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Superstochastic Systems

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To my grandmothers

Contents

Abstract

A general theory of measurement for a statistical system describes states the system might occupy, observables that might be measured by apparatuses and expected values of outcomes obtained upon measurements. We develop a unified measurement theory for stochastic systems with both continuous and discrete degrees of freedom. The theory is geometric as well as time-covariant, thanks to an odd dimensional generalization of symplectic geometry. Observables are superfunctions on a symplectic supermanifold, states belong to a convex cone in the space of superfunctions and expectation values are the result of a fiberwise Hodge pairing between states and observables. Locality of measurements is captured by the sheaf structure of supermanifolds. Our geometric model also comes with a canonical notion of laboratories that describes instantaneous measurements in terms of embedded symplectic hypersurfaces.

Supermanifolds, unlike ordinary manifolds, include anti-commuting Grassmann directions that naturally encode discrete degrees of freedom. The algebra of Grassmann variables is however non-trivial. To interpret the algebra operations physically, we define a complete set of super indicators parameterized by elements of a σ-algebra on a discrete sample space. While the product of indicator superfunctions mimics the boolean algebra of sets for independent events, their statistics like moments reveal intricate combinatorial information related to parity and ordering of events. We characterize the uncertainity in joint measurements by a generalized notion of covariance that relies on the Jordan product of superfunctions. We also comment on some key classical features of our measurement theory.

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I have always dreamed of working on meaningful problems and being able to share the joy of learning with those around me. I am grateful to have been able to find a home and an academic haven during my time here in Davis. This dissertation is a culmination of my journey through graduate school, that has been nothing short of remarkable—largely because of the kindness and support of my professors, friends and family. My parents, despite being halfway across the planet, offered unwavering support that carried me through the highs and lows of this journey. My paternal and maternal grandmother always wished to see me succeed in my endeavors. I would like to dedicate this dissertation to their fond memory.

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facets of quantum entropy. His weekly seminars gave me a glimpse into elegant and rigorous approaches to quantum spin systems and phases of quantum matter, that I otherwise wouldn't have known about. He encouraged me to my participate in the 21st International Congress of Mathematical Physics (ICMP) at Strasbourg, France, where I got a chance to interact with many leading mathematical physicists. Bruno's positivity, friendliness and welcoming demeanor made me feel home at QMAP.

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CHAPTER 1

Introduction

Our understanding of the physical world is based on empirical measurements. For many physical systems, measurements made on identical copies called *samples*, do not yield the same result and are possibly completely random. However, the average measurement result of a large collection of such identical systems, called a *population*, might turn out to be predictable. A *statistical* system is one whose samples are random but population as a whole follows some probability distribution. The origin of the statistical behavior could be traced down to random microscopic interactions within the system or between the system and environment, or random disturbances due to measuring apparatus. We would like to study the dynamics of statistical systems.

Stochastic systems are dynamical probabilistic systems. In the upcoming sections, we describe basic phenomenological aspects of stochastic systems and provide measure-theoretic definitions of stochastic processes and dynamical systems. Some good references on the mathematical foundations of probability theory and stochastic processes are [8, 56, 57]. For a historical overview of the fundamental ideas of probability theory due to Kolmogorov, Doob, Lévy, Khinchin and Feller see [51].

1.1. Stochastic systems

A physical system Ω may be characterized by a set of measurable properties $\mathcal{O}_\Omega$ called observables measured by apparatuses. Let $\mathrm{Spec}(\mathcal{O}_\Omega)$ be the set of all possible readings or *spectra* of independent apparatuses: tuples of real numbers $(x_1, x_2, \ldots) \in \mathrm{Spec}(\mathcal{O}_\Omega)$. A *dynamical* system is one whose measurable properties evolve over time. As the system evolves for some time T, an experimenter obtains a time-parameterized list of readings $\{X(t) \in \mathbb{R} \mid t \in [0, T]\}$ for every apparatus $X \in \mathcal{O}_\Omega$. The same experiment with identical initial conditions could be repeated n times to collect batches of data. These data can be summarized by a family of maps $\{\gamma_i^X : t \mapsto X_i(t) \mid i \in \{1, 2, \ldots, n\}\}$. A scientist could now look at the data in hopes of finding patterns in the way the measured properties $X \in \mathcal{O}_\Omega$ evolve. In an ideal world, there might be a

mathematical function $f : \mathrm{Spec}(\mathcal{O}_\Omega) \times [0,T] \to \mathbb{R}$ satisfying

$$f(\{\gamma_i^X(t)\}, t) = 0\,,$$

that fits all the data for every individual trial $i \in \{1, 2, \ldots, n\}$ of the experiment. For most physical systems however, a statistical law

$$f(\{\mathbb{E}[\gamma^X(t)]\}, t) \to 0, \quad n \to \infty\,,$$

where $\mathbb{E}[\gamma^X(t)]$ is the sample average of all the readings of property X at time t, is the best we can hope to discover. Dynamical systems obeying such statistical laws will be called *stochastic*.

An interesting example of stochastic system is a large collection of microscopic particles suspended in a fluid. The microscopic particles are constantly exchanging energy and momentum upon colliding with surrounding fluid particles, so that individual particles undergo random walks in the fluid called *Brownian motion*. As time passes, particles diffuse and the diffusion obeys Einstein's statistical formula

$$\mathbb{E}[x^2] - 2Dt = 0, \quad D = \mu k_B T\,,$$

where $\mathbb{E}[x^2]$ is the mean squared displacement of the particle, D is the diffusion constant and t is time since the start of diffusion. The diffusion constant D is proportional to the mobility μ, k_B is the Boltzmann constant and T is temperature of the fluid. This model was a big leap in our understanding of the microscopic nature of matter. In particular, Einstein's formula allows one to estimate Avogadro's number $N_A = \frac{R}{k_B}$ (counting the number of suspended microscopic particles) since all other quantities can be experimentally measured.

1.2. Probability spaces

Stochastic systems are often modeled by a probability space $(\Omega, \mathcal{F}, P)$ where Ω is the set of possible outcomes of an experiment called the *sample space*, $\mathcal{F}$ is a σ-algebra of subsets of Ω called *events* and $P : \mathcal{F} \to [0,1]$ is a probability measure on Ω.

Measurable properties of stochastic systems are studied using measuring apparatuses. The set of apparatuses $\mathcal{O}_\Omega$ used to probe the system Ω are now described by random variables. A *random variable* $X \in \mathcal{O}_\Omega$ is a map

$$\mathcal{O}_\Omega \ni X : (\Omega, \mathcal{F}) \to (\mathcal{S}, \Sigma)\,.$$

between two measurable spaces: sample space $(\Omega, \mathcal{F})$ and the *value space* $(\mathcal{S}, \Sigma)$ (sometimes also called *state space*, but we will reserve the *state* nomenclature for generalized probability measures *cf.* Chapter 3 Section 3.4 in this thesis). For continuous physical systems, the value space is often taken to be the Borel space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ or its Cartesian power, where $\mathcal{B}(\mathbb{R})$ is the σ-algebra generated by open subsets of $\mathbb{R}$ called the *Borel σ-algebra* of $\mathbb{R}$. The image $X(\Omega) \subset \mathcal{S}$ of the map X models the range of values that could be measured by the apparatus associated to X in an experiment.

For a realistic physical system, we typically do not have detailed knowledge of its properties and hence its full sample space $(\Omega, \mathcal{F}, P)$. The goal then is to figure out as much information about the sample space (including its probability measure) as possible from the data collected by measuring apparatuses. Measurements using each apparatus will give an empirical probability distribution on its outcomes. We would thus like to assign a probability measure to the measurement results (elements of $X(\Omega) \subset \mathcal{S}$) associated to a random variable $X \in \mathcal{O}_\Omega$. This means we must be able to give $(\mathcal{S}, \Sigma)$ the structure of a probability space using the probability measure P on Ω. For this, we will require the map X be *measurable*

$$X^{-1}(V) \in \mathcal{F}, \quad V \in \Sigma\,,$$

which means inverse image of measurable sets are measurable. This now enables use to define the pushforward measure P_X on $\mathcal{S}$

$$P_X(V) := P(X^{-1}(V)), \quad V \in \Sigma\,.$$

The pushforward measure on $\mathcal{S}$ defined using the map X is called the *probability distribution* of the random variable X. Using the probability distribution P_X, *expectation value* $\mathbb{E}[X]$ of the random variable X can be defined

$$\mathbb{E}[X] := \int_{\mathcal{S}} x \; dP_X(x) = \int_\Omega X(\omega) dP(\omega)\,.$$

A random field $\{X_i : i \in I\}$ is a collection of random variables parameterized by an indexing set I, and defined with respect to a common sample space $(\Omega, \mathcal{F}, P)$ and value space $(\mathcal{S}, \Sigma)$. A stochastic process is a special type of random field. We will now give the simplest definition of a stochastic process, originally due to Khinchin [59].

DEFINITION 1.2.1. *Let $\mathcal{P} = (\Omega, \mathcal{F}, P)$ be a probability space, $(\mathcal{S}, \Sigma)$ a measurable space and T an indexing set. A stochastic process $\boldsymbol{X}$ with outcomes valued in $\mathcal{S}$ is a one-parameter family $\{X_t : t \in T\}$ of random variables on the probability space $\mathcal{P}$, or equivalently a collection of random paths in $\mathcal{S}$*

$$\boldsymbol{X} : \Omega \to \mathrm{Maps}(T, \mathcal{S}) \,.$$

This in turn gives a one-parameter family of probability distributions $\{P_X(t)\}$ on the value space $(\mathcal{S}, \Sigma)$, or equivalently a pushforward measure $P_{\boldsymbol{X}}$ on the path space $\mathcal{S}^T := \mathrm{Maps}(T, \mathcal{S})$. For most applications, the infinite dimensional path space is not practical. One is able to experimentally determine only a collection of finite dimensional marginal distributions $\{P_{t_1 t_2 \dots t_k} : t_i \neq t_j \in T, k \in \mathbb{N}\}$ corresponding to (time-) discretized random paths $\{\boldsymbol{X}\big|_I : I \subset T, |I| < \infty\}$. Kolmogorov's extension theorem (see Appendix D of [8] for details) gives necessary and sufficient "consistency" conditions on $P_{t_1 t_2 \dots t_k}$ that guarantees the existence of a stochastic law or a probability measure $P_{\boldsymbol{X}}$ on the path space $\mathcal{S}^T$.

1.3. Dynamical systems

A dynamical physical system may be characterized by a one-parameter family of measure-preserving maps $\{\phi_t : (\Omega, \mathcal{F}, P) \to (\Omega, \mathcal{F}, P) : t \in T\}$ on the sample space Ω. Given a random variable $X_0 : (\Omega, \mathcal{F}) \to (\mathcal{S}, \Sigma)$, there is an induced stochastic process

$$\boldsymbol{X} = \{X_t := X_0 \circ \phi_t : t \in T\} \,.$$

In classical mechanics, ϕ_t is a one-parameter family of symplectomorphisms on a symplectic sample space

$$\Omega = (M^{2n}, \tau), \quad \mathcal{F} = \mathcal{B}(\tau), \quad P = \frac{\int \omega^{\wedge n} \,\cdot}{\int \omega^{\wedge n}} \,,$$

where ω is the symplectic form on M, $\mathcal{B}(\tau)$ is the Borel σ-algebra generated by the topology τ on a $2n$-dimensional symplectic manifold M and P is the probability measure corresponding to the symplectic volume-form $\omega^{\wedge n}$. The random variable $X_0 : M \to \mathbb{R}$ is a real-valued smooth function on the symplectic manifold M. The one-parameter family of probability distributions $\{P_X(t)\}$ satisfies a conservation law

$$\mathbb{E}[X(t)] = \int_{\mathbb{R}} X(t) dP_{X(t)} = \int_{\mathbb{R}} X_0 dP_{X_0} = \mathbb{E}[X_0] \,.$$

The time-dependent probability distribution $\{P_X(t) : t \in T\}$ is called a *dynamical state*. Stochastic processes $\boldsymbol{X}$ not directly induced from the system's dynamics might still obey generalized conservation laws [25]. In Chapter 3, we will give a general definition of states associated to $*$-algebras.

In this thesis, we describe a class of stochastic systems called *superstochastic systems*, characterized by measurable properties $\mathcal{O}_\Omega$ that arise from the geometry of a supermanifold. The evolution parameter, time, is incorporated as an intrinsic coordinate of the supermanifold, ensuring a fully covariant formulation. In Chapter 4, we define a sample space Ω associated to the algebra of functions $\mathcal{O}_\Omega$ on a supermanifold. The product of functions on a supermanifold is non-commutative. By constructing generalized indicators, we show that the superproduct captures non-trivial combinatorics of discrete sample spaces. In Chapter 5, we show that Hamiltonian superstochastic processes exhibit Markov-like behavior.

Next, we discuss key features of superstochastic systems and explore why supergeometry is relevant to probabilists and statisticians. Each section begins with an overview, followed by pointers to where detailed discussions can be found later in the thesis.

1.4. Grassmann algebra

Superstochastic systems generalize stochastic systems modelled on geometric spaces such as symplectic manifolds. Points $x \in U \subset M$ on a symplectic manifold M of dimension $2n$ can be represented by coordinates $(q^i = q^i(x), p_i = p_i(x))$ with respect to some neighborhood chart (U, ϕ)

$$\begin{array}{rccc} \phi: & U & \xrightarrow{\cong} & V \subset \mathbb{R}^{2n} \\ & x & \mapsto & \phi(x) = (q^i(x), p_i(x))\,. \end{array}$$

The points (q^i, p_i) on a symplectic manifold M are associated to the continuous spectra of mechanical devices like rulers and speedometers used in studying the stochastic system. The coordinate functions $x^A \in \{q^i(x), p_i(x)\}$ are commuting random variables

$$x^A x^B - x^B x^A = 0\,,$$

where the product is the pointwise product of functions.

Many experiments may also either measure discrete properties of the system or use digital instruments like multimeter or oscilloscope. These digital devices output discrete readings. This

calls for a geometric model that accommodates both continuous and discrete variables. Supermanifolds are geometric spaces perfectly suited to model both continuous and discrete spectra. This is because, in addition to commuting coordinates x^A of a manifold, they contain anticommuting Grassmann coordinates θ^α

$$\theta^\alpha\theta^\beta + \theta^\beta\theta^\alpha = 0\,,$$

that mimic properties of discrete Boolean variables. Unlike coordinates on the manifold, which correspond to coordinate functions, the Grassmann coordinates are generators of local sections of a sheaf of supercommuting algebras. This generalization from functions on a manifold to sections of a sheaf is a key step towards a generalized geometric measurement theory.

- In Chapter 2 Section 2.3.2, we give the sheaf-theoretic definition of a supermanifold.
- In Chapter 2 Section 2.1, we discuss microscopic and macroscopic physical systems having continuous and discrete degrees of freedom.
- In Chapter 3 Section 3.6 spectrum is defined from an algebraic point of view. We also give examples of spectra of algebras which are topological spaces on which the algebras are represented as spaces of functions. This defines a geometry on the outcome space of measurements. A good resource on an algebraic approach to geometry is [74], where the authors describe manifold theory in terms of spectra of commutative $\mathbb{R}$-algebras.
- In Chapter 3 Section 3.2, we explain the relationship between algebras of superfunctions and Stone spaces of Boolean algebras.

1.5. Symplectic geometry

Dynamics of superstochastic systems depends on the underlying geometry. It turns out that classical systems have the geometry of a linear symplectic space $(\mathbb{R}^{2n}, \omega)$. The symplectic structure $\omega : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ is an anti-symmetric, non-degenerate, bilinear form on the vector space $\mathbb{R}^{2n}$. For $v, w \in \mathbb{R}^{2n}$, with respect to a suitable basis, the linear symplectic structure can always be written as

$$\omega(v, w) = v^T J w, \quad J = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{-1} & \mathbf{0} \end{pmatrix}.$$

Newtonian mechanics describes motion of particles in space, subject to different forces and constraints. Dynamics of a system described by a Hamiltonian $H = H(q, p, t)$ can be cast in the form

of Hamilton's equations

$$\frac{d}{dt}\begin{pmatrix} q(t) \\ p(t) \end{pmatrix} = J.\,\mathrm{grad}_{q,p}(H)\,, \tag{1.1}$$

where $\mathrm{grad}_{q,p} H := \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix}$ is the phase-space gradient of the Hamiltonian $H(q,p,t)$. The system evolves along the path $(q(t), p(t))$ that solves Equation (1.1).

A symplectic manifold (M^{2n}, ω) is a $2n$-dimensional manifold equipped with smooth assignment $\omega : M \to T^*M \wedge T^*M$ of symplectic forms to every point in M. It can be described as a consistent gluing of linear symplectic spaces. Let $(U_\alpha, \phi_\alpha : U_\alpha \to \mathbb{R}^{2n})$ be an atlas of M. Then the linear symplectic spaces $(\phi_\alpha(U_\alpha), \omega_\alpha)$ are patched together using transition functions $\phi_{\alpha\beta}$

$$\phi_{\alpha\beta} := \phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta), \quad \phi_{\alpha\beta}^*(\omega_\alpha) = \omega_\beta\,,$$

that preserve the symplectic form. In other words, the derivatives of transition functions are gauge transformations of the tangent bundle TM valued in the symplectic group $\mathrm{Sp}(2n, \mathbb{R})$

$$(\phi_{\alpha\beta})_* \in C^\infty\left(\phi_\beta(U_\alpha \cap U_\beta), \mathrm{Sp}(2n, \mathbb{R})\right)\,.$$

The global data of the symplectic structure ω thus constrains the way local patches are glued. The gluing data encodes the symmetries of dynamics (corresponding to the symplectic group $\mathrm{Sp}(2n, \mathbb{R})$). The local patches correspond to the position and momenta of particles as detected by different local measurements. This is the power of symplectic geometry: global symplectic form ω facilitating a global description of dynamics of particles, while being consistent with local observations.

A supersymplectic manifold $(\mathcal{M}, \boldsymbol{\omega})$ is isomorphic (non-canonically) to a metric exterior bundle $(\Lambda E, g_{\mathrm{Hodge}}, \nabla) \overset{\pi_\nabla}{\to} (M, \omega)$ over a symplectic base (M, ω) called the *body* of the supermanifold

$$(\mathcal{M}, \boldsymbol{\omega}) \overset{\nabla}{\cong} (\Lambda E, g_{\mathrm{Hodge}}, \omega), \quad \nabla g = 0\,.$$

The bundle connection ∇ is the data of a splitting of the supermanifold $\mathcal{M}$ and determines the projection map π_∇. For a generalized measurement theory, one needs a convex cone of states. The data of the bundle Hodge metric g_{Hodge} canonically determines such a cone. This is the super

analog of the quantum mechanical Bloch cone and comprises of super Moyal $\star$-squares of sections of the exterior bundle [25].

- In Chapter 2 Section 2.1.1.1, we give details of structural aspects of symplectic geometry relevant for dynamics.
- In Chapter 2 Section 2.4, we explain splittings of supermanifolds, while in Section 2.5 we describe the supersymplectic structure.
- In Chapter 5 Section 5.3, we give the Bloch cone construction on a symplectic supermanifold.

1.6. Non-commutative probability

Observables correspond to measuring devices in an experiment. Experiments typically require multiple devices to be used in tandem to perform simultaneous/joint measurements. However, devices can interfere with each other and affect the outcome of experiments. In quantum mechanics, this is characterized by Heisenberg's uncertainty prinicple

$$\Delta x \Delta p \geq \frac{\hbar}{2}\,,$$

where the minimum uncertainty in the joint measurement of position and momenta is precisely due to their non-commutativity. This motivates why quantum mechanical observables like position and momenta are really self-adjoint operators acting on some Hilbert space of states. Following the success of operator algebras in describing quantum mechanics, there have been attempts at defining other non-commutative probability theories. These theories differ in their algebra of observables. Classical (Kolmogorov) probability theory is based on a commuting algebra of observables, while random matrix theory is based on the non-commuting algebra of matrices [93]. For quantum mechanics, matrix algebras are generalized to operator algebras of operators on a Hilbert space. Free probability theory deals with the most non-commutative algebra possible *viz.* a freely generated algebra [71, 97]. The algebra of superfunctions on a supermanifold can be viewed as a relatively mild deformation of the classical commutative algebra of functions where commutativity is replaced by supercommutativity.

Any algebra $\mathcal{A}$ over a field K is expressible as a quotient of some free algebra by some by an ideal $\mathcal{I} \subset \mathcal{A}$

$$\mathcal{A} = \frac{K[X^1, X^2, \ldots]}{\mathcal{I}} =: \frac{F}{\mathcal{I}}\,,$$

where $X^1, X^2, \ldots \in \mathcal{A}$ are generators. Let us fix the free algebra F and consider all possible ideals $\mathcal{I}$ in F. These ideals may be partially ordered by set inclusion. The ordering of ideals induces an opposite ordering of the non-commuting quotient $F/\mathcal{I}$ algebras. Following is a hierarchy of the extent of non-commutativity in different theories discussed in the previous paragraph

$$\text{Commuting}, \text{Supercommuting}, \ldots < \ldots < \text{Matrix} < \ldots < \text{Quantum/Operator} < \ldots < \text{Free}\,.$$

The gaps in the above chain represents other algebras that might "fit" between the ones mentioned.

- In Chapter 3, we describe the measurement theory associated to a $*$-algebra of observables.
- In Chapter 4 Section 4.6, we describe measurement theory of superobservables in terms special superobservables generalizing indicators in classical probability theory.

CHAPTER 2

Odd symplectic supermanifolds

A supermanifold is a geometric model for a physical system encapsulating finitely many continuous and discrete degrees of freedom. Importantly, it admits a simple local description. But to discuss dynamics, we will need additional structure. In particular, we will equip our supermanifolds with a symplectic structure and we also typically demand that it has an odd number of continuous degrees of freedom generalizing positions, momenta and time. The inclusion of time as a supermanifold coordinate makes dynamics geometric *i.e.* they correspond to unparameterized paths.

Statistical mechanics gives a probabilistic description of our physical world. Statistical mechanics combines classical mechanics of particles with ideas from probability theory to make predictions for large systems of particles. Some good references for the foundations of classical statistical mechanics are [**43**, **58**, **77**, **86**, **87**]. In the next section, we will motivate our geometric model for dynamical statistical systems.

2.1. Configuration spaces

The notion of a configuration space is important in both the classical mechanics of point particles as well as the statistical mechanics of a large collection of particles. A physical system may be characterized by a set of generalized degrees of freedom. As the name suggests, they label quantities that in principle may independently be measured. [1] A simple example is that of a single molecule moving in a three-dimensional space. It has six degrees of freedom, three positions and three momenta. A coin spinning and moving vertically in the air after being tossed, can be viewed as possessing eight degrees of freedom: height, vertical component of linear momentum, three Euler angles and three components of angular momentum. These degrees of freedom all take *values* in real numbers. However, as we shall see later, we could also have degrees of freedom taking values in a discrete space.

[1] In physics parlance, degrees of freedom typically refer to either generalized positions or momenta but not both. We recycle the terminology to refer to both positions and momenta relevant for the geometric model.

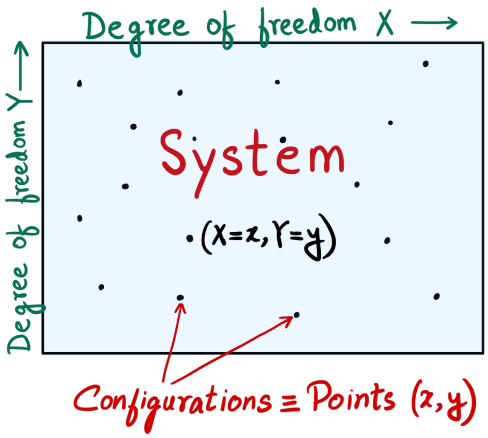
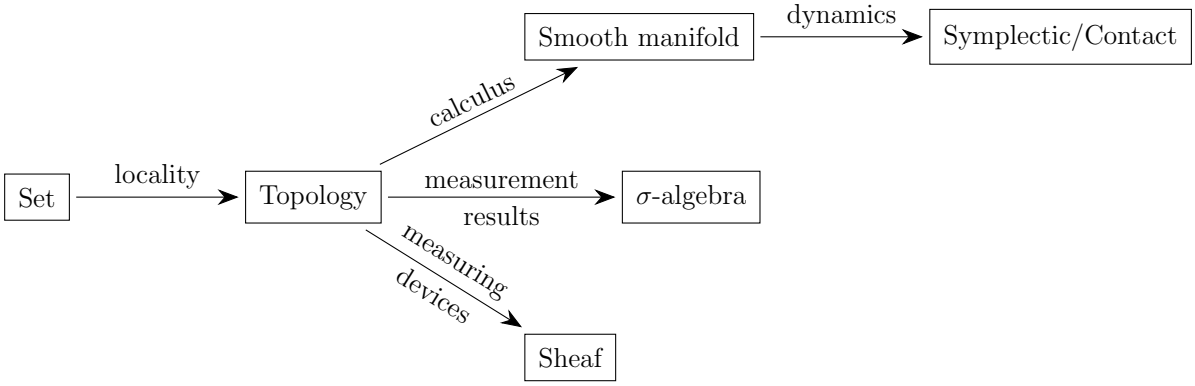


FIGURE 2.1. Configuration space

A *configuration* is defined by an assignment of values to all the degrees of freedom of the system. The set of all configurations attainable by the system will be called its *configuration space*. The word "space" refers to extra structures that may be added on top of this basic set-theoretic foundation. A flowchart of all the structures to be introduced is given below.

Anatomy of a local dynamical system



For every arrow shown above, the structure on the right builds on the structure to its the left.

2.1.1. Structures. The first structures with which we equip our configuration space are a topology and a smooth structure.

A smooth manifold M, for example, is a smooth topological space of dimension n, that has a simple local structure of Euclidean space $\mathbb{R}^n$. A smooth manifold is a particularly appealing configuration space for a physical system. Its key features and their physical significance are outlined below.

- *Topology*: Topology on a topological space is the data of a collection of subsets of the space called *open sets*. Open sets gives a rudimentary notion of nearness and allows us to discuss neighbourhoods of points. More importantly, it describes continuity of maps. Continuous functions as special maps on the topological space whose inverse sends open sets in the target space to open sets in the topological space. Topological spaces are equivalent (or *homeomorphic*) if there is map between them that is a bijection, continuous and has a continuous inverse.
- *Locally Euclidean*: This is the key manifold structure that gives a local description in terms of flat affine charts. There is also a notion of all possible "directions" called *tangent space* at a point. An assignment of tangent vector to every point on the manifold, called a *vector field*, allows us to give a local description of dynamics.
- *Smoothness*: This structure allows us to import the calculus of affine spaces into manifolds. The gluing data of a manifold is the data of transition maps between its charts. The degree of smoothness of functions on the manifold is constrained by the order of differentiability of its transition functions. The shape of the manifold, determined by charts and their gluing data, capture interesting global aspects of dynamics like number of fixed points or existence of periodic orbits. These could be attributed to spatial features of the system or shape of external forces acting on the system.
- *Dimension*: Degrees of freedom of the system could be described by a set of independent directions on the manifold. The dimension n of the manifold, given by the dimension of its charts, corresponds to the number of degrees of freedom. The number of degrees of freedom n chosen to describe a system could be dictated by a variety of factors. For example, a particle could be physically constrained to move along a plane. Or it could be that the experimenter's laboratory is limited by apparatus available to measure different properties of the system.

There are some extra technical requirements on the topology of a topological manifold, like being second-countable and Hausdorff. These make the space well-behaved, such as guaranteeing uniqueness of limits of sequences.

The next step is to augment the manifold structure with some geometry suitable for dynamics. The dynamical degrees of freedom are generalized position and momenta of a phase space. Degrees

of freedom may be classified as *continuous* or *discrete* depending on the space they are valued in. They may also be classified as *microscopic* and *macroscopic/statistical* based on the number of particles in the system. The corresponding configuration spaces will also be called microscopic and macroscopic/statistical respectively. We will next discuss the geometry governing dynamics of continuous degrees of freedom.

2.1.1.1. *Microscopic.* Microscopic degrees of freedom are phase space variables like position and momentum of individual particles of a system. The configuration space is an even (say $2n,\ n \in \mathbb{N}$) dimensional manifold M called a *symplectic manifold*. An excellent introduction to symplectic geometry from a physics perspective can be found in [2,4], while [29,70] have more comprehensive mathematical treatment. We will give a brief summary of some key ideas that are relevant to us.

A symplectic structure is a non-degenerate and closed 2-form ω on M

$$\omega^{\wedge n} \neq 0 = d\omega\,. \tag{2.1}$$

In the above line, $\omega^{\wedge n} \neq 0$ is just a shorthand for the top form $\omega^{\wedge n}$ being nowhere vanishing. In Chapter 1 Section 1.5, we saw how non-degenerate, anti-symmetric bilinear forms give rise to Hamiltonian dynamics. The non-degeneracy of the symplectic form ω lets us use it to determine volumes of regions in the phase space M. To understand the physical significance of the closedness condition, we must first discuss its symmetries given by symplectomorphisms.

Symmetries

Two symplectic manifolds (M, ω_M) and (N, ω_N) are equivalent (or *symplectomorphic*) if there is a diffeomorphism $\varphi : M \to N$ such that

$$\varphi^* \omega_N = \omega_M\,.$$

An infinitesimal (auto)symplectomorphism X on M is given by *symplectic vector field*, one that preserves the symplectic form

$$\mathcal{L}_X \omega = 0 \iff d(\iota_X \omega) = 0\,. \tag{2.2}$$

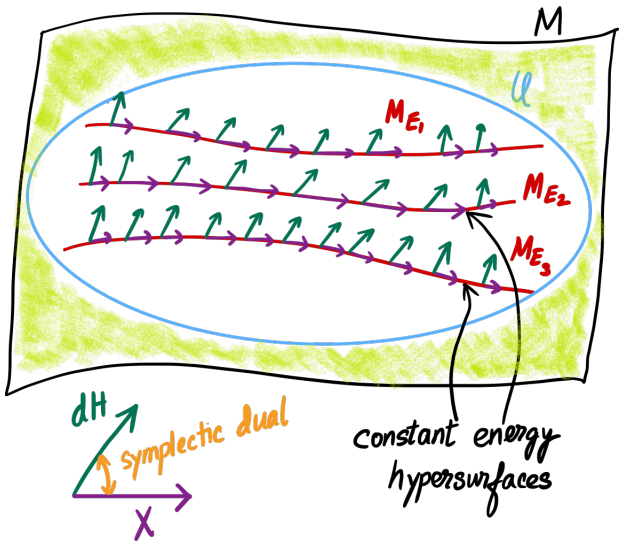


FIGURE 2.2. Symplectic vector field X conormal to differential of a local Hamiltonian H

Since dynamics corresponds to a one-parameter family of autosymplectomorphisms, a symplectic vector field must know about some Hamiltonian, atleast locally. By Poincaré lemma, the closure of ω therefore implies that for every point $x \in M$, there exists a neighborhood $U \subseteq M$ around x such that

$$(\iota_X \omega)|_U = -dH, \quad H \in C^\infty(U)\,, \tag{2.3}$$

for some local Hamiltonian H on M. The differential 1-form dH is said to be the *symplectic dual* of the symplectic vector field X on U. Since X is conormal to dH, X is everywhere tangent to the constant energy hypersurfaces $M_E := H^{-1}(\{E\}) \subseteq U$ of H

$$dH(X) = 0 \iff X \in TM_E, \quad E \in \mathbb{R}\,.$$

This is a geometric statement of Noether's theorem: for every symplectomorphism X generating dynamics, there is a locally conserved quantity H that determines a local foliation of M by constant energy hypersurfaces $\{M_E\}$.

Symplectic vector fields form a subalgebra $\mathrm{symp}(M, \omega)$ of the infinite-dimensional Lie algebra $\mathrm{Vect}(M)$ of smooth vector fields on M

$$\mathcal{L}_{[X,Y]}\omega = [\mathcal{L}_X, \mathcal{L}_Y]\omega = 0\,.$$

In the above line, we have used the fact that Lie derivative is compatible with the Lie bracket of vector fields. The flows of symplectic vector field correspond to paths in the group of autosymplectomorphisms $\mathrm{Symp}(M,\omega)$ that are connected to the identity.

Local flatness

The global closedness condition of the symplectic form drastically constrains its local properties. This is exemplified by the Darboux theorem.

THEOREM 2.1.1 (Darboux). *Let (M,ω) be a symplectic manifold. Then for each point $x \in M$ there is a neighborhood $U \subseteq M$ containing x symplectomorphic to the "flat" symplectic space $(\mathbb{R}^{2n}, \omega_{\mathrm{std}})$*

$$\omega_{\mathrm{std}} := \sum_{i=1}^{n} dp_i \wedge dq^i, \quad (q^i, p_i) \in \mathbb{R}^{2n}\,.$$

A neighborhood $U \subseteq M$ where ω looks like (symplectomorphic to) ω_{std} is called *Darboux neighborhood/chart*. There is an elegant proof of this due to Moser. The basic idea is to construct a smooth isotopy $\{\rho_t : U \to U : t \in [0,1]\}$ of an open neighborhood U around $x \in U \subset M$ such that

$$\omega = \rho_1^* \omega_{\mathrm{std}}, \quad \rho_0 = \mathrm{Id}\ . \tag{2.4}$$

Let $\omega_t := t\omega_{\mathrm{std}} + (1-t)\omega$ be a family of cohomologous symplectic forms connecting ω and ω_{std}. Moser's flow can be generated by a time-dependent vector field X_t which solves the system

$$\frac{d}{dt}(\rho_t^* \omega_t) = 0, \quad \frac{d}{dt}\rho_t = X_t \circ \rho_t\,. \tag{2.5}$$

Applying chain rule followed by Cartan's magic formula $\mathcal{L}_X = d \circ \iota_X + \iota_X \circ d$, we obtain

$$\frac{d}{dt}(\rho_t^* \omega_t) = \rho_t^* \left(\frac{d}{dt}\omega_t + \mathcal{L}_{X_t}\omega_t \right) = \rho_t^* (\omega_{\mathrm{std}} - \omega + d(\iota_{X_t}\omega_t)) = \rho_t^* d(\alpha + \iota_{X_t}\omega_t)\,,$$

where assuming ω_{std} and ω are cohomologous: $[\omega_{\mathrm{std}}] = [\omega] \implies \omega_{\mathrm{std}} - \omega = d\alpha$ for some 1-form $\alpha \in \Omega^1(M)$. The non-degeneracy of ω then implies that there exists a unique solution to X_t in Equation 2.5. Since U has a compact neighborhood, it follows that the flow ρ_t exists for $t = 1$ and thus we have constructed a symplectomorphism ρ_1 (see Equation 2.4) isotopic to the identity. Darboux's theorem therefore implies that there are no non-trivial local deformations of

the standard flat symplectic structure. This is unlike Riemmanian geometry where non-trivial Riemannian curvature of the metric obstructs local flatness.

We saw that symplectic vector fields are locally (symplectically) dual to the differential of local Hamiltonian functions. When the duality extends to the differential of a global Hamiltonian function, the symplectic vector field is called a *Hamiltonian vector field*. This generates the standard Hamiltonian dynamics. Dynamics of a particle corresponds to an integral curve $\gamma : [-\epsilon, \epsilon] \to M$, $\epsilon \in \mathbb{R}^+$ of a Hamiltonian vector field X_H generated by a Hamiltonian function $H : M \to \mathbb{R}$ on the symplectic manifold

$$\iota_{X_H}\omega = -dH, \quad \dot{\gamma(t)} = X_H(\gamma(t))\,. \tag{2.6}$$

Hamiltonian vector fields are symplectic (*cf.* Equation 2.2). This is an elementary result due to Liouville and follows from the definition in Equation 2.6.

THEOREM 2.1.2 (Liouville). *Flows due to hamiltonian vector fields preserve the symplectic form*

$$\mathcal{L}_{X_H}\omega = 0\,.$$

The above equation is easy to check by applying Cartan's magic formula and using Equation 2.6.

Poisson algebra of observables

The space $C^\infty(M)$ of smooth functions on a symplectic manifold M has the structure of a Poisson algebra and hence M is also a Poisson manifold. Some good references on the geometry of Poisson manifolds are [34, 95]. The Poisson structure can capture many important aspects of dynamics of physical system like constraints, gauge symmetries and integrability. A homological approach to constraints and gauge symmetries of dynamical systems is Poisson/classical BRST cohomology [41, 60].

Let $F, G, H \in C^\infty(M)$ be smooth functions on M. The Poisson bracket given by

$$\{F, G\} := X_F(G)\,,$$

is both a Lie bracket (obeying the Jacobi identity) and a derivation with respect to the pointwise product of functions

$$\{F,\{G,H\}\} + \{G,\{H,F\}\} + \{H,\{F,G\}\} = 0, \quad \{F, GH\} = \{F,G\}H + G\{F,H\}\,.$$

The Lie bracket $[X_H, X_G]$ of their corresponding Hamiltonian vector fields X_H, X_G

$$[X_H, X_G] = \mathcal{L}_{X_H} X_G = -\mathcal{L}_{X_H}(\omega^{-1}(dG)) = -\omega^{-1}(d\iota_{X_H} dG) = X_{\{H,G\}}\,,$$

is also a Hamiltonian vector field. Thus, Hamiltonian vector fields form a Lie subalgebra of the Lie algebra of symplectic vector fields

$$\mathrm{ham}(M,\omega) \subset \mathrm{symp}(M,\omega) \subset \mathrm{Vect}(M)\,.$$

Gauge symmetries

Finite-dimensional subalgebras $\mathfrak{g}$ of $\mathrm{ham}(M,\omega)$ capture gauge symmetries of dynamical systems. The Hamiltonian action of the gauge group $G := \exp(\mathfrak{g})$ on M can be described in terms of a moment map

$$\mu : M \to \mathfrak{g}^*\,.$$

Let $\{X_i\}$ be a set of generators of $\mathfrak{g}$. Then we obtain a set of Hamiltonian functions $H_i := \langle \mu, X_i \rangle$ which in turn determine Hamiltonian vector fields X_{H_i}

$$\iota_{X_{H_i}} \omega = -dH_i\,.$$

The Hamiltonian action of G on M

$$\rho : G \to \mathrm{Ham}(M,\omega)$$

is finally obtained by solving the time-1 flow problem for each of the X_{H_i}

$$\phi_t^{H_i} : M \to M, \quad \frac{d}{dt}\phi_t^{H_i} = X_{H_i} \circ \phi_t^{H_i}, \quad \rho(\exp(X_i)) = \phi_1^{H_i}.$$

The action ρ can be extended to a neighborhood of the identity in G by composing generator exponentials

$$\rho\left(\exp\left(\sum_i t_i X_i\right)\right) = \bigcirc_i \rho(\exp(r_i(t_j) X_i))\,,$$

where the coefficients r_i depend on the local coordinates t_i in $\mathfrak{g}$ by the the Baker-Campbell-Hausdorff formula. Next, we will discuss coisotropic submanifolds and the geometry of gauge-fixing.

Gauge fixing and gauge orbits

Let $\mathfrak{g}$ be a k-dimensional Lie algebra generated by a set $\{X_{H_i}\}_{i=1}^k$ of pointwise linearly independent Hamiltonian vector fields. This means they close with respect to the Lie bracket

$$[X_{H_i}, X_{H_j}] = f_{ij}^k X_{H_k} \iff \{H_i, H_j\} = f_{ij}^k H_k \,, \tag{2.7}$$

where $f_{ij}^k \in \mathbb{R}$ are the structure constants. By Frobenius' integrability theorem, there exists a foliation of M by maximal integral submanifolds. This Hamiltonian foliation is special in that it restricts to the codimension-k "gauge-fixing" submanifold $\mathcal{C} \subset M$

$$\mathcal{C} = \{x \in M \mid H_i(x) = 0\} \,,$$

corresponding to the zero locus of the functions H_i. This is because the Hamiltonian vector fields X_{H_i} are tangent to $\mathcal{C}$

$$dH_j(X_{H_i})|_{\mathcal{C}} = X_{H_i}(H_j)|_{\mathcal{C}} = \{H_i, H_j\}|_{\mathcal{C}} = f_{ij}^k H_k|_{\mathcal{C}} = 0 \,, \tag{2.8}$$

generating the "residual" gauge symmetries. Now, the symplectic complement $(T\mathcal{C})^\omega$ defined as

$$TM \supset (T\mathcal{C})^\omega := \ker \omega|_{\mathcal{C}} = \sqcup_{p\in\mathcal{C}} \{v_p \in T_pM \mid \omega(v_p, w_p) = 0 \;\forall\; w_p \in T_p\mathcal{C}\} \,,$$

consists of vector fields on M that annihilate ω when restricted to $\mathcal{C}$. Since ω is non-degenerate on M, the tangency in Equation 2.8 implies that the vector fields in the symplectic complement must be in the span of the symplectic duals of dH_i, or

$$(TC)^\omega = \langle\{X_{H_i}\}\rangle = \mathfrak{g}|_{\mathcal{C}} \subset T\mathcal{C} \,.$$

The submanifold $\mathcal{C}$ is called *coisotropic submanifold* since the symplectic form ω is degenerate when restricted to it. The closure of the symplectic vector fields X_{H_i} under Lie bracket ensures that the symplectic complement $(T\mathcal{C})^\omega$ is a rank-k integrable, null/isotropic (annihilated by ω) distribution in TC. Its leaves are gauge orbits that foliate the gauge fixing submanifold $\mathcal{C}$.

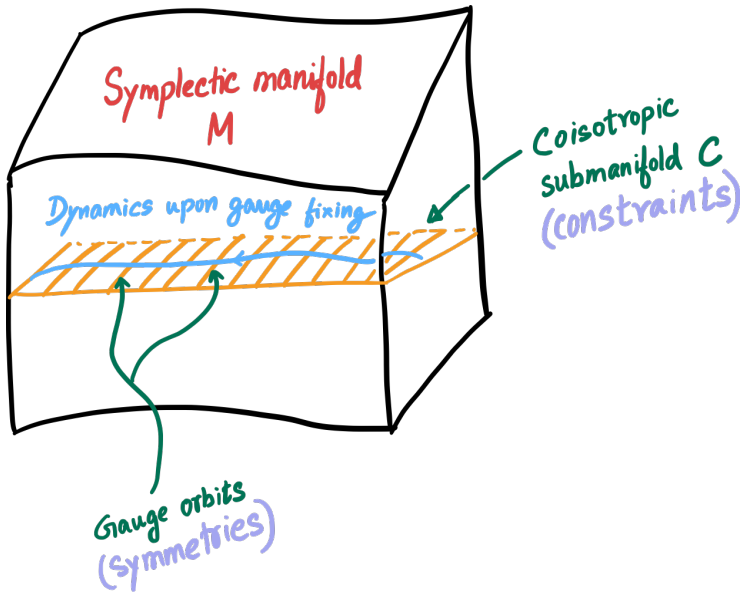


FIGURE 2.3. Dynamics in the presence of gauge symmetries

Constraints from action principle

Lagrangian mechanics describes dynamics as critical points of a Floer action functional on a space of compactly supported paths on a symplectic manifold. Hamiltonian mechanics may be derived by extremizing the action corresponding to the Legendre transform of a Hamiltonian. The structure of Lagrangian dynamics is richer than standard (unconstrained) Hamiltonian dynamics. The space of critical paths extremizing the Floer action functional may not be unique and perhaps might itself be infinite-dimensional. Lagrangians with non-unique dynamics are said to be *singular*. Dynamical systems described by singular Lagrangians $L(q^i, \dot{q}^i, t)$ are characterized by momenta that are not all independent, rather constrained during dynamics. Independent momenta represent the physical/reduced degrees of freedom of the system.

The Legendre transformation from a singular Lagrangian to the canonical Hamiltonian is degenerate: many-to-one map from the configuration space of positions and speeds (q^i, v^i) to a symplectic phase space (M, ω) coordinatized by positions and momenta (q^i, p_i). The upshot is that the canonical Hamiltonian is insufficient to encode all aspects of Lagrangian dynamics: it must be extended by constraints (together with their conjugates) and constraints must be suitably imposed. A geometric approach to encode all the constraints is by restricting to a constraint submanifold of the original symplectic manifold. We will next describe the Poisson geometry of constraint submanifolds.

Let $H \in C^\infty(M \times \mathbb{R})$ be a (possibly time-dependent) Hamiltonian function describing the same dynamics as the Lagrangian $L(q^i, \dot{q}^i, t)$. Let $\{\phi_t^H : M \to M : t \in T\}$ be the flow generated by the Hamiltonian vector field X_H on the symplectic manifold (M, ω). The set $\mathcal{I}$ of all dynamical constraints is given by

$$\mathcal{I} := \{f \in C^\infty(M) \mid \mathcal{L}_{\phi_t^H} f = 0\}\,.$$

Without loss of generality, these may be identified with the (multiplicative) ideal of functions vanishing along dynamical paths

$$\mathcal{I} \cong \{f \in C^\infty(M) \mid (\phi_t^H)^* f = 0\}\,.$$

First class functions are functions in the Poisson normalizer of $\mathcal{I}$: $f \in C^\infty(M)$ that Poisson commute with elements of $\mathcal{I}$ along flows

$$\mathcal{N}(\mathcal{I}) = \{f \in C^\infty(M) \mid \mathcal{L}_{\phi_t^H}\{f, g\} = 0 \;\forall\; g \in \mathcal{I}\}\,.$$

It is easy to check that $\mathcal{N}(\mathcal{I})$ is a Poisson subalgebra of $\mathcal{I}$

$$\{\mathcal{N}(\mathcal{I}), \mathcal{N}(\mathcal{I})\} \subset \mathcal{N}(\mathcal{I})\,.$$

The set $\mathcal{I}'$ of *first class constraints* consists of constraints in $\mathcal{N}(\mathcal{I})$

$$\mathcal{I}' := \mathcal{I} \cap \mathcal{N}(\mathcal{I})\,,$$

and $\mathcal{I}'$ is a Poisson ideal of $\mathcal{N}(\mathcal{I})$. In the phase-space formulation, dynamics occurs on the vanishing locus of $\mathcal{I}$ called the *constraint submanifold* $\mathcal{C}$ [2]

$$\mathcal{C} := (Z(\mathcal{I}), \omega|_{\mathcal{C}})\,,$$

of the dynamical system (M, ω, H) where H is a distinguished time-dependent Hamiltonian function called the *Hamiltonian*, whose Hamiltonian flow is also ϕ_t^H . The constraint submanifold $\mathcal{C}$ has more structure. It knows about the gauge symmetries of the system corresponding to *first class constraints* in $\mathcal{I}'$. In fact, $\mathcal{I}'$ is a Poisson commuting subalgebra of $\mathcal{N}(\mathcal{I})$ when restricted to the

[2]The constraint submanifold $\mathcal{C}$ being referred to here is one obtained by imposing only second class constraints on M. The first class constraints are like gauge symmetries that are often retained (not gauge-fixed) for simpler quantization procedures like in BRST.

constraint submanifold

$$\{\mathcal{I}', \mathcal{I}'\}|_{\mathcal{C}} = 0\,.$$

This is the same coisotropic structure we saw earlier in Equation 2.8. It follows from the same arguments that the Hamiltonian vector fields corresponding to $\mathcal{I}'$ give rise to a foliation of $\mathcal{C}$ by maximal integral null leaves. This motivates us to define constraints in terms of their null foliations also called *characteristic foliations*.

DEFINITION 2.1.3. *The characteristic or null foliation* $\{\mathcal{N}_I\}$ *of a constraint submanifold* $(\mathcal{C}, \omega|_{\mathcal{C}}) \subset (M, \omega)$ *are the maximal leaves of the integrable distribution* $\ker \omega|_C$

$$T\mathcal{N}_I = \ker \omega|_C\,.$$

REMARK 2.1.1. *A rank-k characteristic/null foliation* $\{\mathcal{N}_I\}$ *is induced whenever one restricts to a codimension-k level set* $C_{\{E_a\}} = \{x \in M \mid H_a(x) = E_a,\ E_a \in \mathbb{R}\}$ *of k independent functions* H_a

$$X \in T\mathcal{N}_I \iff X \in \mathrm{Span}(\{X_{H_a}\})\,.$$

A constraint submanifold $\mathcal{C}^{2n-k} \hookrightarrow (M^{2n}, \omega)$ of dimension $2n-k$ equipped with a rank $2(n-k)$ induced closed 2-form $\omega|_{\mathcal{C}}$ is a *presymplectic manifold*. Presympletic structures give a fully geometric approach [46] to the Bergmann-Dirac theory [3, 33] of constraints and gauge symmetries as well Kostant-Souriau theory [62, 90, 104] of geometric pre-quantization.

Integrability

If a Hamiltonian H generates dynamics, and $\{H_i\}$ are independent constraints as before, the constraints must be constants of motion

$$\frac{d}{dt}H_i = \{H, H_i\} = 0\,.$$

Note that the above equation is deemed to hold only on-shell or on the equation of motions. A $2n$-dimensional symplectic manifold M can have at most n independent and pairwise Poisson commuting constants of motion. These are locally either n position coordinates or n momentum coordinates. A dynamical system (M, ω, H), whose dynamics is generated by a distinguished Hamiltonian H, is said to be completely integrable if there exists $n-1$ more independent constants

of motion that also Poisson commute with each other on-shell. A simultaneous level set of these constants of motion is a constraint submanifold with maximum rank of its null foliation, called a *Lagrangian submanifold*. Due to a Liouville-Arnold type theorem [4], dynamics on Lagrangian submanifolds can be completely described in terms of generalized action-angle variables. In particular, for compact energy hypersurfaces, the Lagrangian submanifolds are invariant tori.

Parameterized paths

On any Darboux chart $U \ni (q^i, p_i)$, the Hamiltonian vector field X_H in Equation 2.6 takes the form

$$X_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}\,,$$

and its integral curve $\gamma(t) = (q^i(t), p_i(t))$ satisfies *Hamilton's equation of motion* (*cf.* Equation 1.1). The integral curve $\gamma(t)$ is parameterized by some parameter t that plays the rôle of time. The choice of the parameter t is however completely arbitrary and does not have any of the physical properties one would hope for a time coordinate, such as a notion of future and past. In special relativity for example, time is a part of the geometry and invariantly describes dynamics as well as causal relationships between events. A suitably generalized geometry should handle time the same way as position and momentum. In particular, it should be possible to promote time to a coordinate on a larger manifold. In that manifold, dynamics could be described by *worldlines i.e.* unparameterized curves defined by the points in the image $\gamma := \text{Image}(\gamma(t))$. In the next section, we will take a closer look at the physical significance of time as well as geometries that allow covariant description of dynamics.

2.1.1.2. *Time.* In special relativity, time is always relative to observer's frame of reference. While length and time measurements are observer dependent, the speed of light is constant. This is related to the underlying Poincaré spacetime symmetry group $SO(1,3) \ltimes \mathbb{R}^4$ generated by spacetime translations, spatial rotations and Lorentz boosts. Einstein's 1905 breakthrough was to realize that space and time can be combined into the four-dimensional, Lorentzian, flat Minkowski space $\mathbb{R}^{1,3} := (\mathbb{R}^4, \eta)$, where η is the metric on $\mathbb{R}^4$

$$\eta = \text{diag}(-1,1,1,1) \in \Gamma(\odot^2(T\mathbb{R}^4))\,.$$

The isometries of the Lorentzian metric η are precisely the Poincaré transformations. A Lorentzian geometry is a pseudo-Riemannian geometry with one independent timelike direction. The symmetric bilinear form η can be used to compute distances d^{γ}_{PQ} between pairs of points P, Q along paths $\gamma(t) : [0,1] \to \mathbb{R}^4$ connecting them, according to (bad....)

$$d^{\gamma}_{PQ} = \int_P^Q ||\dot{\gamma}(t)||\, dt\,,$$

where the squared length $||\cdot||^2 := \eta(\cdot,\cdot)$. Thus the Riemannian geometry is a type of G-structure that not only gives a notion of distance between points along curves, but also controls the underlying smooth structure.

Lorentzian geometry also encodes a causal structure. This corresponds to our experience of past and future in the physical world. The parameterization of dynamical paths of particles correspond to the time recorded by clocks carried by individual observers. Each clock has a notion of forward time that orders events tracked by it. Our everyday experience suggests that this ordering of events is consistent across clocks carried by different observers. This means we must have a geometric way to order events that is consistent across all observers' clocks. This is called a *global time-orientation* of the spacetime.

Mathematically, a causal structure is the data of a partial ordering of points based on orientation of paths associated to dynamics of particles in the spacetime. Depending on the type of particle, its path may be restricted by a range of admissible directions. This is best described in terms of ray sub-bundles of the projectivised tangent bundle of $\mathbb{R}^4$. The Lorentzian metric η gives a partition of the projectivised tangent bundle into ray subbundles

$$\mathbb{R}^4 \times \mathbb{RP}^3 \cong P(T\mathbb{R}^4) \stackrel{\eta}{=} \mathcal{T} \sqcup \mathcal{N} \sqcup \mathcal{X}\,,$$

where at any point $p \in \mathbb{R}^4$, $\mathcal{T}_p$ is the space of timelike (positive squared norm) rays, $\mathcal{N}_p$ is the space of null/light (zero norm) rays and $\mathcal{X}_p$ is the space of spacelike (negative square norm) rays. Massive relativistic particles move along $\mathcal{T}$, massless particles like photons follow $\mathcal{N}$ while $\mathcal{X}$ determines the path of particles called tachyons moving faster than the speed of light. The cone $\mathcal{T}_p$ at a point $p \in \mathbb{R}^4$ has two connected components, called *future half-cone* $\mathcal{T}^+_p$ and *past half-cone* $\mathcal{T}^-_p$. Since the bundle $\mathcal{T}$ is trivial and hence orientable, we can make make a consistent choice of a future half-cone

$\mathcal{T}_p^+$ on the whole of $\mathbb{R}^4$

$$\mathcal{T}^+ = \sqcup_{p\in\mathbb{R}^4}\mathcal{T}_p^+ \,.$$

Let V be a continuous, nowhere-vanishing, future-directed timelike vector field (corresponding to a section of $\mathcal{T}^+$). The causal structure on $\mathbb{R}^{1,3}$ can now be described in terms of η and V. An event x preceeds y or $x \leq_\eta y$, if y can be reached from x by traversing an everywhere future-directed, timelike smooth curve $\gamma : [0,1] \to \mathbb{R}^{1,3}$

$$x \leq_\eta y \iff \exists\, \gamma(t),\ \gamma(0) = x,\ \gamma(1) = y,\ \eta(V, \dot\gamma(t)) < 0,\ ||\dot\gamma(t)||^2 > 0\,.$$

These ideas extend to general relativity, except spacetime is now generalized to a curved Lorentzian manifold (M, g). The causal structure is given by a non-trivial bundle of light cones $\mathcal{N}$ in the projectivised tangent bundle $P(TM)$ of the Lorentzian manifold M

$$\begin{array}{ccc} \mathcal{N} & \hookrightarrow & P(TM) \\ & & \downarrow \\ & & (M,g)\,. \end{array}$$

Some good references that have more detailed discussions of these topics are [48, 75, 99, 101].

Symplectic manifolds describes dynamics of particles in terms of parameterized paths. This has some clear limitations

(1) The choice of clock/time-parameterization affects dynamical equations. Dynamical equations for the same system might therefore look different for different observers. Symplectic structure does not capture invari Also, time reparameterization is not a function of positions and momenta of particles. It would be nice to incorporate time as an extra direction into the manifold à la Einstein, to not only make dynamics covariant but allow more general transformations of observers' frames of reference. Another upshot of a covariant treatment of time, is that worldlines of particles could be assigned a notion of *proper time*, much the same way as general relativity.

(2) Dynamical equations, on their own, are symmetric under time reversal and thus know nothing about causality.

(3) A time-dependent Hamiltonian does not admit a simple geometric description as a function on a phase space. We would like to use geometry to describe even dissipative physical systems, that could be modeled by certain time-dependent Hamiltonians.

The above problems may be remedied by considering odd dimensional geometries that generalize the even dimensional symplectic geometry of phase space to a phase-spacetime. These come with distinguished vector fields generating dynamics as well as some generalized symplectic structures to capture phase-space measurements. The following are some interesting cases, depending on the kind of symmetries one would like the dynamics to enjoy.

(1) *Odd symplectic structures*: We equip odd dimensional manifolds Z with a maximally non-degenerate, closed 2-form ω which we call an *odd symplectic form* [25][3] An odd symplectic form ω canonically incorporates dynamics $\gamma(t) : [0, T] \to Z$ in terms of a line subbundle $L \subset TZ$ of the tangent bundle of M

$$\ker \omega = \langle \dot{\gamma}(t) \rangle \, .$$

This structure address problems (1) and (3) described above.

(2) *Contact structures*: These are odd symplectic manifolds (Z, ω) for which one additionally chooses local contact 1-forms $\alpha \in \Omega^1(U)$, $U \subset Z$ such that

$$\omega_U = d\alpha \, ,$$

where the existence of α follows from closedness of ω. These local contact forms must be consistent on overlapping charts

$$t^*_{\alpha\beta} \alpha_\beta = \alpha_\alpha \, ,$$

where $t_{\alpha\beta} : U_\alpha \cap U_\beta \to U_\alpha \cap U_\beta$ is a transition map on the overlap $U_\alpha \cap U_\beta$. Equivalently, a contact manifold (Z, ξ) comes equipped with a hyperplane distribution $\xi \subset TZ$ that is maximally non-integrable. Dynamics correspond to paths everywhere transverse to the distribution. Locally, dynamics $\gamma(t) : [0, T] \to Z$ may be described by the data of a

[3] In [2] Section 5.1.4, the authors give the definition of an odd symplectic manifold but call it contact (which is a special kind of odd symplectic manifold). They use the terminology *exact symplectic* for a co-orientable contact manifold.

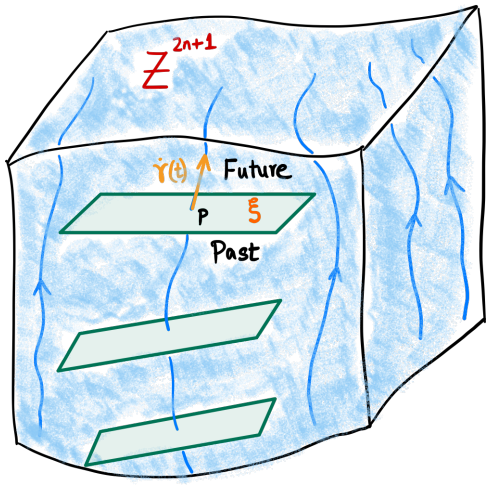


FIGURE 2.4. Geometry of contact flow $\gamma_p(t)$ in a contact manifold (Z, ξ)

nowhere vanishing contact 1-form α such that

$$\ker \alpha = \xi, \quad \ker d\alpha = \langle \dot{\gamma}(t) \rangle, \quad \alpha \wedge (d\alpha)^{\wedge n} \neq 0 \iff \ker \alpha \cap \ker d\alpha = \{0\}\,.$$

The maximal non-integrability of ξ is equivalent to the transversality of the unique dynamical path $\gamma(t)$ with respect to the symplectic distribution $(\xi, d\alpha|_\xi)$. The contact form α faithfully captures four salient features of dynamics of particles

(a) *Extremality*: A contact form α gives an invariant way to describe an action principle $S[\gamma]$ for dynamical paths $\gamma : T \to Z$,

$$S[\gamma] := \int_\gamma \alpha, \quad \delta S = 0\,,$$

where the extremization is over compactly supported variations $V : \gamma \to \gamma^* TZ$ *i.e.* $V(\partial\gamma) = 0$. An extremal path must then satisfy

$$\int_\gamma \mathcal{L}_V \alpha = 0 \; \forall \; V \implies \iota_{\dot{\gamma}(t)} d\alpha = 0\,.$$

(b) *Uniqueness*: Since $d\alpha$ is maximally non-degenerate, there is a unique extremal path $\gamma(t)$ that solves the above equation.

(c) *Timelike*: Because of its transversality with respect to the distribution ξ, the path $\gamma(t)$ is timelike *i.e.* time is monotonically changing during dynamics. The codimension-1

distribution ξ plays the rôle of a light cone, whose coorientations correspond to past and future halves.

(d) *Measurability*: Due to the maximal non-integrability of $\xi = \ker \alpha$, there is an induced canonical volume form on Z

$$\mathrm{dVol}_\alpha := \alpha \wedge (d\alpha)^{\wedge n} \neq 0\,, \tag{2.9}$$

that can in turn be used to define expectation values of classical observables $X \in C^\infty(Z)$

$$\langle X \rangle_\alpha := \frac{\int \mathrm{dVol}_\alpha\, X}{\int \mathrm{dVol}_\alpha}\,. \tag{2.10}$$

A *coorientable contact manifold* is one whose quotient line bundle TZ/ξ is orientable and therefore its contact structure ξ can be defined by a global contact 1-form α. Such contact manifolds can encode causal structures in terms of orientations of the line bundle TZ/ξ. A coorientable contact structure can therefore address all the problems related to the treatment of time mentioned above. A good resource on basic results in contact geometry is [44]. For applications of contact geometry to dynamical systems, the reader is encouraged to look at [20, 23, 35, 50, 68].

(3) *Almost contact/pre-contact structures*: These generalize contact structures by relaxing the condition of maximal non-integrability on the contact distribution $\xi = \ker \alpha$ defined by a 1-form α on an odd dimensional manifold Z. Instead of requiring the exact 2-form $d\alpha|_\xi$ to be nondegenerate (i.e., symplectic), the distribution ξ is equipped with an *almost complex structure* $J \in \mathrm{End}(\xi)$ satisfying $J^2 = -\mathrm{Id}$. Symplectic structures may be recovered as fundamental 2-forms corresponding to choices of J-compatible Riemannian metrics. A great pedagogical reference on contact and symplectic metric geometries is [16]. Almost contact structures have also proved to be an important tool for classification of overtwisted contact structures [17, 24]. Our goal here is to simply highlight that its the basic structure that allows a pointwise notion of phase-spacetime measurements.

Definition 2.1.4. *An almost-contact metric structure on an odd-dimensional manifold Z^{2n+1} is the data of a quartet (α, R, ϕ, g), where:*

- $\alpha \in \Omega^1(Z)$ *is a 1-form,*

- $R \in \mathrm{Vect}(Z)$ *is a vector field such that* $\alpha(R) = 1$,
- $\phi \in \Gamma(\mathrm{End}(TZ))$ *is a (1,1)-tensor field*
- $g \in \odot^2 TZ$ *is a Riemannian metric satisfying*

$$\phi^2 = -\mathrm{Id} + R \otimes \alpha, \quad g(\phi X, \phi Y) = g(X, Y) - \alpha(X)\alpha(Y)\,.$$

The tensor ϕ restricts to an almost-Hermitian structure on $\ker \alpha$, and annihilates R, i.e., $\phi(R) = 0$ and $\alpha \circ \phi = 0$. When $d\alpha|_{\ker \alpha}$ is also nondegenerate, this recovers the standard contact structure. In the generic case however, $d\alpha$ may not have maximum possible rank (here $2n-1$). We therefore make use the almost-complex structure ϕ and the metric g, to define a volume form

$$\mathrm{dVol}_{\alpha, \omega_{\phi,g}} := \alpha \wedge \omega_{\phi,g} \neq 0\,,$$

where $\omega_{\phi,g} := g(X, \phi Y)$ is the fundamental 2-form. This allows us to define expectation values of functions on Z in exactly the same way as shown in Equation 2.10. If the almost-contact metric structure satisfies the *cosymplectic* condition

$$d\alpha = 0, \quad d\omega_{\phi,g} = 0\,,$$

we obtain a foliation of Z by symplectic leaves of the rank-$2n$ characteristic distribution $\ker \alpha$. These leaves represent instantaneous measurement laboratories (see Section 2.6).

A symplectic phase space captures the geomtery of microscopic degrees of freedom, while an odd symplectic phase spacetime captures the geometry of dynamics of these degrees of freedom. It is interesting to view time itself as a special kind of macroscopic degree of freedom of a system, related to the choice of an observer's clock tracking dynamics. Curiously enough, in the next section, we will see that macroscopic degrees of freedom such as temperature, pressure, internal energy etc. of a large collection of gas molecules, are also described by contact structures.

Odd dimensional configuration spaces are all in all better suited for a covariant description of dynamics. Since an odd symplectic structure is sufficient for dynamics, we will therefore formulate our measurement theory with that data.

2.1.1.3. *Macroscopic.* Statistical degrees of freedom are intensive-extensive pairs of thermodynamic parameters like pressure p and volume V, temperature T and entropy S, chemical potential μ and particle number N of a gas of molecules. A thermodynamic phase space [4, 19] is an odd dimensional manifold Z^{2n+1} parameterized by these conjugate variables together with a *potential* like internal energy U, Gibbs free energy G, enthalpy H or Helmholtz free energy A. Thermodynamic processes, corresponding to dynamical paths $\gamma(t)$ in this phase space, which obey the First Law

$$\Delta U - Q - W = 0\,, \tag{2.11}$$

where ΔU represents changes in the internal energy during the process, Q denotes the heat absorbed by the system from the environment and W denotes work done on the system by generalized external forces. When the system is at equilibrium and undergoes a reversible/quasi-static process (slow/gradual changes that does not disturb the equilibrium), heat and work can be locally defined in terms of differential forms on Z

$$dQ = TdS, \quad dW = -PdV + \mu dN\,.$$

Along such processes, the first law in Equation 2.11 also admits a local differential form

$$dU - TdS + PdV - \mu dN = 0\,.$$

To invariantly describe all possible reversible processes in a thermodynamic system, one can equip the thermodynamic phase space Z with a contact structure with local *Gibbs contact 1-form*

$$\alpha = dU - TdS + PdV - \mu dN\,.$$

The contact distribution $\xi = \ker \alpha$ encodes all possible local disturbances of the system that do not destroy its state of equilibrium. A path $\gamma(t) : [0, T] \to Z$ everywhere tangent to ξ

$$\dot{\gamma}(t) \in \xi\,,$$

is called a *Legendrian curve*. Physically, Legendrian curves represent reversible processes at equilibrium. Reversible flows $\{\phi^t_{X_i}\}$ generated by a set $\{X_i \in \mathrm{Vect}(Z)\}$ of pointwise linearly independent

vector fields on Z that close with respect to the Lie bracket

$$[X_i, X_j] = f_{ij}^k X_k \iff X_i \in T\Sigma_I \,,$$

give rise to a foliation of Z by integral submanifolds $\{\Sigma_I\}$ of ξ. A maximal integral submanifold (of dimension n) of Z, tangent to the distribution ξ is called a *Legendrian submanifold*, and represents physical degrees of freedom of a state at equilibrium, whence the equilibrium state is a probability distribution on the Legendrian submanifold. The Legendrian corresponding to a particular macroscopic system relies on the underlying microscopic degrees of freedom (e.g. positions and momenta) and their interactions (e.g. Hamiltonian).

It is possible to describe an equilibrium Legendrian Σ (and a state ρ on it) in terms of generating functions $f(x_i)$ on the contact thermodynamic manifold Z. These generating functions correspond to thermodynamic potentials such as entropy, internal energy, free energy, enthalpy etc depending on half of the thermodynamic variables (must be non-conjugate ones). Geometrically, this is the data of an integrable *Lagrangian distribution* on Z with conormal df. Once a potential is given, expected values of all possible thermodynamic observables X with respect to a state ρ can be computed.

$$\langle X \rangle_{\Sigma,\rho} = \int_\Sigma X\rho \,.$$

EXAMPLE 2.1.5 (Microcanonical ensembles). *A microcanonical foliation* $\mathcal{F}_{\text{micro}} = \{F_{N_0V_0U_0}\}$ *is a collection of* (N,V,U)*-ensembles transverse to a maximal Legendrian* Σ_f *of* $(Z, \xi_{\text{micro}} = \ker \alpha_{\text{mirco}})$*. In local coordinates* $(T,U,p,V,\mu,N,S) \in \mathcal{U} \subset Z$

$$\alpha_{\text{mirco}}|_{\mathcal{U}} = dS - \frac{dU}{T} - \frac{pdV}{T} + \frac{\mu dN}{T}, \quad F_{N_0V_0U_0} = N^{-1}(N_0) \cap V^{-1}(V_0) \cap U^{-1}(U_0), \quad \alpha_{\text{micro}}|_{T\Sigma_f} = 0\,.$$

To describe equilibrium states in terms of N, V *and* U*, we have to ensure that entropy* S *is a function of those variables. In other words, entropy* $S = f(N,V,U)$ *is constant along an ensemble* $F_{N_0V_0U_0}$*. The transverse three-dimensional Legendrian* Σ_f *can also be given in terms of the local generating function* $f(N,V,U)$ *obtained by integrating the PDE*

$$dS = \frac{dU}{T} + \frac{pdV}{T} - \frac{\mu dN}{T}\,, \tag{2.12}$$

along Σ_f. *Plugging in the ansatz* $S = f(N,V,U)$ *in the above line, we obtain the remaining equations of state*

$$\frac{\partial f}{\partial U} = \frac{1}{T}, \quad \frac{\partial f}{\partial V} = \frac{p}{T}, \quad \frac{\partial f}{\partial N} = -\frac{\mu}{T}. \tag{2.13}$$

Thermodynamic systems are said to be homogeneous when entropy S *is a degree-1 homogeneous function of extensive variables* N,V,U. *In that case, the solution to Equation* (2.12) *is given by Euler's formula*

$$S = \frac{\partial S}{\partial U}U + \frac{\partial S}{\partial V}V + \frac{\partial S}{\partial N}N = \frac{U}{T} + \frac{pV}{T} - \frac{\mu N}{T}. \tag{2.14}$$

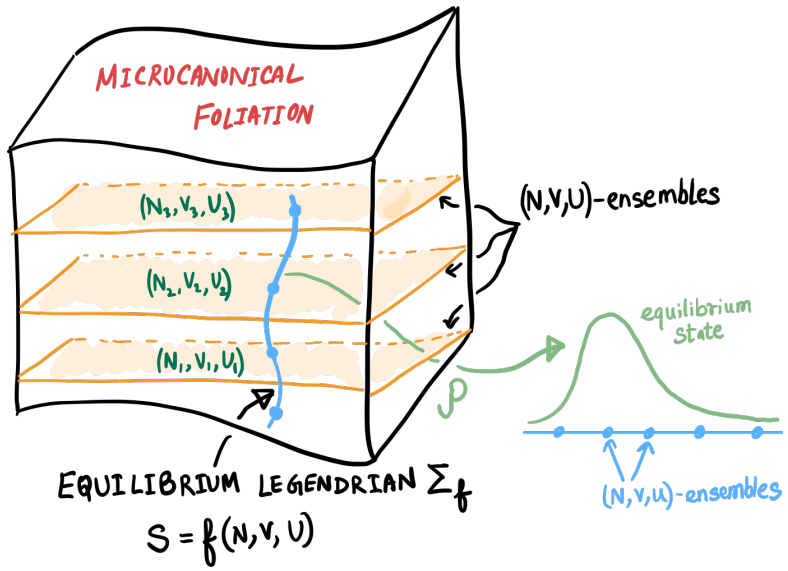


FIGURE 2.5. Mircocanonical ensembles and equilibrium states

The thermodynamic Equations (2.13)–(2.14) *should be interpreted as defining points*

$$(T(N,V,U), U, p(N,V,U), V, \mu(N,V,U), N, S(N,V,U)) \in \Sigma_f \subset Z$$

that characterize the equilibrium state space Σ_f. *The only problem is we do not yet know how the entropy* $S(N,V,U)$ *depends on the ensemble parameters* N,V,U. *The Boltzmann's formula expresses the entropy in terms of the number of microstates* $\Omega(N,V,U)$ *in an* (N,V,U) *ensemble*

$$S(N,V,U) = k_B \log(\Omega(N,V,U)).$$

Thus the generating function S depends on the microscopic details of the system such as Hamiltonian H, particle count N and accessible volume V.

There is a second kind of dynamics on a contact manifold, one that is *transverse* or nowhere tangent to the contact distribution. This generalizes the Hamiltonian dynamics of symplectic manifolds. Some non-equilibrium thermodynamic relaxation processes could also be described in terms of transverse dynamics that start at a Legendrian submanifold and end at another [36]. We will assume an extra orientability criterion to give a simple description of such dynamics. A *co-orientable* contact manifold is one whose quotient line bundle TZ/ξ is trivial. Then there exists a global contact form α, and a nowhere vanishing vector field R_α called a *Reeb vector field*. Let $\omega := d\alpha$ be its symplectic *Levi* 2-form. R_α is in fact uniquely determined by α

$$\iota_{R_\alpha}\omega = 0, \quad \iota_{R_\alpha}\alpha = 1\,. \tag{2.15}$$

Given a contact distribution ξ, the contact form α is not unique. Indeed, a nowhere vanishing function $f \in C^\infty(M)$, produces another contact form $\alpha' = f\alpha$ with the same contact structure. When the contact form α is fixed, the contact manifold Z is called *strict*. The non-integrability condition (2.9) implies a Whitney splitting of the tangent bundle TZ of a strict contact manifold Z

$$TZ = \xi \oplus \langle R_\alpha \rangle\,. \tag{2.16}$$

Different choices of contact form α, with the same contact structure ξ, lead to different directions of the tranverse Reeb vector fields R_α, and hence different dynamics. It is possible to work in a larger ambient strict contact manifold to organize all possible Reeb dynamics corresponding to the contact structure [23].

Contactomorphims are the contact analogues of symplectomorphisms. Two contact manifolds (Y, ξ_Y) and (Z, ξ_Z) are *contactomorphic* if there is a diffeomorphism $\varphi : T \to Z$ such that

$$\varphi_* \xi_Y = \xi_Z\,.$$

From Equation (2.15), it follows that Reeb flows are special contactomorphisms called *strict contactomorphisms* since preserve they also preserve the contact form

$$\mathcal{L}_R \alpha = 0\,.$$

In general, a vector field X whose flow $\phi_t : Z \to Z$ preserves the contact structure

$$(\phi_t)_*\xi = \xi \iff \mathcal{L}_X\alpha = g\alpha, \quad g \in C^\infty(Z)\,,$$

is called a *contact vector field*. On integrating the above action of the contact vector field, we obtain

$$\alpha_t := \phi_t^*\alpha = f_t\alpha\,,$$

where $f_t := \exp\left(\int_0^t g \circ \phi_s \, ds\right)$. Thus contact vector fields are precisely the Reeb vector fields corresponding to choices of different contact forms. Contact vector fields are also like Hamiltonian vector fields, in that each such vector field X_H is uniquely determined by a smooth function $H : Z \to \mathbb{R}$ called a *contact Hamiltonian*

$$\iota_{X_H}\omega = -dH + dH(R_\alpha)\alpha, \quad \iota_{X_H}\alpha = H\,. \tag{2.17}$$

Notice that the first equation in the above line is a deformed version of (2.6). When H is the constant function whose value is 1, then the above equations reduce to (2.15) and $X_H = R_\alpha$.

The normalization condition $\alpha(R) = 1$ mimics that of a unit-speed geodesic vector field associated to a relativistic free particle in a (pseudo-)Riemannian spacetime. Indeed, it is possible to derive geodesics of particles in terms of Reeb flow on a unit sphere bundle [**76**].

EXAMPLE 2.1.6 (Relativistic particle). *Let (M^n, g) be a Riemannian manifold and $^\sharp : T^*M \to TM$, $^\flat : TM \to T^*M$ be the canonical musical isomorphisms given by $v^\flat = g(v, .)$, $g(w^\sharp, v) = w(v)$ for $v \in TM, w \in T^*M$. The total space of the cotangent bundle $\pi : (T^*M, \omega_{\text{can}}) \to M$ is a symplectic manifold equipped with the canonical exact symplectic form $\omega_{\text{can}} = d\theta_{\text{can}}$. In local coordinates $(q^i, p_i) \in T^*M$, the canonical symplectic potential θ_{can} and its symplectic dual/Liouville vector field Y are*

$$\theta_{\text{can}} = \sum_{i=1}^n p_i dq^i, \quad Y = \sum_{i=1}^n p_i \partial_{p_i}\,.$$

The unit cotangent bundle $Z := ST^*M = \{(q, \hat{p}) \mid q \in M,\ \hat{p} \in T_x^*M,\ \hat{p}(\hat{p}^\sharp) = 1\}$ is a hypersurface $Z \overset{i}{\hookrightarrow} T^*M$ transverse to the Liouville vector field Y. The canonial bundle projection is $\pi' := \pi \circ i : Z \to M$.

$$\begin{array}{ccc} ST^*M & \xhookrightarrow{\ i\ } & T^*M \\ & \searrow^{\pi'} & \downarrow \pi \\ & & M \end{array}$$

Z is a contact manifold with induced contact form $\alpha_{x,\hat{p}}(u) := i^*\theta_{\text{can}}$. The splitting

$$TZ = HZ \oplus VZ, \quad R_\alpha = \sum_{i=1}^{n} \hat{p}_i \partial_{q^i}, \quad HZ = \ker(\alpha) = \xi, \quad VZ = \langle R_\alpha \rangle,$$

is the data of a linear (Ehresmann) connection on $\pi' : Z \to M$ and α is its connection 1-form. The parallel sections correspond to geodesics on M. Geodesic flows on M are therefore Reeb flows on ST^*M.

The probabilistic foundations of statistical mechanics rest on the notion of a probability distribution associated to a large system at equilibrium. This distribution is defined over an *ensemble* of microscopic configurations underlying a macroscopic system. Ensemble theory is centered around the equality of two averages at equilibrium: empirical and theoretical; the time average of a large number of measurement results on a system and the ensemble average based on a probability distribution. These ideas can be traced back to the pioneering work of Clausius, Boltzmann and Gibbs, among others. Microscopic physics has profound influence on macroscopic physics. Interesting emergent physical phenomena like thermodynamics, phase transitions, magnetism, superconductivity all arise from the dynamics and interactions of microscopic degrees of freedom of a macroscopic substance.

In the case of thermodynamic systems, a contact macroscopic configuration space emerges from an underlying high dimensional symplectic microscopic configuration space. Each macroscopic equilibrium state (macrostate) corresponds to an ensemble of accessible microscopic configurations (microstates) of particles. When a system reaches equilibrium, macroscopic measurable properties like energy, number of particles, pressure, temperature *etc.* become *steady* or constant in time. However, the system is still dynamical and continues to visit different microstates in the ensemble. The relationship between the equilibrium macrostate and its ensemble of microstates is captured by a probability distribution on the ensemble called the *equilibrium distribution*. At equilibrium, many systems also tend to be *ergodic*. This means that *all* microstates in its ensemble are explored. Assuming ergodicity, the amount of time the system spends at every microstate is proportional to the probability given by the equilibrium distribution.

2.1.1.4. *Discrete.* Degrees of freedom of a manifold configuration space label physical quantities taking values in a continuous space. However, we there exists many systems that possess degrees of freedom taking values in a discrete space. These could be heads and tails of a coin, on and off configurations of a switch or streams of bits on a computer. Even though these systems are discrete, their evolution could be probabilistic. This is the case for most physical systems.

Let us see first see how probabilistic evolution of disrete degrees of freedom might emerge from an otherwise deterministic system. For this, we return to the the spinning coin example from before. We will assume an ideal system; there is no air resistance and the coin is only acted upon by gravity pointing vertically down. This is a Hamiltonian system that can completely described in terms of its translation and rotational kinetic energies, as well as its gravitational potential energy. As the coin spins in the air, when viewed from the top, it continuously switches from heads to tails to heads and so on. This is a continuous change even though there are "quantized" states that correspond to the two perfect orientations of (heads)"up" and (heads)"down". At the end of the toss, the coin lands either heads up or heads down. The referee in a game of football, deciding which team attacks which side's goal, is only interested in the final outcome. For the purposes of the decision, the coin may be described by a discrete random variable X called *side* taking values in the binary set $\{0, 1\}$, where 0 represents heads and 1 represents tails. The probability of heads/tails is dynamical and changes continuously from 0 to 1 as the coin spins in the air. Since the rigid-body dynamics of the coin is Hamiltonian, so is the dynamics of the random variable X. It is interesting to view the discrete set $\{0, 1\}$ as a macroscopic configuration space. The microstates corresponding to each macrostate could be (not necessarily perfectly up or down) orientations of the coin when they are perceived to be up or down. A computer is a macroscopic system that manipulates a large number of coin-like degrees of freedom called bits. These bits are constantly being sent through noisy channels. The result is a probabilistic evolution of bits. There are many other systems that are also error-prone. An imperfect automated assembly plant can for example produce a fully working device with only 70% probability, or a DNA sequence in a cell can get mutated due to random external factors like radiation.

We posit that supermanifolds are ideal for describing systems possessing both continuous and discrete degrees of freedom. Supermanifolds are a natural generalization of manifolds. Supermanifolds admit special anti-commuting (Grassmann odd) directions in addition to the commuting

continuous ones. Odd variables are step-2 nilpotent and are therefore intrinsically binary. This greatly affects the type of functions that live on a supermanifold. Along odd directions, superfunctions look like polynomials with degree at most one in each generator. The elements of an algebra of commuting idempotents also have the same form. The algebra of commuting idempotents is a Boolean algebra that describes binary degrees of freedom. While anti-commuting variables may seem arcane at first, their binary nature allow them to encode discrete degrees of freedom.

A geometric configuration space, on its own, can not be a complete model for a physical system. The "physical" quality is the ability to perform experiments on the system and make measurements. We will next discuss this issue of measurability.

2.2. Measurability

Measurements are key to studying any physical system. In the physical world, a system is meaningful only so far as it can be observed/probed by experiments. Experiments allow determining specific properties of systems using appropriate apparatus or measuring device. While setting up an experiment, an experimenter is faced with three basic questions:

(1) *Which* system to study?

(2) *What* property of the system to study?

(3) *How* to measure? (or which apparatus to use?)

At first thought, the "system" in first question seems silly as we already described relevant geometric models for physical configuration spaces. However, recall that configuration spaces label all possible configurations the system might be in. By "system" we mean the true "configuration" of the system as a function of time. For a microscopic system, this is the trajectory of a point on a symplectic manifold. For a macroscopic system, as discussed in the previous section, we may only have a time-parameterised family of probability distributions over the underlying microstate configuration space. The latter includes the special case of trajectory of a point when the distribution at every instant of time is a degenerate point mass distribution. We will henceforth call a dynamical (time-parameterised) probability distribution on a microscopic configuration space a *state*.

The "property" in the second question could refer to a range of degrees of freedom of the system the experimenter wishes to probe. These could be position, momentum intervals on a microscopic

phase space or pressure, temperature intervals on a thermodynamic phase space. In general, a property is simply a measurable subset of the configuration space and will be called an *event*.

The role of "apparatus" could be played by a measurable function, called *observable*, on the configuration space. Examples include Hamiltonian energy of microscopic systems, state functions and potentials of thermodynamic systems. In probability theory, once a probability measure is fixed, these are called random variables. Note that there are apparatus like ruler, speedometer, thermometer *etc.* that directly measure specific degrees of freedom. These correspond to the coordinate functions of microscopic/macroscopic configuration spaces.

We would now like to give precise mathematical defintions of states, observables and measurements on a configuration space. We would like to have two important properties for a geometric measurement theory

(1) *Local-global principle:* Local observers probe local properties of a globally well-defined system. Local observers must also agree on compatible measurements of the system. These point to a tight relationship between local and global aspects of a physical system.
(2) *Algebra/ring of observables:* Observables need not be functions or maps between some domain and codomain sets. The defining property of observables is that they form an algebra. This is the case for quantum mechanical observables, for example.

The above two ideas naturally lead us to the notion of *sheaves* of rings. Interestingly, the definition of supermanifolds rely on sheaves as well.

2.3. Sheaves of functions

Broadly speaking, functions are morphisms between objects in a suitable category. Functions on a smooth manifold, for example, refer to smooth maps from open sets on the manifold to the set of real numbers. Functions on an open set may be restricted to smaller open sets contained in the original set. It may also be possible to define *ring* operations *i.e.*, addition and multiplication of functions. A presheaf is a space of functions, called *sections*, attached to open sets on a manifold and compatible with restrictions. The germ of sections of a presheaf (defined in terms of a direct limit) at a point is called its *stalk* at that point. Stalks record the behavior of sections in the vicinity of points. A sheaf is a presheaf with an additional *gluing* property: global data is obtained by gluing local data. Following are the definitions of presheaf and sheaf of rings.

DEFINITION 2.3.1. *A presheaf of rings $\mathcal{F} = (X, \mathcal{O}_X)$ is an assignment $\mathcal{F} : U \to \mathcal{F}(U)$ of a family of rings $\mathcal{F}(U)$ to open sets $U \subseteq X$ together with restriction maps $\rho_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$, $V \subseteq U \subseteq X$ that satisfy the restriction axioms*

(1) $\rho_{U,U} = \mathrm{Id}\,,$

(2) $\rho_{V,W} \circ \rho_{U,V} = \rho_{U,W}\,.$

Let $\{U_i\}$ be any open cover for a subset $U \subseteq X$. For a presheaf to be a sheaf, it must also satisfy

(1) locality: If $s, t \in \mathcal{F}(U)$ and $\rho_{U,U_i}(s) = \rho_{U,U_i}(t)\ \forall\, i$, then $s = t$.

(2) gluing: If $s_i \in \mathcal{F}(U_i)$ and $\rho_{U_i,U_i\cap U_j}(s_i) = \rho_{U_j,U_i\cap U_j}(s_j)$ then there exists $s \in \mathcal{F}(U)$ such that $\rho_{U,U_i}(s) = s_i$.

The above two conditions of locality and gluing ensure that the gluing process is *unique* for *compatible* sections. The compatibility is just that sections defined on sets of a cover, agree on pairwise overlaps of the sets.

The rings $C^\infty(U)$ of all smooth maps $U \to \mathbb{R}$, defined on each open set U of a smooth manifold M, form a sheaf and will be denoted as $\mathcal{F} = (M, C^\infty)$. The restriction maps, in this case, correspond to usual restriction of functions

$$\rho_{U,V} : C^\infty(U) \ni f \mapsto f|_V \in C^\infty(V)\,.$$

The restriction and gluing axioms are all satisfied here since sections are just maps between sets. Supermanifolds generalize this sheaf of rings. They are defined by locally adjoining Grassmann anticommuting variables to the ring of smooth functions. Before jumping into supermanifolds, we would like describe two sheaves that are foundational to any measurement theory. To discuss them, we need to add a measurability structure. A topological configuration space M is said to be *measurable* when equipped with a σ-algebra Σ of subsets called *events* an observer/experimenter is able to measure. The σ-algebra generated by open sets of M is called the *Borel σ-algebra* of the topological space M. This is will be our preferred σ-algebra unless otherwise mentioned. We will denote a measurable configuration space by (M, Σ).

2.3.1. States and Observables. States are defined as sections of a sheaf of probability measures [12, 45, 89] on a measurable topological space. We will refer to it as the *state sheaf*: $(M, \Sigma, \mathrm{State})$. Its stalk at any point is the set $\{1\}$. Upon identifying stalks with their underlying

points, points in turn get a state interpretation. Points are thus idealized states of the system called *pure states*. Newtonian mechanics, for example, describes deterministic dynamics of particles or pure states as trajectories on space-velocity configuration spaces. In practice, however, a system is typically statistical or macroscopic and never in a pure state. A realistic model for a physical system will regard a state as being *mixed*, *i.e.* in a statistical mixture of pure states, which is precisely a section of the state sheaf.

Random variables are measurable functions between two measurable spaces. The simplest kind one encounters in introductory probability theory courses are those taking values in real numbers (equipped with the standard Lebesgue measure). Similar to a state sheaf, one can define a sheaf of real-valued random variables [89] on a measurable configuration space (M, Σ). We will call this the *observable sheaf*: $(M, \Sigma, \mathrm{Obs})$. An experiment measures expected values of observables [103] probing certain properties of states. Expected value $\langle X \rangle_{\rho,E}$ of a random variable $X \in \mathrm{Obs}$ measuring an event $E \in \Sigma$ with respect to a state $\rho \in \mathrm{State}$ may be defined as

$$\langle X \rangle_{\rho,E} := \int_E X\rho \,,$$

where $X\rho$ is the pointwise product of two measurable functions and hence is itself measurable. Note that the above integral is defined in the sense of Lebesgue and may not always exist.

EXAMPLE 2.3.2 (Canonical ensemble). *Consider a rigid box of N gas molecules, occupying volume V and placed in contact with a thermal reservoir at temperature T. The equilibrium state of such a system corresponds to a canonical (NVT)-ensemble. The thermodynamic phase space Z, locally isomorphic to $\mathbb{R}^7$, is parameterized by three conjugate pairs, temperature (T) and entropy (S), pressure (p) and volume (V), chemical potential (μ) and particle number (n), as well as the Helmholtz free energy (F). The equilibrium states are points (N, V, T) in a 3-dimensional Legendrian submanifold $\mathcal{L}$ generated by the first jet of of a smooth function $f : Z \to \mathbb{R}$*

$$S = -\frac{\partial f}{\partial T}\bigg|_{N,V}, \quad p = -\frac{\partial f}{\partial V}\bigg|_{N,T}, \quad \mu = \frac{\partial f}{\partial N}\bigg|_{V,T}, \quad F = f \,.$$

The above are the fundamental state equations of the system. Any process $\gamma : [-\epsilon, \epsilon] \to \mathcal{L}$ satisfies the First Law since

$$dF|_{T\mathcal{L}} = (-SdT - pdV + \mu dN)|_{T\mathcal{L}} \,.$$

The equilibrium distribution is given by the Gibbs measure

$$P(i) = \frac{e^{-\beta E_i}}{\sum_j e^{-\beta E_j}} ,$$

where P_i is the probability of the gas to be occupying the microstate i whose Hamiltonian energy is E_i and $\beta = \frac{1}{kT}$ is proportional to the inverse temperature. The Gibbs measure is a section of a state sheaf on the microscopic configuration space. Many other physical quantities of interest are sections of the sheaf of smooth real-valued functions on a measurable configuration space. These include random variables like partition function associated to an ensemble of microstates as well as thermodynamic state functions like enthalpy and Gibbs free energy which are functions on equilibrium Legendrian submanifolds.

2.3.2. Supermanifold. A supermanifold is a *locally ringed superspace*. It consists of a smooth topological manifold M called the *body* together with a sheaf $\mathcal{O}_M$ of supercommutative algebras called its *structure sheaf*. These algebras are labelled by open sets in the body M, and locality refers to the stalks being *local rings i.e.* for each point $x \in M$, the stalk $\mathcal{O}_{M,x}$ at x has a unique maximal ideal $\mathfrak{m}_x \subset \mathcal{O}_{M,x}$ (the set of all nilpotents at x). Let $\Lambda\mathbb{R}^m$ denote the Grassmann algebra over $\mathbb{R}$ generated by m anti-commuting elements $\theta^1, \theta^2, \ldots, \theta^m \in \Lambda\mathbb{R}^m$

$$\Lambda\mathbb{R}^m := \mathbb{R}[\theta^1, \theta^2, \ldots, \theta^m], \quad \theta^i\theta^j + \theta^j\theta^i = 0 .$$

The defining property of supermanifolds, that distinguishes it from general superspaces, is that it locally looks like the tensor product of a Grassmann algebra $\Lambda\mathbb{R}^m$ and the ring of smooth functions on an affine space $\mathbb{R}^n$. This refers to the geometry of an *affine superspace* denoted by $\mathbb{R}^{n|m} = (M, \mathcal{O}_M)$ where

$$M = \mathbb{R}^n, \quad \mathcal{O}_M = C^\infty \otimes_{\mathbb{R}} \Lambda\mathbb{R}^m .$$

To be able to compare two supermanifolds, we need to define suitable maps or morphisms between them. A morphism between two superspaces $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$ is defined as a pair of maps (ϕ, ψ) such that

(1) $\phi : M \to N$ is continuous map between topological spaces,

(2) $\psi : \mathcal{O}_N \to \phi_* \mathcal{O}_M$ is a morphism (commutes with restrictions, preserves grading) of sheaves of rings where for every open set $V \subseteq N$, $\phi_* \mathcal{O}_M\big|_V := \mathcal{O}_M\big|_{\phi^{-1}(V)}$ is the direct image sheaf of $\mathcal{O}_M$ along ϕ,

(3) for every point $x \in M$, the homomorphism of stalks $\psi_x : \mathcal{O}_{N,\phi(x)} \to \mathcal{O}_{M,x}$ preserves locality: $\psi_x\left(\mathfrak{m}_{\phi(x)}\right) \subset \mathfrak{m}_x$.

We are now ready to give the definition of a supermanifold.

DEFINITION 2.3.3. *A supermanifold $\mathcal{M}^{(n|m)} := (M, \mathcal{O}_M)$ of dimension $(n|m)$ is a sheaf $\mathcal{O}_M$ of supercommutative local rings over a smooth manifold M of dimension n, such that for every point $x \in M$ there exists an open neighborhood $U \subseteq M$ on which the sheaf $\mathcal{O}_M$ trivializes as*

$$(U, \mathcal{O}_U) \cong \mathbb{R}^{m|n} . \tag{2.18}$$

Note that the isomorphism above refers to a pair of maps and is in the category of superspaces as described earlier. Let us now turn to some key structures that are special to a supermanifold.

(1) $\mathbb{Z}_2$-*grading*: The "super" in sheaf $\mathcal{O}_M$ of supercommutative algebras refers to a direct sum decomposition into *even* (or grade 0) and *odd* (or grade 1) sections

$$\mathcal{O}_M = \mathcal{O}_{M,\text{even}} \oplus \mathcal{O}_{M,\text{odd}} .$$

(2) *Sheaf of nilpotents*: A *nilpotent* is an element of a ring that is zero when multiplied with itself a finite number of times. On the other hand, a *unit* is an invertible element (inverse x^{-1} of an element x of a unital ring, if it exists, must obey $x^{-1}x = xx^{-1} = 1$) of a unital ring. The set of all nilpotents of the ring form an ideal which we will denote as $\mathfrak{m}$. The set of all units of a ring form a group. An element $x \in \Lambda\mathbb{R}^m$ may be expressed in terms of generators

$$x = x_{\text{body}} + x_{\text{soul}} , \tag{2.19}$$

where $x_{\text{soul}} := \sum_{i=1}^{m} x_i \theta^i + \sum_{i<j} x_{ij} \theta^i \theta^j + \ldots + x_{12\ldots m} \theta^1 \theta^2 \ldots \theta^m$ is the nilpotent part of x, $x_{\text{body}} := x_0 1$ is the term proportional to the identity 1 in the algebra, and the coefficients $x_0, x_i, x_{ij}, \ldots, x_{12\ldots m} \in \mathbb{R}$. If $x_{\text{body}} \neq 0$, x is invertible and hence a unit. Since the quotient

$$\Lambda\mathbb{R}^m / \mathfrak{m} \cong \mathbb{R} \tag{2.20}$$

is a field, $\mathfrak{m}$ is a maximal ideal, and the Grassmann algebra $\Lambda\mathbb{R}^m$ is a local ring. The sheaf of nilpotents $\mathcal{J}_M \subset \mathcal{O}_M$ are generated by the odd sections in $\mathcal{O}_M$

$$\mathcal{J}_M = \mathcal{O}_{M,\text{odd}} \oplus \mathcal{O}^2_{M,\text{odd}}\,,$$

and Equation (2.20) generalizes to an isomorphism of sheaves

$$\mathcal{O}_M/\mathcal{J}_M \cong C^\infty\,. \tag{2.21}$$

To get the above isomorphism, we have used the local trivializations (2.18) and the augmentation/body map $\Lambda\mathbb{R}^m \to \mathbb{R}$ (as in (2.19)) that sends any element in the Grassmann algebra to its body.

(3) *Body map*: By Equation (2.21) and the canonical projection $\mathcal{O}_M \to \mathcal{O}_M/\mathcal{J}_M$, there is a canonical embedding of supermanifolds called the body map ε

$$(M, C^\infty) \overset{\varepsilon}{\hookrightarrow} (M, \mathcal{O}_M)\,. \tag{2.22}$$

The body map ε is a key structural data of a supermanifold and is sometimes explicitly included in its definition.

A prototypical example of supermanifold is the section sheaf $(M, \Gamma(\Lambda E))$ of an exterior bundle ΛE of a smooth real vector bundle E. This is the most common type of superspace used in supergravity and supersymmetric gauge theories to describe fields, constraints and their dynamics [69]. In the next section, we will see a remarkable theorem that says that smooth supermanifolds are equivalent to the sheaf of sections of exterior algebra bundles up to a non-canonical choice. That choice corresponds to a notion of splitting.

2.4. Splittings

We will now review one of the most important structural properties of differentiable real supermanifolds. In [9], Batchelor showed that every real supermanifold can be *split*. This means that the following short exact sequence of sheaves

$$0 \longrightarrow \mathcal{J}_M \longrightarrow \mathcal{O}_M \underset{\pi^*}{\overset{\varepsilon^*}{\rightleftarrows}} C^\infty M \longrightarrow 0\,, \tag{2.23}$$

splits due to the existence of a *section* π^* of ε^*: $\varepsilon^* \circ \pi^* = \mathrm{Id}_{C^\infty M}$. The map π^* defines a surjective morphism $\pi = (\mathrm{Id}, \pi^*)$ of supermanifolds

$$(M, \mathcal{O}_M) \xrightarrow{\pi} (M, C^\infty M)\,. \tag{2.24}$$

Note that ε^* in Equation (2.23) is just the pullback along the body map (2.22) and hence is part of the structural data of the supermanifold $(M, \mathcal{O}_M)$. Because of local triviality of supermanifolds, the sequence splits when the sheaves are restricted to local trivializing charts. Splittability is therefore a global obstruction problem that depends on the way local sections glue to produce global sections. Such global problems are often studied by cohomological methods. This is the approach taken by Batchelor in her doctoral thesis. For a smooth manifold M, she proved the following isomorphism of two Čech/sheaf cohomology groups

$$\hat{H}^1(M, GL(m)) \cong \hat{H}^1(M, \mathrm{Aut}(\Lambda\mathbb{R}^m))\,. \tag{2.25}$$

The group $GL(m)$ of invertible linear transformations of $\mathbb{R}^m$ is the structure group of a real vector bundle $E \to M$. The classes in the first Čech cohomology group $\hat{H}^1(M, GL(m))$ are inequivalent gluing data $\{U_\alpha \subseteq M, g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(m)\}$ for vector bundles over M and therefore completely classifies smooth vector bundles over M. Similarly, $\hat{H}^1(M, \mathrm{Aut}(\Lambda\mathbb{R}^m))$ classifies supermanifolds over M in terms of the structure group $\mathrm{Aut}(\Lambda\mathbb{R}^m)$. Note that the sheaf $\mathcal{M}|_U = C^\infty U \otimes \Lambda\mathbb{R}^m$, defined for every open set $U \subseteq M$ is $\mathbb{Z}$-graded, while a supermanifold comes with a $\mathbb{Z}_2$ grading only. Every $\mathbb{Z}$-grading canonically induces a $\mathbb{Z}_2$-grading based on evenness/oddness of integers. The $\mathbb{Z}$-grading is precisely the data of the map π in Equation (2.23). A supermanifold is said to be *split* when choice of such a map π is made or equivalently a $\mathbb{Z}$-grading is chosen for $\mathcal{M}$. Since the first Čech cohomology groups in Equation (2.25) classify isomorphism classes of vector bundles and split supermanifolds on M respectively, it establishes an equivalence (up to choice of π) between exterior sheaves of vector bundles and real supermanifolds. The main result of Batchelor is presented below.

THEOREM 2.4.1 (Batchelor). *Let $\mathcal{M} = (M, \mathcal{O}_M)$ be a real supermanifold. If E is a real vector bundle over the smoooth manifold M, let ΛE be the associated exterior bundle and let $\Gamma(\Lambda E)$ be the sheaf of sections of ΛE. Then there is a non-canonical isomorphism*

$$(M, \mathcal{O}_M) \cong (M, \Gamma(\Lambda E))\,, \tag{2.26}$$

for some vector bundle E over M.

To discuss $\mathbb{Z}$-gradings in a geometric way, it is sometimes useful to work in terms of the graded supermanifold $\mathrm{Gr}(\mathcal{M}^{n|m}) := (M, \mathrm{Gr}\,\mathcal{O}_M)$

$$\mathrm{Gr}(\mathcal{O}_M) := \oplus\, \mathrm{Gr}^m_{i=0}(\mathcal{O}_M), \quad \mathrm{Gr}_{i>0}(\mathcal{O}_M) := J^i_M/J^{i+1}_M, \quad \mathrm{Gr}_0(\mathcal{O}_M) := \mathcal{O}_M/\mathcal{J}_M\,.$$

Let $\Pi : \mathcal{M} \to \mathrm{Gr}(\mathcal{M})$ be the canonical projection. Batchelor's theorem (2.26) rephrased in terms an isomorphism s between a supermanifold and its graded counterpart

$$\mathcal{M} \underset{\sim}{\overset{\Pi}{\underset{s}{\rightleftarrows}}} \mathrm{Gr}\,\mathcal{M}\ , \quad \Pi \circ s = \mathrm{Id}_{\mathrm{Gr}\,\mathcal{M}}\ . \tag{2.27}$$

Note that the identity morphism $\mathrm{Id}_{\mathrm{Gr}\,\mathcal{M}} = (\mathrm{Id}_M, \mathrm{Id}_{\mathcal{O}_{\mathrm{Gr}\,\mathcal{M}}})$ is in the catgeory of supermanifolds.

Connections. In [64], Koszul showed that the splittings π or equivalently the $\mathbb{Z}$-gradings may be given a geometric description in terms of affine connections on $\mathcal{M}^{n|m}$. Let $T\mathcal{M} := \mathrm{Der}(\mathcal{O}_M)$ be the *tangent sheaf* or sheaf of derivations of superfunctions (sections of $\mathcal{O}_M$) which is a locally free module over $\mathcal{O}_M$. It is a vector bundle $T\mathcal{M} \overset{\pi'}{\to} \mathcal{M}$ corresponding to the canonical projection. It is a Whitney sum

$$T\mathcal{M} = T\mathcal{M}_{\mathrm{even}} \oplus T\mathcal{M}_{\mathrm{odd}}\,,$$

corresponding to the even and odd $\mathcal{O}_M$-submodules of $T\mathcal{M}$. The fibers of $T\mathcal{M}_{\mathrm{odd}}$ are of dimension $(0|m)$, while $T\mathcal{M}_{\mathrm{odd}}$ is a supermanifold of dimension $(n|m)$. Using the grading induced by the commutator of derivations, $T\mathcal{M}_{\mathrm{odd}}$ may be identified with the graded supermanifold $\mathrm{Gr}(\mathcal{M})$. The grade-1 subbundle E of $T\mathcal{M}_{\mathrm{odd}}$

$$T\mathcal{M}_{\mathrm{odd}} \supset E \cong \mathrm{Gr}_1(\mathcal{M})\,,$$

is one of Batchelor's vector bundles (see (2.26)). The data of an affine connection ∇ on $\mathcal{M}$ allows us to flow along geodesics to nearby points in E. The fibers of the pullback bundle $E \overset{\varepsilon^*\pi'}{\to} M$ (pulled back along the body map ε in Equation (2.22)) correspond precisely to geodesics along the odd directions in $\mathcal{M}$. Let $\exp_\nabla : T\mathcal{M} \to \mathcal{M}$ be the exponential map corresponding to the connection ∇. We have the following commutative diagram

$$\begin{array}{ccc} \mathrm{Gr}\,\mathcal{M} \cong T\mathcal{M}_{\mathrm{odd}} & \xrightarrow[\sim]{\exp_\nabla} & \mathcal{M} \\ & \searrow^{\varepsilon^*\pi'} & \downarrow \pi \\ & & M \end{array}$$

The exponential map $\exp_\nabla$ is an isomorphism when restricted to $T\mathcal{M}_{\text{odd}}$ and is a splitting map (cf. s in Equation (2.27)). The non-canonical projection π (cf. Equation (2.24)) is determined in terms of the exponential map

$$\pi = \varepsilon^* \pi' \circ \exp_\nabla^{-1} \Big|_{T\mathcal{M}_{\text{odd}}} .$$

The $\mathbb{Z}$-grading on $\mathcal{M}$ is determined by the action of the Euler vector field $\mathcal{X}$ of the vector bundle $E \overset{\varepsilon^*\pi'}{\to} M$. A superfunction $f \in \mathcal{O}_U$ is homogeneous with degree k if

$$(\exp_\nabla)_* \mathcal{X}|_U(f) = kf \,. \tag{2.28}$$

Koszul gives a simpler description of the Euler vector field without using the exponential map and shows that it is uniquely determined once a connection ∇ is given. Recall that an affine connection $\nabla : \operatorname{Der}\mathcal{O}_M \otimes \operatorname{Der}\mathcal{O}_M \to \operatorname{Der}\mathcal{O}_M$ is a map obeying the Leibniz rule

$$\nabla|_U(X \otimes fY) = X(f)Y + (-1)^{\tilde{f}\tilde{X}} \nabla|_U(X \otimes Y) \,,$$

where $U \subset M$ is open, $f \in \mathcal{O}_U$ and $X, Y \in \operatorname{Der}\mathcal{O}_U$ and both f, X have homogeneous grades $\tilde{f}, \tilde{X}$ respectively. Koszul's result on splittings induced by connections is given below.

THEOREM 2.4.2 (Koszul). *Let ∇ be an affine connection on a supermanifold $\mathcal{M}$. Then there exists a unique derivation $H^\nabla \in \operatorname{Der}\mathcal{O}_M$ such that*

$$\nabla_{H^\nabla} H^\nabla = H^\nabla \,.$$

Note that Koszul's H^∇ plays the role of the Euler vector field $\mathcal{X}$ in Equation (2.28) and uniquely determines a splitting of $\mathcal{M}$. Smooth (C^∞) supermanifolds have smooth manifolds as their bodies. Since smooth manifolds admit partitions of unity, there always exists affine connections on smooth supermanifolds and hence they can be split using connections. We will be working in this smooth setting and will always assume that our supermanifold is split by an affine connection ∇. Following the conventions used in [25], a representative vector bundle $(\mathcal{V}M, \nabla) \overset{\pi(\nabla)}{\to} M$, will be called a *Koszul bundle* for a supermanifold $\mathcal{M}$ if the exterior bundle $\mathcal{E}^\nabla M := (\Lambda\mathcal{V}M, \nabla)$ is isomorphic to the supermanifold $\mathcal{M}$

$$\mathcal{E}^\nabla M \overset{\nabla}{\cong} \mathcal{M} \,.$$

To describe dynamics, we would now like to equip our supermanifold $\mathcal{M}^{n|m}$ with a symplectic structure. For covariance reasons, we want the body manifold M to include phase-space and time directions, hence we will demand n to be odd. In the next section, we will state some basic results about the structure of such symplectic supermanifolds.

2.5. Supersymplectic structures

The symplectic structure of a split supermanifold $\pi : \mathcal{E}^\nabla M \xrightarrow{\pi} M$ with even-dimensional body M, has been characeterized by Rothstein in [85]. It determines the symplectic structure $\boldsymbol{\omega}$ of a split supermanifold $\mathcal{E}^\nabla M$ in terms of a symplectic form ω on its body M and a pseudo-Riemannian bundle metric g on the Koszul bundle $\mathcal{V}^\nabla M \to M$. Let us first give the definition of a supersymplectic manifold.

Definition 2.5.1. *A supersymplectic manifold $(\mathcal{M}^{2n|m}, \boldsymbol{\omega})$ is a supermanifold $\mathcal{M}$, equipped with an even symplectic 2-form $\boldsymbol{\omega} \in \Omega^2(\mathcal{M})_{\text{even}}$ that is closed and non-degenerate*

$$\boldsymbol{\omega}^{\wedge n} \neq 0 = d\boldsymbol{\omega}\,.$$

Let $d^\nabla : \Omega^*(M, \mathcal{V}M) \to \Omega^*(M, \mathcal{V}M)$ be the exterior covariant derivative corresponding to the induced connection ∇ on the vector bundle $\mathcal{V}M$. Using some superdiffeomorphsim $\phi : \mathcal{E}M \to \mathcal{E}M$, it is always possible [85] to bring a symplectic 2-form $\boldsymbol{\omega}$ on $\mathcal{E}M$ to a "quadratic" normal form

$$\phi^*\boldsymbol{\omega} = \pi^*\omega + d\alpha_{g,\nabla}\,, \tag{2.29}$$

where $\omega := \varepsilon^*\boldsymbol{\omega} \in \Omega^2(M)$ is a symplectic form on the body M. The 1-form $\Omega^1(\mathcal{V}M) \ni \alpha_{g,\nabla}(u) := g(d^\nabla u, u)$, $u \in \Gamma(\mathcal{V}M)$ is defined in terms of a bundle (pseudo-Riemannian) metric $\mathrm{Sym}^2(\mathcal{V}M) \ni g := \varepsilon^*\left(\boldsymbol{\omega}\big|_{\ker d\pi}\right)$ (using the canonical isomorphism $V\mathcal{V}M \cong \mathcal{V}M$ along the zero section in $\mathcal{V}M \to M$). Note that, by construction, α is a homogeneous grade-2 (with respect to the adjoint action of the Euler vector field on $\mathcal{E}M$) 1-form. Hence this decomposition is called the "quadratic" form. Rothstein's main theorem is given below.

Theorem 2.5.2 (Rothstein). *Given a manifold M and a vector bundle E, there is a 1-1 correspondence between*

(1) (ω, g, ∇) *where* ω *is a symplectic form on* M*,* g *is a metric on* E *and* ∇ *is a* g*-compatible connection, and*

(2) Supersymplectic forms of quadratic type on $(M, \mathcal{E}M)$*.*

The theorem above is stronger than just giving a normal form for supersymplectic structures. The converse is also true. That is given the data $(M, \omega, E, \nabla, g)$ of a real vector bundle E over M with connection ∇ compatible with the bundle metric g on E, and a symplectic form ω on M, there exists a supersymplectic form on the dual exterior bundle $\mathcal{E}M = \Lambda E^*$. There is an explicit construction that produces a supersymplectic form.

THEOREM 2.5.3 (Rothstein). *Given the data* $(M, \omega, E, \nabla, g)$*, let* $R \in \Gamma(M, \operatorname{End} E^* \otimes \Omega^2(M))$ *be the curvature of* ∇ *and let* R^g *be the contraction*

$$R^g := g_{bc} R^c_{ija} \theta^a \theta^b dx^i dx^j \,.$$

Let $X, Y \in \operatorname{Vect}(M)$ *and* $\Phi, \Psi \in \Gamma(M, E)$ *the following defines a supersymplectic form* $\boldsymbol{\omega} \in \Gamma(M, \Omega^2(\Lambda E^*))$

$$\begin{aligned} \boldsymbol{\omega}(\nabla_X, \nabla_Y) &= \omega(X, Y) + \frac{1}{2} R^g(X, Y) \,, \\ \boldsymbol{\omega}(\Phi, \Psi) &= g(\Phi, \Psi) \,, \\ \boldsymbol{\omega}(\nabla_X, \Psi) &= 0 \,. \end{aligned}$$

Rothstein's normal form preserves the connection that is originally used to split the supermanifold. It is possible to use local superdiffeomorphism to change the connection ∇ (and hence the splitting) locally so that its curvature R is also locally zero. This leads to a super analogue of Darboux theorem for graded supersymplectic manifolds, first proved by Kostant [63].

THEOREM 2.5.4 (Kostant). *Any split supersymplectic manifold* $(\mathcal{E}\mathcal{M}^{2n|m}, \boldsymbol{\omega})$ *admits a covering by an atlas of super Darboux charts* $(\boldsymbol{U}_\alpha, \varphi_\alpha : U_\alpha \to \mathbb{R}^{2n|m} \ni (p_i, q^i, \theta^a)_\alpha)$ *i.e.*

$$\left(\varphi_\alpha^{-1}\right)^* \boldsymbol{\omega} = \sum_{i=1}^{n} dp_{\alpha i} \wedge dq^i_\alpha + \frac{1}{2} \sum_{a=1}^{m} \epsilon_a (d\theta^a_\alpha)^2 \,,$$

where $\epsilon_a \in \{-1, 1\}$*.*

A super Darboux atlas $(\boldsymbol{U}_\alpha, \varphi_\alpha = (p_i, q^i, \theta^a)_\alpha)$ on a split supermanifold $\mathcal{E}M$ may be described in terms of a family of flat, torsion-free, affine connections $\nabla_\alpha := \varphi_\alpha^* d_{\mathbb{R}^{m|n}}$ adapted to the Darboux charts U_α *i.e.*

$$\nabla_\alpha^2 = 0, \quad \nabla_\alpha \boldsymbol{\omega}|_{U_\alpha} = 0\,.$$

A torsion-free, symplectic, affine connection on a supermanifold is called a *Fedosov connection*. Note that, in general, it may not be possible to glue the local flat Fedosov connections ∇_α to get a global connection on $\mathcal{E}M$. We will often split a supermanifold using local Fedosov connections adapted to a super Darboux atlas. This motivates the following notion of a Darboux splitting.

DEFINITION 2.5.5. *A Darboux splitting of a supersymplectic manifold $(\mathcal{M}^{(2n|m)}, \boldsymbol{\omega})$ is the data $(\boldsymbol{U}_\alpha, \nabla_\alpha)$ of a family of flat, Fedosov connections ∇_α defined on each open superdomain $U_\alpha \subseteq \mathcal{M}$, such that $\{\boldsymbol{U}_\alpha\}$ is a cover for the supermanifold $\mathcal{M}$.*

Fedosov structures encode the classical data for deformation quantization problems. Some good references for Fedosov structures on supermanifolds are [6, 72, 96].

2.5.1. Odd Supersymplectic. In Sections 2.1.1.2 and 2.1.1.3, we saw that odd symplectic manifolds like contact manifolds allowed us to treat time on the same footing as phase-space. We would now like to generalize the supersymplectic structure to include time.

DEFINITION 2.5.6. *An odd supersymplectic manifold $(\mathcal{M}^{2n+1|m}, \boldsymbol{\omega})$ is the data of a supermanifold $\mathcal{M}$ with an odd dimensional body and an even, closed 2-form $\boldsymbol{\omega} : T\mathcal{M} \to T^*\mathcal{M}$ having maximum possible rank*

$$\boldsymbol{\omega}^n \neq 0 = d\boldsymbol{\omega}\,.$$

The above definition implies that there exists a one-dimensional kernel (over $\mathcal{O}_M$) spanned by an even vector field. The disjoint union of these kernels gives a rank-1 subsheaf L of the tangent sheaf $T\mathcal{M}$

$$L|_U = \ker(\boldsymbol{\omega}|_U)\,. \tag{2.30}$$

Any nowhere vanishing section $v \in \Gamma(L)$ may be chosen as the generator of supersymplectic dynamics.

2.6. Laboratories

Measurements in a laboratory typically take place at an instant of time. The set of all possible configurations measurable by an experimenter at an instant of time may be characterized by a hypersurface embedded in the supermanifold. To capture instantaneity, the vector field generating dynamics must be transverse to the hypersurface at every point. This motivates the following defintion of a laboratory

DEFINITION 2.6.1. *Let* $\Sigma \overset{i}{\hookrightarrow} (\mathcal{M}, \boldsymbol{\omega})$ *be a codimension-*$(1|0)$ *subsupermanifold. The subsupermanifold* Σ *is a laboratory in* $\mathcal{M}$ *if*

$$\dim \ker(\imath^* \boldsymbol{\omega}) = 0\,.$$

In a laboratory, often measuring devices themselves are dynamical *i.e.* they change with respect to time over the course of an experiment. This could be captured by the geometry of a corank-$(1,0)$ foliation of supersymplectic manifold. To obtain a foliation of $\mathcal{M}$ by symplectic hypersurfaces, the data of a closed 1-form η satisfying the condition

$$\eta \wedge \boldsymbol{\omega}^{\wedge n} \neq 0 \text{ everywhere}\,,$$

suffices. This is a super generalization of the cosymplectic structure discussed in Section 2.1.1.2. On a laboratory Σ, phase-space measurements can be performed by superobservables $X \in i^*\mathcal{M}$ and expected values of measurements $\langle X \rangle_{\Sigma,\rho}$ are given by

$$\langle X \rangle_{\Sigma,\rho} := \frac{\int_{\Sigma_0} \boldsymbol{\omega}_0^{\wedge n}(\rho(X))_0}{\int_{\Sigma_0}(\rho(\mathbb{1}))_0}\,,$$

where $\rho \in (i^*\mathcal{M})^*$ is a state *i.e.* a section in the linear dual of $i^*\mathcal{M}$ with positive body, and the subscript $_0$ refer to the body of the superfunctions.

From an abstract viewpoint, measured foliations can be described by Cartan (maximal abelian) subalgebras of von Neumann algebras [55]. Deep results of Feldman-Moore and Connes [28, 40] show that it is sometimes possible to recover measured foliations from the data of von Neumman algebras. A good concise resource on these connections is [27]. In the next chapter, we will discuss measurement theory from a C^*-algebraic lens. This generality is very useful for a rigorous quantum theory of measurements. With some tweaks, we find that it also serves as a foundation for our measurement theory of superobservables.

CHAPTER 3

Measurement theory

A statistical description of a physical system assigns probabilities to outcomes of measurements. In the previous chapter, we argued that classical measurement theory is based on notions of states, observables, events, as well as a way to pair states with observables to yield expected values. We will now give a *general* algebraic formulation of states, observables, expectations and explain key properties necessary to describe classical measurement theory. This generalization allows a unified treatment of measurements of classical physical systems containing both continuous and discrete degrees of freedom. Historically, quantum mechanics was an important driver in the development of this theory. Basic measurement theory ideas were first conceived by von Neumann (see [**98**]), who developed the theory of operator algebras that model observables in quantum mechanics.

3.1. The problem

In Chapter 4, we will describe the measurement theory given by supercommuting algebras of superfunctions. Just as for commuting observables, we will give a physical interpretation to moments of supercommuting observables. The goal of this chapter is to introduce basic ideas from general measurement theory required to understand superobservables. Given a classical probability space $(\Omega, \mathcal{F}, P)$, one obtains a commuting algebra $\mathcal{A}$ of observables corresponding to measurable complex-valued functions on Ω, a set $\mathbb{1}$ of indicator/characteristic functions supported on elements of $\mathcal{F}$ and a positive linear functional $\rho \in \mathcal{A}^*$ given by the probability measure P. We would like to pose the converse question:

> *Given an algebra of observables $\mathcal{A}$ for a system in a state $\rho \in \mathcal{A}^*$ as well as a family of indicators $\{\mathbb{1}_A : A \in \mathcal{F}\}$, can one define a classical probability space $(\Omega, \mathcal{F}, P)$?*

In non-commutative probability theories, such as random matrix theory, operator algebras or free probability theory, one expects to obtain a classical probability space only for maximal

commuting subalgebras $\mathcal{C}$ of a $*$-algebra $\mathcal{A}$. For a Banach algebra $\mathcal{A}$, this is described by the Gel'fand representation of a maximal commuting subalgebra $\mathcal{C} \subset \mathcal{A}$

$$\pi : \mathcal{C} \to C_0(\Omega)\,, \tag{3.1}$$

where π is an algebra homomorphism, the sample space Ω is a locally compact, Hausdorff, topological space called the spectrum or Gel'fand space of $\mathcal{C}$, and $C_0(\Omega)$ refers to the commuting algebra of continuous complex-valued functions on Ω that vanish at infinity.

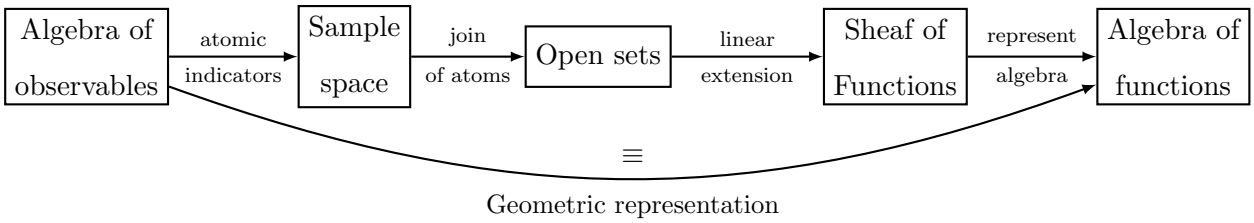
3.2. Sketch of solution

Since superobservables are only mildly non-commutative, we may hope to find a sample space Ω describing all elements of supercommutative $*$-algebra $\mathcal{A}$. In particular, we wish to reconstruct a classical probability space $(\Omega, \mathcal{F}, P)$ consisting of a sample space Ω, a σ-algebra $\mathcal{F}$ consisting of subsets of Ω, and a probability measure P supported on Ω, from the data of the $*$-algebra $\mathcal{A}$ equipped with a state $\rho \in \mathcal{A}^*$, and an indicator map $\mathbb{1} : \mathcal{F} \to \mathcal{O}_\mathcal{A}$ valued in the subspace $\mathcal{O}_\mathcal{A} \subset \mathcal{A}$ of self-adjoint observables. Schematically:

$$(\Omega, \mathcal{F}, P) \underset{?}{\overset{\text{easy}}{\rightleftarrows}} (\mathcal{A}, \mathbb{1}, \rho). \tag{3.2}$$

For a general non-commutative algebra $\mathcal{A}$, it is not clear whether such a sample space Ω exists, or whether $\mathcal{A}$ can be represented as some algebra of functions on Ω. There is a notion of spectrum of a C^* algebra $\mathcal{A}$ described by the set of irreducible, unitary $*$-representations called the *unitary dual* $\hat{\mathcal{A}}$ of $\mathcal{A}$. Elements in the algebra are represented as functions on the space of primitive ideals (equivalently, kernels of irreducible, unitary $*$-representations) of the algebra equipped with the hull-kernel topology. However, this definition of spectrum is too rigid to encode non-trivial spectrum of the nilpotents in the Grassmann algebra.

To answer to the question posed Equation 3.2, we develop a theory of superindicators based on order-theoretic ideas that generalize the commuting Boolean algebra of idempotents. One can view them as super thickenings of idempotents. The support of superindicators give us a local geometric model to serve as a backdrop against which we can understand the non-local features of the algebra of superobservables.



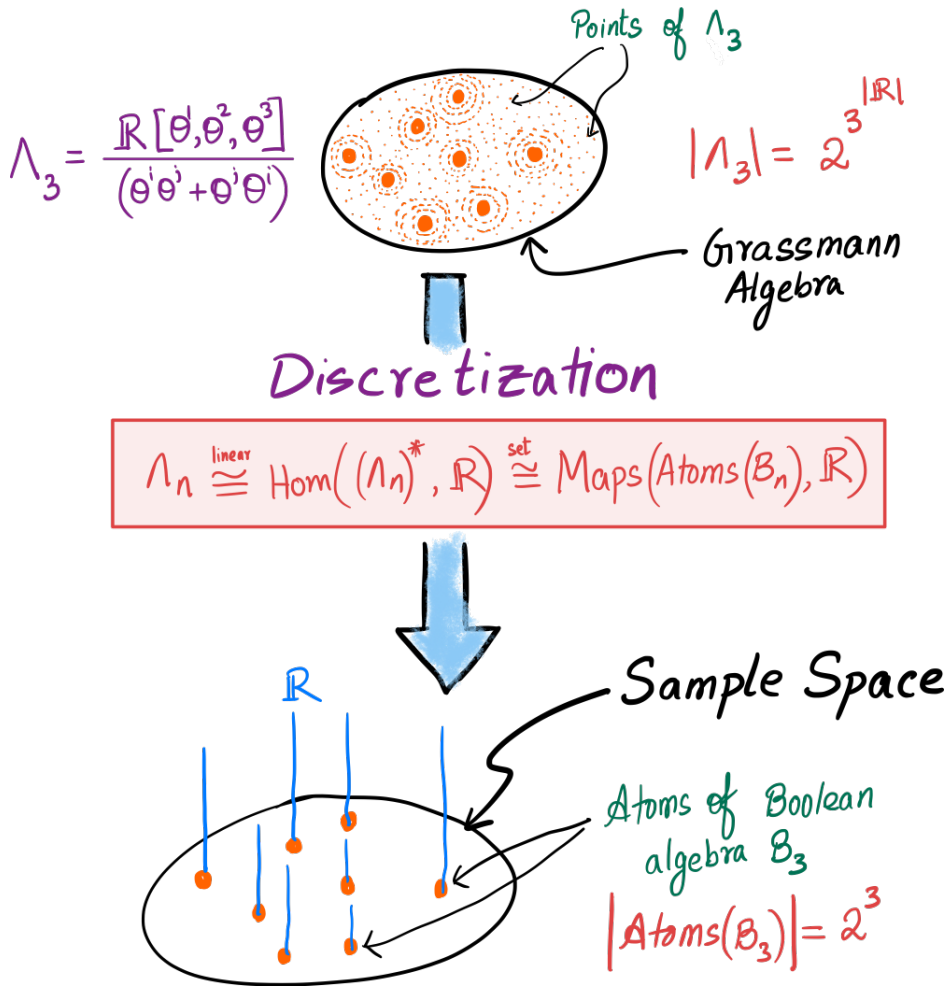


FIGURE 3.1. Superfunction in Λ_n = Function on the discrete set Atoms(B_n)

The basic steps in our recipe are as follows

(1) *Sample space*: We first define sample space and random variables using only the linear structure of the Grassmann algebra $\mathcal{A} = \Lambda_n$, generated by n elements. We choose a basis of $\mathcal{A}$

$$\mathcal{B} = \{e^i := \mathbb{1}_{a_i} : a_i \in \mathcal{F}\}\,,$$

from the given family of indicators $\{\mathbb{1}_A : A \in \mathcal{F}\}$. Points in the sample space Ω are elements of the dual basis $\mathcal{B}^* = \{e_i\}$. The dual basis also gives a notion of spectrum in terms of coordinate functionals. The spectral decomposition of a superobservable $X \in \langle\{e^i\}\rangle$ is obtained by evaluating the dual basis $\mathcal{B}^*$ on it

$$\mathcal{A} \ni X \overset{\sim}{\mapsto} \widetilde{X} = (e_i(X)) \in \mathrm{RV}(\Omega) := \mathrm{Hom}(\mathcal{A}^*, \mathbb{R})\,, \tag{3.3}$$

where $e_i(e^j) = \delta_j^i$, $\mathrm{Hom}(\mathcal{A}^*, \mathbb{R})$ refers to the space of linear maps from $\mathcal{A}^*$ to $\mathbb{R}$. The above allows each superobservable X to play the *rôle* of a random variable $\widetilde{X}$ *i.e.* a real-valued function on the sample space Ω.

(2) *Topology*: We next define a σ-algebra on the sample space Ω using the algebra $\mathcal{A}$. This is a Boolean algebra $\mathscr{B} = (\mathrm{Image}(\mathbb{1}), \leq, \vee, \wedge, \neg)$ generated by the set of indicators $\{\mathbb{1}_A : A \in \mathcal{F}\}$. The three Boolean operations: join ($\vee$), meet $\wedge$ and complement $\neg$ are all defined in terms of the sum and product operations in $\mathcal{A}$. The Boolean algebra $\mathscr{B}$ can be used to encode both points in the sample space Ω as well as its discrete topology, as we will discuss next.

An elementary property of a finite Boolean algebra is that it is *atomistic i.e.* isomorphic to the Boolean algebra of subsets of the set $\mathrm{Atoms}(\mathscr{B})$ of its atoms

$$\mathscr{B} \cong \square(\mathrm{Atoms}(\mathscr{B}))\,,$$

where $\square(A)$ denotes the power set of the set A. Atoms are the minimal elements of a Boolean algebra. They are join-prime/irreducible

$$\mathbb{1}_a \in \mathrm{Atoms}(\mathscr{B}) \iff \mathbb{1}_a = \mathbb{1}_A \vee \mathbb{1}_{A'} \implies \mathbb{1}_A = \mathbb{1}_a \text{ or } \mathbb{1}_{A'} = \mathbb{1}_a\,.$$

The set of atoms $\text{Atoms}(\mathscr{B})$ has cardinality 2^n. For our Boolean algebra $\mathscr{B}$, we will require the atoms to match the basis elements

$$\text{Atoms}(\mathscr{B}) = \mathcal{B} = \{e^i\}\,.$$

Each element of the Boolean algebra $\mathbb{1}_A \in L$ is the join of atoms below it

$$\mathbb{1}_A = \bigvee \{\mathbb{1}_a \in \text{Atoms}(\mathscr{B}) : \mathbb{1}_a \leq \mathbb{1}_A\}\,.$$

The above line is the Boolean analogue of expansion in basis elements of a vector space.

The discrete topology of our finite sample space Ω can be derived from the Boolean algebra $\mathscr{B}$. This is a direct consequence of the Stone duality (holds more generally even for infinite cardinality Boolean algebras by assuming Zorn's lemma). The atoms e^i of $\mathscr{B}$ are in bijection with the singletons $\{e_i\}$ in the power set $\square(\Omega)$ of the sample space Ω. Formally, the atoms are represented as sets by their upper sets

$$\begin{aligned} \text{Atoms}(\mathscr{B}) &\xrightarrow{\cong} \Omega \\ e^i \mapsto e^i\uparrow &\cong \{e_i\}\,. \end{aligned}$$

The upper sets are precisely the maximal filters called *ultrafilters* and dual to maximal (or equivalently prime) order ideals of the Boolean algebra $\mathscr{B}$. Points corresponding to prime or maximal ideals of a ring is typical in spectral theories. The singletons $\{e_i\}$ generate the discrete topology of clopen subsets of Ω. By Stone's representation theorem, each element of the Boolean algebra is represented by a clopen subset of Ω: the set of ultrafilters containing the Boolean element

$$\begin{aligned} \mathscr{B} &\xrightarrow{\cong} \text{Clopen}(\Omega) \\ \mathbb{1}_A &\mapsto \{e_i \in \Omega : \mathbb{1}_A \in e^i\uparrow\}\,. \end{aligned}$$

Such discrete topological spaces whose topologies can be described by Boolean algebras are called *Stone spaces*. Our sample space Ω, equipped with the discrete topology arising from the super Boolean algebra $\mathscr{B}$, is a finite Stone space.

(3) *Algebra of random variables*: There is still more information in $\mathcal{A}$ that is not completely captured by either the sample space Ω or the Boolean algebra $\mathscr{B}$. For that, we pull back both the sum and product structures in $\mathcal{A}$ using the map $\widetilde{\ }$ in Equation 3.3 the product in the algebra $\mathcal{A}$ to $\mathrm{RV}(\Omega)$

$$\widetilde{X} + \widetilde{Y} = \widetilde{X + Y}, \quad \widetilde{X}\widetilde{Y} := \widetilde{XY}\,.$$

The algebra of these random variables is non-local and can not be expressed as pointwise-product. This is apparent from the structure formula

$$\mathbb{1}_a \mathbb{1}_b = \sum_c f_{ab}^c \mathbb{1}_c\,,$$

where $\mathbb{1}_a, \mathbb{1}_b, \mathbb{1}_c \in \mathrm{Atoms}(\mathscr{B})$ and the structure constants $f_{ab}^c \in \mathbb{Q}$. Importantly, the rational values of the structure constants indicate that the algebra of super random variables is enumerative. In Chapter 4, we will compute these structure constants and they will turn out to be extremely useful to give physical meaning to superobservables.

(4) *Probabilities*: We shall recover a probability measure P on Ω from the data of statistics/expectation values of all finite ordered family of superobservables $(X_1, X_2, \ldots, X_n)$ with respect to the given state ρ such that

$$\langle X_1 X_2 \ldots X_n \rangle_\rho = \int_\Omega dP(p)\ \widetilde{X}_1(p)\widetilde{X}_2(p)\ldots\widetilde{X}_n(p)\,.$$

In the right-hand side of the above equation, the superobservables have been identified with real-valued random variables on the sample space Ω, the range of values corresponding to the spectrum defined by the chosen basis $\mathcal{B}$.

The two non-trivial steps in the above list are (2) and (4), which will be the focus of Chapter 4.

We shall now review the ingredients of the aforementioned construction. The presentation in the rest of this chapter is partly based on Tao's free probability theory notes [94], the C^*-algebraic/operator-theoretic approach to quantum statistical mechanics described in [18, 38, 73], the theory of positive operator-valued measures in quantum measurement theory [22], as well as our work [25].

3.3. Algebras of observables

Observables correspond to measurable attributes of a system. These attributes are typically parameterized by real numbers. It may be possible to combine some observables to produce other ones. For example, an optics experiment might use a polarimeter to measure the polarization and a photometer to measure the intensity of a circularly polarized light beam. It is possible to engage both the polarimeter and photometer to simultaneously measure the polarization and intensity of a beam of light instead of measuring either one. This process may be understood in two steps

(1) *Joint measurement*: We are first trying to define an observable vector $X_{P\cap I}$ that *jointly* measures both polarization X_P and intensity X_I,

$$X_{P\cap I}(\omega) := (X_P(\omega), X_I(\omega)), \quad \omega \in \mathcal{S}\,,$$

where $\omega \in \mathcal{S}$ is an element of the sample space $\mathcal{S}$ of all experimental outcomes.

(2) *Interaction/Interference*: Interaction between a collection of measuring devices could affect the final outcome of the measurement. Here simultaneous measurements with polarimeter and photometer could interfere with one another. This can be modeled by a map

$$\cdot : \mathrm{Obs} \times \mathrm{Obs} \to \mathrm{Obs}\,,$$

that produces a new observable $X_P \cdot X_I$ from the observables X_P and X_I, where $X_P \cdot X_I$ need not equal $X_{P\cap I}$.

A circularly polarized beam of light could itself be either left or right circularly polarized. Therefore the polarization observable X_P itself can be decomposed into

$$X_P = X_L + X_R\,,$$

where X_L, X_R are observables corresponding to left and right circularly polarized light respectively. This motivates an algebra structure where observables can be both added and multiplied.

A $*$-algebra is an algebra equipped with an involution $*$ and self-adjoint/hermitian/real elements are those that are preserved by the $*$-action. Observables can be modeled by self-adjoint elements of a $*$-algebra.

A $*$-algebra $\mathcal{A}$ is a $*$-ring that is an associative algebra over a commutative $*$-ring R. A $*$-ring/algebra is a ring/algebra equipped with a map * that is both an antiautomorphism and an involution. For all $X, Y \in \mathcal{A}$, $\lambda \in R$, and $^-$ being the involution in R, the following must be obeyed

- $(X+Y)^* = X^* + Y^*$,
- $(XY)^* = Y^* X^*$,
- $(X^*)^* = X$,
- $(\lambda X)^* = \overline{\lambda} X^*$.

DEFINITION 3.3.1. *Observables are self-adjoint elements of a unital, associative $*$-algebra $\mathcal{A}$ over the complex numbers $\mathbb{C}$.*

We will denote the set of observables by $\mathcal{O}_\mathcal{A} = \mathcal{O}_\mathcal{A}^* \subset \mathcal{A}$. In measure theory, they are real measurable functions, while in probability theory they are real random variables. In quantum mechanics, observables are self-adjoint linear operators acting on a Hilbert space. On a manifold M, classical observables are often taken to be smooth functions. Like measurable real-valued functions, they form a commutative algebra with respect to pointwise addition and multiplication. On the other hand, operators on a Hilbert space form a non-commutative algebra with respect to composition of maps.

Observables do not close with respect to the product in the algebra. It is nonetheless possible to close the algebra $\mathcal{O}_\mathcal{A}$ with respect to a commutative, but non-associative product called the Jordan product $\circ$. For $X, Y \in \mathcal{O}_\mathcal{A}$ their Jordan product $X \circ Y$ is given by

$$\mathcal{O}_\mathcal{A} \ni X \circ Y := \frac{1}{2}(XY + YX) .$$

3.4. States

States are generalized probability distributions associated to measurable properties of a physical system. The set of probability measures on a measurable space has the structure of a convex set. States are identified with positive linear functionals (positive acting on positive elements) in the linear dual $\mathcal{A}^*$ of the $*$-algebra $\mathcal{A}$.

DEFINITION 3.4.1. *States are linear functionals $\mathcal{P}_\mathcal{A} \ni \psi : \mathcal{A} \to \mathbb{C}$ of a unital, associative $*$-algebra $\mathcal{A}$ that obey*

(1) positivity: $\mathcal{P}(X^*X) \geq 0$,

(2) normalization: $\mathcal{P}(\mathbb{I}) = 1$,

(3) *-compatibility: $\mathcal{P}(X^*) = \overline{P(X)}$.

The set of states of an algebra $\mathcal{A}$ as its *state space* $\mathcal{P}_\mathcal{A}$. For a C^*-algebra this is a compact, convex subset of the linear dual $\mathcal{A}^*$ equipped with the weak-* topology.

3.4.1. Purity. Since states are elements of a convex set, it is useful to consider those $\psi \in \mathcal{P}_\mathcal{A}$ that cannot be expressed as a convex combination of two states χ, ξ

$$\psi = a\chi + (1-a)\xi,\ \chi \neq \xi \implies \psi = \chi \text{ or } \xi\,.$$

These are called *extreme/pure states*. States that are in the convex hull (the set of finite convex combinations) of linearly independent pure states are called *mixed*. There are some useful analytic results that tell us that mixed states are quite generic and hence can be used to approximate arbitary states.

3.4.2. Convex representations. Whenever the state space is a compact convex subset of a locally convex topological space, every state (not just pure or mixed) admits an approximation by a net of mixed states. This follows from the Krein-Milman theorem, which shows that such sets are generated by their extreme points, together with the barycentric representation theorems of Choquet and of Bishop-de Leeuw [15, 26, 65]. Some good references on convex structures in operator algebras is [1, 5].

THEOREM 3.4.2 (Krein-Milman). *Let X be a locally convex, Hausdorff topological vector space and $K \subseteq X$ be a compact convex subset. Then every point in K lies in the closure of the convex hull of the set* $\operatorname{ext} K$ *of extreme points of K, i.e.*

$$K = \overline{\operatorname{co}}(\operatorname{ext} K)\,.$$

The above theorem tells us, for example, that mixed states of a C^*-algebra are weakly dense in the space of all states equipped with the weak *-topology. It does not tell us how to represent a state as a convex combination of the pure states. The theorems of Choquet-Bishop-de Leeuw further tell us that the approximation in terms of convex combinations can in fact be strengthened to probability measures on the set of pure states.

THEOREM 3.4.3 (Choquet-Bishop-de Leeuw). *Let X be a locally convex, Hausdorff topological vector space and $K \subseteq X$ be a compact convex subset. For every $x \in K$, there is exists a Borel probability measure μ_x supported on the extreme set* $\operatorname{ext} K$ *of K such that*

$$x = \int_{\operatorname{ext} K} y \, d\mu_x(y) \, .$$

Note that the integral representation in the above theorem is generally not unique and a choice of such a measure may be characterized by the data of a probability simplex called the *Choquet simplex*. In finite dimensions, there is an upper bound on the number of pure states necessary to represent a mixed state. This is an elementary result due to Carathéodory.

THEOREM 3.4.4 (Carathéodory). *Let K be a compact subset of an n-dimensional vector space. Then every point in K is in the convex hull of atmost $n+1$ extreme points.*

Later we will see how to obtain probabilities of outcomes of a measurement from the data of a state and an observable.

3.5. Indicators

In this section, we will review some fairly elementary notions in probability theory like sample space, events and indicators. A good textbook covering the basics of probability theory is [56]. The aim of the discussion below is to highlight some key measure-theoretic properties that will be relevant to us.

An experimenter in a laboratory makes measurements using measuring devices/apparatus. The reading on these apparatus correspond to outputs of measurements. Each distinct list of outputs is an outcome of the experiment. The set of all possible outcomes that could be recorded by an experimenter studying a given physical system is called the *sample space* $\mathcal{S}$ of the system.

It is possible that an individual experimenter will not be able to measure all possible properties of a system at an instant of time. Additionally, there are error bars associated to any measuring apparatus. These considerations behoove us to equip the sample space $\mathcal{S}$ with a σ-algebra Σ. Elements of the σ-algebra are called measurable sets or *events*. A σ-algebra is a set algebra (or

field of sets) closed under countable unions and complements, *i.e.*

$$\bigcup_{i=1}^{\infty} A_i \in \Sigma, \quad A^{\mathsf{C}} \in \Sigma\,,$$

where $A, A_i \in \Sigma$. We have already discussed this measurability problem (see Section 2.2) and a notion of laboratories (see Section 2.6) in the context of configuration spaces in Chapter 2.

Indicators are special observables that model the outcomes observed by the apparatuses available to the experimenter. More generally, they are a family of observables parameterized by subsets $A \in \Sigma$ that resolve the identity in the algebra.

Let $\mathcal{A}$ be a C^*-algebra *i.e.* a complex, associative $*$-algebra that is also a Banach space with respect to the norm $||.||$ such that for $X, Y \in \mathcal{A}$ the following holds

(1) Submultiplicative: $||XY|| \leq ||X|||Y||$,

(2) Unit norm: $||\mathbb{1}|| = 1$,

(3) C^*-condition: $||XX^*|| = ||X||^2$.

Definition 3.5.1. *An indicator is a σ-additive set function $\mathbb{1} : \Sigma \to \mathcal{O}_{\mathcal{A}}$ that satisfies*

$$\mathbb{1}(\phi) = 0, \quad \mathbb{1}(A^{\mathsf{C}}) = \mathbb{I} - \mathbb{1}(A), \quad \mathbb{1}(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mathbb{1}(A_n)$$

for $A \in \Sigma$ and any countable family $A_1, A_2, \ldots$ of disjoint sets in Σ.

Remark 3.5.1. *The convergence of the infinite sum on the right is with respect to the strong operator topology of $\mathcal{A}$. The third property above is called "normality".*

It follows from the above definition that $\mathbb{1}(\Omega) = \mathbb{I}$, which is the identity in the algebra $\mathcal{A}$ and hence this is also called a *resolution of identity*. In classical probability theory, the indicators are characteristic functions obeying the idempotence property

$$\mathbb{1}_A^2 = \mathbb{1}_A\,,$$

and supported on events $A \in \Sigma$ of a sample space Ω. In quantum mechanics, indicators may be defined on the outcome space Ω of a family of commuting self-adjoint operators on a Hilbert space. The indicator map is the data of a positive operator-valued measure

$$\mathbb{1}_A \geq 0\,,$$

where 0 is the zero operator and $\geq$ is the ordering induced by the cone of positive self-adjoint bounded operators on a Hilbert space. In either case, the indicator map is valued in a cone of positive elements. However, for a Grassmann algebra there is *a priori* no notion of positivity. We will therefore use the indicator map $\mathbb{1}$ to define an observer's cone in the Grassmann algebra (see Chapter 4, Section 4.6).

Atoms. Sometimes it is possible to partition a measurable sample space $(\Omega, \mathcal{F})$ into non-empty, minimal, measurable subsets $A_i \in \mathcal{F}$ called *atoms*

$$\Omega = \bigsqcup_i A_i \,. \tag{3.4}$$

Note that minimal means no proper measurable subset. In the above equation, disjoint union could be used since minimality forces atoms to be disjoint. If the atoms A_i further generate the σ-algebra (by countable unions and complements), then every set $A \in \mathcal{F}$ can be uniquely expressed as the disjoint union of atoms contained in it

$$A = \bigsqcup_{\mathcal{F} \ni A_i \subseteq A} A_i \,. \tag{3.5}$$

A σ-algebra satisfying the above properties (Equations 4.16, 3.5) is said to be *atomic*. For example, Borel σ-algebras on finite sets are atomic. The Borel σ-algebra on the real interval $[0, 1]$ is *atomless, i.e.* it does not contain any atom. Any countably generated σ-algebra

$$\mathcal{F} = \langle \{A_n : n \in \mathbb{N}\} \rangle \,,$$

is atomic. This is because for each point $x \in \Omega$, the intersection

$$A_x = \bigcap_{x \in A_n \in \mathcal{F}} A_n \,,$$

is an element of $\mathcal{F}$ by the countable closure property of the σ-algebra $\mathcal{F}$, and is minimal. The atoms A_x now yield a partition Π of Ω

$$\Pi = \{A_x : x \in \Omega\} / \sim \,,$$

where $\sim$ is the equivalence relation defined by $A_x \sim A_y$ if $y \in A_x$. In other words, the equivalence classes $[A_x] \in \Pi$ are disjoint and cover Ω

$$[A_x] \cap [A_y] = \phi, \quad \bigsqcup_{x \in \Omega} [A_x] = \Omega \,.$$

A generic σ-algebra $\mathcal{F}$ contains both atomic and atomless/continuous parts. Let us denote the set of all atoms by

$$\text{Atoms}(\mathcal{F}) = \{A \in \text{Power}(\Omega) : \text{ A is an atom of } \mathcal{F}\} \,.$$

Then its complement can be used to define the continuous σ-subalgebra of $\mathcal{F}$

$$\mathcal{F}_{\text{cont}} := \{A \cap (\Omega \setminus \bigcup \text{Atoms}(\mathcal{F})) : A \in \mathcal{F}\} \,.$$

The original σ-algebra can be expressed as the join of the atomic and continuous subalgebras

$$\mathcal{F} = \sigma(\text{Atoms}(\mathcal{F})) \vee \mathcal{F}_{\text{cont}} \,.$$

Similarly, a probability measure P on $\mathcal{F}$ can also be split

$$P = P_{\text{atom}} + P_{\text{cont}} \,,$$

where $P_{\text{atom}}(A) := P(A \cap (\bigcup \text{Atoms}(\mathcal{F})))$ and $P_{\text{cont}} := P(A \cap (\bigcup \mathcal{F}_{\text{cont}}))$.

A generating set of atoms is to a σ-algebra what basis is to a vector space. Later, we will identify the set of indicators supported on the atomic sets, called *atomic indicators*, with a linearly independent subset of the unital, associative $*$-algebra $\mathcal{A}$. This will be useful to obtain a spectral decomposition for observables measurable by an experimenter's measuring devices (captured by the indicator map $\mathbb{1}$).

Binary codes

There is a useful representation of atoms of a countably generated σ-algebra $\mathcal{F}$ in terms of sequences of zeros and ones, corresponding to whether a point in the sample space Ω does not belong or belongs to a generating set $A_i \in \mathcal{F}$. Let $[A_x]$ be an atom containing $x \in \Omega$. Then the

map

$$\phi : \Omega \to \text{Maps}(\mathbb{N}, \{0, 1\})$$

$$x \mapsto (\mathbb{1}_{A_i}(x)) \,,$$

gives a binary representation of x and consequently of the atom A_x it lies in.

3.6. Spectra

The spectrum of an observable refers to possible outputs of measurements as recorded by measuring devices in a lab. In classical probability theory, these are the range of values taken by a random variable. They can be discrete like the number of heads observed when tossing five coins, or continuous such as the dispersion of light through a prism, splitting white light into its constituent frequencies. Indeed spectroscopy is the area of science devoted to measuring with radiation the atomic and molecular energy levels of systems of particles. If we wish to model observables by elements of a unital, associative $*$-algebra, it ought to be possible to assign a spectrum to every observable.

In a $*$-algebra, the spectrum $\sigma(X)$ of an element $X \in \mathcal{A}$ is defined in terms of non-existence of inverses

$$\sigma(X) := \{a \in \mathbb{C} \mid X - \mathbb{I}a \text{ is not invertible}\} \,,$$

where an element $X \in \mathcal{A}$ is invertible if there exists an element $X^{-1} \in \mathcal{A}$ such that $XX^{-1} = X^{-1}X = \mathbb{I}$. This generalizes the point spectrum/eigenvalues of operators to include continuous spectra. The above definition relies on the algebra (sum and product) structure of $\mathcal{A}$. For nilpotent observables, the spectrum is always trivially $\{0\}$. This is not surprising, nilpotents are as far from projectors/idempotents as possible. Since nilpotents will be an important class of superobservables, we ought to generalize the notion of a spectrum in way that does not rely on the product in the algebra.

In order to have a richer spectrum for nilpotents, we could use the linear structure of $\mathcal{A}$. For a finite dimensional algebra $\mathcal{A}$, one can choose a basis $\mathcal{B} = \{e_i\}$. Then any element $X \in \mathcal{A}$ may be uniquely expanded as

$$X = \sum_i e^i(X) e_i \,,$$

where $\{e^i \in \mathcal{A}^*\}$ is the basis of $\mathcal{A}^*$ dual to $\mathcal{B}$. The set $\{e^i(X) \in \mathbb{C}\}$ of expansion coefficients can be interpreted as the spectrum of the element X. This notion of spectrum depends on the choice of a basis of $\mathcal{A}$, akin to the choice of measuring devices in an experiment. We will next formalize this idea in terms of the data of an indicator map $\mathbb{1}$.

Let $\mathbb{1} : (\Omega, \Sigma) \to \mathcal{O}_\mathcal{A}$ be an indicator map on a sample space Ω equipped with an atomic σ-algebra Σ. Let us also assume that the indicator map $\mathbb{1}$ is generated by linearly independent observables, meaning

$$\mathcal{B}_\mathbb{1} := \{\mathbb{1}_U : U \in \text{Atoms}(\Sigma)\} \quad \text{is a linearly independent set.}$$

For an infinite-dimensonal $*$-algebra $\mathcal{A}$, the set $\text{Atoms}(\Sigma)$ of atoms could be infinite. In that case, the meaning of linear independence is that any finite subset of $\mathcal{B}_\mathbb{1}$ is linearly independent. There are various options for an infinite basis:

(1) a possibly uncountable *Hamel basis* where every element in $\mathcal{A}$ can be expressed as a finite linear combination of basis elements,
(2) a countable *Schauder basis* in a topological vector space where every element admits a covergent expansion in the basis elements,
(3) a possibly uncountable orthonormal basis in a (possibly non-separable) Hilbert space in which every element admits a convergent expansion in the basis elements.

Next, we would like to define a dual basis $\mathcal{B}^*_\mathbb{1}$. The dual atomic indicators

$$\mathcal{B}^*_\mathbb{1} := \{\psi_U \in \mathcal{A}^* \mid \psi_U(\mathbb{1}_V) = \delta_{UV},\ U, V \in \text{Atoms}(\Sigma)\}$$

may be used to define the spectrum of an observable $X \in \langle \text{Image}(\mathbb{1})\rangle$ in the span of the indicator map $\mathbb{1}$.

Definition 3.6.1. *Let $\mathcal{A}$ be a unital, associative $*$-algebra and $\mathbb{1} : (\mathcal{S}, \Sigma) \to \mathcal{O}_\mathcal{A}$ an indicator map on an atomic σ-algebra Σ, generated by linearly independent observables. The spectrum of an observable $X \in \langle \text{Image}(\mathbb{1}_\bullet)\rangle$ is the set $\sigma(X)$ of all possible values of the observables as measured by the atomic indicators*

$$\sigma(X) := \{(\mathbb{1}^*(U))(X) \mid U \in \text{Atoms}(\Sigma)\}\,.$$

3.7. Localization

The ring $\mathcal{O}_p$ of germs of functions at the point $p \in \Omega$ on a topological space Ω is given by the localization

$$\mathcal{O}_p := \left\{ \frac{f}{g} : f, g \in C^1(\Omega) \text{ and } g \neq 0 \text{ on some neighborhood } U \text{ containing } p \right\} / \sim = S_p^{-1} C^1(\Omega) \,,$$

with respect to the multiplicative set S_p of non-vanishing functions at p

$$S_p := \{ f \in C^1(\Omega) : f(p) \neq 0 \} \,.$$

Note that the $\sim$ denotes the equivalence relation where we identify fractions $\frac{f_1}{g_1} \sim \frac{f_2}{g_2}$ if there exists $h \in S_p$ such that

$$\frac{f_1 h}{g_1 h} = \frac{f_2}{g_2} \,.$$

The kernel of the localization map $C^1(\Omega) \to \mathcal{O}_p$ is the unique maximal ideal

$$I_p := \{ f \in C^1(\Omega) : f(p) = 0 \} \,.$$

It follows that the ring of germs $\mathcal{O}_p$ is a local ring

$$\mathcal{O}_p / I_p \cong \mathbb{R} \,.$$

Observe that the restriction of a function $f \in C^1(\Omega)$ to a point p amounts to looking at its image in the local ring $\mathcal{O}_p$. The value of the function at the point p is recovered algebraically from the composition

$$C^1(\Omega) \to \mathcal{O}_p \to \mathcal{O}_p / I_p \,.$$

In general, localization $\mathcal{S}^{-1}\mathcal{R}$ of a ring $\mathcal{R}$ with respect to some multiplicative set $\mathcal{S}_p$, may not lead to a local ring at p. When they do, one obtains a canonical notion of "value" of ring elements.

3.7.1. Spectra of rings. The spectrum of an algebra refers to a dual geometric space: topological space on which the algebra is represented as a sheaf of rings. The spectrum of a ring element corresponds to the range of values in the stalks of the sheaf. There are some remarkable isomorphisms between commutative rings and topological spaces.

- In algebraic geometry, the spectrum of commutative ring $\mathcal{R}$ is a scheme $(\operatorname{Spec}\mathcal{R}, \mathcal{O}_{\operatorname{Spec}\mathcal{R}})$: topological space $\operatorname{Spec}\mathcal{R}$ whose points are prime ideals of $\mathcal{R}$, open sets given by the Zariski topology, and the sheaf $\mathcal{O}_{\operatorname{Spec}\mathcal{R}}$ sends every open set to the localization of the ring $\mathcal{R}$ at the complement of union of prime ideals in the open set.
- Grothendieck duality is the isomorphism between the category AffSch of affine schemes and the category CRing of commutative rings with identity

$$\text{AffSch} \cong \text{CRing}^{\text{op}} ,$$

  where the functor from AffSch to CRing is the global sections functor $\Gamma(\cdot, \mathcal{O}_{\operatorname{Spec}\mathcal{R}})$, while the functor from CRing to AffSch is the prime spectrum functor Spec.
- Gelfand duality is another example of a duality between algebra and geometry that expresses commutative, unital C^*-algebras as algebras of continuous complex-valued functions on compact, Hausdorff spaces

$$\text{CHaus} \cong \text{Comm}\text{C}^*\text{Alg}^{\text{op}} .$$

  The points in the compact, Hausdorff space dual to a commutative C^*-algebra $\mathcal{A}$ correspond to characters (unital $*$-homomorphisms $\mathcal{A} \to \mathbb{C}$) or equivalently maximal ideals of the algebra $\mathcal{A}$, while the open sets are given by the weak*-topology.

Yet another example is the Stone duality between the category Bool of Boolean algebras and the category Stone of compact, Hausdorff, totally disconnected spaces called *Stone spaces*

$$\text{Stone} \cong \text{Bool}^{\text{op}} .$$

Connes' theory of non-commutative spectral geometry [28] is an approach to extend this duality to non-commutative C^*-algebras. Some good resources on these categorical dualities are [54, 67].

The algebra of superfunctions, while still non-commutative, interestingly shares some similarities with the commutative algebra of idempotents. Consider a bijection $f : \Lambda_m \to B_m$ from a Grassmann algebra Λ_m over $\mathbb{Z}_2$ generated by m odd generators θ^i to a Boolean ring B_m generated by free generators $f(\theta^i)$ satisfying

$$f(\theta^i)f(\theta^i) = 1, \quad f(\theta^i)f(\theta^j) + f(\theta^j)f(\theta^i) = 0 .$$

We see that f preserves the anticommutation relation for generators but replaces nilpotence by idempotence. A free Boolean ring is a deformed Grassmann algebra.

3.7.2. Superfunctions to measurable functions. Despite sharing some structural similarity with Boolean algebras, there does not seem to be an obvious spectral theory for Grassmann algebras similar to the case of commutative rings. This is due to the lack of non-trivial prime ideals (the augmentation ideal containing all nilpotents is the unique maximal/prime ideal). In order to view superobservables as random variables, we however need to assign topological spaces to Grassmann algebras. We claim that this can be achieved by defining a generalized set of indicator superfunctions called superindicators. These could encode the supercommuting algebra in terms of non-trivial combinatorics of discrete sets. The construction of superindicators will be given in the next chapter. Here we will give a brief qualitative picture of how the indicator map might help in recovering atomic subsets of a measurable sample space, encoding generalized spectrum of a $*$-algebra.

Let us assume that the image of our indicator map $\mathbb{1} : \Sigma \to \mathcal{O}_{\mathcal{A}}$ spans $\mathcal{A}$, in which case we will call $\mathbb{1}$ *complete*

$$\langle \text{Image}(\mathbb{1}) \rangle = \mathcal{A} \,.$$

From the above definition of spectra, we obtain a map

$$\text{Spec}_\sigma : \mathcal{O}_{\mathcal{A}} \to \text{Maps}(\text{Atoms}(\Sigma), \mathbb{R}) \,,$$

which sends self-adjoint elements of the $*$-algebra to $\mathbb{R}$-valued functions on $\text{Atoms}(\Sigma)$. This is the analog of Equation 3.1, and gives us a notion of spectrum of the space of observables in terms of functions on a topological space. In particular, it gives us a way to reconstruct parts of the sample space $\mathcal{S}$ generated by $\text{Atoms}(\Sigma) \subset \text{Open}(\mathcal{S})$ in terms of evaluations

$$x(f) := \text{Spec}_\sigma(f)(A) \,,\ x \in A \in \text{Atoms}(\Sigma) \,.$$

3.8. Statistics

Experiments often aim to infer statistical properties of physical systems that are characterized by a probability measure P on some measurable sample space $(\Omega, \mathcal{F})$. Following the discussion in the previous section, we could try to use spectral data to reconstruct a sample space Ω. Next, we

would like to use the algebra of observables $\mathcal{A}$ together with the state ρ to determine the sample space probability measure P.

In this way of thinking, everything that we can ever learn about the system is encoded by the underlying probability measure P. While P is typically hard to determine directly, one can empirically determine probability distributions P_X associated to various observables or random variables X. A rich enough family $\{X_i : \Omega \to (\mathcal{S}_i, \Sigma_i) : i \in I\}$ of random variables, such as if the σ-algebra $\mathcal{F}$ is generated by it

$$\mathcal{F} = \langle \{X_i^{-1}(A) \mid A \in \Sigma_i\} \rangle_{\cup,\cap,\mathsf{C}} ,$$

will in principle be sufficient to recover P.

A rich family of such random variables is one given by an algebra $\mathcal{A}$ of observables/random variables. Our task of gleaning as much information about the state of the system from the algebra $\mathcal{A}$ can be broken down into two key steps [94]

(1) *Sample expectations*: Individual random variable probability distributions P_X can be inferred through measurements of their sample moments $\mathbb{E}_\rho[X^n]$

$$\mathbb{E}_\rho[X^n] := \frac{\rho(X^n)}{\rho(\mathbb{I})} .$$

The moments may be packaged into a generating function such as the *characteristic function* $\phi_X(t)$

$$\phi_X(t) := \mathbb{E}_\rho[\exp(itX)] = \rho(\mathbb{I}) \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \mathbb{E}_\rho[X^n] .$$

By Stone-Weierstrass theorem, every continuous function $f(X)$ can be uniformly approximated by polynomials generated by the monomials X^n. Then by the Riesz-Markov-Kakutani theorem, the positive linear functional ρ can be represented by a unique positive Borel measure P_X on the spectrum $\sigma(X)$ such that

$$\mathbb{E}_\rho[X^n] = \int_{\sigma(X)} x^n dP_X(x) .$$

Thus, the data of the characteristic function $\phi_X(t)$ recovers the probability distribution P_X of the random variable X.

Let $\{X_i : i \in I\}$ be a $\mathcal{F}$-generating family of random variables. Then, the joint distributions $\{P_{X_{\mathcal{I}}} : \mathcal{I} \in \text{Power}(I)\}$ are encoded by characteristic functions $\{\phi_{X_{\mathcal{I}}}\}$.

(2) Recovering sample space probability measure by using the joint probability distributions $\{P_{\mathcal{J}} := P_{X_{i_1}X_{i_2}\ldots X_{i_k}} : \{i_1, i_2, \ldots, i_k\} \in \mathcal{J} \subset P(\mathcal{I})\}$ of a collection $\{X_i \in \mathcal{A} : i \in \mathcal{I}\}$ of random variables

$$P(A) = \sum_{\mathcal{J}} P_{\mathcal{J}}(\cap_{i\in\mathcal{J}} A_i), \quad A = \sqcup_{\mathcal{J}} \left(\cap_{i\in\mathcal{J}} X_i^{-1}(A_i)\right), \quad A_i \in \Sigma_i \,.$$

Statistical quantities like mean and standard deviation capture average values and dispersions from mean respectively. Higher moments and mixed statistics and capture more non-local self and cross-correlations among random variables. For superobservables, we will given a physical interpretation of moments based on a set-theoretic interpretation of the superproduct.

CHAPTER 4

Superindicators

In the previous chapter, we gave an outline of our super measurement theory and explained various facets of it like indicators, topology, spectra and statistics. A key idea is that any random variable is a linear combination of atomic indicators. To endow random variables with the product of superobservables, we must find an effective way to encode the product of "atomic indicators". We achieve this by constructing "atomic indicators" as superproducts of prebase indicators.

Indicators are labelled by measurable open sets in the topology of the sample space. Let us first describe a useful labelling for sets in a discrete topology of a binary sample space. This labelling is special to a finite Boolean algebra whose atoms have the geometry of a hyperoctahedron.

4.1. Geometry of free Boolean algebras

We will start by choosing a minimal prebase for the discrete topology of a binary (power of 2 cardinality) sample space. We will see that a minimal prebase assigns signed labellings to corresponding basis elements of the discrete topology of the sample space. The signed representation is more than a notational device. It manifests an action of the signed permutation or hyperoctahedral group.

4.1.1. Boolean prebases. For a Boolean algebra $\mathscr{B}_n$ freely generated by n generators, there is a canonical prebase pB of cardinality $2n$, generating the discrete topology of the associated Stone sample space Ω. The prebase contains clopen sets and therefore can be expressed as a disjoint union

$$pB = P \sqcup P^{\mathsf{C}},$$

where the collection of sets P will be termed the *semi-prebase* and contains half of the prebase elements. We would like to use the semi-prebase P to label special subsets of Ω.

Next we will discuss how order theory can be used to give an algebraic description of finite sets. This is very relevant to us since non-trivial superfunctions are far from usual indicators

that are commuting idempotents. Failure of idempotency implies that there isn't a canonical notion of point-sets associated to superfunctions. An interesting problem in supermanifold theory is to obtain a satisfactory description of supermanifolds in terms of point-sets [7]. The functor of points approach to supermanifolds provides a categorical description. Originally developed for applications in commutative algebraic geometry, one defines X-points of a supermanifold M as the set of all morphisms $M \to X$ in the category of superschemes SupSch. By Yoneda's lemma, the supermanifold M may be completely recovered from its functor of points F_M

$$F_M : \mathrm{SupSch} \to \mathrm{Set}$$
$$X \mapsto \mathrm{Hom}(M, X)\,.$$

While this is helpful in classifying geometric data over M, it does not yield a satisfactory description of supermanifolds in terms of set-theoretic points. Deligne and Morgan's supersymmetry notes [31] has a short discussion on the functor of points approach to supermanifolds, while Dan Freed's notes give an intuitive explanation (with pictures) in terms of probes for "nilpotent clouds" [42]. There are also more classical point-set approaches to supermanifolds due to DeWitt [32], Rogers [83], as well as an attempt to compactify odd directions in a supermanifold by Bruzzo and Pestov [21].

4.1.2. Orders and lattices. A basic property of a collection of sets is the ability to order them based on set inclusion. This is generalized by the notion of a partially ordered set or poset $(P, \leq)$ which is an antisymmetric and transitive binary relation $\leq$ on the set P. Posets $(P, \leq, \vee, \wedge)$ which admit infimum/meet $\wedge$ and supremum/join $\vee$ for every pair of elements are called lattices

$$x \wedge y = \max\{z \in P : z \leq x,\ z \leq y\}, \quad x \vee y = \min\{z \in P : x \leq z,\ y \leq z\}\,.$$

Boolean algebras are special lattices that are completely characterized as being bounded (existence of a least/bottom $\bot$ and greatest/top element $\top$)

$$x \leq \top, \quad x \geq \bot\ \forall\ x \in P\,,$$

complemented (existence of a complement C for every element)

$$x \in P \implies \exists\, x^{\mathsf{C}} \in P, \quad x \vee x^{\mathsf{C}} = \top, \quad x \wedge x^{\mathsf{C}} = \bot\,,$$

and distributive (join and meet distribute over each other)

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \quad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z), \quad x, y, z \in P\,.$$

Some good resources on order and lattice theory are [13, 30]. For basic results in Boolean algebras one can look at [47, 88]. Just as points form the building blocks of point-set topologies, posets form the building blocks of lattices. In fact, it is possible to represent every finite and distributive lattice as a collection of certain subsets of a poset whose join and meet operations are the usual union and intersection of sets. To see why this might be true, let us consider a poset P. An order ideal I of P is a non-empty, directed *i.e.*

$$x, y \in I \implies \exists z \in I,\ x \leq z,\ y \leq z\,,$$

down/lower-subset of P *i.e.*

$$x \leq y \implies x \in I, \quad x \in P, y \in I\,.$$

The set of order ideals of the poset P forms a distributive lattice with join given by union and meet by intersection of sets. What is remarkable is that the converse statement is also true, *i.e.* every finite and distributive lattice is precisely of this form. This is the content of Birkhoff's representation theorem [14]

THEOREM 4.1.1 (Birkhoff). *Every finite, distributive lattice L is isomorphic to a lattice of finite sets. In particular,*

$$L \cong \mathcal{O}(P)\,,$$

where $\mathcal{O}(P)$ is the lattice of order ideals of a poset P and the equivalence $\cong$ means that joins and meets are preserved by the isomorphism.

The above theorem says that lattices are topological: while lattice elements need not be sets (sometimes called pointless sets since they lack a description in terms of points), they behave exactly like sets. It turns out that this extends even to the infinite setting. Boolean algebras (possibly infinite) were shown to be dual to be compact, Hausdorff and totally disconnected spaces (also called Stone spaces) by Stone [92] and (possibly infinite) bounded distributive lattices were shown to be dual to certain ordered Stone spaces (also called Priestley spaces) later by Priestley [80]. These dualities are categorical. Spaces described algebraically in terms of their lattice of open sets

are called locales and offer a pointless approach to topology. Some good references on locales as well categorical aspects of the above dualities are [54, 78].

To define measurable sets in terms of superfunctions, we will take an order-theoretic approach. Order theory will allow us to define a discrete topology from the data of a lattice of superfunctions. Since superfunctions are not-commuting, this lattice can not be a Boolean algebra. Nevertheless, there exists super lattices that are geometric and labelled by a minimal set of prebase elements generating the discrete topology of a binary sample space.

4.1.3. Root systems and generalized posets. Root systems are special finite subsets of the Euclidean space that control the representation theory of semisimple Lie algebras. The elements of a root system are simultaneous eigenvectors corresponding to the adjoint action of a Cartan subalgebra on the Lie algebra it is contained in. A concise reference on root systems from a Lie-theoretic point of view is [53]. Besides applications to Lie algebras, they could be used to geometrically encode order relations of a poset. This insight is due to Reiner and forms a part of his PhD thesis [81]. The poset binary relations are captured by "convex cones" in the root system A_{n-1}, whose Weyl group is the symmetric group S_n that permutes elements of the set. In [82], Reiner generalized posets to their signed/hyperoctahedral analogs by viewing them as "convex cones" in the root system B_n, whose Weyl group is the signed permutation group H_n. To realize the anticommuting algebra of Grassmann variables in terms of an algebra of sets, it will be useful to identify relevant signed posets.

4.1.4. Prebases as signed posets. If we treat P as a discrete (no non-trivial order relations) signed poset, then its lattice $\mathcal{O}(P)$ of order ideals describes the topological base B generated by prebase pB. Stembridge calls $\mathcal{O}(P)$ a Coxeter cone [91], as Weyl groups are just special finite groups in the much broader class of Coxeter reflection groups. The lattice $\mathcal{O}(P)$ is isomorphic to a lattice of signed $\mathbb{Z}_3 = \{0, \pm\}$ labellings

$$B \equiv \mathcal{O}(P) \cong \text{Maps}(P, \{0, +, -\}) \,,$$

where the partial order on the signed n-tuples is componentwise, such that $0 < \pm$. A basis element $A \in B$, generated by the prebase pB looks like

$$A = \bigcap_{P_i \in P} P_i^{\epsilon_i}, \quad \epsilon_i \in \{0, +, -\},$$

where $\mathcal{C}^0 := \Omega$, $\mathcal{C}^+ := \mathcal{C}$, $\mathcal{C}^- := \mathcal{C}^{\mathsf{C}}$ for $C \in P$. Let us choose a labelling for the semi-prebase P by natural numbers

$$P = [n] := \{1, 2 \ldots, n\}.$$

This labelling gives P a natural total order. The prebase pB is then the disjoint union

$$pB = [n] \sqcup -[n].$$

Points in the sample space Ω correspond to the maximal elements of the base lattice $\mathcal{O}(P)$ and can be represented by sequences of n pluses and minuses. The next example illustrates these lattice-theoretic ideas for $n = 3$.

Example 4.1.2 (3-bit topology). *Consider a 3-bit computer. Its sample space Ω_3 is of cardinality 2^3. Its discrete topology is a Boolean algebra $\mathscr{B}_3$, whose atoms (corresponding to points in Ω) can be represented as 3-tuples of $\pm$ signs*

$$\Omega \cong \mathrm{Atoms}(\mathscr{B}_3) = \mathrm{Maps}([3], \{\pm 1\}) = \{+++, ++-, +-+, +--, -++, -+-, --+, ---\}.$$

The elements of the prebase pB can be expressed as 3-tuples that contain exactly one non-zero sign

$$pB = \pm[3] = \{+00, 0+0, 00+\} \sqcup \{-00, 0-0, 00-\}.$$

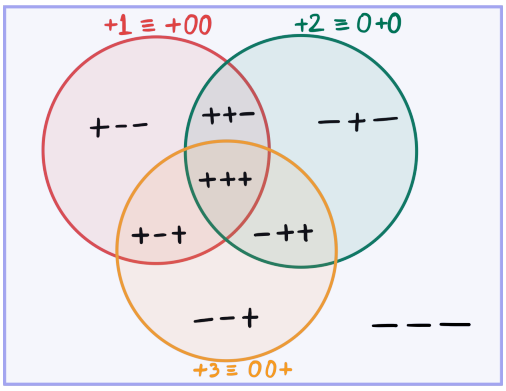
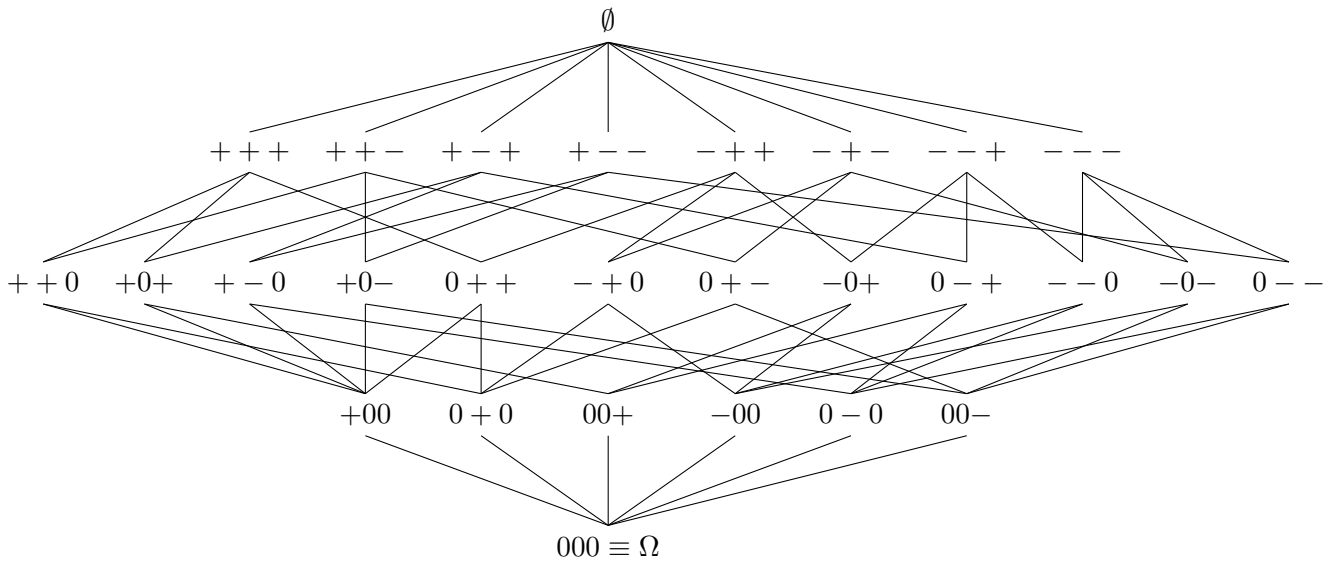
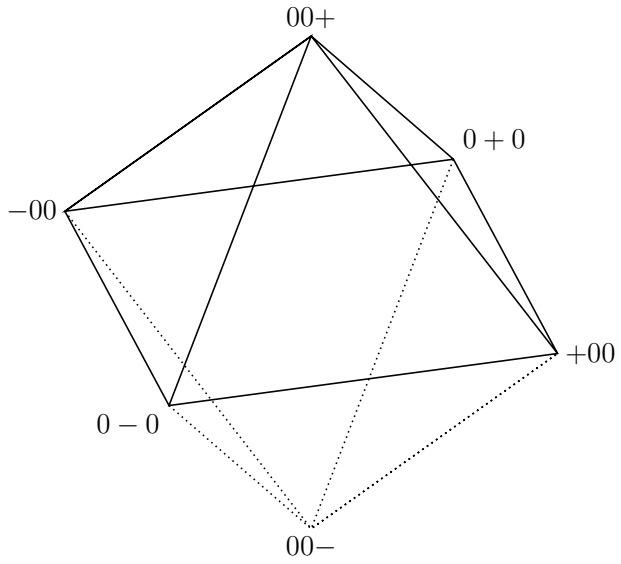


FIGURE 4.1. Venn diagram of 3-bit sample space covered by a semi-prebase $P = \{+1, +2, +3\}$.

The Hasse diagram of the bounded lattice $\mathcal{O}^{\top}([3]) := \mathcal{O}([3]) \oplus \emptyset$ *consisting of basis elements (and a new top element* $\emptyset$*) for the discrete topology is*



The octahedron Oct_3 *realizes this lattice in the simplicial/convex geometry of its faces. This is easy to see if we choose its center to be the origin and vertices to be points at unit (Euclidean) distance from the center lying on the coordinate axis:* $V = \{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\}$.



The facets (vertices, edges, faces, cells, etc.) can be described as convex hulls of certain subsets of V. We can represent each facet by its vertex set and order the facets in terms of set inclusion based on the sets of vertices contained in them. This is called the face lattice of the octahedron Oct_3 *and is lattice-isomorphic to the bounded lattice* $\mathcal{O}^{\top}([3])$ *of basis elements. In particular, we have the bijections*

$$\mathrm{Vertices}(\mathrm{Oct}_3) \text{ or } V \equiv \mathrm{Prebase}(\Omega_3) \text{ or } \mathrm{pB}\,,$$

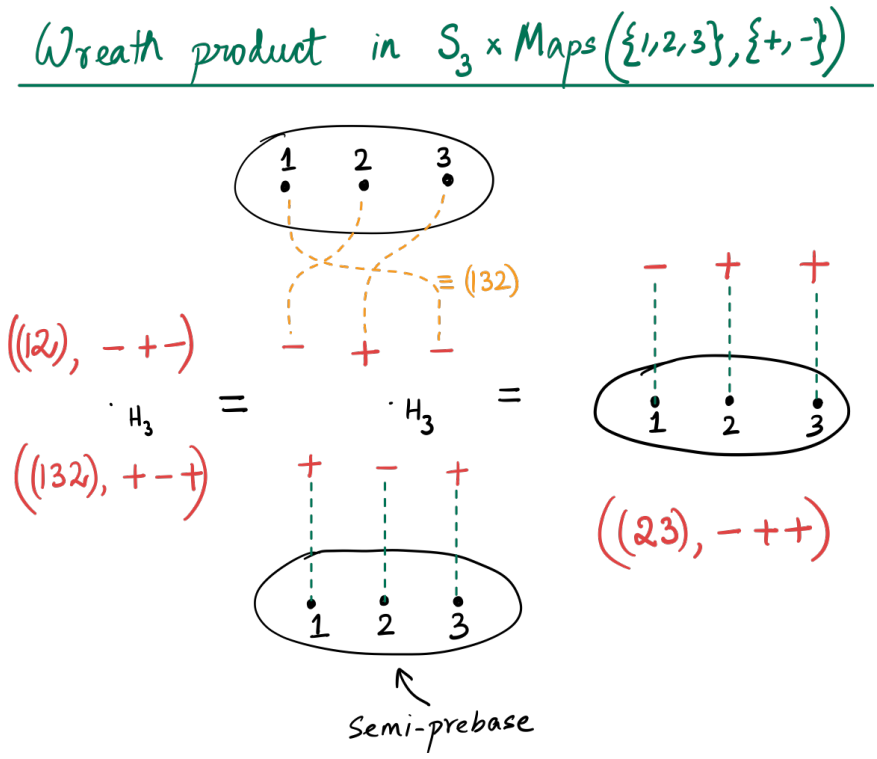
$$\mathrm{Edges}(\mathrm{Oct}_3) \equiv pB \cap pB\,,$$

$$\mathrm{Faces}(\mathrm{Oct}_3) \equiv pB \cap pB \cap pB \equiv \Omega_3\,.$$

The appearance of the octahedral geometry in the previous example is not a coincidence. As mentioned earlier, signed posets and their lattice of order ideals are defined in terms of root systems associated to the group of signed permutations H_n, also called *hyperoctahedral groups*

$$H_n := \mathbb{Z}_2 \wr S_n\,,$$

where $\mathbb{Z}_2 \wr S_n := \mathbb{Z}_2^n \rtimes S_n$ is the wreath product corresponding to the action of S_n on n copies of $\mathbb{Z}_2$, $\mathbb{Z}_2$ is the group describing sign flips of coordinates, while the symmetric group S_n permutes the coordinates. The following sketch illustrates the product in H_3 in terms of 3-tuples of $\pm$-signs labelling elements in a Boolean basis.



In the above example, we see that the permutation $(132) \in S_3$ twists the commuting pointwise product of the signed 3-tuples $-+-$ and $+-+$ to $-++$ instead of $---$. This perspective will be quite useful later to physically interpret the non-local product of atomic indicators that are in bijection with points in the binary sample space.

Crossed polytopes Oct_n generalize the three-dimensional octahedron Oct_3 to n dimensions and their symmetries are described by the hyperoctahedral groups H_n. These special simplicial complexes therefore capture the geometry of atoms of a finite Boolean algebra.

4.1.5. Chains associated to poset labellings. For many computational purposes, it is quite useful to work with ordered prebase pB or ordered semi-prebase B. For example, if one chooses to represent a basis element of the discrete topology by a monomial in the free algebra $\mathbb{R}[P] = \mathbb{R}[x_1, x_2, x_3]$ generated by some signed poset P

$$\mathscr{B}_3 \ni -+- \mapsto x_1^- x_2^+ x_3^- := (1-x_1)x_2(1-x_3)\,,$$

then the monomial $x_1^- x_2^+ x_3^-$ can be unambiguously chosen to represent the basis element $-+-$ only by knowing the underlying labelling of the poset P. This total order in P determines a chain

(totally ordered sublattice) in the order ideal $\mathcal{O}(P)$. For instance, the ordering $-2 < 1 < -3$ of $pB = \pm[3]$, picks out the chain

$$000 < 0-0 < +-0 < +--\,,$$

which is a descending filtration (with respect to set inclusion) of boundary simplices in the octahedron Oct_3.

4.1.6. Poset relabelling symmetry. In [82], Reiner proves that the lattice of order ideals $\mathcal{O}(P)$ of a signed poset P does not depend on how we choose to label (by a total-order) the elements of the semi-prebase P. Formally, this is a statement about the equivariance of $\mathcal{O}(P)$ under the action of the hyperoctahedral group H_n. For any signed permutation $\pi \in H_n$, there is a lattice-isomorphism

$$\mathcal{O}(P) \cong \mathcal{O}(\pi P)\,.$$

In our construction of indicators below, we will use a total order on P to give an explicit description of the lattice $\mathcal{O}(P)$. The H_n-equivariance ensures that such choices simply correspond to the gauge freedom of our description.

4.2. Deforming Boolean to Grassmann

We would now like to give a Boolean description of Grassmann algebras. This is the key idea for our solution to the measurement problem stated in Chapter 3 Section 3.1. For this we will identify and leverage structural similarities between Boolean algebras and Grassmann algebras. While Boolean algebras are generated by commuting idempotents with $\mathbb{Z}_2$-coefficients, Grassmann algebras are generated by anticommuting step-2 nilpotents. This fact leads to many interesting properties that we exploit in our construction of superindicators. For example, the square of a Boolean generator X_{Boolean} is only a $\mathbb{Z}_2$-flip away from that of a Grassmann generator $X_{\text{Grassmann}}$

$$X^2_{\text{Boolean}} = 1 \overset{\text{deform}}{\longrightarrow} X^2_{\text{Grassmann}} = 0\,.$$

In the next sections we will give a geometric description of such deformations. There is a rich geometric deformation theory for algebra of observables on a phase-space leading to quantum

mechanical models. To make contact with ideas from deformation theory, the first step is to represent elements of Grassmann algebras by polynomials.

4.2.1. Polynomial representations. Let V be an $\mathbb{R}$-vector space of dimension n. The Grassmann algebra Λ_n is isomorphic to the exterior algebra ΛV of the vector space V. Importantly, the Grassmann algebra Λ_n is supercommutative. While it can be described as a quotient of a free algebra by an ideal encoding the anticommutativity relations, it is interesting to ask whether it also arises as a deformation of the commuting algebra of polynomials. Two ways to describe Grassmann algebras are:

(1) *Free algebra quotients*: The Grassmann algebra Λ_n is isomorphic to the quotient of the freely generated tensor algebra $T(V) = V^{\otimes \cdot}$ by the ideal $I = \langle v_1 \otimes v_2 + v_2 \otimes v_1 : v_1, v_2 \in V \rangle_{\mathbb{R}}$

$$\Lambda_n \cong T(V)/I\,.$$

(2) *Deformed polynomial algebra*: The Grassmann algebra Λ_n is vector space isomorphic to a polynomial $\mathbb{R}$-algebra generated by commuting idempotents $X_1, X_2, \ldots, X_n$

$$\frac{\mathbb{R}[X_1, X_2, \ldots, X_n]}{(X_i^2 - X_i)} \overset{p}{\cong} \Lambda_n\,.$$

A key observation is that the above polynomial $\mathbb{R}$-algebra contains the Boolean algebra $\mathscr{B}_n$, which in turn gives the following linear isomorphism

$$\mathbb{R}[\mathscr{B}_n] \cong \Lambda_n, \tag{4.1}$$

where

$$\mathscr{B}_n := \frac{\mathbb{Z}_2[X_1, X_2, \ldots, X_n]}{(X_i^2 - X_i)}$$

is obtained by restricting the coefficients in the polynomial algebra to the finite field $\mathbb{Z}_2$. The Boolean algebra $\mathscr{B}_n$ can be equipped with a *convoluted product* defined by pulling back the product in the Grassmann algebra product using Equation (4.1).

The free algebra perspective relies on the existence of special generators obeying a minimal set of non-commutative relations. It is useful when studying quantum mechanics in an operator-theoretic framework. In the polynomial viewpoint, generators are as treated as commuting at the expense of keeping track of a convoluted product. This is could be viewed as geometric since it

mimics the commuting algebra of functions on a topological space and is closer in spirit to a classical measurement theory.

Next, we would like to contrast quantum deformation with superdeformation of polynomials. We will later give explicit formulas in terms of superindicators that further characterizes this superdeformation.

4.2.2. Deformation of polynomial algebras. The deformation quantization program for Poisson manifolds [39,61,102] aims to describe the algebra of quantum observables as a deformation of the commuting algebra of phase-space functions. Famously, Kontsevich proved the formality conjecture which gives a precise geometric characterization of formal, associative, non-commutative deformations of the commuting algebra of functions on a Poisson manifold. It states that there is a quasi-L_∞ isomorphism of two differential graded Lie algebras:

$$T_{\text{poly}}(X) \overset{L_\infty}{\cong} \mathcal{D}_{\text{poly}}(X)\,,$$

where $\mathcal{D}_{\text{poly}}(X)$ is the Hochschild complex of polydifferential operators on the associative algebra of polynomials on X, and $T_{\text{poly}}(X)$ is the graded Lie algebra of polyvector fields on the Poisson manifold X [61]. The isomorphism is called "quasi" since the L_∞ morphism induces isomorphisms only on the cohomologies or gauge-equivalence classes. To make the isomorphism strict, one has to work with polydifferential operators (or $\star$-products) depending formally on a parameter $\hbar$. The essence of this statement is that formal deformations of the associative algebra of functions on X is in one-one correspondence (up to gauge equivalence) with the geometric data of formal polyvector fields on X. For example, the commuting product of phase-space functions on $(\mathbb{R}^{2n}, \omega_{\text{std}})$ deforms to a non-commuting Moyal product $\star_\hbar$

$$f \star_\hbar g = \left(\text{prod} \circ \exp\left(\frac{i\hbar}{2}\Pi\right)\right)(f \otimes g) = fg + \frac{i\hbar}{2}\{f,g\} + O(\hbar^2)\,, \tag{4.2}$$

where $\Pi(f \otimes g) = \{f,g\} = \omega_{\text{std}}^{-1}(df, dg)$ is the leading deformation given by the constant symplectic form ω_{std}. Observe that the $\star$-product is a series consisting of bidifferential operators. The above $\star$-product is equivalent to the product of quantum operators. This is typically expressed in terms of a quantization map $Q : C^\infty(\mathbb{R}^{2n}) \to \text{End}_{\mathbb{C}}(\mathcal{H})$ obeying

$$Q_{\{f,g\}_\star} = [Q_f, Q_g]\,, \tag{4.3}$$

where $\{f,g\}_\star = f\star g - g\star f$ is the $\star$-commutator. For example, let $f = x+p$ and $g = x-p$ be polynomial functions on $(\mathbb{R}^2, \omega = dp\wedge dx)$, then

$$f \star_\hbar g = (x+p)(x-p) + \frac{i\hbar}{2}(-1-1) = x^2 - p^2 - i\hbar\,.$$

The $\star$-product extracts a polynomial by a particular reordering of the operator expression corresponding to the concatenatation of the two input polynomials. The first step is to decompose into monomials

$$(\hat{x}+\hat{p})(\hat{x}-\hat{p}) = \hat{x}^2 - \hat{p}^2 - \hat{x}\hat{p} + \hat{p}\hat{x}\,.$$

Then perform a phase-space Jordan-Lie decomposition for each monomial in the sum. For instance,

$$\hat{x}\hat{p} = \frac{1}{2}(\hat{x}\hat{p}+\hat{p}\hat{x}) + \frac{1}{2}(\hat{x}\hat{p}-\hat{p}\hat{x}) = \hat{x}\bullet\hat{p} + \frac{i\hbar}{2}\,\mathrm{Id}\,,$$

where we used the canonical commutation relation $[\hat{x},\hat{p}] = i\hbar\,\mathrm{Id}$. Proceeding this way for all the monomials, we arrive at an ordering interpretation of the Moyal $\star$-product formula

$$(\hat{x}+\hat{p})(\hat{x}-\hat{p}) = \hat{x}^2 - \hat{p}^2 - (\hat{x}\bullet\hat{p} + \frac{i\hbar}{2}\,\mathrm{Id}) + (\hat{p}\bullet\hat{x} + \frac{i\hbar}{2}\,\mathrm{Id}) - i\hbar\,\mathrm{Id} \overset{Q^{-1}}{\to} x^2 - p^2 - i\hbar\,.$$

Let $\hat{\cdot}$ denote the average of normal and anti-normal orderings that we shall call *Jordan quantization*, so that for momomials

$$\widehat{x^l p^m} = \hat{x}^l \bullet \hat{p}^m\,.$$

The Moyal $\star$-product can be expressed in terms of the Jordan-Lie structure:

$$\widehat{f\star g} = \hat{f}\bullet\hat{g} + \hat{f}\times\hat{g}\,, \tag{4.4}$$

where the Jordan term $\hat{f}\bullet\hat{g}$ captures the commuting product of f and g, while $\hat{f}\times\hat{g} = \frac{1}{2}[\hat{f},\hat{g}]$ is the commutator Lie-bracket encoding $\hbar$-corrections away from Jordan product in the $\star$-product. The above equation of course satisfies Equation (4.3). Note that Equation (4.4) gives us a way to view the $\star$-product as deforming a commutative, but possibly non-associative Jordan product $\bullet$ by the Lie product $\times$. Thus $\star$-algebras can be used to study deformation of Jordan algebras more general than the commuting algebra of functions.

4.2.3. Superproduct in terms of a Poisson bracket. The polynomial algebra perspective will allow us to use commuting algebra intuition as well as the Jordan–Lie structure of quantum

algebras to understand the non-local behavior of the algebra of superobservables. We can in fact describe the superproduct of two polynomials $f, g \in \mathbb{R}[\mathscr{B}_n]$ as a bidifferential operator on the Boolean algebra

$$f \wedge g = \left(\text{prod} \circ \frac{\Pi^{n-1}}{(n-1)!} \right) (f \otimes g) \,, \tag{4.5}$$

where $\text{End}(\mathbb{R}[x_i] \otimes \mathbb{R}[x_i]) \ni \Pi := \sum_{i=1}^{n-1} x_i x_{i+1} \partial_i \wedge \partial_{i+1}$ is a Poisson-bivector whose action picks out the sign of the permutation associated to the ordering of the generators in a monomial in the product $f \wedge g$. This formula follows from the associativity of the Grassmann algebra as well as that of its associated Jordan algebra. The supercommutativity of the superproduct $\wedge$ forces associativity of the corresponding Jordan product $\bullet$. (see Appendix A for a proof).

Comparing Equations (4.2) and (4.5) gives insight into geometric nature of the superproduct: the multiplication operation prod implements the commuting Jordan product of polynomials. The normalized n-th power of Poisson bivector

$$\frac{\Pi^n}{n!} \,,$$

is a bidifferential operator that computes contractions corresponding to n transpositions. The anti-symmetric nature of the Poisson bivector facilitates introduction of signs that records the ordering of variables in monomials. The exponential of the constant Poisson bivector in Equation (4.2) is an ordering gadget: the identity represents the fully ordered expression, while higher order derivatives capture iterated contractions necessary to reorder the product of phase-space variables to the chosen standard ordering. The $(n-1)$-th power of the quadratic Poisson structure in Equation (4.5) performs a total signed reordering of generators in a monomial.

Next, we will look at some special bases of the vector space of a Grassmann algebra corresponding to linear subspaces of a generating vector space. These are ubiquitious for coordinate computations with superfunctions in physics. While not quite superindicators, they are Boolean precursors with rich combinatorial properties.

4.3. Boolean blades

Blades are the simplest incarnations of Boolean algebras sitting inside Grassmann algebras. Blades are monomials built from free generators of a Grassmann algebra. In this section, we

will switch to the complex field $\mathbb{C}$ in order to prepare for our latter computations with hermitian superfields and the Bloch cone of states.

Let $\chi = \Lambda_n \hookrightarrow \Lambda\mathbb{C}^n$ denote our algebra of superobservables. A blade $\theta^I \in \Lambda\mathbb{C}^n$ in the complex Grassmann algebra generated by n degree-1 elements θ^i is

$$\theta^I := (\theta^1)^{i_1}(\theta^2)^{i_2}\cdots(\theta^n)^{i_n}\,,$$

where $I = (i_1 \dots i_m),\ i_k \in \{0,1\}$ is a Boolean-valued multi-index. For example, $\theta^1\theta^3\theta^4 \in \Lambda\mathbb{C}^5$ will be written as θ^{10110}.

4.3.1. Reality. The algebra $\Lambda\mathbb{C}^n$ can be equipped with an (anti)involution $\dagger$ whose action on monomials θ^I is given by

$$(\theta^I)^\dagger := (-1)^{\left\lfloor \frac{|I|}{2} \right\rfloor}\theta^I\,,$$

where $|I| := \sum_k i_k$. The Boolean length $|I|$ of a multi-index I computes the degree of the monomial θ^I and gives a $\mathbb{Z}$-grading on $\Lambda\mathbb{C}^n$. The function

$$t(I) := \left\lfloor \frac{|I|}{2} \right\rfloor$$

counts modulo 2 the number of transpositions required to reverse the order of $|I|$ distinct objects. The space of real superobservables $\mathcal{X}$, with respect to the involution $\dagger$, is the $\mathbb{R}$-span of the 2^n "hermitian" monomials

$$i^{t(I)}\theta^I = \left(i^{t(I)}\theta^I\right)^\dagger\,.$$

A general real superobservable $X \in \Lambda\mathbb{C}^n$ takes the form

$$\begin{aligned} X =& \sum_I i^{t(I)} X_I \theta^I \\ =& X_{00000\cdots0} + X_{00100\cdots0}\theta^1 + \cdots + iX_{10110\cdots0}\theta^1\theta^3\theta^4 + \cdots + i^{\left\lfloor \frac{n}{2} \right\rfloor} X_{11111\cdots1}\theta^1\theta^2\cdots\theta^{n-1}\theta^n\,, \end{aligned} \tag{4.6}$$

where the coefficients X_I are real. Note that the product of $X, Y \in \Lambda\mathbb{C}^n$ obeys

$$(XY)^\dagger = Y^\dagger X^\dagger\,.$$

Some authors [31, 83] employ an involution $*$ such that $(XY)^* = (-1)^{|X||Y|}Y^*X^* = X^*Y^*$ by writing $X^* = i^{|X|}X^\dagger$. The anti-involution $\dagger$ is better suited to the construction of convex cones described in Chapter 5 Section 5.3.

4.3.2. Boolean components. We can view the real superobservable X as a 2^n-tuple

$$(X_{00000\cdots0}, X_{00100\cdots0}, \cdots, X_{10110\cdots0}, \cdots, X_{11111\cdots1}) \in \mathbb{R}^{2^n}\,.$$

This obscures the algebra structure given by both the exterior product and the $\star$ product to be used to define states as $\star$-squares. However, observe that we can also view X as a real valued map

$$X : \square([n]) \to \mathbb{R}\,,$$

with domain the power set $\square([n])$ (the set of all subsets) of n integers $\{1, 2, \ldots, n\}$. The power set $\square([n])$ forms a Boolean algebra with the usual set-theoretic operations of union, intersection and complement. We can use Boolean operations to characterize the exterior and $\star$-product algebras.

4.3.3. Grassmannians. The space of ordered bases called *frames* of a generating n-dimensional vector space V of the Grassmann algebra Λ_n forms a Stiefel manifold $V_n(V)$. Its points are orthonormal n-tuples of vectors in $\mathbb{R}^n$ equipped with the Euclidean inner product

$$g \in V_n(V) \cong SO(n), \quad \mathrm{Gr}(k, V) \overset{i_g}{\hookrightarrow} P(\Lambda_n)\,,$$

where $SO(n)$ is the special orthogonal group, $\mathrm{Gr}(k, V)$ is the Grassmannian consisting of k-dimensional subspaces of V, $P(\Lambda_n)$ is the projectivised Grassmann algebra. The map i_g above is the Plücker embedding corresponding to an element $g \in SO(n)$, and sends subspaces of $\mathbb{R}^n$, spanned by k-subframes of the n-frame $g \in V_n(\mathbb{R}^n)$, to their corresponding exterior products

$$V_n(\mathbb{R}^n) \ni g = (v_1, v_2, \ldots, v_n) \supset (v_1, v_2, \ldots, v_k), \quad \mathrm{Span}(v_1, v_2, \ldots, v_k) \mapsto v_1 \wedge v_2 \wedge \ldots v_k\,.$$

The map i_g is defined for other k-frames that are not subframes of g by extending the above map linearly. Plücker embeddings therefore imbue blades with the geometry of Grassmannians.

4.3.4. Signed deformations. The Grassmann algebra Λ_n is a signed $\mathbb{Z}_3$ deformation of the power set Boolean algebra $\square([n])$

$$\Lambda_n \overset{\text{linear}}{\cong} \mathbb{R} \otimes_{\mathbb{Z}_3} \mathbb{Z}_3[\square([n])] \,.$$

The superproduct of blades can be expressed in terms of the Boolean symmetric difference

$$\theta^I . \theta^J = \epsilon_{IJ} \theta^{I \oplus J} \,,$$

where $\epsilon_{IJ} := \text{sgn}(I.J)\delta_{|IJ|}$ is the Levi-Civita permutation symbol, $\text{sgn}(I \wedge J) := \text{sgn}((s(I), s(J)) \to s(I) \ \oplus \ s(J))$ is the sign of the shuffle-$(|I|, |J|)$ permutation required to rearrange the ordered supports $s(I)$ and $s(J)$ into the ordered symmetric difference

$$s(I) \ \oplus \ s(J) := (s(I) \setminus s(J)) \sqcup (s(J) \setminus s(I)) \,,$$

and $\delta_{|IJ|}$ is the Boolean Kronecker delta function that is 1 iff $s(I) \cap s(J) = \emptyset \iff |IJ| = 0$. The associated Lie and Jordan products of the Grassmann algebra $\Lambda\mathbb{C}^n$ expressed in the blade basis are

$$i\theta^I \times \theta^J = \chi^{IJ}_{\text{odd}} \epsilon_{IJ} \theta^{I \oplus J} \,,$$
$$\theta^I \bullet \theta^J = \chi^{IJ}_{\text{even}} \epsilon_{IJ} \theta^{I \oplus J} .$$

where $\chi^{IJ}_{\text{odd}} = 1 - \chi^{IJ}_{\text{even}} := \frac{1-(-1)^{|I||J|}}{2}$ is a Boolean function that is 1 if the degrees of both I and J are odd and 0 otherwise. Note that the Jordan product is zero iff either $|IJ| \neq 0$ or $|I||J| = 1 \bmod 2$. Thus it depends on both the intersection/incidence and dimensionality/parity of subspaces represented by the blades.

4.3.5. Blades as atoms. We would now like to argue why blades fail to be indicators for a discrete binary sample space. The free Boolean algebra $\mathscr{B}_n$ is isomorphic to the double power set

$$\mathscr{B}_n \cong \square(\square([n])) \,.$$

When the atoms of a free Boolean algebra are given by blades, the top and bottom elements are

$$\top = \sum_{I \in \square([n])} \theta^I \text{ and } \quad \bot = \text{Id} \,.$$

These do not correspond to the desired top $\top = \mathrm{Id}$ multiplicative identity and bottom $\bot = 0$ additive identity 0 of the algebra Λ_n. The desired bottom element 0 is the characteristic function for the empty set, while the desired top element Id is the characteristic function for the entire binary sample space. We therefore need a different approach to generate an atomic basis of the Grassmann algebra Λ_n, this will be based on a resolution of unity.

4.4. Resolution of identity

The multiplicative identity $\mathrm{Id} \in \Lambda_n$ can be decomposed into 2^n Grassmann elements as follows

$$\begin{aligned} \mathrm{Id} = \mathrm{Id}\,.\,\mathrm{Id}\,.\,.\,.\,.\,\mathrm{Id} &= (\mathbb{1}^{+1} + \mathbb{1}^{-1}).(\mathbb{1}^{+2} + \mathbb{1}^{-2})\,.\,.\,.\,(\mathbb{1}^{+n} + \mathbb{1}^{-n}) \\ &= \mathbb{1}^{+1}\mathbb{1}^{+2}\,.\,.\,.\,\mathbb{1}^{+n} + \mathbb{1}^{-1}\mathbb{1}^{+2}\,.\,.\,.\,\mathbb{1}^{+n} + \cdots + \mathbb{1}^{-1}\mathbb{1}^{-2}\,.\,.\,.\,\mathbb{1}^{-n}\,, \end{aligned} \tag{4.7}$$

where

$$\mathbb{1}^{\pm i} := \frac{1}{2} \pm \theta^i\,. \tag{4.8}$$

This is analogous to the Peirce decomposition of associative algebras or sum of products expansion in free Boolean rings, except that the elements $\mathbb{1}^{\pm i}$ are no longer commuting idempotents, but are instead depend on anti-commuting step-2 nilpotents. The atomic superindicators may be taken to be the summands in the expansion of Equation (4.7). Let us note some important features of the superindicators $\mathbb{1}^{\pm i}$:

- They satisfy the non-homogeneous quadratic equation

$$\left(\frac{1}{2}\right)^2 + \left(\mathbb{1}^{\pm i}\right)^2 = \mathbb{1}^{\pm i}\,, \tag{4.9}$$

  which can be viewed as a *superidempotence* condition.

- The ratio $\frac{1}{2}$ in the definition of the superindicator in Equation (4.8) can be interpreted in terms of the cardinality of its support set

$$\mathrm{Card}(\mathrm{supp}(\mathbb{1}^{\pm i})) = \frac{1}{2}\,\mathrm{Card}(\Omega) = 2^{n-1}\,, \tag{4.10}$$

  where the Ω is a binary sample space containing 2^n points.

- Idempotents e generalize projections and are in one-one correspondence with involutions r

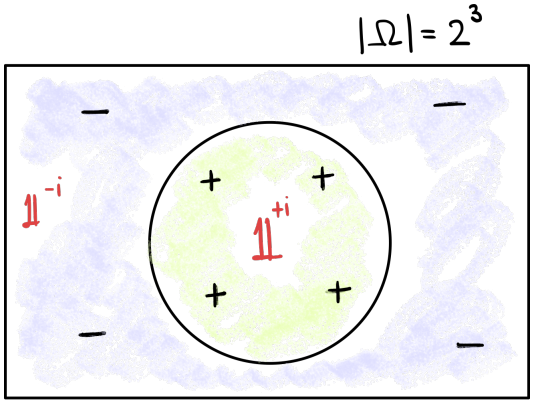
$$r = e_\perp - e = 1 - 2e\,,$$

where $e_\perp := 1 - e$. These involutions generalize reflections. The involutions associated to superidempotents $\mathbb{1}^{-i}$ are given by

$$2\theta^i = \mathbb{1}^i - \mathbb{1}^{-i}\,,$$

and manifestly exhibit a ($\mathbb{Z}_2$) reflection symmetry.

The involution $2\theta^i$ has an interesting set-theoretic interpretation, *viz.* it represents a signed 2-partition of a binary sample space Ω.



Our construction of superindicators in Section 4.6 will be based on this resolution of identity. We will also generalize the superindicators $\mathbb{1}^{\pm i}$ to generic superfunctions.

We are almost ready to describe the theory of superindicators. The last ingredient is an H_n action on Λ_n. As explained in earlier sections, superdeformations are characterized by $\mathbb{Z}_2^n$-actions twisted by permutations. In Section 4.1 we saw that free Boolean algebras have a canonical action of the hyperoctahedral group H_n. By extending this to act on Grassmann algebras, we will interpret signed expressions of moments of superobservables, see Section 4.7.

4.5. Action of the hyperoctahedral group

We would now like to describe an action of the hyperoctahedral group H_n on the Grassmann algebra Λ_n. This is possible, thanks to the polynomial description of Grassmann algebra discussed

in Section 2. The action depends on properties of the isomorphism p in Equation (2). Let us first discuss some important aspects of the map p

(1) The map p sends commuting generators of the polynomial algebra to non-commuting generators of the Grassmann algebra. The image $p(X_i)$ of the indeterminate X_i is generic in the sense that it need not be a blade in Λ_n. The generators $p(X_i)$ are not in general of homogeneous parity and thus neither commutative nor anti-commutative.

(2) To generate all elements of the Grassmann algebra, we require a non-degeneracy condition

$$\prod_{i=1}^{n} \mathrm{soul}(p(X_i)) \neq 0\,.$$

The above implies that each generator $p(X_i)$ must contain a non-zero degree-1 term.

(3) The isomorphism p in (2) is not canonical and depend on the ordering of the generators $p(X_i)$. The orderings are parameterized by elements of the symmetric group S_n.

When Λ_n is viewed as a polynomial algebra in the superindicators $\mathbb{1}^{\pm i}$, its elements have a canonical action $H_n \times \Lambda_n \to \Lambda_n$ of the signed permutation group H_n, based on the wreath product defined in Section 4.1.4. Let $(\pi, \sigma) \in S_n \times \mathbb{Z}_2^n \cong H_n$ be a signed permutation acting on an atomic superindicator $\mathbb{1}^{i_1}\mathbb{1}^{i_2}\ldots\mathbb{1}^{i_n}$, $(i_1, i_2, \ldots, i_n) \in H_n$ according to

$$(\pi, \sigma) \times \mathbb{1}^{i_1}\mathbb{1}^{i_2}\ldots\mathbb{1}^{i_n} \mapsto \mathbb{1}^{\sigma(1)\pi(i_1)}\mathbb{1}^{\sigma(2)\pi(i_2)}\ldots\mathbb{1}^{\sigma(n)\pi(i_n)}\,. \tag{4.11}$$

Since the atomic superindicators form an $\mathbb{R}$-basis for the vector space of the Grassmann algebra Λ_n, the H_n action can be extended linearly to act on all elements of Λ_n.

4.6. Superfunction indicators

We have discussed indicators in the context of $*$-algebras back in Chapter 3 Section 3.5. They are special membership characterizing functions on a sample space. They act as a bridge between the geometry/topology of points in a space and the algebra of functions on the space. In practice, they turn elements of an abstract algebra into more concrete set-functions.

Indicators takes binary/logical values on its support sets depending on whether a point belongs to its support or not. An example of an indicator is a characteristic function $\mathbb{1}_A$ for the subset $A = \mathrm{supp}(\mathbb{1}_A) \subseteq \Omega$

$$\mathbb{1}_A(x \in A) = 1, \quad \mathbb{1}_A(x \notin A) = 0\,.$$

We would like to build indicators from superfunctions. For this, we will use our Stone sample space Ω_n, whose points are identified with atoms in the set $\mathrm{Atoms}(\mathscr{B}_n)$. Following the discussion in Section 4.1, we can describe these atoms in terms of a minimal prebase pB for the discrete topology $\mathscr{B}_n$ on Ω_n. Since prebase pB elements generate atoms by intersection (or more generally meet in the lattice/Boolean terminology), they help make the meet operation manifest in a set algebra. In what follows, the key idea of our construction of superindicators will be to give the definition of a meet operation in terms of the product of superfunctions with supports on prebase elements.

We will next work through the case of Λ_3 before giving the general theory. We will use the signed description of the 3-bit topology introduced in Example 4.1.2 and the superindicators described in Section 4.4.

EXAMPLE 4.6.1 (3-bit (super) computer). *Consider the real subspace of the exterior algebra* $\Lambda\mathbb{C}^3 := \mathbb{C}[\theta^1, \theta^2, \theta^3]/(I)$, *where* (I) *is the ideal generated by the set* $I = \{\theta^1\theta^2 + \theta^2\theta^1, \theta^2\theta^3 + \theta^3\theta^2, \theta^1\theta^3 + \theta^3\theta^1\}$ *of anticommutators. This subspace may be generated from the following 6 real indicator superfunctions:*

$$\mathbb{1}^{\pm i} := \frac{1}{2} \pm \theta^i,\ i \in \{1, 2, 3\}\,.$$

There are 3^3 *indicator superfunctions with supports on the base generated by 3 semi-prebase elements or 6 prebase elements. They are given by the following real exterior products*

$$\begin{aligned}
&\mathbb{1}^{\pm 00} := \mathbb{1}^{\pm 1}, \quad \mathbb{1}^{0\pm 0} := \mathbb{1}^{\pm 2}, \quad \mathbb{1}^{00\pm} := \mathbb{1}^{\pm 3}\,,\\
&\mathbb{1}^{\pm\pm 0} := \mathbb{1}^{\pm 1} \wedge_{\mathrm{real}} \mathbb{1}^{\pm 2}, \quad \mathbb{1}^{\pm 0\pm} := \mathbb{1}^{\pm 1} \wedge_{\mathrm{real}} \mathbb{1}^{\pm 3}, \quad \mathbb{1}^{0\pm\pm} := \mathbb{1}^{\pm 2} \wedge_{\mathrm{real}} \mathbb{1}^{\pm 3}\,,\\
&\mathbb{1}^{\pm\pm\pm} := \mathbb{1}^{\pm 1} \wedge_{\mathrm{real}} \mathbb{1}^{\pm 2} \wedge_{\mathrm{real}} \mathbb{1}^{\pm 3}\,.
\end{aligned}$$

where $\mathbb{1}^0 := 1$ *and the real exterior multiplication* $\wedge_{\mathrm{real}}$ *is the hermitian analog of the exterior multiplication of superfunctions*

$$X \wedge_{\mathrm{real}} Y := X \circ Y + X \times Y\,.$$

The 2^3 *indicators* $\in \{\mathbb{1}^{\pm\pm\pm}\}$ *correspond are the atomic ones and have supports on the points in the binary sample space* Ω_3. *They also form an* $\mathbb{R}$*-basis of the vector subspace of real superfunctions* $(\Lambda\mathbb{C}^3)_{\mathrm{real}}$.

The blades $\theta^I,\ I \in \square([3])$ *can be re-written in terms of these base indicators as*

$$\begin{aligned}
\theta^1 &= \frac{1}{2}(\mathbb{1}^{+00} - \mathbb{1}^{-00}),\ \theta^2 = \frac{1}{2}(\mathbb{1}^{0+0} - \mathbb{1}^{0-0}),\ \theta^3 = \frac{1}{2}(\mathbb{1}^{00+} - \mathbb{1}^{00-})\,,\\
-i\theta^{12} &= \frac{1}{4}(\mathbb{1}^{++0} + \mathbb{1}^{--0} - \mathbb{1}^{+-0} - \mathbb{1}^{-+0}),\ -i\theta^{13} = \frac{1}{4}(\mathbb{1}^{+0+} + \mathbb{1}^{-0-} - \mathbb{1}^{-0+} - \mathbb{1}^{+0-})\,,\\
-i\theta^{23} &= \frac{1}{4}(\mathbb{1}^{0++} + \mathbb{1}^{0--} - \mathbb{1}^{0+-} - \mathbb{1}^{0-+})\,,\\
-i\theta^{123} &= \frac{1}{8}(\mathbb{1}^{+++} + \mathbb{1}^{+--} + \mathbb{1}^{-+-} + \mathbb{1}^{--+} - \mathbb{1}^{-++} - \mathbb{1}^{+-+} - \mathbb{1}^{++-} - \mathbb{1}^{---})\,.
\end{aligned}$$

Consider the action of $\sigma \in \mathbb{Z}_2^{|I|}$*, given by* $|I|$ *sign changes, on each indicator* $\mathbb{1}^I,\ I \in H_n$ *labelled by* $|I|$ *±signs*

$$\mathbb{1}^{\sigma(I)} := \bigwedge_{\text{real}_{i=1}}^{|I|} \mathbb{1}^{\sigma(i).I(i)}\,.$$

Then the above blade expansions can be viewed as a parity-weighted average under the $\mathbb{Z}_2^{|I|}$ *action on the corresponding superindicator:*

$$i^{t(I)}\theta^I = \frac{\text{sgn}(\pi_I)}{2^{|I|}} \sum_{\sigma \in \mathbb{Z}_2^{|I|}} \text{sgn}(\sigma)\mathbb{1}^{\sigma(I)} = \frac{\text{sgn}(\pi_I)}{2^{|I|}} \left(\sum_{\sigma \text{ even}} \mathbb{1}^{\sigma(I)} - \sum_{\sigma \text{ odd}} \mathbb{1}^{\sigma(I)} \right),$$

where the factor $i^{t(I)}$ *ensures reality,* $\text{sgn}(\pi_I)$ *is the sign of the permutation associated to the ordering of the generators in* θ^I*, and* $\text{sgn}(\sigma) := \prod_{i=1}^{|I|} \sigma(i)$ *is the parity of the sign flips. The above expression tells us that each blade corresponds to an even-odd* 2*-partition of the sample space* Ω_3*. For example,*

$$-4i\theta^{23} = \mathbb{1}^{\{+++,-++,+--,---\}} - \mathbb{1}^{\{++-,-+-,+-+,--+\}}\,,$$

where the support set of the first superindicator consists of the even parity $\mathbb{Z}_2^2$*-orbit of* $(0,+,+)$ *while that of the second superindicator consists of the odd parity* $\mathbb{Z}_2^2$*-orbit. These give explicit realizations of Reiner's signed posets on a binary sample space.*

Next we will present a general construction of superfunction Boolean algebras. Our approach relies on the atomic representation of finite Boolean algebras and has two key steps:

$$\text{Semi-prebase} \xrightarrow{\mathsf{C},\wedge} \text{Atoms} \xrightarrow{\vee} \text{Boolean algebra}\,,$$

where in the first step we will define atomic superfunctions from products $\wedge$ and complements C of semi-prebase superfunctions, and in the second step we will obtain a general boolean superfunction as a sum $\vee$ of atomic superfunctions below it.

4.6.1. Super Boolean algebra. Let Ω_n be the Stone sample space of cardinality 2^n. The indicator algebra $\{\mathbb{1}_A : A \in \square(\Omega)\}$ of characteristic functions on Ω is isomorphic to the Boolean algebra $\mathscr{B}_n$. For $A, B \in \square(\Omega)$ the characteristic functions must obey

$$\begin{aligned}
\mathbb{1}_{\emptyset} &= 0 \\
\mathbb{1}_{A^{\mathsf{C}}} &= \mathbb{1}_{\Omega} - \mathbb{1}_A \,, \\
\mathbb{1}_{A\cap B} &= \mathbb{1}_A.\mathbb{1}_B \,, \\
\mathbb{1}_{A\cup B} &= \mathbb{1}_A + \mathbb{1}_B - \mathbb{1}_{A\cap B} \,.
\end{aligned}$$

where the . refers to the standard pointwise product of commuting functions.

We would now like to define a poset embedding

$$\mathcal{E} : B^{\partial} \to \mathscr{B}_n \,,$$

of the dual B^{∂} a hyperoctahedral H_n-lattice B into the Boolean algebra $\mathscr{B}_n$ of subsets of Ω. The dual is needed here since the maximal elements of $B = \mathcal{O}(P)$ correspond to the minimal elements or atoms of $\mathscr{B}_n$. The H_n-lattice is not distributive (it is in fact only locally distributive, see [**82**]) and hence the image $\mathcal{E}(B^{\partial})$ can not be a sublattice of $\mathscr{B}_n$. Instead, the image will be subposet corresponding to a π-system (analogous to a topological basis *i.e.* a collection of sets closed under finite intersections) of the Boolean algebra $\mathscr{B}_n$. One can view the map $\mathcal{E}$ as a kind of calibration map aligning a distinguished lattice of superfunctions with bonafide characteristic functions.

Since atomic indicators span the set of all indicators, and atomic indicators are in turn determined by prebase indicators, we will only need to assign superfunctions to the prebase indicators. This is the essence of our main theorems related to superfunction representation of the Boolean algebra that has two parts:

(1) describe a super H_n-lattice B^{∂}, see Theorem (4.6.4),

(2) describe an emebedding $\mathcal{E}$, see Theorem (4.6.5).

To state our main result, we will need some definitions as well as elementary lemmas about finite Boolean/σ -algebras and their generating sets.

DEFINITION 4.6.2. *Let $(\Omega, \mathcal{F})$ be a measurable sample space with σ-algebra $\mathcal{F}$ on Ω. When the cardinality of Ω is 2^n for some positive integer n, we call $(\Omega, \mathcal{F})$ a binary sample space.*

DEFINITION 4.6.3. *A minimal semi-prebase P for a measurable sample space $(\Omega, \mathcal{F})$ is a generating set for the σ-algebra $\mathcal{F}$ on Ω with minimum possible cardinality.*

LEMMA 4.6.3.1. *Let $(\Omega, \mathcal{F})$ be a measurable sample space with binary expansion of the cardinality of set of atoms $|\mathcal{A}(\mathcal{F})| = \sum_i k_i 2^i$. Then there exists a unique (up to automorphisms) minimal semi-prebase P for $\mathcal{F}$ satisfying*

$$\mathcal{F} \cong \square_{\cup}(\square_{\mathsf{C},\cap}(P)), \qquad |P| = \sum_i i k_i \,.$$

PROOF. Firstly note that finite boolean algebras are completely classified (up to equivalence) by the cardinality k of the set of atoms $\mathcal{A}(\mathcal{F})$. By definition,

$$\mathcal{F} = \square_{\sqcup}(\mathcal{A}(\mathcal{F}))\,.$$

To get a smaller generating set than the set of atoms $\mathcal{A}(\mathcal{F})$, we must use complements and intersections. The set of atoms $\mathcal{A}(\mathcal{F})$ can be totally ordered by their cardinality. We would like to partition $\mathcal{A}(\mathcal{F})$ into subsets whose cardinalities are distinct powers of two. Therefore consider the binary expansion of k:

$$k = a_l 2^l + a_{l-1} 2^{l-1} + \ldots + a_0\,, \quad a_i \in \{0,1\}, \quad a_l \neq 0\,.$$

We may now use the ordering of $\mathcal{A}(\mathcal{F})$ to arrange the atoms into sets $\mathcal{A}_i$ of cardinalities $a_0, 2a_1, 2^2 a_2, \ldots, 2^l a_l$: When $a_0 = 1$, we place an atom of lowest order in the first set $\mathcal{A}_0$. When $a_0 = 0$, the first set is empty. Instead, assume a_i is the first non-zero digit in the binary expansion. Then we place 2^i atoms of lowest order in that set. Proceeding this way, we obtain a partition (unique up to permutations of atoms of the same order) of the desired type:

$$\mathcal{A}(\mathcal{F}) = \sqcup_{i=0}^{l} \mathcal{A}_i\,, \qquad |\mathcal{A}_i| = a_i 2^i\,.$$

The above now implies the following direct-sum decomposition

$$\mathcal{F} \cong \bigoplus_{i=0}^{l} \Box_{\cup}(\Box_{\mathsf{C},\cap}(P_i))\,,$$

where P_i are semi-prebases corresponding to each $\mathcal{A}_i$ and the complements performed by $\Box_{\mathsf{C},\cap}$ are relative to the sets $\cup\mathcal{A}_i$ generated by the corresponding atoms. Finally, $P = \bigcup_{i=0}^{l} P_i$ with cardinality $\sum_{i=0}^{l} a_i i$ is the sought after minimal generating set and is isomorphic to the quotient of a free boolean algebra

$$\mathcal{F} \cong \Box_{\cup}(\Box_{\mathsf{C},\cap}(P)) \cong \Box_{\cup}(\Box_{\mathsf{C},\cap}(P_{\text{free}}))/\mathcal{I}\,.$$

In the above expression, $\mathcal{I}$ is the boolean ideal generated by $\mathcal{B}_i \cap \mathcal{B}_j - \delta_{ij}\mathcal{B}_i$, where we mean for $X, Y \subseteq \Box(\Box(\Omega))$, $\Box(\Box(\Omega)) \ni X \cap Y := \{x \cap y | x \in X, y \in Y\}$. P_{free} refers to the the set P but forgetting all the boolean relations in $\mathcal{I}$. □

Lemma 4.6.3.2. *Let $(\Omega, \Box(\Omega), P)$ be a binary sample space of cardinality 2^n equipped with a minimal semi-prebase P. Every singleton $x \in \Omega$ can be uniquely expressed as an intersection of m distinct prebase elements.*

Proof. As explained earlier, we can label elements of the sample space Ω with strings of m pluses and minuses or equivalently strings of m $+1$'s and -1's. A signed representation of the sample space is therefore a bijection

$$\text{sgn} : \Omega \xrightarrow{\cong} \text{Maps}(\{1, 2, \ldots, n\} \longrightarrow \{+1, -1\})\,. \tag{4.12}$$

It is then easy to check that any element $x \in \Omega$ can be expressed as the following intersection of elements in the prebase $pB = \{\pm 1, \pm 2, \ldots, \pm m\}$,

$$x = \cap_{\hat{i}=1}^{n} \hat{i}\big(\text{sgn}(x)\big)(\hat{i}) =: \cap_{\hat{i}=1}^{n} x_{\hat{i}}\ ,$$

where $x_{\hat{i}} := \hat{i}(\text{sgn}(x))(\hat{i})$. □

Lemma 4.6.3.3. *Let Ω be a binary sample space. Every Ω-subset $U \in \Box(\Omega)$ is uniquely generated by a minimal set of semi-prebase elements.*

Proof. From the previous lemma, we know that any singleton can be uniquely expressed as an intersection of m distinct prebase elements. Clearly any subset of Ω is a disjoint union of

cardinality-1 subsets x or singletons,

$$U = \sqcup_{\mathcal{S} \ni x \in U} x.$$

Therefore we start by taking unions V of all pairs of singletons that lie in some cardinality-2 basis element. For example, this allows us to rewrite

$$U = \sqcup_{\{V \in \mathcal{B} \,\big|\, |V|=2\}} V \sqcup_{x \in U \setminus \sqcup V} x,$$

resulting in a perhaps smaller generating set for U. We can next perform disjoint union of all pairs of cardinality-2 basis elements that are contained in a cardinality-4 basis element

$$U = \sqcup_{\{V \in \mathcal{B} \,\big|\, |V| \in \{1,2,4\}\}} V.$$

This process can be iterated $m-1$ times in total until we reach prebase elements each with cardinality 2^{m-1}. This yields a representation of U in terms of a set of disjoint basis elements with smallest possible cardinality. Note that because of the minimal cardinality $2m$ of the prebase, each base element is itself a unique intersection of prebase elements. The union of all semi-prebase elements, generating each base element by intersections, is the minimal set we are after. □

LEMMA 4.6.3.4. *Let Ω be a binary sample space with a total order on its semi-prebase. Then there is an induced total ordering on the power set Boolean algebra $\square(\Omega)$. For $U, V \in \square(\Omega)$*

$$U \leq_{\hat{B}} V \text{ if } \max(U) \leq \min(V)\,,$$

PROOF. For $U, V \in \square(\Omega)$

$$U \leq_{\hat{B}} V \text{ if } \max(U) \leq \min(V)\,,$$

where the max and min are defined on the minimal set of semi-prebase elements that generate the set. □

We are ready to state our two main results. They give the most general definition of superindicators for a finite Boolean algebra.

THEOREM 4.6.4. *Let Ω be a binary sample space equipped with a totally ordered semi-prebase P and induced partial ordering on $\leq_P$ on the discrete topology $\square(\Omega)$. A super indicator algebra*

is a family of distinct real superfunctions $\{\mathbb{1}^U,\ U \in \mathcal{F}\}$ *parameterized by elements of a* σ*-algebra* $\mathcal{F} \subset \square(\Omega)$ *on* Ω *that satisfies the following*

$$\begin{aligned}
\mathbb{1}^{\emptyset} &= 0\,,\\
\mathbb{1}^{U^{\mathsf{C}}} &= 1 - \mathbb{1}^{U}\,,\\
\mathbb{1}^{U\cap V} &= \mathbb{1}^{U} \wedge_{\text{real}} \mathbb{1}^{V}, \textit{when} \quad U \leq_P V,\ |U \cap V| = \frac{|U||V|}{|\Omega|}\,,\\
\mathbb{1}^{U\cup V} &= \mathbb{1}^{U} + \mathbb{1}^{V} - \mathbb{1}^{U\cap V}\,.
\end{aligned}$$

Proof. This mimics the standard construction of free Boolean algebras from a set of free generators. All we need to check is that the superindicator $\mathbb{1}^{U\cap V}$ defined on the intersection $U \cap V$ can be given by an exterior product of the superindicators of the intersecting sets $\mathbb{1}^U$ and $\mathbb{1}^V$ when

$$U \leq_P V, \quad |U \cap V| = \frac{|U||V|}{|\Omega|}\,.$$

The $U \leq_P V$ condition ensures $\mathbb{1}^U \wedge_{\text{real}} \mathbb{1}^V$ is "symmetric" with respect to the ordering of U and V. The cardinality condition is precisely the definition of freeness/independence of generators and coincides with the properties of a minimal semi-prebase. □

Theorem 4.6.5. *Let* P *be a semi-prebase for the discrete topology* $\square(\Omega)$ *of a binary sample space* Ω *of cardinality* 2^n*. Moreover, let* $\left(\mathbb{1}^{\hat{i}} \in \Lambda\mathbb{C}^n \;\middle|\; \hat{i} \in P\right)$ *be a sequence of real superfunctions whose souls satisfy* $\wedge_{\hat{i}=1}^{m} \text{soul}(\mathbb{1}^{\hat{i}}) \neq 0$*. Then the map*

$$\begin{array}{rccc}
\mathbb{1}: & \square(\Omega) & \longrightarrow & (\Lambda\mathbb{C}^n)_{\text{real}} \\
& \cup & & \cup \\
& U & \longmapsto \mathbb{1}^U & = \displaystyle\sum_{x\in U} \bigwedge_{\text{real}\,\hat{i}=1}^{m} \mathbb{1}^{x_{\hat{i}}}\,,
\end{array}$$

is a Boolean algebra isomorphism where the product of the indicators is taken to be the pointwise product as random variables.

Proof. The Boolean isomorphism follows from the fact that atoms of the power set Boolean algebra are in bijection with the atoms of the super indicator algebra. The mapping of atoms is achieved by choosing semi-prebase superfunctions whose products are non-degenerate. □

4.7. Superstatistics

Statistics refer to expectation values of products of random variables. Expectations of moments (monomial powers) of random variables probe the underlying probability distribution on some sample space. For discrete sample spaces, probability distributions are tuples of probabilities. In order to describe statistics of superobservables, we first need to reexpress the Grassmann algebra product (denoted by a dot) in an atomic superindicator basis

$$\mathbb{1}^I.\mathbb{1}^J = \sum_K f_K^{IJ}\mathbb{1}^K\,, \tag{4.13}$$

where $I, J, K \in \mathbb{Z}_2^n$, and f_K^{IJ} are superindicator structure constants. The goal is to compute f_K^{IJ} for a given list of superindicators. We will do this for the choice of superindicators described in Section 4.4 on a binary sample space Ω_n.

4.7.1. Product of atomic superindicators. We can label superindicators by signed permutations corresponding to the standard ordering and signs appearing in the prebase indicators. First pick an integer $0 < k \leq n$. Then let i_l be k integers labelling prebase elements. The map acting on a signed permutation I, given by

$$H_k \ni I = [i_1, i_2, \ldots, i_k] \mapsto \mathbb{1}^I = \mathbb{1}^{i_1}.\mathbb{1}^{i_2}.\cdots.\mathbb{1}^{i_k}\,,$$

takes values in superindicators only when the i_k in I follow the standard ordering

$$|I(i)| < |I(j)| \iff i < j\,.$$

The following "signed anti-commutation" identity follows from the definition in Equation (4.8)

$$\mathbb{1}^{[i,j]} + \mathbb{1}^{[-j,-i]} = \frac{1}{2}\,. \tag{4.14}$$

The following identities simplify (signed) squares,

$$\mathbb{1}^{[i,-i]} = \mathbb{1}^i.\mathbb{1}^{-i} = \frac{1}{4}\,, \quad \mathbb{1}^{[i,i]} = \mathbb{1}^i.\mathbb{1}^i = \mathbb{1}^{[i]} - \frac{1}{4}\,. \tag{4.15}$$

Together with the above superorthogonality and superidempotence formulæ, we can now reorder products of prebase superindicators $\mathbb{1}^{\pm i}$ to obtain a sum of base superindicators (expressed in

standard order with signed square-free prebase factors). These identities can be applied iteratively to obtain the desired structure constants f_K^{IJ} of Equation (4.13).

In practice, it is easier to work in the blade basis and use the involutive/2-partition identity we saw in Section 4.4 and Example 4.6.1, *viz.*

$$2^{|I|}\theta^I = \sum_{\sigma\in\mathbb{Z}_2^k \text{ even}} \mathbb{1}^{\sigma(I)} - \sum_{\sigma\in\mathbb{Z}_2^k \text{ odd}} \mathbb{1}^{\sigma(I)} = \sum_{\sigma\in\mathbb{Z}_2^k} \operatorname{sgn}(\sigma)\mathbb{1}^{\sigma(I)}\,, \tag{4.16}$$

where we have assumed that the signed permuation $I \in H_k$ is already in standard order and $\theta^{-i} := -\theta^i$ (where $1 \leq i, k \leq n$) . Expanding the left hand side of Equation (4.13) into blades gives

$$\left(\frac{1}{2^n}\right)^2 \sum_{L\in B(\mathscr{B}_n)} p_L^{IJ} 2^{|L|}\theta^L \quad \text{with} \quad p_L^{IJ} = \sum_{\substack{\mathcal{I}\subseteq I,\, \mathcal{J}\subseteq J \\ \mathcal{I}\sqcup\mathcal{J}=L}} \operatorname{sgn}(\mathcal{I},\mathcal{J})\,. \tag{4.17}$$

where $\operatorname{sgn}(\mathcal{I},\mathcal{J})$ is the sign of the underlying permutation performing the $(|\mathcal{I}|, |\mathcal{J}|)$-shuffle (viewed as an element of the standard embedding of $S_n \hookrightarrow H_n$). We can use Equation (4.16) for the blade $2^{|L|}\theta^L$ in the above display, and then further expand each base superindicator $\mathbb{1}^{\sigma(I)}$ as a sum of atomic superindicators according to

$$\mathbb{1}^L = \sum_{\mathbb{Z}_2^n \ni K \supset L} \mathbb{1}^K\,.$$

In the above, by $K \supset L$, we mean any sequence K of n signs which restrict to L. For example, when $n = 3$, $(+,+,-) \supset (+,0,-)$. In this way we obtain an interesting combinatorial formula for the structure constants in Equation (4.13)

$$f_K^{IJ} = \frac{1}{|\Omega_n|^2} \sum_{\pi=\pi_1\sqcup\pi_2\subseteq[n]} \operatorname{sgn}(\pi_1,\pi_2) \quad (-1)^{(I_{\pi_1}+J_{\pi_2}+K_\pi)}\,, \tag{4.18}$$

where $I_{\pi_1} = \sum_{i\in\pi_1}\sum(I(i))$ is the parity of I restricted to the subset $\pi_1 \subseteq [n]$ and similarly for the other binary multi-indices J, K. The above tells us that the superproduct performs a signed count of the 3-partitions π of $[n]$ that depends on the parity and ordering of prebase elements labelled by elements of $[n]$. It should be possible to define generating/partition functions encoding the distribution of such binary set-partitions based on formal power series [82]. It would be interesting to find the analog of Schur-Weyl duality between super-partition algebras and hyperoctahedral groups.

An important application of the structure constants is the Jordan algebra.

EXAMPLE 4.7.1 (Jordan structure for Λ_2). *The four atomic indicators corresponding to a binary sample space of cardinality* 2^2 *are*

$$\begin{aligned}
\mathbb{1}^{++} &= \frac{1}{4} + \frac{\theta^1}{2} + \frac{\theta^2}{2} - i\theta^1\theta^2\,, \\
\mathbb{1}^{+-} &= \frac{1}{4} + \frac{\theta^1}{2} - \frac{\theta^2}{2} + i\theta^1\theta^2\,, \\
\mathbb{1}^{-+} &= \frac{1}{4} - \frac{\theta^1}{2} + \frac{\theta^2}{2} + i\theta^1\theta^2\,, \\
\mathbb{1}^{--} &= \frac{1}{4} - \frac{\theta^1}{2} - \frac{\theta^2}{2} - i\theta^1\theta^2\,,
\end{aligned}$$

The Jordan structure equation is

$$\frac{1}{N^2} + \mathbb{1}^I \bullet \mathbb{1}^J = \frac{1}{N}(\mathbb{1}^I + \mathbb{1}^J) \iff n^I \bullet n^J = 0\,. \tag{4.19}$$

where $N = |\Omega_2| = 2^2$ *is the cardinality of the binary sample space* Ω_2 *and* $\mathbb{1}^I =: \frac{1}{N}(1 + n^I)$. *This means that the superindicator souls* n^I *can be viewed as independent observables in this case.*

The above Jordan structure equation gets more complicated with more Grassmann generators. For example, for Λ_3 we get

$$\frac{1}{N^2} + \mathbb{1}^I \bullet \mathbb{1}^J = \frac{1}{N}(\mathbb{1}^I + \mathbb{1}^J) + \frac{1}{4N}\delta_{\mathrm{sgn}(I),\mathrm{sgn}(J)}\mathrm{sgn}(I)\,(1 - 4\chi_{\text{non-adj sign flip}}(I))\,\mathbb{1}^+\,,$$

where $\mathbb{1}^+ = \sum_{\sigma\in\mathbb{Z}_2^3}\mathrm{sgn}(\sigma)\mathbb{1}^{\sigma(+++)}$, $N = 2^3$ and $\chi_{\text{non-adj sign flip}}(I)$ is the characteristic function for non-adjacent sign change in I. While Equation (4.18) gives an intuitive understanding of what the structure constants are calculating, we need good combinatorial tools like generating functions to compute them.

4.7.2. Moments. Let $X = \sum_I X_I \mathbb{1}^I \in \Lambda_n$ be a superobservable. Its n-th moment is defined to be the expectation value of its n-th power, which can now be expanded as

$$\langle X^n\rangle_\rho = \sum_{I_1 I_2 \ldots I_n} X_{I_1 I_2 \ldots I_n}\langle \mathbb{1}^{I_1} \bullet \mathbb{1}^{I_2} \bullet \cdots \bullet \mathbb{1}^{I_n}\rangle_\rho\,,$$

where the state

$$\rho = \sum_I p_I (\mathbb{1}^I)^* \in \Lambda_n^*\,,$$

is a normalized state with $p_I \geq 0$, $\sum_I p_I = 1$. Moments characterize statistical correlations. Of particular interest is the *Jovariance*

$$\sigma^2_\rho(X) := -\mathrm{Jov}_\rho(X,X) = -\langle (X - \langle X\rangle_\rho)^2\rangle_\rho = \langle X\rangle^2_\rho - \langle X^2\rangle_\rho\,. \tag{4.20}$$

In the above, the quantity $\langle X\rangle_\rho$ has the usual interpretation as the mean value of the observable X in the state ρ. Since we are dealing with a non-commutative product, the Jovariance is not necessarily positive. If the Jordan algebra is however given by Equation (4.19), the Jovariance is

$$\sigma^2_\rho(X) = \left(\sum_I X_I(\frac{1}{2^2} - p_I)\right)^2 = (X, \rho_{\mathrm{soul}})^2_\star\,, \tag{4.21}$$

where $p = \frac{1}{2^n}$ is the uniform counting measure on the sample space, and $(\cdot,\cdot)_\star$ is the Hodge inner product. Hence the Jovariance captures the deviation of the soul of the state ρ from the uniform state Id in the direction of the observable X. Positive definiteness of the symmetric bilinear form $\mathrm{Jov}_\rho(X,Y)$ then implies an uncertainity principle

$$|\mathrm{Jov}(X,Y)| \leq \sqrt{\sigma^2(X)\sigma^2(Y)}.$$

It would be interesting to study moments in Jordan algebras for higher n.

4.7.3. Signed deformations. In Section 2, we explained superdeformations starting from a commuting product. Superindicators manifest hyperoctahedral symmetry and so we can use $\mathbb{Z}_2$ action to characterize the structure of deformation. From the definition of the superindicators in Equation (4.8), it follows that

$$\mathbb{1}^i \wedge \mathbb{1}^j = \mathbb{1}^i.\mathbb{1}^j + \underbrace{\frac{1}{4}(\delta^{-ij} - \delta^{ij})}_{\text{symm deform}} + \underbrace{\frac{\chi(i>j)\epsilon_{ij}}{2}(\mathbb{1}^i.\mathbb{1}^j - \mathbb{1}^{-i}.\mathbb{1}^j - \mathbb{1}^i.\mathbb{1}^{-j} + \mathbb{1}^{-i}.\mathbb{1}^{-j})}_{\text{antisymm deform}}\,,$$

where $\wedge$ denotes the exterior/super product in the Grassmann algebra Λ_n and . the commuting product of functions. The symmetric and anti-symmetric deformations give rise to the following super Jordan and Lie deformation formulas

LEMMA 4.7.1.1. *The super Jordan and Lie structures are signed/hyperoctahedral deformations of the usual commutative product of measurable functions*

(1) $\mathbb{1}^i \times \mathbb{1}^j = \frac{1}{4}\epsilon^{|i||j|}\sum_{\sigma\in\mathbb{Z}_2^2}\mathrm{sgn}(\sigma)\mathbb{1}^{\sigma(i)}.\mathbb{1}^{\sigma(j)}\,,$

(2) $\mathbb{1}^i \circ \mathbb{1}^j = -\frac{1}{4} \sum_{\sigma \in \mathbb{Z}_2^2} \left(\mathrm{sgn}(ij)\delta_{|i||j|} + \mathrm{sgn}(\sigma)(1 - \delta_{|i||j|}) \right) \mathbb{1}^{\sigma(i)}.\mathbb{1}^{\sigma(j)}$.

CHAPTER 5

Superstochastic processes

Stochastic processes are typically described by time-parameterized families of random variables. Another—often equivalent—approach is to keep observables fixed and instead allow the state of the system to evolve. The state picture is well-suited to our supergeometric description.

In Chapter 1, we discussed how stochastic processes generalize the dynamics of points to that of random variables which are functions on a configuration/sample space. Now unlike topological manifolds described by point-set topology, supermanifolds are sheaves of supercommuting algebras that do not have a canonical topological description in terms of point-sets. The data of a supermanifold gives a topology for its (purely even) body only. The odd directions, on the other hand, do not come with a topology, and instead described by the sheaf structure. Therefore, in order to give a point-set description of superstochastic processes in systems with discrete degrees of freedom, we will need a finer supermanifold topology. This could be induced from a collection of distinguished Boolean superfunctions called superindicators.

5.1. Covering spaces and non-local products

In Chapter 4, we developed a theory of superindicators for the real Grassmann algebras Λ_n. This is the supermanifold $\mathbb{R}^{0|n}$, whose body is a point. These superindicators realize Grassmann algebras in terms of real-valued measurable functions on a binary Stone space. We can extend this construction to general supermanifolds. The generalized superindicators that now depend on the body, topologize the odd directions of a supermanifold $\mathcal{M}^{m|n}$, yielding a covering space description:

$$\text{Stone}\,(\mathscr{B}_m) \hookrightarrow \mathcal{M}^{m|n} \xrightarrow{\pi} M\,,$$

where M is the body of $\mathcal{M}$ of dimension m, and the binary fibers $\text{Stone}\,(\mathscr{B}_m)$ encode the Grassmann-odd degrees of freedom.

The product in the sheaf of supercommuting algebra can be interpreted as a non-local product of functions F, G on this covering space,

$$(F \wedge G)(K) = \sum_{I,J} f_K^{IJ} F(I) G(J), \quad I, J, K \in \{\pm\}^n \,,$$

where f_K^{IJ} are the structure constants discussed in Chapter 4 Section 4.7. Just as the Moyal product is a quantum deformation of the pointwise product of phase-space functions, the super/exterior product is a signed deformation of the pointwise product. Superdeformations however differ from quantum deformations in their commutators/Lie-brackets. While the quantum (Moyal) bracket is non-zero, the supercommutator is zero. In this sense, the superproduct is still classical.

States on a supermanifold are bonafide probabilistic measures on associated covering spaces. In the upcoming sections, we will flesh out the details of our probabilistic approach to dynamics of supermanifolds.

5.2. Liouvillian dynamics

Local dynamics of stochastic systems are typically described in terms of stochastic differential equations. These generalize ordinary differential equations by including additional noise terms described by some given random variable. A typical stochastic differential equation is

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dB_t \,,$$

where X_t is the dynamical random variable and B_t is the noise term corresponding to a standard Brownian motion or Wiener process. The solution of the stochastic differential equation is formally obtained by integrating the above equation *à la* Itô. Some good references for stochastic differential equations are [**37**, **100**]. There exists generalizations of stochastic differential equations describing random motion of particles/fields on manifolds [**52**, **100**] or supermanifolds [**84**]

This is only an effective description. Stochastic differential equations describe dynamics of macroscopic systems where one may not have complete knowledge of how the system evolves. Such equations include random noises. This is quite relevant in physical situations where a system's evolution is described by dynamics of its state, and an observer measures time-varying expected values of random variables with respect to the dynamical state.

5.2.1. Microscopic dynamics. We would like to describe symplectic particle dynamics of microscopic systems. For us, the underlying microscopic dynamics is deterministic. Randomness is associated to the preparation and measurement of systems, just as in quantum mechanics. We consider symplectic supermanifolds. These describe phase-spacetimes that include discrete degrees of freedom. Dynamics is generated by a symplectic vector field ρ and the state Ψ_t solves a Liouville equation

$$\mathcal{L}_\rho \Psi_t = 0\,. \tag{5.1}$$

The classical Liouville equation is a continuity equation encoding microscopic conservation of probability. Its quantum mechanical counterpart is the von-Neumman equation

$$i\hbar \frac{d\hat{\rho}}{dt} = [H, \hat{\rho}]\,,$$

giving unitary evolution of a state described by a density matrix $\hat{\rho}$. Instead of evolving states, we can dually discuss dynamics in terms of evolution of observables X_t

$$\mathcal{L}_\rho X_t = 0\,.$$

Since we evolve states, we must next discuss how positivity is preserved under evolution.

5.2.2. Positive evolution. Since we employ Hamiltonian vector fields on a supersymplectic manifold $\mathcal{M}$ to generate dynamics of states, we must define their action on states. Well-posed dynamics must ensure that states in the convex cone $\mathcal{P}$ never leave the cone. One way to achieve this is to declare states Ψ to be "positive squares"

$$\Psi = \Phi \star \Phi$$

of some state function Φ (a real superfield in the dual $\mathcal{X}^*$) with respect to some suitable product $\star$. This can be viewed as a generalized Born rule and corresponds to a surjective squaring map $\bullet^2$

$$\bullet^2 : \mathcal{X}^* \twoheadrightarrow \mathcal{P}.$$

The evolution problem for states can then be defined on state functions as an action of Hamiltonian flows Φ_t^* and pulled back to states by a section (square root) $\surd : \mathcal{P} \hookrightarrow \mathcal{X}^*$ such that $\bullet^2 \circ \surd = \mathrm{id}_{\mathcal{P}}$.

Dynamics of a state $\tilde{\Phi}_t^* := \cdot^2 \circ \Phi_t^* \circ \sqrt{}$ is thus given by traversing the following commutative diagram:

$$\begin{array}{ccc} \mathcal{X}^* & \xrightarrow{\Phi_t^*} & \mathcal{X}^* \\ \uparrow{\scriptstyle \sqrt{}} & & \downarrow{\scriptstyle \cdot^2} \\ \mathcal{P} & \xrightarrow{\tilde{\Phi}_t^*} & \mathcal{P} \end{array}$$

5.3. Convex cone of states

In Chapter 3 Section 3.4, we have discussed states and their convex representations from a $*$-algebra point of view. We are interested in the dynamics of states in a convex cone. Note that pure states are extremal boundary elements that generate the cone. Importantly, dynamics must evolve states such that they do not leave the cone. As discussed in Section 5.2.2, we can try to ensure this property by a Born mechanism, *viz.* expressing states as suitable squares of state functions. Hence we will construct convex cones by employing a positivity constraint, meaning that elements are expressible as generalized squares. There are several ways to do this as discussed below.

5.3.1. Polyhedral cones. Polyhedral cones have finitely many pure states:

(1) *Positive half-space:* One might consider superfields whose bodies are positive and thus can be expressed as squares using the standard multiplication of superfunctions.

(2) *Positive orthant:* Alternatively, one might choose a basis for $\Lambda\mathbb{C}^m$ (which can be identified with the space of superfields) and then study those elements whose components are positive (squaring is defined component-wise).

Both the cones described above are polyhedral (finitely generated). This physically corresponds to a finite number of outcomes in the sample space. The body positive cone encodes a singleton sample space, while the positive orthant encodes a sample space of size 2^m. These cones are either trivial (sample space is a point) or make an *ad hoc* choice of basis that is not canonical to the exterior algebra. Dynamics inside such cones is also more complicated as it is a manifold with corners, and one needs to specify additional boundary conditions. We will later employ such polyhedral cones to model observers who wish to extract a finite list of probabilities from our stochastic systems.

5.3.2. Bloch cones. There is a particularly interesting class of convex cones used to quantum mechanical spinning particles.The pure state space of an n-level quantum system is the space of complex lines in $\mathbb{C}^{n+1}$ *viz.* the complex projective space $\mathbb{CP}^n$ [10]. Odd dimensional spheres S^{2n+1}

are principal $U(1)$-bundles (Hopf fibrations h) over $\mathbb{CP}^n$

$$\begin{array}{ccccc} U(1) & \hookrightarrow & S^{2n+1} & \hookrightarrow & \mathbb{C}^{n+1}. \\ & & \downarrow h & & \\ & & \mathbb{CP}^n & & \end{array}$$

The quantum mechanical convex cone of states $\mathcal{P}_q$ is given by $(n+1)\times(n+1)$ positive semidefinite hermitian matrices ρ. They form an $(n+1)^2$ dimensional convex subset of the Hilbert space $\cong \mathbb{C}^{(n+1)^2}$, with respect to the Hilbert-Schmidt inner product $(A,B)_{\text{HS}} := \text{tr}(A^\dagger B)$, of all linear operators on $\mathbb{C}^{n+1}$. The convex base is isomorphic to the unit trace states $\text{tr}(\rho) = 1$. These are called *density matrices*. Here pure states are those density matrices that act as rank-1 orthogonal projectors on $\mathbb{C}^{n+1}$, namely those satisfying

$$\rho^2 = \rho\,.$$

These are in one-one correspondence with the space of lines $\mathbb{CP}^n$ in $\mathbb{C}^{n+1}$. When $n = 1$, we obtain the Bloch sphere $\mathbb{CP}^1 \cong S^2$ as the pure state space and the closed 3-ball as the convex base.

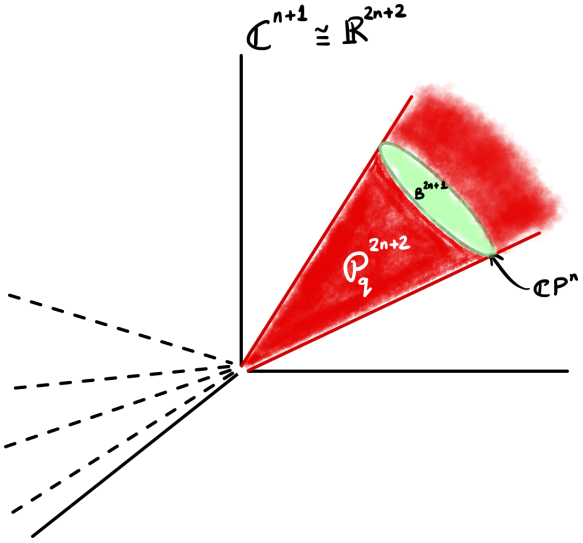


FIGURE 5.1. Bloch cone $\mathcal{P}_q^{2n+2}$

Dynamics in quantum mechanics is governed by an action of unitaries and thus does not run into boundary issues like in the case of polyhedral cones. Since BETTER it provides a simple setting for dynamics and measurement theory, we choose the Bloch cone $\mathcal{P}_q$ to be the state space and describe unitary dynamics of states generated by Hamiltonian vector fields. (In [25] it is shown how to

construct composite probability cones for system with both Grassmann and continuous variables built from standard classical probability cones and the quantum ones.) We re-emphasize that while states are quantum mechanical, the algebra of observables on a supersymplectic manifold is not the standard quantum mechanical one.

In order to describe unitary dynamics of states, we need to equip $\Lambda\mathbb{C}^m$ with an inner product and hence obtain a Hilbert space. In fact there is a canonical inner product on $\Lambda\mathbb{C}^m$ that extends the standard hermitian inner product on $\mathbb{C}^m$ to multivectors.

5.3.3. Hodge inner product. The algebra of superobservables $\mathcal{A}$, *per definitionem*, is a vector space. States live in a convex cone in the linear dual $\mathcal{A}^*$. An inner product on $\mathcal{A}$ would allow us to identify the two spaces. We start by describing the complex Hodge inner product on $\Lambda\mathbb{C}^m$.

Let us identify the linear dual of $\mathbb{C}^m$ with itself using the standard hermitian inner product $\langle \cdot, \cdot \rangle$ of complex vectors. This induces an inner product on the exterior algebra $\Lambda\mathbb{C}^m$ as follows. For $Z = \sum_{\mathcal{I}} c_{\mathcal{I}} z^{\mathcal{I}}$ and $W = \sum_{\mathcal{I}} d_{\mathcal{I}} w^{\mathcal{I}}$ where $\mathcal{I}$ denotes a multi-index $i_1 \cdots i_k$ of any length $k := \mathrm{len}(\mathcal{I}) \leq m$ so that $c_{\mathcal{I}} := c_{i_1 i_2 \cdots i_k} \in \mathbb{C}$ and $z^{\mathcal{I}} := z^{i_1} \wedge z^{i_2} \wedge \cdots \wedge z^{i_k}$ with $z^i \in \mathbb{C}^m$, and the sum over $\mathcal{I}$ runs over distinct values of the multi-index. Then the Hodge inner product is defined in terms of determinants of Gram matrices according to

$$\langle Z, W \rangle_{\mathrm{Hodge}} := \sum_{\mathcal{I}\mathcal{J}} \delta_{\mathrm{len}(\mathcal{I})\mathrm{len}(\mathcal{J})} \overline{c_{\mathcal{I}}} d_{\mathcal{J}} \det \left(\langle z^i, w^j \rangle_{\substack{i \in \mathcal{I} \\ j \in \mathcal{J}}} \right) .$$

We are now ready to map superobservables in $\Lambda\mathbb{C}^m$ (using the Hodge inner product to identify this with $(\Lambda\mathbb{C}^m)^*$) to the Bloch cone $\mathcal{P}_q$ by a surjective squaring map. Exterior multiplication is anti-symmetric and thus not ideal for squaring. However, it is possible to deform the wedge product of two vectors $z, w \in \mathbb{C}^m$ by the hermitian inner product, leading to a new product $\star$ given by

$$z \star w = z \wedge w + \langle z, w \rangle \in \Lambda\mathbb{C}^m.$$

This is precisely the Clifford product and can be naturally extended to multivectors resulting in the Clifford algebra $\mathrm{Cl}(\mathbb{C}^m)$. The Bloch cone will be identified with Clifford product squares.

5.3.4. Clifford algebra, quantization and a squaring map. Positive (semi-definite) hermitian matrices in the Bloch cone $\mathcal{P}_q$ can be identified with squares of self-adjoint elements in the

Clifford algebra $(\mathrm{Cl}_m(\mathbb{C}), \cdot)$. The Clifford algebra material that follows is fairly standard, see for example [66, 79]; to the best of our knowledge Theorem 5.3.1 and Lemma 5.3.1.1 are novel. Clifford algebra *cognoscenti* can safely read only those two results as well as the squaring map given at the end of the section. First let γ^i be generators of $(\mathrm{Cl}_m(\mathbb{C}), \cdot)$ satisfying

$$\{\gamma^i, \gamma^j\} = 2\delta^{ij}\,\mathrm{id}\,,$$

where the Clifford anticommutator $\{\gamma^i, \gamma^j\} := \gamma^i \cdot \gamma^j + \gamma^j \cdot \gamma^i$. We often suppress the identity element id in formulæ. Given an orthonormal basis θ^i of $\mathbb{C}^m$, there is an embedding of $\mathbb{C}^m$ into $\mathrm{Cl}_m(\mathbb{C})$ given by

$$v_i\theta^i =: v \mapsto \gamma_v := v_i\gamma^i.$$

This map can be extended to monomials (identity is sent to identity) and then by linearity to all multivectors.

$$\begin{aligned} \gamma : \Lambda\mathbb{C}^m &\to \mathrm{Cl}(\mathbb{C}^m) \\ v^{\mathcal{I}} &\mapsto \gamma_{v^{\mathcal{I}}}\,, \end{aligned}$$

where $v^{\mathcal{I}} := v^{i_1} \wedge v^{i_2} \wedge \cdots \wedge v^{i_k}$ and $\gamma_{v^{\mathcal{I}}} := \frac{1}{k!}\sum_{\sigma\in S_k} \mathrm{sgn}(\sigma)\, \gamma_{v^{\sigma(i_1)}} \cdot \gamma_{v^{\sigma(i_2)}} \cdot \gamma_{v^{\sigma(i_3)}} \cdot \cdots \cdot \gamma_{v^{\sigma(i_k)}}$, is the totally antisymmetrized product of Clifford vectors. The above map γ is called the quantization map and is an isomorphism of vector spaces. We can pull back the Clifford product using the quantization map to obtain a super-Moyal $\star$-product on superfunctions $u, v \in \Lambda\mathbb{C}^m$,

$$u \star v := \gamma^{-1}(\gamma_u \cdot \gamma_v).$$

THEOREM 5.3.1. *The super-Moyal product $\star$ is a Poisson bivector on $\Lambda\mathbb{C}^m$ defined in terms of the Hodge inner product*

$$u \star v = \delta^{IJ}_{\mathrm{Hodge}} \iota_{\theta^I}(u) \wedge \iota_{\theta^J}(v) = u\left(\wedge \circ \exp\left(\overleftarrow{\partial_i}\delta^{ij}\overrightarrow{\partial_j}\right)\right) v\,,$$

where $\delta^{IJ}_{\mathrm{Hodge}} = \delta^{|I||J|} g^{i_1 j_1} \ldots g^{i_k j_k}$ are the components of the Hodge inner product in an orthonormal basis $\{\theta^i\}$.

We can promote the quantization map to an isometry by defining an inner product on the Clifford algebra. For that, we first equip the algebra with a real structure. Let us define the

anti-involution † defined by $(\gamma_i\gamma_j\cdots\gamma_k)^\dagger := \gamma_k\cdots\gamma_j\gamma_i$ (extending to general Clifford elements by complex linearity so that $\gamma_u^\dagger = \overline{u}^i\gamma_i$). Note that the Clifford algebra is a $\mathbb{Z}$-filtered algebra, *i.e.* for any non-negative integers i and j,

$$\mathrm{Cl}^{\leq i}(\mathbb{C}^m)\cdot\mathrm{Cl}^{\leq j}(\mathbb{C}^m)\subseteq\mathrm{Cl}^{\leq i+j}(\mathbb{C}^m).$$

For a general element $x\in\mathrm{Cl}(\mathbb{C}^m)$, we define $x_0\in\mathrm{Cl}^0(\mathbb{C}^m)\cong\mathbb{C}$ to be its 0-grade element. Alternatively, one could work in any matrix representation of the Clifford algebra $\mathrm{Cl}(\mathbb{C}^m)$ and use the matrix trace tr to compute the body

$$x_0=\frac{\mathrm{tr}(x)}{\mathrm{tr}(1)}\,.$$

Then for $x,y\in\mathrm{Cl}(\mathbb{C}^m)$ we may define the Clifford inner product

$$\langle x,y\rangle_{\mathrm{Cliff}} := (x^\dagger\cdot y)_0\,.$$

LEMMA 5.3.1.1. *The exterior algebra equipped with the Hodge inner product is isomorphic (as inner product spaces) to the Clifford algebra equipped with the Clifford inner product.*

$$\langle u,v\rangle_{\mathrm{Hodge}}=\langle\gamma_u,\gamma_v\rangle_{\mathrm{Cliff}}.$$

The Artin-Wedderburn theorem states that any finite dimensional simple algebra over a field is isomorphic to a matrix algebra over a division ring whose center is the base field. For even m, the Clifford algebra $\mathrm{Cl}(\mathbb{C}^m)$ is a central simple algebra (the center is isomorphic to the base field) and thus isomorphic to a matrix algebra over the base field. In particular, by dimension counting $\mathrm{Cl}(\mathbb{C}^m)\cong M_{2^{\frac{m}{2}}}(\mathbb{C})$ as algebras. For odd m, the center includes pseudoscalars (the even subalgebra is still central simple) and hence $\mathrm{Cl}(\mathbb{C}^m)$ isomorphic to two copies of $M_{2^{\frac{m-1}{2}}}(\mathbb{C})$. A particularly nice basis of the Clifford algebra is given by the gamma group (generalized Dirac matrices) where all elements are hermitian by construction. We want our quantization map to also respect the corresponding hermitian structures. This can be achieved by sending the hermitian orthonormal

generators θ^i of the exterior algebra to gamma matrices Γ^i under the quantization map

$$\gamma : (\Lambda\mathbb{C}^m, \star, \dagger) \to (\mathrm{Cl}(\mathbb{C}^m), \cdot, \dagger)$$
$$\theta^i \mapsto \Gamma^i \,.$$

The quantization map now satisfies all the properties of a $*$-algebra isomorphism with respect to the $\star$ and Clifford products:

(1) $\gamma(u \star v) = \gamma(u) \cdot \gamma(v)$
(2) $\gamma^{-1}(1) = 1$
(3) $\gamma(u^\dagger) = \gamma(u)^\dagger$.

The above map now allows us to embed the Bloch cone in the exterior algebra. Elements in the Bloch cone are precisely $\star$-squares. The sought after squaring map is

$$\cdot^2 : \Lambda\mathbb{C}^m \to \mathcal{P}_q \subset \Lambda\mathbb{C}^m$$
$$\phi \mapsto \phi \star \phi.$$

5.4. Supermeasurements

We would now like to use apply our supermeasurement theory to describe dynamics of a discrete system whose sample space contains 2^2 configurations.

5.4.1. Probability cones. Consider a space of random variables given by the vector space $\mathcal{X} = \mathbb{R}^4$. This corresponds to a cardinality four outcome space, for example that of a pair of coin tosses $\mathcal{S} = \{HH, HT, TH, TT\}$. Elements of $\mathcal{X} = \mathbb{R}^{\mathcal{S}}$ can be neatly packaged as real superfunctions of a pair of Grassmann variables

$$X = x_0 + x_1\theta^1 + x_2\theta^2 - ix_{12}\theta^1\theta^2 \,.$$

To extract probabilities for a set of distinguished outcomes, we introduce a set of linearly independent indicator functions X_s such that

$$\sum_{s \in U} X_s = 1_{\mathcal{S}} \,,$$

where 1_S is the random variable that assigns 1 to all outcomes, *i.e.* the characteristic function on $\mathcal{S}$, and U is a set of disjoint subsets of $\mathcal{S}$ whose union is $\mathcal{S}$, and also such that their expectations $p_s = \langle X_s \rangle$ lie in the interval $[0, 1]$ and sum to unity

$$\sum_{s \in U} p_s = 1 \,.$$

To define expectations we need the data of a convex cone $\mathcal{C}$ in the dual vector space $\mathcal{X}^*$ to the space of random variables. Then, the expectation of $X \in \mathcal{X}$ with respect to $\varpi \in \mathcal{C}$ is given by

$$\langle X \rangle_\varpi = \frac{\varpi(X)}{\varpi(1_\mathcal{S})} \,.$$

5.4.2. Supersquares. To construct such a convex cone $\mathcal{C}$, given the data of a supersymplectic form

$$\Omega = i\eta_{ab}(t) d\theta^a d\theta^b - i\big(H(t)\epsilon_{ba} + \eta'_{ab}(t)\big)\theta^b d\theta^a dt \,,$$

where $a = 1, 2$ on a supertimeline $\mathcal{L}$, we proceed as follows. We need a notion of positivity for sections of the supertimeline $\mathcal{L}$. For that we construct a convex probability cone in each fiber for a given choice of timeline time $t \in \mathscr{L}$. At fixed t, a section determines a function f of the odd fiber coordinates θ^a. Here, one might be tempted to define positive functions of Grassmann coordinates as those that can be written as squares. However, in general, there exists squares, such as

$$-i\theta^1\theta^2\theta^3 = \big(\tfrac{1}{2}\theta^1 - i\theta^2\theta^3\big)^2 \;\text{ and }\; -i\theta^4\theta^5\theta^6 = \big(\tfrac{1}{2}\theta^4 - i\theta^5\theta^6\big)^2 \,,$$

whose sum, here

$$-i(\theta^1\theta^2\theta^3 + \theta^4\theta^5\theta^6) \,,$$

itself cannot be expressed as a square.

This problem, can be solved [25] by modifying the product of superfunctions. For that, one borrows from quantum mechanics an invertible "quantization" map σ from superfunctions $f = f_0 + f_1\theta^1 + f_2\theta^2 - if_{12}\theta^1\theta^2$ to Pauli matrices given by

$$\sigma(f) = \begin{pmatrix} f_0 + f_{12} & f_1 - if_2 \\ f_1 + if_2 & f_0 - f_{12} \end{pmatrix} \,. \tag{5.2}$$

Then, for a pair of superfunctions f and g we may define their "star" product by

$$f \star g = \sigma^{-1}\big(\sigma(f)\sigma(g)\big)\,,$$

where the product of images of σ above is given by usual matrix multiplication. When a superfunction f can be written as

$$f = g \star g\,,$$

we call f a *supersquare*. The set of all supersquares is a convex cone []. In this simple two bit example, a general supersquare is given by a superfunction with positive body,

$$f^{\star 2} = f_0^2 + f_1^2 + f_2^2 + f_{12}^2 + 2f_0(f_1\theta^1 + f_2\theta^2) - 2if_0f_{12}i\theta^1\theta^2\,.$$

Note that $\sigma(f^{*2})$ is a positive, hermitean, 2×2 matrix (in quantum mechanics, normalized, positive hermitean matrices are used to describe states for spins). Also, body positivity means we can normalize $f^{\star 2}$ according to

$$\widehat{f^{\star 2}} := \frac{f^{\star 2}}{||f||^2}\,.$$

Here, the norm of a general real superfunction f is defined by

$$||f|| := \sqrt{(f^{\star 2})_0}\,.$$

Since the space of supersquares is convex, we may take this to be the probability cone $\mathcal{C}$. A real superfunction Φ on the supertimeline $\mathcal{L}$ with the property that $\Phi(t)$ at each time $t \in \mathscr{L}$ is a supersquare, defines a time-dependent probability distribution (in the sense of vectors in a positive convex cone). We will call the elements of $\mathcal{C}$ *states*.

To define the expectation value $\langle X(t)\rangle_{\Phi(t)}$ of a random variable X at time t with respect to $\Phi(t)$, we write

$$\langle X(t)\rangle_{\Phi(t)} := \frac{1}{2}\,\mathrm{Tr}(\sigma(\Phi(t))\sigma(X(t))\,. \tag{5.3}$$

Note that the pairing

$$(f,g)_\star := \frac{\mathrm{Tr}(\sigma(f)\sigma(g))}{\mathrm{Tr}(\sigma(1))}$$

is an inner product for a pair of real superfunctions f, g. This allows us to canonically identify the vector space of real superfunctions with its dual. We will do so in what follows.

5.4.3. Sample space. To extract discrete probabilities, we need to consider a finitely generated polyhedral cone [25]. The probability cone $\mathcal{C}$ defined above is *not* finitely generated [25]; in fact viewed as a space of rays, its boundary is precisely the Bloch sphere of quantum mechanical single particle spin 1/2 states. Hence we look for a set of indicator random variables $\mathscr{I} := \{X_i\}$ that describe a finitely generated cone $\mathcal{C}_{\mathscr{I}}$.

To illustrate these ideas, consider the following set $\mathscr{I}$ of indicators whose dual cone $\mathcal{C}_{\mathscr{I}}$ describes the vertices of a regular tetrahedron, with centroid the origin and containing an insphere of radius 1:

$$X_1 = \frac{1}{4}\left(1 + \frac{2\sqrt{2}\theta^1}{3} + \frac{i\theta^1\theta^2}{3}\right), \qquad X_2 = \frac{1}{4}\left(1 - \frac{\sqrt{2}\theta^1}{3} + \frac{\sqrt{6}\theta^2}{3} + \frac{i\theta^1\theta^2}{3}\right),$$
$$X_3 = \frac{1}{4}\left(1 - \frac{\sqrt{2}\theta^1}{3} - \frac{\sqrt{6}\theta^2}{3} + \frac{i\theta^1\theta^2}{3}\right), \qquad X_4 = \frac{1}{4}\left(1 - i\theta^1\theta^2\right).$$

These superfunctions satisfy the property that the convex hull $\mathcal{C}_{\mathscr{I}}$ of their dual contains the cone of states $\mathcal{C}$. To see this, it suffices to show that $p_i = \langle X_i\rangle_{\mathcal{C}} \geq 0$. A normalized state $\hat{\Psi} \in \mathcal{C}$ looks like

$$\hat{\Psi} = \frac{\Phi^{\star 2}}{||\Phi||^2} = 1 + \frac{2\phi_0\phi_1}{||\Phi||^2}\theta^1 + \frac{2\phi_0\phi_2}{||\Phi||^2}\theta^2 - \frac{2i\phi_0\phi_{12}}{||\Phi||^2}\theta^1\theta^2,$$

where the statefunction $\Phi = \phi_0 + \phi_1\theta^1 + \phi_2\theta^2 - i\phi_{12}\theta^1\theta^2$ and the square of its length $||\Phi||^2 = \phi_0^2 + \phi_1^2 + \phi_2^2 + \phi_{12}^2$. Observe that

$$||\hat{\Psi}_{\text{soul}}||^2 = \frac{4\phi_0^2(\phi_1^2 + \phi_2^2 + \phi_{12}^2)}{||\Phi||^4} \leq 1,$$

which follows from the identity $||\Phi||^4 = (\phi_0^2 - \phi_1^2 - \phi_2^2 - \phi_{12}^2)^2 + 4\phi_0^2(\phi_1^2 + \phi_2^2 + \phi_{12}^2)$. The bound is saturated precisely when $\Phi = \phi_0 + \phi_1\theta^1 + \phi_2\theta^2 - i\phi_{12}\theta^1\theta^2 \in \mathbb{R}^4$ lies on the cone

$$\phi_0^2 - \phi_1^2 - \phi_2^2 - \phi_{12}^2 = ||\Phi||^2 - 2(\phi_1^2 + \phi_2^2 + \phi_{12}^2) = 0 \tag{5.4}$$

over a two-sphere. This two-sphere describes the set of pure states in $\mathcal{C}$. The expectation value of an observable $X = x_0 + x_1\theta^1 + x_2\theta^2 - ix_{12}\theta^1\theta^2$ in the state $\hat{\Psi}$ is

$$\langle X\rangle_{\hat{\Psi}} = x_0 + \frac{2\phi_0}{||\Phi||^2}(x_1\phi_1 + x_2\phi_2 + x_{12}\phi_{12}).$$

It can be checked that the expectation values of all the indicators X_i lie in the interval $[0, \frac{1}{2}]$. Positivity follows by construction, since we chose the cone $C_{\mathscr{J}}$ to be over a tetrahedron whose insphere corresponds to the boundary of $\mathcal{C}$.

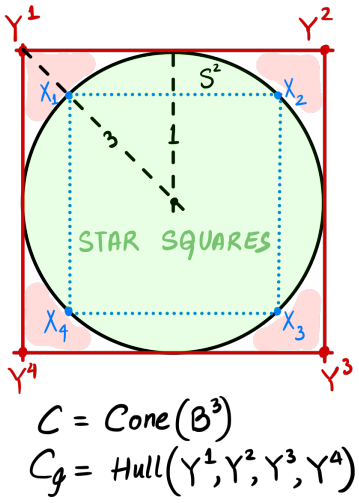


FIGURE 5.2. Configuration of cones: $\mathcal{C} \subset \mathcal{C}_{\mathscr{J}}$

5.5. Superdynamics

We now demonstrate our key ideas with the two-bit system evolving in time. Our basic data is the supermanifold $\mathcal{Z} = \mathbb{R}^{1|2} \ni (t, \theta^a)$ with $a = 1, 2$. For the flat, torsion-free affine connection we take

$$\nabla = d_T$$

in the coordinate system (t, θ^a). Our dynamics shall be described by the action functional of [11]

$$S = \int \tfrac{d\tau}{i} \big(\eta_{ab}(t) \theta^a \dot{\theta}^b - \frac{1}{2} H(t) \epsilon_{ab} \theta^a \theta^b \dot{t} \big) \, .$$

Assuming η is invertible, this is the most general super phase-spacetime dynamics on $\mathcal{Z}$. In the above the dot denotes differentiation with respect to the worldline parameter τ and ϵ_{ab} is the two-dimensional Levi-Civita symbol. Better said, this action is an integral of a hermitean super one-form

$$\xi = i \eta_{ab}(t) \theta^a d\theta^b + \frac{i}{2} H(t) \epsilon_{ab} \theta^a \theta^b dt \, .$$

Here we used that, for a one-form $d\tau$ (given by d_T of a bosonic function), one has $d\tau \dot{\theta} = d\theta$ (this choice of sign ensures that the variation of the above action agrees with the dynamics of the super

symplectic form below). This gives the symplectic form

$$\Omega = d_T\xi = i\eta_{ab}(t)d\theta^a d\theta^b - i\big(H(t)\epsilon_{ba} + \eta'_{ab}(t)\big)\theta^b d\theta^a dt\,.$$

The data $(\mathbb{R}^{1|2}, \Omega)$ defines a super phase-spacetime. Dynamics is determined by the supervector (*conferatur* Equation (2.30))

$$P = \frac{\partial}{\partial t} + \theta^a M^b{}_a \frac{\partial}{\partial \theta^b}\,,$$

where

$$M^b{}_a := -\frac{1}{2}\eta^{bc}(t)\big(H(t)\epsilon_{ac} + \eta'_{ca}(t)\big)\,,$$

and η^{ab} is the inverse of η_{ab}. Also $'$ denotes t-differentiation.

Now we consider a state function Φ given by a hermitean superfield

$$\Phi = \phi + \phi_a\theta^a - \tfrac{i}{2}\varphi\epsilon_{ab}\theta^a\theta^b\,,$$

(where $\epsilon_{12} = 1$) subject to

$$P\Phi = 0\,.$$

Here the four functions of t given by (ϕ, ϕ_a, φ) are all real and must obey

$$\phi' = 0\,, \quad \phi'_a + M^b{}_a\phi_b = 0\,, \quad \varphi' + M^a{}_a\varphi = 0\,. \tag{5.5}$$

Note that $M^a{}_a = -\frac{1}{2}\eta^{ab}\eta'_{ab}$.

We now consider the super laboratory Σ_T given by superhypersurface $t = T$. The pullback of Ω is then

$$\Omega^* = i\eta_{ab}(T)d\theta^a d\theta^b\,. \tag{5.6}$$

Before computing the star product, we solve the Equations (5.5). For that we impose boundary conditions for $\Phi(T)$ along Σ_T given by $\Phi(T) = (\phi^{\mathrm{i}}, \phi^{\mathrm{i}}_a, \varphi^{\mathrm{i}})$. Then

$$\phi(t) = \phi^{\mathrm{i}}\,, \quad \phi_a(t) = U(t,T)^b{}_a\,\phi^{\mathrm{i}}_b\,, \quad \varphi(t) = u(t,T)\varphi(T)\,,$$

where $U(t,T)^b{}_a = \Big[\mathrm{P}\exp\big(-\int_T^t M\big)\Big]^b{}_a$, $u(t,T) = \exp\Big(-\int_T^t M^a{}_a\Big) = \exp\Big(\frac{1}{2}\int_T^t \eta^{ab}\eta'_{ab}\Big)$, and P denotes path ordering.

To compute the vertical star product of state functions we first need the Clifford map γ. For that we introduce orthonormal frames ${e_a}^{\overline{a}}$ for the vertical distribution on $T^*\mathcal{M}$ determined by the Koszul connection, subject to

$$\eta_{ab}(t)d\theta^a d\theta^b = \delta_{\overline{a}\overline{b}}{e_a}^{\overline{a}}(t){e_b}^{\overline{b}}(t)d\theta^a d\theta^b\,.$$

We also define $\theta^{\overline{a}}(t) := {e_a}^{\overline{a}}(t)\theta^a$ so that

$$\Phi = \phi + \phi_{\overline{a}}\theta^{\overline{a}} - \tfrac{i}{2}\overline{\varphi}\,\epsilon_{\overline{a}\overline{b}}\theta^{\overline{a}}\theta^{\overline{b}}\,,$$

where $\phi_{\overline{a}} := {e^a}_{\overline{a}}\phi_a\,,\quad \overline{\varphi} := (\det{}^{-1}e)\varphi$, and $\epsilon_{\overline{12}} = 1$. We may now employ the map σ in Equation (5.2) along each fiber. This gives an invertible map from hermitean superfunctions to hermitean matrices taking values in functions of t,

$$\sigma(\Phi) = \begin{pmatrix} \phi + \overline{\varphi} & \phi_{\overline{1}} - i\phi_{\overline{2}} \\ \phi_{\overline{1}} + i\phi_{\overline{2}} & \phi - \overline{\varphi} \end{pmatrix}\,,$$

such that the star product is computed via matrix multiplication:

$$\Phi \star_\nabla \Phi = \sigma^{-1}\big(\sigma(\Phi)^2\big) = \phi^2 + \phi_{\overline{1}}^2 + \phi_{\overline{2}}^2 + \overline{\varphi}^2 + 2\phi(\phi_{\overline{1}}\theta^{\overline{1}} + \phi_{\overline{2}}\theta^{\overline{2}}) - i\phi\overline{\varphi}\epsilon_{\overline{a}\overline{b}}\theta^{\overline{a}}\theta^{\overline{b}}\,.$$

Hence, once again, a hermitean superfield Ψ is a vertical star square of a non-zero state function precisely when its body is positive.

5.5.1. Measurements. We can now construct expectations of observables. For that let $X = x + x_a\theta^a - \frac{i}{2}\chi\epsilon_{ab}\theta^a\theta^b$ be a hermitean superfield, which we may also expand in the adapted Grassmann coordinates $\theta^{\overline{a}}(t)$,

$$X = x + x_{\overline{a}}\theta^{\overline{a}} - \tfrac{i}{2}\overline{\chi}\,\epsilon_{\overline{a}\overline{b}}\theta^{\overline{a}}\theta^{\overline{b}}\,.$$

The pullback in Equation (5.6) in terms of the adapted $\theta^{\overline{a}}$ is

$$\Omega^* = i\delta_{\overline{a}\overline{b}}d\theta^{\overline{a}}d\theta^{\overline{b}}\,.$$

Hence the expectation of the observable X in the $t = T$ superlaboratory with respect to a state $\Psi = \Phi \star_\nabla \Phi$ is given by (see Equation (5.7))

$$\langle X\rangle_{\Psi,T} = x(T) + \frac{2\phi(\phi_{\overline{1}} x_{\overline{1}} + \phi_{\overline{2}} x_{\overline{2}} + \overline{\varphi\chi})}{\phi^2 + \phi_{\overline{1}}^2 + \phi_{\overline{2}}^2 + \overline{\varphi}^2}\Big|_{t=T} = x(T) + \frac{2\phi(\phi_a \eta^{ab} x_b + \det{}^{-1}\eta\, \varphi\chi)}{\phi^2 + \phi_a\eta^{ab}\phi_b + \det{}^{-1}\eta\, \varphi^2}\Big|_{t=T}\,. \tag{5.7}$$

To simplify the remainder of our discussion, let us assume η and H are t-independent. It follows that $u(T,t) = 1$ and

$$U(T,t) = \cos\left(\tfrac{(t-T)H}{2\sqrt{\det\eta}}\right)\mathrm{Id} + \sqrt{\det\eta}\,\sin\left(\tfrac{(t-T)H}{2\sqrt{\det\eta}}\right)\eta^{-1}\epsilon\,. \tag{5.8}$$

By construction $U^\top \eta\, U = \eta$ so that total probabilities are conserved in the sense that $(\Phi,\Phi)_{\Omega,\Sigma_t}$ is constant. It is then not difficult to see that

$$\frac{d\langle X\rangle_{\Psi,t}}{dt}\bigg|_{t=T} = \langle \mathcal{L}_P X\rangle_{\Psi,T}\,,$$

which is a statement about conservation of probabilities. Note that the supervector P acts on the components (x, x_a, χ) of X according to $(x', (x' + Mx)_a, \chi'))$.

Finally we can consider the evolution of a set of probabilities. For further simplicity we now take the case $\eta = \delta$ (we could always change basis to achieve this). To discuss standard probabilities, we need to introduce some choice of a polyhedral convex cone containing the convex cone of states (see [49] for a related construction). For that recall the indicator functions of Section 5.4.3

$$X_1 = \tfrac{1}{4} + \tfrac{\sqrt{8}\theta^1 + i\theta^1\theta^2}{12}\,, \qquad X_2 = \tfrac{1}{4} + \tfrac{-\sqrt{2}\theta^1 + \sqrt{6}\theta^2 + i\theta^1\theta^2}{12}\,,$$
$$X_3 = \tfrac{1}{4} + \tfrac{-\sqrt{2}\theta^1 - \sqrt{6}\theta^2 + i\theta^1\theta^2}{12}\,, \qquad X_4 = \tfrac{1}{4} + \tfrac{-3i\theta_1\theta_2}{12}\,.$$

We next write a state function in terms of probabilities corresponding to expectations of these indicators. Note that, unlike states, state functions can have vanishing body. From Equation (5.7) we see that any state function with vanishing body $\phi = 0$ here has indicator expectations

$$\langle X_i\rangle = \frac{1}{4}\,, \quad i = 1,\ldots,4\,.$$

Otherwise, for $\phi \neq 0$, we may first normalize Φ along a fixed $t = T$ laboratory such that its star-square has unit body

$$(\Phi,\Phi)_T := \left(\Phi\star_\nabla\Phi\right)_0 = \phi^2 + \phi_a\delta^{ab}\phi_b + \varphi^2 = 1\,, \tag{5.9}$$

which simplifies Equation (5.7). Calling $p_i := \langle X_i \rangle \in [0,1]$ we then have

$$p_1 = \tfrac{1}{4} + \phi\left(\tfrac{\sqrt{8}\phi_1 - \varphi}{6}\right), \qquad p_2 = \tfrac{1}{4} + \phi\left(\tfrac{-\sqrt{2}\phi_1 + \sqrt{6}\phi_2 - \varphi}{6}\right),$$
$$p_3 = \tfrac{1}{4} + \phi\left(\tfrac{-\sqrt{2}\phi_1 - \sqrt{6}\phi_2 - \varphi}{6}\right), \qquad p_4 = \tfrac{1}{4} + \tfrac{\phi\varphi}{2}.$$

Since $\phi \neq 0$, we may solve $(\phi_1, \phi_2, \varphi)$ in terms of (ϕ, p_1, p_2, p_3)

$$\phi_1 = \frac{2p_1 - p_2 - p_3}{\sqrt{2}\phi}, \quad \phi_2 = \frac{\sqrt{3}(p_2 - p_3)}{\sqrt{2}\phi}, \quad \varphi = \frac{3 - 4(p_1 + p_2 + p_3)}{2\phi}.$$

Finally we can find ϕ by plugging the previous list of expressions for ϕ_a, φ in (5.9). While the resulting quartic equation itself has four solutions, we choose the root

$$\phi = \frac{\sqrt{1 + 2\sqrt{-\Delta_{\text{ellipsoid}}}}}{\sqrt{2}},$$

where $\frac{\Delta_{\text{ellipsoid}}}{6} := (p_1 + p_2 + p_3)^2 - (p_1 + p_2 + p_3 + p_1p_2 + p_2p_3 + p_1p_3) + \frac{1}{3}$. This makes sense only when $\Delta_{\text{ellipsoid}} \leq 0$, implying that the supersymplectic probabilities are constrained to lie inside the ellipsoid of Equation (5.4) in $\mathbb{R}^3_{\geq 0}$.

It remains to study the evolution of probabilities. Returning to Equation (5.8) and calling $\mathfrak{c}(t) := \cos\left(\frac{(t-T)H}{2}\right)$, $\mathfrak{s}(t) := \sin\left(\frac{(t-T)H}{2}\right)$, we obtain

$$\phi_1(t) = \quad \mathfrak{c}(t)\phi_1(T) + \mathfrak{s}(t)\phi_2(T),$$

$$\phi_2(t) = -\mathfrak{s}(t)\phi_1(T) + \mathfrak{c}(t)\phi_2(T).$$

Thus

$$\begin{pmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \\ p_4(t) \end{pmatrix} = \begin{pmatrix} \frac{1+2\mathfrak{c}(t)}{3} & \frac{1-\mathfrak{c}(t)+\sqrt{3}\mathfrak{s}(t)}{3} & \frac{1-\mathfrak{c}(t)-\sqrt{3}\mathfrak{s}(t)}{3} & 0 \\ \frac{1-\mathfrak{c}(t)-\sqrt{3}\mathfrak{s}(t)}{3} & \frac{1+2\mathfrak{c}(t)}{3} & \frac{1-\mathfrak{c}(t)+\sqrt{3}\mathfrak{s}(t)}{3} & 0 \\ \frac{1-\mathfrak{c}(t)+\sqrt{3}\mathfrak{s}(t)}{3} & \frac{1-\mathfrak{c}(t)-\sqrt{3}\mathfrak{s}(t)}{3} & \frac{1+2\mathfrak{c}(t)}{3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_1(T) \\ p_2(T) \\ p_3(T) \\ p_4(T) \end{pmatrix}.$$

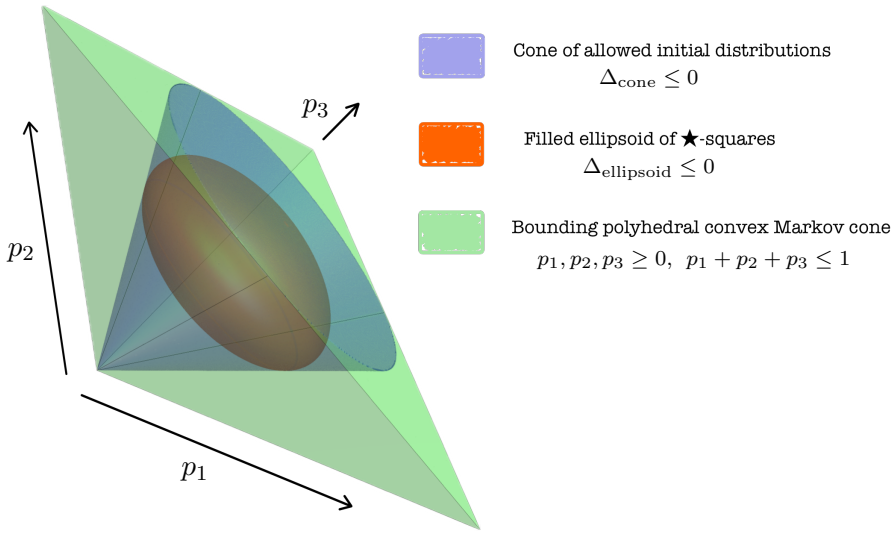


FIGURE 5.3. Dynamics corresponds to a path inside the orange ellipsoid

For probabilities to evolve to probabilities, examining the above transition matrix we see that the initial probabilities must obey the conical constraint

$$\Delta_{\text{cone}} := p_1^2 + p_2^2 + p_3^2 - 2(p_1 p_2 + p_1 p_3 + p_2 p_3) \leq 0\,.$$

This is automatically satisfied for supersymplectic probability distributions, *videlicet* those that satisfy $\Delta_{\text{ellipsoid}} \leq 0$ or lie inside the ellipsoid of Equation (5.4), as depicted in Figure 5.3. While the above transition matrix is not Markov (its entries in any row or column sum to unity without being individually strictly non-negative) its eigenvalues do have magnitudes bounded by unity. It thus defines a Markov-like stochastic process.

APPENDIX A

Associativity of Grassmann Jordan algebra

We will prove associativity of the Grassmann Jordan product by demonstrating it for blades. The Grassmann Jordan associator $[\theta^I : \theta^J : \theta^K]$ is given by

$$[\theta^I : \theta^J : \theta^K] := \theta^I \bullet (\theta^J \bullet \theta^K) - (\theta^I \bullet \theta^J) \bullet \theta^K \, .$$

There are a few cases:

- The associator is trivially 0 when $I \cap J \neq \emptyset$ or $J \cap K \neq \emptyset$ or $I \cap K \neq \emptyset$. Thus we will assume $I \cap J = I \cap K = J \cap K = \emptyset$.
- If atleast two of the multi-indices I, J and K are of odd parity, the Jordan products in the both the terms in the associator are zero.
- If atleast two of the multi-indices are even, then both the terms in the associator are equal to $\theta^I \theta^J \theta^K$ and hence they cancel.

Thus we find that supercommutavity forces the Grassmann Jordan algebra to be associative.

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