frac{1}{2}\int_{\bar{\Omega}} \langle g^{-1}(x)\hat{p}^\epsilon(x), \hat{p}^\epsilon(x)\rangle \hat{\sigma}_s^\epsilon(dx) - 4\epsilon^2 F_\Phi(\hat{\sigma}_s^\epsilon)$$

is independent of s. Replacing $\hat{p}^\epsilon$ by $g(x)\hat{v}^\epsilon$, and using the fact that

$$h(s) = \int_0^1 h(t)dt,$$

we obtain

$$\frac{1}{2}\int_{\bar{\Omega}} \langle g(x)\hat{v}_s^\epsilon(x), \hat{v}_s^\epsilon(x)\rangle \hat{\sigma}_s^\epsilon(dx) - 4\epsilon^2 F_\Phi(\hat{\sigma}_s^\epsilon) = \int_0^1 \left(\frac{1}{2}\int_{\bar{\Omega}} \langle g(x)\hat{v}_t^\epsilon(x), \hat{v}_t^\epsilon(x)\rangle \hat{\sigma}_t^\epsilon(dx) - 4\epsilon^2 F_\Phi(\hat{\sigma}_t^\epsilon)\right)dt.$$

Rearranging, we obtain that

$$\frac{1}{2}\int_{\bar{\Omega}} \langle g(x)\hat{v}_s^\epsilon(x), \hat{v}_s^\epsilon(x)\rangle \hat{\sigma}_s^\epsilon(dx) = \hat{\mathcal{A}}^\epsilon[\hat{\sigma}^\epsilon, \hat{v}^\epsilon] + 4\epsilon^2 F_\Phi(\hat{\sigma}_s^\epsilon) - 8\epsilon^2 \int_0^1 F_\Phi(\hat{\sigma}_t^\epsilon)dt$$

This, together with (6.5.1), implies

$$\begin{aligned} 8\epsilon^2\left(F_\Phi(\hat{\sigma}_s^\epsilon) - \int_0^1 F_\Phi(\hat{\sigma}_t^\epsilon)\right) \leq& \int_{\bar{\Omega}} \langle g(x)\hat{v}_s^\epsilon(x), \hat{v}_s^\epsilon(x)\rangle \hat{\sigma}_s^\epsilon(dx) - \mathrm{dist}_g^2\big(P_0, P_T\big) \\ \leq& 8\epsilon^2\left(\int_0^1 F_\Phi(\hat{\sigma}_t^0)dt + F_\Phi(\hat{\sigma}_s^\epsilon)\right). \end{aligned}$$

Thus

$$\left|\int_{\bar{\Omega}} \langle g(x)\hat{v}_s^\epsilon(x), \hat{v}_s^\epsilon(x)\rangle \hat{\sigma}_s^\epsilon(dx) - \mathrm{dist}_g^2\big(P_0, P_T\big)\right| \leq 8\epsilon^2 \|\Phi\|_{L^\infty}. \tag{6.5.2}$$

Since

$$\begin{aligned} \partial_t \hat{L}^{g,\epsilon}(t,q,v) =& \partial_t \hat{\alpha}^\epsilon(t,q) \\ =& 4\epsilon^2 \int_{\bar{\Omega}} \Big\langle \nabla_{q_2}\Phi(q,q_1), \hat{v}_t^\epsilon(q_2)\Big\rangle \hat{\sigma}_t^\epsilon(dq_2) \\ &- 4\epsilon^2 \int_{\bar{\Omega}^2} \left(\Big\langle \nabla_{q_1}\Phi(q_1,q_2), \hat{v}_t^\epsilon(q_1)\Big\rangle + \Big\langle \nabla_{q_2}\Phi(q_1,q_2), \hat{v}_t^\epsilon(q_2)\Big\rangle\right)\hat{\sigma}_t^\epsilon(dq_1)\hat{\sigma}_t^\epsilon(dq_2), \end{aligned}$$

we conclude that

$$|\partial_t \hat{L}^{g,\epsilon}(t,q,v)| \leq 4\epsilon^2\Big(2\|\nabla_{q_2}\Phi\|_{L^\infty} + \|\nabla_{q_1}\Phi\|_{L^\infty}\Big)|\hat{v}_t^\epsilon|_{g,L^2(\sigma_t^\epsilon)}.$$

This, together with (6.5.2), implies

$$|\partial_t \hat{L}^{g,\epsilon}(t,q,v)| \leq 12\epsilon^2 \|\nabla\Phi\|_{L^\infty}\Big(\mathrm{dist}_g^2\big(P_0,P_T\big) + 8\epsilon^2\|\Phi\|_{L^\infty}\Big)^{\frac{1}{2}}. \tag{6.5.3}$$

6.6 On Lasry-Lions strictly monotone functions

Recall that given $\Phi \in C^2(\mathbb{R}^{2d})$ symmetric, we defined

$$F_\Phi(\mu) := \int_{\mathbb{R}^d} \Phi(x,y)\mu(dy).$$

When $\Phi \in C^2(\mathbb{R}^d)$ is even, we will abuse notation and continuous to write

$$F_\Phi(\mu) := \int_{\mathbb{R}^d} \Phi(x-y)\mu(dy).$$

Let $\mathcal{M}(\mathbb{R}^d)$ be the set of finite signed Borel measures on $\mathbb{R}^d$. In this section, we would like to find examples of $\Phi \in C^2(\mathbb{R}^d)$ such that F_Φ is either Lasry–Lions monotone or Lasry–Lions strictly monotone. We would also like to find conditions on $\Phi \in C^2(\mathbb{R}^d)$ even, such that

$$\mu \in \mathcal{M}(\mathbb{R}^d) \mapsto F_\Phi(\mu) := \int_{\mathbb{R}^d} \Phi(x-y)\mu(dy)$$

is either Lasry–Lions monotone or Lasry–Lions strictly monotone.

This means that either

$$\mu \in \mathcal{P}_2(\mathbb{R}^d) \mapsto \mathcal{F}_\Phi(\mu) := \frac{1}{2}\int_{\mathbb{R}^{2d}} \Phi(x-y)\mu(dx)\mu(dy)$$

is convex or strictly convex.

If $\mu = \mu_1 - \mu_0 \neq 0$ and $\mu_0, \mu_1 \in \mathcal{P}_2(\mathbb{R}^d)$, we set $\mu_t := (1-t)\mu_0 + t\mu_1$. Then

$$\frac{d^2}{dt^2}\mathcal{F}_\Phi(\mu_t) = \int_{\mathbb{R}^{2d}} \Phi(x-y)(\mu_1-\mu_0)(dx)(\mu_1-\mu_0)(dy) \tag{6.6.1}$$

must be either non–negative or positive. This statement is related to Bôchner Lemma on positive definite functions.

6.6.0.1 Example and Polynomials

(i) Set $\Phi_0(\mu) = \frac{1}{2}|x|^2$. Using the fact that $\mu := \mu_1 - \mu_0$ is of null average we have

$$\begin{aligned}
&\int_{\mathbb{R}^{2d}} \Phi(x-y)\mu(dx)\mu(dy) \\
=& \int_{\mathbb{R}^d} \mu(dy) \int_{\mathbb{R}^d} (|x|^2 + |y|^2 - 2\langle x, y\rangle)\mu(dx) \\
=& \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} |x|^2 \mu_1(dx) - \int_{\mathbb{R}^d} |x|^2 \mu_0(dx) - 2\Big\langle \int_{\mathbb{R}^d} x\mu_1(dx) - \int_{\mathbb{R}^d} x\mu_0(dx), y\Big\rangle\Big)\mu(dy) \\
=& -2\int_{\mathbb{R}^d} \Big(\Big\langle \int_{\mathbb{R}^d} x\mu_1(dx) - \int_{\mathbb{R}^d} x\mu_0(dx), y\Big\rangle\Big)\mu(dy) \\
=& -2\Big|\int_{\mathbb{R}^d} x\mu_1(dx) - \int_{\mathbb{R}^d} x\mu_0(dx)\Big|^2
\end{aligned}$$

Hence $-F_{\Phi_0}$ is Lasry–Lions monotone, but fails to be strictly monotone.

(ii) Consider

$$\mathcal{F}_1(\mu) = \frac{1}{2}\int_{\mathbb{R}^{2d}} |x+y|^2 \mu(dx)\mu(dy).$$

As above, we have

$$\begin{aligned}
\frac{d^2}{dt^2}\mathcal{F}_1(\mu_t) =& \int_{\mathbb{R}^d} \mu(dy) \int_{\mathbb{R}^d} (|x|^2 + |y|^2 + 2\langle x, y\rangle)\mu(dx) \\
=& \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} |x|^2 \mu_1(dx) - \int_{\mathbb{R}^d} |x|^2 \mu_0(dx) + 2\Big\langle \int_{\mathbb{R}^d} x\mu_1(dx) - \int_{\mathbb{R}^d} x\mu_0(dx), y\Big\rangle\Big)\mu(dy) \\
=& 2\int_{\mathbb{R}^d} \Big(\Big\langle \int_{\mathbb{R}^d} x\mu_1(dx) - \int_{\mathbb{R}^d} x\mu_0(dx), y\Big\rangle\Big)\mu(dy) \\
=& 2\Big|\int_{\mathbb{R}^d} x\mu_1(dx) - \int_{\mathbb{R}^d} x\mu_0(dx)\Big|^2
\end{aligned}$$

Hence $(q, \mu) \mapsto \int_{\mathbb{R}^d} |x+y|^2 \mu(dy)$ is Lasry–Lions monotone, but fails to be strictly monotone.

(iii) For any $a, b > 0$, the following functions are Lasry–Lions monotone:

$$F_{(a,b)}(q, \mu) = a\int_{\mathbb{R}^d} |x+y|^2 \mu(dy) - b\int_{\mathbb{R}^d} |x+y|^2 \mu(dy) = \int_{\mathbb{R}^d} \Phi_{(a,b)}(x, y)\mu(dy)$$

where, $\Phi_{(a,b)}$ are the polynomials

$$\Phi_{(a,b)}(x, y) = (a-b)(|x|^2 + |y|^2) + 2(a+b)\langle x, y\rangle$$

In particular, $F_{(1/2,1/2)}$ is Lasry–Lions monotone and $\Phi_{(1/2,1/2)} = \langle x, y\rangle$. This is the most important case since the contribution of $|x|^2$ and $|y|^2$ is immaterial. In general, $\Phi_{(a,b)}(x, y)$ is not a function of solely $x - y$.

Lemma 6.6.1. *Let ν be a finite Radon measure on $\mathbb{R}$ and let $\lambda \mapsto \phi_\lambda \in C^2(\mathbb{R}^d, \mathbb{R}^d)$ be a continuous function for the uniform topology and such that there exists $c_0 > 0$ such that $|\phi_\lambda(x)| \le c_0(1 + |x|^2)$ and $|\nabla^2\phi_\lambda(x)| \le c_0$ for all $x \in \mathbb{R}^d$.*

Set

$$\Phi(x, y) = \int_{\mathbb{R}} \langle \phi_\lambda(x), \phi_\lambda(y)\rangle \nu(d\lambda).$$

Then F_Φ is Lasry–Lions monotone.

Proof. Given $\mu_0, \mu_1 \in \mathcal{P}_2(\mathbb{R}^d)$, let set $\mu = \mu_1 - \mu_0$. Notice that we have

$$\begin{aligned} 2\mathcal{F}_\Phi(\mu) &= \int_{\mathbb{R}} \nu(d\lambda) \int_{\mathbb{R}^d} \Big\langle \phi_\lambda(x), \int_{\mathbb{R}^d} \phi_\lambda(y)(\mu_1(dy) - \mu_0(dy)\Big\rangle \mu(dx) \\ &= \int_{\mathbb{R}} \Big| \int_{\mathbb{R}^d} \phi_\lambda(y)(\mu_1(dy) - \mu_0(dy)\Big\rangle \Big|^2 \nu(d\lambda). \end{aligned}$$

□

Remark 6.6.2. *(i) Observe that if ν is an average of Dirac masses, then Lemma 6.6.1 is applicable to*

$$\Phi(x, y) = \sum_{i=1}^{n} \langle \phi_i(x), \phi_i(y)\rangle,$$

provided that there exists $c_0 > 0$ such that for any i, we have $|\phi_i(x)| \le c_0(1 + |x|^2)$ and $|\nabla^2\phi_i(x)| \le c_0$ for all $x \in \mathbb{R}^d$.

(ii) Given $x = (x_1, \cdots, x_d), y = (y_1, \cdots, y_d) \in \mathbb{R}^d$, we use the notation

$$x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}, \quad \langle x^\alpha, y^\beta\rangle = \sum_{i=1}^{d} x_i^{\alpha_i} y_i^{\beta_i}, \quad \forall \alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{N},\ \beta = (\beta_1, \cdots, \beta_d) \in \mathbb{N}.$$

Given $k \in \mathbb{N}$, let N_k be the cardinality of $\{\alpha \in \mathbb{N} : |\alpha|_{\ell^1} = k\}$. We choose a bounded set $(C_\alpha) \subset (0, \infty)$ and define

$$\Phi(x,y) = \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} \sum_{|\alpha|_{\ell^1}=k} \frac{C_\alpha}{N_k} \langle x^\alpha, y^\beta \rangle.$$

Given $\mu_0, \mu_1 \in \mathcal{P}_2(\mathbb{R}^d)$ and setting $\mu = \mu_1 - \mu_0$, we have

$$\int_{\mathbb{R}^{2d}} \Phi(x,y)\mu(dx)\mu(dy) = \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} \sum_{|\alpha|_{\ell^1}=k} \frac{C_\alpha}{N_k} \left| \int_{\mathbb{R}^d} x^\alpha (\mu_1(dx) - \mu_0(dx)) \right|^2 \geq 0.$$

Furthermore,

$$\int_{\mathbb{R}^{2d}} \Phi(x,y)\mu(dx)\mu(dy) = 0$$

implies

$$\int_{\mathbb{R}^d} x^\alpha \mu_0(dx) = \int_{\mathbb{R}^d} x^\alpha \mu_1(dx), \qquad \forall \alpha \in \mathbb{N}. \tag{6.6.2}$$

Since span$\{x^\alpha : \alpha \in \mathbb{N}\}$ is a dense subset of $C(K)$ for any compact set $K \subset \mathbb{R}^d$, (6.6.2) *implies that $\mu_0 = \mu_1$. Thus, F_Φ is Lasry–Lions strictly monotone. Note that the proof does not encompass the case where Φ is a polynomial since we required that $C_\alpha > 0$ for all α.*

6.6.0.2 Non-polynomial potential functions solely depending on $x - y$

Given $a, b \in \mathbb{R}$, if we $w = a + ib$, we set $|w|^2 = a^2 + b^2 \geq 0$.

Let $\mathcal{S}(\mathbb{R}^d)$ be the Schwartz space, which is the set of $f \in C^\infty(\mathbb{R}^d, \mathbb{C})$ such that for any multi–index $\alpha \in \mathbb{N}_0^d$ and any non-negative number N, the functions $x \mapsto (1 + |x|^N)f(x)$ are bounded. The standard topology on $\mathcal{S}(\mathbb{R}^d)$ is in such a way that $(f_n)_n \subset \mathcal{S}(\mathbb{R}^d)$ converges to $f \in \mathcal{S}(\mathbb{R}^d)$ if for any multi–indexes $\alpha, \beta \in \mathbb{N}_0^d$ and any non-negative number N we have

$$\lim_{n\to\infty} \left\| x^\beta \partial^\alpha (f_k - f) \right\|_{L^\infty(\mathbb{R}^d)} = 0.$$

Recall that the Fourier transform of $f \in L^1(\mathbb{R}^d)$ is $\hat{f} \in C_b(\mathbb{R}^d)$, defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i \langle x, \xi \rangle} f(x) dx.$$

Similarly, we define the Fourier transform of any finite signed Borel measure. If both μ and ν are finite signed Borel measures, then $\mu * \nu$ is the signed finite Borel measure defined by

$$\int_{\mathbb{R}^d} h(z)\mu * \nu(dz) = \int_{\mathbb{R}^{2d}} h(x-y)\mu(dx)\nu(dy).$$

The functions $\hat{\mu}$ and $\hat{\nu}$ are bounded continuous and

$$\widehat{\mu * \nu} = \hat{\mu}\hat{\nu}. \tag{6.6.3}$$

The Fourier transform can be extended to a map of $L^2(\mathbb{R}^d)$ into $L^2(\mathbb{R}^d)$. If also a continuous bijection of $\mathcal{S}(\mathbb{R}^d)$ to itself. In particular, if we set

$$g_\lambda(x) = e^{-\lambda\pi|x|^2},$$

then $g_\lambda \in \mathcal{S}(\mathbb{R}^d)$ and

$$\hat{g}_\lambda = \sqrt{\lambda}^{-d} g_{\frac{1}{\lambda}}.$$

If we denote by ℓ the map $x \in \mathbb{R}^d \mapsto -x$, then the Fourier transform is invertible on $L^2(\mathbb{R}^d)$ and its inverse there is $f \mapsto \hat{f} \circ \ell$.

When $f \in \mathcal{S}(\mathbb{R}^d)$ and μ is a finite signed Borel measure, we have Plancherel formula

$$\int_{\mathbb{R}^d} f(x)\mu(dx) = \int_{\mathbb{R}^d} \overline{\hat{f}(\xi)}\hat{\mu}(\xi)d\xi.$$

Thus if we denote by μ_0 the push forward of μ by ℓ, we have

$$\int_{\mathbb{R}^d} f * \mu(x)\mu(dx) = \int_{\mathbb{R}^d} f(z)(\mu * \mu_0)(dz) = \int_{\mathbb{R}^d} \overline{\hat{f}(\xi)}\widehat{\mu * \mu_0}(d\xi).$$

We use (6.6.3) to conclude that since μ is a real valued measure,

$$\int_{\mathbb{R}^d} f * \mu(x)\mu(dx) = \int_{\mathbb{R}^d} \overline{\hat{f}(\xi)}\hat{\mu}(\xi)\hat{\mu}_0(\xi)d\xi = \int_{\mathbb{R}^d} \overline{\hat{f}(\xi)}|\hat{\mu}(\xi)|^2 d\xi. \tag{6.6.4}$$

Now choose a real value function $g \in \mathcal{S}(\mathbb{R}^d)$ such that g is even and $g > 0$. Set $\Phi := \hat{g}$. Since g is even, Φ is a real valued function. Furthermore,

$$\Phi(-x) = \int_{\mathbb{R}^d} e^{2\pi i\langle x,\xi\rangle} g(\xi)d\xi = \int_{\mathbb{R}^d} e^{-2\pi i\langle x,\xi\rangle} g(-\xi)d\xi = \int_{\mathbb{R}^d} e^{-2\pi i\langle x,\xi\rangle} g(\xi)d\xi = \Phi(x).$$

Thus Φ is even. If $\mu_0 \neq \mu_1$ are two probability measures and we set $\mu_t := (1-t)\mu_0 + t\mu_1$, using (6.6.1) and (6.6.4) and the fact that $\hat{\Phi} \circ \ell = g$ we have

$$\frac{d^2}{dt^2}\mathcal{F}_\phi(\mu_t) = \int_{\mathbb{R}^d} g(\xi)|\widehat{\mu_1 - \mu_0}(\xi)|^2 d\xi > 0. \tag{6.6.5}$$

6.7 Explicite formula for inverse map in some cases

Define $F : C(\mathbb{R}^d) \to C(\mathbb{R})$ by

$$F(h)(s) = h\big(\hat{P}_s - \hat{\gamma}_s\big) - h(0), \qquad \forall s \in \mathbb{R}.$$

For $a_1, \cdots, a_n \in \mathbb{R}$ distinct, we set

$$W := span\Big\{h_1, \cdots, h_n\Big\}, \qquad h_i : x \mapsto e^{-a_i|x|^2}.$$

By (6.6.1), to ensure the Lasry–Lions monotonicity condition, we need to impose that $a_1, \cdots, a_n > 0$. This is needed to assert that $h_i \in L^2(\mathbb{R}^d)$ and to compute explicitly the Fourier transform of h_i.

Note that F is a linear map such that the range of its restriction to W is the linear space

$$V := span\Big\{F(h_1), \cdots, F(h_n)\Big\}.$$

Since d$imV = n$, we conclude that $F|_W$ is a bijection and its inverse G_W is a linear map.

Since

$$\ln\big(F(h_i)(s) + 1\big) = -a_i|\hat{P}_s - \hat{\gamma}_s|^2,$$

we conclude that

$$-a_i = \frac{\int_0^1 \ln\big(F(h_i)(s) + 1\big)ds}{\int_0^1 |\hat{P}_s - \hat{\gamma}_s|^2 ds}.$$

Hence

$$h_i(x) = \exp\left(|x|^2 \, \frac{\int_0^1 \ln\big(F(h_i)(s) + 1\big)ds}{\int_0^1 |\hat{P}_s - \hat{\gamma}_s|^2 ds}\right).$$

In other words,

$$G_W(f_i) = \exp\left(|x|^2 \, \frac{\int_0^1 \ln\big(f_i(s)+1\big)ds}{\int_0^1 |\hat{P}_s - \hat{\gamma}_s|^2 ds}\right).$$

This implies that

$$G_W\left(\sum_{i=1}^n \beta_i f_i\right) = \sum_{i=1}^n \beta_i \exp\left(|x|^2 \, \frac{\int_0^1 \ln\big(f_i(s)+1\big)ds}{\int_0^1 |\hat{P}_s - \hat{\gamma}_s|^2 ds}\right)$$

To obtain that $\sum_{i=1}^n \beta_i h_i$ is Lasry–Lions monotone, we assume that $\beta_i \geq 0$ and at least one of the β_i are positive.

6.8 Future work to be done

We state three problems that are open to future work in mean field games.

The first problem is about global uniqueness and boundary rigidity.

Problem 6.8.1. *Find a class of metric and interactions $\tilde{\mathcal{C}} \times \tilde{\mathcal{F}} \subset \mathcal{G}(a,b) \times \mathcal{F}$ such that, if $(g_1, F_1), (g_2, F_2) \in \tilde{\mathcal{C}} \times \tilde{\mathcal{F}}$ is such that*

$$\mathcal{I}_{g_1,F_1} = \mathcal{I}_{g_1,F_2}\,,$$

then there exists a C^{k+1} diffeomorphism $\eta : \overline{\Omega} \to \overline{\Omega}$ fixing the boundary such that

$$g_1(q)\delta_{ij} = g_2(\eta(q)) \sum_{k=1}^d \partial_k \eta_i(q) \partial_k \eta_j(q) \qquad \text{and} \qquad F_1(\mu) = F_2(\eta_* \mu)$$

where $\eta_*\mu$ is the pushforward measure of μ by η.

Notice the conclusions above can be written as

$$g_1 = \eta^* g_2 \qquad \text{and} \qquad F_1 = \eta^* F_2\,,$$

where $\eta^* g$ is now the pullback metric of g, and $\eta^* F$ is also the pullback function of F, with the notions defined as

$$\eta^* g_q(v,w) := g_{\eta(q)}(d\eta_q(v), d\eta_q(w)) \text{ and } \eta^* F(\mu) := F(\eta_* \mu)\,.$$

The second problem is about generic local uniqueness and boundary rigidity.

Problem 6.8.2. *Find a class of metric and interactions* $\tilde{\mathcal{C}} \times \tilde{\mathcal{F}} \subset \mathcal{G}(a,b) \times \mathcal{F}$ *such that we have a dense subset* $\mathcal{D} \subset \tilde{\mathcal{C}} \times \tilde{\mathcal{F}}$ *with the following property: for any* $(g_0, F_0) \in \mathcal{D}$*, there exists an* $\varepsilon > 0$ *such that if* $(g_1, F_1), (g_2, F_2) \in \tilde{\mathcal{C}} \times \tilde{\mathcal{F}}$ *with* $\|g_m - g_0\|_{C^k(\overline{\Omega})} + \|F_m - F_0\|_{C^1(\mathcal{P}_2(\overline{\Omega}))} \leq \varepsilon$ *for* $m = 1, 2$*, and*

$$\mathcal{I}_{g_1,F_1} = \mathcal{I}_{g_2,F_2}\,,$$

then there exists a C^{k+1} *diffeomorphism* $\eta : \overline{\Omega} \to \overline{\Omega}$ *fixing the boundary such that*

$$g_1 = \eta^* g_2 \text{ and } \qquad \text{and} \quad F_1 = \eta^* F_2\,.$$

The last problem is about generic local stability of $\tilde{\mathcal{C}} \times \tilde{\mathcal{F}}$.

Problem 6.8.3. *Find a class of metric and interactions* $\tilde{\mathcal{C}} \times \tilde{\mathcal{F}} \subset \mathcal{G}(a,b) \times \mathcal{F}$ *such that we have a dense subset* $\mathcal{G} \subset \tilde{\mathcal{C}} \times \tilde{\mathcal{F}}$ *with the following property: for any* $(g_0, F_0) \in \mathcal{G}$*, there exists an* $\varepsilon > 0$ *such that if* $(g_1, F_1), (g_2, F_2) \in \tilde{\mathcal{C}} \times \tilde{\mathcal{F}}$ *with* $\|g_m - g_0\|_{C^k(\overline{\Omega})} + \|F_m - F_0\|_{C^1(\mathcal{P}_2(\overline{\Omega})))} \leq \varepsilon$ *for* $m = 1, 2$*, then there exists a* C^{k+1} *diffeomorphism* $\eta : \overline{\Omega} \to \overline{\Omega}$ *fixing the boundary such that*

$$\|g_1 - \eta^* g_2\|_{C^k(\overline{\Omega})} + \|F_1 - \eta^* F_2\|_{C^1(\mathcal{P}_2(\overline{\Omega}))} \leq d(\mathcal{I}_{g_1,F_1}, \mathcal{I}_{g_2,F_2})\,,$$

for some expression d*, which we can view as a metric.*

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