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Authors
Kendall, Bruce E.
Schaffer, William M.
Tidd, Charles W.

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Transient periodicity in chaos

Bruce E. Kendall, W.M. Schaffer and C.W. Tidd
Department of Ecology and Evolutionary Biology, University of Arizona, Tucson, AZ 85721, USA

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Chaotic time series can exhibit rare bursts of "periodic" motion. We discuss one mechanism for this phenomenon of "transient periodicity": the trajectory gets temporarily stuck in the neighborhood of a semiperiodic "semi-attractor" (or "chaotic saddle"). This can provide insight for interpreting such phenomena in empirical time series; it also allows for a novel partition of the phase space, in which the attractor may be viewed as the union of many such chaotic saddles.

An experimental time series can often exhibit what appears to be qualitatively different dynamics at various stages in its evolution. Particularly striking examples are ones in which periodic episodes are interspersed with intervals of much more irregular dynamics. There are a number of ways such a phenomenon can arise: the system could be under the influence of external perturbations, there may be a varying control parameter, or the switching may be intrinsic to the dynamics. A perturbation could knock the trajectory off of an attracting periodic orbit and onto a chaotic repeller [1,2], leading to an episode of irregular dynamics; it could also cause the system to switch between two distinct attractors [3]. Variation in a control parameter can cause bifurcations between periodic and chaotic dynamics; the pattern of observed switching then depends on the nature of the fluctuations in the parameter (monotonic, sinusoidal, etc.). Finally, both the periodic and irregular motions may be part of the asymptotic dynamics, but there is some sort of "dynamical bottleneck" that temporarily traps the trajectory into one or another region of the phase space, with different dynamics associated with the various regions. This is what occurs in intermittency [4]: much of the time series looks periodic, with occasional bursts of large amplitude oscillations occurring when the trajectory finally escapes from the region of periodic motion. In this paper we discuss another type of spontaneous switching which is a generalization of "crisis-induced intermittency" [5]. The latter has only been examined for parameter values close to the associated crisis, with an eye to finding scaling laws, but we have found that the topological structures involved can influence the dynamics over a wide range of parameter values. Furthermore, it has much more in common with transient dynamics and repellers (as we show below) than it does with the classic forms of intermittency (such as type I); for this reason, and to avoid confusion, we prefer to call the phenomenon "transient periodicity".

After describing the phenomenology of transient periodicity, we discuss the topology underlying it, building on the results from ref. [5]. We will dwell at some length on the mechanism by which a periodic episode begins, for this has not been discussed in the literature, and indeed it offers some interesting implications for how we conceive of chaotic attractors in dissipative systems. We will conclude with a discussion of empirical applications of transient periodicity. For conceptual simplicity, the theoretical section will be restricted to two-dimensional maps, but the results are generally applicable to higher dimensional maps and flows. Similarly, we will only consider periodic orbits arising from saddle-node bifurcations, but under the appropriate conditions, other sorts of periodic orbits may produce the same phenomenon.

Transient periodicity is characterized by spontaneous switching between apparently periodic and chaotic dynamics (fig. 1). The "periodic" dynamics are not truly periodic, but are modulated by small chaotic oscillations. Depending on the parameter values of the system, the proportion of the time series exhibiting periodic dynamics can vary tremendously: near the crisis the periodic component constitutes nearly the entire time series, while at the other extreme the time series in almost entirely chaotic, with only rare periodic bursts. For a fixed parameter value, the frequency distribution of the number of iterates spent in a given periodic episode can be approximated by a negative exponential function [6], although the frequency of short episodes can deviate from this: the latter is governed by the way in which the trajectory is injected into the region of transient periodicity, which is a global feature of the system, whereas the former is governed by the local topology. The periodic dynamics are localized in n (where n is the period) distinct regions in the phase space among which the trajectory moves in order (fig. 2). Because the system is deterministic, the last few iterates before the onset of a periodic episode (what we shall call the preimages of the episode) always lie within a well-defined region of the phase space.

We now turn to the topology that underlies this phenomenon. Consider a continuous diffeomorphism f_λ: $\mathbb{R}^2 \to \mathbb{R}^2$ which generates a chaotic attractor for some set of parameter values λ. Furthermore, let there be a hyperbolic fixed point p_0 in the attractor, and let f_λ generate a "protohorseshoe" [7]: there is a region A containing the attractor whose image under f_λ is folded, with $f_\lambda(\mathrm{A}) \subset \mathrm{A}$. For other values of λ there is a true horseshoe [8], in which only a fractal subset of A remains in A under repeated iteration of f_λ. A well-studied example, which we use to generate our illustrative figures, is the Hénon map [9]: $f_{a,b}(x, y) = (1-ax^2+y, bx)$ (fig. 2). We fix $b=0.3$ and allow a to vary as the control parameter.

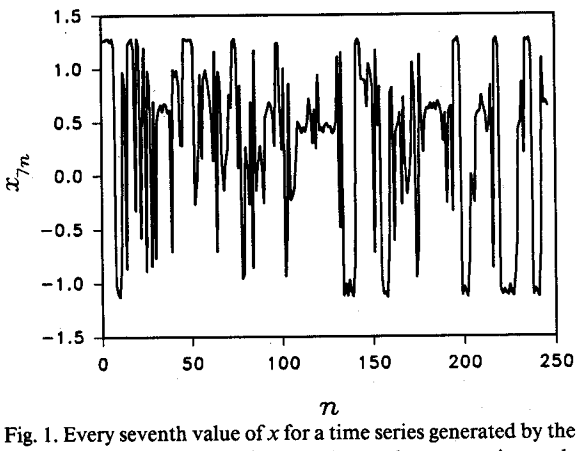

Fig. 1. Every seventh value of x for a time series generated by the Hénon map with $a=1.282$, $b=0.3$. The nearly constant intervals represent episodes of transient periodicity.

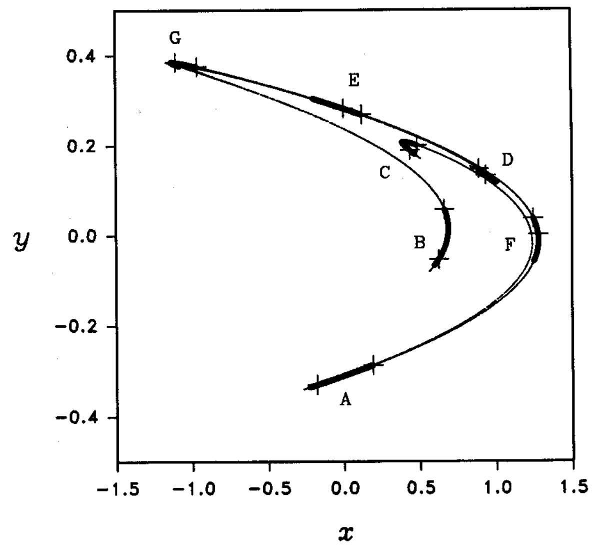

Fig. 2. The Hénon attractor with $a=1.282$, $b=0.3$. Points in the neighborhood of the "period-seven" semi-attractor (see text) are darkened. The letters indicate the order in which the trajectory visits the various segments of the semi-attractor; crosses mark the location of the two nonstable period-seven orbits.

As λ is varied to change the protohorseshoe to a horseshoe (without loss of generality, assume that this transition occurs by increasing λ), there is an infinite sequence of saddle-node and period-doubling bifurcations [10]; a sample "periodic window" is shown in fig. 3. The initial saddle-node bifurcation creates a period-n orbit; as λ is increased, there is a sequence of period-doubling bifurcations creating orbits of period $2n$, $4n$, $8n$, ... Beyond the accumulation point of the period-doubling bifurcations (marked III in fig. 3), the attractor is made up of n distinct pieces. The trajectory hops among the pieces in the same order as it does among the points on the periodic orbit p_n at II, but the motion within each piece is chaotic. This is called a "semiperiodic" attractor [11]. At the right end of the window, there is an "interior crisis" [7], in which each piece of the

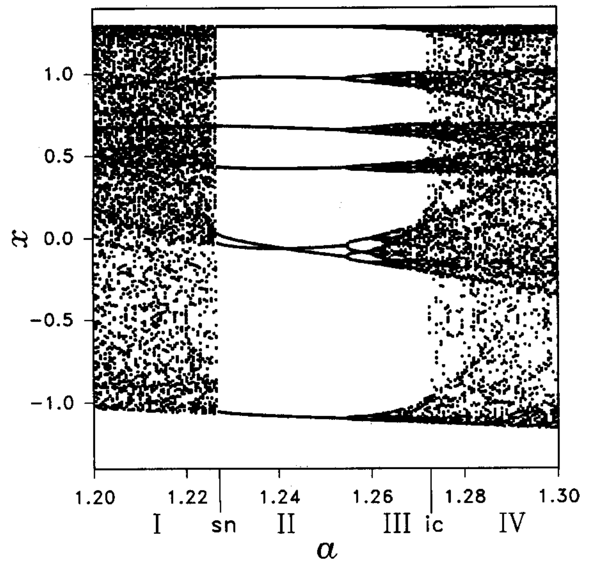

Fig. 3. Part of a bifurcation diagram for the Hénon map with $b=0.3$, showing a period-seven window. sn: saddle-node bifurcation; ic: interior crisis; I–IV delineate the different dynamical regimes discussed in the text.

semiperiodic attractor intersects the stable manifold of the associated nonstable period-n saddle orbit, s_n (compare figs. 4c and 4d). At this point in parameter space, λ_{cn}, a horseshoe is formed for f^n_λ, the nth composition of f_λ, and for $\lambda > \lambda_{cn}$, this horseshoe forms an escape hatch through which trajectories leave the neighborhood of what had been the semiperiodic attractor. The semiperiodic invariant set, which is organized around all the nonstable period kn orbits, $k \in \mathbb{N}$, is now a repeller, and indeed a special type of repeller called a "semi-attractor" [12], as we shall see below. Because the horseshoe has a well defined location in the phase space, there is a constant mean probability of escape per iteration, determined by the size and location of the escape hatch and the invariant probability distribution on the repeller; this is what gives rise to the exponential decay of residence times.

Once the trajectory leaves the vicinity of the semiperiodic repeller, its motion is governed by the large amplitude chaotic attractor that "reappears" when the periodic window ends. The reappearance of the chaotic attractor is not as magical as it might appear from examining bifurcation diagrams like fig. 3: the topological structures that make up the region I attractor are still present in regions II and III. These structures are merely unstable, with almost all trajectories ultimately going to the periodic or semiperiodic attractor. The topology governing the large amplitude fluctuations is little changed in the transition from III to IV: almost every trajectory in region IV eventually lands on the semiperiodic object, just as in III. Indeed the only change as λ passes through λ_{cn} is the fate of the trajectory after it lands on the semiperiodic object. Thus we can say that the semiperiodic object in IV has attracting as well as repelling "directions", and call it a semi-attractor (it has also been called a "chaotic saddle" [13]). To show how the attracting part of the semi-attractor works, we will first discuss the mechanism by which the chaotic transient decays in the periodic window and then show how this extends to region IV.

The easiest way to understand this mechanism is in terms of manifolds. A chaotic attractor contains, and to a large extent is defined by, the unstable manifolds of p_0, $W^u(p_0)$. The stable manifolds of p_0, $W^s(p_0)$, intersect the unstable manifolds infinitely many times, in a homoclinic tangle: it is the folded structure of the stable manifolds that determines how the trajectory hops from point to point on the unstable manifold, and governs the rate of separation of nearby trajectories. These intersections are not, in general, dense on $W^u(p_0)$ (fig. 4a). In particular, when the system is in a periodic window with a stable periodic orbit p_n there is a neighborhood N of p_n which the stable manifold does not penetrate. This follows from the fact that any trajectory that is precisely on $W^s(p_0)$ must ultimately approach p_0, whereas points in N are governed by the local linear dynamics around p_n: *all* trajectories in N ultimately approach the periodic orbit. In fact, this local basin of attraction is defined by the stable manifold of s_n (fig. 4b). The basin is not fully enclosed: it has one or more "tails" that fold around and intersect $W^u(p_0)$. Points in this first intersection will map, under n iterations, to N. As the tail is followed out further, it folds and intersects the unstable manifold repeatedly, so that the union of these intersections is dense on $W^u(p_0)$. A typical trajectory hops from tail to tail until it arrives in the local basin, at which point the local dynamics of p_n take over. At II this is monotonic convergence to p_n; at III, where p_n has gone through its period-doubling sequence, the local dy-

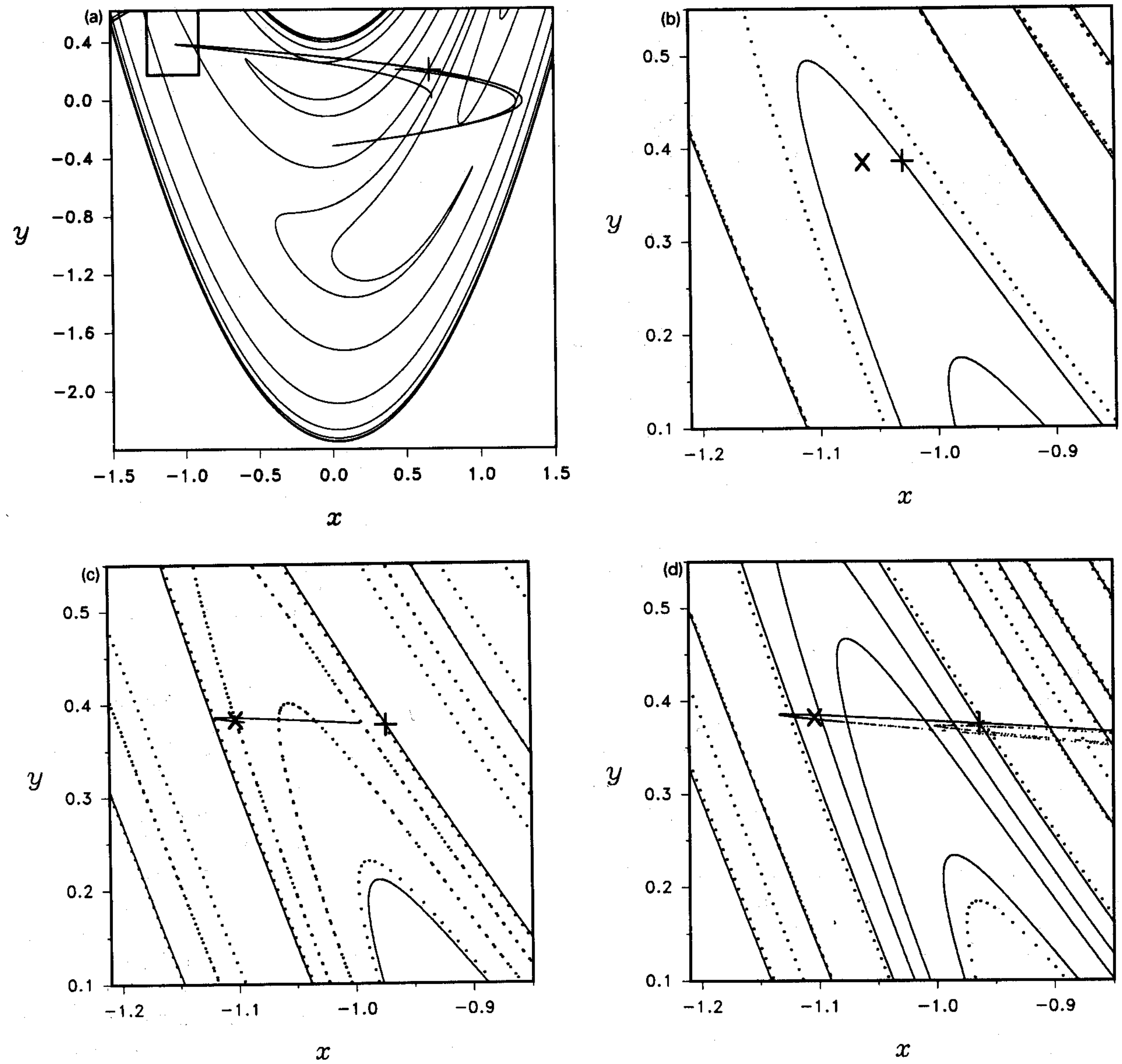

Fig. 4. Manifold structure in the vicinity of a period-seven orbit in the Hénon map, $b=0.3$. (a) Before the saddle-node bifurcation, $a=1.22$ (I): dark line: $W^u(p_0)$; light line: $W^s(p_0)$. The box marks the region of the phase space shown in (b)–(d). (b) Stable periodic orbit, $a=1.23$ (II): (×) p_7; (+) s_7; solid line: $W^s(s_7)$; dotted line: $W^s(p_0)$. (c) Semiperiodic attractor, $a=1.27$ (III); solid line: $W^s(s_7)$; dotted line: $W^s(p_7)$. (d) Transient chaos, $a=1.28$ (IV); solid line: $W^s(s_7)$; dotted line: $W^s(p_0)$.

namics are determined by $W^s(p_n)$ and $W^u(p_n)$ (fig. 4c).

Now let us examine the situation in region IV, past the interior crisis. The neighborhood N of p_n (defined by the primary lobe of $W^s(s_n)$) still exists (fig. 4d), and the associated tails still govern the approach of the trajectory to N (fig. 5). However, at the crisis there is not one but an infinite number of heteroclinic tangencies (tangential intersections between $W^u(p_n)$ and $W^s(s_n)$) involving secondary lobes of $W^s(s_n)$ (this follows from the fact that $\{W^u(p_n) \cap W^s(s_n)\}$ is an invariant set, and all it-

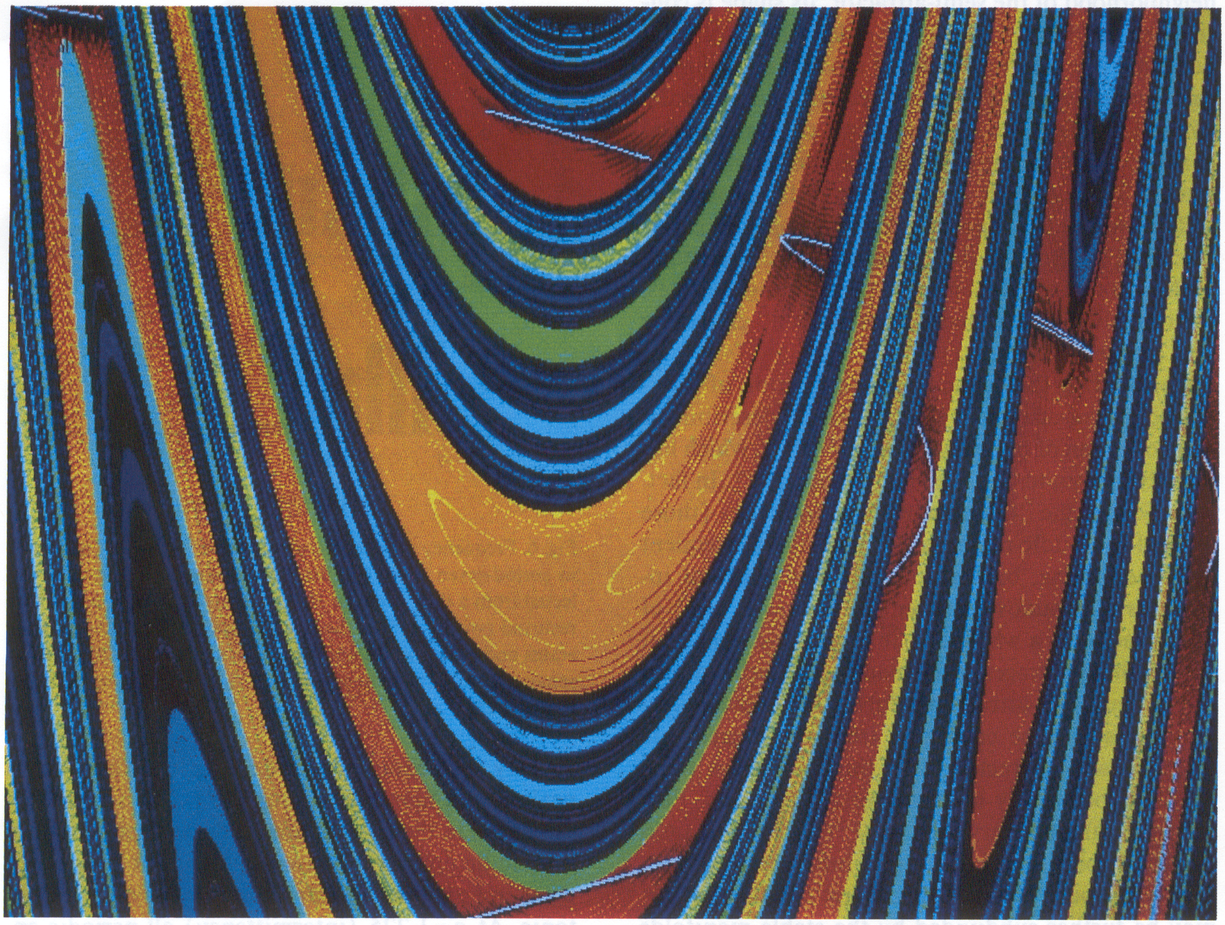

Fig. 5. Preimages of the period-seven semi-attractor in the Hénon map; $a=1.28$. The white lines delineate the semi-attractor, and the colors indicate the number of iterates required to reach the semi-attractor: dark red represents 1, purple represents 14, and intermediate values follow the spectrum. Points requiring more than 14 iterates are left black. By including preimages from off the attractor, the "tails" defined by the stable manifolds (see text) are clearly visible.

erates of one point of intersection must also be points of intersection). The first (and largest) such secondary lobe is shown in fig. 4d. Points below this lobe are in a sense "outside" the region bounded by $W^s(s_n)$, and so are subject to the influence of $W^s(p_0)$ and eventually escape to the rest of the attractor. If we define E to be the union of all regions outside the secondary lobes, we find that $E \cap N$ is dense on $W^n(p_n)$, so almost every trajectory eventually escapes from N.

Let us review what we have. A trajectory starting near p_0 will have the following evolutions for the various regimes labeled in fig. 3:

(I) chaos [attractor];

(II) chaos [repeller]→periodicity [attractor];

(III) chaos [repeller]→semiperiodicity [attractor];

(IV) chaos [?]→semiperiodicity [repeller]→chaos→periodicity→....

What should we call the chaotic object in IV? It is clearly not an attractor, for trajectories escape from it just as they do in II. The actual attractor is the union of the chaotic and semiperiodic sets. But the "chaotic attractor" in I is itself the union of all the semiperiodic repellers formed in earlier saddle-node bifurcations. While $W^s(p_0)$ does extend into N, the neighborhood of the semi-attractor, its shape is constrained by $W^s(s_n)$. A trajectory landing in N must negotiate the labyrinth defined by $W^s(s_n)$ before it can escape from N. For many practical purposes the dynamics within the semi-attractor can be regarded as independent of the dynamics on the rest of the attractor. Thus, at least in the region of the proto-horseshoe, where the rigorous concepts of strange attractors [14] are not fully applicable, we may think of the attractor as the union of these strange saddles, each of which has a "basin of attraction" extending into all the others.

Considering the attractor as a collection of semiperiodic semi-attractors is analogous to describing it as a collection of nonstable periodic orbits, which has already proven to be a useful approach [15]. In fact, its picks out the "fundamental" periodicities (those formed by saddle-node, rather than the subsequent period-doubling, bifurcations). The dynamical invariants (Lyapunov exponents, dimension, etc.) of the attractor as a whole are, in ergodic systems, just the weighted averages of the measures associated with the various semi-attractors. Since some of these local invariants may be estimated from the escape rate from the semi-attractor [16], this may prove to be a rather robust way to characterize attractors. Furthermore, the primary lobe of $W^s(s_n)$ may be used to crudely partition the attractor into regions of "period-n" versus "chaotic" dynamics; the chaotic part may be further subdivided by the stable manifolds of other periodic orbits that arose in other saddle-node bifurcations. A symbolic dynamics can then be constructed, based on the dominant semiperiodic motions. Among other things, one could then construct a matrix of transition probabilities among the various periods, both as a characterization of the attractor and as a useful tool for prediction and control of the system.

If transient periodicity is to be useful in an empirical setting, we must be able to distinguish it from the various types of intermittency and externally induced transitions outlined in the introductory paragraph. Perhaps of most interest is the contrast with type I intermittency, which exhibits two major phenomenological differences from transient periodicity. The first is the distribution of residence times (fig. 6): in contrast with the exponential tail of transient periodicity, type-I intermittency is characterized by a maximum episode length, with a strong peak at that maximum value [17]. The second difference is in the modulation of the trajectory within a periodic episode: whereas in transient periodicity these are chaotic, in intermittency they are monotonic. At $a=1.225$ (intermittency) all periodic episodes of length greater than five orbits have a significant positive correlation of x with iterate number; at $a=1.28$ (transient periodicity) there are no such significant correlations. The former distinction is robust to observational noise (fig. 7); however, the differences are erased by only modest levels of "dynamical noise", or perturbations to the system. What do we mean by "modest"? Recall that in the periodic window, if the trajectory is perturbed sufficiently far from the periodic or semiperiodic attractor, then it leaves the local domain of attraction and moves on the chaotic semi-attractor for a while before returning to the attractor. It turns out that as the magnitude and/or the frequency of the noise is increased, there is a rather abrupt transition from chaotic excursion being rare to chaotic excursions being frequent [18]. The precise location of this transition depends of course on the details of the topology, but

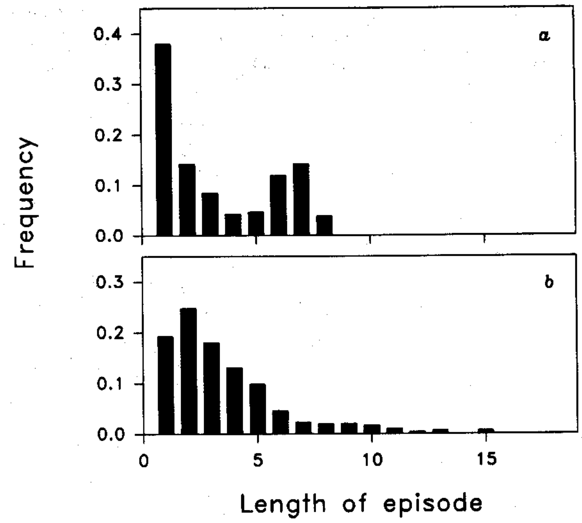

Fig. 6. Frequency distribution of "periodic" episodes (measured in period-seven orbits) in the Hénon map. Episodes were collected from a time series of 10 000 points by finding points that recurred (within 15% of the size of the attractor) after seven iterates and counting the number of such points that followed in sequence. (a) Intermittency; $a=1.225$. (b) Transient periodicity: $a=1.28$.

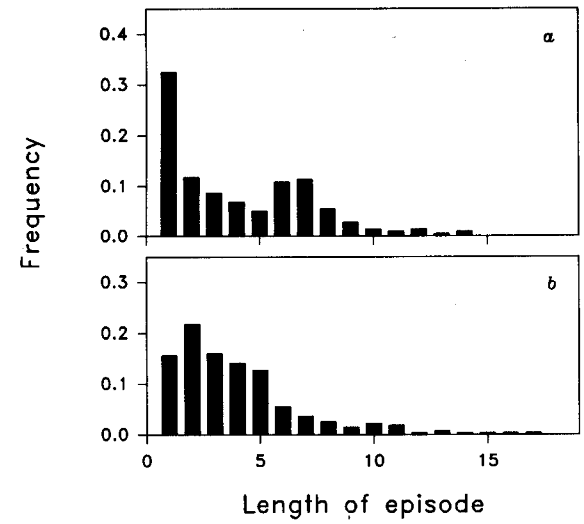

Fig. 7. Same as fig. 6, but with simulated observational noise. The time series was corrupted by adding random variates drawn from a Gaussian distribution with zero mean and 0.05 standard deviation.

typically occurs for noise magnitudes less than one percent of the size of the attractor. When there is this much noise in the system, then the escape rate from transient periodic episodes is dominated by the noise-driven perturbations across the bounding manifolds, rather than the escape hatches discussed above. In intermittency, there is not such a well defined boundary in the phase space to mark the region of periodicity, but there is region of near recurrence; again in the presence of these modest noise levels, the escape rate from the region of recurrence is dominated by the noise rather than the topology. Nevertheless, there are many experimental examples of intermittency in which the monotonic trend of the trajectory through a periodic episode is quite clear, suggesting that the noise level will often be low enough to clearly distinguish between the two phenomena.

In a bifurcation phenomenon the distribution of residence times will depend as much on the dynamics of the varying parameter as on the dynamics of the system, and no a priori predictions can be made. Depending on the nature of the bifurcation, there may or may not be consistent localization in the phase space of the transitions between regimes. Furthermore, invoking a bifurcation introduces a whole new layer of complexity to one's understanding of the system, and should probably be avoided unless there is direct evidence that a parameter is varying in concert with the dynamical shifts.

How would one go about using this in empirical applications? The easiest situation is when there is a relatively simple model which gives qualitatively similar dynamics as the data. The bifurcation structure of the model can be examined for semiperiodic attractors that correspond (in the phase space) to the periodic episodes found in the data. This is the approach taken in ref. [19] to analyze measles epidemics, which in some cities show episodes of biennial outbreaks interspersed with more irregular dynamics. In the model, the biennial dynamics correspond to a four-piece semiperiodic attractor which is destabilized when it collides with a large amplitude chaotic repeller; the trajectory subsequently switches back and forth between the "periodic" and chaotic regimes.

When the bifurcation structure of the system cannot be determined either experimentally or numerically, more inferential techniques must be used. If there are enough periodic episodes, the exponential distribution of residence times can be sought. As an example of this approach, consider epilepsy. Some variants of this disease are good candidates for analysis in terms of transient periodicity because the seizures, which are characterized by a highly coherent signal in the brain electrical activity, seem to begin and end spontaneously. Some patients have multiple seizures in a short time period, especially during sleep, so it should be possible to construct a frequency distribution of seizure lengths. Another approach, if the data are well enough resolved and the dimension of the underlying attractor is not too high, is to examine the preimages of the periodic regions to find evidence of the "tails" discussed above.

Determining the mechanisms underlying the switching between periodic and chaotic dynamics is important for understanding the system under study, but it is critical from a prediction and control standpoint. For example, suppose that the chaotic episodes were undesirable, and one wished to restrict the system to the periodic regime. If the system were exhibiting perturbations onto a chaotic repeller one would use very different techniques (noise reduction) than if it were subject to transient periodicity (change the control parameter to bring the system into the periodic window). We feed that the concepts of transient periodicity may be useful in a wide variety of such applications.

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