Title
Parabolic flows on complex manifolds

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Publication Date
2012

UNIVERSITY OF CALIFORNIA, SAN DIEGO

Parabolic Flows on Complex Manifolds

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Mathematics

by

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2012

The dissertation of Matthew Gill is approved, and it is acceptable in quality and form for publication on microfilm:

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Chair

University of California, San Diego

2012

DEDICATION

To my parents and grandparents.

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ACKNOWLEDGEMENTS

I am grateful to Ben Weinkove for getting me involved in geoemetric analysis, for numereous helpful discussions, and for guiding me through the majority of graduate school. I would also like to thank Valentino Tosatti for always being available to look over my work and offer suggestions for improvement. I also thank the referee for [22] from Communations in Analysis and Geometry for a careful reading of the first version of the paper and for making a number of helpful comments. Finally, I thank the San Diego ARCS foundation for their support and funding.

Chapter 2, in full, is a reprint of the material as it appears in Communications in Analysis and Geometry volume 19, no. 2, 2011. Gill, Matthew, International Press 2011. The disseratation author was the author of this paper.

Chapter 3, in full, is currently being prepared for submission for publication of the material. Gill, Matthew. The dissertation author was the author of this material.

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PUBLICATIONS

Gill, M., “Convergence of the parabolic complex Monge-Ampère equation on compact Hermitian manifolds”, *Communications in Analysis and Geometry*, 19 (2011), no. 2, 277 – 304.

Gill, M., “Collapsing of products along the Kähler-Ricci flow”, preprint, arXiv: 1203.3781.

ABSTRACT OF THE DISSERTATION

Parabolic Flows on Complex Manifolds

by

Matthew Gill
Doctor of Philosophy in Mathematics

University of California San Diego, 2012

Professor Ben Weinkove, Chair

We prove C^∞ convergence for suitably normalized solutions of the parabolic complex Monge-Ampère equation on compact Hermitian manifolds. This provides a parabolic proof of a recent result of Tosatti-Weinkove.

Additionally, let $X = M \times E$ where M is an m-dimensional Kähler manifold with negative first Chern class and E is an n-dimensional complex torus. We obtain C^∞ convergence of the normalized Kähler-Ricci flow on X to a Kähler-Einstein metric on M. This strengthens a convergence result of Song-Weinkove and confirms their conjecture.

Chapter 1

Introduction

In 1981, Hamilton introduced the Ricci flow

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} \tag{1.1}$$

to classify three-manifolds with positive Ricci curvature and four-manifolds with positive curvature operator [29, 30]. Later, Hamilton proposed a program to prove the Poincaré and Geometrization conjectures via the Ricci flow with surgery [31]. Building on Hamilton's program, Perelman developed several new and powerful tools which he used to solve these two famous conjectures [39, 40, 41]. Since then, the Ricci flow has become one of the most important objects of study in geometric analysis.

Considering the Ricci flow starting at a Kähler metric ω_0 on a complex manifold, the Ricci flow can be written in terms of $(1,1)$-forms as

$$\frac{\partial}{\partial t}\omega = -\operatorname{Ric}(\omega), \qquad \omega|_{t=0} = \omega_0 \tag{1.2}$$

and is known as the Kähler-Ricci flow. The problem of finding a unique Kähler metric whose Ricci form represents the first Chern class was known as the Calabi conjecture. Calabi reduced the problem to finding a unique solution to the complex Monge-Ampère equation

$$\log \frac{\left(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi\right)^n}{\omega_0^n} = F, \qquad \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi > 0, \tag{1.3}$$

and showed that if a solution φ exists, it is unique up to the addition of a constant [4].

To prove the conjecture, Yau developed *a priori* estimates for a solution of the complex Monge-Ampère equation (1.3). Independently, both Yau [81] and Aubin [1] proved the existence of a unique Kähler-Einstein metric on a manifold with negative first Chern class. Using the estimates of Yau and Aubin, Cao showed that the Kähler-Ricci flow produces the unique Kähler-Einstein metric on a manifold with zero and negative first Chern class [6].

The Kähler-Ricci flow has since become a major field of study in geometric analysis. The existence of a Kähler-Einstein metric on a manifold with positive first Chern class is still an open question which has been related to algebraic stability [2, 5, 9, 15, 38, 43, 44, 45, 46, 47, 48, 50, 52, 68, 70, 73, 74, 82, 83]. There have also been studies on extending the flow to the more general Hermitian setting [22, 36, 65, 66, 67, 21, 77]. In particular, chapter 2 contains a reprinting of [22] and the proof of the following theorem:

Theorem 1.0.1. *Let (M, g) be a compact Hermitian manifold of complex dimension n with $\operatorname{Vol}(M) = \int \omega^n = 1$. Let F be a smooth function on M. There exists a smooth solution φ to the parabolic complex Monge-Ampère equation* (2.1) *for all time. Let*

$$\tilde{\varphi} = \varphi - \int_M \varphi \, \omega^n. \tag{1.4}$$

Then $\tilde{\varphi}$ converges in C^∞ to a smooth function $\tilde{\varphi}_\infty$. Moreover, there exists a unique real number b such that the pair $(b, \tilde{\varphi}_\infty)$ is the unique solution to (2.3).

Very recently, the Kähler-Ricci flow has been conjectured by Song and Tian to behave as an analytic version of the Minimal Model Program in algebgraic geometry (please see chapter 3 for a larger discussion on the Minimal Model Program). Much work has been done on this subject [18, 19, 20, 23, 55, 56, 57, 58, 60, 61, 62, 59, 64], and in particular chapter 3 contains a reprinting of [23] and the proof of the following theorem:

Theorem 1.0.2. *Let (M, g_M) be an m complex dimensional Kähler manifold with negative first Chern class where g_M is its Kähler-Einstein metric. Let (E, g_E) be an*

n dimensional complex torus with flat metric g_E. Let g_0 be any Hermitian metric on $X = M \times E$ and ω_0 its associated $(1,1)$ form. Let $\omega(t)$ be the solution to the normalized Kähler-Ricci flow

$$\frac{\partial}{\partial t}\omega = -\operatorname{Ric}(\omega) - \omega \tag{1.5}$$

with initial Kähler metric $\omega(0) = \omega_0$. Then

(a) $\omega(t)$ converges to $\pi_M^* \omega_M$ in $C^\infty(X, \omega_0)$ as $t \to \infty$.

(b) For any $z \in M$, let $E(z) = \pi_M^{-1}(z)$ denote the fiber above z. Then $e^t \omega(t)|_{E(z)} \to \omega_{flat}|_{E(z)}$ in $C^\infty(E(z), \omega_E)$ as $t \to \infty$, where ω_{flat} is a $(1,1)$-form on X with $[\omega_{flat}] = [\omega_0]$ whose restriction to each fiber is a flat Kähler metric.

Chapter 4 discusses potential future projects leading from the contents of chapter 2, chapter 3, and [77].

Chapter 2

Convergence of the parabolic complex Monge-Ampère equation on compact Hermitian manifolds

2.1 Introduction

Let (M, g) be a compact Hermitian manifold of complex dimension n and ω be the real $(1,1)$ form $\omega = \sqrt{-1}\sum_{i,j} g_{i\bar{j}} dz^i \wedge dz^{\bar{j}}$. Let F be a smooth function on M. We consider the parabolic complex Monge-Ampère equation

$$\frac{\partial \varphi}{\partial t} = \log \frac{\det (g_{i\bar{j}} + \partial_i \partial_{\bar{j}} \varphi)}{\det g_{i\bar{j}}} - F, \quad g_{i\bar{j}} + \partial_i \partial_{\bar{j}} \varphi > 0 \tag{2.1}$$

with initial condition $\varphi(x, 0) = 0$.

The study of this type of Monge-Ampère equation originated in proving the Calabi conjecture. The proof of the conjecture reduced to assuming that ω is Kähler and finding a unique solution to the elliptic Monge-Ampère equation

$$\log \frac{\det (g_{i\bar{j}} + \partial_i \partial_{\bar{j}} \varphi)}{\det g_{i\bar{j}}} = F, \quad g_{i\bar{j}} + \partial_i \partial_{\bar{j}} \varphi > 0. \tag{2.2}$$

Calabi showed that if a solution to (2.2) exists, it is unique up to adding a constant to φ [4]. Yau used the continuity method to show that if

$$\int_M e^F \omega^n = \int_M \omega^n$$

then (2.2) admits a smooth solution [81]. The proof of Yau required *a priori* C^∞ estimates for φ.

Cao used Yau's estimates to show that in the Kähler case, (2.1) has a smooth solution for all time that converges to the unique solution of (2.2) [6].

Since not every complex manifold admits a Kähler metric, one can naturally study the Monge-Ampère equations (2.1) and (2.2) on a general Hermitian manifold. Fu and Yau discussed physical motivation for studying non-Kähler metrics in a recent paper [21].

Cherrier studied (2.2) in the general Hermitian setting in the eighties, and showed that in complex dimension 2 or when ω is balanced (i.e. $d(\omega^{n-1}) = 0$), there exists a unique normalization of F such that (2.2) has a unique solution [11]. Precisely, Cherrier proved that under the above conditions, given a smooth function F on M, there exists a unique real number b and a unique function φ solving the Monge-Ampère equation

$$\log \frac{\det(g_{i\bar{j}} + \partial_i \partial_{\bar{j}} \varphi)}{\det g_{i\bar{j}}} = F + b, \quad g_{i\bar{j}} + \partial_i \partial_{\bar{j}} \varphi > 0 \tag{2.3}$$

such that $\int_M \varphi \, \omega^n = 0$.

Recently, Guan and Li proved that (2.2) has a solution on a Hermitian manifold with the added condition

$$\partial \bar{\partial} \omega^k = 0$$

for $k = 1, 2$. They applied this result to finding geodesics in the space of Hermitian metrics. Related work can be found in [3], [8], [10], [14], [27], [28], [37], [49], and [51].

Tosatti and Weinkove gave an alternate proof of Cherrier's result in [75]. In a very recent paper [76], they showed that the balanced condition is not necessary and the result holds on a general Hermitian manifold. Dinew and Kolodziej studied (2.2) in the Hermitian setting with weaker conditions on the regularity of F [13].

In this chapter we prove the following theorem.

Theorem 2.1.1. *Let* (M, g) *be a compact Hermitian manifold of complex dimension* n *with* $\mathrm{Vol}(M) = \int \omega^n = 1$*. Let* F *be a smooth function on* M*. There exists* *a smooth solution φ to the parabolic complex Monge-Ampère equation* (2.1) *for all time. Let*

$$\tilde{\varphi} = \varphi - \int_M \varphi \, \omega^n. \tag{2.4}$$

Then $\tilde{\varphi}$ converges in C^∞ to a smooth function $\tilde{\varphi}_\infty$. Moreover, there exists a unique real number b such that the pair $(b, \tilde{\varphi}_\infty)$ is the unique solution to (2.3)*.*

We remark that Theorem 2.1.1 gives a parabolic proof of the result due to Tosatti and Weinkove in [76].

The flow (2.1) could be considered as an analogue to Kähler-Ricci flow for Hermitian manifolds. In the special case that $-\sqrt{-1}\partial\bar{\partial}\log\det g = \sqrt{-1}\partial\bar{\partial}F$ (such an F always exists under the topological condition $c_1^{BC}(M) = 0$, for example) then taking $\sqrt{-1}\partial\bar{\partial}$ of the flow (2.1) yields

$$\frac{\partial \omega'}{\partial t} = \sqrt{-1}\partial\bar{\partial}\log\det g'$$

with initial condition $\omega'(0) = \omega$. In general, the right hand side is the first Chern form, but if we assume Kähler, it becomes $-\operatorname{Ric}(\omega')$.

When (M, g) is Kähler, Székelyhidi and Tosatti showed that a weak plurisubharmonic solution to (2.2) is smooth using the parabolic flow (2.1) [69]. Their result suggests that the flow could be used to prove a similar result in the Hermitian case. In a recent paper [67], Streets and Tian consider a different parabolic flow on Hermitian manifolds and suggest geometric applications for the flow.

We now give an outline of the proof of the main theorem and discuss how it differs from previous results. In sections 2.2 through 2.5, we build up theorems that eventually show that φ is smooth. Like in Yau's proof, we derive lower order estimates and then apply Schauder estimates to attain higher regularity for the solution.

In section 2.2 we use the maximum principle to show that the time derivative of φ is uniformly bounded. We define the normalization

$$\tilde{\varphi} = \varphi - \int_M \varphi \, \omega^n.$$

We chose to assume that the volume of M is one to simplify the notation of this normalization and the following calculations. Then using the zeroth order estimate from [76], we prove that $\tilde{\varphi}$ is uniformly bounded.

Section 2.3 contains a proof of the second order estimate. Specifically, we derive that

$$\operatorname{tr}_g g' \leq C_1 e^{C_2(\sup_{M\times[0,T)}\tilde{\varphi}-\inf_{M\times[0,T)}\tilde{\varphi})} e^{\left(e^{A\left(\sup_{M\times[0,T)}\tilde{\varphi}-\inf_{M\times[0,T)}\tilde{\varphi}\right)}-e^{A\left(\sup_{M\times[0,T)}\tilde{\varphi}-\tilde{\varphi}\right)}\right)} \tag{2.5}$$

where $[0,T)$ is the maximum interval of existence for φ and C_1, C_2, and A are uniform constants. This estimate is not as sharp as the estimate

$$\operatorname{tr}_g g' \leq C e^{\left(e^{A(\sup_M \varphi - \inf_M \varphi)} - e^{A(\sup_M \varphi - \varphi)}\right)}$$

from Guan and Li or the estimate

$$\operatorname{tr}_g g' \leq C e^{A(\varphi - \inf_M \varphi)}$$

from Tosatti and Weinkove in the case $n = 2$ or ω balanced. Cherrier also produced a different estimate. These estimates are from the elliptic case, but they suggest that (2.5) could be improved. The proof of (2.5) follows along the method of Tosatti and Weinkove in [75], but there are extra terms to control that arrive in the parabolic case.

In section 2.4, we derive a Hölder estimate for the time dependent metric $g'_{i\bar{j}}$. This estimate provides higher regularity using a method of Evans [16] and Krylov [32]. To prove the Hölder estimate, we apply a theorem of Lieberman [34], a parabolic analogue of an inequality from Trudinger [78]. The method follows closely with the proof of the analogous estimate in [75], but differs in controlling the extra terms that arise from the time dependence of φ.

We show that φ is smooth and also prove the long time existence of the flow (2.1) in section 2.5. The proof uses a standard bootstrapping argument.

Section 2.6 uses analogues of lemmas from Li and Yau [33] to prove a Harnack inequality for the equation

$$\frac{\partial u}{\partial t} = g'^{i\bar{j}} \partial_i \partial_{\bar{j}} u$$

where $g'^{i\bar{j}} \partial_i \partial_{\bar{j}}$ is the complex Laplacian with respect to g'. This differs from the equation

$$\left(\triangle - q(x,t) - \frac{\partial}{\partial t}\right) u(x,t) = 0$$

considered by Li and Yau, where $\triangle$ is the Laplace-Beltrami operator.

In Section 2.7, we apply these lemmas to show that time derivative of $\tilde{\varphi}$ decays exponentially. Precisely, we show that

$$\left|\frac{\partial \tilde{\varphi}}{\partial t}\right| \leq C e^{-\eta t}$$

for some $\eta > 0$. From here we show that $\tilde{\varphi}$ converges to a smooth function $\tilde{\varphi}_\infty$ as t tends to infinity. In fact, the convergence occurs in C^∞ and $\tilde{\varphi}_\infty$ is part of the unique pair $(b, \tilde{\varphi}_\infty)$ solving the elliptic Monge-Ampère equation

$$\log \frac{\det(g_{i\bar{j}} + \partial_i \partial_{\bar{j}} \tilde{\varphi}_\infty)}{\det g_{i\bar{j}}} = F + b$$

where

$$b = \int_M \left(\log \frac{\det(g_{i\bar{j}} + \partial_i \partial_{\bar{j}} \tilde{\varphi}_\infty)}{\det g_{i\bar{j}}} - F \right) \omega^n.$$

This provides an alternate proof of the main theorem in [76].

2.2 Preliminary estimates

By standard parabolic theory, there exists a unique smooth solution φ to (2.1) on a maximal time interval $[0, T)$, where $0 < T \leq \infty$.

We show that the time derivatives of φ and its normalization $\tilde{\varphi}$ are bounded. This fact will be used in the second order estimate.

Lemma 2.2.1. *For φ a solution of* (2.1) *and $\tilde{\varphi}$ as in* (2.4)*,*

$$\left|\frac{\partial \varphi}{\partial t}\right| \leq C, \quad \left|\frac{\partial \tilde{\varphi}}{\partial t}\right| \leq C \tag{2.6}$$

where C depends only on the initial data.

Proof. Differentiating (2.1) with respect to t gives

$$\frac{\partial \varphi_t}{\partial t} = g'^{i\bar{j}} \partial_i \partial_{\bar{j}} \varphi_t, \tag{2.7}$$

where $\varphi_t = \frac{\partial \varphi}{\partial t}$. So by the maximum principle,

$$\left|\frac{\partial \varphi}{\partial t}(x, t)\right| \leq C \sup_{x \in M} \left|\frac{\partial \varphi}{\partial t}(x, 0)\right|. \tag{2.8}$$

From the definition of $\tilde{\varphi}$,

$$\left|\frac{\partial \tilde{\varphi}}{\partial t}\right| \leq \left|\frac{\partial \varphi}{\partial t}\right| + \int \left|\frac{\partial \varphi}{\partial t}\right| \omega^n \leq 2C. \tag{2.9}$$

□

We show that $\tilde{\varphi}$ is bounded in $M \times [0, T)$ using the main theorem of [76].

Lemma 2.2.2. *For φ a solution to* (2.1) *and $\tilde{\varphi}$ the normalized solution, there exists a uniform constant C such that*

$$\sup_{M \times [0,T)} |\tilde{\varphi}| \leq C$$

where $[0, T)$ is the maximum interval of existence for φ.

Proof. We can rearrange (2.1) to

$$\log \frac{\det g'_{i\bar{j}}}{\det g_{i\bar{j}}} = F - \frac{\partial \varphi}{\partial t} \tag{2.10}$$

Since $\left|\frac{\partial \varphi}{\partial t}\right|$ is bounded by Lemma 2.2.1, this is equivalent to the complex Monge-Ampère equation of the main theorem in [76]. This implies that

$$\sup_M \varphi(., t) - \inf_M \varphi(., t) \leq C \tag{2.11}$$

for some C depending only on (M, g) and F.

Fix (x, t) in $M \times [0, T)$. Since $\int_M \tilde{\varphi}\ \omega^n = 0$, there exists (y, t) such that $\tilde{\varphi}(y, t) = 0$. Then

$$|\tilde{\varphi}(x, t)| = |\tilde{\varphi}(x, t) - \tilde{\varphi}(y, t)| = |\varphi(x, t) - \varphi(y, t)| \leq C. \tag{2.12}$$

Thus $\tilde{\varphi}$ is a bounded function on $M \times [0, T)$. □

2.3 The second order estimate

In this section $\triangle = g^{i\bar{j}}\partial_i\partial_{\bar{j}}$ will denote the complex Laplacian corresponding to g. Similarly, write $\triangle' = g'^{i\bar{j}}\partial_i\partial_{\bar{j}}$ for the complex Laplacian for the time dependent metric g'. We prove an estimate on $\mathrm{tr}_g\, g' = g^{i\bar{j}}g'_{i\bar{j}} = n + \triangle\tilde{\varphi}$.

Lemma 2.3.1. *For φ a solution to* (2.1) *and $\tilde{\varphi}$ the normalized solution, we have the following estimate*

$$\operatorname{tr}_g g' \leq C_1 e^{C_2(\sup_{M\times[0,T)}\tilde{\varphi} - \inf_{M\times[0,T)}\tilde{\varphi})} e^{\left(e^{A\left(\sup_{M\times[0,T)}\tilde{\varphi} - \inf_{M\times[0,T)}\tilde{\varphi}\right)} - e^{A\left(\sup_{M\times[0,T)}\tilde{\varphi} - \tilde{\varphi}\right)}\right)}$$

where $[0,T)$ is the maximum interval of existence for φ and C_1, C_2, and A are uniform constants. Hence there exists a uniform constant C such that $\operatorname{tr}_g g' \leq C$ and also

$$\frac{1}{C} g \leq g' \leq Cg.$$

Proof. This proof follows along with the notation and method featured in [75]. For brevity we omit some of the calculations and refer the reader to [75] and [24]. Let E_1 and E_2 denote error terms of the form

$$|E_1| \leq C_1 \operatorname{tr}_{g'} g$$

$$|E_2| \leq C_2 (\operatorname{tr}_{g'} g)(\operatorname{tr}_g g')$$

where C_1 and C_2 are constants depending only on the initial data. We call such a constant depending only on (M,g) and $\sup_M F$ a uniform constant. We remark that by the flow equation (2.1) and estimate (2.6), an error term of type E_1 is also of type E_2 and a uniform constant is of type E_1. In general, C will denote a uniform constant whose definition may change from line to line. For a function φ on M, we write φ_i for the ordinary derivative

$$\varphi_i = \partial_i \varphi.$$

Similarly, φ_t will denote the time derivative of φ. If f is a function on M, we write ∂f for the vector of ordinary derivatives of f.

We define the quantity

$$Q = \log \operatorname{tr}_g g' + e^{A\left(\sup_{M\times[0,T)}\tilde{\varphi} - \tilde{\varphi}\right)} \tag{2.13}$$

We note that the form of Q differs here than in [75] and Yau's estimate [81] and Aubin's estimate [1]. They consider a quantity of the form $\log \operatorname{tr}_g g' - A\varphi$. The exponential in the definition of Q helps to control a difficult term in the analysis.

Fix $t' \in [0,T)$. Then let (x_0,t_0) be the point in $M \times [0,t']$ where Q attains its maximum. Notice that if $t_0 = 0$ the result is immediate, so we assume $t_0 > 0$. To start the proof, we need to perform a change of coordinates made possible by the following lemma from [24].

Lemma 2.3.2. *There exists a holomorphic coordinate system centered at x_0 such that for all i and j,*

$$g_{i\bar{j}}(x_0) = \delta_{ij}, \quad \partial_j g_{i\bar{i}}(x_0) = 0, \tag{2.14}$$

and also such that the matrix $\varphi_{i\bar{j}}(x_0,t_0)$ is diagonal.

Applying $\triangle' - \frac{\partial}{\partial t}$ to Q,

$$\left(\triangle' - \frac{\partial}{\partial t}\right) Q = \frac{\triangle' \operatorname{tr}_g g'}{\operatorname{tr}_g g'} - \frac{|\partial \operatorname{tr}_g g'|^2_{g'}}{(\operatorname{tr}_g g')^2} - \frac{\triangle \frac{\partial \varphi}{\partial t}}{\operatorname{tr}_g g'} + A\frac{\partial \tilde{\varphi}}{\partial t} e^{A(\sup_{M\times[0,T)} \tilde{\varphi} - \tilde{\varphi})} + \triangle' e^{A(\sup_{M\times[0,T)} \tilde{\varphi} - \tilde{\varphi})}. \tag{2.15}$$

First we will control the first and third terms in (2.15) simultaneously. We apply the complex Laplacian $\triangle$ to the complex Monge-Ampère equation:

$$\begin{aligned}\triangle \frac{\partial \varphi}{\partial t} &= -g^{k\bar{l}} g'^{p\bar{j}} g'^{i\bar{q}} \partial_k g'_{p\bar{q}} \partial_{\bar{l}} g'_{i\bar{j}} + g^{k\bar{l}} g'^{i\bar{j}} \partial_k \partial_{\bar{l}} g'_{i\bar{j}} + g^{k\bar{l}} g^{p\bar{j}} g^{i\bar{q}} \partial_k g_{p\bar{q}} \partial_{\bar{l}} g_{i\bar{j}} \\ &\quad - g^{k\bar{l}} g^{i\bar{j}} \partial_k \partial_{\bar{l}} g_{i\bar{j}} - \triangle F \\ &= \sum_{i,k} g'^{i\bar{i}} \varphi_{i\bar{i}k\bar{k}} - \sum_{i,j,k} g'^{i\bar{i}} g'^{j\bar{j}} \partial_k g'_{i\bar{j}} \partial_{\bar{k}} g'_{j\bar{i}} + E_1. \end{aligned} \tag{2.16}$$

For the first term in (2.15), following a calculation in [75] (see equation (2.6) in [75]) gives

$$\triangle' \operatorname{tr}_g g' = \sum_{i,k} g'^{i\bar{i}} \varphi_{i\bar{i}k\bar{k}} - 2\operatorname{Re}\left(\sum_{i,j,k} g'^{i\bar{i}} \partial_{\bar{i}} g_{j\bar{k}} \varphi_{k\bar{j}i}\right) + E_2. \tag{2.17}$$

We will now handle the $2\operatorname{Re}\left(\sum_{i,j,k} g'^{i\bar{i}} \partial_{\bar{i}} g_{j\bar{k}} \varphi_{k\bar{j}i}\right)$ term in (2.17) using a trick from [24]. Using Lemma 2.3.2, at the point (x_0,t_0),

$$\sum_{i,j,k} g'^{i\bar{i}} \partial_{\bar{i}} g_{j\bar{k}} \varphi_{k\bar{j}i} = \sum_i \sum_{j \neq k} g'^{i\bar{i}} \partial_{\bar{i}} g_{j\bar{k}} \partial_k g'_{i\bar{j}} + E_1. \tag{2.18}$$

Hence,

$$\left|2\,\mathrm{Re}\left(\sum_{i,j,k} g'^{i\bar{i}}\partial_{\bar{i}} g_{j\bar{k}}\varphi_{k\bar{j}i}\right)\right| \leq \sum_i \sum_{j\neq k} g'^{i\bar{i}} g'^{j\bar{j}} \partial_k g'_{i\bar{j}} \partial_{\bar{k}} g'_{j\bar{i}} + \sum_i \sum_{j\neq k} g'^{i\bar{i}} g'_{j\bar{j}} \partial_{\bar{i}} g_{j\bar{k}} \partial_i g_{k\bar{j}} + E_1 \leq \sum_i \sum_{j\neq k} g'^{i\bar{i}} g'^{j\bar{j}} \partial_k g'_{i\bar{j}} \partial_{\bar{k}} g'_{j\bar{i}} + E_2. \tag{2.19}$$

Putting together (2.16), (2.17), and (2.19) gives

$$\triangle' \operatorname{tr}_g g' - \triangle \frac{d\varphi}{dt} \geq \sum_{i,j} g'^{i\bar{i}} g'^{j\bar{j}} \partial_j g'_{i\bar{j}} \partial_{\bar{j}} g'_{j\bar{i}} + E_2. \tag{2.20}$$

Now we will control the $\frac{|\partial \operatorname{tr}_g g'|^2_{g'}}{(\operatorname{tr}_g g')^2}$ term in (2.15). By Lemma 2.3.2 we have at (x_0, t_0),

$$\partial_i \operatorname{tr}_g g' = \partial_i \triangle\varphi = \partial_i \sum_j \varphi_{j\bar{j}} = \sum_j \partial_j \varphi_{i\bar{j}} = \sum_j \partial_j g'_{i\bar{j}} - \sum_j \partial_j g_{i\bar{j}}. \tag{2.21}$$

So

$$\frac{|\partial \operatorname{tr}_g g'|^2_{g'}}{\operatorname{tr}_g g'} = \frac{1}{tr_g g'} \sum_{i,j,k} g'^{i\bar{i}} \partial_j g'_{i\bar{j}} \partial_{\bar{k}} g'_{k\bar{i}} - \frac{2}{tr_g g'} \mathrm{Re}\left(\sum_{i,j,k} g'^{i\bar{i}} \partial_j g_{i\bar{j}} \partial_{\bar{k}} g'_{k\bar{i}}\right) + E_1. \tag{2.22}$$

As in Yau's second order estimate, we use Cauchy-Schwarz on the first term in (2.22) (see [75] equation (2.15) for the exact calculation).

$$\frac{1}{tr_g g'} \sum_{i,j,k} g'^{i\bar{i}} \partial_j g'_{i\bar{j}} \partial_{\bar{k}} g'_{k\bar{i}} \leq \sum_{i,j} g'^{i\bar{i}} g'^{j\bar{j}} \partial_j g'_{i\bar{j}} \partial_{\bar{j}} g'_{j\bar{i}}. \tag{2.23}$$

To deal with the second term in (2.22), since (x_0, t_0) is the maximum point of Q, $\partial_{\bar{i}} Q = 0$ implies

$$\frac{1}{\operatorname{tr}_g g'} \sum_k \partial_{\bar{i}} g'_{k\bar{k}} = A \partial_{\bar{i}} \varphi e^{A(\sup_{M\times[0,T)} \tilde{\varphi} - \tilde{\varphi})}. \tag{2.24}$$

Using equations (2.24) and (2.21) we can bound the difficult term:

$$\begin{aligned}
&\left|\frac{2}{\operatorname{tr}_g g'}\operatorname{Re}\left(\sum_{i,j,k} g'^{i\bar{i}}\partial_j g_{i\bar{j}}\partial_{\bar{k}} g'_{k\bar{i}}\right)\right| \\
&\quad = \left|\frac{A}{\operatorname{tr}_g g'} e^{A(\sup_{M\times[0,T)}\tilde{\varphi}-\tilde{\varphi})} 2\operatorname{Re}\left(\sum_{i,j,k} g'^{i\bar{i}}\partial_j g_{i\bar{j}}\partial_{\bar{i}}\varphi\right)\right| + E_1 \\
&\quad \leq A^2|\partial\varphi|^2_{g'} e^{A(\sup_{M\times[0,T)}\tilde{\varphi}-\tilde{\varphi})} + \frac{C(\operatorname{tr}_{g'} g)}{(\operatorname{tr}_g g')^2} e^{A(\sup_{M\times[0,T)}\tilde{\varphi}-\tilde{\varphi})} + E_1 \\
&\quad \leq A^2|\partial\varphi|^2_{g'} e^{A(\sup_{M\times[0,T)}\tilde{\varphi}-\tilde{\varphi})} + C(\operatorname{tr}_{g'} g) e^{A(\sup_{M\times[0,T)}\tilde{\varphi}-\tilde{\varphi})} + E_1, \qquad (2.25)
\end{aligned}$$

where for the last inequality we used the fact that $\operatorname{tr}_g g'$ is bounded from below away from zero by the flow equation (2.1) and estimate (2.6).

Plugging (2.23) and (2.25) into (2.22) gives

$$\begin{aligned}
\frac{|\partial \operatorname{tr}_g g'|^2_{g'}}{(\operatorname{tr}_g g')^2} &\leq \frac{1}{(tr_g g')^2}\sum_{i,j} g'^{i\bar{i}} g'^{j\bar{j}} \partial_j g'_{i\bar{j}} \partial_{\bar{j}} g'_{j\bar{i}} + A^2|\partial\varphi|^2_{g'} e^{A(\sup_{M\times[0,T)}\tilde{\varphi}-\tilde{\varphi})} \\
&\quad + C(\operatorname{tr}_{g'} g) e^{A(\sup_{M\times[0,T)}\tilde{\varphi}-\tilde{\varphi})} + E_1. \qquad (2.26)
\end{aligned}$$

By combining (2.20) and (2.26) with (2.15) at the point (x_0, t_0), we get the inequality

$$\begin{aligned}
0 &\geq \frac{1}{tr_g g'}\left(\sum_{i,j} g'^{i\bar{i}} g'^{j\bar{j}} \partial_j g'_{i\bar{j}} \partial_{\bar{j}} g'_{j\bar{i}} + E_2\right) - \frac{1}{\operatorname{tr}_g g'}\sum_{i,j} g'^{i\bar{i}} g'^{j\bar{j}} \partial_j g'_{i\bar{j}} \partial_{\bar{j}} g'_{j\bar{i}} \\
&\quad - A^2|\partial\varphi|^2_{g'} e^{A(\sup_{M\times[0,T)}\tilde{\varphi}-\tilde{\varphi})} - \operatorname{tr}_{g'} g e^{A(\sup_{M\times[0,T)}\tilde{\varphi}-\tilde{\varphi})} + E_1 \\
&\quad + A\frac{\partial\tilde{\varphi}}{\partial t} e^{A(\sup_{M\times[0,T)}\tilde{\varphi}-\tilde{\varphi})} + \left(-An + A\operatorname{tr}_{g'} g + A^2\,|\partial\varphi|^2_{g'}\right) e^{A(\sup_{M\times[0,T)}\tilde{\varphi}-\tilde{\varphi})} \\
&\geq -A(C+n) e^{A(\sup_{M\times[0,T)}\tilde{\varphi}-\tilde{\varphi})} + (A-1)\operatorname{tr}_{g'} g e^{A(\sup_{M\times[0,T)}\tilde{\varphi}-\tilde{\varphi})} - C_1 \operatorname{tr}_{g'} g \\
&\geq -A(C+n) e^{A(\sup_{M\times[0,T)}\tilde{\varphi}-\tilde{\varphi})} + (A-1-C_1)\operatorname{tr}_{g'} g. \qquad (2.27)
\end{aligned}$$

Taking A large enough so that

$$(A-1-C_1) > 0$$

implies that at (x_0, t_0),

$$\operatorname{tr}_{g'} g(x_0, t_0) \leq C e^{A(\sup_{M\times[0,T)}\tilde{\varphi} - \inf_{M\times[0,T)}\tilde{\varphi})}. \qquad (2.28)$$

Then

$$\begin{aligned}\operatorname{tr}_g g'(x_0,t_0) &\leq \frac{1}{(n-1)!}\left(\operatorname{tr}_{g'} g\right)^{n-1}\frac{\det g'}{\det g}\\ &= \frac{1}{(n-1)!}\left(\operatorname{tr}_{g'} g\right)^{n-1} e^{F-\frac{\partial\varphi}{\partial t}}\\ &\leq Ce^{A(n-1)(\sup_{M\times[0,T)}\tilde{\varphi}-\inf_{M\times[0,T)}\tilde{\varphi})}. \end{aligned} \tag{2.29}$$

For all (x,t) in $M\times[0,t']$,

$$\begin{aligned}&\log\operatorname{tr}_g g'(x,t)+e^{A(\sup_{M\times[0,T)}\tilde{\varphi}-\tilde{\varphi}(x,t))}\\ &\qquad\leq \log\left(Ce^{A(n-1)(\sup_{M\times[0,T)}\tilde{\varphi}-\inf_{M\times[0,T)}\tilde{\varphi})}\right)+e^{A(\sup_{M\times[0,T)}\tilde{\varphi}-\inf_{M\times[0,T)}\tilde{\varphi})}\end{aligned}$$

and so

$$\operatorname{tr}_g g' \leq C_1 e^{C_2(\sup_{M\times[0,T)}\tilde{\varphi}-\inf_{M\times[0,T)}\tilde{\varphi})} e^{\left(e^{A\left(\sup_{M\times[0,T)}\tilde{\varphi}-\inf_{M\times[0,T)}\tilde{\varphi}\right)}-e^{A\left(\sup_{M\times[0,T)}\tilde{\varphi}-\tilde{\varphi}\right)}\right)}. \tag{2.30}$$

□

2.4 The Hölder estimate for the metric

The estimate in this section is local, so it suffices to work in a domain in $\mathbb{C}^n$. To fix some notation, define the parabolic distance function between two points (x,t_1) and (y,t_2) in $\mathbb{C}^n\times[0,T)$ to be $|(x,t_1)-(y,t_2)|=\max(|x-y|,|t_1-t_2|^{1/2})$.

For a domain $\Omega\in\mathbb{C}^n\times[0,T)$ and a real number $\alpha\in(0,1)$, define for a function φ on $\mathbb{C}^n\times[0,T)$,

$$[\varphi]_{\alpha,(x_0,t_0)}=\sup_{(x,t)\in\Omega\setminus\{(x_0,t_0)\}}\frac{|\varphi(x,t)-\varphi(x_0,t_0)|}{|(x,t)-(x_0,t_0)|^{\alpha}}$$

and

$$[\varphi]_{\alpha,\Omega}=\sup_{(x,t)\in\Omega}[\varphi]_{\alpha,(x,t)}. \tag{2.31}$$

We will show that

$$[g'_{i\bar{j}}]_{\alpha,\Omega}\leq C$$

for an appropriate choice of Ω. The smoothness of φ and $\tilde{\varphi}$ will follow. Given the Hölder bound for the metric and the second order estimate for $\tilde{\varphi}$, we can differentiate the flow and apply Schauder estimates to achieve higher regularity.

Lemma 2.4.1. *Let* φ *be a solution to the flow* (2.1) *and* $g'_{i\bar{j}} = g_{i\bar{j}} + \varphi_{i\bar{j}}$. *Fix* $\varepsilon > 0$. *Then there exists* $\alpha \in (0,1)$ *and a constant* C *depending only on the initial data and* ε *such that*

$$[g'_{i\bar{j}}]_{\alpha,\Omega} \leq C \tag{2.32}$$

where $\Omega = M \times [\varepsilon, T)$.

We apply a method due to Evans [16] and Krylov [32]. The proof itself is essentially contained in [34] and [25], but only in the case where the manifold is $\mathbb{R}^n$. Hence we produce a self-contained proof in the notation of this problem. The method of this proof follows closely with the analogous estimate in [75] and [54]. The main issue is applying the correct Harnack inequality to get the estimate.

Proof. Let $B \in \mathbb{C}^n$ be an open ball about the origin. Fix a point $t_0 \in [\varepsilon, T)$. To prove (2.32) it suffices to show that for sufficiently small $R > 0$ there exists a uniform C and $\delta > 0$ such that

$$\sum_{i=1}^{n} \operatorname{osc}_{Q(R)}(\varphi_{\gamma_i\bar{\gamma}_i}) + \operatorname{osc}_{Q(R)}(\varphi_t) \leq CR^{\delta}$$

where $\{\gamma_i\}$ is a basis for $\mathbb{C}^n$ and $Q(R)$ is the parabolic cylinder

$$Q(R) = \{(x,t) \in B \times [0,T) | |x| \leq R, t_0 - R^2 \leq t \leq t_0\}.$$

We rewrite the flow as

$$\frac{\partial \varphi}{\partial t} = \log\det g'_{i\bar{j}} + H \tag{2.33}$$

where $H = -\log\det g_{i\bar{j}} - F$. We define the operator Φ on a matrix A by

$$\Phi(A) = \log\det A,$$

then (2.33) becomes

$$\frac{\partial \varphi}{\partial t} = \Phi(g') + H. \tag{2.34}$$

By the concavity of Φ, for all (x, t_1) and (y, t_2) in $B \times [0, T)$,

$$\sum \frac{\partial \Phi}{\partial a_{i\bar{j}}}(g'(y,t_2)) \left(g'_{i\bar{j}}(x,t_1) - g'_{i\bar{j}}(y,t_2)\right) \geq \frac{\partial \varphi}{\partial t}(x,t_1) - \frac{\partial \varphi}{\partial t}(y,t_2) - H(x) + H(y).$$

The Mean Value Theorem applied to H shows that

$$\frac{\partial \varphi}{\partial t}(x,t_1)-\frac{\partial \varphi}{\partial t}(y,t_2)+\sum \frac{\partial \Phi}{\partial a_{i\bar{j}}}(g'(y,t_2))\left(g'_{i\bar{j}}(y,t_2)-g'_{i\bar{j}}(x,t_1)\right) \le C|x-y|. \quad (2.35)$$

Now we must recall a lemma from linear algebra.

Lemma 2.4.2. *There exists a finite number N of unit vectors $\gamma_\nu = (\gamma_{\nu 1}, \dots, \gamma_{\nu n}) \in \mathbb{C}^n$ and real-valued functions β_ν on $B \times [0,T)$, for $\nu = 1, 2, \dots, N$ with*

(i) $0 < C_1 \le \beta_\nu \le C_2$

(ii) $\gamma_1, \dots, \gamma_N$ containing an orthonormal basis of $\mathbb{C}^n$

such that

$$\frac{\partial \Phi}{\partial a_{i\bar{j}}}(g'(y,t_2)) = \sum_{\nu=1}^{N} \beta_\nu(y,t_2)\gamma_{\nu i}\overline{\gamma_{\nu j}}.$$

We define for $\nu = 1, \dots, N$,

$$w_\nu = \partial_{\gamma_\nu}\partial_{\bar{\gamma}_\nu}\varphi = \sum_{i,j=1}^{n} \gamma_{\nu i}\overline{\gamma_{\nu j}}\varphi_{i\bar{j}}.$$

We write $w_0 = -\frac{\partial \varphi}{\partial t}$ and $\beta_0 = 1$. Then using the linear algebra lemma, (2.35) can be rewritten as

$$\sum_{\nu=0}^{N} \beta_\nu(y,t_2)\left(w_\nu(y,t_2) - w_\nu(x,t_1)\right) \le C|x-y|. \quad (2.36)$$

Letting γ be an arbitrary unit vector in $\mathbb{C}^n$, we differentiate the flow (2.1) along γ and $\bar{\gamma}$:

$$\begin{aligned} \frac{\partial \varphi_{\gamma\bar{\gamma}}}{\partial t} &= \frac{\partial^2 \Phi}{\partial a_{i\bar{j}}\partial a_{k\bar{l}}}(g')g'_{i\bar{j}\gamma}g'_{k\bar{l}\bar{\gamma}} + \frac{\partial \Phi}{\partial a_{i\bar{j}}}(g')g'_{i\bar{j}\gamma\bar{\gamma}} + H_{\gamma\bar{\gamma}} \\ &\le g'^{i\bar{j}}g'_{i\bar{j}\gamma\bar{\gamma}} + H_{\gamma\bar{\gamma}} \quad (2.37) \end{aligned}$$

where on the last line we used the concavity of Φ and the fact that $\frac{\partial \Phi}{\partial a_{i\bar{j}}}(g') = g'^{i\bar{j}}$. Applying $\frac{\partial}{\partial t}$ to (2.34) gives

$$\frac{\partial \varphi_t}{\partial t} = g'^{i\bar{j}}\varphi_{i\bar{j}t}. \quad (2.38)$$

From (2.37) and (2.38) we have a bounded function h (depending on $g'^{i\bar{j}}$ which is bounded by Theorem 3.1) such that

$$-\frac{\partial w_\nu}{\partial t} + g'^{i\bar{j}}\partial_i\partial_{\bar{j}}w_\nu \ge h. \quad (2.39)$$

Recall that t_0 is a fixed point in $[\varepsilon, T)$. Pick $R > 0$ small enough such that $t_0 - 5R^2 > t_0/2$. We define another parabolic cylinder

$$\Theta(R) = \{(x,t) \in B \times [0,T) | |x| < R, t_0 - 5R^2 \le t \le t_0 - 4R^2\}.$$

For $s = 1, 2$ and $\nu = 0, 1, \ldots, N$, let

$$M_{s\nu} = \sup_{Q(sR)} w_\nu, \ m_{s\nu} = \inf_{Q(sR)} w_\nu,$$

and

$$\psi(sR) = \sum_{\nu=0}^{N} \left(M_{s\nu} - m_{s\nu}\right).$$

We let l be an integer such that $0 \le l \le N$ and $v = M_{2l} - w_l$. To continue we need Theorem 7.37 from [34]. We say that $v \in W^{2,1}_{2n+1}$ if $v_x, v_{ij}, v_{i\bar{j}}, v_{\bar{i}\bar{j}}$, and v_t are in L^{2n+1}. We restate the theorem as follows.

Lemma 2.4.3. *Suppose that $v(x,t) \in W^{2,1}_{2n+1}$ satisfies*

$$-\frac{\partial v}{\partial t} + g'^{i\bar{j}} \partial_i \partial_{\bar{j}} v \le f$$

and $v \ge 0$ on $Q(4R)$. Then there exists a constant C and a $p > 0$ depending only on the bounds of $g'^{i\bar{j}}$ and the eigenvalues of $g'^{i\bar{j}}$ such that

$$\frac{1}{R^{2n+2}} \left(\int_{\Theta(R)} v^p \right)^{1/p} \le C \left(\inf_{Q(R)} v + R^{\frac{2n}{2n+1}} ||f||_{n+1} \right).$$

Since v satisfies $-\frac{\partial v}{\partial t} + g'^{i\bar{j}} \partial_i \partial_{\bar{j}} v \le -h$, we can apply the Harnack inequality to get

$$\frac{1}{R^{2n+2}} \left(\int_{\Theta(R)} (M_{2l} - w_l)^p \right)^{1/p} \le C \left(M_{2l} - M_l + R^{\frac{2n}{2n+1}} \right). \tag{2.40}$$

For every (x, t_1) and (y, t_2) in $Q(2R)$, (2.36) gives

$$\beta_l(y,t_2) \left(w_l(y,t_2) - w_l(x,t_1)\right) \le CR + \sum_{\nu \ne l} \beta_\nu \left(w_\nu(x,t_1) - w_\nu(y,t_2)\right).$$

The definition of m_{2l} allows us to choose (x, t_1) in $Q(2R)$ such that $w_l(x, t_1) \le m_{2l} + \varepsilon$. Since ε is arbitrary,

$$w_l(y,t_2) - m_{2l} \le CR + C_2 \sum_{\nu \ne l} \left(M_{2\nu} - w_\nu(y,t_2)\right).$$

After integrating over $\Theta(R)$ and applying (2.40), we have

$$\begin{aligned}\frac{1}{R^{2n+2}}\left(\int_{\Theta(R)}(w_l-m_{2l})^p\right)^{1/p} &\le \frac{1}{R^{2n+2}}\left(\int_{\Theta(R)}\left(CR+C_2\sum_{\nu\neq l}(M_{2\nu}-w_\nu)\right)^p\right)^{1/p}\\ &\le C_3R+C_4\sum_{\nu\neq l}\frac{1}{R^{2n+2}}\left(\int_{\Theta(R)}(M_{2\nu}-w_\nu)^p\right)^{1/p}\\ &\le C_5\sum_{\nu\neq l}(M_{2\nu}-M_\nu)+C_6R^{\frac{2n}{2n+1}} \qquad (2.41)\end{aligned}$$

where on the last line we used the fact that $R<1$ is small. Adding (2.40) and (2.41) yields

$$\begin{aligned}M_{2l}-m_{2l} &\le C_7\sum_{\nu=0}^{N}(M_{2\nu}-M_\nu)+C_8R^{\frac{2n}{2n+1}}\\ &\le C_7\sum_{\nu=0}^{N}(M_{2\nu}-M_\nu+m_\nu-m_{2\nu})+C_8R^{\frac{2n}{2n+1}}\\ &= C_7\left(\psi(2R)-\psi(R)\right)+C_8R^{\frac{2n}{2n+1}}.\end{aligned}$$

Summing over l shows that

$$\psi(2R)\le C_9\left(\psi(2R)-\psi(R)\right)+C_{10}R^{\frac{2n}{2n+1}}$$

and thus for some $0<\lambda<1$,

$$\psi(R)\le\lambda\psi(2R)+C_{11}R^{\frac{2n}{2n+1}}.$$

Applying a standard iteration argument (see Chapter 8 in [25]) shows that

$$\psi(R)\le CR^\delta$$

for some $\delta>0$, completing the proof. □

2.5 Long time existence and smoothness of the normalized solution

In this section we show that the solution φ and its normalization $\tilde{\varphi}$ are smooth and exist for all time, hence proving part of Theorem 2.1.1. The proof uses a standard bootstrapping argument.

Lemma 2.5.1. *Let* (M, g) *be a Hermitian manifold and* F *a smooth function on* M*. Let* φ *be a solution to the flow*

$$\frac{\partial \varphi}{\partial t} = \log \frac{\det(g_{i\bar{j}} + \varphi_{i\bar{j}})}{\det(g_{i\bar{j}})} - F$$

and let $\tilde{\varphi} = \varphi - \int_M \varphi\, \omega^n$*. Then there are uniform* C^∞ *estimates for* $\tilde{\varphi}$ *on* $[0, T)$*. Moreover,* $T = \infty$*.*

Proof. Differentiating the flow with respect to z^k gives

$$\frac{\partial \varphi_k}{\partial t} = g'^{i\bar{j}} \partial_i \partial_{\bar{j}} \varphi_k - F_k - \frac{\partial}{\partial z^k} \log \det g_{i\bar{j}}. \tag{2.42}$$

Lemma 2.3.1 implies that the above equation is uniformly parabolic. Lemma 2.4.1 shows that the coefficients in the above equation are Hölder continuous with exponent α. Using the Schauder estimate (see Theorem 4.9 in [34], for example) gives a uniform parabolic $C^{2+\alpha}$ bound on φ_k. Similarly, one obtains a uniform parabolic $C^{2+\alpha}$ estimate for $\varphi_{\bar{k}}$.

But the better differentiability of φ allows us to differentiate the flow again and obtain a uniformly parabolic equation with Hölder continuous coefficients. The Schauder estimate will give a uniform parabolic $C^{2+\alpha}$ estimate on $\varphi_{kl}, \varphi_{k\bar{l}}$, and $\varphi_{\bar{k}\bar{l}}$. Repeated application shows that $\tilde{\varphi}$ is uniformly bounded in C^∞. Hence $\tilde{\varphi}$ and thus φ are smooth. We note that φ is not necessarily bounded in C^0. The above iterations only provide regularity for the derivatives of φ.

To see that $T = \infty$, suppose that for $T < \infty$, $[0, T)$ is the maximal interval for the existence of the solution. Since $\tilde{\varphi}$ is smooth, we can apply short time existence to extend the flow for $\tilde{\varphi}$ to $[0, T + \varepsilon)$, a contradiction.

□

2.6 The Harnack inequality

We begin this section by proving lemmas analogous to those of Li and Yau [33] for the equation $\frac{\partial u}{\partial t} = g'^{i\bar{j}} \partial_i \partial_{\bar{j}} u$ for a positive function u on a Hermitian manifold (see [80] for the proof of these lemmas in the Kähler case). The lemmas lead to a Harnack inequality, which in turn shows that the time derivative of $\tilde{\varphi}$ decays exponentially. This allows us to prove the convergence of $\tilde{\varphi}$ as t tends to infinity.

In this section, we again use the notation $u_t = \frac{\partial u}{\partial t}$ and $u_i = \partial_i u$ for the ordinary derivatives of a function u on M.

Let u be a positive function on M. Consider the heat type equation

$$u_t = g'^{i\bar{j}} u_{i\bar{j}}$$

where $g'_{i\bar{j}}$ denotes the time dependent metric $g_{i\bar{j}} + \varphi_{i\bar{j}}$. Define $\tilde{\varphi} = \varphi - \int_M \varphi \omega^n$.

Define $f = \log u$ and $F = t(|\partial f|^2 - \alpha f_t)$ where $1 < \alpha < 2$. We remark that this F is different from the one in equation (2.1). Then

$$g'^{i\bar{j}} f_{i\bar{j}} - f_t = -|\partial f|^2$$

where ∂f is the vector containing the ordinary derivatives of f and

$$|\partial f|^2 = g'^{i\bar{j}} \partial_i f \partial_{\bar{j}} f.$$

Also write

$$\langle X, Y \rangle = g'^{i\bar{j}} X_i Y_{\bar{j}}$$

for the inner product of two vectors X and Y with respect to $g'_{i\bar{j}}$.

We now prove an estimate that will be useful in applying the maximum principle to F.

Lemma 2.6.1. *There exist constants C_1 and C_2 depending only on the bounds of the metric g' such that for $t > 0$,*

$$g'^{k\bar{l}} F_{k\bar{l}} - F_t \geq \frac{t}{2n} \left(|\partial f|^2 - f_t \right)^2 - 2 \operatorname{Re} \langle \partial f, \partial F \rangle - \left(|\partial f|^2 - \alpha f_t \right) - C_1 t |\partial f|^2 - C_2 t.$$

Proof. First calculate $F = -t g'^{i\bar{j}} f_{i\bar{j}} - t(\alpha - 1) f_t$. Then

$$(g'^{i\bar{j}} f_{i\bar{j}})_t = \frac{1}{t^2} F - \frac{1}{t} F_t - (\alpha - 1) f_{tt} \tag{2.43}$$

and

$$\begin{aligned} F_t &= |\partial f|^2 - \alpha f_t + t \left(g'^{i\bar{j}} f_{ti} f_{\bar{j}} + g'^{i\bar{j}} f_i f_{t\bar{j}} + \left(\frac{\partial}{\partial t} g'^{i\bar{j}} \right) f_i f_{\bar{j}} - \alpha f_{tt} \right) \\ &= |\partial f|^2 - \alpha f_t + 2t \operatorname{Re} \langle \partial f, \partial(f_t) \rangle + t \left(\frac{\partial}{\partial t} g'^{i\bar{j}} \right) f_i f_{\bar{j}} - \alpha t f_{tt}. \end{aligned} \tag{2.44}$$

We calculate $g'^{k\bar{l}}F_{k\bar{l}}$ to get the desired estimate.

$$g'^{k\bar{l}}F_{k\bar{l}} = tg'^{k\bar{l}}\Big[\left(g'^{i\bar{j}}\right)_{k\bar{l}} f_i f_{\bar{j}} + \left(g'^{i\bar{j}}\right)_k f_{i\bar{l}} f_{\bar{j}} + \left(g'^{i\bar{j}}\right)_k f_i f_{\bar{j}\bar{l}} + \left(g'^{i\bar{j}}\right)_{\bar{l}} f_{ik} f_{\bar{j}} + g'^{i\bar{j}} f_{ik\bar{l}} f_{\bar{j}}$$
$$+ g'^{i\bar{j}} f_{ik} f_{\bar{j}\bar{l}} + \left(g'^{i\bar{j}}\right)_{\bar{l}} f_i f_{\bar{j}k} + g'^{i\bar{j}} f_{i\bar{l}} f_{\bar{j}k} + g'^{i\bar{j}} f_i f_{\bar{j}k\bar{l}} - \alpha f_{tk\bar{l}}\Big]. \tag{2.45}$$

Now we control all of the above terms using the bounds on the metric obtained in Lemma 2.3.1 and the higher order bounds from Lemma 2.5.1. For the first term of (2.45),

$$\left| tg'^{k\bar{l}} \left(g'^{i\bar{j}}\right)_{k\bar{l}} f_i f_{\bar{j}} \right| \le C_1 t |\partial f|^2.$$

Let $\varepsilon > 0$. We bound the second and third terms of (2.45) with the inequalities

$$\left| tg'^{k\bar{l}} \left(g'^{i\bar{j}}\right)_k f_{i\bar{l}} f_{\bar{j}} \right| \le \frac{t}{\varepsilon} |\partial f|^2 + t\varepsilon |\partial\bar{\partial} f|^2$$

and

$$\left| tg'^{k\bar{l}} \left(g'^{i\bar{j}}\right)_k f_i f_{\bar{j}\bar{l}} \right| \le \frac{t}{\varepsilon} |\partial f|^2 + t\varepsilon |D^2 f|^2$$

where

$$|\partial\bar{\partial} f|^2 = g'^{k\bar{l}} g'^{i\bar{j}} f_{i\bar{l}} f_{\bar{j}k}, \quad |D^2 f|^2 = g'^{k\bar{l}} g'^{i\bar{j}} f_{ik} f_{\bar{j}\bar{l}}.$$

Term six is equal to $t|D^2 f|^2$ and term eight equals $t|\partial\bar{\partial} f|^2$. The fifth and ninth terms of (2.45) combine to give

$$\begin{aligned} tg'^{k\bar{l}} g'^{i\bar{j}} f_{ik\bar{l}} f_{\bar{j}} + tg'^{k\bar{l}} g'^{i\bar{j}} f_i f_{\bar{j}k\bar{l}} &= 2t \operatorname{Re} \left\langle \partial f, \partial (g'^{k\bar{l}} f_{k\bar{l}}) \right\rangle - tg'^{i\bar{j}} \left(g'^{k\bar{l}}\right)_i f_{k\bar{l}} f_{\bar{j}}. \\ &\quad - tg'^{i\bar{j}} \left(g'^{k\bar{l}}\right)_{\bar{j}} f_i f_{k\bar{l}} \\ &\ge 2t \operatorname{Re} \left\langle \partial f, \partial (g'^{k\bar{l}} f_{k\bar{l}}) \right\rangle - \frac{t}{\varepsilon} |\partial f|^2 - t\varepsilon |\partial\bar{\partial} f|^2. \end{aligned}$$

We use the definition of F to show

$$\begin{aligned} & tg'^{k\bar{l}} g'^{i\bar{j}} f_{ik\bar{l}} f_{\bar{j}} + tg'^{k\bar{l}} g'^{i\bar{j}} f_i f_{\bar{j}k\bar{l}} \\ &\qquad \ge -2 \operatorname{Re} \langle \partial f, \partial F \rangle - 2t(\alpha - 1) \operatorname{Re} \langle \partial f, \partial (f_t) \rangle - \frac{t}{\varepsilon} |\partial f|^2 \\ &\qquad\quad - t\varepsilon |\partial\bar{\partial} f|^2. \end{aligned} \tag{2.46}$$

Applying equation (2.44) to (2.46) gives

$$\begin{aligned}
&tg'^{k\bar{l}}g'^{i\bar{j}}f_{ik\bar{l}}f_{\bar{j}} + tg'^{k\bar{l}}g'^{i\bar{j}}f_i f_{\bar{j}k\bar{l}} \\
&\qquad \geq -2\operatorname{Re}\langle \partial f, \partial F\rangle - (\alpha-1)F_t + (\alpha-1)\left(|\partial f|^2 - \alpha f_t\right) \\
&\qquad\quad + t(\alpha-1)\left(\frac{\partial}{\partial t}g'^{i\bar{j}}\right) f_i f_{\bar{j}} - t\alpha(\alpha-1)f_{tt} - \frac{t}{\varepsilon}|\partial f|^2 - t\varepsilon|\partial\bar{\partial} f|^2 \\
&\qquad \geq -2\operatorname{Re}\langle \partial f, \partial F\rangle - (\alpha-1)F_t + (\alpha-1)\left(|\partial f|^2 - \alpha f_t\right) \\
&\qquad\quad - C_2 t|\partial f|^2 - t\alpha(\alpha-1)f_{tt} - \frac{t}{\varepsilon}|\partial f|^2 - t\varepsilon|\partial\bar{\partial} f|^2.
\end{aligned}$$

The final term of (2.45) becomes, using (2.43)

$$\begin{aligned}
-\alpha t g'^{k\bar{l}} f_{tk\bar{l}} &= \alpha t\left(\frac{\partial}{\partial t}g'^{k\bar{l}}\right) f_{k\bar{l}} - \alpha t\frac{\partial}{\partial t}\left(g'^{k\bar{l}}f_{k\bar{l}}\right) \\
&\geq -\frac{Ct}{\varepsilon} - t\varepsilon|\partial\bar{\partial} f|^2 - \frac{\alpha}{t}F + \alpha F_t + t\alpha(\alpha-1)f_{tt}.
\end{aligned}$$

We put all of the above in to (2.45), which shows that

$$\begin{aligned}
g'^{k\bar{l}}F_{k\bar{l}} \geq F_t &- 2\operatorname{Re}\langle \partial f, \partial F\rangle - \left(|\partial f|^2 - \alpha f_t\right) + t(1-4\varepsilon)|\partial\bar{\partial} f|^2 \\
&+ t(1-2\varepsilon)|D^2 f|^2 - t\left(C_1 + C_2 + \frac{6}{\varepsilon}\right)|\partial f|^2 - \frac{Ct}{\varepsilon}.
\end{aligned}$$

Taking ε sufficiently small and applying the arithmetic-geometric mean inequality

$$|\partial\bar{\partial} f|^2 \geq \frac{1}{n}\left(g'^{k\bar{l}}f_{k\bar{l}}\right)^2 = \frac{1}{n}\left(|\partial f|^2 - f_t\right)^2,$$

we see that

$$g'^{k\bar{l}}F_{k\bar{l}} - F_t \geq \frac{t}{2n}\left(|\partial f|^2 - f_t\right)^2 - 2\operatorname{Re}\langle \partial f, \partial F\rangle - \left(|\partial f|^2 - \alpha f_t\right) - Ct|\partial f|^2 - Ct.$$

□

Using the previous lemma, we derive an estimate which will be used to prove the Harnack inequality.

Lemma 2.6.2. *There exist constants C_1 and C_2 depending only on the bounds of the metric g' such that for $t > 0$,*

$$|\partial f|^2 - \alpha f_t \leq C_1 + \frac{C_2}{t}.$$

Proof. Fix $T > 0$ and let (x_0, t_0) in $M \times [0, T]$ be where F attains its maximum. Note that we can take $t_0 > 0$. Then at (x_0, t_0), from the previous lemma,

$$\frac{t_0}{2n}\left(|\partial f|^2 - f_t\right)^2 - \left(|\partial f|^2 - \alpha f_t\right) \leq C_1 t_0 |\partial f|^2 + C_2 t_0. \tag{2.47}$$

First we assume that $f_t(x_0, t_0) \geq 0$, then the α in the above inequality can be dropped to give

$$\frac{t_0}{2n}\left(|\partial f|^2 - f_t\right)^2 - \left(|\partial f|^2 - f_t\right) \leq C_1 t_0 |\partial f|^2 + C_2 t_0.$$

We factor the above to get

$$\frac{1}{2n}\left(|\partial f|^2 - f_t\right)\left(|\partial f|^2 - f_t - \frac{2n}{t_0}\right) \leq C_1 |\partial f|^2 + C_2.$$

Hence,

$$|\partial f|^2 - f_t \leq C_3 |\partial f| + C_4 + \frac{C_5}{t_0}.$$

There exists a constant C_6 such that

$$C_3 |\partial f| \leq \left(1 - \frac{1}{\alpha}\right) |\partial f|^2 + C_6.$$

We plug this in to the previous inequality, showing that

$$\frac{1}{\alpha}|\partial f|^2 - f_t \leq C_7 + \frac{C_5}{t_0}. \tag{2.48}$$

At the point (x_0, t_0), we have

$$F(x_0, t_0) = t_0\left(|\partial f|^2(x_0, t_0) - \alpha f_t(x_0, t_0)\right) \leq C_8 t_0 + C_5.$$

Hence for all x in M,

$$F(x, T) \leq F(x_0, t_0) \leq C_8 t_0 + C_5 r \leq C_8 T + C_5$$

completing the proof for this case.

Now we consider the case where $f_t(x_0, t_0) < 0$. Using (2.47) at the point (x_0, t_0),

$$\frac{t_0}{2n}|\partial f|^4 - |\partial f|^2 \leq C_1 t_0 |\partial f|^2 + C_2 t_0 - \alpha f_t.$$

We factor the above to get

$$|\partial f|^2\left(\frac{1}{2n}|\partial f|^2 - \frac{1}{t_0} - C_1\right) \leq C_2 - \frac{\alpha}{t_0} f_t.$$

Hence,

$$|\partial f|^2 \le C_3 + \frac{C_4}{t_0} - \frac{1}{2} f_t. \tag{2.49}$$

We use (2.47) again and the condition that $f_t(x_0, t_0) < 0$ to see that

$$\frac{t_0}{2n} f_t^2 + \alpha f_t \le C_1 t_0 |\partial f|^2 + |\partial f|^2 + C_2 t_0.$$

By factoring the above, we show that

$$\frac{1}{2n} (-f_t) \left(-f_t - \frac{2n\alpha}{t_0} \right) \le C_1 |\partial f|^2 + \frac{1}{t_0} |\partial f|^2 + C_2.$$

And so

$$-f_t \le C_5 + \frac{C_6}{t_0} + \frac{1}{2} |\partial f|^2. \tag{2.50}$$

We plug (2.50) in to (2.49), arriving at

$$|\partial f|^2 \le C_3 + \frac{C_4}{t_0} + \frac{C_5}{2} + \frac{C_6}{2t_0} + \frac{1}{4} |\partial f|^2.$$

This provides the following estimate for $|\partial f|^2$:

$$|\partial f|^2 \le C_7 + \frac{C_8}{t_0}. \tag{2.51}$$

Similarly, we can show that

$$-\alpha f_t \le C_9 + \frac{C_{10}}{t_0}. \tag{2.52}$$

We add (2.51) and (2.52) to obtain the estimate

$$|\partial f|^2 - \alpha f_t \le C_{11} + \frac{C_{12}}{t_0}.$$

Repeating the argument after (2.48) completes this case and hence the proof.

□

We use the previous lemma to derive a Harnack inequality similar to that of Li and Yau in the case of a Hermitian manifold.

Lemma 2.6.3. *For* $0 < t_1 < t_2$*,*

$$\sup_{x \in M} u(x, t_1) \le \inf_{x \in M} u(x, t_2) \left(\frac{t_2}{t_1} \right)^{C_2} \exp \left(\frac{C_3}{t_2 - t_1} + C_1 (t_2 - t_1) \right)$$

where C_1, C_2 *and* C_3 *are constants depending only on the bounds of the metric* g'*.*

Proof. Let $x, y \in M$, and define γ to be the minimal geodesic (with respect to the initial metric $g_{i\bar{j}}$) with $\gamma(0) = y$ and $\gamma(1) = x$. Define a path $\zeta : [0,1] \to M \times [t_1, t_2]$ by $\zeta(s) = (\gamma(s), (1-s)t_2 + st_1)$. Then using Lemma 2.6.2,

$$\begin{aligned}
\log \frac{u(x,t_1)}{u(y,t_2)} &= \int_0^1 \frac{d}{ds} f(\zeta(s))\ ds \\
&= \int_0^1 \left(\langle \dot{\gamma}, 2\partial f\rangle - (t_2 - t_1) f_t\right)\ ds \\
&\leq \int_0^1 -\frac{t_2 - t_1}{\alpha}\left(|\partial f| - \frac{\alpha|\dot{\gamma}|}{(t_2 - t_1)}\right)^2 + \frac{\alpha|\dot{\gamma}|^2}{(t_2 - t_1)} \\
&\quad + C_1(t_2 - t_1) + C_2\frac{t_2 - t_1}{t}\ ds \\
&\leq \int_0^1 \frac{C_{19}}{t_2 - t_1} + C_{17}(t_2 - t_1) + C_{18}\frac{t_2 - t_1}{t}\ ds \\
&= \frac{C_3}{t_2 - t_1} + C_1(t_2 - t_1) + C_2 \log\left(\frac{t_2}{t_1}\right)
\end{aligned}$$

Exponentiating both sides completes the proof. □

2.7 Convergence of the flow

With the Harnack inequality, we complete the proof of the main theorem by showing the convergence of $\tilde{\varphi}$ (cf. [6]).

Proof. Define $u = \frac{\partial \varphi}{\partial t}$. Then

$$\frac{\partial u}{\partial t} = g'^{i\bar{j}} \partial_i \partial_{\bar{j}} u.$$

Let m be a positive integer and define

$$\xi_m(x,t) = \sup_{y \in M} u(y, m-1) - u(x, m-1+t)$$

$$\psi_m(x,t) = u(x, m-1+t) - \inf_{y \in M} u(y, m-1).$$

These functions satisfy the heat type equations

$$\frac{\partial \xi_m}{\partial t} = g'^{i\bar{j}}(m-1+t)\partial_i \partial_{\bar{j}} \xi_m$$

$$\frac{\partial \psi_m}{\partial t} = g'^{i\bar{j}}(m-1+t)\partial_i \partial_{\bar{j}} \psi_m.$$

First consider the case where $u(x, m-1)$ is not constant. Then ξ_m is positive for some x in M at time $t = 0$. By the maximum principle, ξ_m must be positive for all x in M when $t > 0$. Similarly, ψ_m is positive everywhere when $t > 0$. Hence we can apply Lemma 2.6.3 with $t_1 = \frac{1}{2}$ and $t_2 = 1$ to get

$$\sup_{x\in M} u(x, m-1) - \inf_{x\in M} u\left(x, m - \frac{1}{2}\right) \leq C\left(\sup_{x\in M} u(x, m-1) - \sup_{x\in M} u(x, m)\right)$$

$$\sup_{x\in M} u\left(x, m - \frac{1}{2}\right) - \inf_{x\in M} u\left(x, m-1\right) \leq C\left(\inf_{x\in M} u(x, m) - \inf_{x\in M} u(x, m-1)\right).$$

We define the oscillation $\theta(t) = \sup_{x\in M} u(x,t) - \inf_{x\in M} u(x,t)$. Adding the above inequalities gives

$$\theta(m-1) + \theta\left(m - \frac{1}{2}\right) \leq C\left(\theta(m-1) - \theta(m)\right).$$

Rearranging and setting $\delta = \frac{C-1}{C} < 1$ yields

$$\theta(m) \leq \delta\theta(m-1).$$

By induction,

$$\theta(t) \leq Ce^{-\eta t}$$

where $\eta = -\log\delta$. Note that if $u(x, m-1)$ is constant, this inequality is still true.

Fix (x,t) in $M \times [0, \infty)$. Since

$$\int_M \frac{\partial\tilde{\varphi}}{\partial t}\omega^n = 0,$$

there exists a point y in M such that $\frac{\partial\tilde{\varphi}}{\partial t}(y,t) = 0$.

$$\begin{aligned}
\left|\frac{\partial\tilde{\varphi}}{\partial t}(x,t)\right| &= \left|\frac{\partial\tilde{\varphi}}{\partial t}(x,t) - \frac{\partial\tilde{\varphi}}{\partial t}(y,t)\right| \\
&= \left|\frac{\partial\varphi}{\partial t}(x,t) - \frac{\partial\varphi}{\partial t}(y,t)\right| \\
&\leq Ce^{-\eta t}.
\end{aligned}$$

Consider the quantity $Q_2 = \tilde{\varphi} + \frac{C}{\eta}e^{-\eta t}$. Then by construction,

$$\frac{\partial Q_2}{\partial t} \leq 0.$$

Since Q_2 is bounded and monotonically decreasing, it tends to a limit as $t \to \infty$, call it $\tilde{\varphi}_\infty$. But

$$\lim_{t\to\infty} \tilde{\varphi} = \lim_{t\to\infty} Q_2 - \lim_{t\to\infty} \frac{C}{\eta} e^{-\eta t} = \tilde{\varphi}_\infty.$$

To show that the convergence of $\tilde{\varphi}$ to $\tilde{\varphi}_\infty$ is actually C^∞, suppose not. Then there exists a time sequence $t_m \to \infty$ such that for some $\varepsilon > 0$ and some integer k,

$$||\tilde{\varphi}(x, t_m) - \tilde{\varphi}_\infty||_{C^k} > \varepsilon, \quad \forall m. \tag{2.53}$$

However, since $\tilde{\varphi}$ is bounded in C^∞ there exists a subsequence $t_{m_j} \to \infty$ such that $\tilde{\varphi}(x, t_{m_j}) \to \tilde{\varphi}'_\infty$ as $j \to \infty$ for some smooth function $\tilde{\varphi}'_\infty$. By (2.53), $\tilde{\varphi}'_\infty \neq \tilde{\varphi}_\infty$. This is a contradiction, since $\tilde{\varphi} \to \tilde{\varphi}_\infty$ pointwise. Hence the convergence of $\tilde{\varphi}$ to $\tilde{\varphi}_\infty$ is C^∞.

We observe that $\tilde{\varphi}$ solves the parabolic flow

$$\frac{\partial \tilde{\varphi}}{\partial t} = \log \frac{\det(g_{i\bar{j}} + \partial_i \partial_{\bar{j}} \tilde{\varphi})}{\det g_{i\bar{j}}} - F - \int_M \frac{\partial \varphi}{\partial t}\, \omega^n.$$

Taking t to infinity, we see that $\tilde{\varphi}_\infty$ solves the elliptic Monge-Ampère equation

$$\log \frac{\det(g_{i\bar{j}} + \partial_i \partial_{\bar{j}} \tilde{\varphi}_\infty)}{\det g_{i\bar{j}}} = F + b$$

where

$$b = \int_M \left(\log \frac{\det(g_{i\bar{j}} + \partial_i \partial_{\bar{j}} \tilde{\varphi}_\infty)}{\det g_{i\bar{j}}} - F \right) \omega^n.$$

This combined with Lemma 2.5.1 completes the proof of Theorem 2.1.1, and also provides a parabolic proof of the main theorem in [76]. □

Chapter 2, in full, is a reprint of the material as it appears in Communications in Analysis and Geometry volume 19, no. 2, 2011. Gill, Matthew, International Press 2011. The disseratation author was the author of this paper.

Chapter 3

Collapsing of products along the Kähler-Ricci flow

3.1 Introduction

Let M be an m-dimensional Kähler manifold with negative first Chern class and let E be an n-dimensional complex torus. Independently from Yau and Aubin, there exists a unique Kähler-Einstein metric g_M on M [81, 1]. Fix a flat metric g_E on E. Recall that we can associate a $(1,1)$-form ω to a Kähler metric g by defining

$$\omega = \frac{\sqrt{-1}}{2\pi} g_{i\bar{j}} dz^i \wedge dz^{\bar{j}}. \tag{3.1}$$

Throughout this paper, we will relate Kähler metrics g, g_M, ... with their Kähler forms ω, ω_M, ... using the obvious notation. We will also refer to ω as a Kähler metric since ω and g uniquely determine each other. Additionally, a uniform constant $C, C', \ldots$ will be a constant depending only on the initial data whose definition my change from line to line.

Let $X = M \times E$ and define projection maps $\pi_M : X \to M$ and $\pi_E : X \to E$. Let ω_0 be any Kähler metric on X and consider the normalized Kähler-Ricci flow

$$\frac{\partial}{\partial t}\omega = -\operatorname{Ric}(\omega) - \omega, \qquad \omega_{t=0} = \omega_0. \tag{3.2}$$

Observe that

$$\operatorname{Ric}(\pi_M^* \omega_M + \pi_E^* \omega_E) = -\pi_M^* \omega_M.$$

Hence $c_1(X) = -[\pi_M^*\omega_M] \leq 0$ and the flow (3.2) exists for all time by the work of Tsuji [79] and Tian-Zhang [72]. Notice that in general ω_0 is not a product. In the case when ω_0 is a product, the work of Cao shows that the flow exists for all time and converges smoothly to a Kähler-Einstein metric on M [6]. We prove the following theorem.

Theorem 3.1.1. *Let $\omega(t)$ be the solution to the normalized Kähler-Ricci flow* (3.2) *with initial Kähler metric ω_0 on $X = M \times E$. Then*

(a) $\omega(t)$ converges to $\pi_M^*\omega_M$ in $C^\infty(X, \omega_0)$ as $t \to \infty$.

(b) For any $z \in M$, let $E(z) = \pi_M^{-1}(z)$ denote the fiber above z. Then $e^t\omega(t)|_{E(z)} \to \omega_{flat}|_{E(z)}$ in $C^\infty(E(z), \omega_E)$ as $t \to \infty$, where ω_{flat} is a $(1,1)$-form on X with $[\omega_{flat}] = [\omega_0]$ whose restriction to each fiber is a flat Kähler metric.

We remark that this theorem holds for any compact Kähler manifold that admits a flat metric, which includes certain quotients of complex tori. This theorem strengthens a convergence result of Song and Weinkove and confirms their conjecture [62]. They prove that when $m = n = 1$, the convergence in (a) takes place in $C^\beta(X, \omega_0)$ for any β between 0 and 1, and that the convergence in (b) takes place in $C^0(E(z), \omega_E)$. They conjecture that the convergence in this case is in fact C^∞. This problem originates from the work of Song and Tian [56]. They considered the normalized Kähler-Ricci flow on an elliptic surface $f : X \to \Sigma$ where some of the fibers may be singular. It was shown that the solution of the flow converges to a generalized Kähler-Einstein metric on the base Σ in $C^{1,1}$. This result was generalized to the fibration $f : X \to X_{can}$ where X is a nonsingular algebraic variety with semi-ample canonical bundle and X_{can} is its canonical model [57]. Theorem 3.1.1 is a step towards strengthening this convergence result to C^∞. We remark that Gross, Tosatti and Zhang have studied a similar manifold as in Theorem 1.1, but considered the case where the Kähler class of the metric tends to the boundary of the Kähler cone instead of evolving by the Kähler-Ricci flow [26]. Fong and Zhang have examined the rate of collapse of the fibers of a similar manifold along the Kähler-Ricci flow in a recent preprint [20].

Theorem 3.1.1 is related to viewing the Kähler-Ricci flow with surgery as an analytic Minimal Model Program (MMP) as conjectured by Song and Tian and proved in the weak sense [58]. The idea of the MMP is that after several blow-downs and flips, a projective algebraic variety becomes either a minimal model or a Mori fiber space (an algebraic fibration $f : X \to B$ where the generic fibers are Fano). Recent results due to Song and Weinkove show that the Kähler-Ricci flow performs blow-downs as canonical surgical contractions in complex dimension 2 [60] and in the case of the blow-up of orbifold points [61]. Song and Yuan have given an example of the flow performing a flip [64]. Specific examples of collapsing along the flow have been investigated by Song and Weinkove in the case of a Hirzebruch surface [59] and by Fong in the case of a projective bundle over a Kähler-Einstein manifold [18].

After performing blow-downs and flips, the Kähler-Ricci flow is conjectured to produce either a minimal model or a Mori fiber space. If we continue the flow on a Mori fiber space, the flow is expected to collapse the fibers in finite time. An example of this was examined by Song, Székelyhidi and Weinkove [55]. The rate of collapse of the diameter was improved by Fong under an assumption on the Ricci curvature [19]. If we continue the flow on a minimal model, the flow exists for all time because the canonical class is nef. In this case, the rescaled flow may collapse in infinite time. This is the case considered in [56, 57, 62, 20] and in this paper.

In section 2, we derive several estimates following [62]. Section 3 contains new higher order estimates for the case of a degenerating metric using only the maximum principle. If the metric is not degenerating, then the work in section 3 most likely gives an alternate proof of the results in [63]. For other examples of where higher order estimates were obtained using only the maximum principle, see [7, 12, 35]. In section 4, we obtain the convergence of ω, completing the proof of the main theorem.

3.2 Estimates

First we establish reference metrics and reduce the flow to a parabolic complex Monge-Ampère equation. The Kähler class of ω evolves as

$$[\omega(t)] = e^{-t}[\omega_0] + \left(1 - e^{-t}\right)[\omega_M].$$

This can be verified by substituting in to the normalized Kähler-Ricci flow. Note that we have written ω_M in place of $\pi_M^*\omega_M$ to simplify notation and we will continue to do so for the remainder of this paper.

We define a family of reference metrics $\hat{\omega}_t$ in the class of $\omega(t)$ by

$$\hat{\omega}_t = e^{-t}\omega_0 + \left(1 - e^{-t}\right)\omega_M.$$

Pick a smooth volume form Ω on X such that

$$\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\Omega = \omega_M, \qquad \int_X \Omega = \binom{m+n}{m}\int_X \omega_M^m \wedge \omega_0^n. \tag{3.3}$$

This is possible since ω_M represents the negative of the first Chern class of X. Consider the parabolic complex Monge-Ampère equation

$$\frac{\partial}{\partial t}\varphi = \log\frac{e^{nt}\left(\hat{\omega}_t + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi\right)^{m+n}}{\Omega} - \varphi, \qquad \hat{\omega}_t + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi > 0, \qquad \varphi_{t=0} = 0. \tag{3.4}$$

Then the solution φ to (3.4) exists for all time and $\omega(t) = \hat{\omega}_t + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi$ solves the normalized Kähler-Ricci flow (3.2).

We derive uniform estimates for the Kähler potential φ. The result of Lemma 3.2.1 and Lemma 3.2.2 were proved in more general settings in the work of Song and Tian [56]. See also [20] in the case of a holomorphic submersion $X \to \Sigma$. Following the notation in [62], we provide a proof for the reader's convenience.

Lemma 3.2.1. *There exists $C > 0$ such that $X \times [0, \infty)$,*

(a) $|\varphi| \leq C$.

(b) $|\dot{\varphi}| \leq C$.

(c) $\frac{1}{C}\hat{\omega}_t^{m+n} \leq \omega^{m+n} \leq C\hat{\omega}_t^{m+n}$.

Proof. We begin by calculating

$$e^{nt}\hat{\omega}_t^{m+n} = e^{-mt}\omega_0^{m+n} + \binom{m+n}{1}e^{-(m-1)}\left(1-e^{-t}\right)\omega_0^{m+n-1}\wedge\omega_M + \dots \\ + \binom{m+n}{m}\left(1-e^{-t}\right)^m\omega_0^n\wedge\omega_M^m. \tag{3.5}$$

This equation implies that

$$\frac{1}{C}\Omega \leq e^{nt}\hat{\omega}_t^{m+n} \leq C\Omega. \tag{3.6}$$

To obtain the upper bound for φ, assume that φ attains a maximum at a point (z_0, t_0) with $t_0 > 0$. At that point, the maximum principle implies

$$0 \leq \frac{\partial}{\partial t}\varphi \leq \log\frac{e^{nt}\hat{\omega}_t^{m+n}}{\Omega} - \varphi \leq \log C - \varphi. \tag{3.7}$$

Thus we find $\varphi \leq \log C$, giving the upper bound. Similarly, we obtain a lower bound giving (a).

To prove (b), we calculate the evolution equation of $\dot{\varphi}$ to be

$$\left(\frac{\partial}{\partial t} - \Delta\right)\dot{\varphi} = \mathrm{tr}_\omega\left(\omega_M - \hat{\omega}_t\right) + n - \dot{\varphi}. \tag{3.8}$$

Note that by the definition of $\hat{\omega}_t$ there exists a constant $C_0 > 1$ such that $\omega_M \leq C_0\hat{\omega}_t$ (however it is not true that there exists $C_0 > 0$ such that $\frac{1}{C_0}\hat{\omega}_t \leq \omega_M$ since ω_M is degenerate). Then at the maximum of the quantity $Q_1 = \dot{\varphi} - (C_0 - 1)\varphi$,

$$\begin{aligned} 0 &\leq \left(\frac{\partial}{\partial t} - \Delta\right)Q_1 = \mathrm{tr}_\omega\left(\omega_M - \hat{\omega}_t\right) + n - \dot{\varphi} - \left(C_0 - 1\right)\dot{\varphi} + \left(C_0 - 1\right)\Delta\varphi \\ &\leq \left(C_0 - 1\right)\mathrm{tr}_\omega\hat{\omega}_t + n - C_0\dot{\varphi} + \left(C_0 - 1\right)\mathrm{tr}_\omega\left(\omega - \hat{\omega}_t\right) \\ &\leq n + \left(C_0 - 1\right)\left(m + n\right) - C_0\dot{\varphi}. \end{aligned} \tag{3.9}$$

Hence Q_1 is bounded above, and so is $\dot{\varphi}$ by (a).

To obtain the lower bound for $\dot{\varphi}$, we define the quantity $Q_2 = \dot{\varphi} + (m+1)\varphi$. Working at a point where Q_2 achieves a minimum,

$$\begin{aligned} 0 &\geq \left(\frac{\partial}{\partial t} - \Delta\right)Q_2 = \mathrm{tr}_\omega\left(\omega_M - \hat{\omega}_t\right) + n - \dot{\varphi} + \left(1 + m\right)\dot{\varphi} - \left(m + 1\right)\mathrm{tr}_\omega\left(\omega - \hat{\omega}_t\right) \\ &\geq m\left(\mathrm{tr}_\omega\hat{\omega}_t + \dot{\varphi} - \left(m + n + 1\right)\right). \end{aligned} \tag{3.10}$$

Using the arithmetic-geometric mean inequality and (3.6),

$$e^{-\frac{(\dot{\varphi}+\varphi)}{m+n}} = \left(\frac{\Omega}{e^{nt}\omega^{m+n}}\right)^{\frac{1}{m+n}} \leq C\left(\frac{\hat{\omega}_t^{m+n}}{\omega^{m+n}}\right)^{\frac{1}{m+n}} \leq C\,\mathrm{tr}_\omega\,\hat{\omega}_t \leq C - \dot{\varphi}. \tag{3.11}$$

This gives a uniform lower bound for $\dot{\varphi}$ at (z_0, t_0), and hence a uniform lower bound for $\dot{\varphi}$.

Finally, for (c), using (a), (b) and (3.4) we have

$$\frac{1}{C} \leq \frac{e^{nt}\omega^{m+n}}{\Omega} \leq C, \tag{3.12}$$

completing the proof of the lemma. □

Recall that we say two metrics ω_1 and ω_2 are uniformly equivalent if there exists a constant $C > 0$ such that $\frac{1}{C}\omega_2 \leq \omega_1 \leq C\omega_2$. We now show that ω is uniformly equivalent to $\hat{\omega}_t$. Although the following lemma is known in more generality (see [56], [20]), we provide a proof for the reader's convenience. We introduce another family of reference metrics

$$\tilde{\omega}_t = \omega_M + e^{-t}\omega_E. \tag{3.13}$$

By writing $\tilde{\omega}_0 = \omega_M + \omega_E$ and $\tilde{\omega}_t = e^{-t}\tilde{\omega}_0 + (1 - e^{-t})\,\omega_M$, it is easy to see that $\hat{\omega}_t$ and $\tilde{\omega}_t$ are uniformly equivalent. We choose $\tilde{\omega}_t$ so that its curvature tensor vanishes on E which will be useful for the remainder of this paper.

Lemma 3.2.2. *The metrics ω and $\tilde{\omega}_t$ are uniformly equivalent, i.e. there exists $C > 0$ such that on $X \times [0, \infty)$,*

$$\frac{1}{C}\tilde{\omega}_t \leq \omega \leq C\tilde{\omega}_t. \tag{3.14}$$

We remark that since $\hat{\omega}_t$ is uniformly equivalent to $\tilde{\omega}_t$, we also have the following corollary.

Corollary 3.2.3. *The metrics ω and $\hat{\omega}_t$ are uniformly equivalent.*

Now we will prove the above lemma using a method similar to Song and Weinkove. The main difference in the proof is that we need to be careful with the curvature tensor of $\tilde{\omega}_t$ due to the increase in dimension.

Proof. By Lemma 3.2.1 part (c), the lemma will follow by bounding $\operatorname{tr}_{\tilde{\omega}_t} \omega$ from above. We begin with the evolution equation for the quantity $\log \operatorname{tr}_{\tilde{\omega}_t} \omega$ from [62]. This is analogous to Cao's [6] second order estimate, which is the parabolic version of an elliptic estimate from Yau [81] and Aubin [1]:

$$\left(\frac{\partial}{\partial t} - \Delta\right) \log \operatorname{tr}_{\tilde{\omega}_t} \omega \leq -\frac{1}{\operatorname{tr}_{\tilde{\omega}_t} \omega} g^{\bar{l}k} R(\tilde{g}_t)_{k\bar{l}}{}^{\bar{j}i} g_{i\bar{j}}. \tag{3.15}$$

To control the Riemann curvature tensor of $\tilde{g}$, we choose product normal coordinates for g_M and g_E. In these coordinates,

$$R(\tilde{g}_t)_{k\bar{l}i\bar{j}} = \begin{cases} R(g_M)_{k\bar{l}i\bar{j}} & : 1 \leq i, j, k, l \leq m \\ 0 & : \text{else} \end{cases} \tag{3.16}$$

We recall that an inequality of tensors $T_{k\bar{l}i\bar{j}} \leq S_{k\bar{l}i\bar{j}}$ in the Griffiths sense is defined as follows. For any vectors X and Y of type $T^{1,0}$, we have $T_{k\bar{l}i\bar{j}} X^k \overline{X^l} Y^i \overline{Y^j} \leq S_{k\bar{l}i\bar{j}} X^k \overline{X^l} Y^i \overline{Y^j}$. Since $\operatorname{Rm}(g_M)$ (the Riemann curvature tensor of g_M, $R_{k\bar{l}i\bar{j}}$) is a fixed tensor on M, for every X and Y on M,

$$\left| R(g_M)_{k\bar{l}i\bar{j}} X^k \overline{X^l} Y^i \overline{Y^j} \right|^2_{g_M} \leq |\operatorname{Rm}(g_M)|^2_{g_M} |X|^2_{g_M} |Y|^2_{g_M}. \tag{3.17}$$

This gives the following inequality in the Griffiths sense

$$-R(g_M)_{k\bar{l}i\bar{j}} \leq C_1 (g_M)_{k\bar{l}} (g_M)_{i\bar{j}}. \tag{3.18}$$

Applying (3.16) and (3.18) to (3.15) gives

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta\right) \log \operatorname{tr}_{\tilde{\omega}_t} \omega &\leq \frac{1}{\operatorname{tr}_{\tilde{\omega}_t} \omega} \sum_{i,j,l,k,p,q=1}^{m} C_1 g^{\bar{l}k} g_{i\bar{j}} \tilde{g}_t^{\bar{q}i} \tilde{g}_t^{\bar{j}p} (g_M)_{k\bar{l}} (g_M)_{p\bar{q}} \\ &= C_1 \frac{1}{\operatorname{tr}_{\tilde{\omega}_t} \omega} (\operatorname{tr}_\omega \omega_M) \sum_{i=1}^{m} g_{i\bar{i}} \\ &\leq C_1 \frac{1}{\operatorname{tr}_{\tilde{\omega}_t} \omega} (\operatorname{tr}_\omega \omega_M) (\operatorname{tr}_{\tilde{\omega}_t} \omega) \\ &= C_1 \operatorname{tr}_\omega \omega_M. \end{aligned} \tag{3.19}$$

Recall that there exists $C_0 > 1$ such that $\omega_M \leq C_0 \hat{\omega}_t$. Now we define the quantity $Q_3 = \log \operatorname{tr}_{\tilde{\omega}_t} \omega - (C_0C_1+1)\varphi$. Then at the maximum of Q_3,

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right) Q_3 &\leq C_1 \operatorname{tr}_\omega \omega_M - (C_0C_1+1)\dot{\varphi} + (C_0C_1+1) \operatorname{tr}_\omega (\omega - \hat{\omega}_t) \\ &\leq (C_0C_1+1)(m+n) - (C_0C_1+1)\dot{\varphi} - \operatorname{tr}_\omega \hat{\omega}_t \\ &\leq C - \frac{1}{C} \operatorname{tr}_{\tilde{\omega}_t} \omega. \end{aligned} \tag{3.20}$$

To get the last line we use the fact that $\dot{\varphi}$ is bounded from Lemma 3.2.1 part (b), that $\tilde{\omega}_t$ and $\hat{\omega}_t$ are uniformly equivalent, and Lemma 3.2.1 part (c). Using Lemma 3.2.1 part (a) and the maximum principle shows that Q_3 is bounded, hence so is $\operatorname{tr}_{\tilde{\omega}_t} \omega$. □

By choosing product normal coordinates for g_M and g_E, $\partial_k(\tilde{g}_t)_{i\bar{j}} = 0$ for all i, j and k and for all $t \geq 0$. This implies that the Christoffel symbols for $\tilde{\omega}_t$ do not depend on t, hence we may write $\tilde{\nabla}$ for both $\nabla_{\tilde{g}_t}$ and $\nabla_{\tilde{g}_0}$ without ambiguity. This also implies that the curvature tensor $R(\tilde{g}_t)_{i\bar{j}k}{}^{l}$ does not depend on time. Using these facts, we prove the following lemma which we will make heavy use of for the remainder of the paper. We remark that the proof of the following lemma uses the product structure of the manifold in a very strong way.

Lemma 3.2.4. *Let* $\operatorname{Rm}(\tilde{g}_0)$ *denote the Riemann curvature tensor of* $\tilde{g}_0$, $R(\tilde{g}_0)_{i\bar{j}k}{}^{l}$. *Then there exists a uniform* $C(k) > 0$ *for* $k = 0, 1, 2, \ldots$ *such that on* $X \times [0, \infty)$,

$$|\tilde{\nabla}^k_{\mathbb{R}} \operatorname{Rm}(\tilde{g}_0)|^2 \leq C(k), \tag{3.21}$$

where $|\cdot|$ *denotes the norm with respect to* $g(t)$ *and where* $\tilde{\nabla}_{\mathbb{R}}$ *is the covariant derivative with respect to* $\tilde{g}_0$ *as a Riemannian metric.*

Proof. Recall that $\tilde{g}_t$ is a product metric on $X = M \times E$. Using the fact that $\operatorname{Rm}(\tilde{g}_t)$ does not depend on time and Lemma 3.2.2,

$$|\tilde{\nabla}^k_{\mathbb{R}} \operatorname{Rm}(\tilde{g}_0)|^2 = |\nabla^k_{\tilde{g}_t,\mathbb{R}} \operatorname{Rm}(\tilde{g}_t)|^2_g \leq C|\nabla^k_{\tilde{g}_t,\mathbb{R}} \operatorname{Rm}(\tilde{g}_t)|^2_{\tilde{g}_t}. \tag{3.22}$$

Then because g_E is a flat metric on E,

$$|\tilde{\nabla}^k_{\mathbb{R}} \operatorname{Rm}(\tilde{g}_0)|^2 \leq C|\nabla^k_{\tilde{g}_t,\mathbb{R}} \operatorname{Rm}(\tilde{g}_t)|^2_{\tilde{g}_t} = C|\nabla^k_{g_M,\mathbb{R}} \operatorname{Rm}(g_M)|^2_{g_M} \leq C(k). \tag{3.23}$$

□

We will now bound the first derivative of the metric ω following the method of [62].

Lemma 3.2.5. *There exists a uniform $C > 0$ such that on $X \times [0, \infty)$,*

$$S := |\tilde{\nabla} g|^2 \leq C \quad \textit{and} \quad |\tilde{\nabla} g|^2_{\tilde{g}_0} \leq C \tag{3.24}$$

where $|\cdot|$ and $|\cdot|_{\tilde{g}_0}$ denote the norms with respect to $g(t)$ and $\tilde{g}_0$ respectively. Moreover,

$$\left(\frac{\partial}{\partial t} - \Delta\right) S \leq -\frac{1}{2}|\operatorname{Rm}(g)|^2 + C' \tag{3.25}$$

for some uniform $C' > 0$ and where $\operatorname{Rm}(g)$ denotes the Riemann curvature tensor of g, $R_{i\bar{j}k}{}^{l}$.

Proof. We will derive the evolution equation of S using a formula of Phong-Sesum-Sturm [43]. We follow the notation of [43, 62]. Let $\Psi^k_{ij} = \Gamma^k_{ij} - \tilde{\Gamma}^k_{ij} = g^{\bar{l}k}\tilde{\nabla}_i g_{j\bar{l}}$, where Γ and $\tilde{\Gamma}$ are the Christoffel symbols for $g(t)$ and $\tilde{g}_0$ respectively. Then we have

$$S = |\Psi|^2 = g^{\bar{j}i} g^{\bar{l}k} g_{p\bar{q}} \Psi^p_{ik} \overline{\Psi^q_{jl}}. \tag{3.26}$$

Before computing the evolution equation of S, we need the evolution equation of Ψ^k_{ij}.

$$\frac{\partial}{\partial t}\Psi^k_{ij} = \frac{\partial}{\partial t}\left(g^{\bar{l}k}\partial_i g_{j\bar{l}} - \tilde{g}^{\bar{l}k}\partial_i \tilde{g}_{j\bar{l}}\right) = g^{\bar{l}k}\partial_i\left(-R_{j\bar{l}} - g_{j\bar{l}}\right) = -\nabla_i R_j{}^k. \tag{3.27}$$

We also compute the rough Laplacian of Ψ^k_{ij}:

$$\Delta \Psi^k_{ij} = g^{\bar{q}p}\nabla_p \nabla_{\bar{q}} \Psi^k_{ij} = \nabla^{\bar{q}}\left(R(\tilde{g}_0)_{i\bar{q}j}{}^k - R_{i\bar{q}j}{}^k\right) = \nabla^{\bar{q}} R(\tilde{g}_0)_{i\bar{q}j}{}^k - \nabla_i R_j{}^k. \tag{3.28}$$

Hence we have

$$\left(\frac{\partial}{\partial t} - \Delta\right)\Psi^k_{ij} = -\nabla^{\bar{q}} R(\tilde{g}_0)_{i\bar{q}j}{}^k. \tag{3.29}$$

Now we calculate the evolution of S.

$$\begin{aligned}
\frac{\partial}{\partial t} S &= \frac{\partial}{\partial t}\left(g^{\bar{j}i} g^{\bar{l}k} g_{p\bar{q}} \Psi^p_{ik}\overline{\Psi^q_{jl}}\right) \\
&= -\left(-R^{\bar{j}i} - g^{\bar{j}i}\right) g^{\bar{l}k} g_{p\bar{q}} \Psi^p_{ik}\overline{\Psi^q_{jl}} - g^{\bar{j}i}\left(-R^{\bar{l}k} - g^{\bar{l}k}\right) g_{p\bar{q}}\Psi^p_{ik}\overline{\Psi^q_{jl}} \\
&\quad + g^{\bar{j}i} g^{\bar{l}k}\left(-R_{p\bar{q}} - g_{p\bar{q}}\right)\Psi^p_{ik}\overline{\Psi^q_{jl}} \\
&\quad + 2\operatorname{Re}\left(g^{\bar{j}i} g^{\bar{l}k} g_{p\bar{q}}\left(\Delta\Psi^p_{ik} - \nabla^{\bar{s}} R(\tilde{g}_0)_{i\bar{s}k}{}^p\right)\overline{\Psi^q_{jl}}\right)
\end{aligned} \tag{3.30}$$

Taking the Laplacian of S,

$$\Delta S = |\nabla\Psi|^2 + |\bar{\nabla}\Psi|^2 + g^{\bar{j}i}g^{\bar{l}k}g_{p\bar{q}}\left((\Delta\Psi^p_{ik})\,\overline{\Psi^q_{jl}} + \Psi^p_{ik}\,\overline{(\bar{\Delta}\Psi^q_{jl})}\right). \tag{3.31}$$

We have the following commutation formula:

$$\overline{(\bar{\Delta}\Psi^q_{jl})} = \Delta\Psi^q_{jl} + R_j{}^r\Psi^q_{rl} + R_l{}^r\Psi^q_{jr} - R_r{}^q\Psi^r_{jl}. \tag{3.32}$$

Substituting (3.32) into (3.31) and combining with (3.30), we obtain

$$\left(\frac{\partial}{\partial t} - \Delta\right)S = S - |\nabla\Psi|^2 - |\bar{\nabla}\Psi|^2 - 2\,\mathrm{Re}\left(g^{\bar{j}i}g^{\bar{l}k}g_{p\bar{q}}\nabla^{\bar{s}}R(\tilde{g}_0)_{i\bar{s}k}{}^p\overline{\Psi^q_{jl}}\right) \tag{3.33}$$

Now we need to control the final term in (3.33) to complete the proof. By choosing normal coordinates for $\tilde{g}_0$,

$$\begin{aligned} 2\,\mathrm{Re}\left(g^{\bar{j}i}g^{\bar{l}k}g_{p\bar{q}}\nabla^{\bar{s}}R(\tilde{g}_0)_{i\bar{s}k}{}^p\overline{\Psi^q_{jl}}\right) &= 2\,\mathrm{Re}\Big(g^{\bar{j}i}g^{\bar{l}k}g_{p\bar{q}}g^{\bar{s}r}\Big(\tilde{\nabla}_r R(\tilde{g}_0)_{i\bar{s}k}{}^p - \Psi^a_{ir}R(\tilde{g}_0)_{a\bar{s}k}{}^p \\ &\quad - \Psi^a_{kr}R(\tilde{g}_0)_{i\bar{s}a}{}^p + \Psi^p_{ar}R(\tilde{g}_0)_{i\bar{s}k}{}^a\Big)\overline{\Psi^q_{jl}}\Big). \end{aligned} \tag{3.34}$$

We bound the first term in (3.34) using Lemma 3.2.4:

$$\left|2\,\mathrm{Re}\left(g^{\bar{j}i}g^{\bar{l}k}g_{p\bar{q}}g^{\bar{s}r}\tilde{\nabla}_r R(\tilde{g}_0)_{i\bar{s}k}{}^p\overline{\Psi^q_{jl}}\right)\right| \leq C|\tilde{\nabla}\,\mathrm{Rm}(\tilde{g}_0)|^2 + CS \leq C + CS. \tag{3.35}$$

Similarly for the remaining terms in (3.34),

$$\left|2\,\mathrm{Re}\left(g^{\bar{j}i}g^{\bar{l}k}g_{p\bar{q}}g^{\bar{s}r}R(\tilde{g}_0)_{a\bar{s}k}{}^p\Psi^a_{ir}\overline{\Psi^q_{jl}}\right)\right| \leq C|\,\mathrm{Rm}(\tilde{g}_0)|^2 S \leq CS. \tag{3.36}$$

Using (3.34), (3.35) and (3.36), we obtain the estimate

$$\left|2\,\mathrm{Re}\left(g^{\bar{j}i}g^{\bar{l}k}g_{p\bar{q}}\nabla^{\bar{s}}R(\tilde{g}_0)_{i\bar{s}k}{}^p\overline{\Psi^q_{jl}}\right)\right| \leq C' + CS. \tag{3.37}$$

We combine (3.37) with (3.33) to obtain

$$\left(\frac{\partial}{\partial t} - \Delta\right)S \leq C' + CS - |\nabla\Psi|^2 - |\bar{\nabla}\Psi|^2. \tag{3.38}$$

Define the quantity $Q_4 = S + A\,\mathrm{tr}_{\tilde{\omega}_t}\,\omega$ where A is a large constant to be determined later. The evolution equation of $\mathrm{tr}_{\tilde{\omega}_t}\,\omega$ is (see [62]),

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta\right)\mathrm{tr}_{\tilde{\omega}_t}\,\omega &= -\,\mathrm{tr}_{\tilde{\omega}_t}\,\omega - g^{\bar{l}k}R(\tilde{g}_t)_{k\bar{l}}{}^{\bar{j}i}g_{i\bar{j}} - g^{\bar{l}k}\tilde{g}_t^{\bar{j}i}g^{\bar{q}p}\tilde{\nabla}_i g_{k\bar{q}}\tilde{\nabla}_{\bar{j}}g_{p\bar{l}} \\ &\leq -g^{\bar{l}k}R(\tilde{g}_t)_{k\bar{l}}{}^{\bar{j}i}g_{i\bar{j}} - g^{\bar{l}k}\tilde{g}_t^{\bar{j}i}g^{\bar{q}p}\tilde{\nabla}_i g_{k\bar{q}}\tilde{\nabla}_{\bar{j}}g_{p\bar{l}}. \end{aligned} \tag{3.39}$$

Using (3.39) and (3.38) we have

$$\left(\frac{\partial}{\partial t} - \Delta\right) Q_4 \leq C' + CS - |\nabla\Psi|^2 - |\bar{\nabla}\Psi|^2 - Ag^{\bar{l}k} R(\tilde{g}_t)_{k\bar{l}}{}^{\bar{j}i} g_{i\bar{j}} - Ag^{\bar{l}k} \tilde{g}_t^{\bar{j}i} g^{\bar{q}p} \tilde{\nabla}_i g_{k\bar{q}} \tilde{\bar{\nabla}}_{\bar{j}} g_{p\bar{l}}. \tag{3.40}$$

To handle the fourth term in (3.40), we again work in product normal coordinates for g_M and g_E. Using the same argument to control the curvature as in Lemma 3.2.2 and the fact that g and $\tilde{g}_t$ are uniformly equivalent,

$$\left| g^{\bar{l}k} R(\tilde{g}_t)_{k\bar{l}}{}^{\bar{j}i} g_{i\bar{j}} \right| \leq C''(\operatorname{tr}_\omega \tilde{\omega}_t)(\operatorname{tr}_{\tilde{\omega}_t} \omega) \leq C''. \tag{3.41}$$

We combine (3.40), (3.41) and again use the uniform equivalence of g and $\tilde{g}_t$, giving

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta\right) Q_4 &\leq C' + CS - |\nabla\Psi|^2 - |\bar{\nabla}\Psi|^2 + AC'' - \frac{A}{C'''} S \\ &\leq -S - |\bar{\nabla}\Psi|^2 + C \end{aligned} \tag{3.42}$$

where on the last line we choose A large enough so that $C - A/C''' \leq -1$ and throw away the term $|\nabla\Psi|^2$. Also ignoring the term $|\bar{\nabla}\Psi|^2$ gives an upper bound for Q_4 by the maximum principle. Using Lemma 3.2.2 then shows that S is bounded above as well. Since $g \leq C\tilde{g}_0$ we also have an upper bound for $|\tilde{\nabla} g|^2_{\tilde{g}_0}$.

Now we derive (3.25). Notice that by definition $|\bar{\nabla}\Psi|^2 = |\operatorname{Rm}(g) - \operatorname{Rm}(\tilde{g}_0)|^2$ where we use $\operatorname{Rm}(\tilde{g}_0)$ for the Riemann curvature tensor of $\tilde{g}_0$, $R(\tilde{g}_0)_{i\bar{j}k}{}^{l}$. By Lemma 3.2.4,

$$|\operatorname{Rm}(g)|^2 \leq 2|\operatorname{Rm}(g) - \operatorname{Rm}(\tilde{g}_0)|^2 + 2|\operatorname{Rm}(\tilde{g}_0)|^2 \leq 2|\bar{\nabla}\Psi|^2 + C. \tag{3.43}$$

Substituting (3.43) into (3.42) along with the bound on S gives (3.25).

□

Following [62], we bound the curvature tensor of g.

Lemma 3.2.6. *There exists a uniform $C > 0$ such that on $X \times [0, \infty)$,*

$$|\operatorname{Rm}(g)|^2 \leq C. \tag{3.44}$$

Proof. We have the following evolution equation for curvature along the Kähler-Ricci flow (see [62]):

$$\left(\frac{\partial}{\partial t} - \Delta\right) |\operatorname{Rm}(g)| \leq \frac{C_0}{2} |\operatorname{Rm}(g)|^2 - \frac{1}{2} |\operatorname{Rm}(g)|. \tag{3.45}$$

Define the quantity $Q = |\operatorname{Rm}(g)| + (C_0 + 1)S$. Then using (3.25), (3.45) and the maximum principle, we have the estimate

$$\left(\frac{\partial}{\partial t} - \Delta\right) Q \leq -\frac{1}{2}|\operatorname{Rm}(g)|^2 + C, \tag{3.46}$$

obtaining a bound for $|\operatorname{Rm}(g)|^2$. □

Using Shi's derivative estimates, we obtain bounds for the derivatives of curvature. For a proof of the following lemma, please see [53] (or [62] Theorem 2.15).

Lemma 3.2.7. *There exists uniform $C(k)$ for $k = 0, 1, 2, \ldots$ such that on $X \times [0, \infty)$,*

$$|\nabla^k_{\mathbb{R}} \operatorname{Rm}(g)|^2 \leq C(k), \tag{3.47}$$

where $\nabla_{\mathbb{R}}$ is the covariant derivative with respect to g as a Riemannian metric.

3.3 Higher order estimates for the metric $\omega(t)$

We will now use the curvature bounds and the maximum principle to obtain higher order estimates for g. Examples of higher order estimates using similar quantities and the maximum principle can be found in [7, 12, 35].

Lemma 3.3.1. *There exists uniform $C(k) > 0$ for $k = 0, 1, 2, \ldots$ such that on $X \times [0, \infty)$,*

$$|\tilde{\nabla}^k g|^2 \leq C(k). \tag{3.48}$$

Proof. We observe that a uniform bound on $|\tilde{\nabla}\Psi|^2$ will give a uniform bound on $|\tilde{\nabla}\tilde{\nabla} g|^2$. We begin by calculating

$$\begin{aligned}
\frac{\partial}{\partial t}|\tilde{\nabla}\Psi|^2 = & \frac{\partial}{\partial t}\left(g^{\bar{s}r} g^{\bar{j}i} g^{\bar{l}k} g_{p\bar{q}} \tilde{\nabla}_r \Psi^p_{ik} \overline{\tilde{\nabla}_s \Psi^q_{jl}}\right) \\
= & -\left(-R^{\bar{s}r} - g^{\bar{s}r}\right) g^{\bar{j}i} g^{\bar{l}k} g_{p\bar{q}} \tilde{\nabla}_r \Psi^p_{ik} \overline{\tilde{\nabla}_s \Psi^q_{jl}} \\
& - g^{\bar{s}r}\left(-R^{\bar{j}i} - g^{\bar{j}i}\right) g^{\bar{l}k} g_{p\bar{q}} \tilde{\nabla}_r \Psi^p_{ik} \overline{\tilde{\nabla}_s \Psi^q_{jl}} \\
& - g^{\bar{s}r} g^{\bar{j}i}\left(-R^{\bar{l}k} - g^{\bar{l}k}\right) g_{p\bar{q}} \tilde{\nabla}_r \Psi^p_{ik} \overline{\tilde{\nabla}_s \Psi^q_{jl}} \\
& + g^{\bar{s}r} g^{\bar{j}i} g^{\bar{l}k}\left(-R_{p\bar{q}} - g_{p\bar{q}}\right) \tilde{\nabla}_r \Psi^p_{ik} \overline{\tilde{\nabla}_s \Psi^q_{jl}} \\
& + 2\operatorname{Re}\left(g^{\bar{s}r} g^{\bar{j}i} g^{\bar{l}k} g_{p\bar{q}} \tilde{\nabla}_r \left(\Delta \Psi^p_{ik} - \nabla^{\bar{b}} R(\tilde{g}_0)_{i\bar{b}k}{}^p\right) \overline{\tilde{\nabla}_s \Psi^q_{jl}}\right).
\end{aligned} \tag{3.49}$$

Applying the Laplacian to $|\tilde{\nabla}\Psi|^2$,

$$\begin{aligned}
\Delta|\tilde{\nabla}\Psi|^2 = {} & |\nabla\tilde{\nabla}\Psi|^2 \\
& + |\bar{\nabla}\tilde{\nabla}\Psi|^2 + g^{\bar{s}r}g^{\bar{j}i}g^{\bar{l}k}g_{p\bar{q}}\left(\left(\Delta\tilde{\nabla}_r\Psi^p_{ik}\right)\overline{\tilde{\nabla}_s\Psi^q_{jl}} + \tilde{\nabla}_r\Psi^p_{ik}\overline{\left(\bar{\Delta}\tilde{\nabla}_s\Psi^q_{jl}\right)}\right) \\
= {} & |\nabla\tilde{\nabla}\Psi|^2 + |\bar{\nabla}\tilde{\nabla}\Psi|^2 + 2\,\mathrm{Re}\left(g^{\bar{s}r}g^{\bar{j}i}g^{\bar{l}k}g_{p\bar{q}}\left(\Delta\tilde{\nabla}_r\Psi^p_{ik}\right)\overline{\tilde{\nabla}_s\Psi^q_{jl}}\right) \\
& + R^{\bar{s}r}g^{\bar{j}i}g^{\bar{l}k}g_{p\bar{q}}\tilde{\nabla}_r\Psi^p_{ik}\overline{\tilde{\nabla}_s\Psi^q_{jl}} + g^{\bar{s}r}R^{\bar{j}i}g^{\bar{l}k}g_{p\bar{q}}\tilde{\nabla}_r\Psi^p_{ik}\overline{\tilde{\nabla}_s\Psi^q_{jl}} \\
& + g^{\bar{s}r}g^{\bar{j}i}R^{\bar{l}k}g_{p\bar{q}}\tilde{\nabla}_r\Psi^p_{ik}\overline{\tilde{\nabla}_s\Psi^q_{jl}} - g^{\bar{s}r}g^{\bar{j}i}g^{\bar{l}k}R_{p\bar{q}}\tilde{\nabla}_r\Psi^p_{ik}\overline{\tilde{\nabla}_s\Psi^q_{jl}},
\end{aligned} \tag{3.50}$$

where on the last line we use a commutation formula similar to (3.32). Putting together (3.49) and (3.50), we obtain the evolution equation

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta\right)|\tilde{\nabla}\Psi|^2 = {} & 2|\tilde{\nabla}\Psi|^2 - |\nabla\tilde{\nabla}\Psi|^2 - |\bar{\nabla}\tilde{\nabla}\Psi|^2 \\
& - 2\,\mathrm{Re}\left(g^{\bar{s}r}g^{\bar{j}i}g^{\bar{l}k}g_{p\bar{q}}\tilde{\nabla}_r\nabla^{\bar{b}}R(\tilde{g}_0)_{i\bar{b}k}{}^p\overline{\tilde{\nabla}_s\Psi^q_{jl}}\right) \\
& + 2\,\mathrm{Re}\left(g^{\bar{s}r}g^{\bar{j}i}g^{\bar{l}k}g_{p\bar{q}}\left(\tilde{\nabla}_r\Delta - \Delta\tilde{\nabla}_r\right)\Psi^p_{ik}\overline{\tilde{\nabla}_s\Psi^q_{jl}}\right).
\end{aligned} \tag{3.51}$$

Choose coordinates so that $\tilde{g}_0$ is the identity and $\partial_i\tilde{g}_0 = 0$ and $\partial_{i_1}\partial_{i_2}\tilde{g}_0 = 0$ at a point as in [70]. To deal with the fourth term in (3.51), we calculate

$$\begin{aligned}
\tilde{\nabla}_r\nabla^{\bar{b}}R(\tilde{g}_0)_{i\bar{b}k}{}^p = {} & \tilde{\nabla}_r g^{\bar{b}a}\Big(\tilde{\nabla}_a R(\tilde{g}_0)_{i\bar{b}k}{}^p - \Psi^\alpha_{ia}R(\tilde{g}_0)_{\alpha\bar{b}k}{}^p - \Psi^\alpha_{ka}R(\tilde{g}_0)_{i\bar{b}\alpha}{}^p \\
& + \Psi^p_{\alpha a}R(\tilde{g}_0)_{i\bar{b}k}{}^\alpha\Big) + g^{\bar{b}a}\Big(\tilde{\nabla}_r\tilde{\nabla}_a R(\tilde{g}_0)_{i\bar{b}k}{}^p - \tilde{\nabla}_r\Psi^\alpha_{ia}R(\tilde{g}_0)_{\alpha\bar{b}k}{}^p \\
& - \Psi^\alpha_{ia}\tilde{\nabla}_r R(\tilde{g}_0)_{\alpha\bar{b}k}{}^p - \tilde{\nabla}_r\Psi^\alpha_{ka}R(\tilde{g}_0)_{i\bar{b}\alpha}{}^p - \Psi^\alpha_{ka}\tilde{\nabla}_r R(\tilde{g}_0)_{i\bar{b}\alpha}{}^p \\
& + \tilde{\nabla}_r\Psi^p_{\alpha a}R(\tilde{g}_0)_{i\bar{b}k}{}^\alpha + \Psi^p_{\alpha a}\tilde{\nabla}_r R(\tilde{g}_0)_{i\bar{b}k}{}^\alpha\Big).
\end{aligned} \tag{3.52}$$

We now bound all of the terms arising from (3.52) using Lemmas 3.2.4 and 3.2.5. For the first term in (3.52),

$$\begin{aligned}
\left|2\,\mathrm{Re}\left(g^{\bar{s}r}g^{\bar{j}i}g^{\bar{l}k}g_{p\bar{q}}\tilde{\nabla}_r g^{\bar{b}a}\tilde{\nabla}_a R(\tilde{g}_0)_{i\bar{b}k}{}^p\overline{\tilde{\nabla}_s\Psi^q_{jl}}\right)\right| & \leq C|\tilde{\nabla}g||\tilde{\nabla}\,\mathrm{Rm}(\tilde{g}_0)||\tilde{\nabla}\Psi| \\
& \leq C|\tilde{\nabla}\Psi|^2 + C.
\end{aligned} \tag{3.53}$$

We bound the second, and similarly the third and fourth terms in (3.52):

$$\begin{aligned}
\left|2\,\mathrm{Re}\left(g^{\bar{s}r}g^{\bar{j}i}g^{\bar{l}k}g_{p\bar{q}}\tilde{\nabla}_r g^{\bar{b}a}\Psi^\alpha_{ia}R(\tilde{g}_0)_{\alpha\bar{b}k}{}^p\overline{\tilde{\nabla}_s\Psi^q_{jl}}\right)\right| & \leq C|\tilde{\nabla}g||\,\mathrm{Rm}(\tilde{g}_0)||\tilde{\nabla}\Psi| \\
& \leq C|\tilde{\nabla}\Psi|^2 + C
\end{aligned} \tag{3.54}$$

Calculating similarly for the remaining terms in (3.52), we obtain the following bound for the fourth term of (3.51):

$$2\,\mathrm{Re}\left(g^{\bar{s}r}g^{\bar{j}i}g^{\bar{l}k}g_{p\bar{q}}\tilde{\nabla}_r\nabla^{\bar{b}}R(\tilde{g}_0)_{i\bar{b}k}{}^{p}\overline{\tilde{\nabla}_s\Psi^q_{jl}}\right) \leq C|\tilde{\nabla}\Psi|^2 + C. \tag{3.55}$$

Using the same coordinates as above, we compute the commutation relation for

$$\left(\tilde{\nabla}_r\Delta - \Delta\tilde{\nabla}_r\right)\Psi^p_{ik}$$

to handle the last term in (3.51),

$$\begin{aligned}\tilde{\nabla}_r\Delta\Psi^p_{ik} &= \tilde{\nabla}_r\left(g^{\bar{b}a}\nabla_a\nabla_{\bar{b}}\Psi^p_{ik}\right)\\ &= \partial_r g^{\bar{b}a}\left(\partial_a\partial_{\bar{b}}\Psi^p_{ik} - \Gamma^\alpha_{ia}\partial_{\bar{b}}\Psi^p_{\alpha k} - \Gamma^\alpha_{ka}\partial_{\bar{b}}\Psi^p_{i\alpha} + \Gamma^p_{\alpha a}\partial_{\bar{b}}\Psi^\alpha_{ik}\right)\\ &\quad + g^{\bar{b}a}\Big(\partial_r\partial_a\partial_{\bar{b}}\Psi^p_{ik} - \partial_r\Gamma^\alpha_{ia}\partial_{\bar{b}}\Psi^p_{\alpha k} - \Gamma^\alpha_{ia}\partial_r\partial_{\bar{b}}\Psi^p_{\alpha k} - \partial_r\Gamma^\alpha_{ka}\partial_{\bar{b}}\Psi^p_{i\alpha}\\ &\quad - \Gamma^\alpha_{ka}\partial_r\partial_{\bar{b}}\Psi^p_{i\alpha} + \partial_r\Gamma^p_{\alpha a}\partial_{\bar{b}}\Psi^\alpha_{ik} + \Gamma^p_{\alpha a}\partial_r\partial_{\bar{b}}\Psi^\alpha_{ik}\Big).\end{aligned} \tag{3.56}$$

$$\begin{aligned}\Delta\tilde{\nabla}_r\Psi^p_{ik} &= g^{\bar{b}a}\nabla_a\nabla_{\bar{b}}\tilde{\nabla}_r\Psi^p_{ik}\\ &= g^{\bar{b}a}\Big(\partial_r\partial_a\partial_{\bar{b}}\Psi^p_{ik} - \Gamma^\beta_{ra}\partial_\beta\partial_{\bar{b}}\Psi^p_{ik} \qquad (3.57)\\ &- \Gamma^\beta_{ia}\partial_r\partial_{\bar{b}}\Psi^p_{\beta k} - \Gamma^\beta_{ka}\partial_r\partial_{\bar{b}}\Psi^p_{i\beta} + \Gamma^p_{\beta a}\partial_r\partial_{\bar{b}}\Psi^\beta_{ik}\\ &- \partial_a R(\tilde{g}_0)_{i\bar{b}r}{}^{\alpha}\Psi^p_{\alpha k} - R(\tilde{g}_0)_{i\bar{b}r}{}^{\alpha}\partial_a\Psi^p_{\alpha k} + \Gamma^\beta_{ia}R(\tilde{g}_0)_{\beta\bar{b}r}{}^{\alpha}\Psi^p_{\alpha k}\\ &+ \Gamma^\beta_{ra}R(\tilde{g}_0)_{i\bar{b}\beta}{}^{\alpha}\Psi^p_{\alpha k} - \Gamma^\alpha_{\beta a}R(\tilde{g}_0)_{i\bar{b}r}{}^{\beta}\Psi^p_{\alpha k} + \Gamma^\beta_{\alpha a}R(\tilde{g}_0)_{i\bar{b}r}{}^{\alpha}\Psi^p_{\beta k}\\ &+ \Gamma^\beta_{ka}R(\tilde{g}_0)_{i\bar{b}r}{}^{\alpha}\Psi^p_{\alpha\beta} - \Gamma^p_{\beta a}R(\tilde{g}_0)_{i\bar{b}r}{}^{\alpha}\Psi^\beta_{\alpha k} - \partial_a R(\tilde{g}_0)_{k\bar{b}r}{}^{\alpha}\Psi^p_{i\alpha}\\ &- R(\tilde{g}_0)_{k\bar{b}r}{}^{\alpha}\partial_a\Psi^p_{i\alpha} + \Gamma^\beta_{ka}R(\tilde{g}_0)_{\beta\bar{b}r}{}^{\alpha}\Psi^p_{i\alpha} + \Gamma^\beta_{ra}R(\tilde{g}_0)_{k\bar{b}\beta}{}^{\alpha}\Psi^p_{i\alpha}\\ &- \Gamma^\alpha_{\beta a}R(\tilde{g}_0)_{k\bar{b}r}{}^{\beta}\Psi^p_{i\alpha} + \Gamma^\beta_{ia}R(\tilde{g}_0)_{k\bar{b}r}{}^{\alpha}\Psi^p_{\beta\alpha} + \Gamma^\beta_{\alpha a}R(\tilde{g}_0)_{k\bar{b}r}{}^{p}\Psi^p_{i\beta}\\ &- \Gamma^p_{\beta a}R(\tilde{g}_0)_{k\bar{b}r}{}^{\alpha}\Psi^\beta_{i\alpha} + \partial_a R(\tilde{g}_0)_{\alpha\bar{b}r}{}^{p}\Psi^\alpha_{ik} + R(\tilde{g}_0)_{\alpha\bar{b}r}{}^{p}\partial_a\Psi^\alpha_{ik}\\ &- \Gamma^\beta_{\alpha a}R(\tilde{g}_0)_{\beta\bar{b}r}{}^{p}\Psi^\alpha_{ik} - \Gamma^\beta_{ra}R(\tilde{g}_0)_{\alpha\bar{b}\beta}{}^{p}\Psi^\alpha_{ik} + \Gamma^p_{\beta a}R(\tilde{g}_0)_{\alpha\bar{b}r}{}^{\beta}\Psi^p_{ik}\\ &- \Gamma^\beta_{ia}R(\tilde{g}_0)_{\alpha\bar{b}r}{}^{p}\Psi^\alpha_{\beta k} - \Gamma^\beta_{ka}R(\tilde{g}_0)_{\alpha\bar{b}r}{}^{p}\Psi^\alpha_{i\beta} + \Gamma^\alpha_{\beta a}R(\tilde{g}_0)_{\alpha\bar{b}r}{}^{p}\Psi^\beta_{ik}\Big).\end{aligned} \tag{3.58}$$

Putting these together and making use of our choice of coordinates,

$$
\begin{aligned}
\Big(\tilde{\nabla}_r\Delta - \Delta\tilde{\nabla}_r\Big)\Psi^p_{ik} = \tilde{\nabla}_r g^{\bar{b}a}\Big(&\tilde{\nabla}_a(R_{i\bar{b}k}{}^p - R(\tilde{g}_0)_{i\bar{b}k}{}^p) - \Psi^\alpha_{ia}(R_{\alpha\bar{b}k}{}^p - R(\tilde{g}_0)_{\alpha\bar{b}k}{}^p) \\
&- \Psi^\alpha_{ka}(R_{i\bar{b}\alpha}{}^p - R(\tilde{g}_0)_{i\bar{b}\alpha}{}^p) + \Psi^p_{\alpha a}(R_{i\bar{b}k}{}^\alpha - R(\tilde{g}_0)_{i\bar{b}k}{}^\alpha)\Big) \\
&+ g^{\bar{b}a}\Big(- \tilde{\nabla}_r\Psi^\alpha_{ia}(R_{\alpha\bar{b}k}{}^p - R(\tilde{g}_0)_{\alpha\bar{b}k}{}^p) \\
&- \tilde{\nabla}_r\Psi^\alpha_{ka}(R_{i\bar{b}\alpha}{}^p - R(\tilde{g}_0)_{i\bar{b}\alpha}{}^p) \\
&+ \tilde{\nabla}_r\Psi^p_{\alpha a}(R_{i\bar{b}k}{}^\alpha - R(\tilde{g}_0)_{i\bar{b}k}{}^\alpha) + \Psi^\beta_{ra}\tilde{\nabla}_\beta(R_{i\bar{b}k}{}^p - R(\tilde{g}_0)_{i\bar{b}k}{}^p) \\
&+ \tilde{\nabla}_a R(\tilde{g}_0)_{i\bar{b}r}{}^\alpha\Psi^p_{\alpha k} + R(\tilde{g}_0)_{i\bar{b}r}{}^\alpha\tilde{\nabla}_a\Psi^p_{\alpha k} - \Psi^\beta_{ia}R(\tilde{g}_0)_{\beta\bar{b}r}{}^\alpha\Psi^p_{\alpha k} \\
&- \Psi^\beta_{ra}R(\tilde{g}_0)_{i\bar{b}\beta}{}^\alpha\Psi^p_{\alpha k} + \Psi^\alpha_{\beta a}R(\tilde{g}_0)_{i\bar{b}r}{}^\beta\Psi^p_{\alpha k} - \Psi^\beta_{\alpha a}R(\tilde{g}_0)_{i\bar{b}r}{}^\alpha\Psi^p_{\beta k} \\
&- \Psi^\beta_{ka}R(\tilde{g}_0)_{i\bar{b}r}{}^\alpha\Psi^p_{\alpha\beta} + \Psi^p_{\beta a}R(\tilde{g}_0)_{i\bar{b}r}{}^\alpha\Psi^\beta_{\alpha k} + \tilde{\nabla}_a R(\tilde{g}_0)_{k\bar{b}r}{}^\alpha\Psi^p_{i\alpha} \\
&+ R(\tilde{g}_0)_{k\bar{b}r}{}^\alpha\tilde{\nabla}_a\Psi^p_{i\alpha} - \Psi^\beta_{ka}R(\tilde{g}_0)_{\beta\bar{b}r}{}^\alpha\Psi^p_{i\alpha} - \Psi^\beta_{ra}R(\tilde{g}_0)_{k\bar{b}\beta}{}^\alpha\Psi^p_{i\alpha} \\
&+ \Psi^\alpha_{\beta a}R(\tilde{g}_0)_{k\bar{b}r}{}^\beta\Psi^p_{i\alpha} - \Psi^\beta_{ia}R(\tilde{g}_0)_{k\bar{b}r}{}^\alpha\Psi^p_{\beta\alpha} - \Psi^\beta_{\alpha a}R(\tilde{g}_0)_{k\bar{b}r}{}^p\Psi^p_{i\beta} \\
&+ \Psi^p_{\beta a}R(\tilde{g}_0)_{k\bar{b}r}{}^\alpha\Psi^\beta_{i\alpha} - \tilde{\nabla}_a R(\tilde{g}_0)_{\alpha\bar{b}r}{}^p\Psi^\alpha_{ik} - R(\tilde{g}_0)_{\alpha\bar{b}r}{}^p\tilde{\nabla}_a\Psi^\alpha_{ik} \\
&+ \Psi^\beta_{\alpha a}R(\tilde{g}_0)_{\beta\bar{b}r}{}^p\Psi^\alpha_{ik} + \Psi^\beta_{ra}R(\tilde{g}_0)_{\alpha\bar{b}\beta}{}^p\Psi^\alpha_{ik} - \Psi^p_{\beta a}R(\tilde{g}_0)_{\alpha\bar{b}r}{}^\beta\Psi^p_{ik} \\
&+ \Psi^\beta_{ia}R(\tilde{g}_0)_{\alpha\bar{b}r}{}^p\Psi^\alpha_{\beta k} + \Psi^\beta_{ka}R(\tilde{g}_0)_{\alpha\bar{b}r}{}^p\Psi^\alpha_{i\beta} - \Psi^\alpha_{\beta a}R(\tilde{g}_0)_{\alpha\bar{b}r}{}^p\Psi^\beta_{ik}\Big).
\end{aligned}
\tag{3.59}
$$

Using (3.59) and Lemmas 3.2.4, 3.2.5 and 3.2.7, we can bound all the terms resulting from the final term of (3.51). Starting with the first term from (3.59):

$$
2\,\mathrm{Re}\left(g^{\bar{s}r}g^{\bar{j}i}g^{\bar{l}k}g_{p\bar{q}}\tilde{\nabla}_r g^{\bar{b}a}\tilde{\nabla}_a R_{i\bar{b}k}{}^p\overline{\tilde{\nabla}_s\Psi^q_{jl}}\right) \le C|\tilde{\nabla}g||\tilde{\nabla}\,\mathrm{Rm}(g)||\tilde{\nabla}\Psi|. \tag{3.60}
$$

We bound $|\tilde{\nabla}\,\mathrm{Rm}(g)|$ by observing that

$$
\left(\tilde{\nabla}_a - \nabla_a\right)R_{i\bar{l}p}{}^r = \Psi^\alpha_{ia}R_{\alpha\bar{l}p}{}^r + \Psi^\alpha_{pa}R_{i\bar{l}\alpha}{}^r - \Psi^r_{\alpha a}R_{i\bar{l}p}{}^\alpha, \tag{3.61}
$$

and so

$$
\begin{aligned}
|\tilde{\nabla}\,\mathrm{Rm}(g)|^2 &\le 2|(\tilde{\nabla} - \nabla)\,\mathrm{Rm}(g)|^2 + 2|\nabla\,\mathrm{Rm}(g)|^2 \\
&\le C|\Psi|^2 + C|\,\mathrm{Rm}(g)|^2 + 2|\nabla\,\mathrm{Rm}(g)|^2 \\
&\le C
\end{aligned}
\tag{3.62}
$$

where to get the last inequality we use Lemmas 3.2.5 and 3.2.7. Substituting (3.62) into (3.60) gives the bound

$$2\,\mathrm{Re}\left(g^{\bar{s}r}g^{\bar{j}i}g^{\bar{l}k}g_{p\bar{q}}\tilde{\nabla}_r g^{\bar{b}a}\tilde{\nabla}_a R_{i\bar{b}k}{}^{p}\overline{\tilde{\nabla}_s\Psi^q_{jl}}\right) \leq C|\tilde{\nabla}\Psi| \leq C|\tilde{\nabla}\Psi|^2 + C \tag{3.63}$$

For the second term from (3.59), using Lemmas 3.2.4 and 3.2.5,

$$\begin{aligned}2\,\mathrm{Re}\left(g^{\bar{s}r}g^{\bar{j}i}g^{\bar{l}k}g_{p\bar{q}}\tilde{\nabla}_r g^{\bar{b}a}R(\tilde{g}_0)_{i\bar{b}k}{}^{p}\overline{\tilde{\nabla}_s\Psi^q_{jl}}\right) &\leq C|\tilde{\nabla}g||\tilde{\nabla}\,\mathrm{Rm}(\tilde{g}_0)||\tilde{\nabla}\Psi| \\ &\leq C|\tilde{\nabla}\Psi|^2 + C. \end{aligned}\tag{3.64}$$

Similarly, we bound the remaining terms arising from (3.59) and obtain the estimate

$$|2\,\mathrm{Re}\left(g^{\bar{s}r}g^{\bar{j}i}g^{\bar{l}k}g_{p\bar{q}}\left(\tilde{\nabla}_r\Delta - \Delta\tilde{\nabla}_r\right)\Psi^p_{ik}\overline{\tilde{\nabla}_s\Psi^q_{jl}}\right)| \leq C|\tilde{\nabla}\Psi|^2 + C. \tag{3.65}$$

Substituting (3.55) and (3.65) into (3.51),

$$\left(\frac{\partial}{\partial t} - \Delta\right)|\tilde{\nabla}\Psi|^2 \leq C_2|\tilde{\nabla}\Psi|^2 + C. \tag{3.66}$$

By the definition of Ψ,

$$\nabla_l\Psi^k_{ij} - \tilde{\nabla}_l\Psi^k_{ij} = -\Psi^\alpha_{li}\Psi^k_{\alpha j} - \Psi^\alpha_{lj}\Psi^k_{i\alpha} + \Psi^k_{l\alpha}\Psi^\alpha_{ij}. \tag{3.67}$$

Using this with the Lemma 3.2.5, we have

$$|\tilde{\nabla}\Psi|^2 \leq 2|\nabla\Psi|^2 + 2|\tilde{\nabla}\Psi - \nabla\Psi|^2 \leq 2|\nabla\Psi|^2 + C. \tag{3.68}$$

Define the quantity $Q_1 = |\tilde{\nabla}\Psi|^2 + 2(C_1+1)|\Psi|^2$. Then using (3.38), (3.66), (3.68) and Lemma 3.2.5,

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right)Q_1 &\leq C_1|\tilde{\nabla}\Psi|^2 + C + 2(C_1+1)\left(C + C|\Psi|^2 - |\nabla\Psi|^2 - |\bar{\nabla}\Psi|^2\right) \\ &\leq -|\tilde{\nabla}\Psi|^2 + C.\end{aligned}\tag{3.69}$$

This gives a uniform bound for $|\tilde{\nabla}\Psi|^2$ and hence a uniform bound for $|\tilde{\nabla}g|^2$.

Now we may proceed inductively to derive estimates of any order. As in the case when $k = 1$, it will suffice to bound $|\tilde{\nabla}^k\Psi|^2$ by induction. Computing as in (3.51), the evolution equation of $|\tilde{\nabla}^k\Psi|^2$ is

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right)|\tilde{\nabla}^k\Psi|^2 &= (k+1)|\tilde{\nabla}^k\Psi|^2 - |\nabla\tilde{\nabla}^k\Psi|^2 - |\bar{\nabla}\tilde{\nabla}^k\Psi|^2 - 2\,\mathrm{Re}\left\langle\tilde{\nabla}^k T, \tilde{\nabla}^k\Psi\right\rangle \\ &\quad + 2\,\mathrm{Re}\left\langle\left(\tilde{\nabla}^k\Delta - \Delta\tilde{\nabla}^k\right)\Psi, \tilde{\nabla}^k\Psi\right\rangle,\end{aligned}\tag{3.70}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product with respect to g and where T is the tensor $T_{ij}^k = \nabla^{\bar{b}} R_{i\bar{b}j}{}^k$. We work in coordinates where $\tilde{g}_0$ is the identity and $\partial_i \tilde{g}_0 = 0, \partial_{i_1}\partial_{i_2}\tilde{g}_0 = 0, \ldots, \partial_{i_1}\partial_{i_2}\ldots\partial_{i_{k+1}}\tilde{g}_0 = 0$ at a point as in [70]. Using these coordinates, $\tilde{\Gamma} = 0, \ \ldots, \ \tilde{\nabla}^k\tilde{\Gamma} = 0$ and $\Gamma = \Psi, \ldots, \tilde{\nabla}^k\Gamma = \tilde{\nabla}^k\Psi$. Proceeding as we did to obtain (3.55), we bound the fourth term in (3.70) by $C|\tilde{\nabla}^k\Psi|^2 + C$ since all lower order derivatives of Ψ are bounded by induction. As in (3.59), the final term is made up of terms involving derivatives of curvature tensors and derivatives of Ψ of order less than or equal to k. All terms here are good, since a k-th order derivative of Ψ is what we are estimating, and by induction lower order derivatives of Ψ are bounded. Derivatives of order less than or equal to k of $\mathrm{Rm}(g)$ are bounded by induction and Lemma 3.2.7 since differentiation with respect to g and $\tilde{g}_0$ differ by terms involving lower order derivatives of Ψ as in (3.62). Any derivatives of $\mathrm{Rm}(\tilde{g}_0)$ are bounded by Lemma 3.2.4. As above, we obtain the estimate

$$\left(\frac{\partial}{\partial t} - \Delta\right)|\tilde{\nabla}^k\Psi|^2 \le C_k|\tilde{\nabla}^k\Psi|^2 + C. \tag{3.71}$$

We define the quantity $Q_k = |\tilde{\nabla}^k\Psi|^2 + 2(C_k+1)|\tilde{\nabla}^{k-1}\Psi|^2$. We have the inequality

$$\begin{aligned} |\tilde{\nabla}^k\Psi|^2 &\le 2|\nabla\tilde{\nabla}^{k-1}\Psi|^2 + 2|(\nabla - \tilde{\nabla})\tilde{\nabla}^{k-1}\Psi|^2 \\ &\le 2|\nabla\tilde{\nabla}^{k-1}\Psi|^2 + C \end{aligned} \tag{3.72}$$

since $(\nabla - \tilde{\nabla})\tilde{\nabla}^{k-1}\Psi$ is made up of terms involving Ψ and $\tilde{\nabla}^{k-1}\Psi$ and hence is bounded by the induction hypothesis. Then using this and (3.71), we have

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta\right) Q_k &\le C_k|\tilde{\nabla}^k\Psi|^2 + C + 2(C_k+1)\left(C - |\nabla\tilde{\nabla}^{k-1}\Psi|^2\right) \\ &\le -|\tilde{\nabla}^k\Psi|^2 + C \end{aligned} \tag{3.73}$$

giving us a bound for $|\tilde{\nabla}^k\Psi|^2$. □

Because of the symmetries of the metric tensor $g_{i\bar{j}}$, we obtain the following lemma bounding the barred derivatives of the metric.

Lemma 3.3.2. *There exists uniform $C(k) > 0$ for $k = 0, 1, 2, \ldots$ such that on $X \times [0,\infty)$,*

$$|\bar{\tilde{\nabla}}^k g|^2 \le C(k). \tag{3.74}$$

Using Lemmas 3.3.1 and 3.3.2, we construct estimates for all possible covariant derivatives of the metric.

Lemma 3.3.3. *There exists uniform* $C(k) > 0$ *for* $k = 0, 1, 2, \ldots$ *such that on* $X \times [0, \infty)$*,*

$$|\tilde{\nabla}_{\mathbb{R}}^k g|^2 \leq C(k), \tag{3.75}$$

where $\tilde{\nabla}_{\mathbb{R}}$ *is the covariant derivative with respect to* $\tilde{g}_0$ *as a Riemannian metric.*

Proof. Let $\mathbf{a} = (a_1, a_2, \ldots, a_k)$ be a k-tuple with symbolic entries z or $\bar{z}$. We define $\tilde{\nabla}^{a_i}$ to be the operator $\tilde{\nabla}$ if $a_i = z$ or $\bar{\tilde{\nabla}}$ if $a_i = \bar{z}$. Then we define $\tilde{\nabla}^{\mathbf{a}}$ to be the operator $\tilde{\nabla}^{a_1} \ldots \tilde{\nabla}^{a_k}$ (if $\mathbf{a}$ is a 0-tuple, define $\tilde{\nabla}^{\mathbf{a}}$ to be the identity). To prove the lemma, it suffices to bound the quantity $|\tilde{\nabla}^{\mathbf{a}} g|^2$.

We will proceed by induction on k. The case where $k = 1$ is handled by Lemmas 3.3.1 and 3.3.2. For the general k we may assume that there exists an index l such that $a_l = z$, otherwise we are done by Lemma 3.3.2. Choose l to be the greatest index such that $a_l = z$ and define $\mathbf{a}'$ to be the $(l-1)$-tuple containing the first $l - 1$ entries of $\mathbf{a}$. If $l = k$, we observe that a bound on $|\tilde{\nabla}^{\mathbf{a}} g|^2$ will follow from a bound on $|\tilde{\nabla}^{\mathbf{a}'} \Psi|^2$.

We will introduce some notation: if A and B are tensors, let $A * B$ denote any linear combination of products of A and B formed by contractions with the metric g. If l is not equal to k, by commuting the covariant derivatives, we have

$$\begin{aligned}
\tilde{\nabla}^{\mathbf{a}} g &= \tilde{\nabla}^{\mathbf{a}'} \tilde{\nabla} \bar{\tilde{\nabla}}^{k-l} g \\
&= \tilde{\nabla}^{\mathbf{a}'} \left(\bar{\tilde{\nabla}} \tilde{\nabla} \bar{\tilde{\nabla}}^{k-l-1} g + \mathrm{Rm}(\tilde{g}_0) * \bar{\tilde{\nabla}}^{k-l-1} g \right) \\
&= \tilde{\nabla}^{\mathbf{a}'} \left(\bar{\tilde{\nabla}}^{k-l} \tilde{\nabla} g + \bar{\tilde{\nabla}}^{k-l-1} \mathrm{Rm}(\tilde{g}_0) * g + \ldots + \mathrm{Rm}(\tilde{g}_0) * \bar{\tilde{\nabla}}^{k-l-1} g \right).
\end{aligned} \tag{3.76}$$

Hence a bound on $|\tilde{\nabla}^{\mathbf{a}} g|^2$ follows from a bound on $|\tilde{\nabla}^{\mathbf{a}'} \bar{\tilde{\nabla}}^{k-l} \Psi|^2$ since the other terms are bounded by Lemma 3.2.4 and induction. We will now complete the proof by bounding $|\tilde{\nabla}^{\mathbf{a}'} \Psi|^2$ for a general $(k-1)$-tuple $\mathbf{a}'$.

Notice that if every entry of $\mathbf{a}'$ is z or if every entry of $\mathbf{a}'$ is $\bar{z}$, the proof is complete by Lemmas 3.3.1 and 3.3.2. Now let r be the greatest index such that $a'_r = \bar{z}$ and define $\mathbf{a}''$ to be the $(r-1)$-tuple containing the first $r - 1$ entries of $\mathbf{a}'$.

If $r = k-1$, then

$$|\tilde{\nabla}^{\mathbf{a}'}\Psi|^2 = |\tilde{\nabla}^{\mathbf{a}''}\bar{\tilde{\nabla}}\Psi|^2 = |\tilde{\nabla}^{\mathbf{a}''}(\mathrm{Rm}(g) - \mathrm{Rm}(\tilde{g}_0))|^2 \le |\tilde{\nabla}^{\mathbf{a}''}\mathrm{Rm}(g)|^2 + |\tilde{\nabla}^{\mathbf{a}''}(\mathrm{Rm}(\tilde{g}_0)|^2. \tag{3.77}$$

Notice that the second term in the right hand side of (3.77) is bounded by Lemma 3.2.4. We observe that $\tilde{\nabla}^{\mathbf{a}''}\mathrm{Rm}(g)$ differs from $\nabla^{\mathbf{a}''}\mathrm{Rm}(g)$ only by terms involving $\mathrm{Rm}(g), \ldots, \nabla_{\mathbb{R}}^{k-3}\mathrm{Rm}(g)$ and $\Psi, \ldots, \tilde{\nabla}_{\mathbb{R}}^{k-3}\Psi$. By induction and Lemma 3.2.7, we have a bound for $\tilde{\nabla}^{\mathbf{a}''}\mathrm{Rm}(g)$ and hence

$$|\tilde{\nabla}^{\mathbf{a}'}\Psi|^2 \le C. \tag{3.78}$$

If $r < l-1$, we commute the covariant derivatives,

$$\begin{aligned}
\tilde{\nabla}^{\mathbf{a}'}\Psi &= \tilde{\nabla}^{\mathbf{a}''}\bar{\tilde{\nabla}}\tilde{\nabla}^{l-1-r}\Psi \\
&= \tilde{\nabla}^{\mathbf{a}''}\left(\tilde{\nabla}\bar{\tilde{\nabla}}\tilde{\nabla}^{l-r-2}\Psi + \mathrm{Rm}(\tilde{g}_0) * \tilde{\nabla}^{l-r-2}\Psi\right) \\
&= \tilde{\nabla}^{\mathbf{a}''}\left(\tilde{\nabla}^{l-r-1}\bar{\tilde{\nabla}}\Psi + \tilde{\nabla}^{l-r-2}\mathrm{Rm}(\tilde{g}_0) * \Psi + \ldots + \mathrm{Rm}(\tilde{g}_0) * \tilde{\nabla}^{l-r-2}\Psi\right).
\end{aligned} \tag{3.79}$$

Notice that the norm of the first term of (3.79) is bounded as in (3.78) and the norms of the other terms are bounded by induction and Lemma 3.2.4, completing the proof. □

3.4 Convergence

In this section we will complete the proof of the main theorem by showing that $\omega(t)$ converges smoothly to ω_M as $t \to \infty$. Fix $z \in M$ and define a function ρ_z on $E(z) := \pi_M^{-1}(z)$ by

$$\omega_0|_{E(z)} + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\rho_z > 0, \quad \mathrm{Ric}\left(\omega_0|_{E(z)} + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\rho_z\right) = 0, \quad \int_{E(z)} \rho_z\omega_0^n = 0. \tag{3.80}$$

Note that since ρ_z varies smoothly with z, we may define a smooth function $\rho(z, e)$ on X. Then

$$\omega_{flat} := \omega_0 + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\rho \tag{3.81}$$

determines a closed $(1,1)$-form on X with $[\omega_{flat}] = [\omega_0]$. Also, ω_{flat} may not be a metric on X, but $\omega_{flat}|_{E(z)}$ is a flat Kähler metric on each fiber.

We will now prove the following estimate for φ, which will give us the convergence of $\omega(t)$.

Lemma 3.4.1. *There exists uniform $C > 0$ such that on $X \times [0, \infty)$,*

$$|\varphi| \leq C(1+t)e^{-t}. \tag{3.82}$$

Proof. This proof follows similarly as in [62]. To simplify notation, let b_k denote the binomial coefficient $b_k = \binom{m+n}{k}$. Then using (3.3) and the fact that $[\omega_{flat}] = [\omega_0]$,

$$\Omega = b_m \omega_M^m \wedge \omega_{flat}^n. \tag{3.83}$$

We define the quantity $Q = \varphi - e^{-t}\rho$ and calculate its evolution

$$\begin{aligned} \frac{\partial}{\partial t} Q &= \log \frac{e^{nt}\left(\hat{\omega}_t + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi\right)^{m+n}}{b_m \omega_M^m \wedge \omega_{flat}^n} - \varphi + e^{-t}\rho \\ &= \log \frac{e^{nt}\left(e^{-t}\omega_{flat} + \left(1-e^{-t}\right)\omega_M + \sqrt{-1}\partial\bar{\partial}Q\right)^{m+n}}{b_m \omega_M^m \wedge \omega_{flat}^n} - Q. \end{aligned} \tag{3.84}$$

Now let $Q_1 = e^t Q - At$ where A is a constant to be determined later. Suppose Q_1 attains its maximum at a point (z_0, t_0) with $t_0 > 0$, then at that point

$$\begin{aligned} 0 \leq \frac{\partial}{\partial t} Q_1 &\leq e^t \log \frac{e^{nt}\left(e^{-t}\omega_{flat} + \left(1-e^{-t}\right)\omega_M\right)^{m+n}}{b_m \omega_M^m \wedge \omega_{flat}^n} - A \\ &= e^t \log \frac{e^{nt}\left(b_m e^{-nt}(1-e^{-t})^m \omega_M^m \wedge \omega_{flat}^n + \ldots + e^{-(m+n)t}\omega_{flat}^{m+n}\right)}{b_m \omega_M^m \wedge \omega_{flat}^n} - A \\ &\leq e^t \log\left(1 + C_1 e^{-t} + \ldots + C_m e^{-mt}\right) - A \\ &\leq C - A. \end{aligned} \tag{3.85}$$

If we choose $A > C$, we obtain a contradiction and hence Q_1 must attain its maximum at $t = 0$. This gives the estimate $\varphi \leq C\,(1+t)\,e^{-t}$, and we can similarly obtain a lower bound. □

We may now complete the proof of the main theorem.

Proof. Using Lemma 3.3.3, Lemma 3.4.1 and the definition of $\omega(t)$, we immediately see that $\omega(t) \to \omega_M$ in C^∞ as $t \to \infty$ proving part (a).

We will restrict Lemma 3.2.5 to $E(z)$ using a method similar to that in [74]. Choose complex coordinates $x^{m+1}, \ldots, x^{m+n}$ on E so that g_E is the identity and $g|_E$ is diagonal with entries $\lambda_{m+1}, \ldots, \lambda_{m+n}$. Then choose complex coordinates $x^1, \ldots, x^m$ on X such that at a point p the space spanned by $\frac{\partial}{\partial x^1}|_p, \ldots, \frac{\partial}{\partial x^m}|_p$ is orthogonal to the space spanned by $\frac{\partial}{\partial x^{m+1}}|_p, \ldots, \frac{\partial}{\partial x^{m+n}}|_p$ with respect to g. In this coordinate system, g is diagonal with entries $\lambda_1, \ldots, \lambda_{m+n}$, and so

$$\begin{aligned} \left|\nabla_E g|_{E(z)}\right|^2_{g|_{E(z)}} &= \sum_{i,j,k=m+1}^{m+n} \frac{1}{\lambda_i\lambda_j\lambda_k} \tilde{\nabla}_k g_{i\bar{j}}|_{E(z)} \overline{\tilde{\nabla}_k g_{i\bar{j}}|_{E(z)}} \\ &\le \sum_{i,j,k=1}^{m+n} \frac{1}{\lambda_i\lambda_j\lambda_k} \tilde{\nabla}_k g_{i\bar{j}} \overline{\tilde{\nabla}_k g_{i\bar{j}}} \\ &= |\tilde{\nabla} g|^2 \le C. \end{aligned} \tag{3.86}$$

By restricting the uniform equivalence of g and $\tilde{g}_t$ to $E(z)$, we see that $g|_{E(z)}$ is uniformly equivalent to $e^{-t}g_E$. Using this fact coupled with (3.86) we estimate the derivative of $e^t g|_{E(z)}$.

$$\begin{aligned} |\nabla_E e^t g|_{E(z)}|^2_{g_E} &= e^{2t} g_E^{\bar{j}i} g_E^{\bar{l}k} g_E^{\bar{q}p} \nabla_{E,i}(g|_{E(z)})_{k\bar{q}} \overline{\nabla_{E,j}(g|_{E(z)})_{l\bar{p}}} \\ &\le Ce^{-t}(g_E)^{\bar{j}i}(g_E)^{\bar{l}k}(g_E)^{\bar{q}p} \nabla_{E,i}(g|_{E(z)})_{k\bar{q}} \overline{\nabla_{E,j}(g|_{E(z)})_{l\bar{p}}} \\ &= Ce^{-t}\left|\nabla_E g|_{E(z)}\right|^2_{g|_{E(z)}} \\ &\le C'e^{-t}. \end{aligned} \tag{3.87}$$

Similarly, we obtain estimates for the k-th order derivative of $e^t g|_{E(z)}$:

$$|\nabla_E^k e^t g|_{E(z)}|^2_{g_E} \le Ce^{-kt}. \tag{3.88}$$

We constructed g_{flat} to be a flat metric when restricted to the complex torus $E(z)$, and so it is given by a constant Hermitian metric on $\mathbb{C}^n$. Using a standard coordinate system for $E(z)$, we see that $\nabla_E^k g_{flat} = 0$ for all k, thus

$$|\nabla_E^k (e^t g|_{E(z)} - g_{flat}|_{E(z)})|^2_{g_E} \le Ce^{-kt}. \tag{3.89}$$

It remains to show that $e^t g|_{E(z)} \to g_{flat}|_{E(z)}$ in $C^0(E(z))$. Define a function ψ on $E(z)$ by

$$\psi = e^{-t}\varphi|_{E(z)} - \rho_z. \tag{3.90}$$

Letting Δ_E denote the Laplacian with respect to g_E,

$$\Delta_E\psi = \mathrm{tr}_{g_E}(e^t g|_{E(z)} - g_{flat}|_{E(z)}). \tag{3.91}$$

Combining (3.89) with $k = 1$ and (3.91) gives the estimate

$$|\nabla_E \Delta_E \psi|^2_{g_E} \leq Ce^{-t}. \tag{3.92}$$

Since $\int_E \Delta_E \psi \omega_E^n = 0$, for each time t there exists a point $y(t)$ in $E(z)$ so that $\psi(y(t), t) = 0$. Applying the Mean Value Theorem with (3.92) shows that

$$|\Delta_E\psi(x,t)|^2_{g_E} = |\Delta_E\psi(x,t) - \Delta_E\psi(y(t),t)|^2_{g_E} \to 0 \tag{3.93}$$

as $t \to \infty$. (3.89), (3.91) and (3.93) show that $e^t g|_{E(z)} \to g_{flat}|_{E(z)}$ in C^∞ on $E(z)$, completing the proof of the main theorem. □

Chapter 3, in full, is currently being prepared for submission for publication of the material. Gill, Matthew. The dissertation author was the author of this material.

Chapter 4

Future work

4.1 Evolution by the Chern-Ricci form

Let M be a complex manifold, let $(g_0)_{i\bar{j}}$ be a Hermitian metric on M, and let $\omega_0 = \sqrt{-1}(g_0)_{i\bar{j}} dz^i \wedge dz^{\bar{j}}$ be the associated $(1,1)$-form. Consider the flow, called the Chern-Ricci flow in [77],

$$\frac{\partial}{\partial t}\omega = -\operatorname{Ric}(\omega), \quad \omega|_{t=0} = \omega_0 \tag{4.1}$$

where $\operatorname{Ric}(\omega)$ is the Chern-Ricci form given in local coordinates by

$$\operatorname{Ric}(\omega) = -\sqrt{-1}\partial\bar{\partial}\log\det g. \tag{4.2}$$

We remark that if ω is Kähler, then the Chern-Ricci flow becomes the Kähler-Ricci flow. Since the Chern-Ricci form is a closed real $(1,1)$-form, it gives rise to a cohomology class denoted $c_1^{BC}(M)$ in the Bott-Chern cohomology group

$$H^{1,1}_{BC}(M,\mathbb{R}) = \frac{\{\text{closed real } (1,1) \text{ forms}\}}{\{\sqrt{-1}\partial\bar{\partial}\psi, \psi \in C^\infty(M,\mathbb{R})\}}. \tag{4.3}$$

Given the condition $c_1^{BC}(M) = 0$, we may find a smooth function F such that

$$\sqrt{-1}\partial\bar{\partial}\log\det g_0 = \sqrt{-1}\partial\bar{\partial}F. \tag{4.4}$$

If φ is the solution to the parabolic complex Monge-Ampère equation

$$\frac{\partial}{\partial t}\varphi = \log\frac{\det((g_0)_{i\bar{j}} + \partial_i\partial_{\bar{j}}\varphi)}{\det(g_0)_{i\bar{j}}} - F, \; (g_0)_{i\bar{j}} + \partial_i\partial_{\bar{j}}\varphi > 0, \; \varphi|_{t=0} = 0, \tag{4.5}$$

as in Theorem 2.1.1, then $\omega(t) = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi(t)$ solves the Chern-Ricci flow (4.1). This gives the following corollary to Theorem 2.1.1:

Corollary 4.1.1. *If $c_1^{BC}(M) = 0$, then for any Hermitian metric ω_0, there exists a solution $\omega(t)$ to the Chern-Ricci flow* (4.1) *for all time and the metrics $\omega(t)$ converge smoothly as $t \to \infty$ to a Hermitian metric ω_∞ with* $\mathrm{Ric}(\omega_i nfty) = 0$.

In general, the Chern-Ricci flow is more complicated if the first Bott-Chern class is nonzero. In this case studied by Tosatti and Weinkove [77], we replace g_0 in the Monge-Ampère equation by a smoothly varying family of metrics $\tilde{g}_t$ and also modify F. They prove a number of interesting results on the Chern-Ricci flow. First, they show that the Chern-Ricci flow has a unique solution on a maximal time interval $[0, T)$, where the higher order estimates of chapter 2 are used in this proof. Additionally, if we define

$$\alpha_t = \omega_0 - t\,\mathrm{Ric}(\omega_0) \tag{4.6}$$

then

$$T = \sup\{t \geq 0 | \exists\psi \in C^\infty(M), \alpha_t + \sqrt{-1}\partial\bar{\partial}\psi > 0\}. \tag{4.7}$$

This is analogous to the result on the maximal time interval for the Kähler-Ricci flow of Tian-Zhang [72]. In the case of a complex surface, Tosatti and Weinkove provide several interesting geometric results for the Chern-Ricci flow:

Theorem 4.1.2. *(Tosatti-Weinkove) Let M be a compact complex surface, ω_0 a $\partial\bar{\partial}$-closed Hermitian metric. Then the Chern-Ricci flow* (4.1) *exists until either the volume of M goes to zero, or the volume of a curve of negative self-intersection goes to zero.*

Theorem 4.1.3. *(Tosatti-Weinkove) Let M be a compact complex surface, ω_0 a $\partial\bar{\partial}$-closed Hermitian metric, and let $[0, T)$ be the maximual existence time of the Chern-Ricci flow* (4.1). *Then*

(a) If $T = \infty$, then M is minimal

(b) If $T < \infty$ and $\mathrm{Vol}(M, \omega(t)) \to 0$ as $t \to T^-$, then M is either birational to a ruled surface or it is a surface of class VII (and in this case it cannot be an Inoue surface)

(c) If $T < \infty$ and $\mathrm{Vol}(M, \omega(t))$ *stays positive as $t \to T^-$, then M contains* (-1)*-curves.*

Furthermore, if M is minimal than $T = \infty$ unless M is $\mathbb{CP}^2$, a ruled surface, a Hopf surface or a surface of class VII with $b_2 > 0$, in which cases (*b*) *holds.*

Moreoever, they conjecture that the Chern-Ricci flow on $\partial\bar{\partial}$-closed metrics behaves like the Kähler-Ricci flow on Kähler surfaces and examine the flow on comlex manifolds with negative first Chern class and also on Hopf manifolds.

Theorem 2.1.1 and [77] create several potential products in complex geoemtry. One can construct examples of non-Kähler solutions to the Chern-Ricci flow and also continue building up the theory of the Chern-Ricci flow as a Hermitian analogue of the Kähler-Ricci flow.

4.2 The Minimal Model Program

As mentioned in the introduction to Chapter 2, there has been a lot of progress in viewing the Kähler-Ricci flow as an analytic minimal model program with many questions still unanswered. In the case of collapsing in infinite time, Theorem 3.1.1 provides a start in showing that the fibers collapse smoothly along the Kähler-Ricci flow. The author plans to extend these convergence results to more general settings, for example when the canonical class of the base is big and nef. Additionally, one can examine the case of a holomorphic fibration as considered in [20] or the more general case considered in [57].

In the case of finite time collapsing, for example on a Mori fiber space (an algebraic fibration $f : X \to B$ where the generic fibers are Fano), less is known about the smoothness of convergence. On a projective bundle, Song-Székelyhidi-Weinkove showed that if the initial metric satisfies the condition $[\omega_0] - Tc_1(X) = [\pi^*\omega_B]$ for some Kähler metric ω_B on B, then the $\mathbb{P}^n$ fibers collapse in finite time under the Kähler-Ricci flow [55]. Moreover, they showed that there is a subsequence in time $\{t_i\}$ so that $(X, \omega(t_i))$ converges to (B, d_B) in the Gromov-Hausdorff sense. Fong improved the rate of collapse under an assumption on the Ricci curvature [19].

Perelman has shown that if ω_0 represents the first Chern class and the fibers are Fano (as in a Mori fiber space), the scalar curvature and diameter remain bounded along the normalized Kähler-Ricci flow [42, 52]. Considering the unnormalized Kähler-Ricci flow, this shows that the flow contracts to a point in the Gromov-Hausdorff sense. Because of this, Perelman's techniques may be useful in establishing smooth convegence for the flow.

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