ANDREI ROGERS

A Note on the Temporal Decomposition of Interpoint Transition Matrices

Center for Planning & Development Research: Institute of Urban & Regional Development
University of California, Berkeley

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1. INTRODUCTION

A problem of common interest to social scientists is the decomposition of a transition matrix which describes interpoint movement during an n unit time interval into the corresponding "average" transition matrix for a single unit time interval. That is, given a transition matrix, G, with respect to a time period of n units, find a transition matrix P such that:

$$P^n = G \,. \tag{1}$$

A convenient solution method is to find a collinearity transformation of G such that:

$$G = NVN^{-1} \,, \tag{2}$$

where V is a diagonal matrix and N is a suitably chosen non-singular square matrix of the same order as G. Because of the associative property of matrix multiplication, such a transformation enables us to examine a number of polynomial functions of G. For, in general, if $f(\cdot)$ is a polynomial function then it can be shown that (Fraser, *et al.*, [1; p. 67]):

$$f(G) = Nf(V)N^{-1} \tag{3}$$

and thus, for example,

$$\begin{aligned} G^2 &= (NVN^{-1})(NVN^{-1}) \\ &= NV(N^{-1}N)VN^{-1} \\ &= NV^2N^{-1} \,. \end{aligned}$$

Since our temporal decomposition problem, in effect, is to find $G^{1/n}$, we may use Equation (3) to establish the relationship:

$$G^{1/n} = NV^{1/n}N^{-1} \,, \tag{4}$$

which in view of the diagonal nature of V, is relatively simple to solve.

2. ESTIMATING V AND N

A well-known result in matrix algebra establishes that at least one solution for V, in Equation (2), is:

* The author is Assistant Professor of City and Regional Planning at the University of California at Berkeley.

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$$V = \begin{pmatrix} v_1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & v_2 & \cdot & & & \\ \cdot & & \cdot & & & \\ \cdot & & & \cdot & & \\ \cdot & & & & \cdot & \\ 0 & \cdot & \cdot & \cdot & \cdot & v_n \end{pmatrix} \tag{5}$$

where $v_1, v_2, \ldots v_n$ are the characteristic roots of the determinental equation:

$$|| vI - G || = 0 . \tag{6}$$

Moreover, it is also established that the columns of N are the characteristic vectors which correspond to the characteristic roots of Equation (6) (Fraser, *et al.*, [1; pp. 66–70]). That is, they are the column vectors of N where:

$$GN = NV . \tag{7}$$

Since the set of linear equations represented by Equation (7) are homogeneous, a unique solution does not exist. However, any solution will satisfy our purposes. One such solution may be constructed by using the cofactors of the first row of $(v_i I - G)$ for the i^{th} column of N.

3. EXAMPLE

At this point, an example may serve to clarify the above discussion. Consider the following interregional population model:

$$\underline{w}_{t+1} = \underline{w}_t(B - D + M) = \underline{w}_t G , \tag{8}$$

where

$\underline{w}_t$ = a row vector whose i^{th} element denotes the total population of region i at time t;

B, D = diagonal matrices whose i^{th} diagonal elements denote the crude birth and death rates of the i^{th} region;

M = a migration transition matrix whose elements, m_{ij}, denote the proportion of people who were in the i^{th} region at time t and in the j^{th} region at time $t + 1$.

For illustrative purposes, consider the following two regions: California and the Rest of the United States. Fitting the model to these two regions, for the 1955–1960 period, we have (in thousands):

$$(15{,}199;\ 163{,}040) = (12{,}988;\ 152{,}082) \begin{pmatrix} 1.0215 & .0627 \\ .0127 & 1.0667 \end{pmatrix} \tag{9}$$

Now, for the G in Equation (9), we need to find a P such that:

$$P^5 = G .$$

Solving the characteristic equation of Equation (6) for the two characteristic roots v_1 and v_2 we have:

$$\begin{Vmatrix} v - 1.0215 & - .0627 \\ - .0127 & v - 1.0667 \end{Vmatrix} = 0$$

which yields roots $v_1 = 1.0802$ and $v_2 = 1.0080$.

Hence

$$V = \begin{pmatrix} 1.0802 & 0 \\ 0 & 1.0080 \end{pmatrix}.$$

The two cofactors of the first row of $(v_1 I - G)$ are

$$n_{11} = .0135$$
$$n_{21} = .0127\,.$$

Analogously, for $(v_2 I - G)$ we have

$$n_{12} = -\,.0587$$
$$n_{22} = .0127\,.$$

Hence we may assemble the matrix

$$N = \begin{pmatrix} .0135 & -.0587 \\ .0127 & .0127 \end{pmatrix}$$

and find its inverse

$$N^{-1} = \begin{pmatrix} 13.8504 & 64.0173 \\ -13.8504 & 14.7229 \end{pmatrix}.$$

As a check on our arithmetic, it is a simple matter to compute NVN^{-1} and confirm that the result is indeed the matrix G.

With the collinear transform of G we now are in a position to find $G^{1/5}$. The first step is to derive $V^{1/5}$. Since V is a diagonal matrix we need only to find $\sqrt[5]{v_{ii}}$ which in our example yields:

$$V^{1/5} = \begin{pmatrix} 1.0155 & 0 \\ 0 & 1.0016 \end{pmatrix}.$$

Recalling Equation (4), we have:

$$\begin{aligned} P = G^{1/5} &= NV^{1/5}N^{-1} \\ &= \begin{pmatrix} .0135 & -.0587 \\ .0127 & .0127 \end{pmatrix} \begin{pmatrix} 1.0155 & 0 \\ 0 & 1.0016 \end{pmatrix} \begin{pmatrix} 13.8504 & 64.0173 \\ -13.8504 & 14.7229 \end{pmatrix} \\ &= \begin{pmatrix} 1.0028 & .0113 \\ .0028 & 1.0128 \end{pmatrix}. \end{aligned}$$

Again, as a check on our arithmetic, we compute P^5 and confirm that the result is the matrix G.

4. CONCLUSION

This note describes a convenient method for temporally decomposing an interpoint transition matrix. An example drawn from demography illustrates a practical application. Together with recent efforts to estimate interpoint transition matrices solely on the basis of distributional time series (Madansky [2], Telser [4], Theil [5]), this technique should yield valuable insights into interpoint flows. For example, in interregional migration analysis, an area with a dearth of data concerning place-to-place movements, the possibility of deriving reasonably accurate estimates of transition structures should significantly improve population forecasting efforts (Rogers and Miller [3]).

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