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Inverse Problems for Hyperbolic PDEs and Lorentzian Manifolds

A dissertation submitted in partial satisfaction of the requirements

for the degree Doctor of Philosophy

in

Mathematics

by

Yuchao Yi

Committee in charge:

Professor Ioan Bejenaru, Chair

Professor Bo Li

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Professor William Young

Professor Andrej Zlatoš

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The Dissertation of Yuchao Yi is approved, and it is acceptable in quality and form for publication on microfilm and electronically.

University of California San Diego

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DEDICATION

In memory of my mother.

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Chapter 4, in full, has been submitted for publication as it may appear in joint work with Yang Zhang. The dissertation author was the primary researcher and author of this material.

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ABSTRACT OF THE DISSERTATION

Inverse Problems for Hyperbolic PDEs and Lorentzian Manifolds

by

Yuchao Yi

Doctor of Philosophy in Mathematics

University of California San Diego, 2026

Professor Ioan Bejenaru, Chair

This thesis studies inverse problems for hyperbolic equations and Lorentzian manifolds, with the goal of recovering coefficients of a PDE or the geometry of an unknown region from boundary measurements. It develops microlocal and geometric methods adapted to Lorentzian settings. The first part concerns inverse problems for hyperbolic PDEs, including nonlinear and partial data problems; the second part concerns geometric rigidity questions for Lorentzian manifolds using time separation and scattering data.

We first study a bounded time inverse scattering problem for a semilinear Dirac equation in $(0,T)\times\mathbb{R}^3$ with smooth nonlinearity $F(x,z)$. We prove that the solution map from initial data at $t=0$ to the solution at time $t=T$ determines $F(x,z)$ for $x\in\mathbb{R}^3$ and $|z|\leq M$, under natural a priori knowledge of low-order derivatives of F at $z=0$. A key ingredient is a collision simulation mechanism that compensates for the absence of a physical boundary.

Next, we analyze the restricted Dirichlet-to-Neumann map $\Lambda_{g,A,q}^{U,V}$ for wave equations with electromagnetic potential on Lorentzian manifolds with timelike boundary, where the source and observation sets are disjoint. Using gliding rays, we recover the conformal class of the boundary metric, the tangential component of the potential, and boundary jets up to the natural gauge at recoverable boundary points. If the metric and time orientation are known on one of the boundary sets, we obtain recovery on a larger part of the boundary.

We then introduce a variation of the Calderón problem in which one assumes access to a family of Dirichlet-to-Neumann maps associated with an open set of magnetic or electromagnetic perturbations. In the Riemannian case this yields uniqueness of the metric without gauge equivalence; in the Lorentzian case generic perturbations allow recovery of null geodesic trajectories and hence the conformal class.

Finally, we prove rigidity results for real analytic Lorentzian manifolds from time separation and scattering data, showing that these data determine the manifold up to analytic isometry under suitable non-trapping and lightlike cut point assumptions.

Chapter 1

Overview and main results

In Section 1.1, we give a quick overview of the field of inverse problem, and go over some of the most important inverse problems in Section 1.1.1 and Section 1.1.2. The examples start with elliptic and Riemannian setting, which paves the road for the more complicated hyperbolic and Lorentzian setting. We then summarize the main results of this thesis in Section 1.2.

1.1 Overview of inverse problems

Inverse problems study the recovery of hidden parameters or geometry from different types of indirect observations. A common setting is to view the forward model as a map

$$\mathcal{F}: \text{ unknown coefficients/geometry} \longrightarrow \text{measured data},$$

and to ask the following questions:

1. **(Uniqueness)** Is $\mathcal{F}$ injective?

2. **(Reconstruction)** How to reconstruct the unknown?

3. **(Stability)** How do measurement errors in the data affect the reconstruction?

Unlike the forward problem, inverse problems are often ill-posed because non-uniqueness is built into many natural measurement setups. A particularly important source of non-uniqueness is the presence of *gauge equivalences*: transformations of the unknown that leave the forward model and the data invariant. Typical examples include changes of coordinates: two descriptions related by a diffeomorphism may represent the same underlying physical medium or spacetime, so boundary measurements cannot distinguish between them. In other words, the difference of the reconstruction results may simply be caused by the choice of different coordinate systems. Thus, a basic task in inverse problems is to identify the correct equivalence class of unknowns determined by the data and to prove recovery only modulo these natural gauges.

In this thesis, we roughly divide inverse problems into *PDE-based inverse problems* and *geometric inverse problems*. In PDE-based inverse problems, the unknown may be a coefficient, potential, or nonlinearity in a partial differential equation, and the data are typically boundary measurements or scattering data. In geometric inverse problems, the unknown is usually a geometric structure, for example the metric or the causal structure, and the data are encoded by geodesic dynamics, distance functions, or some types of integrated information. These questions are motivated by applications in medical imaging, geophysics, physics, and problems occurred in many other fields.

1.1.1 The Calderón problem

The most iconic PDE-based inverse problem is electrical impedance tomography (EIT), also known as the Calderón problem. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain representing a conductor, and let $\gamma \in C^\infty(\overline{\Omega})$ be the conductivity distribution (we may assume it is strictly positive). Given a boundary voltage (Dirichlet data) f, the electric potential u is modeled by

$$\nabla \cdot (\gamma \nabla u) = 0 \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = f. \tag{1.1.1}$$

The induced boundary current (Neumann data) is $\gamma \partial_\nu u|_{\partial\Omega}$, and the voltage-to-current map, also known more generally as the Dirichlet-to-Neumann (DN) map, is defined by

$$\Lambda_\gamma : f \mapsto \gamma \partial_\nu u|_{\partial\Omega}. \tag{1.1.2}$$

The Calderón problem asks whether the boundary measurements Λ_γ determine the conductivity γ.

The mathematical study of this problem began with the foundational note of Alberto Calderón, who formulated this inverse boundary value problem of determining an unknown conductivity from the associated Dirichlet-to-Neumann map in [22][1]. An important breakthrough was made by Sylvester and Uhlmann in [141], where they proved global uniqueness for smooth isotropic conductivities in dimensions $n \geq 3$. More importantly, they introduced Complex Geometric Optics (CGO) solutions, establishing the modern analytic framework for the subject. In two dimensions, a major advance was achieved by Nachman, who proved global uniqueness together with a constructive reconstruction method for smooth conductivities in [110].

So far, the conductivity has been a function in the region, indicating that the conductivity does not depend on the specific direction of the current. We refer to such setting as *isotropic*. However, in many physical media the conductivity is *anisotropic*, meaning that the current responds differently to electric fields in different directions. Mathematically one replaces the scalar conductivity $\gamma(x)$ by a symmetric, positively determined matrix field $\sigma(x) \in \mathbb{R}^{n\times n}$ and considers

$$\nabla\cdot(\sigma\nabla u) = 0 \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = f, \tag{1.1.3}$$

with the corresponding Dirichlet-to-Neumann map

$$\Lambda_\sigma f = \nu\cdot\sigma\nabla u|_{\partial\Omega}.$$

A fundamental feature of the anisotropic problem is a natural gauge invariance: if $\Phi : \Omega \to \Omega$ is a diffeomorphism with $\Phi|_{\partial\Omega} = \mathrm{Id}$, then the pushforward conductivity $\Phi_*\sigma$ produces the same boundary measurements, i.e. $\Lambda_{\Phi_*\sigma} = \Lambda_\sigma$. This suggests that one can at best hope to recover σ up to such boundary-fixing change of variables. Thus the *anisotropic Calderón problem* asks whether the boundary measurements Λ_σ determines the anisotropic conductivity σ up to boundary-fixing diffeomorphisms.

When the dimension n is at least 3, a useful geometric reformulation is obtained by writing (1.1.3) as a Laplace equation for a Riemannian metric. Indeed, if we set $\sigma = |g|^{1/2}g^{-1}$, then one can interpret

[1] A reprinted version is [11].

(1.1.3) as an equation of the form

$$\Delta_g u = 0, \quad u|_{\partial\Omega} = f,$$

where $g = |\sigma|^{1/(n-2)}\sigma^{-1}$ is well-defined since $n \geq 3$ and Δ_g is the Laplace-Beltrami operator. In this case, we denote the DN map as Λ_g. The anisotropic Calderón problem can thus be generalized to Riemannian manifolds, which we refer to as the *Riemannian Calderón problem.*

Conjecture 1.1.1 (Riemannian Calderón Problem)**.** *Let (M_1, g_1) and (M_2, g_2) be two n-dimensional Riemannian manifolds with boundary for $n \geq 3$, such that $\partial M_1 = \partial M_2$. If $\Lambda_{g_1} = \Lambda_{g_2}$, then there exist a boundary-fixing diffeomorphism $\Phi : M_1 \to M_2$ such that $g_1 = \Phi^* g_2$.*

Remark 1.1.2. *The isotropic conductivity γ is a special case in the sense that the corresponding metric $g = \gamma^{2/(n-2)}\delta$ is conformally flat with respect to the standard global coordinate system in $\mathbb{R}^n$. This also explains why isometry gauge (that is, determine only up to boundary-fixing diffeomorphism) does not show up in the isotropic case. Indeed, for a fixed open region $\Omega \subset \mathbb{R}^n$ and $n \geq 3$, if $\gamma_1\delta = \Phi^*(\gamma_2\delta)$ for some boundary-fixing diffeomorphism Φ on Ω, then by Liouville's Theorem it must be a Möbius transformation (everything is assumed to be smooth here). Boundary-fixing then implies Φ is the identity map. In other words, isotropic Calderón problem for $n \geq 3$ says if two metrics are conformally flat with respect to a common global coordinate system (in particular they are in the same conformal class), then same DN maps implies the metrics are the same.*

The Riemannian Calderón problem is still widely open. All the results so far for $n \geq 3$ either assumes analyticity [98, 94], or has a priori assumption that the two metrics belong to the same conformal class that satisfies certain structural assumptions [38] (including the isotropic case previously mentioned).

Even though the formulation in Conjecture 1.1.1 seems to work for $n = 2$ as well, there are some important distinctions between $n = 2$ and $n \geq 3$. Recall that when the underlying manifold is a fixed subregion of $\mathbb{R}^n$ for $n \geq 3$, the Riemannian interpretation is equivalent to the conductivity formulation via $\sigma = |g|^{1/2}g^{-1}$. However, when $n = 2$ and g_1 and g_2 belong to the same conformal class, straightforward computation shows $|g_1|^{1/2}g_1^{-1} = |g_2|^{1/2}g_2^{-1}$. This not only means they correspond to the same conductivity, but also implies that $\Delta_{g_1} u = 0$ if and only if $\Delta_{g_2} u = 0$. Moreover, if the conformal factor relating the two metrics is 1 on the boundary, then $\Lambda_{g_1} = \Lambda_{g_2}$. As a result, 2-dimensional Riemannian Calderón problem has an extra conformal gauge comparing to the higher dimensional case. The 2d Riemannian Calderón problem has been proved in [94].

Theorem 1.1.3 ([94] Theorem 1.1)**.** *Let (M, g) be a 2-dimensional Riemannian surface with boundary. Then Λ_g determines the conformal class of the metric.*

Remark 1.1.4. *In [94], they actually proved a stronger result than stated here. Let $\Gamma \subset \partial M$ be any non-empty open set, and suppose the Dirichlet data is only supported in Γ. They proved that the restricted DN map, namely the DN map that only takes in Dirichlet data supported in Γ, determines the conformal class of the metric. The proof relies heavily on the fact that any 2-dimensional Riemannian surface is locally flat via harmonic coordinates, which does not hold for higher dimensions, see also Remark 1.1.2.*

There is a natural generalization of the Calderón problem to the Lorentzian setting (for basic Lorentzian geometry we refer to Section 2.1). Specifically, if we consider a Lorentzian manifold (M, g) with timelike boundary of dimension $n + 1$, the wave equation is well-defined as

$$\Box_g u = 0, \quad u|_{\partial M} = f, \quad \mathrm{supp}(u) \subset J^+(\mathrm{supp}(f)),$$

where $\Box_g$ is the Laplace-Beltrami operator, and J^+ denotes the causal future. The DN map is similarly defined as

$$\Lambda_g : f \mapsto \partial_\nu u|_{\partial M}.$$

Clearly, any boundary-fixing diffeomorphism Φ is a gauge, namely $\Lambda_{\Phi^* g} = \Lambda_g$, and we similarly have the following *Lorentzian Calderón problem.*

Conjecture 1.1.5 (Lorentzian Calderón problem)**.** *Let (M_1, g_1) and (M_2, g_2) be two $n + 1$-dimensional Lorentzian manifolds with timelike boundary for $n \geq 2$, such that $\partial M_1 = \partial M_2$. If $\Lambda_{g_1} = \Lambda_{g_2}$, then there exist a boundary-fixing diffeomorphism $\Phi : M_1 \to M_2$ such that $g_1 = \Phi^* g_2$.*

Unlike the Riemannian setting, there has been substantial progress for the Lorentzian Calderón problem. Historically, the first major conceptual input on the hyperbolic side is the *boundary control method*, introduced by Belishev [17] and developed with Kurylev to reconstruct a Riemannian manifold from boundary wave data [19]; in particular, for *ultra-static* spacetimes $(\mathbb{R} \times M_0, -dt^2 + g_0)$ this yields recovery of the spatial metric g_0 (hence the Lorentzian metric) from the hyperbolic DN map, modulo the usual boundary-fixing coordinate gauge. Beyond such static settings, a key obstruction is that sharp unique continuation generally fails for arbitrary Lorentzian geometries, and early uniqueness results therefore imposed analyticity in the time variable: Eskin proved that if the lower order terms depend real-analytically on time (and under a geometric control condition), the time-dependent DN map on part of the boundary determines the coefficients up to diffeomorphisms and gauge transformations [42]. More recently, Alexakis, Feizmohammadi and Oksanen were able to remove time-analyticity by proving a new unique continuation result in the exterior of double null cones and deriving uniqueness under spacetime curvature bounds (and the a priori assumption that the metrics are in a fixed conformal class) [5]; they subsequently strengthened this framework to include perturbations near Minkowski and weaker curvature hypotheses [6]. Finally, Oksanen, Rakesh and Salo proved the following: if a globally hyperbolic metric agrees with Minkowski outside a compact set and has the same hyperbolic DN map as Minkowski, then the metric must be Minkowski up to a boundary-fixing diffeomorphism [111].

In this thesis, for PDE-based inverse problems, we will mainly focus on hyperbolic PDEs on Lorentzian manifolds, which are all closely related to the Lorentzian Calderón problem. For more details on the Calderón problem, we refer to surveys [144, 125] and the book [47]. This inverse problem has become a prototype for determining coefficients and geometry from boundary measurements, which motivates a wide class of inverse problems for various other PDEs.

1.1.2 The boundary rigidity problem

Let (M, g) be a compact smooth Riemannian manifold with nonempty boundary ∂M. For $x, y \in M$, denote by $d_g(x, y)$ the Riemannian distance between x and y measured by g, and consider the *boundary distance function*

$$d_g\big|_{\partial M \times \partial M} : \ \partial M \times \partial M \to \mathbb{R}^{\geq 0}.$$

The *boundary rigidity problem* asks whether $d_g|_{\partial M\times\partial M}$ determines the metric g in the interior. As in many geometric inverse problems, there is an intrinsic gauge equivalence: if $\Phi : M \to M$ is a boundary-fixing diffeomorphism, then the pullback metric $\Phi^* g$ has the same boundary distance function, since lengths of curves with fixed endpoints on ∂M are preserved under such a boundary-fixing change of coordinates. Thus, we say a Riemannian manifold is *boundary rigid* if the boundary distance function determines it up to boundary-fixing isometry.

Unfortunately, there are many Riemannian manifolds that are not boundary rigid. For example, suppose Ω is a domain in $\mathbb{R}^n$ and $U \subset \Omega$ is some subregion in the interior. Consider a smooth cutoff function χ that is 1 on U and vanishes outside a small neighborhood of U. Equip Ω with the conformally flat metric $(1+C\chi)\delta$ for some large constant C. Then we have deliberately caused the geodesics to travel extremely slow inside U. In particular, if C is sufficiently large, then for any two boundary points $x, y \in \partial\Omega$, the shortest path between them should avoid U entirely. This example illustrates the fact that the boundary distance function may not contain information of the entire metric, simply because the path that achieved minimal distance may not pass through certain region: the information related to that region is trapped inside. As a result, there is of course no way of determining the metric in the trapped region.

A natural question is, what condition would then guarantee boundary rigidity? Michel proposed that the following special class of compact Riemannian manifolds should be boundary rigid in [102].

Definition 1.1.6. *A compact Riemannian manifold with boundary is simple if the following conditions are satisfied:*

1. *the boundary is strictly convex* [2]*;*

2. *there are no conjugate points;*

3. *geodesics are non-trapping (that is, any geodesic eventually leaves the manifold).*

Conjecture 1.1.7 (Michel [102])**.** *Simple Riemannian manifolds are boundary rigid.*

Michel's conjecture is solved in 2-dimension by Pestov and Uhlmann in [117]. Higher dimensional

[2] Strict convexity is said to hold at a boundary point p if for all sufficiently close $q \in \partial M$, there exists a geodesic from p to q in the interior (except the two end points) with length $d_g(p,q)$. The boundary is strictly convex if strict convexity holds at all boundary points.

case has been proved by Stefanov, Uhlmann and Vasy in [137] if (M, g) satisfies any of the following extra assumptions:

1. (M, g) has non-positive sectional curvature;

2. (M, g) has non-negative sectional curvature;

3. (M, g) has no focal points.

The conjecture in its most general form still remains open when $n \geq 3$.

A closely related problem is the *lens rigidity problem*, which we briefly state here. Let us consider a simple manifold (see Definition 1.1.6) (M, g) to avoid too many notations. Let $x \in \partial M$ be any boundary point and $v \in S_x M$ a unit length inward pointing tangent vector (that is, $g(v, \nu) < 0$ for the unit outward pointing vector ν). Because of the simplicity assumption, the geodesic with initial data (x, v) will eventually leave the manifold, at location $y \in \partial M$ with outward pointing direction $w \in S_y M$. The map $\mathcal{S}(x, v) = (y, w)$ is called the *scattering relation*. One may also record the length of the geodesic from (x, v) to (y, w) as $\ell(x, v)$, commonly known as the *length function*. Together, $(\mathcal{S}, \ell)$ is referred to as the *lens data* of (M, g), and the *lens rigidity problem* asks whether lens data uniquely determine the Riemannian manifold up to boundary-fixing isometry. In general, lens data include more information than the boundary distance function, because if $x, y \in \partial M$ are connected by more than 1 geodesics, then all of these will be captured by lens data while only the length of the shortest one will be recorded by the boundary distance function. On the other hand, the two types of data are equivalent for simple manifolds, see for example [56]. Lens data naturally appears in many other settings. As an example, consider a static Lorentzian metric, namely $(\mathbb{R} \times M, -dt^2 + g)$ with (M, g) being a compact Riemannian manifold with boundary. If a lightlike geodesic travels from $(t, x) \in \mathbb{R} \times \partial M$ in the direction of $(1, v)$ to $(s, y) \in \mathbb{R} \times \partial M$ in the direction of $(1, w)$, then the projection of this lightlike geodesic is a unit length geodesic in M connecting x and y. Moreover, the lens data is given by $\mathcal{S}(x, v) = (y, w)$ and $\ell(x, v) = |s - t|$.

Another type of problem that naturally arises from the boundary rigidity problem is the study of the *geodesic ray transform*. Consider a function f supported in a bounded domain Ω of $\mathbb{R}^n$. Suppose for any straight line γ passing through Ω, we denote the *X-ray transform* of f as

$$Xf(\gamma) = \int_\gamma f.$$

The inverse problem for the X-ray transform asks whether this operation is invertible, stable, and reconstructible. In medical imaging, the function f can be thought of as the density distribution of the patient's body, and the X-ray transform corresponds to the change of intensity of the X-ray after going through patient's body, which is affected by the density distribution along the trajectory of the X-ray. Mathematically, this can be considered on any compact Riemannian manifold with boundary, where the integral is with respect to any geodesic γ, and the corresponding operator

$$If(\gamma) = \int_\gamma f$$

is called the *geodesic ray transform*. It can be generalized to m-tensors as

$$I_m\omega(\gamma) = \int_\gamma \omega_{\gamma(s)}(\dot\gamma(s), \cdots, \dot\gamma(s))ds,$$

and the case $m = 2$ is closely related to the linearized boundary distance function. Specifically, let $g(\epsilon)$ be a smooth perturbation of the background metric $g(0) = g_0$, and for simplicity assume g_0 is simple. Then the linearized boundary distance function is precisely

$$\left.\frac{d}{d\epsilon}\right|_0 d_{g(\epsilon)}(x, y) = I_2h(\gamma) = \int_\gamma h_{\gamma(s)}(\dot\gamma(s), \dot\gamma(s))ds,$$

where h is the symmetric 2-tensor corresponding to the perturbation of $g(\epsilon)$ at $\epsilon = 0$, and γ is the unique geodesic with respect to g_0 connecting x and y [3].

There are many other geometric inverse problems, including scattering rigidity (whether scattering relation $\mathcal{S}$ uniquely determines the manifold), marked length spectrum rigidity (the analogue of lens rigidity problem on closed manifolds), and so on. For the current progress of the aforementioned problems as well

[3] For simple manifolds, any two boundary points are connected by a unique geodesic, see for example [56] for details

as other geometric inverse problems, we refer to [138, 56, 134, 115] for more details.

The references mentioned above mainly focus on Riemannian manifolds, and the corresponding problems for Lorentzian manifolds are much more complicated. For instance, in the Riemannian setting, it is well-known that the distance function induces a metric structure which agrees with the manifold topology. However, this never holds in the Lorentzian setting (for basic Lorentzian geometry we refer to Section 2.1). Consider (M, g) a (time-orientable) Lorentzian manifold with boundary with signature $(-, +, \cdots, +)$, the Lorentzian distance function, also known as the *time separation function*, is defined as follows: if $y \in J^+(x)$, then $d_g(x, y)$ is the sup of the length of all future pointing piecewise smooth causal curves from x to y; otherwise $d_g(x, y) = 0$. In other words, it measures the maximum proper time difference rather than the spatial distance between two causally related points. As a result, we shall refer to the Lorentzian boundary rigidity problem as the *boundary time separation rigidity problem*.

Similarly, the lens rigidity problem is also more restricted. To discuss the Lorentzian lens rigidity problem, we start from another special property of the Lorentzian metrics: the invariance of causal structure under conformal change. Specifically, if g_1 and g_2 are in the same conformal class, then γ is a lightlike geodesic with respect to g_1 if and only if it is a lightlike geodesic with respect to g_2 up to reparameterization, hence they share the same light cones. As a result, in many Lorentzian inverse problems, one can only obtain information related to lightlike geodesics (especially when conformal change is a gauge, namely one only expects to recover the causal structure rather than the full metric). The lens data for lightlike geodesics, however, is exactly the same as just the scattering relation itself, simply because the length of any lightlike geodesic is 0. In other words, a natural question to consider for the Lorentzian manifold is the *lightlike scattering rigidity problem*, which asks whether scattering relation for lightlike geodesics determine the conformal class of the metric. Correspondingly, the *light ray transform* refers to

$$L_m\omega(\gamma) = \int_\gamma \omega_{\gamma(s)}(\dot{\gamma}(s), \cdots, \dot{\gamma}(s))ds$$

where ω is an m-tensor and γ is a lightlike geodesic. The inversion of lightlike transform is also an important problem in the field of Lorentzian geometric inverse problems.

Details on previous results can be found in Chapter 6, where we also prove some rigidity results for analytic Lorentzian manifolds related to time separation function and scattering relations.

1.2 Main results

In this section we briefly summarize the main results in this thesis, with more detailed explanation in the introduction part of Chapter 3 - 6. For definitions related to Lorentzian geometry and microlocal analysis, we refer to Chapter 2.

1.2.1 Inverse problem for nonlinear Dirac equation

This part corresponds to my work in [156], published in *Inverse Problems*. The main result is Theorem 1.2.3.

We start with an inverse problem for a nonlinear hyperbolic equation. Indeed, for forward problems (that is, to prove well-posedness of solution given initial data), nonlinear equations are much harder than the linear version. For inverse problem, however, some nonlinear equations are actually much easier to deal with than the linear counterpart. Intuitively, nonlinearity allows the interaction of two waves, creating richer information for recovering the parameters of the physical system. Specifically, we present an inverse problem for semilinear Dirac equation.

The Dirac equation, introduced by Paul Dirac in [37], is fundamental in quantum mechanics and quantum field theory for describing fermions. Its nonlinear variants, which can be seen either as self-interacting spinors (e.g. the cubic term $\langle \beta u, u \rangle u$) or as reductions of the coupled Dirac-Maxwell (and ultimately Einstein-Cartan-Sciama–Kibble) systems, have attracted extensive mathematical study. In particular, extensive symmetry classifications and explicit solutions were developed by Fushchich and Zhdanov [50], while Esteban and Séré [44] proved the existence of stationary (time-harmonic) states via variational methods. A recent centenary review of Dirac-equation derivations appears in [127], and the cubic self-interaction induced by spacetime torsion in Einstein-Cartan-Sciama-Kibble gravity is discussed in [63]. Finally, the full Maxwell-Dirac coupling has been shown to be well-posed and structurally rich in the works of Chadam [24], Radford [119], Glassey and Strauss [52], and Sparber and Markowich [128].

In Chapter 3, we consider an inverse problem for 4 by 4 nonlinear Dirac equations of the following form

$$\begin{cases} i\partial_t u + i\alpha\cdot\nabla u - \beta u = F(x,u) \text{ on } (0,T)\times\mathbb{R}^3 \\ \\ u(0,x) = \phi(x) \end{cases} \tag{1.2.1}$$

where the notation mean the following:

1. $u : [0,T]\times\mathbb{R}^3 \to \mathbb{C}^4$ is the solution of (1.2.1), with the variables denoted by (t,x);

2. For $\xi\in\mathbb{R}^3$, $\alpha\cdot\xi = \sum_{j=1}^3 \xi_j\alpha_j$, and α_j are 4×4 Pauli matrices defined by

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \\ \sigma_j & 0 \end{pmatrix}, \sigma_1 = \begin{pmatrix} 0 & 1 \\ \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ \\ 0 & -1 \end{pmatrix};$$

3. β is the 4×4 matrix

$$\beta = \begin{pmatrix} I_2 & 0 \\ \\ 0 & -I_2 \end{pmatrix}$$

   and I_2 is the 2×2 identity matrix;

4. $F : \mathbb{R}^3\times\mathbb{C}^4 \to \mathbb{C}^4$ is a smooth nonlinearity with $F(x,0)=0$, and its partial derivatives of all orders are bounded on sets of the form $\mathbb{R}^3\times K$, where $K\subset\mathbb{C}^4$ compact. We denote the variables as (x,z).

We now phrase the main question. Assume that given any initial data $u(0)=\phi$, we are able to read the solution $u(T)$ at time T; in other words, we have complete knowledge of the map $u(0)\mapsto u(T)$. Does this uniquely determine the nonlinearity F? We will give an affirmative answer for data in the appropriate space of functions to be specified below.

To precisely state our main result, we first invoke a local existence result from [121], adapted to our setting.

Theorem 1.2.1. *([121] Theorem 6.3.1) If $s>3/2$, then there is a $T>0$ and a unique solution $u\in C([0,T];H^s(\mathbb{R}^3))$ to the semilinear initial value problem defined by the partial differential equation* (1.2.1) *together with the initial condition*

$$u(0,x) = \phi(x) \in H^s(\mathbb{R}^3).$$

The time T can be chosen uniformly for ϕ from bounded subset of $H^s(\mathbb{R}^d)$. Consequently, there is a $T^* \in]0,\infty]$ and a maximal solution $u \in C([0,T^*[; H^s(\mathbb{R}^d))$. If $T^* < \infty$, then

$$\lim_{t\to T^*} \|u(t)\|_{H^s(\mathbb{R}^d)} = \infty.$$

The above theorem allows us to define bounded time solution map for sufficiently small initial data.

Definition 1.2.2. *Given $s > 3/2$ and $\delta > 0$, denote*

$$H^s_\delta := \{\phi \in H^s(\mathbb{R}^3) : ||\phi||_{H^s} < \delta\}.$$

By Theorem 1.2.1 there exists some $T > 0$ such that for any initial data $\phi \in H^s_\delta$, (1.2.1) *has a unique solution u up to time T. Then the time T solution map is defined on H^s_δ by*

$$S(\phi) = u(T).$$

That is, given any sufficiently small initial data in H^s_δ, the time T solution map returns the solution at time T. We can now state the main result.

Theorem 1.2.3 (Y 25 [156])**.** *Given smooth nonlinearity F_1 and F_2, suppose for some $T > 0$ they each have a well-defined time T solution map S_1, S_2 on the same domain H^s_δ. Then $S_1 = S_2$ implies $\partial_z^3 F_1(x,z) \equiv \partial_z^3 F_2(x,z)$ for all $(x,z) \in \mathbb{R}^3 \times B_M$ where $B_M \subset \mathbb{C}^4$ is the closed ball centered at 0 with radius M, and*

$$M = \sup\{||\phi||_{L^\infty} : \phi \in H^s_\delta \cap C^\infty(\mathbb{R}^3)\}.$$

In addition, if for all $x \in \mathbb{R}^3$,

$$\partial_z F_1(x,0) \equiv \partial_z F_2(x,0), \quad \partial_z^2 F_1(x,0) \equiv \partial_z^2 F_2(x,0),$$

then $F_1(x,z) = F_2(x,z)$ for all $(x,z) \in \mathbb{R}^3 \times B_M$.

Although our result is proved for smooth small-data solutions, it applies directly to several physical models governed by semilinear Dirac equations. In Einstein-Cartan-Sciama-Kibble gravity, coupling the Dirac field to torsion produces exactly the cubic, four-fermion self-interaction studied here. In condensed matter, “Dirac materials” such as graphene or topological insulators, effective Dirac equations with nonlinear corrections capture electron-electron and electron-phonon interactions, so one could, at least in principle, use bounded time scattering data to recover the interaction law. Similar semilinear Dirac systems also arise in nonlinear optics as envelope equations for spinor-like wavepackets in photonic lattices.

We now briefly outline the general idea. Kerylev, Lassas and Uhlmann in [83] introduced a general framework of higher order linearization and microlocal analysis for nonlinear hyperbolic PDEs. They utilize the nonlinear wave interaction to create a collision point in the interior, which reveals the causal structure and can be tracked by microlocal analysis. However, a main difficulty in this chapter is that the forward operator is a *first-order, matrix-valued* hyperbolic system rather than the scalar wave equation. Microlocally, the principal symbol is non-diagonal and becomes singular on the characteristic set, so the symbols arising from higher-order linearization satisfy *matrix* transport equations along bicharacteristics, with initial data constrained to subspaces of $\mathbb{C}^4$ determined by the Dirac symbol. Moreover, unlike the scalar wave equation which has vanishing subprincipal symbol, the Dirac equation after linearization contains a genuinely non-trivial subprincipal symbol, which involves the unknown nonlinearity term. Since no special structural form of the nonlinearity $F(x, z)$ is assumed, it is subtle to extract pointwise information on F from a single wave interaction.

To overcome these problems, we construct a sequence of three-wave interactions concentrated along a fixed bicharacteristic whose interaction points approach the initial time slice $t = 0$. In the limit, this produces an effective boundary-type interaction, allowing one to read off the relevant transported interaction symbol at time T from the bounded-time solution map $u(0) \mapsto u(T)$. Analyzing the resulting leading-order term and its matrix structure yields recovery of the relevant derivatives of F (and hence F on the stated region, up to the natural low-order ambiguity).

1.2.2 Partial data inverse problem with disjoint source and observation sets

This part corresponds to a joint work with Yang Zhang in [158]. It is submitted for publication. The main results are Theorem 1.2.7, Theorem 1.2.6 and Theorem 1.2.9.

Next, we come back to the linear hyperbolic equations and study the DN map for linear wave equation when the source region and the observation region are disjoint subsets of the boundary. The study of partial data problem is crucial, as the data in the real world is usually not global. For instance, seismic waves passing through the Earth can be used to reconstruct the underground structure, but it is not physical to collect data on the entire surface of the Earth. Moreover, in many real world applications, the location where observation is made can be completely disjoint from the location where the source of the data is created. This is very common in, for example, cosmology, where the wave collected on the Earth came from some other locations of the universe. We will use the microlocal information of this restricted DN map for some boundary recovery results.

Consider a time-oriented Lorentzian manifold (M, g) of dimension n for $n \geq 3$, with a timelike, strictly null-convex boundary ∂M, see Definition 1.2.4. Suppose g has the signature $(-, +, \cdots, +)$. We consider the wave operator with a first-order perturbation

$$\square_{g,A,q} \;=\; \frac{1}{\sqrt{|\det g|}}\big(\partial_j - iA_j\big)\Big(\sqrt{|\det g|}\, g^{jk}\big(\partial_k - iA_k\big)\Big) \;+\; q,$$

where A is a smooth 1-form and q is a smooth scalar potential on M. Let u be the outgoing solution of

$$\begin{aligned} \square_{g,A,q} u = 0 \quad &\text{in } M^\circ, \\ u = f \quad &\text{on } \partial M, \\ \operatorname{supp}(u) \subseteq J^+&(\operatorname{supp}(f)), \end{aligned} \tag{1.2.2}$$

where $J^+(\operatorname{supp}(f))$ denotes the causal future of support of f, for more details see Section 4.2. Here by outgoing, we mean the microlocal solution that we choose has singularities propagating along forward bicharacteristics, for more details see Section 4.2.5. The associated outgoing *Dirichlet-to-Neumann map* (DN map) is defined as

$$\Lambda_{g,A,q} : f \mapsto (\partial_\nu u - i\langle A, \nu\rangle u)|_{\partial M},$$

where ν denotes the unit outward normal vector field to ∂M. Let U and V be open subsets of ∂M. The goal of this chapter is to study the inverse problem of recovering g and A up to gauge transformations, from partial boundary data. More precisely, we impose the Dirichlet boundary conditions $u = f$ on U and measure the corresponding Neumann derivative on V. For this purpose, we define the *restricted DN map* as

$$\Lambda^{U,V}_{g,A,q} : f \mapsto (\partial_\nu u - i\langle A, \nu\rangle u)|_V.$$

We shall mainly focus on the case where $U \cap V = \emptyset$, unless stated otherwise. In this case, as the second term is only supported in $U \cap V$, the DN map is simply given by

$$\Lambda^{U,V}_{g,A,q}(f) = \partial_\nu u|_V.$$

We will work with the following class of Lorentzian manifolds.

Definition 1.2.4. *Let $n \geq 3$ and (M, g) be a smooth connected n-dimensional Lorentzian manifold with non-empty boundary ∂M, where g has the signature $(-, +, \ldots, +)$. We say (M, g) is admissible if*

1. *there exists a smooth, proper, surjective function $\tau\colon M \to \mathbb{R}$, such that $d\tau$ is everywhere timelike;*

2. *∂M is timelike, i.e., the induced metric $\bar{g} := g|_{T\partial M \times T\partial M}$ is Lorentzian;*

3. *∂M is* strictly null-convex*, i.e., if ν denotes the outward pointing unit normal vector field on ∂M, then*

$$I\!I(V, V) = g(\nabla_V \nu, V) > 0,$$

for all null vectors $V \in T_p\partial M$, where $I\!I$ is the second fundamental form.

We call such τ a temporal function. The definition above is adapted from [66, Definition 1.1], with some minor differences: here we assume the temporal function is smooth and the boundary is strictly null-convex. With the existence of the temporal function, the well-posedness of (1.2.2) follows from [70, Theorem 24.1.1], see also [5].

Recall $T^*\partial M\setminus 0$ is naturally decomposed into the elliptic, glancing, and hyperbolic sets. More explicitly, the glancing set contains lightlike covectors in $T^*\partial M$ and the hyperbolic set contains timelike covectors in $T^*\partial M$. The glancing set plays an important role in our analysis, which can be further decomposed into the diffractive part, the gliding part, and higher-order glancing sets, according to the behavior of the null-bicharacteristics near the boundary. Gliding rays are corresponding to the gliding part. Roughly speaking, they are generalized bicharacteristics that hit the boundary tangentially, with second order contact, lying locally in ∂M. For more details see Chapter 4. Now let $S^*\partial M \coloneqq (T^*\partial M\setminus 0)/\mathbb{R}^+$. As we can only hope to determine the geometry at boundary points that are visible to measurements, we introduce the following definition for covectors in the glancing set. For definition of forward/backward gliding rays, see Chapter 4.

Definition 1.2.5. *Let (M, g) be an admissible Lorentzian manifold with dimension $n \geq 3$. Let $\mathcal{G}$ be the glancing set. A glancing covector $(x', \xi') \in \mathcal{G}$, viewed as an element of $S^*\partial M$, is called a recoverable direction if*

- *$x' \in U$ and the corresponding forward gliding ray passes through V; or*
- *$x' \in V$ and the corresponding backward gliding ray passes through U.*

The point x' is called a recoverable point if

- *for $n = 3$, both directions in glancing set are recoverable;*
- *for $n > 3$, at least one direction in glancing set is recoverable.*

We denote by $\tilde{U}$ the set of recoverable points in U and by $\tilde{V}$ the set of recoverable points in V. We refer to them as recoverable sets.

We remark that in our setting, the projection of a gliding ray onto the base manifold is in fact a null-geodesic on ∂M with respect to the induced metric $\bar{g}$, see Lemma 4.2.16. Let $\mathcal{L}^{\pm}_{\partial M}(p)$ denote the future (or past) light cone from $p \in \partial M$, defined as the union of all forward (or backward) null-geodesics in $(\partial M, \bar{g})$ emanating from p. For definition of forward/backward null-geodesics, see Chapter 4. When $n > 3$, the set $\tilde{U}$ consists of the points in U whose future light cone in $(\partial M, \bar{g})$ intersects V. Equivalently, these are points in U that lie on the past light cone in $(\partial M, \bar{g})$ of some point in V. A similar argument applies to $\tilde{V}$. Consequently, the recoverable sets are explicitly given by

$$\tilde{U} = (\bigcup_{q \in V} \mathcal{L}^-_{\partial M}(q)) \cap U \quad \text{and} \quad \tilde{V} = (\bigcup_{p \in U} \mathcal{L}^+_{\partial M}(p)) \cap V.$$

In addition, we consider the future pointing lens relation L, also referred to as the scattering relation in the literature. Let $(x', \xi') \in T^*\partial M$ be timelike and future pointing. Let $(x', \xi) \in T^*M$ be the unique inward-pointing lightlike covector with orthogonal projection (x', ξ'). There exists a unique forward null-bicharacteristic Γ starting from (x', ξ). If Γ intersects the boundary again at some $(y', \eta) \in L^*M$, then under our assumptions the intersection must be transversal. Let $(y', \eta') \in T^*M$ be the orthogonal projection of (y', η). We define $L(x', \xi') = (y', \eta')$. Using that L is an even map, one can define L for past pointing timelike covector, for more details see [139].

Recall $\bar{g}$ denotes the restriction of g to $T\partial M \times T\partial M$, i.e., $\bar{g} = g|_{T\partial M \times T\partial M}$. We assume no a priori information is known on the open disjoint boundary sets U or V. The following theorem summarizes what can be reconstructed from the restricted Dirichlet-to-Neumann map $\Lambda^{U,V}_{g,A,q}$.

Theorem 1.2.6 (Y-Zhang 25 [158])**.** *Let (M, g) be an admissible Lorentzian manifold of dimension $n \geq 3$ and U, V be disjoint open subsets of ∂M. Then from the restricted DN map $\Lambda^{U,V}_{g,A,q}$, one can reconstruct*

(1) the recoverable sets $\tilde{U}$ and $\tilde{V}$;

(2) the conformal class of $\bar{g}$ on $\tilde{U}$ and $\tilde{V}$;

(3) for each recoverable direction (x', ξ') on U or V, the lens relation $L|_{\mathcal{N}}$ up to time orientation, where $\mathcal{N}$ is a small conic timelike neighborhood of (x', ξ');

(4) the magnetic potential $A|_{T\partial M}$ on $\tilde{U}$ and $\tilde{V}$.

We note that there is a natural obstruction to reconstructing g, A, q from the restricted DN map $\Lambda^{U,V}_{g,A,q}$. Specifically, there exist gauge transformations that keep $\Lambda^{U,V}_{g,A,q}$ invariant, see Lemma 4.2.7. One can only expect to reconstruct these quantities up to such a gauge transformation. Without prior knowledge of

U or V, Theorem 4.1.3 shows we can reconstruct the induced $\bar{g}$ on the boundary for points in the recoverable sets, up to conformal factors. Moreover, we can reconstruct the 1-form A on the boundary for these points, up to gauge transformations. Indeed, we may pick local coordinates (x', x^n) near such a boundary point and suppose it is the semi-geodesic normal coordinates for some metric, see Lemma 4.2.5. Then by Lemma 4.2.5, there exists a boundary-preserving diffeomorphism Ψ such that $\Psi^* g$ has (x', x^n) as the semi-geodesic normal coordinates as well. Then by [139, Lemma 2.5], in these coordinates we can modify the 1-form $\Psi^* A$ using a gauge transformation such that $(\Psi^* A)_n(x', 0) = 0$. Note that $\Psi^* A|_{T\partial M} = A|_{T\partial M}$ as Ψ fixes the boundary. This implies we reconstruct $\Psi^* A$ on the boundary and therefore A on the boundary up to gauge transformations.

One may extend the definition of lens relation by using broken null-bicharacteristics reflecting finitely many times at ∂M; see Definition 4.2.8 and 4.2.9. Indeed, let $(x', \xi') \in T^*\partial M$ be defined as above. For consistency we write (x', ξ') as $(x'_0, {\xi^0}')$. We consider the unique broken null-bicharacteristic starting from (x'_0, ξ^0), reflecting successively at $x'_k \in \partial M$ with $(x'_k, \xi^k) \in L^* M$, for $k = 1, \ldots, K$. Let $(x'_k, {\xi^k}') \in T^*\partial M$ be the orthogonal projection of (x'_k, ξ^k). Then we define the k-th lens relation L^k as $L^k(x'_0, {\xi^0}') = (x'_k, {\xi^k}')$. By Proposition 4.3.5, when $x'_0 \in U$ and $x'_k \in V$, the graph of such L^k is the canonical relation for the restricted DN map, although the exact value of k may not be recovered, see Section 4.4.

Based on this constructive result, we prove the following determination result.

Theorem 1.2.7 (Y-Zhang 25 [158])**.** *Let (M, g_1) and (M, g_2) be admissible Lorentzian manifolds of dimension $n \geq 3$. Let U, V be disjoint open subsets of ∂M. Suppose*

$$\Lambda^{U,V}_{g_1, A_1, q_1} f = \Lambda^{U,V}_{g_2, A_2, q_2} f$$

for any distribution f with compact support in U. Then $\bar{g}_1 = e^{\varphi_0} \bar{g}_2$ in the same recoverable sets $\tilde{U}$ and $\tilde{V}$, where $\varphi_0 \in C^\infty(\tilde{U} \cup \tilde{V})$ is locally constant. In particular, if $W_1 \subseteq \tilde{U}$ and $W_2 \subseteq \tilde{V}$ are connected components joined by some forward gliding ray or broken bicharacteristic, then there exists a constant c such that

$$\varphi_0|_{W_1} \equiv \frac{c}{n-2} \quad \textit{and} \quad \varphi_0|_{W_2} \equiv \frac{c}{n}.$$

Furthermore, there exists $\varphi \in C^\infty(M)$ with $\varphi|_{\partial M} = \varphi_0$ and a local diffeomorphism ψ near ∂M that fixes it pointwise, such that the jets of g_1 and $e^\varphi \psi^* g_2$ coincide in $\tilde{U}$ and $\tilde{V}$.

Compared to Theorem 1.2.6, this theorem is a uniqueness result that further describes the conformal factor between $\bar{g}_1$ and $\bar{g}_2$ when the corresponding restricted DN maps coincide. Moreover, it proves that the jets of the metric on the boundary are determined up to gauge transformations at points in the recoverable sets.

Next, assume the induced metric and time orientation are known on U (respectively on V). Then the structure on a subset of V (respectively of U) can be reconstructed. To state the result, we introduce the following definition.

Definition 1.2.8. *We say x' is V-recoverable, if there exists a forward broken null-geodesic from x' to some $y' \in V$. Similarly, y' is called U-recoverable, if there exists a backward broken null-geodesic from y' to some $x' \in U$. We denote by $\tilde{U}_0$ the set of V-recoverable points in U and by $\tilde{V}_0$ the set of U-recoverable points in V.*

More explicitly, let $\mathcal{L}^{\pm,b}(p)$ denote the union of all forward (or backward) broken null-geodesics starting from p in (M, g). Then $\tilde{U}_0$ and $\tilde{V}_0$ can be written as

$$\tilde{U}_0 = (\textstyle\bigcup_{q \in V} \mathcal{L}^{-,b}(q)) \cap U \quad \text{and} \quad \tilde{V}_0 = (\textstyle\bigcup_{p \in U} \mathcal{L}^{+,b}(p)) \cap V.$$

We emphasize that both $\tilde{U}_0$ and $\tilde{V}_0$ are open subsets, since under our assumptions broken null-geodesics are stable under small perturbations. In particular, one has $\tilde{U} \subseteq \tilde{U}_0$ and $\tilde{V} \subseteq \tilde{V}_0$ by Lemma 4.2.18. In this setting, we state the following theorem.

Theorem 1.2.9 (Y-Zhang 25 [158])**.** *Let (M, g) be an admissible Lorentzian manifold of dimension $n \geq 3$ and U, V be open disjoint subsets of ∂M. Given the induced metric $\bar{g}$ and the time orientation on $\tilde{U}_0$, from the restricted DN map $\Lambda^{U,V}_{g,A,q}$, one can reconstruct $\bar{g}$ and the time orientation on $\tilde{V}_0$. The same result holds with $\tilde{U}_0$ and $\tilde{V}_0$ interchanged.*

This theorem implies that with additional information on U, we can reconstruct g on the boundary at points in the U-recoverable sets of V, up to gauge transformations. Indeed, with the induced metric $\bar{g}$

reconstructed in the U-recoverable sets, we may choose local coordinates (x', x^n) near a boundary point. As before, by Lemma 4.2.5, there exists a boundary-preserving diffeomorphism Ψ such that $\Psi^* g$ has (x', x^n) as the semi-geodesic normal coordinates. Since $\Psi^* g|_{T\partial M \times T\partial M} = \bar{g}$, we actually reconstruct $\Psi^* g$ on the boundary in these coordinates and thus g on the boundary up to gauge transformations.

1.2.3 Calderón problem with electromagnetic perturbations

This part corresponds to my work in [157], published in *International Mathematics Research Notices (IMRN)*. The main results are Theorem 1.2.12, Theorem 1.2.13 and Theorem 1.2.10.

Recall that the Calderón problem introduced in Section 1.1.1, which is only solved under some strong structural assumption or for some specific metrics. As the last PDE-based inverse problem in this thesis, we introduce a related problem to the classic Calderón problem. Instead of the information of one DN map, we introduce a family of DN maps indexed by perturbation of electromagnetic fields, and ask if such a family of DN maps is enough to recover the metric information. We will deal with both the Riemannian and the Lorentzian setting.

We start with the Riemannian case. Consider the magnetic Laplace-Beltrami operator $\Delta_{g,A}$ where A is a smooth 1-form. In local coordinates, it has the form

$$\Delta_{g,A} = |g|^{-\frac{1}{2}}(\partial_j - iA_j)|g|^{\frac{1}{2}} g^{jk}(\partial_k - iA_k).$$

The 1-form A is usually referred to as the magnetic potential. It appears frequently in physics, for example when a charged quantum particle moves in a magnetic field. Denote the DN map

$$\Lambda_{g,A} : f \mapsto (\partial_\nu - i\langle A, \nu\rangle)u|_{\partial M}$$

where u is the unique solution of

$$\Delta_{g,A} u = 0 \quad \text{in } M^\circ, \quad u|_{\partial M} = f. \tag{1.2.3}$$

In this case, if $\Psi : M \to M$ is a boundary fixing diffeomorphism, and $d\omega$ is a smooth exact 1-form such that $\omega|_{\partial M} = 0$, then

$$\Lambda_{g,A} = \Lambda_{\Psi^* g, \Psi^* A + d\omega}. \tag{1.2.4}$$

Specifically, if u is the solution with respect to g and A, then $e^{i\omega}\Psi^* u$ is the solution with the same Dirichlet boundary data respect to $\Psi^* g$ and $\Psi^* A + d\omega$. We show that g can be fully determined without any gauge equivalence if the DN map with magnetic potential is given for all sufficiently small magnetic potentials.

Theorem 1.2.10 (Y 25 [157]). *Let M be a smooth connected compact manifold with boundary of dimension $n \geq 3$, and g_1, g_2 are two Riemannian metrics on M. Suppose $\Lambda_{g_1,A} = \Lambda_{g_2,A}$ for all A in a neighborhood of 0 with respect to C^k topology for some $k \geq 0$, then $g_1 = g_2$.*

The following result is a direct consequence of Theorem 1.2.10.

Corollary 1.2.11. *For $j = 1, 2$, let (M_j, g_j) be a smooth connected compact Riemannian manifold with boundary of dimension $n \geq 3$. Suppose there exists a boundary fixing diffeomorphism $\Psi : M_1 \to M_2$ such that $\Lambda_{g_1,A} = \Lambda_{g_2,(\Psi^{-1})^* A}$ for all smooth 1-form A on M_1 in a neighborhood of 0 with respect to C^k topology for some $k \geq 0$, then $g_1 = \Psi^* g_2$.*

The same problem can be considered when the metric is Lorentzian. Consider a Lorentzian manifold (M, g) of dimension $n \geq 3$ with a timelike boundary ∂M. We say the manifold is (lightlike) non-trapping if any inextendible lightlike geodesic $\gamma : [a, b] \to M$ eventually leaves M in finite time, that is, $-\infty < a \leq b < \infty$ and $\gamma(a), \gamma(b) \in \partial M$. For simplicity, we assume the manifold is non-trapping and admissible (see Definition 1.2.4), in particular, we consider the following electromagnetic wave equation

$$\begin{cases} \square_{g,A,q} u = 0 \quad \text{in } M^\circ, \\ u|_{\partial M} = f, \\ \operatorname{supp}(u) \subset J^+(\operatorname{supp}(f)), \end{cases} \tag{1.2.5}$$

where J^+ denotes the causal future, and $\Box_{g,A,q}$ is the wave operator with first order perturbation

$$\Box_{g,A,q} = |g|^{-\frac{1}{2}}(\partial_j - iA_j)|g|^{\frac{1}{2}} g^{jk}(\partial_k - iA_k) + q.$$

Here A is a smooth 1-form and q is a smooth function. We emphasize that in the Lorentzian case, for example in Lorentzian formulation of electrodynamics, A is almost always the electromagnetic potential instead of the magnetic potential. It satisfies $dA = F$ where F is the electromagnetic field tensor. In this chapter, in the Lorentzian case we shall refer to A as the electromagnetic potential.

The wave equation (1.2.5) is well-posed for $f \in H^1_{loc}(\partial M)$ by [70, Theorem 24.1.1]. We define the DN map as

$$\Lambda_{g,A,q} : H^1_{loc}(\partial M) \ni f \mapsto (\partial_\nu - i\langle A, \nu\rangle)u|_{\partial M} \in \bar{L}^2_{loc}(M^\circ) \tag{1.2.6}$$

where ν is the unit outward normal vector on the boundary, and $\bar{L}^2_{loc}(M^\circ)$ refers to the space of restrictions to M° of locally L^2 functions on M (see [70, Theorem 24.1.1] and [70, Page 113]). Again if $\Psi : M \to M$ is a boundary fixing diffeomorphism and $d\omega$ is a smooth exact 1-form such that $\omega|_{\partial M} = 0$, then

$$\Lambda_{g,A,q} = \Lambda_{\Psi^* g, \Psi^* A + d\omega, \Psi^* q}. \tag{1.2.7}$$

The existence of the potential term allows for another gauge equivalence. Namely, let ϕ be a smooth function on M such that $\phi|_{\partial M} = \partial_\nu \phi|_{\partial M} = 0$, then

$$\Lambda_{g,A,q} = \Lambda_{e^{-2\phi}g, A+d\omega, e^{2\phi}(q - q_\phi)}, \quad q_\phi = e^{\frac{n-2}{2}\phi}\Box_g e^{\frac{2-n}{2}\phi}. \tag{1.2.8}$$

We refer to [140] for details of the gauge equivalences.

The Lorentzian Calderón problem asks if the metric g is uniquely determined by the DN map $\Lambda_{g,A,q}$ up to these natural gauges. Suppose the electromagnetic field is allowed to perturb, namely we have the information of $\Lambda_{g,A,q}$ for a family of different A. In this case, one can use this perturbation to reconstruct the conformal class of the metric g.

Theorem 1.2.12 (Y 25 [157]). *Suppose (M, g) is non-trapping and admissible. Let $\mathcal{B}_\delta(\mathcal{A})$ be the set of* *smooth 1-forms that are δ close to some fixed smooth 1-form $\mathcal{A}$ in the C^k topology for some k. For any $U \subset M^\circ$ open, denote the set of perturbations of $\mathcal{A}$ in U as*

$$\mathcal{B}_\delta(\mathcal{A}; U) = \{A \in \mathcal{B}_\delta(\mathcal{A}) : supp(A - \mathcal{A}) \subset U\}.$$

If L is the lens relation (see Definition 5.3.2) and $L(x', \xi') = (y', \eta')$, then the unique null-geodesic corresponding to $(y', \eta'; x', \xi')$ is given by

$$\begin{aligned} \gamma = \{x \in M^\circ : \forall U \ni x \text{ open}, \exists A \in \mathcal{B}_\delta(\mathcal{A}; U) \text{ s.t.} \\ \sigma(\Lambda_{g,A,q})|_{(y',\eta';x',\xi')} \neq \sigma(\Lambda_{g,\mathcal{A},q})|_{(y',\eta';x',\xi')}\}. \end{aligned} \tag{1.2.9}$$

Moreover, the set of such A is generic in $\mathcal{B}_\delta(\mathcal{A}; U)$. In particular, the conformal class of g can be constructed from $\{\Lambda_{g,A,q} : A \in \mathcal{B}_\delta(\mathcal{A})\}$.

Besides perturbing the electromagnetic potential, a similar problem can be considered when the perturbation is with respect to other parameters in the equation. In particular, we study the case where the metric itself is allowed to perturb. Consider the wave equation

$$\begin{cases} (\Box_g + q_g)\, u = 0 \\ u|_{\partial M} = f \\ \text{supp}(u) \subset J^+(\text{supp}(f)) \end{cases} \tag{1.2.10}$$

where the potential term q_g may depend on g locally. That is, $q_g(x)$ depends on g only on an arbitrarily small neighborhood of x. For instance, the potential term in the Yamabe equation

$$\left(\Box_g - \frac{n-2}{4(n-1)} R_g\right) u = 0 \tag{1.2.11}$$

is related to scalar curvature R_g, which depends on g. Simple computation shows that a conformal change of g which is the identity map on the boundary does not change the DN map. On the other hand, the

Klein-Gordon equation

$$\left(\Box_g + m^2\right) u = 0 \tag{1.2.12}$$

provides an example where the potential term is independent of g. We show that the conformal class of the metric can also be recovered using perturbation of the metric.

Theorem 1.2.13 (Y 25 [157])**.** *Suppose (M, g) is non-trapping and admissible, denote Λ_g the DN map of* (1.2.10) *with q_g depending on g locally. If L is the lens relation (see Definition 5.3.2) and $L(x', \xi') = (y', \eta')$, then the unique null-geodesic associated with $(y', \eta'; x', \xi')$ can be constructed by the following equivalence conditions.*

1. *γ passes through $x \in M^\circ$.*

2. *For any $U \ni x$ open neighborhood and conormal distribution f singular at (x', ξ'), there exists smooth symmetric 2-tensor h compactly supported in U such that (y', η') is in the wavefront set of $\partial_\epsilon|_0 \Lambda_{g-\epsilon h}(f)$.*

Moreover, the set of such h is generic in the space of smooth symmetric 2-tensors compactly supported in U with respect to C^k topology for any k. In particular, let $\mathcal{B}_\delta(g)$ be the set of smooth Lorentzian metrics that are δ close to g in C^k topology for some k, then the conformal class of g can be constructed from $\{\Lambda_{g'} : g' \in \mathcal{B}_\delta(g)\}$.

Clearly, one actually recovers the conformal class of all the metrics in $\mathcal{B}_\delta(g)$.

1.2.4 Rigidity problems for analytic Lorentzian manifolds

This part corresponds to a joint work with Yang Zhang in [159]. It is submitted for publication. The main results are Theorem 1.2.14, Theorem 1.2.15, Theorem 1.2.17 and Theorem 1.2.18.

Finally, we move on to Lorentzian geometric inverse problems. We prove boundary time separation rigidity and near-lightlike timelike scattering rigidity for analytic Lorentzian manifolds.

Let (M, g) be a Lorentzian manifold with timelike boundary ∂M, where g has the signature $(-, +, \cdots, +)$. A geodesic is said to be *non-trapping* if it exits M in finite time, so that no information can remain confined in the interior. The *Lorentzian distance function*, or more commonly known as the *time separation function*, is defined by

$$d(x, y) = \sup\{L(\alpha) : \alpha \text{ future pointing piecewise smooth causal curve from } x \text{ to } y\},$$

whenever $y \in J^+(x)$, and $d(x, y) = 0$ otherwise. Here L denotes the Lorentzian arc length, see Section 6.2.2. Then, we consider the following boundary rigidity problem.

Problem 1 (Lorentzian boundary rigidity). *For $j = 1, 2$, let (M_j, g_j) be spacetimes with timelike boundaries. Suppose there exists a boundary diffeomorphism $\varphi_0 : \partial M_1 \to \partial M_2$ such that*

$$d_1(x, y) = d_2(\varphi_0(x), \varphi_0(y)), \quad \forall x, y \in \partial M_1.$$

Can φ_0 be extended to a global isometry between the two manifolds?

Another commonly studied rigidity problem is the scattering rigidity problem. Since many definitions coincide with those for Riemannian manifolds, here we may regard (M, g) either as a compact Riemannian manifold with boundary, or as a Lorentzian manifold with timelike boundary. We denote the inward $(-)$ and outward $(+)$ pointing vectors on the boundary by

$$\partial_\pm TM = \{(x, v) \in \partial TM : \pm g(v, \nu) > 0\},$$

where ν is the unit outward pointing normal vector field on ∂M. Clearly,

$$\partial TM = \partial_- TM \cup \partial_+ TM \cup T\partial M, \quad \overline{\partial_\pm TM} = \partial_\pm TM \cup T\partial M.$$

There are two similar but different definitions of scattering information in the literature: one records the information when the geodesic first exits M°; the other records the information when the geodesic completely leaves M. See Figure 1.1.

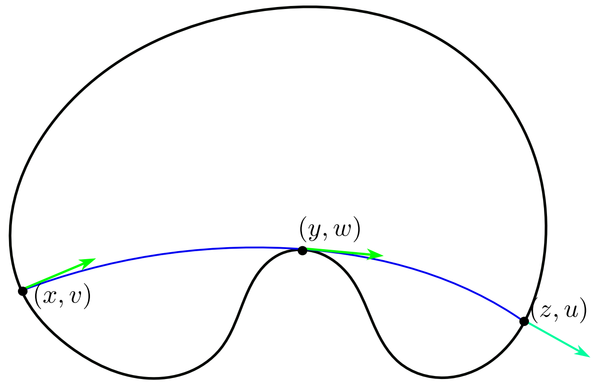

Figure 1.1: For $(x, v) \in \partial_- TM$, denote by γ the corresponding geodesic Let (y, w) be the point and direction at which γ exits M° for the first time (equivalently the first time γ reaches ∂M). Let (z, u) be the point and direction at which γ completely exits M. The complete scattering relation records both transitions $(x, v) \to (z, u)$ and $(y, w) \to (z, u)$. The interior scattering relation, in contrast, records only the first intersection with the boundary, i.e., $(x, v) \to (y, w)$. The second transition $(y, w) \to (z, u)$ is not recorded by the interior scattering relation, as at (y, w) the geodesic is tangential to the boundary.

For $(x, v) \in \partial_- TM$, denote by γ the corresponding geodesic. We define the *interior travel time* as

$$\tau^{in}(x, v) = \sup\{t > 0 : \gamma((0, t)) \subset M^\circ\}.$$

If $\tau^{in}(x, v)$ is finite, we define the *interior scattering relation* and the *interior length function* as

$$S^{in}(x, v) = (\gamma(\tau^{in}(x, v)), \dot{\gamma}(\tau^{in}(x, v))),$$
$$\ell^{in}(x, v) = \tau^{in}(x, v) \cdot |g(v, v)|^{1/2}.$$

The pair (S^{in}, ℓ^{in}) is called the *interior lens data*, and the pair (S^{in}, τ^{in}) is called the *interior travel time data*. Note that the domain of the interior scattering information is a subset of $\partial_- TM$, excluding trapping and boundary tangential directions.

Now let $(\tilde{M}, \tilde{g})$ be an extension of (M, g). For any $(x, v) \in \overline{\partial_- TM}$, as before, let γ be the corresponding geodesic in $\tilde{M}$. We define the *complete travel time* as

$$\tau(x, v) = \sup\{t \geq 0 : \gamma([0, t]) \subset M\}.$$

If $\tau(x,v)$ is finite, we define the *complete scattering relation* and the *complete length function* as

$$S(x,v) = (\gamma(\tau(x,v)), \dot\gamma(\tau(x,v))),$$
$$\ell(x,v) = \tau(x,v)\cdot |g(v,v)|^{1/2}.$$

The pair (S,ℓ) is called the *complete lens data*, and the pair (S,τ) is called the *complete travel time data.* Note that the domain of the complete scattering information is a subset of $\overline{\partial_- TM}$, which may include boundary tangential directions when they are non-trapping. Moreover, they do not depend on the choice of extension.

For both Riemannian manifolds with boundaries and Lorentzian manifolds with timelike boundaries, the unit outward normal vector field ν on the boundary is well-defined. We say that $(x,v)\in T\partial M$ is a *strictly convex direction*, if the second fundamental form is positive, i.e.,

$$\mathrm{I\!I}(v,v) := g(\nabla_v \nu, v) > 0. \tag{1.2.13}$$

In particular, if all boundary tangential vectors are strictly convex directions, we say the boundary is strictly convex.[4] In this case, the interior and complete scattering information coincide, since any geodesic in M intersects the boundary transversally. Moreover, in a Lorentzian manifold with timelike boundary, if all boundary tangential lightlike vectors are strictly convex directions, then we say the manifold is strictly null-convex (see [66]). In this case, the interior and complete scattering relations for lightlike geodesics also coincide.

For $j=1,2$, suppose (M_j,g_j) are either both compact Riemannian manifolds with boundaries or both Lorentzian manifolds with timelike boundaries. Furthermore, suppose there exists a boundary isometry $\varphi_0 : \partial M_1 \to \partial M_2$, that is,

$$\bar g_1 = \varphi_0^* \bar g_2, \qquad \text{with } \bar g_j = g_j|_{T\partial M_j\times T\partial M_j},$$

[4] More commonly, for a Riemannian manifold with boundary, strict convexity refers to the property that any two sufficiently close boundary points are connected by a distance minimizing geodesic that stays in the interior except at its endpoints. This property implies $\mathrm{I\!I}$ is positive semi-definite. Conversely, a positive definite $\mathrm{I\!I}$ ensures this property. See, for example, [56, Section 3.2]. In this chapter, we will only use the notion of a strictly convex direction.

where $\bar{g}_j$ is the boundary metric. In boundary normal coordinates, one may write (x, v) as (x, v', v^n), where the boundary is locally given by $x^n = 0$ and the interior corresponds to $x^n > 0$. By abuse of notation, we view $(\varphi_0)_*$ as

$$(\varphi_0)_* : \partial TM_1 \to \partial TM_2, \quad (x, v', v^n) \mapsto (\varphi_0(x), (\varphi_0)_* v', v^n),$$

which maps unit normal directions to unit normal directions. In particular, as φ_0 is a boundary isometry, under these notations we have

$$g_1|_{\partial TM_1 \times \partial TM_1} = \varphi_0^*(g_2|_{\partial TM_2 \times \partial TM_2}). \tag{1.2.14}$$

Then we have the following two scattering rigidity problems.

Problem 2 (Interior scattering rigidity)**.** *For $j = 1, 2$, let (M_j, g_j) be either both compact Riemannian manifolds with boundaries or both Lorentzian manifolds with timelike boundaries. Suppose there exists a boundary isometry $\varphi_0 : \partial M_1 \to \partial M_2$ such that*

$$(\varphi_0)_* \circ S_1^{in} = S_2^{in} \circ (\varphi_0)_*$$

in boundary normal coordinates and on the domain of S_1^{in} (which, under the non-trapping assumption, is typically all of $\partial_- TM_1$ in the Riemannian case). Can φ_0 be extended to a global isometry?

Problem 3 (Complete scattering rigidity)**.** *For $j = 1, 2$, let (M_j, g_j) be either both compact Riemannian manifolds with boundaries or both Lorentzian manifolds with timelike boundaries. Suppose there exists a boundary isometry $\varphi_0 : \partial M_1 \to \partial M_2$ such that*

$$(\varphi_0)_* \circ S_1 = S_2 \circ (\varphi_0)_*$$

in boundary normal coordinates and on the domain of S_1 (which, under the non-trapping assumption, is typically all of $\overline{\partial_- TM_1}$ in the Riemannian case). Can φ_0 be extended to a global isometry?

For inverse problems, one often omits the subindex j and states the result in a *deterministic* way, without reference to a particular manifold. Thus, Problem 2 (or 3) asks whether the metric g on the entire M can be determined by the given data

$$g|_{\partial TM \times \partial TM} \quad \text{and} \quad S^{in}(\text{or } S).$$

We emphasize that the metric is given on ∂TM instead of $T\partial M$ because $(\varphi_0)_*$ preserves the length of any vector in ∂TM_j, see (1.2.14).

We are now ready to describe the three main rigidity results for analytic Lorentzian manifolds, concerning the determination of an unknown region from exterior time separation data, the boundary rigidity problem in the presence of timelike boundary, and the scattering rigidity problem from information near the light cone.

We first study the case in which the unknown region is embedded in a larger analytic globally hyperbolic Lorentzian manifold, and we assume that lightlike geodesics do not have cut points there. We show that, given the time separation function on a thin exterior layer of the unknown region, the unknown region can be determined up to isometry.

Theorem 1.2.14 (Y-Zhang 25 [159]). *For $j = 1, 2$, let (N_j, g_j) be an analytic globally hyperbolic Lorentzian manifold of dimension $n \geq 3$, and let d_j be the corresponding time separation function. Let $K_j \subset\subset \tilde{M}_j \subset N_j$ be such that K_j is a compact subset and $\tilde{M}_j$ is an open neighborhood of K_j. Denote by $K_j^c = \tilde{M}_j \setminus K_j$. Suppose there exists a diffeomorphism $\varphi_0 : K_1^c \to K_2^c$ such that*

$$d_1(x, y) = d_2(\varphi_0(x), \varphi_0(y)), \quad \forall x, y \in K_1^c,$$

and such that φ_0 extends continuously to a bijection between ∂K_1 and ∂K_2. Suppose further every lightlike geodesic segment in K_j has no cut point. Then there exists an analytic isometry $\varphi : \tilde{M}_1 \to \tilde{M}_2$ such that $\varphi|_{K_1^c} = \varphi_0$.

We emphasize that no regularity assumption is imposed on the unknown region K_j, for $j = 1, 2$; it may be any compact subset. The exterior region K_j^c may also be an arbitrarily thin layer. To prove this theorem, we use the complete travel time data for lightlike geodesics with respect to a smaller neighborhood of K_j, for $j = 1, 2$. This smaller neighborhood is more convenient to work with, since the boundary of K_j may have low regularity. We reconstruct the lightlike complete travel time data from the exterior time separation function, see Section 6.3.2. Then in Proposition 6.3.1, inspired by [153], we show that this lightlike information establishes a correspondence between the two neighborhoods, which preserves the geometric structure and can be extended to an analytic isometry that matches the given exterior identification. For more details, see Section 6.3.

Next, we study the boundary rigidity problem for analytic globally hyperbolic spacetimes with timelike boundaries. For the definition of globally hyperbolic spacetimes with timelike boundaries, see Section 6.2.1. Here by an analytic spacetime with timelike boundary, we mean a smooth manifold with boundary whose transition maps are analytic, the metric is analytic, and the boundary is timelike. Certainly, the boundary and boundary metric are analytic with respect to the induced analytic structure. We require that after passing to a small analytic collar neighborhood, see Section 6.2.3, the extension is still globally hyperbolic and contains no lightlike cut points. We also require the existence of a strictly convex direction on the boundary, for the recovery of the jet of boundary metric.

Theorem 1.2.15 (Y-Zhang 25 [159]). *For $j = 1, 2$, let (M_j, g_j) be an analytic globally hyperbolic spacetime with timelike boundary of dimension $n \geq 3$, and suppose lightlike geodesics are non-trapping. Assume there exists an analytic collar neighborhood extension (see Section 6.2.3), which is globally hyperbolic and contains no lightlike cut points. Suppose there exists an analytic diffeomorphism $\varphi_0 : \partial M_1 \to \partial M_2$ such that*

$$d_1(x, y) = d_2(\varphi_0(x), \varphi_0(y)), \quad \forall x, y \in \partial M_1.$$

Moreover, assume for any connected component C of ∂M_1, there exists a causal direction $(x, v) \in TC$ such that (x, v) and $(\varphi_0(x), (\varphi_0)_* v)$ are strictly convex directions in (M_1, g_1) and (M_2, g_2), respectively. Then there exists an analytic isometry $\varphi : M_1 \to M_2$ such that $\varphi|_{\partial M} = \varphi_0$.

Comparing to the exterior rigidity problem in Theorem 1.2.14, boundary rigidity is more subtle due to the presence of the boundary. Indeed, since we do not assume the timelike boundary to be strictly convex, the distance maximizing curve may no longer be a geodesic. To prove this theorem, by Proposition 6.3.1, it suffices to reconstruct the lightlike complete travel time data with respect to M_j, for $j = 1, 2$, from the boundary time separation function d_j. For this purpose, we first determine the jet of the metric on the boundary, allowing for an analytic extension. The exterior time separation function $\tilde{d}_j$ can be constructed in this extension, from the knowledge of d_j. As the maximal distance is not necessarily realized by a causal geodesic, we use both $\tilde{d}_j$ and d_j to recover the lightlike complete travel time data, for more details see Section 6.4. Note that we require no lightlike cut points in the extension to avoid the case that a boundary point of M becomes a cut point in the extended manifold.

In [90], the authors show that the universal covering of an analytic Lorentzian spacetime (M, g) is determined by the time separation function restricted to a small timelike submanifold contained in M, assuming that (M, g) is geodesically complete modulo scalar curvature singularities. As only local information is assumed, one can only expect uniqueness up to an isometry of the universal covering space, not of the manifold itself. In contrast, in this chapter, we study the boundary rigidity problem for globally hyperbolic Lorentzian manifolds with timelike boundary. As a manifold with boundary, it is always geodesically incomplete. Moreover, since we have global information, we can completely determine the metric up to isometry. In [131], the local lightlike scattering rigidity near the boundary for analytic metrics is established. For a more comprehensive overview of previous results, see Section 6.1.2.

Finally, we study the scattering rigidity problem for analytic Lorentzian manifolds with timelike boundaries. In this setting, we do not impose any causality conditions. To state the result, we first introduce a mild non-conjugacy condition, which comes from [135].

Definition 1.2.16. *Let (M, g) be a semi-Riemannian manifold with boundary. Consider a boundary tangential direction $(x, v) \in T\partial M$. Let γ be the corresponding geodesic. The direction (x, v) is said to satisfy the* non-conjugacy condition*, if γ is non-trapping and x is not conjugate to any point in $\gamma \cap \partial M$.*

Note that in a Lorentzian manifold with strictly null-convex timelike boundary, any boundary tangential timelike vector sufficiently close to the light cone will satisfy this condition. Recall that, by abuse of notation, we can view the differential of a boundary isometry φ_0 as a map from ∂TM_1 to ∂TM_2, in boundary normal coordinates. For simplicity, we denote by

$$JM = \{(x,v) \in TM \setminus 0 : g(v,v) \leq 0\}$$

the set of causal vectors in M. Let $\partial_\pm JM$ denote the inward $(-)$ and outward $(+)$ pointing causal vectors on the boundary. We show that, under proper assumptions, both the interior and complete scattering rigidity hold.

Theorem 1.2.17 (Y-Zhang 25 [159])**.** *For $j = 1,2$, let (M_j, g_j) be an analytic Lorentzian manifold of dimension $n \geq 3$ with analytic timelike boundary. Assume all lightlike geodesics are non-trapping. Then, for sufficiently small conic neighborhoods $\mathcal{U}_j \subset \partial TM_j$ containing ∂LM_j, the interior scattering relation S_j^{in} is well-defined in $\mathcal{V}_j^{in} \coloneqq \mathcal{U}_j \cap \partial_- JM_j$. Moreover, assume for each connected component C of ∂M_1, there exists $(x,v) \in TC \cap \overline{\mathcal{V}_1^{in}}$ such that the non-conjugacy condition holds. If there exists a boundary isometry $\varphi_0 : \partial M_1 \to \partial M_2$ such that*

$$(\varphi_0)_*(\mathcal{V}_1^{in}) = \mathcal{V}_2^{in},$$

$$(\varphi_0)_* \circ S_1^{in}|_{\mathcal{V}_1^{in}} = S_2^{in}|_{\mathcal{V}_2^{in}} \circ (\varphi_0)_*,$$

then there exists an analytic isometry $\varphi : M_1 \to M_2$ such that $\varphi|_{\partial M_1} = \varphi_0$.

Theorem 1.2.18 (Y-Zhang 25 [159])**.** *For $j = 1,2$, let (M_j, g_j) be an analytic Lorentzian manifold of dimension $n \geq 3$ with analytic timelike boundary. Assume all lightlike geodesics are non-trapping. Then, for sufficiently small conic neighborhoods $\mathcal{U}_j \subset \partial TM_j$ containing ∂LM_j, the complete scattering relation S_j is well-defined in $\mathcal{V}_j \coloneqq \mathcal{U}_j \cap \overline{\partial_- JM_j}$. Moreover, assume for each connected component C of ∂M_1, there exists $(x,v) \in TC \cap \mathcal{V}_1$ such that the non-conjugacy condition holds. If there exists a boundary isometry $\varphi_0 : \partial M_1 \to \partial M_2$ such that*

$$(\varphi_0)_*(\mathcal{V}_1) = \mathcal{V}_2,$$

$$(\varphi_0)_* \circ S_1|_{\mathcal{V}_1} = S_2|_{\mathcal{V}_2} \circ (\varphi_0)_*,$$

then there exists an analytic isometry $\varphi : M_1 \to M_2$ *such that* $\varphi|_{\partial M_1} = \varphi_0$.

The proof of Theorem 1.2.17 and Theorem 1.2.18 relies on certain boundary determination results, which are developed in Section 6.5 and are inspired by the ideas in [135]. To establish rigidity from both the interior and complete scattering relations, we study their connections in Section 6.6. In particular, in Section 6.6.3, we prove that both the lightlike interior and complete travel time data can be recovered from either the interior or the complete scattering relation for timelike vectors sufficiently close to the light cone, see Lemma 6.6.4 - 6.6.6. All results in Section 6.6 do not rely on any causality or analyticity assumptions on the manifold, or on any convexity assumptions on the timelike boundary. Instead, we derive these results using geometric arguments and the first variation formula for the travel time in Section 6.6.2. Then, by applying Proposition 6.3.1 again, we complete the proof of these two theorems in Section 6.7.

Chapter 2

Preliminaries

2.1 Lorentzian geometry

Lorentzian geometry is the pseudo-Riemannian analogue of Riemannian geometry adapted to spacetime models: one studies a smooth manifold equipped with a nondegenerate metric of signature $(-,+,\ldots,+)$, so tangent vectors split into timelike, null, and spacelike according to the sign of $g(v,v)$. This single change from positive definite to indefinite creates genuinely new phenomena. For instance, there is no canonical distance function, null directions form a light cone in each tangent space, and geometry is organized around causality (which points can influence which). Time orientation gives a consistent notion of future and past, enabling the definition of causal relations and causal sets such as $I^{\pm}(p)$ and $J^{\pm}(p)$. From the PDE viewpoint, Lorentzian metrics naturally produce hyperbolic operators, most importantly the wave operator $\Box_g$, whose characteristic set is the null cone. As a result, geometric features such as null geodesics and causal diamonds control propagation, well-posedness, and boundary value problems. In inverse and rigidity questions, Lorentzian geometry supplies the correct language to formulate measurements and gauge invariances (for example conformal invariance of the causal structure), while global hyperbolicity provides the structural assumptions under which hyperbolic dynamics and boundary control methods behave cleanly. For a thorough study of Lorentzian geometry we refer to [16, 113].

2.1.1 Notation and conventions

Let M be an n dimensional smooth manifold for $n \geq 2$. A metric tensor is a real symmetric bilinear map

$$g : T_pM \times T_pM \to \mathbb{R}$$

that varies smoothly for $p \in M$. Given any orthogonal basis $v_1, \cdots, v_n \in T_pM$, the signature of g is the triplet (r, s, t), which are the numbers (counted with multiplicity) of positive, negative, and zero eigenvalues of g with respect to the basis. By Sylvester's law of inertia, the signature is independent of the basis chosen. The metric tensor is said to be non-degenerate if $t = 0$, in which case we simply denote the signature as (r, s). A Riemannian metric is a metric tensor with signature $(n, 0)$, in other words positive definite. A Lorentzian metric is a metric tensor with signature $(1, n-1)$ or $(n-1, 1)$. For consistency, we will use the convention $(n-1, 1)$ throughout the thesis, equivalently we sometimes use the notation $(-, +, \cdots, +)$.

Since the metric is not positive definite, the tangent vectors can be grouped into three separate classes depending on its sign. We say $v \in T_pM \backslash 0$ is *timelike/spacelike/lightlike* if $g(v, v)$ is negative/positive/zero. A lightlike vector is also called a *null* vector. A vector is a *causal* vector if it is either lightlike or timelike. The set of lightlike vectors at p is called the *light cone/null cone* at p because it is scaling invariant and forms a cone shape in T_pM. A non-lightlike vector v is a *unit vector* if $|g(v, v)| = 1$.

Non-degeneracy allows the inverse metric to be defined on the cotangent space, which we denote by g^{-1}. Metric tensor gives a natural correspondence between tangent space and cotangent space: if $X \in T_pM$ and $\omega \in T_p^*M$, then

$$X^\flat = g(X, \cdot) \in T_p^*M, \quad \omega^\sharp = g^{-1}(\omega, \cdot) \in T_pM.$$

Thus, a covector is timelike/spacelike/lightlike/causal if the corresponding vector satisfies the requirement.

Given a Lorentzian manifold (M, g), we write ∇ for the Levi-Civita connection of g (which is the unique affine connection that is compatible with the metric and is torsion free). If γ is a smooth curve on (M, g) such that $\nabla_{\dot\gamma}\dot\gamma = 0$, then γ is called a *geodesic*. Simple computation shows that $g(\dot\gamma, \dot\gamma)$ is preserved along the geodesic, hence it is called a *timelike/spacelike/lightlike/causal geodesic* if $\dot\gamma$ is timelike/spacelike/lightlike/causal along the geodesic.

Example 2.1.1 (Minkowski metric)**.** *The model Lorentzian manifold is the n-dimensional Minkowski spacetime*

$$(\mathbb{R} \times \mathbb{R}^n, \eta), \qquad \eta = -dt^2 + dx_1^2 + \cdots + dx_{n-1}^2,$$

with global coordinates $(t, x) \in \mathbb{R} \times \mathbb{R}^{n-1}$. A nonzero vector $v = v^0 \partial_t + v^i \partial_{x_i}$ is timelike/null/spacelike according as $\eta(v, v) = -(v^0)^2 + \sum_{i=1}^{n-1} (v^i)^2$ is negative/zero/positive. The null cone is given by $\{(v_0)^2 = \sum_{i=1}^{n-1} (v^i)^2\}$. The geodesics are straight lines, and a geodesic is timelike/spacelike/lightlike if the slope is larger than/smaller than/equal to 1.

Example 2.1.2 (Static metric)**.** *A Lorentzian metric g on a product manifold $M = \mathbb{R} \times \Sigma$ is called* static *if it is independent of t and has no mixed $dt\,dx$ terms, i.e. it can be written in the form*

$$g = -\beta(x)\, dt^2 + h(x),$$

where $\beta \in C^\infty(\Sigma)$ satisfies $\beta(x) > 0$ and h is a Riemannian metric on Σ. The scaling β is sometimes referred to as the lapse*. A static metric is* ultrastatic *if the lapse is identically one,*

$$g = -dt^2 + h(x).$$

Ultrastatic spacetimes provide a convenient normalization (often achievable locally by a change of the time variable) and serve as a common reference class in hyperbolic boundary value problems.

Example 2.1.3 (Acoustic metric)**.** *A basic class of Lorentzian metrics arising from fluid dynamics are* acoustic metrics*, which encode the propagation of sound waves (small perturbations) in a moving compressible flow. In their simplest form on $M = \mathbb{R} \times \Sigma$ (with $\Sigma \subset \mathbb{R}^{n-1}$), given a background velocity field $v(t, x)$, sound speed $c(t, x) > 0$, and (optionally) a positive conformal factor $\Omega(t, x)$, one considers*

$$g = \Omega(t, x)^2 \Big(- \big(c(t, x)^2 - |v(t, x)|^2\big)\, dt^2 - 2\, v(t, x) \cdot dx\, dt + |dx|^2 \Big),$$

where $|v|$ and $v \cdot dx$ are computed with respect to a chosen Riemannian metric on Σ (typically the Euclidean *metric in applications). The corresponding d'Alembertian* $\Box_g$ *governs the linearized acoustic field, and in particular the null cone of* g *describes the local sound cone: sound rays are null curves of* g. *The metric is Lorentzian precisely when the flow is* subsonic *in the sense* $|v| < c$.

2.1.2 Hypersurface, induced geometry, and Lorentzian manifold with boundaries

Suppose $\Sigma \subset M$ is a smooth hypersurface, and g_Σ the restriction of the metric tensor g on $T\Sigma$. If (M, g) is a Riemannian manifold, then (Σ, g_Σ) is again a Riemannian manifold. However, this does not in general hold for Lorentzian manifolds. Consider the hyperplanes $P_1 = \{t = 0\}$, $P_2 = \{x_1 = 0\}$ and $P_3 = \{t = x_1\}$ in n-dimensional Minkowski spacetime, then it is easy to see that the metric tensor restricted to P_1, P_2, P_3 has signature $(n-1, 0, 0), (n-2, 1, 0), (n-2, 0, 1)$, respectively. In other words, the reduced manifold is Riemannian for P_1, Lorentzian for P_2, and the metric tensor is degenerate on P_3. As a result, we say (Σ, g_Σ) is a *timelike/spacelike/lightlike hypersurface* if the induced metric tensor is everywhere Lorentzian/Riemannian/degenerate on Σ. Timelike/spacelike/lightlike submanifold can be defined in the same way, but we will not use them in this thesis.

Just like hypersurfaces, the boundary of a Lorentzian manifold with boundary can have different geometric behaviors. We say the boundary is a *timelike/spacelike/lightlike* boundary, if the induced metric tensor is everywhere timelike/spacelike/degenerate on the boundary. Among those, the most common one is the Lorentzian manifold with timelike boundary. In particular, if (N, h) is a Riemannian manifold with boundary, then the corresponding static spacetime is a Lorentzian manifold with timelike boundary $\mathbb{R}_t \times \partial N$.

Example 2.1.4 (Minkowski cylinder)**.** *The standard prototype for Lorentzian manifold with timelike boundary is the Minkowski cylinder* $(\mathbb{R}_t \times B, \eta)$ *where* $B \subset \mathbb{R}^{n-1}$ *is the unit ball and* η *is the Minkowski metric.*

2.1.3 Causal structure and time orientation

Recall that the set of timelike and lightlike vectors are the causal vectors, and on $T_pM\backslash 0$ the set of causal vectors always consists of two connected components. We say that (M, g) is *time-oriented* if there exists a continuous timelike vector field T on M. In other words, one can make a globally continuous choice of the connected components. A causal vector v is called *future-directed* if $g(v,T) < 0$ and *past-directed* if $g(v,T) > 0$. A piecewise C^1 curve $\gamma : I \to M$ is *timelike* (resp. *causal*) if $\dot\gamma$ is timelike (resp. causal) wherever defined; it is *future-directed* if $\dot\gamma$ is future-directed wherever defined.

For $p, q \in M$ we write

$$p \ll q \quad\Longleftrightarrow\quad \text{there exists a future-directed timelike curve from } p \text{ to } q,$$

$$p \le q \quad\Longleftrightarrow\quad \text{there exists a future-directed causal curve from } p \text{ to } q \text{ (or } p = q).$$

The *chronological* and *causal* future/past sets are

$$I^{\pm}(p) := \{q \in M : p \ll q \text{ (resp. } q \ll p)\}, \qquad J^{\pm}(p) := \{q \in M : p \le q \text{ (resp. } q \le p)\}.$$

One has $I^{\pm}(p) \subset J^{\pm}(p)$ and $I^{\pm}(p)$ are open; moreover the relations $\ll$ and $\le$ are transitive in the sense that $p \ll q \le r$ or $p \le q \ll r$ implies $p \ll r$, and $p \le q \le r$ implies $p \le r$. A set $A \subset M$ is *achronal* if it contains no distinct points $p, q \in A$ with $p \ll q$.

The causal structure depends only on the conformal class of g. Indeed, if $\Omega \in C^\infty(M)$ satisfies $\Omega > 0$ and $\hat g := \Omega^2 g$, then for all $p \in M$ and $v \in T_pM$

$$\hat g(v,v) = \Omega(p)^2 g(v,v).$$

In particular, the decomposition into timelike/null/spacelike vectors is the same for g and $\hat g$, so the light cones $\mathcal{C}_p$ are identical. Since $\hat g(T,T) = \Omega^2 g(T,T) < 0$, the same vector field T also time-orients $\hat g$, and future-directedness is unchanged. Consequently, a curve is timelike/causal/future-directed for g if and only if it is so for $\hat g$, and hence the relations $\ll, \le$ and the sets $I^{\pm}(p), J^{\pm}(p)$ are identical for g and $\hat g$.

While timelike and spacelike geodesics are not conformally invariant, null geodesics are conformally invariant as *unparametrized* curves. More precisely, a null g-geodesic becomes a $\hat g$-geodesic after a reparametrization.

Lemma 2.1.5. *Let $\Omega > 0$ be smooth and $\hat g = \Omega^2 g$. If γ is a null g-geodesic with tangent field $V = \dot\gamma$ affinely* *parametrized for g, then γ is a $\hat{g}$-pregeodesic:*

$$\hat{\nabla}_V V = f\, V \quad \textit{for some scalar function } f \textit{ along } \gamma.$$

In particular, there exists a reparametrization of γ which is affinely parametrized for $\hat{g}$.

Proof. Write $\varphi := \log \Omega$. The Levi–Civita connections satisfy the conformal change formula

$$\hat{\nabla}_X Y = \nabla_X Y + d\varphi(X)Y + d\varphi(Y)X - g(X,Y)\, \nabla_g \varphi.$$

Taking $X = Y = V$ along γ and using $\nabla_V V = 0$ (geodesic) and $g(V,V) = 0$ (null) gives

$$\hat{\nabla}_V V = 2\, d\varphi(V)\, V,$$

so γ is a $\hat{g}$-pregeodesic. Choosing a new parameter s with $\frac{ds}{d\lambda} = \exp(-2\varphi(\gamma(\lambda)))$ (where λ is the g-affine parameter) makes the right-hand side vanish and yields an affine $\hat{g}$-geodesic parametrization. ☐

As immediate consequences, any statement formulated purely in terms of the relations $\ll$ and $\leq$ (for example, the absence of closed causal curves, or the equality/compactness of causal diamonds $J^+(p) \cap J^-(q)$) is conformally invariant.

Conversely, determining the causal structure is equivalent to determining the conformal class of the metric.

Lemma 2.1.6. *Let g_1 and g_2 be two Lorentzian metrics on n-dimensional smooth manifold M, such that $g_1(v,v) = 0$ if and only if $g_2(v,v) = 0$. Then $g_1 = cg_2$ for some nonzero smooth function c, and $c > 0$ if $n \geq 3$.*

Proof. Fix a unit timelike vector $v \in T_pM$ with respect to g_1, then $g_2(v,v) \neq 0$. Let $w \in T_pM$ be such that $g_1(v,w) = 0$ and $g_1(w,w) = 1$, we reduce the problem to 2-dimensional on $\operatorname{span}\{v,w\}$. By construction, $g_1(v \pm w, v \pm w) = 0$, so $g_2(v \pm w, v \pm w) = 0$. Then $g_2(v,w) = \frac{1}{4}(g_2(v+w,v+w) - g_2(v-w,v-w)) = 0$. Set $c = g_1(v,v)/g_2(v,v)$, simple computation shows that $cg_2 = g_1$ on $\operatorname{span}\{v,w\}$. Doing this for all such w

implies $cg_2 = g_1$ on T_pM, and $c > 0$ if $n \geq 3$ because g_1 and g_2 both have signature $(n-1, 1)$. $\square$

2.1.4 Causality conditions and global hyperbolicity

Let (M, g) be a time-oriented Lorentzian manifold. We will only define some of the most common causality conditions here without going into too much details, for a thorough examination of this topic, we refer to [16, 113].

We now introduce *global hyperbolicity*. We recall the standard hierarchy of causality assumptions are:

- (M, g) is *chronological* if it has no closed timelike curves, equivalently $p \notin I^+(p)$ for all $p \in M$.
- (M, g) is *causal* if it has no closed causal curves, equivalently $p \notin J^+(p) \setminus \{p\}$ for all $p \in M$.
- (M, g) is *strongly causal* if for every $p \in M$ and every neighborhood U of p there exists a neighborhood $V \subset U$ of p, such that any causal curve with two ends inside of V is itself inside of U.

Definition 2.1.7 (Global hyperbolicity)**.** *The spacetime* (M, g) *is called* globally hyperbolic *if it is strongly causal and all causal diamonds* $J(p, q) := J^+(p) \cap J^-(q)$ *are compact.*

Equivalently, it is well-known that one may reduce strong causality to just causality, that is: (M, g) is globally hyperbolic if it is causal and $J(p, q)$ is compact for all $p, q \in M$. In globally hyperbolic spacetimes, $J^{\pm}(p)$ are closed subsets of M and causal propagation enjoys many good compactness properties.

An equivalent definition is the existence of a *Cauchy hypersurface*. A subset $\Sigma \subset M$ is a Cauchy hypersurface if every inextendible timelike curve (meaning it does not have future or past end point) in M intersects Σ exactly once. If Σ is a smooth embedded hypersurface, it is necessarily spacelike. A spacetime is globally hyperbolic if and only if it admits a Cauchy hypersurface; moreover, one may choose Σ smooth, and there exists a smooth *temporal function* $t : M \to \mathbb{R}$ with timelike gradient such that each level set $\Sigma_s := \{t = s\}$ is a smooth spacelike Cauchy hypersurface. In particular, M is diffeomorphic to $\mathbb{R} \times \Sigma$ and one can write

$$g = -\beta(dt^2 + h_t),$$

where $\beta > 0$ and h_t is a smooth family of Riemannian metrics on Σ. Since the relations $\ll, \leq$ and the sets $J^{\pm}(p)$ depend only on the light cones, causality conditions and global hyperbolicity are invariant under conformal rescalings $g \mapsto \Omega^2 g$ with $\Omega > 0$. In particular, if (M, g) is globally hyperbolic and $\hat{g} = \Omega^2 g$, then $(M, \hat{g})$ is globally hyperbolic as well.

Finally, global hyperbolicity can be similarly defined on manifolds with boundaries, we refer to [3] and Chapter 6 for details.

2.2 Microlocal analysis

Microlocal analysis is the refinement of classical PDE analysis in which one studies a distribution not only by its location in the base manifold, but also by the *directions of oscillation* in cotangent space. Concretely, a singularity is encoded as a subset of $T^*X \setminus 0$ rather than merely a subset of X: the *wavefront set* $\mathrm{WF}(u)$ records at which points x and in which covector directions ξ the distribution u fails to be smooth. Pseudodifferential operators provide the natural calculus for localizing in both x and ξ via their symbols, and they capture elliptic regularity by showing that ellipticity is precisely the condition for microlocal invertibility. For operators of real principal type (such as the wave operator), microlocal analysis explains how singularities move: $\mathrm{WF}(u)$ propagates along the Hamiltonian flow of the principal symbol, i.e. along bicharacteristics, giving a precise geometric description of propagation of singularities. In many inverse and geometric problems, this viewpoint is indispensable because it converts analytic questions about solutions into geometric questions on T^*X and canonical relations, and it provides the language of Fourier integral operators to describe solution operators, DN maps, and scattering relations. For more details, we refer to [69, 40, 39, 65].

2.2.1 Symbol and quantization

We start with a standard differential operator

$$P = \sum_{|\alpha|=0}^{N} a_\alpha(x) D_x^\alpha$$

where $D_x = -i\partial_x$ and D_x^α is the multi-index notation for derivatives. For any $u \in C_c^\infty(\mathbb{R}^n)$, we have

$$Pu = \frac{1}{(2\pi)^n} \int \int e^{i(x-y)\xi} \sum_\alpha a_\alpha(x)\xi^\alpha u(y) d\xi dy.$$

In other words, the operator has the form of an oscillatory integral

$$\int e^{i\phi(x,y,\xi)} a(x,y,\xi) d\xi,$$

where ϕ is a non-degenerate phase function and a is called the symbol of the operator. We now introduce the generalization on manifolds.

Let X be a smooth n-dimensional manifold without boundary. We write $T^*X \setminus 0$ for the cotangent bundle with the zero section removed. Locally, in coordinates $x = (x^1, \dots, x^n)$ on X and dual variables $\xi \in \mathbb{R}^n$, we use $\langle \xi \rangle := (1 + |\xi|^2)^{1/2}$.

Definition 2.2.1 (Symbol class)**.** *For $m \in \mathbb{R}$, the standard symbol class $S^m(\mathbb{R}_x^n \times \mathbb{R}_\xi^n)$ consists of all $a(x,\xi) \in C^\infty$ such that for every pair of multi-indices α, β there exists $C_{\alpha\beta}$ with*

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha\beta} \langle \xi \rangle^{m-|\beta|}.$$

On a manifold X, we define $S^m(T^*X)$ by requiring that in every coordinate chart, the local representative belongs to $S^m(\mathbb{R}^n \times \mathbb{R}^n)$, and that changes of coordinates preserve the class. We write $S^{-\infty} := \bigcap_{m \in \mathbb{R}} S^m$ for the smoothing symbols.

A symbol $a \in S^m$ is *classical* (polyhomogeneous), written $a \in S^m_{\mathrm{cl}}$, if it admits an asymptotic expansion

$$a(x,\xi) \sim \sum_{j=0}^\infty a_{m-j}(x,\xi), \qquad |\xi| \to \infty,$$

where each a_{m-j} is homogeneous of degree $m - j$ in ξ for $|\xi| \ge 1$.

Definition 2.2.2 (Quantization)**.** *Fix a cutoff $\chi \in C_c^\infty(\mathbb{R}_\xi^n)$ with $\chi(\xi) = 1$ near $\xi = 0$. For $a \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$*

we define the (Kohn–Nirenberg) quantization

$$Op(a)u(x) := (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}\, a(x,\xi)\, u(y)\, dy\, d\xi, \qquad u \in \mathcal{S}(\mathbb{R}^n).$$

On a manifold X, pseudodifferential operators are defined by localizing to coordinate charts and patching with a partition of unity. We denote by $\Psi^m(X)$ (resp. $\Psi^m_{\mathrm{cl}}(X)$) the space of operators obtained from symbols in $S^m(T^*X)$ (resp. $S^m_{\mathrm{cl}}(X)$); operators in $\Psi^{-\infty}(X)$ are called smoothing operators.

If $A \in \Psi^m(X)$, then the *principal symbol* is defined as the unique equivalence class

$$\sigma_m(A) \in S^m(T^*X)/S^{m-1}(T^*X)$$

such that the quantization of any a in the equivalence class agrees with A modulo $\Psi^{m-1}(X)$. In particular, when $A \in \Psi^m_{\mathrm{cl}}(X)$, $\sigma_m(A)$ is just the leading order term in the expansion of the full symbol. We say an m-th order pseudodifferential operator A is elliptic at (x,ξ) if

$$|\sigma_m(A)(x,\lambda\xi)| \geq C\lambda^m, \quad \lambda > \lambda_0$$

for some C and $\lambda_0 > 0$. An operator is *elliptic* if it is elliptic at all covectors. An equivalence definition of ellipticity is the existence of *parametrices*: there exists $B_1, B_2 \in \Psi^{-m}(X)$ such that $AB_1 - Id, B_2A - Id \in \Psi^{-\infty}(X)$. Namely, the operator is invertible up to smoothing operators.

The class of pseudodifferential operators can be characterized as the ones with the specific phase function $\phi(x,y,\xi) = (x-y)\cdot\xi$. In order to discuss about Fourier integral operators which have different phase functions, we recall some basic symplectic geometry on the cotangent space below.

2.2.2 Basic symplectic geometry on cotangent space

Let X be a smooth manifold. A symplectic structure is a 2-form ω that is closed and non-degenerate. In particular, the cotangent bundle T^*X carries a standard symplectic structure. Consider the standard

Liouville contact form $\alpha \in \Omega^1(T^*X)$, defined at $(x,\xi) \in T^*X$ by

$$\alpha_{(x,\xi)}(V) = \xi\big(d\pi(V)\big), \qquad \pi : T^*X \to X,$$

the *canonical symplectic form* is given by

$$\omega := d\alpha.$$

In local coordinates (x,ξ), one has $\alpha = \sum_j \xi_j \, dx^j$ and $\omega = \sum_j d\xi_j \wedge dx^j$. We will be working with this symplectic structure in this thesis.

Later on, the Fourier integral operators we will be talking about map functions between manifolds X and Y, thus one naturally works on the product

$$T^*Y \times T^*X,$$

equipped with the symplectic form

$$\omega_{Y \times X} := \pi_Y^* \omega_Y - \pi_X^* \omega_X,$$

i.e. the product of (T^*Y, ω_Y) with $(T^*X, -\omega_X)$. The difference is chosen instead of the sum because this twisted symplectic 2-form makes the graph of a *symplectomorphism* a *Lagrangian submanifold*. We now explain what symplectomorphism and Lagrangian submanifolds are.

A submanifold $\Lambda \subset (T^*X, \omega)$ is *isotropic* if $\omega|_{T\Lambda} = 0$. It is *Lagrangian* if it is isotropic and has maximal possible dimension, equivalently

$$\omega|_{T\Lambda} = 0 \quad \text{and} \quad \dim \Lambda = n = \tfrac{1}{2} \dim T^*X.$$

A basic example is the zero section $X \hookrightarrow T^*X$. Another key example is the *conormal bundle*: if $S \subset X$ is an embedded submanifold, its conormal bundle

$$N^*S := \{(x,\xi) \in T^*X : \ x \in S, \ \xi|_{T_xS} = 0\}$$

is a Lagrangian submanifold of T^*X. More generally, if $\phi(x, y, \xi)$ is a nondegenerate phase function, then the critical set of ϕ

$$\{(x, \partial_x \phi; y, -\partial_y \phi) : \partial_\xi \phi = 0\}$$

produces a Lagrangian submanifold in the twisted symplectic manifold $T^*X \times T^*Y$; this is the geometric backbone of Fourier integral operators. Clearly, when $\phi = (x - y) \cdot \xi$ and x, y belong to the same manifold X, the corresponding Lagrangian submanifold is simply the diagonal $\Delta_{T^*X} = \{(x, \xi; x, \xi)\}$ of $T^*X \times T^*X$.

A *canonical transformation* or *symplectomorphism* is a diffeomorphism $\kappa : U \to V$ between open subsets $U \subset T^*X \setminus 0$ and $V \subset T^*Y \setminus 0$ satisfying $\kappa^* \omega_Y = \omega_X$. Its graph

$$\Gamma_\kappa := \{(\kappa(x, \xi), (x, \xi))\} \subset T^*Y \setminus 0 \times T^*X \setminus 0$$

is a conic Lagrangian submanifold of $(T^*Y \times T^*X,\ \omega_{Y \times X})$. Canonical relations are the geometric data underlying Fourier integral operators: an operator A is an FIO associated to C when its Schwartz kernel is an oscillatory integral whose wavefront set is contained in (and typically equals) C.

Finally, given $p \in C^\infty(T^*X)$, the *Hamiltonian vector field* H_p is defined by

$$\omega(H_p, \cdot) = dp(\cdot).$$

The corresponding integral curves are called the Hamiltonian flow, which is crucial in the study of propagation of singularities. In local coordinates,

$$H_p = \sum_{j=1}^{n} \frac{\partial p}{\partial \xi_j} \frac{\partial}{\partial x^j} - \frac{\partial p}{\partial x^j} \frac{\partial}{\partial \xi_j}.$$

The associated Poisson bracket is

$$\{p, q\} := \omega(H_p, H_q) = H_p(q).$$

If $A, B \in \Psi_{\mathrm{cl}}(X)$, then the principal symbol of $i[A, B]$ is precisely $\{\sigma_m(A), \sigma_{m'}(B)\}$.

2.2.3 Lagrangian distribution and principal symbol

We are now ready to introduce the class of *Lagrangian distributions*, which contains the class of Fourier integral operators. First, we recall the definition of density bundles, as they are crucial for intrinsic definition of principal symbols.

The density bundle on X is denoted Ω_X. In local coordinates, it is a collection of functions on coordinate charts which become multiplied by the absolute value of the Jacobian determinant in the change of coordinates. A *half-density* is a smooth section of a chosen square root bundle $\Omega_X^{1/2}$ (well-defined intrinsically as a real line bundle); locally it is written $u(x)\,|dx|^{1/2}$ and transforms by the square root of the absolute Jacobian. The main advantage is that many microlocal objects (especially FIOs) become coordinate-invariant when equipped with a half-density bundle.

Specifically, given manifolds X, Y, an operator

$$A : C_c^\infty(X; \Omega_X^{1/2}) \to \mathcal{D}'(Y; \Omega_Y^{1/2})$$

is represented by a Schwartz kernel

$$K_A \in \mathcal{D}'(Y \times X;\ \Omega_Y^{1/2} \otimes \Omega_X^{1/2}),$$

so that for $u \in C_c^\infty(X; \Omega_X^{1/2})$ one has

$$(Au)(y) = \int_X K_A(y, x)\, u(x),$$

where $K_A(y, x)\, u(x)$ is a density in x (hence can be integrated), and the result is a half-density in y.

Definition 2.2.3 (Lagrangian distribution). *Let $\Lambda \subset T^*X \setminus 0$ be a conic Lagrangian submanifold. A* Lagrangian distribution *of order m associated to Λ (with values in half-densities) is locally an oscillatory integral of the form*

$$K(x) = (2\pi)^{-N} \int_{\mathbb{R}^N} e^{i\phi(x,\theta)}\, a(x, \theta)\, |dx|^{1/2}\, d\theta,$$

where ϕ is a nondegenerate phase function locally parameterizing Λ and a is a symbol in θ. A Fourier Integral Operator (FIO) *is an operator $A : \mathcal{D}'(X; \Omega_X^{1/2}) \to \mathcal{D}'(Y; \Omega_Y^{1/2})$ whose Schwartz kernel is a Lagrangian distribution. The corresponding Lagrangian submanifold Λ is called the* canonical relation *for the FIO.*

The *principal symbol* of a Lagrangian distribution is a section of a natural line bundle over the Lagrangian; if one works with half-densities, it is canonically a *half-density along* Λ, up to the standard Maslov phase ambiguity. Concretely, for our purposes it suffices to remember:

- the principal symbol is an equivalence class in

$$S^l(\Lambda; \Omega_X^{1/2} \otimes M_\Lambda)/S^{l-1}(\Lambda; \Omega_X^{1/2} \otimes M_\Lambda)$$

  defined intrinsically on Λ, where M_Λ is the Maslov bundle (see [69] for definition and details);

- it is invariant under changes of phase function and local coordinate system representation of the same Lagrangian.

We also refer to the preliminary part of Chapter 4 for a more detailed explanation.

2.2.4 Wavefront set

Microlocal singularities of a distribution are encoded by its *wavefront set*, which refines the singular support by recording directions of non-smoothness in cotangent space.

Definition 2.2.4 (Wavefront set)**.** *Let $u \in \mathcal{D}'(X)$, $WF(u)$ is a subset of $T^*X \setminus 0$ satisfying the following condition: $(x, \xi) \notin WF(u)$ if and only if there exists $\chi \in C_c^\infty(X)$ with $\chi(x) \neq 0$ and a conic neighborhood Γ of ξ such that for every $N \in \mathbb{N}$, there exists C_N such that*

$$|\widehat{\chi u}(\xi)| \leq C_N \langle \xi \rangle^{-N} \qquad \textit{for all } \xi \in \Gamma.$$

Wavefront set indeed encodes more detailed information about the singularity of the distribution, since $\text{singsupp}(u) = \pi(\text{WF}(u)) \subset X$ where π is the natural projection on the cotangent bundle. We include here some basic properties of the wavefront set:

- (*Conic*) $\mathrm{WF}(u)$ is closed and conic in the fibers: $(x, \xi) \in \mathrm{WF}(u) \Rightarrow (x, \lambda\xi) \in \mathrm{WF}(u)$ for all $\lambda > 0$.
- (*Smoothness*) $\mathrm{WF}(u) = \emptyset$ iff $u \in C^\infty(X)$.
- (*Localization*) If $\chi \in C_c^\infty(X)$, then $\mathrm{WF}(\chi u) \subset \mathrm{WF}(u)$ and $\mathrm{singsupp}(\chi u) \subset \mathrm{supp}\chi \cap \mathrm{singsupp}(u)$.
- (*Differentiation*) If P is a differential operator with smooth coefficients, then $\mathrm{WF}(Pu) \subset \mathrm{WF}(u)$.

One may characterize the wavefront set via testing by Pseudodifferential operators:

$$(x_0, \xi_0) \notin \mathrm{WF}(u) \quad \Longleftrightarrow \quad \exists\, A \in \Psi^0(X) \text{ elliptic at } (x_0, \xi_0) \text{ such that } Au \in C^\infty(X).$$

In particular, pseudodifferential operators are *microlocal*:

$$A \in \Psi^m(X) \quad \Longrightarrow \quad \mathrm{WF}(Au) \subset \mathrm{WF}(u).$$

Moreover, if A is elliptic on a conic set $\Gamma \subset T^*X \setminus 0$, then $Au \in C^\infty$ microlocally on Γ implies $u \in C^\infty$ microlocally on Γ (elliptic regularity).

More generally, the wavefront set of the Schwartz kernel of a FIO reveals how singularities are moved by the operator. If A is a Fourier integral operator associated to a canonical relation C, then for $u \in \mathcal{D}'(X)$,

$$\mathrm{WF}(Au) \subset C \circ \mathrm{WF}(u) := \{(y, \eta) : \exists (x, \xi) \in \mathrm{WF}(u) \text{ with } (y, \eta; x, \xi) \in C\}.$$

In the special case $A \in \Psi^m(X)$, the relation C is (microlocally) the diagonal, and one recovers $\mathrm{WF}(Au) \subset \mathrm{WF}(u)$.

2.2.5 Propagation of singularities

Finally, in this section we briefly talk about one of the most important microlocal property of hyperbolic equations: the propagation of singularities.

Let $P \in \Psi^m(X)$ be a properly supported pseudodifferential operator with real principal symbol

$p = \sigma_m(P)$. The *characteristic set* of P is

$$\mathrm{Char}(P) := \{(x, \xi) \in T^*X \setminus 0 : p(x, \xi) = 0\}.$$

On T^*X we consider the Hamiltonian vector field H_p associated to p with respect to the canonical symplectic form. Integral curves of H_p contained in $\mathrm{Char}(P)$ are called *(null)-bicharacteristics*. Since $H_p(p) = \{p, p\} = 0$, the flow of H_p preserves $\mathrm{Char}(P)$.

We say that P is of *real principal type* on an open conic set $\Gamma \subset \mathrm{Char}(P)$ if p is real-valued and

$$dp \neq 0 \quad \text{on } \Gamma.$$

Equivalently, H_p is nonvanishing on Γ, so bicharacteristics form a smooth one-dimensional foliation of Γ. Typical examples include the wave operator $\Box_g$ on a Lorentzian manifold, whose principal symbol is $p(x, \xi) = g^{\mu\nu}(x)\xi_\mu\xi_\nu$; in that case $\mathrm{Char}(\Box_g)$ is the null cone in $T^*X \setminus 0$.

Theorem 2.2.5 (Propagation of singularities)**.** *Let $u \in \mathcal{D}'(X)$ and assume $Pu \in C^\infty(X)$ microlocally on an open conic set $\Gamma \subset \mathrm{Char}(P)$. Then the wavefront set of u restricted to Γ is a union of bicharacteristics:*

$$\mathit{WF}(u) \cap \Gamma \textit{ is invariant under the Hamilton flow of } H_p.$$

Equivalently, one can formulate this as the preservation of regularity: if $(x_0, \xi_0) \in \Gamma$ and there exists a bicharacteristic segment through (x_0, ξ_0) on which u is microlocally smooth at one point, then u is microlocally smooth at all points along that segment (as long as it stays in Γ). In particular, when Pu is smooth, singularities of u can only propagate along the null-bicharacteristic flow.

To fully characterize the microlocal behavior, we include the *elliptic regularity* result outside of the characteristic set. If $(x_0, \xi_0) \notin \mathrm{Char}(P)$ (i.e. $p(x_0, \xi_0) \neq 0$), then P is elliptic at (x_0, ξ_0) and microlocal elliptic regularity gives

$$(x_0, \xi_0) \notin \mathrm{WF}(Pu) \quad \Longrightarrow \quad (x_0, \xi_0) \notin \mathrm{WF}(u).$$

Thus if $Pu \in C^\infty(X)$, then singularities can only occur on $\mathrm{Char}(P)$, and within $\mathrm{Char}(P)$ they propagate along H_p when P is of real principal type.

For $P = \Box_g$ on a Lorentzian manifold (M, g), the bicharacteristics of $p(x, \xi) = g^{\mu\nu}\xi_\mu\xi_\nu$ project to null geodesics in M (up to reparametrization). Hence, if $\Box_g u$ is smooth, the singularities of u propagate along null geodesics. This is the microlocal content of finite speed of propagation and underlies geometric optics parametrices and the FIO nature of solution operators.

Part I

Inverse problems for hyperbolic PDEs

Chapter 3

Inverse problem for nonlinear Dirac equation

The content of this chapter corresponds to my work [156]. The main result is Theorem 3.1.3.

3.1 Introduction

The Dirac equation, introduced by Paul Dirac in [37], is fundamental in quantum mechanics and quantum field theory for describing fermions. Its nonlinear variants, which can be seen either as self-interacting spinors (e.g. the cubic term $\langle \beta u, u\rangle u$) or as reductions of the coupled Dirac-Maxwell (and ultimately Einstein-Cartan-Sciama–Kibble) systems, have attracted extensive mathematical study. In particular, extensive symmetry classifications and explicit solutions were developed by Fushchich and Zhdanov [50], while Esteban and Séré [44] proved the existence of stationary (time-harmonic) states via variational methods. A recent centenary review of Dirac-equation derivations appears in [127], and the cubic self-interaction induced by spacetime torsion in Einstein-Cartan-Sciama-Kibble gravity is discussed in [63]. Finally, the full Maxwell-Dirac coupling has been shown to be well-posed and structurally rich in the works of Chadam [24], Radford [119], Glassey and Strauss [52], and Sparber and Markowich [128].

In this chapter, we consider an inverse problem for 4 by 4 nonlinear Dirac equations of the following form

$$\begin{cases} i\partial_t u + i\alpha \cdot \nabla u - \beta u = F(x,u) \text{ on } (0,T) \times \mathbb{R}^3 \\ \\ u(0,x) = \phi(x) \end{cases} \tag{3.1.1}$$

where the notations mean the following:

1. $u : [0,T] \times \mathbb{R}^3 \to \mathbb{C}^4$ is the solution of (3.1.1), with the variables denoted by (t,x);

2. For $\xi \in \mathbb{R}^3$, $\alpha \cdot \xi = \sum_{j=1}^{3} \xi_j \alpha_j$, and α_j are 4×4 Pauli matrices defined by

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};$$

3. β is the 4×4 matrix

$$\beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}$$

   and I_2 is the 2×2 identity matrix;

4. $F : \mathbb{R}^3 \times \mathbb{C}^4 \to \mathbb{C}^4$ is a smooth nonlinearity with $F(x,0) = 0$, and its partial derivatives of all orders are bounded on sets of the form $\mathbb{R}^3 \times K$, where $K \subset \mathbb{C}^4$ compact. We denote the variables as (x,z).

We now phrase the main question. Assume that given any initial data $u(0) = \phi$, we are able to read the solution $u(T)$ at time T; in other words, we have complete knowledge of the map $u(0) \mapsto u(T)$. Does this uniquely determine the nonlinearity F? We will give an affirmative answer for data in the appropriate space of functions to be specified below.

To precisely state our main result, we first invoke a local existence result from [121], adapted to our setting.

Theorem 3.1.1. *([121] Theorem 6.3.1) If $s > 3/2$, then there is a $T > 0$ and a unique solution $u \in C([0,T]; H^s(\mathbb{R}^3))$ to the semilinear initial value problem defined by the partial differential equation* (3.1.1) *together with the initial condition*

$$u(0,x) = \phi(x) \in H^s(\mathbb{R}^3).$$

The time T can be chosen uniformly for ϕ from bounded subset of $H^s(\mathbb{R}^d)$. Consequently, there is a $T^* \in]0, \infty]$ and a maximal solution $u \in C([0, T^*[; H^s(\mathbb{R}^d))$. If $T^* < \infty$, then

$$\lim_{t \to T^*} \|u(t)\|_{H^s(\mathbb{R}^d)} = \infty.$$

The above theorem allows us to define bounded time solution map for sufficiently small initial data.

Definition 3.1.2. *Given $s > 3/2$ and $\delta > 0$, denote*

$$H^s_\delta := \{\phi \in H^s(\mathbb{R}^3) : ||\phi||_{H^s} < \delta\}.$$

By Theorem 3.1.1 there exists some $T > 0$ such that for any initial data $\phi \in H^s_\delta$, (3.1.1) *has a unique solution u up to time T. Then the time T solution map is defined on H^s_δ by*

$$S(\phi) = u(T).$$

That is, given any energy-wise sufficiently small initial data, the time T solution map returns the solution at time T. We can now state the main result.

Theorem 3.1.3. *Given smooth nonlinearities F_1 and F_2, suppose for some $T > 0$ they each have a well-defined time T solution map S_1, S_2 on the same domain H^s_δ. Then $S_1 = S_2$ implies $\partial_z^3 F_1(x, z) \equiv \partial_z^3 F_2(x, z)$ for all $(x, z) \in \mathbb{R}^3 \times B_M$ where $B_M \subset \mathbb{C}^4$ is the closed ball centered at 0 with radius M, and*

$$M = \sup\{||\phi||_{L^\infty} : \phi \in H^s_\delta \cap C^\infty(\mathbb{R}^3)\}.$$

In addition, if for all $x \in \mathbb{R}^3$,

$$\partial_z F_1(x, 0) \equiv \partial_z F_2(x, 0), \quad \partial_z^2 F_1(x, 0) \equiv \partial_z^2 F_2(x, 0), \tag{3.1.2}$$

then $F_1(x, z) = F_2(x, z)$ for all $(x, z) \in \mathbb{R}^3 \times B_M$.

Although our result is proved for smooth small-data solutions, it applies directly to several physical models governed by semilinear Dirac equations. In Einstein-Cartan-Sciama-Kibble gravity, coupling the Dirac field to torsion produces exactly the cubic, four-fermion self-interaction studied here. In condensed matter, "Dirac materials" such as graphene or topological insulators, effective Dirac equations with nonlinear corrections capture electron-electron and electron-phonon interactions, so one could, at least in principle, use bounded time scattering data to recover the interaction law. Similar semilinear Dirac systems also arise in nonlinear optics as envelope equations for spinor-like wavepackets in photonic lattices.

For simplicity of the notations, in the remaining part of the chapter we shall only use F instead of F_1 and F_2, and prove that $\partial_z^3 F$ can be determined from the solution map. Specifically, whenever we say A determines B, we mean the following: for $i = 1, 2$, let A_i, B_i be some quantities given by nonlinearity F_i, then $A_1 = A_2$ implies $B_1 = B_2$.

Below we make several remarks about our result and its consequences.

Remark 3.1.4.

1. *Note that if $\partial_z^3 F(x, z)$ has been determined on $\mathbb{R}^3 \times B_M$, then for any $x \in \mathbb{R}^3$, $F(x, z)$ is determined up to a degree 2 polynomial in z whose constant term is 0. With the extra assumption on the first and second derivatives, one can then determine the nonlinearity. We provide an example to illustrate the theorem in terms of the most important nonlinearity for the Dirac equation, namely the cubic nonlinearity. Suppose there exists some smooth nonlinearity F with vanishing first and second derivative at $z = 0$, and whose time T solution map coincides with that of the cubic nonlinearity. Then, F must match with the cubic nonlinearity in a neighborhood of 0.*

2. *If one can take $s < 3/2$ by imposing additional assumptions on F such that the bounded solution map is well-defined for small data (for instance, see [41] and [100]), then there is no uniform bound on the L^∞ norm in $H^s_\delta \cap C^\infty(\mathbb{R}^3)$. In this case, $M = +\infty$, that is F can be determined on the entire $\mathbb{R}^3 \times \mathbb{C}^4$. (For example, for $\phi \in C_c^\infty(\mathbb{R}^3)$, we can consider $\phi_\epsilon(x) = \epsilon^{-k}\phi(x/\epsilon)$ for $k + s < n/2$, then $||\phi_\epsilon||_{H^s} \to 0$ while $||\phi_\epsilon||_{L^\infty} \to \infty$).*

3. *As a direct result of Theorem 3.1.3, if we have* (3.1.2) *and an additional assumption that both nonlin-*

earities are analytic in z, then $F_1 = F_2$ on $\mathbb{R}^3 \times \mathbb{C}^4$.

Now we explain the main ingredients and challenges. The method used in this chapter involves perturbative techniques around smooth solutions, adapted from the framework of higher order linearization and microlocal analysis established in [83] by Kurylev, Lassas and Uhlmann. This approach leverages nonlinear wave interactions to recover information about the derivatives of nonlinearity through carefully constructed wave collisions. After [82, 83], there has been significant progress on inverse problem for nonlinear hyperbolic equations, see for example [152, 68, 12, 26, 147, 95, 143, 77, 150, 81, 154, 25, 96, 151, 14, 123, 46, 67] and for an overview of the recent progress in [149, 85].

Our work presents additional challenges compared to previous works which stem from the following two features of the problems: 1) Dirac equation is a first order PDE system rather than a scalar equation; 2) we do not assume any specific structural condition for the nonlinearity. More specifically, we encounter the following difficulties:

- **Restrictions on initial data:** Although the Dirac equation remains of real principal type (see Section 3.3 for details), it differs from the scalar case (see for example [82, 96, 68, 152, 124]) in that its principal part is a non-diagonal matrix that is singular on the characteristic set. As we will show, the principal symbol of the higher order linearization must satisfy a matrix-coefficient transport equation along bicharacteristics, with initial values restricted to specific subspaces of $\mathbb{C}^4$ based on each bicharacteristic. We emphasize that this phenomenon also does not happen to all systems, it arises when the principal symbol of the operator is not the zero matrix on the characteristic set (see Section 3.4). For instance, this does not happen to system of wave equation, as the principal symbol is simply the principal symbol of wave operator times identity matrix, which vanishes on the characteristic set.

- **Non-trivial propagation:** Another problem is caused by the fact that Dirac operator is of degree 1. For comparison we consider a semilinear wave equation $\Box u = f(u)$ with unknown nonlinearity f. The terms in higher order linearization satisfy equations of the form

$$(\Box - f'(u))w = g.$$

In this context, multiplication by $f'(u)$ is 2 degree lower than $\Box$, which, according to symbol calculus, does not appear in the transport equation satisfied by the principal symbol of w. As a result, w has constant principal symbol along bicharacteristics. For the Dirac equation, however, multiplication by $\partial_z F$ is on the subprincipal level because Dirac operator is a degree 1 operator. Consequently F is always involved in the transport equation (see Section 3.3.3), leading to non-trivial propagation along bicharacteristic. Moreover, being a system again makes things more complicated as a first order linear system of ODE whose matrix coefficient depends on the variable usually does not have an explicit solution.

- **No structure assumption on F:** There have been some works on systems of PDE and they also encountered the problem of non-trivial propagation. In [26] and [25], the problem was reduced to collecting information about the non-Abelian broken ray transform, followed by inversion of a novel non-Abelian broken light ray transform. However, this approach relies on the specific structure of nonlinearity. For general smooth nonlinearity F, when performing 3 wave interaction and let the three incoming rays converge to the same one, the information at the collision point will be of the form $\partial_z^3 F(x, u, Bv, Bv, Bv)$ for some parallel transport operator B (see [25]) and vector v. In general it is hard to convert it into usable information, which is of the form BG where G doesn't depend on nonlinearity. It is also worth mentioning that another direct consequence of not having structural assumption is that perturbation around zero solution does not recover information of the nonlinearity away from 0, and perturbation around non-trivial solutions further complicates the aforementioned transport equation.

The main strategy for overcoming these challenges is to construct a sequence of collisions that approaches the initial time $t = 0$. Let us outline the general idea below. We fix a bicharacteristic from time 0 to time T, and again use higher order linearization and microlocal analysis to observe how information propagates along this bicharacteristic line. For suitable initial value at time 0, its final value at time T would be determined by the solution map. If, on the other hand, the starting position of the information is at a point on the bicharacteristic that is very close to the actual beginning point at $t = 0$, we can expect its outcome at time T to be very close to the outcome that would result from initiating the information at the actual starting point. This observation suggests the main idea of the proof: we perform 3-wave interactions multiple times along a fixed bicharacteristic, allowing the collision point to approach the boundary at $t = 0$. The limiting case would correspond to the propagation of collision data from $t = 0$ to $t = T$. With some linear algebra the third derivative of F at the boundary can be fully determined.

The organization of the chapter is as follows. In Section 3.2 we perform asymptotic analysis to compute the equations satisfied by each linearization term. In Section 3.3 we present some results about distribution multiplication and propagation of singularity for future use. In Section 3.4 we define and prove the bijectivity of initial data to final data map. In Section 3.5, we perform a 3-wave interaction and compute the equation satisfied by the principal symbol of the third order linearization term. In Section 3.6 we obtain a one parameter family of collisions and compute the limiting case. In Section 3.7 we combine the previous results and finish the proof of Theorem 3.1.3 with some linear algebra arguments.

3.2 Asymptotic analysis

To simplify the notation, starting now we will perform the computations for $F(z)$ instead of $F(x, z)$, thus all derivatives below are taken with respect to z. As one will see the computations up to the end of Section 3.6 are exactly the same when F also depends on x. We will come back to $F(x, z)$ in Section 3.7 when we prove Theorem 3.1.3.

We now introduce perturbations around small smooth initial data. Consider initial data of the form $\phi_\epsilon = \phi + \sum_1^3 \epsilon_j \phi_j$ where $\phi \in H^s_\delta \cap C^\infty$, we only require $\phi_j \in H^s$ such that $\phi_\epsilon \in H^s_\delta$ for sufficiently small ϵ_j. In fact, we will impose specific type of singularity to ϕ_j later.

Let u, u_ϵ denote the solution to (3.1.1) with initial data ϕ, ϕ_ϵ, respectively. By Theorem 3.1.1, u is smooth in x for $t \in [0, T]$ and continuous in t. Since u satisfies (3.1.1), a standard bootstrap argument shows that u is smooth in both t and x.

Let w_j be the solution of the following linear Dirac equation

$$\begin{cases} i\partial_t w_j + i\alpha\cdot\nabla w_j - \beta w_j - F'(u)w_j = 0 \text{ on } (0,T)\times\mathbb{R}^3, \\ w_j(0) = \phi_j. \end{cases}$$

If w solves the following non-homogeneous linear Dirac equation

$$\begin{cases} i\partial_t w + i\alpha\cdot\nabla w - \beta w - F'(u)w = f \text{ on } (0,T)\times\mathbb{R}^3, \\ w(0) = 0, \end{cases} \tag{3.2.1}$$

we denote $Q(f) = w$, note that Q would depend on u. Write the Taylor expansion of F around u as

$$\begin{aligned} F(u_\epsilon) &= F(u) + F'(u)(u_\epsilon - u) + F^{(2)}(u, u_\epsilon - u, u_\epsilon - u) \\ &\quad + F^{(3)}(u, u_\epsilon - u, u_\epsilon - u, u_\epsilon - u) + O(|u_\epsilon - u|^4) \end{aligned}$$

where

$$F^{(k)}(u, v_1, ..., v_k) = \sum_{|\gamma|=k} \frac{1}{\gamma!}(\partial^\gamma F)(u) v_1^{\gamma_1} ... v_k^{\gamma_k}.$$

Then $(u_\epsilon - u - \sum \epsilon_j w_j)(0) = 0$ and

$$\begin{aligned} &(i\partial_t + i\alpha\cdot\nabla - \beta - F'(u))(u_\epsilon - u - \sum \epsilon_j w_j) \\ &= F(u_\epsilon) - F(u) - F'(u)(u_\epsilon - u) \\ &= F^{(2)}(u, u_\epsilon - u, u_\epsilon - u) + F^{(3)}(u, u_\epsilon - u, u_\epsilon - u, u_\epsilon - u) + O(|u_\epsilon - u|^4). \end{aligned}$$

That is,

$$\begin{aligned} u_\epsilon - u = \sum \epsilon_j w_j + Q\Big(&F^{(2)}(u, u_\epsilon - u, u_\epsilon - u) \\ &+ F^{(3)}(u, u_\epsilon - u, u_\epsilon - u, u_\epsilon - u) + O(|u_\epsilon - u|^4)\Big). \end{aligned}$$

By repeatedly substituting $u_\epsilon - u$ with the right hand side and use the linearity of Q and symmetry of

$F^{(2)}, F^{(3)}$, we have

$$u_\epsilon = u + \sum \epsilon_j w_j + \sum_{i\le j} \epsilon_i \epsilon_j w_{ij} + \sum_{i\le j\le k} \epsilon_i\epsilon_j\epsilon_k w_{ijk} + O(|\epsilon|^4) \tag{3.2.2}$$

where for $i < j < k$,

$$\begin{aligned}
w_{ij} =& Q(2F^{(2)}(u, w_i, w_j)),\\
w_{ii} =& Q(F^{(2)}(u, w_i, w_i)),\\
w_{ijk} =& Q\left(6F^{(3)}(u, w_i, w_j, w_k) + 2F^{(2)}(u, w_{ij}, w_k)\right.\\
&\left.+2F^{(2)}(u, w_{ik}, w_j) + 2F^{(2)}(u, w_{jk}, w_i)\right),\\
w_{iij} =& Q\left(3F^{(3)}(u, w_i, w_i, w_j) + 2F^{(2)}(u, w_{ii}, w_j)\right.\\
&\left.+2F^{(2)}(u, w_{ij}, w_i)\right),\\
w_{iii} =& Q\left(F^{(3)}(u, w_i, w_i, w_i) + 2F^{(2)}(u, w_{ii}, w_i)\right).
\end{aligned} \tag{3.2.3}$$

The next perturbation result from [121], again adapted to current scenario, shows that the above terms are well-defined and the remaining term in (3.2.2) is of $|\epsilon|^4$ order in $C([0, T]; H^s(\mathbb{R}^3))$.

Theorem 3.2.1. *([121] Theorem 6.5.2) If $u \in C([0, T]; H^s(\mathbb{R}^3))$ is a solution of* (3.1.1)*, then the map $\psi \mapsto w$ from initial data to solution for* (3.1.1) *is smooth from a neighborhood of $u(0)$ in $H^s(\mathbb{R}^3)$ to $C([0, T]; H^s(\mathbb{R}^3))$, and the first, second and third derivatives of $\phi_\epsilon \mapsto u_\epsilon$ with respect to $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3)$ at $\epsilon = 0$ are given by w_i, w_{ij} and w_{ijk} satisfying the above equations. Derivatives of each order are uniformly bounded on the neighborhood.*

By (3.2.2), the solution map satisfies

$$S(\phi_\epsilon) = u_\epsilon(T) = u(T) + \sum \epsilon_j w_j(T) + \sum_{i\le j} \epsilon_i\epsilon_j w_{ij}(T) + \sum_{i\le j\le k} \epsilon_i\epsilon_j\epsilon_k w_{ijk}(T) + O(|\epsilon|^4).$$

By Theorem 3.2.1, the solution map $S : H^s_\delta \to H^s(\mathbb{R}^3)$ is smooth on a neighborhood of ϕ, we denote its Fréchet derivatives as

$$\partial_{\epsilon^\gamma}(S(\phi_\epsilon))|_{\epsilon=0}$$

for multi-index γ. For example, $\phi_j \mapsto \partial_{\epsilon_j}(S(\phi + \epsilon_j \phi_j))|_{\epsilon_j=0}$ is a bounded linear operator satisfying

$$\lim_{\epsilon_j \to 0} \frac{||S(\phi + \epsilon_j \phi_j) - S(\phi) - \partial_{\epsilon_j} S(\phi + \epsilon_j \phi_j)|_{\epsilon_j=0}||_{H^s(\mathbb{R}^3)}}{\epsilon_j ||\phi_j||_{H^s(\mathbb{R}^3)}} = 0.$$

Same for the higher order derivatives. Then by Theorem 3.2.1, we have the following proposition.

Proposition 3.2.2. *Let the initial data be of the form $\phi_\epsilon = \phi + \sum_{j=1}^{3} \epsilon_j \phi_j$ for $\phi \in H^s_\delta$, $\phi_j \in H^s(\mathbb{R}^3)$. Then w_j, w_{ij} and w_{ijk} defined in* (3.2.3) *are well-defined in $H^s(\mathbb{R}^3)$. The first, second and third derivatives of the solution map give the solution maps of the corresponding linearization terms:*

$$\partial_{\epsilon_j}(S(\phi_\epsilon))|_{\epsilon=0} = w_j(T),$$

$$\partial_{\epsilon_i,\epsilon_j}(S(\phi_\epsilon))|_{\epsilon=0} = w_{ij}(T),$$

$$\partial_{\epsilon_i,\epsilon_j,\epsilon_k}(S(\phi_\epsilon))|_{\epsilon=0} = w_{ijk}(T).$$

3.3 Microlocal analysis

In this section we mainly study the operator

$$P = i\partial_t + i\alpha \cdot \nabla - \beta - F'(u).$$

The definition of real principal type operator for systems of pseudodifferential operators is given in [36].

Definition 3.3.1. *([36] Definition 3.1) An $N \times N$ system P of pseudodifferential operators on X with principal symbol $p(x,\xi)$ is of real principal type at $(y,\eta) \in T^*X \backslash 0$ if there exists an $N \times N$ symbol $\tilde{p}(x,\xi)$ such that*

$$\tilde{p}(x,\xi)p(x,\xi) = q(x,\xi) \cdot Id_N$$

in a neighborhood of (y,η), where $q(x,\xi)$ is a scalar symbol of real principal type. We say that P is of real principal type in T^*X if it is at every point.

The matrices satisfies the following relations:

$$\alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{jk} I_4, \quad \alpha_j \beta = -\beta \alpha_j, \quad \beta^2 = I_4.$$

Hence P is of real principal type with $\tilde{p} = \tau - \alpha \cdot \xi$ and $q = \tau^2 - |\xi|^2$. $\tilde{P} = -i\partial_t + i\alpha \cdot \nabla$ is a quantization of $\tilde{p}$, and

$$\tilde{P}P = \Box - (-i\partial_t + i\alpha \cdot \nabla)(\beta + F'(u))$$

whose principal part is the wave operator. When we say the Hamiltonian vector field related to p or P we mean the Hamiltonian vector field H_q generated by the scalar symbol q.

3.3.1 Notations

We first introduce some notations.

$$\begin{aligned}
\operatorname{Char}(P) &= \{(t, x, \tau, \xi) : \det(-\tau - \alpha \cdot \xi) = 0, \xi \neq 0\} \\
&= \{(t, x, \tau, \xi) : (\tau^2 - |\xi|^2)^2 = 0, \xi \neq 0\} \\
&= L\mathbb{R}^4 \backslash 0.
\end{aligned}$$

the set of lightlike covectors. For some $T' < 0$, let $y_j^{T'} = (T', x_j^{T'})$ and η_j a light-like covector. $\gamma_{y,\eta}$ is the geodesic passing through y in the direction of $\eta^\sharp$. Denote

$$\begin{aligned}
&\mathcal{V}(y_j^{T'}, \eta_j, \epsilon) = \{\eta \in T^*_{y_j^{T'}}\mathbb{R}^{n+1} : |\eta - \eta_j| < \epsilon, |\eta| = |\eta_j|\}, \\
&K(y_j^{T'}, \eta_j, \epsilon) = \{\gamma_{y_j^{T'}, \eta}(s) : \eta \in L^+_{y_j^{T'}}\mathbb{R}^4 \cap \mathcal{V}(y_j^{T'}, \eta_j, \epsilon), s \in (0, \infty)\}, \\
&\Sigma(y_j^{T'}, \eta_j, \epsilon) = \{(y_j^{T'}, r\eta^\flat) \in T^*\mathbb{R}^{n+1} : \eta \in \mathcal{V}(y_j^{T'}, \eta_j, \epsilon), r \neq 0\}.
\end{aligned}$$

Let $\Lambda(y_j^{T'}, \eta_j, \epsilon)$ be the Lagrangian manifold that is the flowout of lightlike covectors in $\Sigma(y_j^{T'}, \eta_j, \epsilon)$ via H_q in the future direction. We shall use $\mathcal{V}_j, K_j, \Sigma_j, \Lambda_j$ for simplicity, and denote

$$\Gamma_j(s) = (y_j, \eta_j) + 2s(\tau_j, -\xi_j, 0, 0) \tag{3.3.1}$$

the bicharacteristic where $\Gamma_j(0) = (y_j, \eta_j) = (0, x_j, \tau_j, \xi_j)$, $\Gamma_j(\frac{T'}{2\tau_j}) = (y_j^{T'}, \eta_j)$. The end point is $\Gamma_j(\frac{T}{2\tau_j}) = (y_j^T, \eta_j) = (T, x_j^T, \tau_j, \xi_j)$.

To perform multiple wave interactions, we introduce the following notations

$$K_{ij} = K_i \cap K_j, \quad K_{123} = K_1 \cap K_2 \cap K_3,$$
$$\Lambda_{ij} = N^* K_{ij}, \quad \Lambda_{123} = N^* K_{123}.$$

If K_1, K_2, K_3 intersect transversally, then conormal fiber of K_{123} is spanned by the conormal fiber of each K_j so would be a timelike subspace, hence K_{123} is a spacelike 1 dimensional submanifold. Let q be a point in K_{123} and (q, ζ) a lightlike direction in Λ_{123}, then the corresponding bicharacteristic only intersect Λ_{123} once. As $\epsilon \mapsto 0$, we see that K_j goes to Γ_j and K_{123} converges to a point.

3.3.2 Lagrangian Distributions and Intersecting Lagrangians

Given a conic Lagrangian submanifold Λ of $T^*\mathbb{R}^4 \backslash 0$, $\mathcal{I}^\mu(\Lambda; \mathbb{C}^4)$ denotes all the corresponding Lagrangian distributions associated with Λ of order μ taking value in $\mathbb{C}^4$. The wavefront set of such distribution is referred to the union of the wavefront set of each coordinate, and the wavefront set of any such distribution would be in Λ. The principal symbol of a Lagrangian distributions is invariantly defined on the cotangent bundle and takes value in $\mathbb{C}^4 \otimes \Omega^{1/2} \otimes \mathcal{L}$ where $\Omega^{1/2}$ is the half density bundle and $\mathcal{L}$ is the Maslov bundle. The principal symbol of $w \in \mathcal{I}^\mu(\Lambda; \mathbb{C}^4)$ is usually denoted by $\sigma_\mu(w)$, and most of the time we shall omit the μ when the level of the principal symbol of clear. Furthermore, when determining distorted plane wave, we shall let it be a classical conormal distribution $\mathcal{I}^\mu_{cl}(\Lambda; \mathbb{C}^4)$, meaning the principal symbol is homogeneous.

Let Λ_0, Λ_1 be two cleanly intersecting conic Lagrangian submanifolds, then $\mathcal{I}^{p,l}(\Lambda_0, \Lambda_1; \mathbb{C}^4)$ is the set of all paired Lagrangian distributions associated with (Λ_0, Λ_1) taking value in $\mathbb{C}^4$. If $w \in \mathcal{I}^{p,l}(\Lambda_0, \Lambda_1; \mathbb{C}^4)$, then $\mathrm{WF}(w) \in \Lambda_0 \cup \Lambda_1$, and if $A \in \Psi^0(\Lambda_0; M_{4\times 4}(\mathbb{C}))$ pseudodifferential operator of order 0 on Λ_0 taking value in 4×4 complex matrices, and $\mathrm{WF}(A) \cap \Lambda_1 = \emptyset$, then $Aw \in \mathcal{I}^{p+l}(\Lambda_0; \mathbb{C}^4)$. Similarly if $B \in \Psi^0(\Lambda_1; M_{4\times 4}(\mathbb{C}))$ and $\mathrm{WF}(B) \cap \Lambda_0 = \emptyset$, then $Bw \in \mathcal{I}^p(\Lambda_1; \mathbb{C}^4)$. The principal symbol of w on Λ_0 and Λ_1 can thus be defined, and they satisfy a compatibility condition that depends only on the geometry of the Lagrangian submanifolds.

We shall omit the $\mathbb{C}^4$ in $\mathcal{I}^\mu(\Lambda; \mathbb{C}^4)$ and $\mathcal{I}^{p,l}(\Lambda_0, \Lambda_1; \mathbb{C}^4)$ for simplicity. The specific definition of Lagrangian distributions can be found in chapter 25 of [75], as well as [69]. And for paired Lagrangian distributions we refer to [101], [53] and [58].

Finally, we also include some distribution multiplication results here for future use.

Lemma 3.3.2. *([96] Lemma 3.3) Let $u \in \mathcal{I}^\mu(\Lambda_1)$, $v \in \mathcal{I}^{\mu'}(\Lambda_2)$. Then we can write $w = uv$ as $w = w_1 + w_2$ where*

$$w_1 \in \mathcal{I}^{\mu,\mu'+1}(\Lambda_{12}, \Lambda_1), \quad WF(w_1) \cap \Lambda_2 = \emptyset,$$
$$w_2 \in \mathcal{I}^{\mu',\mu+1}(\Lambda_{12}, \Lambda_2), \quad WF(w_2) \cap \Lambda_1 = \emptyset.$$

Moreover, for any $(q, \zeta) \in \Lambda_{12} \backslash (\Lambda_1 \cup \Lambda_2)$, we can write $\zeta = \zeta_1 + \zeta_2$ in a unique way such that $\zeta_i \in N_q^* K_i, i = 1, 2$. Microlocally away from $\Lambda_1 \cup \Lambda_2$, $uv \in \mathcal{I}^{\mu+\mu'+1}(\Lambda_{12})$ and the principal symbol of uv satisfies

$$\sigma_{\Lambda_{12}}(uv)(q, \zeta) = (2\pi)^{-1} \sigma_{\Lambda_1}(u)(q, \zeta_1) \cdot \sigma_{\Lambda_2}(v)(q, \zeta_2).$$

Lemma 3.3.3. *([96] Lemma 3.6) Assume that $u \in \mathcal{I}^\mu(\Lambda_3)$, $v \in \mathcal{I}^{p,l}(\Lambda_{12}, \Lambda_1)$ are compactly supported near K_{123}. For $\epsilon > 0$ sufficiently small, we can write $w = uv$ as $w = w_0 + w_1 + w_2$ where*

$$w_0 \in \mathcal{I}^{\mu+p+l-1/2}(\Lambda_{123}), \quad w_1 \in \mathcal{D}'(\mathbb{R}^4; \cup_{i=1}^3 \Lambda_i(\epsilon)),$$
$$w_2 \in \mathcal{I}^{p,\mu+1}(\Lambda_{13}, \Lambda_1) + \mathcal{I}^{\mu,p+1}(\Lambda_{13}, \Lambda_3) + \mathcal{I}^{p,l}(\Lambda_{12}, \Lambda_1).$$

Here $\Lambda^{(1)}(\epsilon)$ is a conic ϵ-neighborhood of $\cup_{i=1}^3 \Lambda_i$. Moreover, for $q \in K_{123}$ and $\zeta \in N_q^* K_{123} \backslash (\cup_{i=1}^3 \Lambda_i)$, we can *write* $\zeta = \zeta_1 + \zeta_2 + \zeta_3$ *uniquely for* $\zeta_i \in N_q^* K_i$*. The principal symbol of* w_0 *satisfies*

$$\sigma_{\Lambda_{123}}(w_0)(q, \zeta) = (2\pi)^{-1} \sigma_{\Lambda_3}(u)(q, \zeta_3) \cdot \sigma_{\Lambda_{12}}(v)(q, \zeta_1 + \zeta_2).$$

3.3.3 Causal inverse

We first invoke a simplified version of propagation of singularity theorem for systems that will be used later, specifically we left out the polarization part as it will not be used.

Theorem 3.3.4. *([36] Theorem 4.2) Let* P *be an* $N \times N$ *system of pseudodifferential operators on a manifold* X *and let* $u \in \mathcal{D}'(X, \mathbb{C}^n)$*. Assume that* P *is of real principal type at* $(y, \eta) \in Char(P)$ *and that* $y \notin WF(Pu)$*. Then, over a neighborhood of* (y, η) *in* $Char(P)$*,* $WF(u)$ *is a union of bicharacteristics of* P*.*

As for the symbol calculus, we have the following theorem.

Theorem 3.3.5. *([60] Theorem 3.1) Let* P *be an* $N \times N$ *real principal type system of pseudodifferential operators of order* m *on a manifold* X*. Assume* Λ *a homogeneous Lagrangian submanifold of* $T^*X \backslash 0$ *such that* $\Lambda \in Char(P)$*. If* $A \in \mathcal{I}^{\mu}(X, \Lambda; \Omega_X^{1/2} \otimes \mathbb{C}^N)$ *and* $a \in S^{\mu+n/4}(X, \Lambda; \Omega_X^{1/2} \otimes \mathbb{C}^N)$ *is a principal symbol of* A *such that* $pa = 0$*, it follows that* $B = PA \in \mathcal{I}^{m+\mu-1}(X, \Lambda; \Omega_X^{1/2} \otimes \mathbb{C}^N)$ *has principal symbol* $b \in S^{m+\mu+n/4-1}(X, \Lambda; \Omega_X^{1/2} \otimes \mathbb{C}^N)$ *satisfying*

$$(\mathcal{L}_{H_q} + \frac{1}{2}\{\tilde{p}, p\} + i\tilde{p}p^s)a = i\tilde{p}b.$$

Here $\mathcal{L}_{H_q}$ *is the Lie derivative with respect to the Hamiltonian vector field* H_q*,* p *and* $\tilde{p}$ *are principal symbols of* P *and* $\tilde{P}$ *respectively, and* p^s *is the subprincipal symbol of* P*.*

In particular, since in our case P is independent of y, the transport equation is

$$[\mathcal{L}_{H_q} - i(\tau - \alpha \cdot \xi)(\beta + F'(u))]a = i(\tau - \alpha \cdot \xi)b$$

where $q = \tau^2 - |\xi|^2$ and H_q is the Hamiltonian vector field of the standard wave operator

$$H_q = 2\tau\frac{\partial}{\partial t} - 2\xi \cdot \frac{\partial}{\partial x}.$$

Now we compute the causal inverse of P, recall that $Q(f)$ is the solution to (3.2.1).

Proposition 3.3.6. *If $f \in \mathcal{I}^{\mu}(\Lambda)$, and $w = Q(f)$, then $w \in \mathcal{I}^{\mu-1/2,-1/2}(\Lambda, \Lambda^g)$ where Λ^g is the future flowout of $\Lambda \cap L^+\mathbb{R}^4$. On $\Lambda \backslash \Lambda^g$,*

$$\sigma_\Lambda(w) = p^{-1}\sigma(f).$$

On Λ^g the principal symbol satisfies

$$\begin{cases} (\mathcal{L}_{H_q} + i\tilde{p}p^s)\sigma_{\Lambda^g}(w) = 0 \text{ on } \Lambda^g \backslash \partial\Lambda^g \\ \sigma_{\Lambda^g}(w) = C\tilde{p}\sigma(f) \text{ on } \partial\Lambda^g \end{cases}$$

for some constant C that only depends on the geometry of Λ and Λ^g.

Proof. Note that if $w = Q(f)$, then $\Box w - \tilde{P}(\beta + F'(u))w = \tilde{P}f$ and $w_t = f - \alpha \cdot \nabla w - i(\beta + F'(u))w = 0$ on $(L^+\text{supp}(f))^c$. By [13] and [101], $\Box w - \tilde{P}(\beta + F'(u))$ has a causal inverse $Q' \in \mathcal{I}^{-3/2,-1/2}(N^*\text{Diag}, \Lambda_g)$ where

$$N^*\text{Diag} = \{(z, \zeta, z', \zeta') \in T^*\mathbb{R}^8 \backslash 0 : z = z', \zeta = \zeta'\}$$

and Λ_g is the Lagrangian obtained by flowing out $N^*\text{Diag} \cap (L\mathbb{R}^4 \times T^*\mathbb{R}^4)$ under H_q lifted from the left factor (see section 3.1 of [95]). Thus $Q' : \mathcal{I}^{\mu'}(\Lambda) \to \mathcal{I}^{\mu'-3/2,-1/2}(\Lambda, \Lambda^g)$. Because $w = Q(f) = Q'(\tilde{P}f)$, plug in $\mu' = \mu + 1$ we have $w \in \mathcal{I}^{\mu-1/2,-1/2}(\Lambda, \Lambda^g)$.

To compute the principal symbol, notice that p is invertible on $\Lambda \backslash \partial\Lambda^g$, so on the principal level $p\sigma_\Lambda(w) = \sigma(f)$ implies the first equation. The transport equation in the second equation is by Theorem 3.3.5. To compute the initial data, notice that on $\Lambda \backslash \partial\Lambda^g$,

$$p^{-1}\sigma(f) = q^{-1}\tilde{p}\sigma(f),$$

and as $q = 0$ parametrizes the light cone, by compatibility condition $\sigma_{\Lambda^g}(w) = C\tilde{p}\sigma(f)$ with some nonzero C that only depends on the geometry of the intersecting submanifolds. We can also obtain the initial value using the fact that $w = Q'(\tilde{p}f)$ and computations from proof of Proposition 6.6 in [101]. □

Proposition 3.3.7. *If $f \in \mathcal{I}^{p,l}(\Lambda_0, \Lambda_1)$, and $w = Q(f)$, then $w \in \mathcal{I}^{p,l-1}(\Lambda_0, \Lambda_1)$. On $\Lambda_0 \backslash \Lambda_1$,*

$$\sigma_{\Lambda_0}(w) = p^{-1}\sigma_{\Lambda_0}(f).$$

Proof. $Q' \in \mathcal{I}^{-3/2,-1/2}(N^*\mathrm{Diag}, \Lambda_g)$ so by [96] Lemma 3.4

$$Q' : \mathcal{I}^{p',l'}(\Lambda_0, \Lambda_1) \to \mathcal{I}^{p'-1,l'-1}(\Lambda_0, \Lambda_1).$$

Then $w \in \mathcal{I}^{p,l-1}(\Lambda_0, \Lambda_1)$ follows from $w = Q(f) = Q'(\tilde{P}f)$. Away from the characteristic set p is invertible on $\Lambda_0 \backslash \Lambda_1$ so the equation follows. □

3.4 Distorted plane wave and first order linearization

First consider $P' = i\partial_t + i\alpha \cdot \nabla - \beta$, let $f_j \in \mathcal{I}_{cl}^{\mu+1/2}(\Sigma_j)$ and $\tilde{w}_j$ the solution of

$$\begin{cases} P'\tilde{w}_j = f_j \\ \tilde{w}_j = 0 \text{ on } (L^+ supp(f_j))^c. \end{cases}$$

Note that P' can be thought of as a special case of P where the $F'(u)$ term is 0 everywhere. Then by Proposition 3.3.6, $\tilde{w}_j \in \mathcal{I}^{\mu,-1/2}(\Sigma_j, \Lambda_j)$. Specifically if we only consider time 0 to time T, then $\tilde{w}_j \in \mathcal{I}^{\mu}(\Lambda_j)$ and the principal symbol satisfies

$$\begin{cases} (\mathcal{L}_{H_q} - i(\tau - \alpha \cdot \xi)\beta)\sigma(\tilde{w}_j) = 0 \text{ on } \Lambda_j \\ \sigma(\tilde{w}_j) = C\tilde{p}\sigma(f_j) \text{ on } \Sigma_j \cap \Lambda_j, \end{cases} \tag{3.4.1}$$

which doesn't contain any nonlinearity so is fully known here. Let $\phi_j(x) = \tilde{w}_j(0, x)$, by making μ sufficiently negative, $\phi_j \in H^s(\mathbb{R}^3)$ and hence perturbation result holds. Let w_j solve

$$\begin{cases} Pw_j = 0 \\ w_j(0) = \phi_j. \end{cases}$$

Note that $P(w_j - \tilde{w}_j) = F'(u)\tilde{w}_j$, and initial data vanishes at 0. By Theorem 3.3.4, and the fact that $\mathrm{WF}(\tilde{w}_j) \subset \Lambda_j$, we have $\mathrm{WF}(w_j - \tilde{w}_j) \subset \Lambda_j$, hence $\mathrm{WF}(w_j) \subset \Lambda_j$.

Let $\mathcal{R}_0$ be the restriction operator at time 0, that is $\mathcal{R}_0(w(t, x)) = w(0, x)$. We know $\mathrm{WF}(w_j), \mathrm{WF}(\tilde{w}_j) \in \Lambda_j$, and Λ_j does not intersect the conormal bundle of the time slice $\mathbb{R}^3_0 = \{(0, x) : x \in \mathbb{R}^3\}$, so $\mathcal{R}_0 \in \mathcal{I}^{1/4}(\mathbb{R}^3_0, \mathbb{R}^4; C_0)$ is a Fourier integral operator of order 1/4 (see [39] Chapter 5.1) where the corresponding canonical relation is given by

$$C_0 = \{(x, \xi; 0, x, \tau, \xi)\}.$$

(It is worth mentioning that $\mathcal{R}_0$ can act on w_j as a FIO because we can extend w_j backward in time such that it is a distribution on an some $(-\epsilon, \epsilon) \times \mathbb{R}^3$.) For any $(0, x, \tau, \xi) \in \Lambda_j$,

$$\sigma(\phi_j)(x, \xi) = \sigma(\mathcal{R}_0)(x, \xi; 0, x, \tau, \xi)\sigma(w_j)(0, x, \tau, \xi),$$

$$\sigma(\phi_j)(x, \xi) = \sigma(\mathcal{R}_0)(x, \xi; 0, x, \tau, \xi)\sigma(\tilde{w}_j)(0, x, \tau, \xi).$$

Since $\sigma(\mathcal{R}_0)(x, \xi; 0, x, \tau, \xi) \neq 0$ (see [39] Chapter 5.1), we have $w_j \in \mathcal{I}^\mu(\Lambda_j)$ and $\sigma(w_j)(0, x, \tau, \xi) = \sigma(\tilde{w}_j)(0, x, \tau, \xi)$.

Using Proposition 3.3.6 and the fact that P' is fully known, we can determine what $\sigma(\tilde{w}_j)(0, x_j, \tau_j, \xi_j)$ is by constructing appropriate f_j. Specifically, we rewrite (3.4.1) along Γ_j as

$$\begin{cases} \frac{d}{ds}\sigma(\tilde{w}_j)(\Gamma_j(s)) = A_0(\Gamma_j(s))\sigma(\tilde{w}_j)(\Gamma_j(s)) \\ \sigma(\tilde{w}_j)(\Gamma_j(\frac{T'}{2\tau_j})) = C\tilde{p}\sigma(f_j)(\Gamma_j(\frac{T'}{2\tau_j})) \end{cases} \tag{3.4.2}$$

an ODE along Γ_j where

$$A_0(y, \eta) = i(\tau - \alpha \cdot \xi)\beta.$$

Note that here A_0 doesn't depend on y, hence it stays constant along $\Gamma_j(s)$, $A_0 = A_0(\Gamma_j(s)) = i(\tau_j - \alpha\cdot\xi_j)\beta$. Then we simply have $\sigma(\tilde{w}_j)(\Gamma_j(s)) = \exp(sA_0)\sigma(\tilde{w}_j)(\Gamma_j(s))$. However there is some restriction on the initial data in (3.4.2), namely it can only be in the kernel of $\tilde{p}(\Gamma_j) = -\tau_j - \alpha\cdot\xi_j$. Moreover, we see that the image of $A_0(\Gamma_j(s))$ is precisely in $\ker(-\tau_j - \alpha\cdot\xi_j)$, meaning $\sigma(\tilde{w}_j)(\Gamma_j(s)) \in \ker(-\tau_j - \alpha\cdot\xi_j)$ for all s.

The result of the above analysis is that, we can assign any $v_j \in \ker(-\tau_j - \alpha\cdot\xi_j)$ as the initial data for $\sigma(w_j)$ at $\Gamma_j(0) = (y_j, \eta_j)$, by choosing initial data f_j such that $C\tilde{p}\sigma(f_j)(\Gamma_j(\frac{T'}{2\tau_j})) = \exp(\frac{T'}{2\tau_j}A_0)v_j$.

According to Theorem 3.3.5, $\sigma(w_j)$ along the bicharacteristic Γ_j solves the ODE

$$\begin{cases} \frac{d}{ds}\sigma(w_j)(\Gamma_j(s)) = A(\Gamma_j(s))\sigma(w_j)(\Gamma_j(s)) \text{ for } 0 < s < \frac{T}{2\tau_j} \\ \sigma(w_j)(\Gamma_j(0)) = v_j \in \ker(-\tau_j - \alpha\cdot\xi_j) \end{cases} \tag{3.4.3}$$

where

$$A(y,\eta) = i(\tau - \alpha\cdot\xi)(\beta + F'(u(y))).$$

Similar argument can also show that the initial data to end data map, $v_j \mapsto \sigma(w_j)(y_j^T, \eta_j)$, is bijective on $\ker(-\tau_j - \alpha\cdot\xi_j)$.

From Proposition 3.2.2, $w_j(T) = \partial_{\epsilon_j}|_{\epsilon=0}\mathcal{S}(\phi + \sum \epsilon_j\phi_j)$, again use the fact that $\mathrm{WF}(w_j) \subset \Lambda_j$, we have

$$\sigma(w_j)(y_j^T, \eta_j) = \sigma(\mathcal{R}_T)(x_j^T, \xi_j; y_j^T, \eta_j)^{-1}\sigma(\partial_{\epsilon_j}|_{\epsilon=0}\mathcal{S}(\phi + \sum \epsilon_j\phi_j))(x_j^T, \xi_j)$$

is determined by the solution map. Note that all the computations hold when F also depends on x. In conclusion, we have the following result.

Lemma 3.4.1. *For $i = 1, 2$, consider the ODE*

$$\begin{cases} \frac{d}{ds}w_i = A_i(\Gamma_j(s))w_i \text{ for } 0 < s < \frac{T}{2\tau_j} \\ w_i(0) = v \end{cases}$$

where

$$A_i(y,\eta) = i(\tau - \alpha\cdot\xi)(\beta + F_i'(x, u(y))).$$

Then the initial data to final data map

$$\begin{aligned} W_i : & ker(-\tau_j - \alpha \cdot \xi_j) \to ker(-\tau_j - \alpha \cdot \xi_j), \\ & v \mapsto w_i(\frac{T}{2\tau_j}) \end{aligned}$$

is bijective. Moreover, the same assumption as in Theorem 3.1.3 implies $W_1 = W_2$*.*

Note that W_i is indeed a bijection on the entire $\mathbb{C}^4$, but as we can only set up initial data in $\ker(-\tau_j - \alpha \cdot \xi_j)$, we can only say the two W_i agree on that part. Finally, since we chose f_j to be a classical conormal distribution, $\sigma(w_j)$ is also going to be homogeneous because of (3.4.1) and (3.4.3).

3.5 Third order linearization

Let $\Gamma_1, \Gamma_2, \Gamma_3$ intersect at a point y_c, and let K_1, K_2, K_3 intersect transversally. Choose some future pointed light-like direction $\eta_0 = \sum_{j=1}^3 k_j \eta_j \in \Lambda_{123} \backslash \cup_{j=1}^3 \Lambda_j$, $k_j \neq 0$. We shall compute the principal symbol of w_{12} and w_{123}, the result for other w_α's are similar. According to the asymptotic analysis we have

$$\begin{aligned} w_{12} =& Q(2F^{(2)}(u, w_1, w_2)) \\ w_{123} =& Q(6F^{(3)}(u, w_1, w_2, w_3) + 2F^{(2)}(u, w_{12}, w_3) \\ & + 2F^{(2)}(u, w_{13}, w_2) + 2F^{(2)}(u, w_{23}, w_1)). \end{aligned}$$

Note that even though w_j's are vectors of distributions, each coordinate of $F^{(2)}(u, w_1, w_2)$ and other similar terms are just linear combinations of multiplication of distributions, hence Lemma 3.3.2 and Lemma 3.3.3 can be used directly.

We start by analyzing w_{12}. By Lemma 3.3.2,

$$F^{(2)}(u, w_1, w_2) \in \mathcal{I}^{\mu,\mu+1}(\Lambda_{12}, \Lambda_1) + \mathcal{I}^{\mu,\mu+1}(\Lambda_{12}, \Lambda_2),$$

and by Proposition 3.3.7

$$w_{12} \in \mathcal{I}^{\mu,\mu}(\Lambda_{12}, \Lambda_1) + \mathcal{I}^{\mu,\mu}(\Lambda_{12}, \Lambda_2).$$

Moreover for $(y, \zeta) \in \Lambda_{12} \backslash (\Lambda_1 \cup \Lambda_2)$

$$\sigma_{\Lambda_{12}}(w_{12})(y, \zeta) = \pi^{-1} p^{-1}(\eta) F^{(2)}(u(y), \sigma(w_1)(y, \zeta_1), \sigma(w_2)(y, \zeta_2))$$

where $\zeta = \zeta_1 + \zeta_2$ for $\zeta_j \in N_y^* K_j$.

Now we can analyze w_{123}. By Lemma 3.3.3, on $\Lambda_{123} \backslash \cup_{j=1}^3 \Lambda_j$

$$F^{(2)}(u, w_{12}, w_3) \in \mathcal{I}^{3\mu-1/2}(\Lambda_{123})$$

and the principal symbol satisfies

$$\sigma_{\Lambda_{123}}(F^{(2)}(u, w_{12}, w_3)) = (2\pi)^{-2} F^{(2)}(u(y^c),$$
$$p^{-1}(k_1\eta_1 + k_2\eta_2) F^{(2)}(u(y^c), \sigma(w_1)(y^c, k_1\eta_1), \sigma(w_2)(y^c, k_2\eta_2)), \sigma(w_3)(y^c, k_3\eta_3)).$$

On the other hand, by Lemma 3.3.2 and Lemma 3.3.3, on $\Lambda_{123} \backslash \cup_{j=1}^3 \Lambda_j$ we have

$$F^{(3)}(u, w_1, w_2, w_3) \in \mathcal{I}^{3\mu+1/2}(\Lambda_{123}),$$

and the principal symbol satisfies

$$\begin{aligned}\sigma_{\Lambda_{123}}(F^{(3)}(u, w_1, w_2, w_3))(y^c, \eta_0) =& (2\pi)^{-2} F^{(3)}(u(y^c), \sigma(w_1)(y^c, k_1\eta_1), \\ & \sigma(w_2)(y^c, k_2\eta_2), \sigma(w_3)(y^c, k_3\eta_3)).\end{aligned}$$

Hence the top order singularity for the right hand side of

$$\begin{cases} Pw_{123} = 6F^{(3)}(u, w_1, w_2, w_3) + 2F^{(2)}(u, w_{12}, w_3) \\ \qquad\qquad + 2F^{(2)}(u, w_{13}, w_2) + 2F^{(2)}(u, w_{23}, w_1) \\ w_{123}(0) = 0 \end{cases}$$

is given by $6F^{(3)}(u, w_1, w_2, w_3)$. By Proposition 3.3.6 we have

$$\begin{cases} (\mathcal{L}_{H_q} + i\tilde{p}p^s)\sigma(w_{123}) = 0 \text{ along } \Gamma_0 \\ \sigma(w_{123})(y^c, \eta_0) = C\tilde{p}\sigma_{\Lambda_{123}}(F^{(3)}(u, w_1, w_2, w_3))(y^c, \eta_0). \end{cases}$$

Similar to the first order linearization it is an ODE of the form

$$\begin{cases} \frac{d}{ds}\sigma(w_{123})(\Gamma_0(s)) = A(\Gamma_0(s))\sigma(w_{123})(\Gamma_0(s)) \text{ for } c < s < \frac{T}{2\tau_0} \\ \sigma(w_{123})(\Gamma_0(c)) = C\tilde{p}\sigma_{\Lambda_{123}}(F^{(3)}(u, w_1, w_2, w_3))(\Gamma_0(c)) \end{cases} \tag{3.5.1}$$

where c is the time such that $\Gamma_0(c) = (y_c, \eta_0)$. Again by Proposition 3.2.2 and same argument as $\mathcal{R}_0$, we have that $\sigma(\mathcal{R}_T)(x_0, \xi_0; y_0^T, \eta_0) \neq 0$ and

$$\sigma(w_{123})(y_0^T, \eta_0) = \sigma(\mathcal{R}_T)(x_0, \xi_0; y_0^T, \eta_0)^{-1}\sigma(\partial_{\epsilon_{123}}|_{\epsilon=0}\mathcal{S}(\phi + \sum \epsilon_j\phi_j))(x_0, \xi_0), \tag{3.5.2}$$

so $\sigma(w_{123})(y_0^T, \eta_0)$ is determined by the solution map.

3.6 Limit of collisions to boundary

We now construct a sequence of collisions along a fixed bicharacteristic and let the collision point getting closer and closer to time 0. First fix some $\eta_j = (1, \xi_j)$, $j = 0, 1, 2, 3$, future pointing light-like such that $\eta_0 = \sum k_j\eta_j$ with $k_j \neq 0$. Let

$$\Gamma_0(s) = (2s, x_0 - 2s\xi_0, 1, \xi_0)$$

travel from $(y_0, \eta_0) = (0, x_0, 1, \xi_0)$ to $(y_0^T, \eta_0) = (T, x_0^T, 1, \xi_0)$, and denote $y^c = (2c, x_0 - 2c\xi_0)$ the point at time $s = c$. Similarly for $j = 1, 2, 3$ denote

$$\Gamma_j^c(s) = (2s, x_j^c - 2s\xi_j, 1, \xi_j)$$

such that at some time T_j^c, $\Gamma_j^c(T_j^c) = (y^c, \eta_j)$, that is the projection is a geodesic from $y_j^c = (0, x_j^c)$ to y^c, see Figure 3.1.

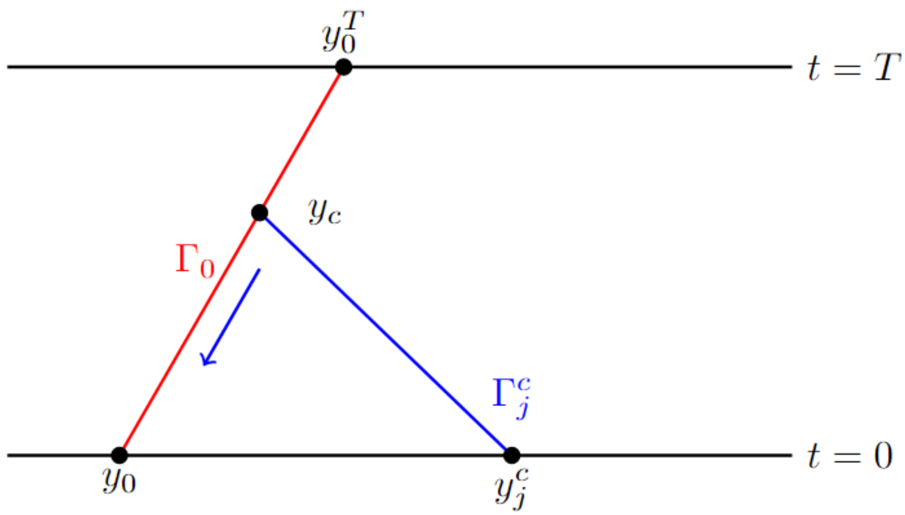

Figure 3.1: Γ_j^c intersects Γ_0 at point y_c. As $c \mapsto 0$ the collision point approaches to y_0, and Γ_j^c goes to a single point.

Note that $\Gamma_j^c(c) = (y^c, \eta_j)$, and as $c \to 0$ we have

$$y^c \to y_0, \quad y_j^c \to y_0,$$

and Γ_j^c becomes just the point y_0. Intuitively, if we consider the scenario to be that $\Gamma_1^c, \Gamma_2^c, \Gamma_3^c$ intersect at y^c and leaves the point along Γ_0 from y^c to y_0^T, then the limiting case is just collision at the boundary point y_0. For each collision point y^c, we will construct corresponding initial data ϕ_j^c, and so the solutions now depend on c, that is the linearization terms are now w_j^c, w_{ij}^c and w_{123}^c.

Using result from first order linearization (3.4.3), we can setup initial data such that $\sigma(w_j^c)$ satisfies the ODE

$$\begin{cases} \frac{d}{ds}\sigma(w_j^c)(\Gamma_j^c(s)) = A(\Gamma_j^c(s))\sigma(w_j^c)(\Gamma_j^c(s)) \text{ for } 0 < s < c \\ \sigma(w_j^c)(\Gamma_j^c(0)) = v_j \text{ in } \ker(-\tau_j - \alpha \cdot \xi_j). \end{cases}$$

We want to show that as $c \to 0$, $\sigma(w_j^c)(\Gamma_j^c(c)) \to v_j$. When c is small we have Γ_j^c is entirely in a neighborhood of y_0, and as $F'(u)$ is smooth, we have for the max matrix norm $|A(\Gamma_j^c(s))| \leq N$ for all $0 \leq s \leq c$ for some constant N. Note that

$$\begin{aligned} |\sigma(w_j^c)(\Gamma_j^c(s)) - v_j| &= |\int_0^c \frac{d}{ds}(\sigma(w_j^c)(\Gamma_j^c(s')))ds'| \\ &= |\int_0^c A(\Gamma_j^c(s'))\sigma(w_j^c)(\Gamma_j^c(s'))ds'| \\ &\leq N \int_0^c |\sigma(w_j^c)(\Gamma_j^c(s'))|ds'. \end{aligned}$$

By Grönwall's inequality

$$|\frac{d}{ds}\sigma(w_j^c)(\Gamma_j^c(s))| \leq N|\sigma(w_j^c)(\Gamma_j^c(s))| \implies |\sigma(w_j^c)(\Gamma_j^c(s))| \leq e^{Ns}|v_j|.$$

Hence

$$|\sigma(w_j^c)(\Gamma_j^c(s)) - v_j| \leq (e^{Nc} - 1)|v_j|$$

which converges to 0 as $c \to 0$.

Similarly the third order linearization tells us that $\sigma(w_{123}^c)$ satisfies

$$\begin{cases} (\mathcal{L}_{H_q} + i\tilde{p}p^s)\sigma(w_{123}^c) = 0 \text{ along } \Gamma_0 \\ \sigma(w_{123}^c)(y^c, \eta_0) = C\tilde{p}\sigma_{\Lambda_{123}}(F^{(3)}(u, w_1^c, w_2^c, w_3^c))(y^c, \eta_0). \end{cases}$$

Again we can write it as an ODE like (3.5.1) and denote $w^c(s) = \sigma(w^c_{123})(\Gamma_0(s))$ for simplicity

$$\begin{cases} \frac{d}{ds} w^c(s) = A(\Gamma_0(s)) w^c(s) \text{ for } c < s < T/2 \\ w^c(c) = C'(1 - \alpha \cdot \xi_0) F^{(3)}(u(y^c), \sigma(w^c_1)(\Gamma^c_1(c)), \\ \qquad\qquad \sigma(w^c_2)(\Gamma^c_2(c)), \sigma(w^c_3)(\Gamma^c_3(c))), \end{cases} \tag{3.6.1}$$

where $C' = C \prod_{j=1}^{3} k_j^{\mu+1/2} \neq 0$ because $\sigma(w^c_j)$ is homogeneous. Now consider another ODE

$$\begin{cases} \frac{d}{ds} w(s) = A(\Gamma_0(s)) w(s) \text{ for } 0 < s < T/2 \\ w(0) = C'(1 - \alpha \cdot \xi_0) F^{(3)}(\phi(x_0), v_1, v_2, v_3). \end{cases} \tag{3.6.2}$$

We want to show that $w^c(T/2) \to w(T/2)$ as $c \to 0$. First of all, by previous computations it is obvious that as $c \to 0$,

$$w^c(c) \to w(0).$$

Note that w and w^c essentially solve the same ODE with different initial data at time c, where

$$w(c) = w(0) + \int_0^c w'(s) ds = w(0) + \int_0^c A(\Gamma_0(s)) w(s) ds.$$

In a neighborhood of Γ_0 we have $|A(\Gamma_0(s))| \leq \tilde{N}$ since it is smooth. Thus use Grönwall's inequality again we obtain

$$\begin{aligned} |w(c) - w^c(c)| &\leq |w(c) - w(0)| + |w(0) - w^c(c)| \\ &\leq (e^{\tilde{N}c} - 1)|w(0)| + |w(0) - w^c(c)| \\ &\to 0 \end{aligned}$$

as $c \to 0$. Hence

$$|w(T/2) - w^c(T/2)| \leq |w(c) - w^c(c)| + |\int_c^{T/2} A(\Gamma_0(s))(w(s) - w^c(s)) ds|$$

$$\leq |w(c) - w^c(c)| + (e^{MT/2} - e^{\tilde{N}c})|w(c) - w^c(c)|$$

$$\to 0$$

as $c \to 0$. By (3.5.2),

$$w(T/2) = \lim_{c \longrightarrow 0} \sigma(\mathcal{R}_T)(x_0, \xi_0; y_0^T, \eta_0)^{-1} \sigma(\partial_{\epsilon_{123}}|_{\epsilon=0} \mathcal{S}(\phi + \sum \epsilon_j \phi_j^c))(x_0, \xi_0)$$

meaning the solution of (3.6.2) at time $T/2$ can be determined by the solution map. Finally, note that (3.6.2) is the same ODE as the one in Lemma 3.4.1, the initial data can also be determined by the injectivity result proved there. Again all the computations are the same when F also depends on x. For any $z < M$ we can find some $\phi \in H^s_\delta \cap C^\infty(\mathbb{R}^3)$ such that $\phi(x_0) = z$, so we obtain the following lemma.

Lemma 3.6.1. *For $j = 1, 2, 3$, let ξ_j be linearly independent lightlike covectors. Under the same assumption as in Theorem 3.1.3, we have*

$$(1 - \alpha \cdot \xi_0)\partial_z^3 F_1(x, z, v_1, v_2, v_3) = (1 - \alpha \cdot \xi_0)\partial_z^3 F_2(x, z, v_1, v_2, v_3)$$

for all $v_j \in ker(-1 - \alpha \cdot \xi_j)$, $\xi_0 = \sum_{j=1}^3 k_j \xi_j$ where $k_j \neq 0$, $x \in \mathbb{R}^3$ and $|z| \leq M$.

3.7 Proof of main theorem

Here is a linear algebra lemma that will be used later.

Lemma 3.7.1. *Let K_1, K_2 be some linear subspace of $\mathbb{C}^n$ passing through the origin such that $K_1 \cap K_2 = \{0\}$, then for any $v_1, v_2 \in \mathbb{C}^n$, $v_1 + K_1 \cap v_2 + K_2$ has at most 1 element.*

Proof. Suppose $w, w' \in v_1 + K_1 \cap v_2 + K_2$. Since K_1 is a linear subspace, $w, w' \in v_1 + K_1$ implies $w = v_1 + k_1$ and $w' = v_1 + k_2$ where $k_1, k_2 \in K$, hence $w - w' = k_1 - k_2 \in K_1$. Similarly, $w - w' \in K_2$. Thus $w = w'$ because $K_1 \cap K_2 = \{0\}$. □

Now we are ready to prove Theorem 3.1.3. We continue to use $F^{(k)}(x, z)$ to denote derivatives with respect to z.

Proof. By Lemma 3.6.1, for linearly independent lightlike covectors $\eta_j = (1, \xi_j)$, $j = 1, 2, 3$, and $\eta_0 = \sum k_j \eta_j$ with $k_j \neq 0$, we can determine

$$(1 - \alpha \cdot \xi_0) F^{(3)}(x, z, v_1, v_2, v_3)$$

for $v_j \in \ker(-1 - \alpha \cdot \xi_j)$, $x \in \mathbb{R}^3$, $|z| \leq M$. Since $v \in \ker(1 - \alpha \cdot \xi_0)^{\perp} \iff v \in \ker(1 + \alpha \cdot \xi_0) \iff (1 - \alpha \cdot \xi_0)v = 2v$,

$$F^{(3)}(x, z, v_1, v_2, v_3) \in \frac{1}{2}(1 - \alpha \cdot \xi_0) F^{(3)}(x, z, v_1, v_2, v_3) + \ker(1 - \alpha \cdot \xi_0).$$

Similarly use some other appropriate η_0' instead of η_0 we can determine $(1 - \alpha \cdot \xi_0') F^{(3)}(x, z, v_1, v_2, v_3)$ and hence

$$F^{(3)}(x, z, v_1, v_2, v_3) \in \frac{1}{2}(1 - \alpha \cdot \xi_0') F^{(3)}(x, z, v_1, v_2, v_3) + \ker(1 - \alpha \cdot \xi_0').$$

We now choose the appropriate η_1, η_2, η_3 and η_0. The principal symbol of P at $(t, x, \tau, \xi) \in L\mathbb{R}^4$ (we shall omit (t, x) since principal symbol doesn't depend on them) for $\xi = (\xi^1, \xi^2, \xi^3)$ is

$$p(\tau, \xi) = \begin{pmatrix} -\tau & 0 & -\xi^3 & -\xi^1 + i\xi^2 \\ 0 & -\tau & -\xi^1 - i\xi^2 & \xi^3 \\ -\xi^3 & -\xi^1 + i\xi^2 & -\tau & 0 \\ -\xi^1 - i\xi^2 & \xi^3 & 0 & -\tau \end{pmatrix}$$

and the kernel is given by

$$\ker(p(\tau, \xi)) = \operatorname{span}\{(\xi^3, \xi^1 + i\xi^2, -\tau, 0), (-\xi^1 + i\xi^2, \xi^3, 0, \tau)\}.$$

Consider now $\eta_1 = (1, 1, 0, 0), \eta_2 = (1, 0, 1, 0), \eta_3 = (1, 0, 0, 1)$. Then

$$\mathcal{M} = \{(1, a, b, c) : a + b + c = a^2 + b^2 + c^2 = 1, a, b, c \neq 1\}$$

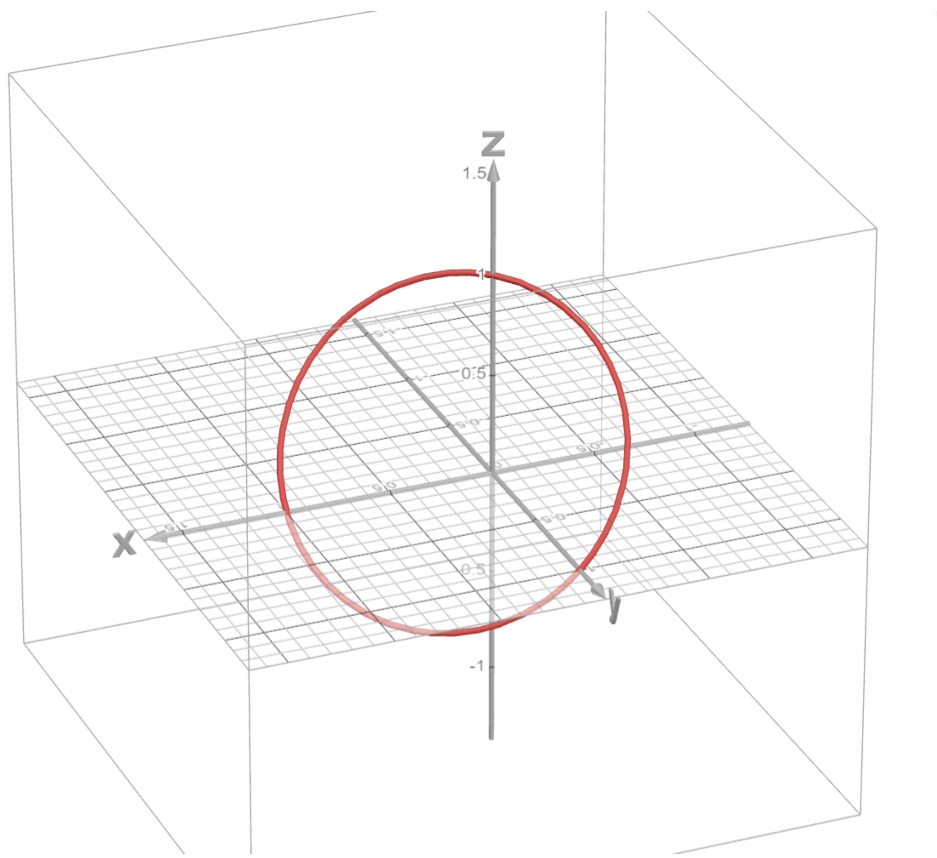

Figure 3.2: The set of available directions for η_0 when η_j are given by $(1, e_j)$.

forms a 1-dimensional set of available directions for η_0, see Figure 3.2. For example, we can take $\eta_0 = (1, -\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$, $\eta_0' = (1, \frac{2}{3}, -\frac{1}{3}, \frac{2}{3})$, and use the fact that $1 - \alpha \cdot \xi = p(-1, \xi)$ to obtain

$$\ker(1 - \alpha \cdot \xi_0) = \operatorname{span}\{(2, -1 + 2i, 3, 0), (1 + 2i, 2, 0, -3)\}$$
$$\ker(1 - \alpha \cdot \xi_0') = \operatorname{span}\{(2, 2 - i, 3, 0), (-2 - i, 2, 0, -3)\}$$

which have trivial intersection, and hence by Lemma 3.7.1 one can determine $F^{(3)}(x, z, v_1, v_2, v_3)$.

So far, we are able to determine any $F^{(3)}(x, z, v_1, v_2, v_3)$ for $v_j \in \ker(p(\eta_j))$ where $\eta_j = (1, e_j)$ and $e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)$. On the other hand, the same thing can be done if one replaces some of the η_j with $\eta_j' = (1, -e_j)$. That is, we are able to determine any $F^{(3)}(x, z, v_1, v_2, v_3)$ for $v_j \in \ker(p(\eta_j)) \cup \ker(p(\eta_j'))$. Straightforward computation shows that $\ker(p(\eta_j)) + \ker(p(\eta_j')) = \mathbb{C}^4$, thus by linearity we can determine $F^{(3)}(x, z, v_1, v_2, v_3)$ for any $v_j \in \mathbb{C}^4$, $x \in \mathbb{R}^3, |z| \leq M$.

If we have the additional assumptions that $F'(x, 0)$ and $F^{(2)}(x, 0)$ are already determined, then along with the fact that $F(x, 0) = 0$, $F(x, z)$ can be determined for any $x \in \mathbb{R}^3, |z| \leq M$. $\square$

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Chapter 4

Partial data inverse problem with disjoint source and observation sets

The content of this chapter corresponds to a joint work with Yang Zhang [158]. The main results are Theorem 4.1.4, Theorem 4.1.3 and Theorem 4.1.6.

4.1 Introduction

We consider a time-oriented Lorentzian manifold (M, g) of dimension n for $n \geq 3$, with a timelike, strictly null-convex boundary ∂M, see Definition 4.1.2. Suppose g has the signature $(-1, 1, \ldots, 1)$. We consider the wave operator with a first-order perturbation

$$\square_{g,A,q} \;=\; \frac{1}{\sqrt{|\det g|}}\big(\partial_j - iA_j\big)\Big(\sqrt{|\det g|}\,g^{jk}\big(\partial_k - iA_k\big)\Big) \;+\; q,$$

where A is a smooth 1-form and q is a smooth scalar potential on M. Let u be the outgoing solution of

$$\begin{aligned} \square_{g,A,q} u = 0 \quad &\text{in } M^\circ, \\ u = f \quad &\text{on } \partial M, \\ \operatorname{supp}(u) \subseteq J^+&(\operatorname{supp}(f)), \end{aligned} \tag{4.1.1}$$

where $J^+(\operatorname{supp}(f))$ denotes the causal future of support of f, for more details see Section 4.2. Here by outgoing, we mean the microlocal solution that we choose has singularities propagating along forward bicharacteristics, for more details see Section 4.2.5. The associated outgoing *Dirichlet-to-Neumann map* (DN map) is defined as

$$\Lambda_{g,A,q} : f \mapsto (\partial_\nu u - i\langle A, \nu\rangle u)|_{\partial M},$$

where ν denotes the unit outward normal vector field to ∂M. Let U and V be open subsets of ∂M. The goal of this chapter is to study the inverse problem of recovering g and A up to gauge transformations, from partial boundary data. More precisely, we impose the Dirichlet boundary conditions $u = f$ on U and measure the corresponding Neumann derivative on V. For this purpose, we define the *restricted DN map* as

$$\Lambda^{U,V}_{g,A,q} : f \mapsto (\partial_\nu u - i\langle A, \nu\rangle u)|_V = \partial_\nu u|_V - i(\langle A, \nu\rangle f)|_{U\cap V}.$$

Throughout the chapter, we shall mainly focus on the case where $U \cap V = \emptyset$, unless stated otherwise. In this case the DN map is simply given by

$$\Lambda^{U,V}_{g,A,q}(f) = \partial_\nu u|_V.$$

Boundary determination problems have been studied in Riemannian settings, with well-established results on uniqueness, stability, and constructive reconstruction methods, see [92, 133, 146] for cases with strictly convex boundaries, and [153, 64] for cases with concave points and analytic structures. In particular, for smooth metrics in dimensions $n \geq 3$, not only the full jet of the metric at the boundary but also the metric in the interior near a strictly convex point can be recovered, which leads to global rigidity results under foliation conditions, see [136, 145, 137]. Besides, there are also many other works addressing lens and scattering rigidity, including [21, 27, 30, 28, 31, 55, 102, 104, 105, 117, 132, 135].

In contrast, the case of general Lorentzian metrics remains less understood, with comparatively fewer results on boundary determination from partial data. The recovery of stationary metrics from the time separation function is studied in [9] for two-dimensional product Lorentzian manifolds, in [90] for universal covering spaces of real-analytic Lorentzian manifolds, and in [148] in dimensions three and higher. The recovery of stationary metrics from the scattering relation (also referred to as the lens relation) is studied in [130], where the problem is reduced to boundary or lens rigidity of magnetic systems on the base via null geodesics; it is further considered in [106] via timelike geodesics and MP-systems. Most recently, it is proved in [131] that the jets of smooth Lorentzian metrics at the boundary near a lightlike strictly convex point can be recovered from the scattering relation, up to a gauge transformation.

These Lorentzian rigidity problems are closely related to the Lorentzian Calderón problems, which are analogues for wave equations of the classical Calderón problems for elliptic equations. In a classical Calderón problem, one considers recovery of a conductivity or a Riemannian manifold (M, g) from the DN map associated with a Laplace–Beltrami equation. Results for real-analytic Riemannian metrics can be found in [94, 93, 98], and for metrics with fixed conformal structure can be found in [38, 48, 23]. For more results, see, for example, surveys [144, 125, 78].

In a Lorentzian Calderón problem, one considers recovery of a Lorentzian manifold (M, g) from the DN map associated with a wave equation. For ultra-static manifolds with stationary metrics, or for metrics real-analytic in t, this problem has been extensively studied in the literature, including [51, 7, 17, 42, 76, 84, 85, 89, 88], based on Gelfand problems, inverse spectral methods, or the boundary control method. In [5] and [6], the assumption of real-analyticity in t is replaced by bounds on the Lorentzian curvature, and the Lorentzian Calderón problem is solved in a fixed conformal class. In [111], global determination from a smaller amount of DN measurements is proved when the metric is globally hyperbolic in $\mathbb{R}^{1+n}$ and agrees with the Minkowski metric outside a compact set. For examples of non-isometric but conformally related Lorentzian metrics that produce identical restricted Dirichlet-to-Neumann maps, see [99]. These examples are related to the conformal gauge invariance of the wave operator, indicating that recovery of the metric must be understood up to such gauge transformations.

While interior determination in a general Lorentzian manifold remains open, boundary determination has been studied in [139]. Such boundary determination is typically the first step towards interior recovery, as it provides essential geometric information on the boundary. It is shown in [139] that for an admissible Lorentzian manifold, the DN map determines the jets of g, A, q on the boundary up to a gauge transformation in a stable way. Here the reconstruction is local or semilocal, but with sources and measurements on the same sets. This allows one to use the pseudodifferential part of the DN map for the reconstruction. In this chapter, we consider boundary determination from the DN map, where the sources and measurements are supported in disjoint subsets of the boundary. This setting models scenarios in mathematical physics and applied fields, where full boundary access is not available. For example, in geophysical imaging, sensors may need to be placed far from the sources due to physical constraints, such as in oil exploration as mentioned in [89]. These partial data problems pose additional challenges. Our goal is to investigate what geometric information, particularly jets of the metric and the magnetic potential on the boundary, can still be recovered from such incomplete measurements.

Main results

In this chapter, we consider admissible Lorentzian manifolds defined as follows.

Definition 4.1.1. *Let $n \geq 3$ and (M, g) be a smooth connected n-dimensional Lorentzian manifold with non-empty boundary ∂M, where g has the signature $(-, +, \ldots, +)$. We say (M, g) is admissible if*

1. *there exists a smooth, proper, surjective function $\tau \colon M \to \mathbb{R}$, such that $d\tau$ is everywhere timelike;*
2. *∂M is timelike, i.e., the induced metric $\bar{g} := g|_{T\partial M \times T\partial M}$ is Lorentzian;*
3. *∂M is* strictly null-convex*, i.e., if ν denotes the outward pointing unit normal vector field on ∂M, then*

$$II(V, V) = g(\nabla_V \nu, V) > 0,$$

for all null vectors $V \in T_p \partial M$, where II is the second fundamental form.

We call such τ a temporal function. The definition above is adapted from [66, Definition 1.1], with some minor differences: here we assume the temporal function is smooth and the boundary is strictly null-convex. With the existence of the temporal function, the well-posedness of (4.1.1) follows from [70, Theorem 24.1.1], see also [5].

Recall $T^*\partial M \setminus 0$ is naturally decomposed into the elliptic, glancing, and hyperbolic sets. More explicitly, the glancing set contains lightlike covectors in $T^*\partial M$ and the hyperbolic set contains timelike covectors in $T^*\partial M$. The glancing set plays an important role in our analysis, which can be further decomposed into the diffractive part, the gliding part, and higher-order glancing sets, according to the behavior of the null-bicharacteristics near the boundary. Gliding rays are corresponding to the gliding part. Roughly speaking, they are generalized bicharacteristics that hit the boundary tangentially, with second order contact, lying locally in ∂M. For more details see Section 4.2.5. Now let $S^*\partial M \coloneqq (T^*\partial M \setminus 0)/\mathbb{R}^+$. As we can only hope to determine the geometry at boundary points that are visible to measurements, we introduce the following definition for covectors in the glancing set. For definition of forward/backward gliding rays, see Section 4.2.5.

Definition 4.1.2. *Let (M, g) be an admissible Lorentzian manifold with dimension $n \geq 3$. Let $\mathcal{G}$ be the glancing set defined as in (4.2.5). A glancing covector $(x', \xi') \in \mathcal{G}$, viewed as an element of $S^*\partial M$, is called a recoverable direction if*

- *$x' \in U$ and the corresponding forward gliding ray passes through V; or*
- *$x' \in V$ and the corresponding backward gliding ray passes through U.*

The point x' is called a recoverable point if

- *for $n = 3$, both directions in glancing set are recoverable;*
- *for $n > 3$, at least one direction in glancing set is recoverable.*

We denote by $\tilde{U}$ the set of recoverable points in U and by $\tilde{V}$ the set of recoverable points in V. We refer to them as recoverable sets.

We remark that in our setting, the projection of a gliding ray onto the base manifold is in fact a null-geodesic on ∂M with respect to the induced metric $\bar{g}$, see Lemma 4.2.16. Let $\mathcal{L}^{\pm}_{\partial M}(p)$ denote the future (or past) light cone from $p \in \partial M$, defined as the union of all forward (or backward) null-geodesics in $(\partial M, \bar{g})$ emanating from p. For definition of forward/backward null-geodesics, see Section 4.2.3. When $n > 3$, the set $\tilde{U}$ consists of the points in U whose future light cone in $(\partial M, \bar{g})$ intersects V. Equivalently, these are points in U that lie on the past light cone in $(\partial M, \bar{g})$ of some point in V. A similar argument applies to $\tilde{V}$. Consequently, the recoverable sets are explicitly given by

$$\tilde{U} = (\textstyle\bigcup_{q \in V} \mathcal{L}^{-}_{\partial M}(q)) \cap U \quad \text{ and } \quad \tilde{V} = (\textstyle\bigcup_{p \in U} \mathcal{L}^{+}_{\partial M}(p)) \cap V.$$

In addition, we consider the future pointing lens relation L, also referred to as the scattering relation in the literature. We describe it below, see also Definition 4.2.3 and Remark 4.2.4 about future pointing lens relation. Let $(x', \xi') \in T^*\partial M$ be timelike and future pointing. Let $(x', \xi) \in T^*M$ be the unique inward-pointing lightlike covector with orthogonal projection (x', ξ'). There exists a unique forward null-bicharacteristic Γ starting from (x', ξ). If Γ intersects the boundary again at some $(y', \eta) \in L^*M$, then under our assumptions the intersection must be transversal. Let $(y', \eta') \in T^*M$ be the orthogonal projection of (y', η). We define $L(x', \xi') = (y', \eta')$. Using that L is an even map, one can define L for past pointing timelike covector, for more details see [139].

Recall $\bar{g}$ denotes the restriction of g to $T\partial M \times T\partial M$, i.e., $\bar{g} = g|_{T\partial M \times T\partial M}$. We assume no a priori information is known on the open disjoint boundary sets U or V. The following theorem summarizes what can be reconstructed from the restricted Dirichlet-to-Neumann map $\Lambda^{U,V}_{g,A,q}$.

Theorem 4.1.3. *Let (M, g) be an admissible Lorentzian manifold of dimension $n \geq 3$ and U, V be disjoint open subsets of ∂M. Then from the restricted DN map $\Lambda^{U,V}_{g,A,q}$, one can reconstruct*

(1) the recoverable sets $\tilde{U}$ and $\tilde{V}$;

(2) the conformal class of $\bar{g}$ on $\tilde{U}$ and $\tilde{V}$;

(3) for each recoverable direction (x', ξ') on U or V, the lens relation $L|_{\mathcal{N}}$ up to time orientation, where $\mathcal{N}$ is a small conic timelike neighborhood of (x', ξ');

(4) the magnetic potential $A|_{T\partial M}$ on $\tilde{U}$ and $\tilde{V}$.

We note that there is a natural obstruction to reconstructing g, A, q from the restricted DN map $\Lambda_{g,A,q}^{U,V}$. Specifically, there exist gauge transformations that keep $\Lambda_{g,A,q}^{U,V}$ invariant, see Lemma 4.2.7. One can only expect to reconstruct these quantities up to such a gauge transformation. Without prior knowledge of U or V, Theorem 4.1.3 shows we can reconstruct the induced $\bar{g}$ on the boundary for points in the recoverable sets, up to conformal factors. Moreover, we can reconstruct the 1-form A on the boundary for these points, up to gauge transformations. Indeed, we may pick local coordinates (x', x^n) near such a boundary point and suppose it is the semi-geodesic normal coordinates for some metric, see Lemma 4.2.5. Then by Lemma 4.2.5, there exists a boundary-preserving diffeomorphism Ψ such that $\Psi^* g$ has (x', x^n) as the semi-geodesic normal coordinates as well. Then by [139, Lemma 2.5], in these coordinates we can modify the 1-form $\Psi^* A$ using a gauge transformation such that $(\Psi^* A)_n(x', 0) = 0$. Note that $\Psi^* A|_{T\partial M} = A|_{T\partial M}$ as Ψ fixes the boundary. This implies we reconstruct $\Psi^* A$ on the boundary and therefore A on the boundary up to gauge transformations.

One may extend the definition of lens relation by using broken null-bicharacteristics reflecting finitely many times at ∂M; see Definition 4.2.8 and 4.2.9. Indeed, let $(x', \xi') \in T^*\partial M$ be defined as above. For consistency we write (x', ξ') as $(x'_0, {\xi^0}')$. We consider the unique broken null-bicharacteristic starting from (x'_0, ξ^0), reflecting successively at $x'_k \in \partial M$ with $(x'_k, \xi^k) \in L^*M$, for $k = 1, \ldots, K$. Let $(x'_k, {\xi^k}') \in T^*\partial M$ be the orthogonal projection of (x'_k, ξ^k). Then we define the k-th lens relation L^k as $L^k(x'_0, {\xi^0}') = (x'_k, {\xi^k}')$. By Proposition 4.3.5, when $x'_0 \in U$ and $x'_k \in V$, the graph of such L^k is the canonical relation for the restricted DN map, although the exact value of k may not be recovered, see Section 4.4.

Based on this constructive result, we prove the following determination result.

Theorem 4.1.4. *Let (M, g_1) and (M, g_2) be admissible Lorentzian manifolds of dimension $n \geq 3$. Let U, V be disjoint open subsets of ∂M. Suppose*

$$\Lambda_{g_1,A_1,q_1}^{U,V} f = \Lambda_{g_2,A_2,q_2}^{U,V} f$$

for any distribution f with compact support in U. Then $\bar{g}_1 = e^{\varphi_0} \bar{g}_2$ in the same recoverable sets $\tilde{U}$ and $\tilde{V}$, where $\varphi_0 \in C^\infty(\tilde{U} \cup \tilde{V})$ is locally constant. In particular, if $W_1 \subseteq \tilde{U}$ and $W_2 \subseteq \tilde{V}$ are connected components *joined by some forward gliding ray or broken bicharacteristic, then there exists a constant c such that*

$$\varphi_0|_{W_1} \equiv \frac{c}{n-2} \quad \text{and} \quad \varphi_0|_{W_2} \equiv \frac{c}{n}.$$

Furthermore, there exists $\varphi \in C^\infty(M)$ with $\varphi|_{\partial M} = \varphi_0$ and a local diffeomorphism ψ near ∂M that fixes it pointwise, such that the jets of g_1 and $e^\varphi \psi^* g_2$ coincide in $\tilde{U}$ and $\tilde{V}$.

Compared to Theorem 4.1.3, this theorem is a uniqueness result that further describes the conformal factor between $\bar{g}_1$ and $\bar{g}_2$ when the corresponding restricted DN maps coincide. Moreover, it proves that the jets of the metric on the boundary are determined up to gauge transformations at points in the recoverable sets.

Next, assume the induced metric and time orientation are known on U (respectively on V). Then the structure on a subset of V (respectively of U) can be reconstructed. To state the result, we introduce the following definition.

Definition 4.1.5. *We say x' is V-recoverable, if there exists a forward broken null-geodesic from x' to some $y' \in V$. Similarly, y' is called U-recoverable, if there exists a backward broken null-geodesic from y' to some $x' \in U$. We denote by $\tilde{U}_0$ the set of V-recoverable points in U and by $\tilde{V}_0$ the set of U-recoverable points in V.*

More explicitly, let $\mathcal{L}^{\pm,b}(p)$ denote the union of all forward (or backward) broken null-geodesics starting from p in (M, g). Then $\tilde{U}_0$ and $\tilde{V}_0$ can be written as

$$\tilde{U}_0 = (\textstyle\bigcup_{q \in V} \mathcal{L}^{-,b}(q)) \cap U \quad \text{and} \quad \tilde{V}_0 = (\textstyle\bigcup_{p \in U} \mathcal{L}^{+,b}(p)) \cap V.$$

We emphasize that both $\tilde{U}_0$ and $\tilde{V}_0$ are open subsets, since under our assumptions broken null-geodesics are stable under small perturbations. In particular, one has $\tilde{U} \subseteq \tilde{U}_0$ and $\tilde{V} \subseteq \tilde{V}_0$ by Lemma 4.2.18. In this setting, we state the following theorem.

Theorem 4.1.6. *Let (M, g) be an admissible Lorentzian manifold of dimension $n \geq 3$ and U, V be open disjoint subsets of ∂M. Given the induced metric $\bar{g}$ and the time orientation on $\tilde{U}_0$, from the restricted DN* *map $\Lambda^{U,V}_{g,A,q}$, one can reconstruct $\bar{g}$ and the time orientation on $\tilde{V}_0$. The same result holds with $\tilde{U}_0$ and $\tilde{V}_0$ interchanged.*

This theorem implies that with additional information on U, we can reconstruct g on the boundary at points in the U-recoverable sets of V, up to gauge transformations. Indeed, with the induced metric $\bar{g}$ reconstructed in the U-recoverable sets, we may choose local coordinates (x', x^n) near a boundary point. As before, by Lemma 4.2.5, there exists a boundary-preserving diffeomorphism Ψ such that $\Psi^* g$ has (x', x^n) as the semi-geodesic normal coordinates. Since $\Psi^* g|_{T\partial M \times T\partial M} = \bar{g}$, we actually reconstruct $\Psi^* g$ on the boundary in these coordinates and thus g on the boundary up to gauge transformations.

4.1.1 Idea of the proof

The full data DN map $\Lambda_{g,A,q}$ on an admissible Lorentzian manifold is known to be associated with two canonical relations: the diagonal and the graph of the lens relation. The first corresponds to its pseudodifferential operator part, and the second to its Fourier Integral Operator part. In [139], the jet bundles of g, A and q on the boundary are stably recovered up to gauge transformations using the full symbol of the pseudodifferential part of $\Lambda_{g,A,q}$. However, when the source region U and the observation region V are disjoint, we no longer have access to the pseudodifferential part of the DN map. Instead, we fully rely on the Fourier Integral Operator part of the DN map to recover information. We perform symbolic construction of solutions to (4.1.1) with conormal distribution f to compute the principal symbol of $\Lambda^{U,V}_{g,A,q}$ on the Fourier Integral Operator part. We show that the absolute value of the principal symbol connects the determinant of the metric on U and V via broken bicharacteristics, while the argument of the principal symbol reveals information about the light ray transform of A along broken null-geodesics.

In order to make use of the information provided by the principal symbol of the DN map, we utilize gliding rays, which are essentially null-bicharacteristics for the metric restricted to the boundary, see Lemma 4.2.16. Gliding rays are important in our proof for two main reasons:

1. the union of gliding rays acts as light observation sets, which provides information about the structure of the light cone on the boundary;
2. the gliding rays are entirely in $T^*\partial M$, where we have more control comparing to the interior.

The use of gliding rays also allows us to have little restriction on U and V, for example, they may be arbitrarily small and away from each other. The amount of information that can be recovered then only depends on the existence of gliding rays connecting them.

Therefore, we first use the propagation of singularities to identify gliding rays on the boundary. From these gliding rays, we reconstruct the light cone and hence the conformal class of the metric at points in the recoverable sets on the boundary. We prove that around the glancing set, we can also construct the lens relation up to time orientation. This allows us to use the main result in [131] to determine the jet bundle of the metric up to gauge transformations.

We then use the principal symbol of $\Lambda_{g,A,q}^{U,V}$ to determine the conformal factor and to construct the 1-form A restricted to boundary tangential directions.

1. The magnitude of the principal symbol gives information of the determinant of the metric. We use a geometric argument to show that the determinant relation provided by the magnitude of the principal symbol is enough to guarantee that the conformal factor is locally constant. This is a deterministic result.

2. The argument of the principal symbol contains the light ray transform of the 1-form A. We use the fact that broken bicharacteristics converge uniformly to gliding rays on admissible Lorentzian manifolds. Since we have the light ray transform of A along broken null-geodesics, the limit gives the integral of A along the projection of gliding rays. We show that this is enough to construct A on boundary tangential directions.

Finally, when we already have a priori information of the boundary metric on U, we show that more can be constructed. Instead of the recoverable sets, which by definition is the region connected by gliding rays, we show that we can construct the metric in the region connected by broken bicharacteristics. Note that this region is larger than recoverable sets, as gliding ray can be uniformly approximated by broken bicharacteristics on bounded time intervals, see Section 4.2.5.

4.1.2 Outline of the chapter

In Section 4.2, we introduce tools and results required. Specifically, in Section 4.2.1, we recall density bundles; in Section 4.2.2, we discuss microlocal analysis; in Section 4.2.3, we discuss the basics of Lorentzian manifold; in Section 4.2.4, we discuss gauge equivalence; and in Section 4.2.5, we discuss properties of broken bicharacteristics and gliding rays.

In Section 4.3, we compute the principal symbol of $\Lambda_{g,A,q}^{U,V}$ for disjoint U and V. In Section 4.3.1, we introduce some notations; in Section 4.3.2, we recall the optics construction of the solution; and in Section 4.3.3, we symbolically construct the solution and compute the principal symbol.

In Section 4.4, we construct the conformal class of the boundary metric via weak lens relation; in Section 4.5, we upgrade the weak lens relation to lens relation up to time orientation near the glancing set; in Section 4.6, we show the conformal factor is locally constant; in Section 4.7, we construct the 1-form A on boundary tangential directions; finally in Section 4.8, we combine the previous results and prove Theorem 4.1.3, Theorem 4.1.4 and Theorem 4.1.6.

4.2 Preliminaries

4.2.1 Operators on half-densities

For $s \in \mathbb{R}$, the *s-density bundle* is the real line bundle $\Omega_M^s \to M$, whose transition map is defined as follows: for any coordinate charts $F_i : U_i \to \mathbb{R}^n$, the transition map is given by $\tau_{ij}(p, v) = (p, |\det(F_i \circ F_j^{-1})'|^{-s} v)$. Just like differential forms on a manifold, one can view s-densities as

$$\Omega_M^s = \{\omega : \Lambda^n M \to \mathbb{R} : \omega(\mu v) = |\mu|^s v,\ v \in \Lambda^n M,\ \mu \in \mathbb{R}\},$$

where M has dimension n and $\Lambda^n M$ is the n-th exterior power of tangent bundle. In what follows, we will mainly use half-densities $\Omega_M^{1/2}$ and we write

$$\omega(v_1, \ldots, v_n) := \omega(v_1 \wedge \cdots \wedge v_n).$$

Hence if $w_1, \dots, w_n$ is another set of vectors and A is the matrix satisfying $v_1 \wedge \cdots \wedge v_n = (\det A) w_1 \wedge \cdots \wedge w_n$, then

$$\omega(v_1, \dots, v_n) = |\det A|^{1/2} \omega(w_1, \dots, w_n).$$

We will also repeatedly use the notation $|dx|^{1/2}$ for the standard coordinate half-density, where x is a local coordinate system, and $dx = dx^1 \wedge \cdots \wedge dx^n$. For more details on densities, see [65].

Let $\mathcal{D}'(M; \Omega_M^{1/2})$ denote the set of half-density valued distributions on M, defined as the dual space of smooth half-densities $C^\infty(M; \Omega_M^{1/2})$. Let $|g|^{1/4} \in C^\infty(M; \Omega_M^{1/2})$ such that in local coordinates x, it has the form

$$|g|^{1/4} = |g|_x^{1/4} |dx|^{1/2}$$

with $|g|_x$ being the absolute value determinant of g in coordinate x. One can convert an operator on functions to one acting on half-densities by conjugating a half-density. The natural choice is given by conjugating $|g|^{1/4}$:

$$P_{g,A,q} = |g|^{1/4} \Box_{g,A,q} |g|^{-1/4},$$

so that $P_{g,A,q}(u|g|^{1/4}) = (\Box_{g,A,q} u)|g|^{1/4}$. In local coordinates, we have

$$\begin{aligned} |dx|^{-1/2} P_{g,A,q}(u(x)|dx|^{1/2}) &= |g|_x^{-1/4} (\partial_j - iA_j) \left[|g|_x^{1/2} g^{jk} (\partial_k - iA_k)(|g|_x^{-1/4} u) \right] + qu \\ &= \left[g^{jk} \partial_j \partial_k + \partial_j g^{jk} \partial_k - 2i A_j g^{jk} \partial_k + \text{lower order terms} \right] u. \end{aligned}$$

Thus the principal symbol, sub-principal symbol and the corresponding Hamiltonian vector field for the operator $P_{g,A,q}$ are

$$\begin{aligned} p &= -g^{jk} \xi_j \xi_k, \\ c &= i \partial_j g^{jk} \xi_k + 2 A_j g^{jk} \xi_k - \frac{1}{2i} \sum_l \frac{\partial^2}{\partial x^l \partial \xi_l} (-g^{jk} \xi_j \xi_k) = -A(H_p), \\ H_p &= -2 g^{jk} \xi_j \partial_{x^k} + \partial_{x^l} g^{jk} \xi_j \xi_k \partial_{\xi_l} \end{aligned} \tag{4.2.1}$$

where we lift A as a 1-form on T^*M.

4.2.2 Lagrangian distributions and principal symbols

In this subsection, we denote by M a smooth manifold without boundary. We follow [69, Definition 3.2.2], with a small modification in notation.

Definition 4.2.1. *Let $\Lambda \subset T^*M\backslash 0$ be a conic Lagrangian submanifold. By $I^m(\Lambda; \Omega_M^{1/2})$ we shall denote the set of all $w \in \mathcal{D}'(M; \Omega_M^{1/2})$ such that $w = \sum_{j\in J} w_j$ with the supports of w_j locally finite and*

$$\langle w_j, v\rangle = (2\pi)^{-(n+2N_j)/4} \int\int e^{i(\phi_j(x,\theta)-\pi N_j/4)} a_j(x,\theta)v(x)dxd\theta, \quad v \in C^\infty(M).$$

In the above equation,

- *X_j is a local coordinate patch and dx is the Lebesgue measure with respect to it;*
- *ϕ_j is a non-degenerate phase function defined in a conic neighborhood U_j of $X_j \times (\mathbb{R}^{N_j}\backslash 0)$, such that the critical set $\{(x,\theta) : \partial_\theta\phi_j(x,\theta) = 0\}$ parameterizes Λ locally via the diffeomorphism*

$$(x,\theta) \mapsto (x, \partial_x\phi_j(x,\theta));$$

- *$a_j \in S^{\mu_j}(\mathbb{R}^n \times \mathbb{R}^{N_j})$ is a symbol of order $\mu_j = m+(n-2N_j)/4$, with supp$(a_j) \subset \{(x,t\theta) : t \geq 1, (x,\theta) \in V\}$, where V is a compact subset of the image U_j in $\mathbb{R}^n \times \mathbb{R}^{N_j}$; see [69, Definition 1.1.1] for symbol classes.*

In particular, if $\Lambda = N^*K$ is the conormal bundle of a smooth submanifold $K \subset M$, then $I^m(N^*K; \Omega_M^{1/2})$ is called the set of half-density valued conormal distributions.

It is well-known that given a Lagrangian distribution, its principal symbol can be invariantly defined as a half-density on the Lagrangian submanifold. To define the principal symbol, consider $w \in I^m(\Lambda; \Omega_M^{1/2})$. Suppose in a local coordinate patch, it has the form

$$w = \int e^{i\phi(x,\theta)}a(x,\theta)d\theta|dx|^{1/2}, \tag{4.2.2}$$

where for simplicity we ignore the factor $(2\pi)^{-(n+2N)/4}$ throughout the chapter, as it does not affect the computation at all. Now let $C = \{(x,\theta) : \partial_\theta \phi = 0\}$ be the critical set and $F(x,\theta) = (x,\phi_x)$ be the diffeomorphism from C to an open subset of Λ. Let L be the Maslov bundle on Λ. Then the principal symbol $\sigma(w) \in S^{m+n/4}(\Lambda;\Omega_\Lambda^{1/2} \otimes L)$ is locally computed via method of stationary phase, see for example [39, 69], as

$$(F^{-1})^* \left(e^{i\pi N/4} a(x,\theta) \left| \frac{D(\lambda, \partial_\theta \phi)}{D(x,\theta)} \right|^{-1/2} |d\lambda|^{1/2} \right),$$

where λ is a local coordinates on C and $N = sgn\left(\frac{D(\lambda,\partial_\theta\phi)}{D(x,\theta)}\right)$ is the Maslov index. The principal symbol is unique modulo lower order symbols, i.e.,

$$\sigma(w) \in S^{m+n/4}(\Lambda;\Omega_\Lambda^{1/2} \otimes L)/S^{m+n/4-1}(\Lambda;\Omega_\Lambda^{1/2} \otimes L)$$

is unique as an equivalence class.

We shall mostly work with classical symbols, i.e., $a(x,\theta)$ can be written asymptotically as

$$\chi(\theta)a(x,\theta) + (1-\chi(\theta)) \sum_{j=m+(n-2N)/4}^{-\infty} a_j(x,\theta),$$

where χ is a compactly supported smooth cutoff equal to 1 around 0, and a_j is homogeneous of degree j in θ. In this case, the principal symbol is uniquely defined in $S^{m+n/4}(\Lambda;\Omega_\Lambda^{1/2} \otimes L)$, and we refer to such Lagrangian distributions as classical Lagrangian distributions, denoted by $I^m_{cl}(\Lambda;\Omega_\Lambda^{1/2})$.

We will repeatedly encounter half-density valued Lagrangian distributions of the form $w = v|g|^{1/4}$, where $v \in \mathcal{D}'(M)$ and $|g|^{1/4} \in C^\infty(M;\Omega_M^{1/2})$ is the half-density defined previously. Then in local coordinates, with w given in (4.2.2), v has the form

$$v(x) = \int e^{i\phi(x,\theta)} b(x,\theta) d\theta = \int e^{i\phi(x,\theta)} |g|_x^{-1/4} a(x,\theta) d\theta.$$

We shall denote

$$\sigma_x(v) = (F^{-1})^* \left(e^{i\pi N/4} b(x,\theta) \left| \frac{D(\lambda, \partial_\theta \phi)}{D(x,\theta)} \right|^{-1/2} |d\lambda|^{1/2} \right)$$

with the subscript x reminding that it depends on coordinate choice. It is easy to see that $\sigma(w) = \sigma_x(v)|g|_x^{1/4}$.

In the following, let X and Y be smooth manifolds. Suppose F is an operator mapping half-density valued distributions on X to half-density valued distributions on Y, whose Schwartz kernel k is a Lagrangian distribution associated with $\Lambda \subset (T^*Y \setminus 0) \times (T^*X \setminus 0)$. Then F is called a Fourier Integral Operator and the Lagrangian submanifold Λ corresponds to a canonical relation C from $T^*Y \setminus 0$ to $T^*X \setminus 0$, see [70, Definition 21.2.12]. One can also call Λ the twisted canonical relation. If, in addition, C is the graph of a conical transformation $\chi : T^*X \setminus 0 \to T^*Y \setminus 0$, i.e., $C = \{(\chi(q); q) : q \in T^*X\}$, then we say C is a canonical graph and F an FIO associated with a conical graph. In the special case where $X = Y$ and Λ is the diagonal of $T^*X \times T^*X$, we say F is a pseudodifferential operator.

For an FIO, the principal symbol $\sigma(F)$ is usually defined as that of its Schwartz kernel k. Throughout this chapter, we will focus on FIOs associated with canonical graphs. Suppose $Fu = v$ for suitable Lagrangian distributions $u \in I^{k+N/2-n/4}(X; \Lambda_1)$ and $v \in I^{k+N/2-n/4+m}(Y; \Lambda_2)$. Then the principal symbols of k, u, v satisfy $\sigma(v) = \sigma(k) \times \sigma(u)$, where the multiplication of half-densities can be found in [70] and [71, Theorem 25.2.3]. This implies that we may identify $\sigma(F)$ as a map from $S^k(\Lambda_1; \Omega^{1/2}_{\Lambda_1} \otimes L_{\Lambda_1})$ to $S^{k+m}(\Lambda_2; \Omega^{1/2}_{\Lambda_2} \otimes L_{\Lambda_2})$, given by

$$\sigma(u) \mapsto \sigma(k) \times \sigma(u). \tag{4.2.3}$$

Thus, by abuse of language, we shall use $\sigma(F)$ to denote this map, which sends the principal symbol of u to that of v.

Finally, let P be a differential operator acting on half-density valued distributions and $p(x, \xi)$ be its principal symbol. Then we define the characteristic set $\mathrm{Char}(P) = \{(x, \xi) \in T^*M : p(x, \xi) = 0\}$. Let $w \in I^m(\Lambda; \Omega^{1/2}_M)$ be such that the Lagrangian submanifold Λ is contained in the characteristic set of P, that is, $\Lambda \subset \mathrm{Char}(P)$. Suppose $Pw \in C^\infty(M; \Omega^{1/2}_M)$. Recall H_p is the Hamiltonian vector field of p, and c the sub-principal symbol of P. Then the principal symbol $\sigma(w)$ satisfies

$$(\mathcal{L}_{H_p} + ic)\sigma(w) = 0, \tag{4.2.4}$$

where $\mathcal{L}_{H_p}$ is the Lie derivative of H_p on half-densities over Λ, see [40, Section 5.3].

For more details on the definition of Lagrangian distributions, principal symbols and FIO calculus, we refer to [65, 39, 69, 70, 71, 40].

4.2.3 Lorentzian geometry

Recall (M, g) is an admissible Lorentzian manifold, see Definition 4.1.2. For $p, q \in M$, we denote by $p < q$ if $p \neq q$ and p is joined to q by a future-pointing causal curve. We use $p \leq q$ if $p = q$ or $p < q$. The causal future of p is $J^+(p) = \{q \in M : p \leq q\}$ and that for any set $A \subseteq M$ is $J^+(A) = \cup_{p \in A} J^+(p)$. For $\eta \in T_p^*M$, the corresponding vector of η is denoted by $\eta^\sharp \in T_pM$. The corresponding covector of $v \in T_pM$ is denoted by $v^\flat \in T_p^*M$. We denote by

$$L_pM = \{v \in T_pM \setminus 0 : \ g(v, v) = 0\}$$

the set of lightlike vectors at $p \in M$ and similarly by L_p^*M the set of lightlike covectors. The sets of future-pointing (or past-pointing) lightlike vectors are denoted by L_p^+M (or L_p^-M), and those of future-pointing (or past-pointing) lightlike covectors are denoted by $L_p^{*,+}M$ (or $L_p^{*,-}M$). Recall the principal symbol p and the Hamiltonian vector field H_p in (4.2.1) for $P_{g,A,q}$. Note that $P_{g,A,q}$ and $\square_{g,A,q}$ share the same principal symbol and thus the same Hamiltonian vector field. In the Lorentzian manifold (M, g), the characteristic set $\mathrm{Char}(P_{g,A,q})$ is actually the set of lightlike covectors with respect to g.

The set $T^*\partial M \setminus 0$ is naturally decomposed into three different regions by the metric in the following way. The elliptic region $\mathcal{E}$, the glancing region $\mathcal{G}$ and the hyperbolic region $\mathcal{H}$ are defined by

$$\begin{aligned}
\mathcal{E} &= \{(x, \xi') \in T^*\partial M \setminus 0 : |\pi^{-1}(x, \xi') \cap L^*M| = 0\}, \\
\mathcal{G} &= \{(x, \xi') \in T^*\partial M \setminus 0 : |\pi^{-1}(x, \xi') \cap L^*M| = 1\}, \\
\mathcal{H} &= \{(x, \xi') \in T^*\partial M \setminus 0 : |\pi^{-1}(x, \xi') \cap L^*M| = 2\}.
\end{aligned} \tag{4.2.5}$$

If $(x, \xi') \in \mathcal{H}$, we thus have two lightlike preimage $(x, \xi_\pm)$, where $+/-$ corresponds to $\xi_\pm^\sharp$ being inward/outward-pointing. Let $\Gamma_{x,\xi_\pm}$ be the integral curves generated by H_p passing through $(x, \xi_\pm)$. We call them the (null-)bicharacteristics for $P_{g,A,q}$ or $\square_{g,A,q}$. Let $\gamma_{x,\xi_\pm}$ be the projection of $\Gamma_{x,\xi_\pm}$ onto M. Note that $\gamma_{x,\xi_\pm}$ is simply the unique null-geodesic passing through x with direction $\dot\gamma_{x,\xi_\pm} = -2\xi_\pm^\sharp$. In the semi-geodesic normal coordinates (see Lemma 4.2.5), the unique inward/outward-pointing preimages of (x', ξ') are $(x, \xi_\pm)$, where we write $x = (x', 0)$ and $\xi_\pm$ are given by

$$\xi_\pm = (\xi', \pm \xi_n), \qquad \text{with } \xi_n := \sqrt{-g^{\alpha\beta}\xi_\alpha \xi_\beta}.$$

The ∂_{x^n} component of the Hamiltonian vector field at $(x', 0, \xi_\pm)$ is thus $\mp 2\xi_n \partial_{x^n}$. This implies Γ_{x,ξ_+} travels in the opposite direction as the parameter increases, that is, $\Gamma_{x,\xi_+}((-s, 0)) \subset M^\circ$ for some $s > 0$. On the other hand, Γ_{x,ξ_-} travels in the same direction as the parameter increases, that is, $\Gamma_{x,\xi_-}((0, s)) \subset M^\circ$ for some $s > 0$. Roughly speaking, when s increases, a bicharacteristic $\Gamma(s)$ starting from the boundary transversally will always leave from some outward-pointing (x, ξ_-) and then hits the boundary transversally at some inward-pointing (y, η_+).

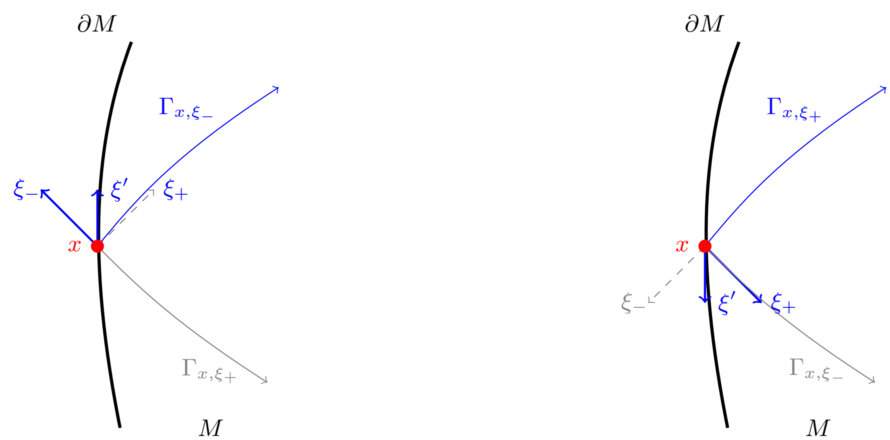

Figure 4.1: If (x', ξ') is past-pointing (left), then the unique forward null-bicharacteristic leaves from its outward preimage (x, ξ_-) and travels forward in time; if (x', ξ') is future-pointing (right), then the unique forward null-bicharacteristic leaves from its inward preimage (x, ξ_+) and travels backward in time. In conclusion, the null-bicharacteristic is forward when it lies in the casual future of x, regardless of the flow direction.

Recall that if the time orientation is given, then a vector v is called future-pointing if it is in the future-pointing light cone. A covector ξ is called future-pointing if the corresponding vector $\xi^\sharp$ is. Recall we assume solutions to the wave equation are only supported in the causal future of the initial data. In other words, information travels along the null-bicharacteristics into the causal future. For every boundary covector $(x', \xi') \in \mathcal{H}$, its preimages $(x, \xi_\pm)$ give us four choices of the null-bicharacteristic, depending on whether it is contained in the interior or exterior of M and whether it is in the causal future or causal past of x. Here recall we write $x = (x', 0)$. Thus, there exists a unique null-bicharacteristic starting from either (x, ξ_+) or (x, ξ_-), and lying in the interior of M as well as the causal future of x'. We call it the unique *forward null-bicharacteristic* corresponding to (x', ξ'). When (x', ξ') is future-pointing, the forward null-bicharacteristic is the one traveling in the opposite direction as the parameter increases, starting from (x, ξ_+); when (x', ξ') is past-pointing, the forward null-bicharacteristic is the one traveling in the opposite direction as the parameter increases, starting from (x, ξ_-), see Figure 4.1. We say γ is the unique *forward null-geodesic* corresponding to (x', ξ'), if it is the projection of the unique forward null-bicharacteristic. Note that the natural parameterization of γ would give us $\dot{\gamma}(0) = -2(\xi')^\sharp$ by (4.2.1). We remark that our definition of the forward null-bicharacteristic corresponds to the future-pointing null-bicharacteristic defined in [139], but it flows in the opposite direction. This is because we chose $p = -g^{jk}\xi_j\xi_k$, whereas their principal symbol is $p = \frac{1}{2}g^{jk}\xi_j\xi_k$.

Note that under the strictly null-convex assumption, if a null-geodesic starting from the interior hits the boundary, then it must do so transversally, see [66, Proposition 2.4]. This implies that null-bicharacteristics starting from the interior only intersect the boundary in the hyperbolic set. In addition, we state the following proposition, using the ideas from [15] and [124, Proposition 7.1].

Proposition 4.2.2. *Suppose ∂M is timelike and strictly null-convex. Let $\Lambda \subset T^*M^o \setminus 0$ be a smooth conic Lagrangian submanifold contained in the characteristic set $\mathrm{Char}(P_{g,A,q})$. Suppose Λ intersects ∂M. Then the intersection is transversal and therefore*

$$\Lambda' \coloneqq \Lambda|_{\partial M},$$

is a smooth conic Lagrangian submanifold of $T^*\partial M$. We may extend Λ across ∂M as a smooth conic Lagrangian submanifold of a larger Lorentzian manifold M_e. Moreover, if $u \in I^{\mu-1/4}(\Lambda)$ is a Lagrangian distribution, then its restriction

$$u_0 \in I^{\mu}(\Lambda')$$

is also a Lagrangian distribution.

Proof. By [66, Proposition 2.4], the Hamiltonian vector field H_p is transversal to ∂M, as its projection to

M, i.e., the null vector field, hits ∂M transversally. Since $\Lambda \subset \mathrm{Char}(P_{g,A,q})$, one has H_p is tangent to Λ and therefore Λ intersects ∂M transversally. As observed in [15], Λ can be extended across the boundary as the union of integral curves of H_p, starting from Λ in M^o. Then the same proof in [124, Proposition 7.1] indicates the rest of the statements. $\square$

Now we define the lens relation.

Definition 4.2.3. *Let $(x', \xi') \in \mathcal{H}$, and let Γ be the unique forward bicharacteristic corresponding to (x', ξ'). Denote by $\mathcal{H}'$ the set of $(x', \xi') \in \mathcal{H}$ such that Γ hits the boundary again at some (y, η). Then the forward lens relation L is defined by*

$$L : \mathcal{H}' \to \mathcal{H}, \quad (x', \xi') \to (y', \eta'),$$

where $(y', \eta') = \pi(y, \eta)$ with the natural projection $\pi : T^*M|_{\partial M} \to T^*\partial M$. In particular, L is invertible on its range, and L^{-1} is the backward lens relation. For all $k \in \mathbb{Z}$, L^k is called the k-th lens relation.

Remark 4.2.4. *We emphasize that L can be defined locally near (x', ξ'). Moreover, L is called the forward lens relation, as y' is always in the causal future of x'. Throughout this chapter, we always use L to denote the forward lens relation, although for convenience we shall refer to it simply as the lens relation.*

4.2.4 Gauge equivalence

There exist some natural gauge transformations such that the restricted DN map $\Lambda_{g,A,q}^{U,V}$ is invariant. In the following, we first introduce the semi-geodesic normal coordinates on Lorentzian manifold with timelike boundary.

Lemma 4.2.5 ([139] Lemma 2.3)**.** *For every $x \in \partial M$, there exists $\epsilon > 0$, a neighborhood N of x in M, and a diffeomorphism $\Psi : \partial M \cap N \times [0, T) \to N$ such that*

1. *$\Psi(x', 0) = x'$ for all $x' \in \partial M \cap N$;*

2. *$\Psi(x', x^n) = \gamma_{x'}(x^n)$ where $\gamma_{x'}(x^n)$ is the unit speed geodesic issued from x' normal to ∂M.*

If we write $x' = (x^1, \ldots, x^{n-1})$ as local boundary coordinates on ∂M, in the coordinate system (x', x^n), the *metric g takes the form*

$$g = g_{\alpha\beta} dx^\alpha \otimes dx^\beta + dx^n \otimes dx^n, \quad \alpha, \beta \leq n-1.$$

For any two metrics g_1 and g_2, there exists a boundary fixing diffeomorphism $\Psi : M \to M$, $\Psi|_{\partial M} = \mathrm{Id}_{\partial M}$, such that (x', x^n) is the semi-geodesic normal coordinates for g_1 if and only if it is the semi-geodesic normal coordinates for $\Psi^* g_2$.

Now we state the gauge equivalence, following the same argument as in [139].

Lemma 4.2.6. *Let (M, g) be an n-dimensional admissible Lorentzian manifold with boundary. Let A be a smooth 1-form and q be a smooth function on M. Suppose U and V are open connected subsets of ∂M such that $U \cap V = \emptyset$. Suppose $\Psi : M \to M$ is a diffeomorphism such that $\Psi|_{U\cup V} = \mathrm{Id}|_{U\cup V}$, then*

$$\Lambda^{U,V}_{g,A,q} = \Lambda^{U,V}_{\Psi^* g, \Psi^* A, \Psi^* q}.$$

Proof. The proof follows directly from [139, Lemma 2.1]. □

Another class of gauge transformations is given by conformal changes, but unlike the case of full boundary data (see [139, Lemma 2.2]), we have fewer restrictions here.

Lemma 4.2.7. *Let (M, g) be an n-dimensional admissible Lorentzian manifold with boundary. Let A be a smooth 1-form and q be a smooth function on M. Suppose U and V are open connected subsets of ∂M such that $U \cap V = \emptyset$. Let φ and ψ be smooth functions satisfying*

$$\varphi|_U \equiv \frac{c}{n-2}, \quad \varphi|_V \equiv \frac{c}{n}, \quad \psi|_{U\cup V} = 0,$$

for some constant c. Then

$$\Lambda^{U,V}_{g,A,q} = \Lambda^{U,V}_{e^{-2\varphi} g, A - d\psi, e^{2\varphi}(q+q_\varphi)}$$

where $q_\varphi = e^{\frac{n-2}{2}\varphi} \Box_g e^{\frac{2-n}{2}\varphi}$.

Proof. Same computation as in [139, Lemma 2.2] shows that if $v = e^{-\frac{c}{2}} e^{\frac{n-2}{2}\varphi} e^{-i\psi} u$ and u, f satisfies

$\square_{g,A,q} u = 0$, $u|_{\partial M} = f$, then

$$\square_{e^{-2\varphi}g, A-d\psi, e^{2\varphi}(q+q_\varphi)} v = 0, \quad v|_{\partial M} = f.$$

By our assumptions on φ, we have $\nu_{e^{-2\varphi}g} = e^{\varphi}\nu_g = e^{\frac{c}{n}}\nu_g$ on V. Use the fact that $U \cap V = \emptyset$, which means $\Lambda^{U,V}_{g,A,q} f = \partial_\nu u|_V$, we have

$$\Lambda^{U,V}_{e^{-2\varphi}g, A-d\psi, e^{2\varphi}(q+q_\varphi)} f = e^{\frac{c}{n}}\partial_\nu v|_V = e^{\frac{c}{n}-\frac{c}{2}+\frac{n-2}{2n}c}\partial_\nu u|_V = \Lambda^{U,V}_{g,A,q} f.$$

$\square$

4.2.5 Broken bicharacteristics and gliding rays

We first define the broken null-geodesics (see [66, Definition 2.7]) and the broken bicharacteristics (see [70, Definition 24.2.2]).

Definition 4.2.8. *Let $I \subset \mathbb{R}$ be an open connected interval. A piecewise smooth curve $\gamma : I \to M$ is called a* broken null-geodesic *if*

1. *for all open intervals $J \subset I$ with $\gamma(J) \cap \partial M = \emptyset$, $\gamma|_J$ is an affinely parametrized null-geodesic in (M°, g);*

2. *if $s \in I$ and $\gamma(s) \in \partial M$, then $\gamma|_{(s-\epsilon,s]}$ and $\gamma|_{[s,s+\epsilon)}$ are null-geodesics with $\gamma(s \pm (0,\epsilon)) \subset M^\circ$ for small $\epsilon > 0$, and we have $\pi(\dot\gamma(s+0))^\flat = \pi(\dot\gamma(s-0))^\flat$.*

Definition 4.2.9. *Let $I \subset \mathbb{R}$ be an open connected interval and $B \subset \mathbb{R}$ be a discrete subset. A* broken bicharacteristic *of $P_{g,A,q}$ is a map $\Gamma : I\backslash B \to T^*M^\circ$ such that*

1. *if $J \subset I\backslash B$ is an interval, then $\Gamma|_J$ is a bicharacteristic of $P_{g,A,q}$ in M°;*

2. *if $s \in B$, then $\Gamma(s \pm 0)$ exist and belong to $T^*M\backslash 0|_{\partial M}$ with $\pi(\Gamma(s+0)) = \pi(\Gamma(s-0))$.*

For notation simplicity, when $s \in B$, we define $\Gamma(s) = \pi(\Gamma(s+0)) = \pi(\Gamma(s-0)) \in T^*\partial M$. Thus, the broken bicharacteristic is defined on the entire I and takes value in $T^*M^\circ \sqcup T^*\partial M$. We shall use $\Gamma|_{\partial M} \subset T^*\partial M$ to denote these boundary reflection points. Note that broken bicharacteristic is not continuous, since when it reaches the boundary, it needs to switch from $(x, \xi_{\pm})$ to $(x, \xi_{\mp})$. A common way of fixing the non-continuous property is to define the compressed bundle and compressed broken bicharacteristic, which we do not go into details here, see [70, Definition 24.3.7].

It is easy to see that the projection of a broken bicharacteristic is a broken null-geodesic. By [66, Proposition 2.12], all inextendible broken null-geodesics are tame. Recall the following definition of tameness.

Definition 4.2.10 ([66, Definition 2.11])**.** *Let $a \geq -\infty$ and $b \leq \infty$. We say an inextendible broken null-geodesic $\gamma : I \to \mathbb{R}$ is tame for $a < t < b$, if for all $a < a'$, $b < b'$, one has*

$$t(\gamma(I)) \cap (a, a') \neq \emptyset, \quad t(\gamma(I)) \cap (b', b) \neq \emptyset.$$

If γ is tame for $-\infty < t < \infty$, we simply say γ is tame.

Proposition 4.2.11 ([66] Proposition 2.12)**.** *Let (M, g) be an n-dimensional admissible Lorentzian manifold with boundary. Let $\gamma : I \to M$ be an inextendible broken null-geodesic. Then $I = \mathbb{R}$, and γ is tame.*

Next we define gliding rays. Suppose ϕ is a boundary defining function such that $\phi > 0$ in M°. We introduce the following terminology for the glancing set, see [70, Definition 24.3.2].

Definition 4.2.12. *The glancing set $\mathcal{G}^k$ of order at least $k \geq 2$ is defined by*

$$\{p = 0 \text{ and } H_p^j \phi = 0 \text{ for } 0 \leq j < k\}.$$

Note that $\mathcal{G} = \mathcal{G}^2 \supset \mathcal{G}^3 \supset \cdots \supset \mathcal{G}^\infty$. The glancing set $\mathcal{G}^2 \backslash \mathcal{G}^3$ of order precisely 2 is the union $\mathcal{G}_d \cup \mathcal{G}_g$ of the diffractive part $\mathcal{G}_d$ where $H_p^2 \phi > 0$ and the gliding part $\mathcal{G}_g$ where $H_p^2 \phi < 0$.

The definition is invariant on the choice of ϕ. In our case, all glancing covectors are in fact gliding, that is $\mathcal{G} = \mathcal{G}_g$.

Lemma 4.2.13. *If ∂M is strictly null-convex, then $\mathcal{G} = \mathcal{G}_g$.*

Proof. One can explicitly compute $H_p^2 \phi$ in local coordinate:

$$\frac{1}{4} H_p^2 \phi = \frac{1}{4}(-2g^{jk}\xi_j \partial_k + \partial_i g^{jk} \xi_j \xi_k \partial_{\xi_i})(-2g^{lm}\xi_l \partial_m \phi)$$

$$= g^{jk}g^{lm}\xi_j\xi_l\partial^2_{km}\phi + g^{jk}\partial_k g^{lm}\xi_j\xi_l\partial_m\phi - \frac{1}{2}g^{lm}\partial_l g^{jk}\xi_j\xi_k\partial_m\phi.$$

On the other hand, by some scaling, we can assume $\nabla\phi = -\nu$ is the inward unit normal vector, then

$$-g(\nabla_{\xi^\sharp}\nu, \xi^\sharp) = g^{jk}\xi_j\xi_l\partial_m\phi(\partial_k g^{lm} + \Gamma^l_{ik}) + g^{jk}g^{lm}\xi_j\xi_l\partial^2_{km}\phi,$$

and the first term becomes

$$\begin{aligned}
&g^{jk}\xi_j\xi_l\partial_m\phi(\partial_k g^{lm} - \frac{1}{2}g^{im}g_{kp}\partial_i g^{lp} - \frac{1}{2}g^{im}g_{ip}\partial_k g^{lp} + \frac{1}{2}g_{ik}g^{lp}\partial_p g^{im})\\
=\,& \xi_l\xi_j\partial_m\phi(g^{jk}\partial_k g^{lm} - \frac{1}{2}g^{im}\partial_i g^{jl} - \frac{1}{2}g^{jk}\partial_k g^{lm} + \frac{1}{2}g^{lp}\partial_p g^{jm})\\
=\,& \xi_l\xi_j\partial_m\phi(g^{jk}\partial_k g^{lm} - \frac{1}{2}g^{im}\partial_i g^{jl}).
\end{aligned}$$

Hence by switching indices, and using the strict null-convexity, we have

$$H_p^2\phi = -4g(\nabla_{\xi^\sharp}\nu, \xi^\sharp) < 0.$$

□

Since the glancing set coincides with the gliding set, we define the gliding vector field and gliding rays as follows, see [70, Definition 24.3.6] and discussions there for more details.

Definition 4.2.14. *Let ϕ be a boundary defining function. The vector field*

$$H_p^G = H_p + \frac{H_p^2\phi}{H_\phi^2 p}H_\phi$$

tangent to $\mathcal{G}$ is called the gliding vector field*. The gliding ray is the integral curve generated by the gliding vector field.*

Again the definition is invariant on the choice of ϕ. Similar to bicharacteristics and null-geodesics, we say a gliding ray is forward (resp. backward) with respect to a glancing direction if it lies in the causal future (resp. causal past) of the base point.

We next state a uniform convergence result relating broken bicharacteristics and gliding rays. First we explain what we mean by uniform convergence. The metric g canonically identifies $T^*\partial M$ with the subspace

$$(T\partial M)^\flat = \{v^\flat : v \in T\partial M\} \subset T^*M|_{\partial M}$$

via the map

$$T^*\partial M \ni \xi' \mapsto ((\xi')^\sharp)^\flat,$$

where the first $\sharp$ is taken with respect to $g|_{T\partial M\times T\partial M}$ so that $(\xi')^\sharp \in T\partial M$, and the second $\flat$ is with respect to g. We will henceforth use this identification without further mention. In semi-geodesic normal coordinates, this map is simply

$$(x', \xi') \mapsto (x', 0, \xi', 0).$$

A gliding ray can thus be viewed as a path in $T^*M|_{\partial M}$. The uniform convergence we refer to is the uniform convergence in T^*M with respect to some arbitrarily chosen Riemannian metric on T^*M. In the following, let $\Gamma^b_{y',\eta'}$ denote the broken null-bicharacteristic corresponding to $(y', \eta') \in \mathcal{H}$ and let $\Gamma^{gl}_{x',\xi'}$ denote the gliding ray corresponding to $(x', \xi') \in \mathcal{G}$.

Lemma 4.2.15. *Let $(x'_j, {\xi^j}') \in \mathcal{H}$ be a sequence converging to $(x', \xi') \in \mathcal{G}$. Let $\Gamma^b_j(t)$ be the broken null-bicharacteristic such that $\Gamma^b_j(0) = (x'_j, {\xi^j}')$ and let $\Gamma^{gl}(t)$ be the gliding ray such that $\Gamma^{gl}(0) = (x', \xi')$. Then for any fixed $T > 0$, the broken null-bicharacteristic segment $\Gamma^b_j([-T, T])$ converges uniformly to the gliding ray segment $\Gamma^{gl}([-T, T])$. In particular, any inextendible gliding ray can be defined on the entire $\mathbb{R}$.*

Proof. See Lemma 24.3.5 of [70] and the discussion after it. □

Recall the restricted metric $\bar{g} = g|_{T\partial M\times T\partial M}$ is Lorentzian on the boundary. We prove that the projection of gliding rays on the boundary are precisely the null-geodesics given by the boundary metric $\bar{g}$, and they are also tame.

Lemma 4.2.16. *Let (M, g) be an n-dimensional admissible Lorentzian manifold with boundary. Let $\gamma^{gl} : I \to M$ be the projection of some inextendible gliding ray Γ^{gl}. Then γ^{gl} is a null geodesic on ∂M with respect to $\bar{g}$, and moreover it is tame.*

Proof. Use the identification of $T^*\partial M \subset T^*M|_{\partial M}$, the gliding vector field is the Hamiltonian vector field of $\tilde{p} = p|_{T^*\partial M}$. On the other hand, in the semi-geodesic normal coordinates, $\tilde{p} = -g^{\alpha\beta}\xi_\alpha\xi_\beta = -\bar{g}^{\alpha\beta}\xi_\alpha\xi_\beta$. Thus, gliding rays are the null-bicharacteristics with respect to $\tilde{p}$, and their projections are the null-geodesics in $(\partial M, \bar{g})$. An inextendible gliding ray corresponds to an inextendible boundary null-geodesic $\gamma^{gl} : I \to \partial M$. Clearly $\gamma^{gl}(s)$ has no limiting point as $s \to \sup I$, since otherwise can extend out as a null-geodesic (same for $s \to \inf I$). Since M has a temporal function, by [62, Proposition 6.4.9] the Lorentzian submanifold $(\partial M, \bar{g})$ is stably causal, see also [103]. Since a stably causal spacetime must be strongly causal, then any inextendible causal curve in $(\partial M, \bar{g})$ can not be imprisoned in a compact set, according to [16, Proposition 3.13]. Using the assumption that the temporal function is proper, we can prove the boundary null-geodesics are tame. ☐

Example 4.2.17. *Consider the simple case $g = -f(t)dt^2 + dx^2$ on an infinite unit cylinder, where $f > 0$. Let $\phi = 1 - |x|^2$ be the boundary defining function. Then the gliding vector field is*

$$H_p^G = 2f(t)\tau\partial_t - f'(t)\tau^2\partial_\tau + a \cdot \partial_x + b \cdot \partial_\xi.$$

The time coordinate of gliding ray is thus given by

$$\begin{cases} \dot{t} = 2f(t)\tau \\ \dot{\tau} = -f'(t)\tau^2 \end{cases} \implies \ddot{t} = \frac{1}{2}\frac{f'(t)}{f(t)}\dot{t}^2.$$

A direct computation shows for some $C > 0$,

$$\dot{t}(s) = C\sqrt{f(t(s))}.$$

If the gliding ray is trapped on one end, say, as $s \to \infty$, then $t(s)$ would be bounded and close to some constant for all large s. Then $\dot{t}(s)$ must be bounded below by a positive constant for all large s. This contradicts with $t(s)$ being bounded, and therefore the gliding rays escape any compact region.

Since we can only observe the broken bicharacteristic when it hits the boundary, we need to show that the uniform convergence in Lemma 4.2.15 can be detected with boundary information. That is, because of strict null-convexity, the boundary points along the broken bicharacteristics can approximate the gliding ray.

Lemma 4.2.18. *Let (M, g) be an n-dimensional admissible Lorentzian manifold with boundary. Suppose $(x'_j, \xi^{j'}) \in \mathcal{H}$ is a sequence converging to $(x', \xi') \in \mathcal{G}$. Let $\Gamma^b_j(t)$ be the broken null-bicharacteristic such that $\Gamma^b_j(0) = (x'_j, \xi^{j'})$. Let $I \subset \mathbb{R}$ be an open connected interval containing 0. Recall we denote by $\Gamma^b_j(I)|_{\partial M} \subset T^*\partial M$ the set of boundary points. Let $\Gamma^{gl}(t)$ be the gliding ray such that $\Gamma^{gl}(0) = (x', \xi')$. Then*

1. *for any fixed $t \in I$, there exists $s_j \in I$ such that $s_j \to t$, $\Gamma^b_j(s_j) \in \Gamma^b_j(I)|_{\partial M}$, and $\Gamma^b_j(s_j) \to \Gamma^{gl}(t)$;*
2. *if (y', η') is a limiting point of $(y'_j, \eta^{j'}) \in \Gamma^b_j(\mathbb{R})|_{\partial M}$, then $(y', \eta') = \Gamma^{gl}(t)$ for some $t \in \mathbb{R}$. That is, the boundary points of this sequence of broken null-bicharacteristics can only converge to some point in Γ^{gl}.*

Proof. We first prove (1). Note that we can assume I is bounded, since it is a local result. We will show that when $j \to \infty$, the maximum time Γ^b_j spent in the interior between two consecutive reflections goes to 0 on I. Let $s' \in I$ be some time such that $\Gamma^b_j(s') \in T^*\partial M$, which is the same as $\gamma^b_j(s') \in \partial M$, where γ^b_j is the corresponding broken null-geodesic. We can always extend the manifold to be M_e such that ∂M is now in the interior by [66, Lemma 2.2]. Let γ be the null-geodesic in M_e such that $(\gamma(0), \dot\gamma(0)^\flat) = \Gamma^{gl}(s')$. In semi-geodesic normal coordinates, by strict null-convexity we have

$$\frac{d}{ds^2}\bigg|_{s=0} (x^n(\gamma(s))) = -g(\nabla_{\dot\gamma(0)}\nu, \dot\gamma(0)) < 0.$$

Since $\dot\gamma(0)$ is tangential to the boundary, we have that $\gamma((0, \varepsilon)) \subset M_e \backslash M$ for arbitrarily small ε. For any such ε, let $W_\varepsilon \subset M_e \backslash M$ be a neighborhood of $\gamma(\varepsilon)$, then for any null-geodesic $\tilde\gamma$ such that $(\tilde\gamma(0), \dot{\tilde\gamma}(0))$ sufficiently close to $(\gamma(0), \dot\gamma(0))$, we have $\tilde\gamma(\varepsilon) \in W_\varepsilon$. In other words, $\tilde\gamma$ intersects with ∂M in time smaller than ε. If j is sufficiently large, by Lemma 4.2.15, $\Gamma^b_j(s')$ would be sufficiently close to $\Gamma^{gl}(s')$, meaning $\gamma^b_j(s' + \delta) \in \partial M$ for some $\delta < \varepsilon$. Since the convergence from $\Gamma^b_j(I)$ to $\Gamma^{gl}(I)$ is uniform and I is compact, we obtain that for any $\varepsilon > 0$, there exists J such that for all $j \geq J$, $\Gamma^b_j(s') \in T^*\partial M$ implies $\Gamma^b_j(s'') \in T^*\partial M$ for some $|s' - s''| < \varepsilon$.

Now let $t \in I^\circ$, $\delta > 0$. There exists $\varepsilon > 0$ such that under the fixed Riemannian metric on T^*M,

$\mathrm{dist}(\Gamma^{gl}(t), \Gamma^{gl}(s)) < \varepsilon/2$ for any $|s-t| < \delta$. By previous argument, pick J sufficiently large, for all $j \geq J$, there exists some s_j such that $\Gamma^b_j(s_j) \in T^*\partial M$ and $|s_j - t| < \delta$ (can assume δ sufficiently small with respect to chosen t such that $(t-\delta, t+\delta) \subset I$). By Lemma 4.2.15, can let J be sufficiently large such that the distance between Γ^b_j and Γ^{gl} is less than $\varepsilon/2$ on I. Then

$$\mathrm{dist}(\Gamma^{gl}(t), \Gamma^b_j(s_j)) \leq \mathrm{dist}(\Gamma^{gl}(t), \Gamma^{gl}(s_j)) + \mathrm{dist}(\Gamma^{gl}(s_j), \Gamma^b_j(s_j)) < \varepsilon.$$

Now we prove (2). Suppose (y', η') is a limiting point of a sequence of $(y'_j, \eta'_j) \in B^{\mathbb{R}}_j$, but it is not in Γ^{gl}. Consider some smooth, surjective, proper temporal function τ, then there exists some bounded time interval $[a, b]$ such that $y'_j, y' \in \tau^{-1}([a, b])$. There exists sufficiently large N such that $\tau(\Gamma^{gl}(N)) > b$ and $\tau(\Gamma^{gl}(-N)) < a$ by Lemma 4.2.16. Then for sufficiently large j, by Lemma 4.2.15, $\tau(\Gamma^b_j(N)) > b$, $\tau(\Gamma^b_j(-N)) < a$, and $\Gamma^b_j([-N, N])$ uniformly close to $\Gamma^{gl}([-N, N])$. Hence (y'_j, η'_j) is away from (y', η'), contradicting (y', η') limiting point. $\square$

In the following, we will denote broken bicharacteristics as Γ^b, broken null-geodesics as γ^b, gliding rays as Γ^{gl}, and the projection of gliding rays as γ^{gl}.

4.3 DN Map as an FIO

In this section, we study the microlocal property of the DN map $\Lambda^{U,V}_{g,A,q}$, assuming U and V are disjoint. We will first recall the optics construction in [139], which shows that the DN map is an elliptic Fourier Integral Operator. Then we will use microlocal analysis to perform a symbolic construction of the solution as in [101], and explicitly compute the principal symbol.

4.3.1 Notation

We first setup some common notations that will be used in both constructions. Let $\Lambda'_0 \subset \mathcal{H}$ be a conic Lagrangian submanifold of $T^*\partial M$. We denote by

$$\Lambda = \bigcup_{(x',\xi')\in\Lambda'_0} \Gamma^b_{x',\xi'}$$

the union of forward broken null-bicharacteristics $\Gamma^b_{x',\xi'}$ starting from $(x',\xi') \in \Lambda'_0$. Recall that we define the broken null-bicharacteristic to be a path in $T^*M^\circ \sqcup T^*\partial M$ and τ is the temporal function. By assuming the base projection of Λ'_0 is sufficiently small and focusing only on a compact region $\tau^{-1}([-T,T])$, we may decompose Λ into

$$\Lambda = \left(\bigsqcup_{j=0}^{N} \Lambda_j\right) \sqcup \left(\bigsqcup_{j=0}^{N} \Lambda'_j\right)$$

where each $\Lambda_j \subset T^*M^\circ$ is union of null-bicharacteristics from boundary to boundary, and each $\Lambda'_j \subset T^*\partial M$ is the reflection part that connects Λ_j with Λ_{j+1}. Specifically,

$$\pi(\Lambda_j|_{\partial M}) = \Lambda'_j \sqcup \Lambda'_{j+1}, \quad L(\Lambda'_j) = \Lambda'_{j+1},$$

where $\pi : T^*M|_{\partial M} \to T^*\partial M$ and L is the lens relation. The set Λ_0, as the flowout of forward bicharacteristics from Λ'_0, is a Lagrangian submanifold of T^*M°. The reflection part $\Lambda'_j = L^j(\Lambda'_0)$ is also a Lagrangian submanifold, since L on an admissible Lorentzian manifold locally gives canonical graph[1]. For simplicity, let $U_j \subset \partial M$ be an open neighborhood of the base projection of Λ'_j such that $\pi(\Lambda_j|_{U_j}) = \Lambda'_j$ and $\pi(\Lambda_j|_{U_{j+1}}) = \Lambda'_{j+1}$. We may assume Λ'_0 is sufficiently small such that U_j are then disjoint from each other.

By [66, Lemma 2.2], one can extend (M,g) to a larger Lorentzian manifold $(M_e,\tilde{g})$, such that M is contained in the interior of M_e and $\tilde{g}|_M = g$. Recall that Λ_j denotes the union of null-bicharacteristics intersecting the boundary transversally. Then we may extend Λ_j across the boundary into T^*M_e and by abuse of notation, we continue to write this extension as Λ_j. We require that $\Lambda_j \cap \Lambda_k = \emptyset$. By slightly extending M, we may assume that bicharacteristics in Λ_j do not return to M, to avoid complications.

[1]The graph of L is locally a canonical graph when the boundary is strictly null-convex, since all null-bicharacteristics hits boundary transversally, see [66, Proposition 2.4] and [139, Theorem 4.2].

Likewise, we can extend A and q to $\tilde{A}$ and $\tilde{q}$ on M_e. We emphasize that this extension is purely for the convenience of constructing solutions, and that the result is independent of the specific extension chosen.

In addition, we denote by Γ^b a forward broken bicharacteristic in Λ, with boundary points

$$(x_0', \xi^{0\prime}) \to (x_1', \xi^{1\prime}) \to \cdots \to (x_N', \xi^{N\prime}), \quad (x_j', \xi^{j\prime}) \in \mathcal{H} \cap T^*U_j. \tag{4.3.1}$$

In the following, we will perform optics construction near Γ^b, and a symbolic construction on Λ, with Λ contained in a neighborhood of Γ^b.

4.3.2 Optics construction

In this subsection, we briefly recall the optics construction near the chosen broken bicharacteristic Γ^b, and recall Theorem 4.2 of [139], for more details see [139]. Let $f \in \mathcal{E}'(U_0)$ be such that $\mathrm{WF}(f) \subset \mathcal{H}$ is a conic neighborhood of $(x_0', \xi^{0\prime})$. The solution near x_0' has the form

$$u(x) = (2\pi)^{-n} \int e^{i\phi(x,\theta)} a(x,\theta) \hat{f}(\theta) d\theta.$$

Here in semi-geodesic normal coordinates, the phase function ϕ solves the Eikonal equation

$$g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + (\partial_n \phi)^2 = 0. \tag{4.3.2}$$

The amplitude is of the form $a \sim \sum_{j=0}^{\infty} a_j(x,\theta)$ for a_j homogeneous of degree $-j$ in θ, and they inductively solve

$$Ta_0 = 0, \quad a_0|_{x_n=0} = \chi,$$

$$iTa_j = -\square_{g,A,q} a_{j-1}, \quad a_j|_{x_n=0} = 0, \quad j \geq 1,$$

where $\chi(x,\theta)$ is a conic cutoff around $(x_0', \xi^{0\prime})$ that is homogeneous of degree 0 in θ, and T is a transport operator along bicharacteristics

$$T = 2g^{jk}\partial_j\phi(\partial_k - iA_k) + \Box_g \phi.$$

To extend out the solution u, one can take a timelike surface transversal to Γ^b, take the restriction of u on that surface, and construct the optics solution again. Repeated this construction, one can thus obtain the optics solution along the entire bicharacteristic until it hits x_1'.

Around x_1', the solution is the sum of incident wave and reflection wave $u = u_1^{\text{inc}} + u_1^{\text{ref}}$. The two solutions satisfy

$$\begin{aligned} u^{\text{inc}}(x) &= (2\pi)^{-n} \int e^{i\phi(x,\theta)}(a_0^{\text{inc}} + a_1^{\text{inc}} + R^{\text{inc}})(x,\theta)\hat{f}(\theta)d\theta, \\ u^{\text{ref}}(x) &= (2\pi)^{-n} \int e^{i\phi^{\text{ref}}(x,\theta)}(a_0^{\text{ref}} + a_1^{\text{ref}} + R^{\text{ref}})(x,\theta)\hat{f}(\theta)d\theta. \end{aligned}$$

Both phase functions satisfy the Eikonal equation, and both symbols satisfy the transport equation, as stated above. They are related by

$$\begin{aligned} &\phi|_{U_1} = \phi^{\text{ref}}|_{U_1}, \quad \partial_\nu\phi|_{U_1} = -\partial_\nu\phi^{\text{ref}}|_{U_1} \\ &a_0^{\text{ref}}|_{U_1} = -a_0^{\text{inc}}|_{U_1}, \quad a_1^{\text{ref}}|_{U_1} = -a_1^{\text{inc}}|_{U_1}. \end{aligned}$$

The DN map $\Lambda_{g,A,q}^{U_0,U_1}$ gives

$$\Lambda_{g,A,q}^{U_0,U_1} f = (2\pi)^{-n} \int e^{i\phi}(2i(\partial_\nu\phi)a_0^{\text{inc}} + 2i(\partial_\nu\phi)a_1^{\text{inc}} + \partial_\nu(a_0^{\text{inc}} + a_0^{\text{ref}}) + a_{-1})\hat{f}(\theta)d\theta.$$

Hence, it is proved in [139, Theorem 4.2] that the restricted DN map $\Lambda_{g,A,q}^{U_0,U_1}$ is an elliptic Fourier Integral Operator of order 1 associated with the canonical graph

$$\{(L(x',\xi'); x',\xi') : (x',\xi') \in \mathcal{H} \cap T^*U_0\} \subset T^*U_1 \times T^*U_0.$$

Furthermore, one can inductively construct solutions u_j^{inc} and u_j^{ref} along Γ^b, for $j = 2, \ldots, N$. Note that by the proof of [139, Theorem 4.2], the map $f \mapsto u_1^{\mathrm{inc}}|_{U_1}$ is an elliptic Fourier Integral Operator of order 0 and so is the map $F_j : -u_{j-1}^{\mathrm{ref}}|_{U_{j-1}} \mapsto u_j^{\mathrm{inc}}|_{U_j}$, for $j = 2, \ldots, N$. A similar argument together with the transversal intersection calculus of Fourier Integral Operators indicates $\Lambda_{g,A,q}^{U_0,U_k}$ is an elliptic Fourier Integral Operator associated with the canonical graph

$$\{(L^k(x', \xi'); x', \xi') : (x', \xi') \in \mathcal{H} \cap T^*U_0\} \subset T^*U_k \times T^*U_0.$$

We emphasize again that the above result is essentially proved in [139, Theorem 4.2]. Since it is elliptic, we can obtain the following propagation of singularity result for the global DN map $\Lambda_{g,A,q}$ (that is when $U = V = \partial M$).

Lemma 4.3.1. *Suppose $WF(f) = \{(x', t\xi') : t > 0\}$, and view $WF(\Lambda_{g,A,q}f)$ as subset of $S^*\partial M = T^*\partial M \backslash 0/\mathbb{R}^+$. We have the following statements.*

- *If $WF(f) \subset \mathcal{E}$, then $WF(\Lambda_{g,A,q}f) \backslash WF(f)$ is empty;*
- *If $WF(f) \subset \mathcal{G}$, then $WF(\Lambda_{g,A,q}f) \backslash WF(f)$ is empty or a continuous set of points;*
- *If $WF(f) \subset \mathcal{H}$, then $WF(\Lambda_{g,A,q}f) \backslash WF(f)$ is a discrete set of points given by the boundary points of the corresponding forward broken bicharacteristic.*

Proof. For the elliptic region, by [70, (24.2.4)] and [70, Theorem 18.3.27], we have

$$\mathcal{E} \cap \mathrm{WF}(\Lambda_{g,A,q}f) \subset \mathcal{E} \cap \mathrm{WF}(f).$$

For the glancing region, we use Lemma 4.2.13 and the propagation of singularity result [8, Theorem 0.4]. The above two results show that $\mathrm{WF}(\Lambda_{g,A,q}f) \backslash \mathrm{WF}(f)$ contains a point of the glancing set if and only if it contains the entire gliding ray. In particular, if $\mathrm{WF}(f)$ is disjoint from $\mathcal{G}$, then by Lemma 4.2.16 and the fact that $\mathrm{supp}(u) \subset J^+(\mathrm{supp}(f))$, $\mathrm{WF}(\Lambda_{g,A,q}f)$ is disjoint from $\mathcal{G}$.

For the hyperbolic region, suppose $\mathrm{WF}(f) \subset \mathcal{H}$. By the optics construction and the fact that each $\Lambda_{g,A,q}^{U_0,U_k}$ is an elliptic FIO with canonical relation given by powers of lens relation, $\mathrm{WF}(\Lambda_{g,A,q}f) \backslash \mathrm{WF}(f)$ is the boundary points of the corresponding forward broken bicharacteristics from $\mathrm{WF}(f)$. If $\mathrm{WF}(f)$ is disjoint from $\mathcal{H}$, Proposition 4.2.11 and [70, Theorem 24.2.1] show that $\mathrm{WF}(\Lambda_{g,A,q} f)$ is disjoint from $\mathcal{H}$.

$\square$

Throughout the chapter, we shall view the wavefront set as both a conic subset of the cotangent bundle ($T^*\partial M \backslash 0$) and a subset of the spherical bundle ($S^*\partial M$). When referring to a point in the wavefront set, we always view the wavefront set as a subset of the spherical bundle.

4.3.3 Microlocal construction of solution

In this subsection, we construct solutions microlocally so that it is easier to analyze the evolution of principal symbols globally.

Let $f \in I^{\mu}_{cl}(\Lambda_0')$ and $\tilde{f} = f|\bar{g}|^{1/4} \in I^{\mu}_{cl}(\Lambda_0'; \Omega^{1/2}_{\partial M})$, where $\bar{g} = g|_{T\partial M \times T\partial M}$. We shall perform a symbolic construction of the solution $\tilde{u}_0 = u_0|\tilde{g}|^{1/4} \in I^{\mu-1/4}_{cl}(\Lambda_0; \Omega^{1/2}_{M_e})$ as in [101]. Specifically, we inductively construct

$$\tilde{v}_k = v_k|\tilde{g}|^{1/4} \in I^{\mu-1/4-k}_{cl}(\Lambda_0; \Omega^{1/2}_{M_e})$$

such that the following equations are solved on the principal level

$$\begin{aligned} &P_{\tilde{g},\tilde{A},\tilde{q}}(\tilde{v}_0 + \cdots + \tilde{v}_k) \in I^{\mu+3/4-k}_{cl}(\Lambda_0; \Omega^{1/2}_{M_e}) \quad && \text{in } M_e, \\ &f - (v_0 + \cdots + v_k)|_{U_0} \in I^{\mu-1-k}_{cl}(\Lambda_0') && \text{on } U_0. \end{aligned} \tag{4.3.3}$$

With this construction, let $\tilde{u}_0 = u_0|\tilde{g}|^{1/4}$ be some asymptotic sum of $\tilde{v}_0 + \tilde{v}_1 + \cdots$. Then it solves

$$\begin{aligned} &\square_{\tilde{g},\tilde{A},\tilde{q}} u_0 \in C^\infty(M_e), \\ &u_0|_{U_0} - f \in C^\infty(U_0). \end{aligned}$$

For this purpose, we start with some $\tilde{v}_0 = v_0|\tilde{g}|^{1/4} \in I^{\mu-1/4}_{cl}(\Lambda_0; \Omega^{1/2}_{M_e})$. By equation (4.2.4) and (4.2.1), we choose v_0 such that

$$0 = i\sigma(P_{\tilde{g},\tilde{A},\tilde{q}} \tilde{v}_0) = \left(\mathcal{L}_{H_p} - iA(H_p)\right)\sigma(\tilde{v}_0).$$

Note that this is a first order differential equation for $\sigma(\tilde{v}_0)$ on Λ_0. The initial condition is given by $\sigma(\tilde{f})$ via the following lemma.

Lemma 4.3.2. *Let Λ_0, Λ_0', and U_0 be defined above. Suppose $v|_{U_0} = f$, where $v \in I_{cl}^{\mu-1/4}(\Lambda_0)$ and $f \in I_{cl}^{\mu}(\Lambda_0')$. Let (x', x^n) be the semi-geodesic normal coordinates and $\pi : \Lambda_0|_{U_0} \to \Lambda_0'$ denote the natural projection. For $(x,\xi) \in \Lambda_0|_{U_0}$, let $w_1, \ldots, w_n \in T_{(x,\xi)}\Lambda_0$ be such that $dx^n(w_n) \neq 0$. Let $\tilde{v} = v|\tilde{g}|^{1/4}$ and $\tilde{f} = f|\bar{g}|^{1/4}$ be the corresponding half-densities. Then there exists $a_j \in \mathbb{R}$ such that*

$$dx^n(w_j + a_j w_n) = 0 \quad \text{for } j = 1, \ldots, n-1,$$

and the principal symbols satisfy

$$\sigma(\tilde{v})(w_1, \ldots, w_n) = e^{i\pi/4}\sigma(\tilde{f})(\pi_*(w_1 + a_1 w_n), \ldots, \pi_*(w_{n-1} + a_{n-1}w_n))|dx^n(w_n)|^{1/2}.$$

Proof. To see how $\sigma(\tilde{v})$ is related to $\sigma(\tilde{f})$, we consider the local representations of v and f. Indeed, in the semi-geodesic normal coordinates, locally f can be written as an oscillatory integral of the form

$$f(x') = \int e^{i\phi_0(x',\theta)} a_0(x',\theta) d\theta + l.o.t.,$$

where ϕ_0 is non-degenerate, a_0 is homogeneous in θ, and $l.o.t.$ denotes lower order terms. Here by non-degenerate, we mean the differentials $d_{x'}(\partial_{\theta_j}\phi_0)$, for $j = 1, \ldots, N$, are linearly independent, see [69, Section 1.2]. Note that ϕ_0 parameterizes Λ_0' in the sense that $\xi' = \partial_{x'}\phi_0(x',\theta)$, for $(x',\xi') \in \Lambda_0'$ and $\partial_\theta\phi_0(x',\theta) = 0$. With $\Lambda_0' \subset \mathcal{H}$, the covector ξ' is timelike and therefore we have $g^{\alpha\beta}\partial_{x^\alpha}\phi_0\partial_{x^\beta}\phi_0 < 0$, for $1 \leq \alpha, \beta \leq n-1$. Thus, there exists a smooth function $\phi(x,\theta)$ locally solving the Eikonal equation in (4.3.2), such that

$$\phi(x', 0, \theta) = \phi_0(x', \theta) \quad \text{and} \quad \partial_{x^n}\phi(x', 0, \theta) \neq 0.$$

Such ϕ parameterizes Λ_0 locally. With $f(x') = v(x', 0)$, we may write

$$v(x) = \int e^{i\phi(x,\theta)} a(x,\theta) d\theta + l.o.t.,$$

where $a(x', 0, \theta) = a_0(x', \theta)$ is the leading amplitude. We emphasize, near $x^n = 0$, the phase function ϕ is also non-degenerate, as $d_x(\partial_{\theta_j}\phi_0)$, for $j = 1, \dots, N$, are linearly independent at $x^n = 0$.

Let λ' be local coordinates on $C_0 = \{(x', \theta) : \partial_\theta \phi(x', 0, \theta) = 0\}$, then by stationary phase computation the coordinate dependent principal symbol of f is given by

$$\sigma_{x'}(f) = (F_0^{-1})^* \left(e^{i\pi N/4} a(x', 0, \theta) \left| \frac{D(\lambda', \partial_\theta \phi(\cdot, 0, \cdot))}{D(x', \theta)} \right|^{-1/2} |d\lambda'|^{1/2} \right)$$

where $F_0 : C_0 \to \Lambda_0', (x', \theta) \mapsto (x', 0, \phi_{x'}(x', 0, \theta), 0)$ is locally a diffeomorphism, and N is the signature. Note that

$$C = \{(x, \theta) : \partial_\theta \phi = 0\} = C_0 \sqcup \{(x, \theta) : \partial_\theta \phi = 0, x^n \neq 0\},$$

so one can extend λ' to C such that (λ', x^n) form local coordinates around C_0. This gives, on ∂M,

$$\sigma_x(v) = (F^{-1})^* \left(e^{i\pi(N+1)/4} a(x', 0, \theta) \left| \frac{D(\lambda', \partial_\theta \phi(\cdot, 0, \cdot))}{D(x', \theta)} \right|^{-1/2} |d\lambda' \wedge dx^n|^{1/2} \right)$$

where $F : C \to \Lambda_0, (x, \theta) \mapsto (x, \phi_x)$ and we used the fact that on ∂M

$$\frac{D(\lambda', x^n, \partial_\theta \phi)}{D(x', x^n, \theta)} = \begin{bmatrix} \frac{D(\lambda', \partial_\theta \phi(\cdot, 0, \cdot))}{D(x', \theta)} & \left. \frac{D(\lambda', \partial_\theta \phi)}{D x^n} \right|_{x^n = 0} \\ 0 & 1 \end{bmatrix}.$$

First, suppose $w_1, \dots, w_{n-1}$ are tangent to $\Lambda_0|_{U_0}$, which is equivalent to $dx^n(w_j) = 0$. Then $dx^n(F_*^{-1} w_j) = (F^{-1})^* dx^n(w_j) = dx^n w_j = 0$. We also have $F_*^{-1} w_j = (F_0^{-1})_*(\pi_* w_j)$ because $F^{-1}(\Lambda_0|_{U_0}) = C_0 = F_0^{-1}(\Lambda_0')$. Then we have

$$\sigma_x(v)(w_1, \dots, w_n)$$

$$
\begin{aligned}
&= e^{i\pi(N+1)/4} a(x',0,\theta) \left| \frac{D(\lambda', \partial_\theta \phi(\cdot,0,\cdot))}{D(x',\theta)} \right|^{-1/2} |d\lambda' \wedge dx^n|^{1/2} (F_*^{-1} w_1, \dots, F_*^{-1} w_n) \\
&= e^{i\pi(N+1)/4} a(x',0,\theta) \left| \frac{D(\lambda', \partial_\theta \phi(\cdot,0,\cdot))}{D(x',\theta)} \right|^{-1/2} \\
&\qquad\qquad \cdot |d\lambda'|^{1/2} ((F_0^{-1})_* \pi_* w_1, \dots, (F_0^{-1})_* \pi_* w_{n-1}) |dx^n(w_n)|^{1/2} \\
&= e^{i\pi/4} \sigma_{x'}(f)(\pi_* w_1, \dots, \pi_* w_{n-1}) |dx^n(w_n)|^{1/2}.
\end{aligned}
$$

In semi-geodesic normal coordinates, $|g|_x = |\bar{g}|_{x'}$, so

$$
\sigma(\tilde{v})(w_1, \dots, w_n) = e^{i\pi/4} \sigma(\tilde{f})(\pi_* w_1, \dots, \pi_* w_{n-1}) |dx^n(w_n)|^{1/2}.
$$

Now consider more general $w_1, \dots, w_{n-1}$, let w_n be such that $dx^n(w_n) \neq 0$. By setting $a_j = -\frac{dx^n(w_j)}{dx^n(w_n)}$ we have $dx^n(w_j + a_j w_n) = 0$. We can thus apply the previous result and get

$$
\begin{aligned}
\sigma(\tilde{v})(w_1, \dots, w_n) &= \sigma(\tilde{v})(w_1 + a_1 w_n, \dots, w_{n-1} + a_{n-1} w_n, w_n) \\
&= e^{i\pi/4} \sigma(\tilde{f})(\pi_*(w_1 + a_1 w_n), \dots, \pi_*(w_{n-1} + a_{n-1} w_n)) |dx^n(w_n)|^{1/2}
\end{aligned}
$$

where the first equal sign comes from

$$
w_1 \wedge \cdots \wedge w_n = (w_1 + a_1 w_n) \wedge \cdots \wedge (w_{n-1} + a_{n-1} w_n) \wedge w_n.
$$

$\square$

In Lemma 4.3.2, we can set $w_n = H_p$, as it is always transversal to the boundary on broken bicharacteristics. Indeed, one has $|dx^n(H_p)| = 2|\xi_n| > 0$. Recall that $\xi_n = \sqrt{-g^{\alpha\beta}\xi_\alpha\xi_\beta}$ is already non-negative, here we use $|\xi_n|$ to emphasize this. Thus, $\sigma(\tilde{v}_0)$ solves

$$
\begin{cases}
\left(\mathcal{L}_{H_p} - iA(H_p)\right) \sigma(\tilde{v}_0) = 0 \quad \text{on } \Lambda_0, \\
\sigma(\tilde{v}_0)(w_1, \dots, w_n) = e^{i\pi/4} \sigma(\tilde{f})(\pi_*(w_1 + a_1 w_n), \dots, \pi_*(w_{n-1} + a_{n-1} w_n)) |dx^n(w_n)|^{1/2},
\end{cases}
$$

where (x', x^n) is the semi-geodesic normal coordinates, w_j are vectors at $\Lambda_0|_{U_0}$ such that $dx^n(w_n) \neq 0$, and $a_j = -\frac{dx^n(w_j)}{dx^n(H_p)}$. Since f is set to be classical, the transport equation above has a unique homogeneous solution, for example see [40, Section 6.4] and [101, Section 6].

Now we choose any $\tilde{v}_0$ such that $\sigma(\tilde{v}_0)$ solves the transport equation. By [71, Theorem 25.2.4], we have $h_1 = P_{\tilde{g},\tilde{A},\tilde{q}}\tilde{v}_0 \in I_{cl}^{\mu+3/4}(\Lambda_0; \Omega_{M_e}^{1/2})$. By Proposition 4.2.2, the restriction $v_0|_{U_0}$ is a Lagrangian distribution with respect to Λ_0', and by our construction $f - v_0|_{U_0} \in I_{cl}^{\mu-1}(\Lambda_0')$ in U_0. Then we consider $\tilde{v}_1 = v_1|\tilde{g}|^{1/4} \in I_{cl}^{\mu-5/4}(\Lambda_0; \Omega_{M_e}^{1/2})$ to solve

$$\begin{cases} P_{\tilde{g},\tilde{A},\tilde{q}}\tilde{v}_1 = -h_1 \mod I_{cl}^{\mu-1/4}(\Lambda_0; \Omega_{M_e}^{1/2}), \\ v_1|_{U_0} = f - v_0|_{U_0} \mod I_{cl}^{\mu-2}(\Lambda_0'), \end{cases}$$

by (4.3.3). We again use (4.2.1), (4.2.4) and Lemma 4.3.2 to solve the equation for $\sigma(\tilde{v}_1)$. The lower order terms $\tilde{v}_k$ can be constructed inductively, and the construction is complete.

Next, we consider the restriction of u_0 to U_1 to construct reflections. Note that $u_0|_{U_1} \in I_{cl}^{\mu}(\Lambda_1')$ is a Lagrangian distribution again by Proposition 4.2.2. Thus, we can inductively and symbolically construct $\tilde{u}_k \in I_{cl}^{\mu-1/4}(\Lambda_k; \Omega_{M_e}^{1/2})$ in the same way to solve

$$\begin{cases} P_{\tilde{g},\tilde{A},\tilde{q}}\tilde{u}_k \in C^\infty(M_e), \\ u_k|_{U_k} + u_{k-1}|_{U_k} \in C^\infty(U_k), \end{cases}$$

along the flowout Λ_k. Putting everything together, restricting the solutions to $\tau^{-1}([-T,T]) \subset M$, we have $\tilde{u} = \sum_{j=0}^{N} \tilde{u}_j$ solving

$$\begin{cases} \square_{g,A,q} u \in C^\infty(M^\circ \cap \tau^{-1}([-T,T])), \\ u|_{\partial M} = f + C^\infty(\partial M \cap \tau^{-1}([-T,T])). \end{cases}$$

In other words, u differs from the actual solution only by a smooth function, so the actual solution is also a Lagrangian distribution sharing the same principal symbol as u. Next we analyze what the principal symbol of u is.

Proposition 4.3.3. *Let Λ_{beg} be a small conic Lagrangian submanifold in $\mathcal{H}$ away from $\mathcal{G}$. Denote by*

$$\Lambda_{flow} = \cup_{(y',\eta')\in\Lambda_{beg}}\Gamma_{y',\eta'}$$

its flowout Lagrangian along forward bicharacteristics. Denote $\pi : T^*M|_{\partial M} \to T^*\partial M$ the natural projection. Let

$$\Lambda_{end} = \{\pi(\Gamma_{y',\eta'}(t)) : \Gamma_{y',\eta'}(t) \in T^*M|_{\partial M}, \quad \Gamma_{y',\eta'}(0) = (y',\eta') \in \Lambda_{beg}, \quad t \neq 0\}$$

be the projection of ending points of these bicharacteristics. When Λ_{beg} is small enough, the projection of Λ_{beg} and Λ_{end} on ∂M are disjoint. Then we have the lens relation

$$L_{\Lambda_{flow}} : \Lambda_{beg} \ni (y',\eta') \mapsto (x',\xi') \in \Lambda_{end}$$

along bicharacteristics. Suppose $\tilde{v} = v|g|^{1/4} \in I_{cl}(\Lambda_{flow}; \Omega_M^{1/2})$ satisfies

$$\mathcal{L}_{H_p}\sigma(\tilde{v}) = iA(H_p)\sigma(\tilde{v}).$$

Denote by $\tilde{f}_1 = v|_{U_1}|\bar{g}|^{1/4} \in I_{cl}(\Lambda_{beg}; \Omega_{\partial M}^{1/2})$ and $\tilde{f}_2 = v|_{U_2}|\bar{g}|^{1/4} \in I_{cl}(\Lambda_{end}; \Omega_{\partial M}^{1/2})$, where $U_1, U_2 \subset \partial M$ are disjoint small neighborhoods around $\Lambda_{beg}, \Lambda_{end}$, respectively. Then we have

$$|\xi_n|^{1/2}\sigma(\tilde{f}_2)|_{(x',\xi')} = \exp\left(i\int_\gamma A(\dot{\gamma})\right) \cdot |\eta_n|^{1/2}(L^{-1}_{\Lambda_{flow}})^*(\sigma(\tilde{f}_1)|_{(y',\eta')}),$$

where the null-geodesic γ is the projection of forward bicharacteristic from (y',η') to (x',ξ') (note that $\int_\gamma A(\dot{\gamma})$ is independent of parametrization of γ), and

$$\xi_n = \sqrt{-\bar{g}^{\alpha\beta}\xi_\alpha\xi_\beta}, \quad \eta_n = \sqrt{-\bar{g}^{\alpha\beta}\eta_\alpha\eta_\beta}.$$

To prove this proposition, we first state a Lemma, relating the pushforward of vectors along the flow and the pushforward of vectors via lens relation.

Lemma 4.3.4. *Let $L = L_{\Lambda_{flow}}$ and $\Lambda_{beg}, \Lambda_{flow}, \Lambda_{end}$ be as defined in Proposition 4.3.3. Denote by Φ_t the flow of H_p along Λ_{flow}. Let $\pi : \Lambda_{flow}|_{\partial M} \to \Lambda_{beg} \sqcup \Lambda_{end}$ be the natural projection. Suppose $L(\bar{y}', \bar{\eta}') = (\bar{x}', \bar{\xi}')$ such that $\Phi_T(\pi^{-1}(\bar{y}', \bar{\eta}')) = \pi^{-1}(\bar{x}', \bar{\xi}')$ for some $T \in \mathbb{R}$. Then for $w \in T_{(\bar{y}',\bar{\eta}')}\Lambda_{beg}$, we have*

$$L_* w = \pi_*((\Phi_T)_*(\pi_*^{-1} w) + aH_p),$$

where $a \in \mathbb{R}$ is the unique number such that

$$dx^n(\Phi_{T*}(\pi_*^{-1} w) + aH_p) = 0.$$

Proof. For $(y', \eta') \in \Lambda_{beg}$, let $t(y', \eta')$ be the arriving time, that is,

$$\Phi_{t(y',\eta')}(y', 0, \eta_\pm) = (x', 0, \xi_\mp) \in \pi^{-1}(\Lambda_{end}).$$

Then $L = \pi \circ \Phi_{t(\cdot)} \circ \pi^{-1}$. Note that $t(\bar{y}', \bar{\eta}') = T$. Denote by $\tilde{w} = (\Phi_T)_* \pi_*^{-1} w$ and it suffices to show

$$(\Phi_{t(\cdot)-T})_* \tilde{w} = \tilde{w} + aH_p,$$

for some $a \in \mathbb{R}$. Now let $N = \{\Phi_T(y', 0, \eta_\pm) : (y', \eta') \in \Lambda_{beg}\} = \Phi_T(\pi^{-1}(\Lambda_{beg}))$ and $\tilde{N} = \{\Phi_{t(y',\eta')}(y', 0, \eta_\pm) : (y', \eta') \in \Lambda_{beg}\} = \pi^{-1}(\Lambda_{end})$. Clearly we have $\pi^{-1}(\bar{x}', \bar{\xi}') \in N \cap \tilde{N}$. Then $\tilde{N}$ can be viewed as a graph of N with coordinates (p, t) where $p \in N$ and $\Phi_t(p) \in \tilde{N}$. The result then immediately follows by computing in local coordinates. Indeed, in local coordinates, the pushforward of ∂_{z^1} from $\{(z, t) : t = 0\}$ to $\{(z, t) : t = f(z)\}$ via $(\cdot, f(\cdot))$ is simply $\partial_{z^1} + \frac{\partial f}{\partial z^1}\partial_t$, and here ∂_t is given by H_p. Finally, a is uniquely determined by the fact that $\tilde{w} \in TN$ and $(\Phi_{t(\cdot)-T})_* \tilde{w} \in T\tilde{N}$. ☐

Proof of Proposition 4.3.3. Assume the notations in the proof of Lemma 4.3.4. Solving the Lie derivative equation, one gets

$$\Phi_T^*(\sigma(\tilde{v})|_{(x',0,\xi_\pm)}) = \exp\left(i \int_\gamma A(\dot{\gamma})\right) \sigma(\tilde{v})|_{(y',0,\eta_\mp)}.$$

Let $w_1, \cdots, w_{n-1} \in T_{(y',\eta')}\Lambda_{beg}$. Then we directly compute

$$
\begin{aligned}
&|\xi_n|^{1/2}\sigma(\tilde{f}_2)|_{(x',\xi')}(L_* w_1, \cdots, L_* w_{n-1})\\
=&|\xi_n|^{1/2}\sigma(\tilde{f}_2)|_{(x',\xi')}(\pi_*((\Phi_T)_*(\pi_*^{-1}w_1) + a_1 H_p), \cdots, \pi_*((\Phi_T)_*(\pi_*^{-1}w_{n-1}) + a_{n-1}H_p))\\
=&C\sigma(\tilde{v})|_{(x',0,\xi_\pm)}(H_p, (\Phi_T)_*(\pi_*^{-1}w_1) + a_1 H_p, \cdots, (\Phi_T)_*(\pi_*^{-1}w_{n-1}) + a_{n-1}H_p)\\
=&C\sigma(\tilde{v})|_{(x',0,\xi_\pm)}((\Phi_T)_* H_p, (\Phi_T)_*(\pi_*^{-1}w_1), \cdots, (\Phi_T)_*(\pi_*^{-1}w_{n-1}))\\
=&C\Phi_T^*(\sigma(\tilde{v})|_{(x',0,\xi_\pm)})(H_p, \pi_*^{-1}w_1, \cdots, \pi_*^{-1}w_{n-1})\\
=&C\exp(i\int_\gamma A(\dot{\gamma}))\cdot\sigma(\tilde{v})|_{(y',0,\eta_\mp)}(H_p, \pi_*^{-1}w_1, \cdots, \pi_*^{-1}w_{n-1})\\
=&\exp(i\int_\gamma A(\dot{\gamma}))\cdot|\eta_n|^{1/2}\sigma(\tilde{f}_1)|_{(y',\eta')}(w_1, \cdots, w_{n-1}),
\end{aligned}
$$

where $C = \frac{e^{-i\pi/4}}{\sqrt{2}}$, and we used Lemma 4.3.2 and Lemma 4.3.4. $\square$

Next, let $(x'_k, \xi^{k\prime}) \in \Lambda'_k$ be connected by a forward broken bicharacteristic Γ^b starting from $(x_0, \xi^{0\prime})$, for $k = 0, 1, \ldots, N-1$. By Proposition 4.3.3, we thus have for any k,

$$
\begin{aligned}
&|\xi_n^k|^{1/2}\sigma(u_{k-1}|_{U_k}|\bar{g}|^{1/4})|_{(x'_k,\xi^{k\prime})}\\
=&\exp\left(i\int_{\gamma^b(x'_{k-1}\to x'_k)} A(\dot{\gamma}^b)\right)\cdot|\xi_n^{k-1}|^{1/2}(L^{-1})^*\left(\sigma(u_{k-1}|_{U_{k-1}}|\bar{g}|^{1/4})|_{(x'_{k-1},\xi^{k-1\prime})}\right)
\end{aligned}
$$

where $\xi_n^k = \sqrt{-\bar{g}^{\alpha\beta}\xi_\alpha^k\xi_\beta^k}$, γ^b is the projection of Γ^b, and L is the lens relation which maps Λ'_k to Λ'_{k+1}. By the construction, $u_k|_{U_k} + u_{k-1}|_{U_k} \in C^\infty(U_k)$ and $u_0|_{U_0} = f + C^\infty(U_0)$, so by induction we have

$$
\begin{aligned}
&|\xi_n^k|^{1/2}\sigma(u_{k-1}|_{U_k}|\bar{g}|^{1/4})|_{(x'_k,\xi^{k\prime})}\\
=&(-1)^{k-1}\exp\left(i\int_{\gamma^b(x'_0\to x'_k)} A(\dot{\gamma}^b)\right)\cdot|\xi_n^0|^{1/2}(L^{-k})^*\left(\sigma(\tilde{f})|_{(x'_0,\xi^{0\prime})}\right).
\end{aligned}
$$

Note that in semi-geodesic normal coordinates, $\sigma(\partial_{x^n}\psi)|_{(x,\xi)} = i\xi_n\sigma(\psi)$. Then we have

$$
\begin{aligned}
&\sigma(\partial_\nu u|_{U_k}|\bar{g}|^{1/4})|_{(x'_k,\xi^{k\prime})}\\
&= \sigma(-\partial_{x^n}u_{k-1}|_{U_k}|\bar{g}|^{1/4})|_{(x'_k,\xi^{k\prime})} + \sigma(-\partial_{x^n}u_k|_{U_k}|\bar{g}|^{1/4})|_{(x'_k,\xi^{k\prime})}
\end{aligned}
$$

$$= -(\pm i|\xi_n^k|)\sigma(u_{k-1}|_{U_k}|\bar{g}|^{1/4})|_{(x_k', \xi^{k\prime})} - (\mp i|\xi_n^k|)\sigma(u_k|_{U_k}|\bar{g}|^{1/4})|_{(x_k', \xi^{k\prime})}$$

$$= \mp 2i|\xi_n^k|\sigma(u_{k-1}|_{U_k}|\bar{g}|^{1/4})|_{(x_k', \xi^{k\prime})}$$

$$= \pm 2i(-1)^k \exp\left(i\int_{\gamma^b(x_0' \to x_k')} A(\dot{\gamma}^b)\right) \cdot |\xi_n^k|^{1/2}|\xi_n^0|^{1/2}(L^{-k})^*\left(\sigma(\tilde{f})|_{(x_0', \xi^{0\prime})}\right),$$

where the sign is $+/-$ if $(x_0', \xi^{0\prime})$ is past/future-pointing. We thus proved the following result.

Proposition 4.3.5. *View $\Lambda_{g,A,q}^{U_0,U_k}$ as an operator on half-density valued distributions via*

$$\Lambda_{g,A,q}^{U_0,U_k}(f|\bar{g}|^{1/4}) := (\Lambda_{g,A,q}^{U_0,U_k} f)|\bar{g}|^{1/4}.$$

Suppose $L^k(x_0', \xi^{0\prime}) = (x_k', \xi^{k\prime})$ for some $k \geq 1$, $x_0' \in U_0$, $x_k' \in U_k$, and L is the lens relation. Then microlocally in a conic neighborhood of $(x_k', \xi^{k\prime}; x_0', \xi^{0\prime}) \in T^*U_k \times T^*U_0$, the operator $\Lambda_{g,A,q}^{U_0,U_k}$ is an elliptic Fourier integral operator of order 1, whose associated canonical relation is given by

$$\{(L^k(x', \xi'); x', \xi') : (x', \xi') \in T^*U_0 \cap \mathcal{H}\}.$$

Its principal symbol, regarded as a map in the sense of (4.2.3), is given by

$$\sigma(\Lambda_{g,A,q}^{U_0,U_k})|_{(x_k', \xi^{k\prime}; x_0', \xi^{0\prime})} = \pm 2i(-1)^k \exp\left(i\int_{\gamma^b} A(\dot{\gamma}^b)\right)|\xi_n^k|^{1/2}|\xi_n^0|^{1/2}(L^{-k})^*,$$

where

- *γ^b is the forward broken null-geodesic that is the projection of the forward broken bicharacteristic connecting*

  $$(x_0', \xi^{0\prime}) \to (x_1', \xi^{1\prime}) \to \cdots \to (x_k', \xi^{k\prime})$$

  (note that $\int_{\gamma^b} A(\dot{\gamma}^b)$ is independent of parametrization of γ^b);

- *the sign is $+/-$ if $(x_0', \xi^{0\prime})$ is a past/future-pointing covector.*

Remark 4.3.6. *Note that the way we turn DN map into an operator on half-density valued distributions* *depends on the metric. Specifically, this means that one should always be careful which of the following situations they are in:*

1. *If* $\Lambda^{U,V}_{g_1,A_1,q_1} = \Lambda^{U,V}_{g_2,A_2,q_2}$ *are given as* ***operators on distributions****, then to make them equivalent as operators on half-densities, we can let*

$$\Lambda^{U,V}_{g_j,A_j,q_j}(\omega) := \Lambda^{U,V}_{g_j,A_j,q_j}(\omega\eta^{-1})\eta$$

for a fixed half-density η *and* $j = 1, 2$.

2. *Similarly, if* $\Lambda^{U,V}_{g_1,A_1,q_1} = \Lambda^{U,V}_{g_2,A_2,q_2}$ *are given as* ***operators on half-density valued distributions****, then to make them equivalent as operators on distributions, we can let*

$$\Lambda^{U,V}_{g_j,A_j,q_j}(f) := \Lambda^{U,V}_{g_j,A_j,q_j}(f\eta)\eta^{-1}$$

for a fixed half-density η *and* $j = 1, 2$.

Remark 4.3.7. *Since the canonical relation for* $\Lambda^{U_0,U_k}_{g,A,q}$ *is a canonical graph in a small conic neighborhood of* $(x'_k, \xi^{k\prime}; x'_0, \xi^{0\prime})$*, its adjoint operator is also an elliptic FIO of order 1 whose canonical relation is given by the backward k-th lens relation. Moreover, from Proposition 4.3.5, its principal symbol is*

$$\sigma((\Lambda^{U_0,U_k}_{g,A,q})^*)|_{(x'_0,\xi^{0\prime};x'_k,\xi^{k\prime})} = \mp 2i(-1)^k \exp\left(-i\int_{\gamma^b} A(\dot{\gamma}^b)\right)|\xi^k_n|^{1/2}|\xi^0_n|^{1/2}(L^k)^*.$$

As a result, the normal operator $\mathcal{N}_k = (\Lambda^{U_0,U_k}_{g,A,q})^*\Lambda^{U_0,U_k}_{g,A,q}$ *is a pseudodifferential operator of order 2, whose principal symbol is*

$$\sigma(\mathcal{N}_k)(x'_0, \xi^{0\prime}) = 4|\xi^0_n||\xi^k_n|.$$

This agrees with [139, Remark 4.8].

Now we explore what information can be obtained from principal symbol of $\Lambda^{U_0,U_k}_{g,A,q}$, recall that in our case, the operators are equivalent when acting on distributions. Suppose we know the conformal class of $\bar{g}$ on $U_0 \cup U_k$, then we can choose some Lorentzian metric h on $U_0 \cup U_k$ in that conformal class, and $\bar{g} = e^{\varphi}h$.

Taking absolute value of the principal symbol to get rid of affects from A and time orientation, we obtain a formula to relate the conformal factor at x_0' and x_k':

$$
\begin{aligned}
&(n-2)\varphi(x_0') - n\varphi(x_k') \\
=&4\log\left[\frac{1}{2}|h(\xi^{0\prime},\xi^{0\prime})h(\xi^{k\prime},\xi^{k\prime})|^{-1/4}\left|\frac{|h(x_k')|_{x'}^{1/4}\sigma_{x'}(\Lambda_{g,A,q}^{U_0,U_k}f)|_{(x_k',\xi^{k\prime})}}{(L^{-k})^*(|h(x_0')|_{x'}^{1/4}\sigma_{x'}(f)|_{(x_0',\xi^{0\prime})})}\right|\right].
\end{aligned}
\tag{4.3.4}
$$

It is easy to see that the right hand side is coordinate invariant since both terms in the fraction are principal symbols of half-density valued distributions, and it is scaling invariant with respect to $\xi^{0\prime}$. Moreover, it gives the same value for any $f \in I_{cl}^{\mu}(\Lambda_0')$ as long as $(x_0', \xi^{0\prime}) \in \Lambda_0'$, because we are essentially using the principal symbol of $\Lambda_{g,A,q}^{U_0,U_k}$. In the case where f is not a classical distribution, the right hand side becomes quotient of equivalence classes, the equation then holds as $|\xi^{0\prime}| \to \infty$.

On the other hand, the argument of the principal symbol gives

$$
\exp\left(i\int_{\gamma^b} A(\dot{\gamma}^b)\right) = \pm i(-1)^k \arg\left(\frac{\sigma_{x'}(\Lambda_{g,A,q}^{U_0,U_k}f)|_{(x_k',\xi^{k\prime})}}{(L^{-k})^*(\sigma_{x'}(f)|_{(x_0',\xi^{0\prime})})}\right)
\tag{4.3.5}
$$

where $\arg(z) = \frac{z}{|z|}$, and the sign is $+/-$ if $(x_0', \xi^{0\prime})$ is future/past-pointing. We emphasize again that the left hand side is independent of parametrization of γ^b.

4.4 Weak Lens Relation and Conformal Class of the Boundary Metric

We first define what a weak lens relation is. Intuitively, it is a map that assigns a suitable subset $B_U \subset T^*U \cap \mathcal{H}$ to a corresponding subset $B_V \subset T^*V \cap \mathcal{H}$, such that singularities in B_U flow along a forward broken bicharacteristic to points in B_V.

Definition 4.4.1. *The* weak lens relation *is the map*

$$
\tilde{L} : B_U \mapsto B_V \subset T^*V \cap \mathcal{H}, \quad \textit{for suitable } B_U \in T^*U \cap \mathcal{H},
$$

satisfying the following conditions:

1. *$(x', \xi') \in B_U$ and $(y', \eta') \in B_V$ implies $y' \in J^+(x')$;*

2. *if $(x', \xi') \in B_U$ and $\Gamma^b_{x',\xi'}$ is its corresponding forward broken bicharacteristic, then $\Gamma^b_{x',\xi'}|_V = B_V$;*

3. *B_U is maximal in the sense that there can not be another $\tilde{B}_U \supsetneq B_U$ satisfies the above two conditions.*

It is not hard to see that the map is well-defined and bijective, as each B_U and B_V correspond to a unique broken bicharacteristic. To see why it is weaker than the standard lens relation L, given some (y', η') in some B_U, there is no guarantee that $L(y', \eta')$ is in B_U or B_V. Moreover, B_U and B_V are given as unordered sets, whereas if one knows the lens relation, then points in B_U and B_V can be ordered.

Proposition 4.4.2. *$\tilde{L}$ can be constructed from $\Lambda^{U,V}_{g,A,q}$.*

Proof. We first consider $\tilde{L}$ as a map on the spherical bundle $S^*\partial M = (T^*\partial M \backslash 0)/\mathbb{R}^+$, that is, we recover the direction relations. As mentioned before, we can view wavefront set as a subset of $S^*\partial M$. For any $(x', \xi') \in S^*\partial M$, set some f such that $\mathrm{WF}(f) = \{(x', \xi')\}$. By Lemma 4.3.1, $\mathrm{WF}(\Lambda^{U,V}_{g,A,q} f)$ can only be one of the following three cases:

1. $(x', \xi') \in \mathcal{E}$, $\mathrm{WF}(\Lambda^{U,V}_{g,A,q} f) = \emptyset$;

2. $(x', \xi') \in \mathcal{H}$, $\mathrm{WF}(\Lambda^{U,V}_{g,A,q} f) = \emptyset$ if the corresponding forward broken bicharacteristic does not pass through V, or a discrete set of points if it passes through V;

3. $(x', \xi') \in \mathcal{G}$, $\mathrm{WF}(\Lambda^{U,V}_{g,A,q} f) = \emptyset$ or the union of several continuous curves.

We collect all (x', ξ') with a discrete set of singularities, these are the hyperbolic directions in S^*U whose corresponding forward broken bicharacteristic passes through V. Label the corresponding f as $f_{x',\xi'}$, and define $\tilde{L}$ on $S^*\partial M$ via

$$\tilde{L}^{-1}(\mathrm{WF}(\Lambda^{U,V}_{g,A,q} f_{x',\xi'})) = \{(z', \zeta') \in S^*U : \mathrm{WF}(\Lambda^{U,V}_{g,A,q} f_{z',\zeta'}) = \mathrm{WF}(\Lambda^{U,V}_{g,A,q} f_{x',\xi'})\}.$$

We have thus determined $\tilde{L}$ as a map on $S^*\partial M$.

Pick some arbitrary Riemannian metric on $T^*\partial M$, so that $S^*\partial M$ has a natural section given by covectors of length 1. For each (x', ξ') in case (2) of length 1, let $(y', \eta') \in \mathrm{WF}(\Lambda_{g,A,q}^{U,V} f_{x',\xi'})$ be of length 1. We know (x', ξ') is connected with $(y', \lambda\eta')$ for some $\lambda > 0$ by the forward broken bicharacteristic. To find out what λ is, choose a sufficiently small conic neighborhood $N_1 \subset T^*U$ of (x', ξ'). Then the union of $\mathrm{WF}(\Lambda_{g,A,q}^{U,V} f_{z',\zeta'})$ for $(z', \zeta') \in N_1$ forms disjoint neighborhoods around points in $\mathrm{WF}(\Lambda_{g,A,q}^{U,V} f_{x',\xi'})$, denote the one around (y', η') as $N_2 \subset T^*V$. When N_1 is sufficiently small, and $(y', \lambda\eta')$ is the k-th reflection of (x', ξ'), then points in N_2 are the k-th reflection of points in N_1. We can thus extend λ to be a function on N_1 homogeneous of degree 0 in phase variable. By Lemma 4.4.3, $\lambda : N_1 \to \mathbb{R}^+$ can be computed by requiring the graph of lens relation to be a canonical relation (that is, symplectic form vanishes). $\square$

Lemma 4.4.3. *If $L : N_1 \to N_2$ is a lens relation for $N_1, N_2 \subset T^*\partial M \cap \mathcal{H}$ conic subsets, and $L' : N_1' \to N_2'$ its corresponding map on $S^*\partial M$ via $L'([x, \xi]) = [L(x, \xi)]$, then L and L' uniquely determines each other.*

Proof. Suppose L' is given, but there exists L_1 and L_2 such that both define L'. Then there exists $\phi : N_1 \to \mathbb{R}^+$ with $\phi(x, k\xi) = \phi(x, \xi)$ for $k > 0$ such that $L_1(x, \xi) = (y, \eta)$, $L_2(x, \xi) = (y, \phi(x, \xi)\eta)$. Use the fact that lens relation is invertible and homogeneous of degree 1 on phase variable,

$$T := L_1^{-1} \circ L_2 : N_1 \to N_1, (x, \xi) \mapsto (x, \phi(x, \xi)\xi).$$

L_1 and L_2 are symplectomorphisms, so T is also a symplectomorphism. In particular, use coordinate (x, ξ, y, η) for the graph, the tangent vector of the graph at some $(x, \xi, x, \phi(x, \xi)\xi)$ consists of

$$X_i = \partial_{x^i} + \partial_{y^i} + \sum_j \frac{\partial \phi}{\partial x^i} \xi_j \partial_{\eta_j}, \quad Y_i = \partial_{\xi_i} + \phi \partial_{\eta_i} + \sum_j \frac{\partial \phi}{\partial \xi_i} \xi_j \partial_{\eta_j}.$$

Then we have

$$(d\xi_k \wedge dx^k - d\eta_k \wedge dy^k)(X_i, Y_i) = -1 + \phi + \frac{\partial \phi}{\partial \xi_i} \xi_i = 0,$$

add up all i gets

$$n(-1 + \phi) + \nabla_\xi \phi \cdot \xi = 0.$$

But ϕ is homogeneous of degree 0 on ξ, meaning $\nabla_\xi \phi \cdot \xi = 0$, so $\phi \equiv 1$. In other words, L can be explicitly computed from L' by the requirement that symplectic form vanishes. The other direction is trivial. $\square$

Remark 4.4.4. *Note that during the proof of Proposition 4.4.2, we do not need to know how small N_1 should be, since we are essentially only using the pushforward of $T_{(x',\xi')}(T^*\partial M)$ via the k-th lens relation at (x',ξ'). In particular, the weak lens relation gives the k-th lens relation on sufficiently small N_1, this fact will be used later. We shall emphasize that we would not be able to know the exact value of k just from weak lens relation.*

We can then use the weak lens relation to recover the conformal class of the boundary metric on the recoverable set.

Proposition 4.4.5. *Given $\Lambda^{U,V}_{g,A,q}$, then the recoverable set $\tilde{U}\cup\tilde{V}$ can be determined. Moreover, the conformal class of the boundary metric $\bar{g} = g|_{T\partial M\times T\partial M}$ can be constructed on $\tilde{U}\cup\tilde{V}$.*

Proof. By Proposition 4.4.2, $\tilde{L}$ can be constructed, we will use it to construct $\tilde{U},\tilde{V}$ and the glancing set on $\tilde{U}\cup\tilde{V}$. Note that once the glancing set is constructed, it is the light cone of $\bar{g} = g|_{T\partial M\times T\partial M}$, so the conformal class of $\bar{g}$ is determined on recoverable sets.

By definition, the domain and range of $\tilde{L}$ are subsets of $\mathcal{H}$, so any limiting point stays in $\mathcal{H}\cup\mathcal{G}$. By abuse of notation, we say $(x',\xi')\in T^*U$ is in domain (range) of $\tilde{L}$, if there exists some $B_U \ni (x',\xi')$ ($B_V \ni (x',\xi')$) in the domain of $\tilde{L}$ (range of $\tilde{L}$), and we denote $\tilde{L}(x',\xi') = \tilde{L}(B_U)$ ($\tilde{L}^{-1}(x',\xi') = \tilde{L}^{-1}(B_V)$). Pick a sequence of (x'_j,ξ'_j) in domain of $\tilde{L}$ that converges to some (x',ξ') not in domain of $\tilde{L}$, and let $B_j = \tilde{L}(x'_j,\xi'_j)$ and Γ^b_j the broken bicharacteristic from (x'_j,ξ'_j) (certainly we only consider those limit such that $\xi'\neq 0$). Note that either $(x',\xi')\in\mathcal{H}$, and Γ^b_j converges uniformly on bounded time interval to Γ^b, which is the forward broken bicharacteristic from (x',ξ'), by smoothness of Hamiltonian vector field; or $(x',\xi')\in\mathcal{G}$, and Γ^b_j converges uniformly on bounded time interval to Γ^{gl}, the forward gliding ray from (x',ξ'), by Lemma 4.2.15. By Proposition 4.2.11, broken bicharacteristics are all tame and have finitely many boundary points in any compact set. In the first case, the limiting points of B_j can only be $\Gamma^b|_V$, which must be empty, because otherwise (x',ξ') would be in the domain of $\tilde{L}$; in the second case, the limiting points of B_j can be $\Gamma^{gl}|_V$, which is non-empty and forms a continuous set of points if and only if Γ^{gl} passes through V.

Conversely, if $(x', \xi') \in \mathcal{G}$ and the corresponding gliding ray passes through V, then by uniform convergence of broken bicharacteristic in Lemma 4.2.18, any sufficiently close hyperbolic point would be in the domain of $\tilde{L}$. As a result, we can identify all the $(x', \xi') \in \mathcal{G}$ whose forward gliding ray passes through V. In particular, this means we can recover $\tilde{U}$. Similarly we can also recover $\tilde{V}$ using $\tilde{L}^{-1}$.

Moreover, by assumption of recoverable point, if dimension is 2+1 we can recover 2 such directions in the glancing set at each recoverable point, which is the entire glancing set; in higher dimensions, if a gliding ray passes through V then all nearby gliding rays also pass through V, so we recover an open set of directions in glancing set for each recoverable point, which then recovers the entire glancing set at the point by analyticity of light cone. □

4.5 Upgrade Weak Lens Relation to Lens Relation

Weak lens relation can be upgraded to lens relation when the boundary is strictly null-convex and when the conformal class on the boundary is determined. From Proposition 4.4.2 and Proposition 4.4.5, we may assume $\tilde{L}, \tilde{U}, \tilde{V}$ and the conformal class of $\bar{g}|_{\tilde{U}\cup\tilde{V}}$ are given. Also as a result of knowing the conformal class, $\mathcal{G}, \mathcal{H}$ and $\mathcal{E}$ are given on $\tilde{U} \cup \tilde{V}$.

Proposition 4.5.1. *Let L be the (future-pointing) lens relation for g. Suppose $\tilde{L}, \tilde{U}, \tilde{V}$, and the conformal class of $\bar{g} = g|_{T(\tilde{U}\cup\tilde{V})\times T(\tilde{U}\times\tilde{V})}$ are given (thus $\mathcal{G}|_{\tilde{U}\cup\tilde{V}}$ is also determined). Then for any recoverable direction $(x', \xi') \in \mathcal{G}|_{\tilde{U}\cup\tilde{V}}$, there exists a conic neighborhood W' of (x', ξ'), $W = W' \cap \mathcal{H}$, such that $L|_W$ can be determined up to time orientation. That is, one can construct some L' on W such that*

$$L' = L|_W \quad \text{or} \quad L' = L^{-1}|_W.$$

Proof. Assume $x' \in \tilde{U}$, similar argument works for $\tilde{V}$. By assumption, the forward gliding ray passes through V. Lemma 4.2.18 shows that there exists a conic neighborhood $\tilde{W}$ of (x', ξ') such that the forward broken bicharacteristics from $\tilde{W} \cap \mathcal{H}$ passes through V. We will shrink $\tilde{W}$ when necessary, and determine the lens relation on the hyperbolic part of a subset $W' \subset \tilde{W}$, $(x', \xi') \in W'$.

Smoothly pick $\partial_t \in T\partial M$ around x' such that it is timelike, this also guarantees that ∂_t has consistent time orientation, which is possible since the light cone has been determined. Consider a smooth hypersurface $N \subset \partial M$ containing x' such that TN is spacelike (again possible because light cone has been determined). Locally around x' one can solve an ODE and obtain a smooth temporal function t around x' such that $t|_N = 0$, and we can shrink $\tilde{W}$ such that t is defined in a neighborhood of the projection of $\tilde{W}$ onto ∂M. We emphasize that ∂_t is timelike, but we do not know if it is future-pointing or past-pointing. By Lemma 4.2.15, by choosing W' conic neighborhood of (x', ξ') sufficiently small, for any $(y', \eta') \in W' \cap \mathcal{H}$, $L(y', \eta')$ and $L^{-1}(y', \eta')$ must be in $\tilde{W} \subset T^*\tilde{U}$ (here we also use the fact that for directions sufficiently close to a glancing vector, time spent in the interior is close to 0, see the proof of Lemma 4.2.18 (1)). Moreover, its forward broken bicharacteristic passes through V, so there exists a unique subset $B \subset T^*U$ such that $(y', \eta') \in B \in \mathrm{Domain}(\tilde{L})$. Since $L(y', \eta') \in T^*\tilde{U}$, $L(y', \eta') \in B$ by definition of weak lens relation, same for $L^{-1}(y', \eta')$. Note that any two points in B must satisfy that one of them is in the causal future of the other, which means points in B can be ordered by this. This order can be recovered up to reversing by the value of t from smallest to largest, and (y', η') must be in between $L(y', \eta')$ and $L^{-1}(y', \eta')$. If we consistently order them from smallest to largest for any such (y', η'), we would be able to recover either L or L^{-1}. $\square$

We then use the following theorem to determine the jet bundle.

Theorem 4.5.2 ([131] Theorem 2.1)**.** *Let g and $\hat{g}$ be two Lorentzian metrics defined near some $x_0 \in \partial M$ so that $\hat{g} = \mu_0 g$ on $T\partial M \times T\partial M$ locally with some $0 < \mu_0 \in C^\infty(\partial M)$. Let $(x_0, v_0) \in T\partial M$ be lightlike for g. Assume that ∂M is strictly convex with respect to g in the direction of (x_0, v_0). Assume that either*

(i) $\hat{\mathcal{S}}^{\#} = \mathcal{S}^{\#}$ in a neighborhood of (x_0, v_0^b), or

(ii) $\hat{\mathcal{S}} = \mathcal{S}$ in a neighborhood of (x_0, v_0), and $\mu_0 = const.$

Then there exists $\mu(x) > 0$ with $\mu = \mu_0$ on ∂M, and a local diffeomorphism ψ near ∂M preserving it pointwise, so that the jets of g and $\mu\psi^*\hat{g}$ coincide on ∂M near x_0.

The $\mathcal{S}^\sharp$ in the theorem is the lens relation L. Moreover, this deterministic result is independent of whether the lens relation is forward or backward.

4.6 Determine Conformal Factor

Suppose both (g_1, A_1, q_1) and (g_2, A_2, q_2) have the same DN map $\Lambda^{U,V}$, by Proposition 4.4.5 $\bar{g}_1$ and $\bar{g}_2$ share the same $\tilde{U}, \tilde{V}$ and conformal class on $T(\tilde{U} \cup \tilde{V})$. Then for some fixed h in the same conformal class as $\bar{g}_1$ and $\bar{g}_2$, we have $\bar{g}_1 = e^{\varphi_1} h$, $\bar{g}_2 = e^{\varphi_2} h$. From equation (4.3.4), if a forward broken null-geodesic goes from $z' \in \tilde{U}$ to $x' \in \tilde{V}$, then

$$(n-2)\varphi_1(z') - n\varphi_1(x') = (n-2)\varphi_2(z') - n\varphi_2(x').$$

In other words, $\bar{g}_1 = e^{\varphi} \bar{g}_2$ with $\varphi = \varphi_1 - \varphi_2$, for any forward broken null-geodesic connecting $z' \in U$ and $x' \in V$,

$$(n-2)\varphi(z') = n\varphi(x').$$

We will show that this condition actually implies φ is locally constant on $\tilde{U} \cup \tilde{V}$. When dimension of M is 3, the main idea is to use light cone in tangent space to approximate the light cone formed by boundary null-geodesics.

Proposition 4.6.1. *If M has dimension 3, and for any forward null-geodesic from some $z' \in U$ to $x' \in V$, φ satisfies $(n-2)\varphi(z') = n\varphi(x')$, then φ is locally constant on $\tilde{U} \cup \tilde{V}$.*

Proof. The assumption gives the following result: if $\tilde{L}$ is the weak lens relation and $\tilde{L}(B_U) = B_V$, then $\varphi|_{\pi(B_U)} = \frac{c}{n-2}$ and $\varphi|_{\pi(B_V)} = \frac{c}{n}$ for some constant c, where $\pi : T^*\partial M \to \partial M$. For $x' \in \tilde{U} \cup \tilde{V}$, let γ^{gl} be the projection of a gliding ray that passes through U, V and x'. By Lemma 4.2.18 (1), for any $z' \in \gamma^{gl} \cap U$, there exists a sequence of broken null-geodesic with initial point x' such that a sequence of boundary points converge to z', so $\varphi(x') = \varphi(z')$. Similarly, for any $y' \in \gamma^{gl} \cap V$, $(n-2)\varphi(x') = n\varphi(y')$. Thus, $\varphi|_{\gamma^{gl} \cap U} = \frac{c}{n-2}$ and $\varphi|_{\gamma^{gl} \cap V} = \frac{c}{n}$ for some constant c.

Locally focus on a small connected neighborhood of some point $x' \in W \subset \tilde{U}$. In dimension 3, by the definition of recoverable point, for any $z' \in W$, both gliding rays of z' passes through V. Note that the gliding rays are the same for two metrics in W since they are conformally equivalent. Use the fact that the projection of gliding ray is the null-geodesic of boundary metric $\bar{g}_1$ and $\bar{g}_2$, we thus convert the assumption to φ being constant along any boundary null-geodesic. To show φ is locally constant at x', suffice to show for any y' sufficiently close to x', a boundary null-geodesic of y' intersects with a boundary null-geodesic of x' in W.

We shall use the light cone in the tangent space to approximate the light cone generated by geodesics. Consider some local coordinate $\phi : W \to N \subset \mathbb{R}^2$, $\phi(x') = 0$, and $h = (\phi^{-1})^* \bar{g}_1$ on N. Let $N' \subset N$ be a small neighborhood of 0 such that the null-geodesic of h for any point $p \in N'$ is uniformly close to the straight line passing through p in the same direction in N. Then the two null-geodesics of p are uniformly approximated by a cross formed by two straight lines centered at p. It is clear that when N' is sufficiently small, the cross centered at p intersects with the cross centered at 0; since the approximation is uniform and the geodesics are connected, we have the union of two null-geodesics from p would intersect the union of two null-geodesics from 0. This proves that for any y' sufficiently close to x', the null-geodesics intersect so $\varphi(y') = \varphi(x')$, hence φ is locally constant on $\tilde{U}$. The proof for points in $\tilde{V}$ is the same. Clearly, if $x' \in U$ is connected to $y' \in V$ by a forward null-geodesic, then the constant c_1 around x' and the constant c_2 around y' need to satisfy $(n-2)c_1 = nc_2$. $\square$

When the dimension is higher, for every point in the recoverable set, we have an open set of recoverable directions in $\mathcal{G}$. We can not simply use the same argument as in dimension 3, because the set of null-geodesics do not naturally form a cross anymore.

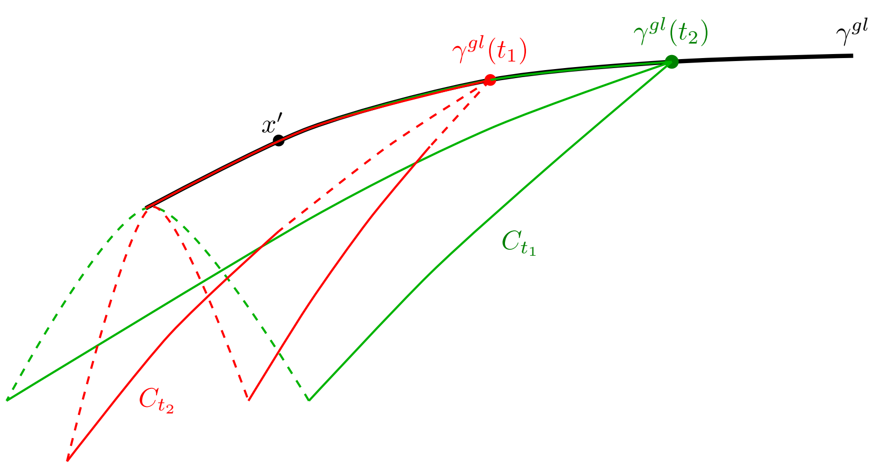

Figure 4.2: Dimension at least 4: in $\tilde{U}$, we use that fact that φ would stay constant along any piecewise boundary null-geodesic curve where each piece of boundary null-geodesic when fully extended passes through V. We show that the union of such paths from x' has non-empty interior. Same argument works for $\tilde{V}$.

Proposition 4.6.2. *If M has dimension at least 4, and for any forward null-geodesic from some $z' \in U$ to $x' \in V$ satisfies $(n-2)\varphi(z') = n\varphi(x')$, then φ is locally constant on $\tilde{U} \cup \tilde{V}$.*

Proof. Again the first part is the same as in dimension 3, we have for any boundary null-geodesic γ^{gl} passing through both U and V, $\varphi|_{\gamma^{gl} \cap U} = \frac{c}{n-2}$ and $\varphi|_{\gamma^{gl} \cap V} = \frac{c}{n}$ for some constant c.

From now on, we look at the boundary manifold $(\partial M, \bar{g})$, for example when we say null-geodesic we refer to the geodesic in the boundary with respect to $\bar{g}$. Fix some $x' \in \tilde{U}$, there exists some (x', ξ') recoverable direction. We will be working in a sufficiently small geodesically convex neighborhood W of x', and use the fact that any two points in W that are connected by a null-geodesic can not be connected by a piecewise lightlike path with fixed time orientation that is not a null-geodesic (because otherwise they would be connected by a timelike geodesic).

Let γ^{gl} be a null-geodesic passing through x' corresponding to a recoverable direction. Denote C_t the union of inextendible null-geodesics in W that passes through $\gamma^{gl}(t)$, and will pass through V when fully extended, denote $\mathcal{C}(x') := \cup C_t$. For $t_1 \neq t_2$, suppose there exists some $y' \in C_{t_1} \cap C_{t_2}$ such that $y' \notin \gamma^{gl}$. Then any two points of y', $\gamma^{gl}(t_1)$ and $\gamma^{gl}(t_2)$ are connected by some null-geodesic. The three are obviously not on the same null-geodesic since $y' \notin \gamma^{gl}$, so it contradicts geodesic convexity of W. As a result, we have for any $t_1 \neq t_2$, $C_{t_1} \cap C_{t_2} = \gamma^{gl}$ is of dimension 1. On the other hand, we know C_t is an $n-2$ dimensional submanifold for all $t \neq 0$, and it changes continuous as t varies. Thus as t goes from some t_1 to t_2, C_t changes continuously from an $n-2$ dimensional submanifold to another $n-2$ dimensional submanifold such that at any two distinct time, the two submanifolds only intersect at γ^{gl}. This implies $\mathcal{C}(x')$ has non-empty interior (since ∂M has dimension $n-1$). Moreover, φ is constant along γ^{gl} and on each C_t, so $\varphi|_{\mathcal{C}(x')}$ is constant.

We can similarly construct $\mathcal{C}(z')$ for any z' in a neighborhood of x'. Note that $\mathcal{C}(z')$ changes continuously and has non-empty interior. Hence for any z' sufficiently close to x', $\mathcal{C}(z') \cap \mathcal{C}(x') \neq \emptyset$, implying φ is constant around x'. Same proof works for points in $\tilde{V}$.

□

4.7 Determine 1-Form

Now we use (4.3.5) to recover the 1-form on $\tilde{U} \cup \tilde{V}$, assuming the conformal class of $\bar{g}$ has been recovered on $\tilde{U} \cup \tilde{V}$. As in the proof of Proposition 4.4.5, for every recoverable direction $(x', \xi') \in T^*\tilde{U} \cap \mathcal{G}$, one can identify its corresponding forward gliding ray Γ^{gl} on both T^*U and T^*V using weak lens relation (for details, see proof of the proposition). Denote γ^{gl} the projection of Γ^{gl}, let $\phi : [-1, 1] \to \gamma^{gl}$ be a parametrization of γ^{gl} around x' such that $\phi(0) = x'$ and $\phi([-1, 1]) \subset \tilde{U}$. Since the conformal class of $\bar{g}$ is already recovered, one can recover $(\xi')^\sharp$ up to a positive scaling. This means we can determine the direction of $\dot{\gamma}^{gl}(0)$, since up to positive rescaling, $\dot{\gamma}^{gl}(0) = -2(\xi')^\sharp$, see (4.2.1) and Section 4.2.3. We may thus assume $\phi'(0)$ is in the same direction as $\dot{\gamma}^{gl}(0)$, otherwise just reverse the parametrization ϕ. We show that for any $t \in [-1, 1]$, we can explicitly construct

$$\exp\left(i \int_0^t A(\phi'(s))ds\right).$$

Fix some t, let $z' = \phi(t)$ and (z', ζ') the corresponding point on Γ^{gl}. By Proposition 4.5.1, the lens relation L can be recovered up to time orientation on a conic neighborhood of glancing set. Then for each point of $\Gamma^{gl}((x', \xi') \to (z', \zeta'))$, we can find neighborhood of uniform size on which L is known. Let $(x'_j, \xi'_j) \in T^*U \cap \mathcal{H}$ such that $(x'_j, \xi'_j) \to (x', \xi')$, by Lemma 4.2.15, the forward broken bicharacteristic Γ^b_j from (x'_j, ξ'_j) converges uniformly to Γ^{gl} on bounded time interval. By Lemma 4.2.18 (1), we can find a sequence of (z'_j, ζ'_j) on Γ^b_j such that $(z'_j, \zeta'_j) \to (z', \zeta')$. As a result, for j sufficiently large, all the boundary points of Γ^b_j between (x'_j, ξ'_j) and (z'_j, ζ'_j) are in the neighborhood where L is known up to time orientation. That is, let k_j be such that $(z'_j, \zeta'_j) = L^{k_j}(x'_j, \xi'_j)$, then one can recover k_j up to a sign determined by time orientation.

Denote (y'_j, η'_j) some arbitrary point on $\Gamma^b_j \cap T^*V$, by equation (4.3.5) we know

$$\exp\left(i \int_{\gamma^b_j(x'_j \to y'_j)} A(\dot{\gamma}^b_j)\right) = \pm i(-1)^{l_j} e^{i\theta_j}, \quad \exp\left(i \int_{\gamma^b_j(z'_j \to y'_j)} A(\dot{\gamma}^b_j)\right) = \pm i(-1)^{m_j} e^{i\omega_j}$$

where $|l_j - m_j| = |k_j|$ is known, $e^{i\theta_j}$ and $e^{i\omega_j}$ are the argument terms that can be explicitly computed. In particular, this means we can explicitly compute the quotient

$$\exp\left(i\int_{\gamma_j^b(x_j'\to z_j')} A(\dot{\gamma}_j^b)\right) = (-1)^{|k_j|}e^{i(\theta_j-\omega_j)}.$$

Note that the uncertainty caused by time orientation is eliminated after taking the quotient. By Lemma 4.2.15, the following limit converges and can be computed:

$$\lim_{j\to\infty}(-1)^{|k_j|}e^{i(\theta_j-\omega_j)} = \exp\left(i\int_{\gamma^{gl}(x'\to z')} A(\dot{\gamma}^{gl})\right) = \exp\left(i\int_0^t A(\phi'(s))ds\right).$$

Since $t \in [-1,1]$ is arbitrary, we can then take derivative and obtain $A(\phi'(0))$. Moreover, we can compute this for any gliding ray that goes from U to V. By assumption of recoverable set, this means 2 gliding rays for dimension 3, and an open set of glancing directions for higher dimensions. In both cases, one can construct $A(v)$ for a set of v that spans the entire $T_{x'}\tilde{U}$. The construction for points in $\tilde{V}$ is the same, in particular we proved the following result.

Proposition 4.7.1. *Given the DN map $\Lambda^{U,V}_{g,A,q}$, let $T\tilde{U}\cup T\tilde{V}\subset T\partial M$ be the set of tangential vectors on $\tilde{U}\cup\tilde{V}$, then $A|_{T\tilde{U}\cup T\tilde{V}}$ can be computed.*

4.8 Proof of Main Theorems

Proof of Theorem 4.1.3: By Proposition 4.4.2, one can construct the weak lens relation $\tilde{L}$. Then by Proposition 4.4.5, the recoverable sets $\tilde{U}\cup\tilde{V}$ can be determined, and the conformal class of the boundary metric $\bar{g} = g|_{T\partial M\times T\partial M}$ can be constructed on $\tilde{U}\cup\tilde{V}$. Moreover, by Proposition 4.5.1, we can recover the lens relation L up to time orientation in a neighborhood of the glancing set. Finally, by Proposition 4.7.1, $A|_{T(\tilde{U}\cup\tilde{V})}$ can be constructed from the principal symbol of $\Lambda^{U,V}_{g,A,q}$. □

Proof of Theorem 4.1.4: By Theorem 4.1.3, the two metrics share the same recoverable sets $\tilde{U}$ and $\tilde{V}$, the same conformal class on recoverable sets, and the same lens relation around glancing sets up to time orientation. By [131, Theorem 2.1], this is enough to determine the jet bundle of the metric on the recoverable sets, because choice of the time orientation does not affect the metric. Finally, by Proposition 4.6.1 and

Proposition 4.6.2, the conformal factor satisfies the relation stated in the theorem. $\square$

We now consider the case where the metric on U or V is a priori given. In this case, we show that the boundary metric can be explicitly constructed on a larger set than the recoverable set.

Proof of Theorem 4.1.6: First assume the boundary metric $\bar{g}$ is given on $\tilde{U}_0$. Let $y' \in \tilde{V}_0$, recall that by definition this means there exists some $x' \in \tilde{U}_0$ and γ^b broken null-geodesic such that x' and y' are connected by γ^b. Then there exists some Γ^b broken null-bicharacteristic such that the projection of Γ^b is γ^b, and Γ^b connects (x', ξ') and (y', η') for some $\xi' \in T^*_{x'}\partial M$ and $\eta' \in T^*_{y'}\partial M$. By Proposition 4.3.5, choose some f whose wavefront set contains (x', ξ'), the absolute value of the principal symbol gives

$$|\bar{g}(y')|_{y'}^{-1/4}|\eta_n|^{1/2} = |\bar{g}(x')|_{x'}^{-1/4}|\xi_n|^{-1/2}\frac{\sigma_{y'}(\Lambda^{U,V}_{g,A,q}f)|_{(y',\eta')}}{(L^{-k})^*\sigma_{x'}(f)|_{(x',\xi')}},$$

where

$$|\xi_n| = \sqrt{-\bar{g}^{\alpha\beta}\xi_\alpha\xi_\beta} = (-\bar{g}(\xi', \xi'))^{1/2}, \quad |\eta_n| = \sqrt{-\bar{g}^{\alpha\beta}\eta_\alpha\eta_\beta} = (-\bar{g}(\eta', \eta'))^{1/2}.$$

Note that the right hand side is fully computable since we are given $\bar{g}$ on $\tilde{U}_0$. Thus we can compute

$$C(y', \eta') := |\bar{g}(y')|_{y'}^{-1}\bar{g}(\eta', \eta')$$

for any (y', η') such that the corresponding backward broken null-bicharacteristic passes through U. In particular, if this holds for (y', η'), then it holds for a conic neighborhood of (y', η'). Note that for any (y', ζ') in that open set, they share the same $|\bar{g}(y')|_{y'}$. Choose a basis $\zeta^1, \cdots, \zeta^{n-1}$ in the open neighborhood, we may assume the determinant $|\bar{g}(y')|_{y'}$ is with respect to this basis. Since $\zeta^\alpha + \zeta^\beta$ is also in the neighborhood for any $\alpha, \beta = 1, \ldots, n-1$. We thus have

$$C(y', \zeta^\alpha) = |\bar{g}(y')|_{y'}^{-1}\bar{g}(\zeta^\alpha, \zeta^\alpha), \quad C(y', \zeta^\alpha + \zeta^\beta) = |\bar{g}(y')|_{y'}^{-1}\bar{g}(\zeta^\alpha + \zeta^\beta, \zeta^\alpha + \zeta^\beta).$$

The above information is enough for us to construct the matrix

$$C(y') := |\bar{g}(y')|_{y'}^{-1}\bar{g}^{-1}(y').$$

As a result, for all $n \geq 3$, the metric at y' can be explicitly computed as

$$\bar{g}(y') = \left(\det(C(y'))^{-\frac{1}{n}}C(y')\right)^{-1}.$$

The same method works when the metric is given on $\tilde{V}_0$. Note that in this case we can similarly construct

$$C(x') := |\bar{g}(x')|_{x'}\bar{g}^{-1}(x'),$$

and so the metric is given by

$$\bar{g}(x') = \left(\det(C(x'))^{\frac{1}{n-2}}C(x')\right)^{-1}.$$

Finally, the fact that time orientation can be determined is because (y', η') and (x', ξ') have the same time orientation if they are connected by a broken bicharacteristic. $\square$

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Chapter 4, in full, has been submitted for publication as it may appear in joint work with Yang Zhang. The dissertation author was the primary researcher and author of this material.

Chapter 5

Calderón problem with electromagnetic perturbations

The content of this chapter corresponds to my work [157]. The main results are Theorem 5.1.3, Theorem 5.1.5 and Theorem 5.1.1.

5.1 Introduction

Suppose (M, g) is a smooth, connected, compact Riemannian manifold with boundary of dimension $n \geq 3$. If Δ_g is the Laplace-Beltrami operator, then the Dirichlet-to-Neumann map (DN map) is defined as

$$\Lambda_g : C^\infty(\partial M) \ni f \mapsto \partial_\nu u|_{\partial M} \in C^\infty(\partial M) \tag{5.1.1}$$

where u is the unique solution of

$$\Delta_g u = 0 \quad \text{in } M^\circ, \quad u|_{\partial M} = f. \tag{5.1.2}$$

Here M° is the interior of M. There exists a natural gauge equivalence for the DN map when $n \geq 3$. Namely, if $\Psi : M \to M$ is a diffeomorphism such that $\Psi|_{\partial M}$ is the identity map, then

$$\Lambda_g = \Lambda_{\Psi^* g}. \tag{5.1.3}$$

The well-known anisotropic Calderón problem asks if the metric g is determined by the DN map Λ_g up to boundary fixing isometry (see [98, Conjecture A]), we also refer to it as the Riemannian Calderón problem. The problem is widely open in dimension $n \geq 3$, see Section 5.1.1 for some history of the problem.

In this chapter, we consider the magnetic Laplace-Beltrami operator $\Delta_{g,A}$ where A is a smooth 1-form. In local coordinates, it has the form

$$\Delta_{g,A} = |g|^{-\frac{1}{2}}(\partial_j - iA_j)|g|^{\frac{1}{2}} g^{jk}(\partial_k - iA_k).$$

The 1-form A is usually referred to as the magnetic potential. It appears frequently in physics, for example when a charged quantum particle moves in a magnetic field. Denote the DN map

$$\Lambda_{g,A} : f \mapsto (\partial_\nu - i\langle A, \nu\rangle)u|_{\partial M}$$

where u is the unique solution of

$$\Delta_{g,A} u = 0 \quad \text{in } M^\circ, \quad u|_{\partial M} = f. \tag{5.1.4}$$

In this case, if $\Psi : M \to M$ is a boundary fixing diffeomorphism, and $d\omega$ is a smooth exact 1-form such that $\omega|_{\partial M} = 0$, then

$$\Lambda_{g,A} = \Lambda_{\Psi^* g, \Psi^* A + d\omega}. \tag{5.1.5}$$

This is referred to as the natural gauge equivalence. Specifically, if u is the solution with respect to g and A, then $e^{i\omega}\Psi^* u$ is the solution with the same Dirichlet boundary data respect to $\Psi^* g$ and $\Psi^* A + d\omega$.

In this chapter, we show that g can be fully determined without any gauge equivalence if the DN map with magnetic potential is given for all sufficiently small magnetic potentials.

Theorem 5.1.1. *Let M be a smooth connected compact manifold with boundary of dimension $n \geq 3$, and g_1, g_2 are two Riemannian metrics on M. Suppose $\Lambda_{g_1,A} = \Lambda_{g_2,A}$ for all A in a neighborhood of 0 with respect to C^k topology for some $k \geq 0$, then $g_1 = g_2$.*

The following result is a direct consequence of Theorem 5.1.1.

Corollary 5.1.2. *For $j = 1, 2$, let (M_j, g_j) be a smooth connected compact Riemannian manifold with boundary of dimension $n \geq 3$. Suppose there exists a boundary fixing diffeomorphism $\Psi : M_1 \to M_2$ such that $\Lambda_{g_1,A} = \Lambda_{g_2,(\Psi^{-1})^*A}$ for all smooth 1-form A on M_1 in a neighborhood of 0 with respect to C^k topology for some $k \geq 0$, then $g_1 = \Psi^* g_2$.*

The same problem can be considered when the metric is Lorentzian. Consider a Lorentzian manifold (M, g) of dimension $n \geq 3$ with a timelike boundary ∂M. We say the manifold is (lightlike) non-trapping if any inextendible lightlike geodesic $\gamma : [a, b] \to M$ eventually leaves M in finite time, that is, $-\infty < a \leq b < \infty$ and $\gamma(a), \gamma(b) \in \partial M$. For simplicity, we assume the manifold is non-trapping and admissible (see Definition 5.3.1), in particular, we consider the following electromagnetic wave equation

$$\begin{cases} \Box_{g,A,q} u = 0 \quad \text{in } M^\circ, \\ u|_{\partial M} = f, \\ \text{supp}(u) \subset J^+(\text{supp}(f)), \end{cases} \tag{5.1.6}$$

where J^+ denotes the causal future, and $\Box_{g,A,q}$ is the wave operator with first order perturbation

$$\Box_{g,A,q} = |g|^{-\frac{1}{2}}(\partial_j - iA_j)|g|^{\frac{1}{2}} g^{jk}(\partial_k - iA_k) + q.$$

Here A is a smooth 1-form and q is a smooth function. We emphasize that in the Lorentzian case, for example in Lorentzian formulation of electrodynamics, A is almost always the electromagnetic potential instead of the magnetic potential. It satisfies $dA = F$ where F is the electromagnetic field tensor. In this chapter, in the Lorentzian case we shall refer to A as the electromagnetic potential.

The wave equation (5.1.6) is globally well-posed for $f \in H^1_{loc}(\partial M)$ by [70, Theorem 24.1.1]. We define the DN map as

$$\Lambda_{g,A,q} : H^1_{loc}(\partial M) \ni f \mapsto (\partial_\nu - i\langle A, \nu\rangle)u|_{\partial M} \in \bar{L}^2_{loc}(M^\circ) \tag{5.1.7}$$

where ν is the unit outward normal vector on the boundary, and $\bar{L}^2_{loc}(M^\circ)$ refers to the space of restrictions to M° of locally L^2 functions on M (see [70, Theorem 24.1.1] and [70, Page 113]). Again if $\Psi : M \to M$ is a boundary fixing diffeomorphism and $d\omega$ is a smooth exact 1-form such that $\omega|_{\partial M} = 0$, then

$$\Lambda_{g,A,q} = \Lambda_{\Psi^* g, \Psi^* A + d\omega, \Psi^* q}. \tag{5.1.8}$$

The existence of the potential term allows for another gauge equivalence. Namely, let ϕ be a smooth function on M such that $\phi|_{\partial M} = \partial_\nu \phi|_{\partial M} = 0$, then

$$\Lambda_{g,A,q} = \Lambda_{e^{-2\phi} g, A + d\omega, e^{2\phi}(q - q_\phi)}, \quad q_\phi = e^{\frac{n-2}{2}\phi} \Box_g e^{\frac{2-n}{2}\phi}. \tag{5.1.9}$$

We refer to [140] for details of the gauge equivalences.

The Lorentzian Calderón problem asks if the metric g is uniquely determined by the DN map $\Lambda_{g,A,q}$ up to these natural gauges. Suppose the electromagnetic field is allowed to perturb, namely we have the information of $\Lambda_{g,A,q}$ for a family of different A. In this case, one can use this perturbation to reconstruct the conformal class of the metric g.

Theorem 5.1.3. *Suppose (M, g) is non-trapping and admissible. Let $\mathcal{B}_\delta(\mathcal{A})$ be the set of smooth 1-forms that are δ close to some fixed smooth 1-form $\mathcal{A}$ in the C^k topology for some k. For any $U \subset M^\circ$ open, denote the set of perturbations of $\mathcal{A}$ in U as*

$$\mathcal{B}_\delta(\mathcal{A}; U) = \{A \in \mathcal{B}_\delta(\mathcal{A}) : supp(A - \mathcal{A}) \subset U\}.$$

If L is the lens relation (see Definition 5.3.2) and $L(x', \xi') = (y', \eta')$, then the unique null-geodesic corre-

sponding to $(y', \eta'; x', \xi')$ is given by

$$\gamma = \{x \in M^\circ : \forall U \ni x \text{ open}, \exists A \in \mathcal{B}_\delta(\mathcal{A}; U) \text{ s.t.} \\ \sigma(\Lambda_{g,A,q})|_{(y',\eta';x',\xi')} \neq \sigma(\Lambda_{g,\mathcal{A},q})|_{(y',\eta';x',\xi')}\}. \tag{5.1.10}$$

Moreover, the set of such A is generic in $\mathcal{B}_\delta(\mathcal{A}; U)$. In particular, the conformal class of g can be constructed from $\{\Lambda_{g,A,q} : A \in \mathcal{B}_\delta(\mathcal{A})\}$.

Remark 5.1.4. *For the definition of principal symbol $\sigma(\Lambda_{g,A,q})$, see Section 5.3.1. A Lorentzian manifold is called non-trapping if any inextendible null-geodesic has two end points in the boundary. When the manifold is non-trapping and admissible (see Definition 5.3.1), all null-geodesics in the interior would eventually reach the boundary transversally, which makes the computation easier. We emphasize that the idea of the proof works for more general Lorentzian manifolds:*

1. *For example, the strictly convexity can be relaxed, then besides Proposition 5.3.3, one also needs to analyze the propagation of singularity along gliding rays or diffractive rays.*
2. *The non-trapping condition can also be removed, and one can only construct the null-geodesics that are non-trapping.*
3. *If the DN map is the partial data version as in [158], then one can construct all the broken null-geodesics that travels from source region to observation region.*

Finally, just like Theorem 5.1.1, since we work on a fixed base manifold, the perturbation of A eliminates the possibility of having boundary fixing isometry as a gauge equivalence.

Intuitively, the theorem states that perturbations of the electromagnetic potential in a test region can help determine whether the null-geodesic passes through it or not. We emphasize that the theorem is robust in the sense that if $x \in \gamma$, then a generic local perturbation of the electromagnetic potential in a neighborhood of x would reveal this fact. Moreover, one does not need to have any a priori information about $\mathcal{A}$, the only thing required is the ability to add perturbation at various locations. It is worth mentioning that if $A - \mathcal{A} = d\omega$ and $\omega|_{\partial M} \equiv 0$, then by the gauge equivalence the DN map does not change. In fact the solution simply changes from u to $e^{i\omega}u$. In physics, this corresponds to $\mathcal{A} + d\omega$ and $\mathcal{A}$ represent the same electromagnetic field tensor F. On the other hand, they still represent the same electromagnetic field F if $A - \mathcal{A}$ is a closed 1-form, but we no longer have the simple change in solutions as in the exact case. As a result, a change in closed 1-form may be used to detect the trajectory of null-geodesics. Specifically, in the proof we will show that the criterion is given by the shift in phase, in physics this is closely related to the well-known phenomenon called the Aharonov–Bohm effect, we refer to Remark 5.3.4 for more details. See also [1, 2].

Besides perturbing the electromagnetic potential, a similar problem can be considered when the perturbation is with respect to other parameters in the equation. In particular, we study the case where the metric itself is allowed to perturb. Consider the wave equation

$$\begin{cases} (\Box_g + q_g)\, u = 0 \\ u|_{\partial M} = f \\ \operatorname{supp}(u) \subset J^+(\operatorname{supp}(f)) \end{cases} \tag{5.1.11}$$

where the potential term q_g may depend on g locally. That is, $q_g(x)$ depends on g only on an arbitrarily small neighborhood of x. For instance, the potential term in the Yamabe equation

$$\left(\Box_g - \frac{n-2}{4(n-1)} R_g\right) u = 0 \tag{5.1.12}$$

is related to scalar curvature R_g, which depends on g. Simple computation shows that a conformal change of g which is the identity map on the boundary does not change the DN map. On the other hand, the Klein-Gordon equation

$$\left(\Box_g + m^2\right) u = 0 \tag{5.1.13}$$

provides an example where the potential term is independent of g. We show that the conformal class of the metric can also be recovered using perturbation of the metric.

Theorem 5.1.5. *Suppose* (M, g) *is non-trapping and admissible, denote* Λ_g *the DN map of* (5.1.11) *with*

q_g depending on g locally. If L is the lens relation (see Definition 5.3.2) and $L(x', \xi') = (y', \eta')$, then the unique null-geodesic associated with $(y', \eta'; x', \xi')$ can be constructed by the following equivalence conditions.

1. *γ passes through $x \in M^\circ$.*

2. *For any $U \ni x$ open neighborhood and conormal distribution f singular at (x', ξ'), there exists smooth symmetric 2-tensor h compactly supported in U such that (y', η') is in the wavefront set of $\partial_\epsilon|_0 \Lambda_{g-\epsilon h}(f)$.*

Moreover, the set of such h is generic in the space of smooth symmetric 2-tensors compactly supported in U with respect to C^k topology for any k. In particular, let $\mathcal{B}_\delta(g)$ be the set of smooth Lorentzian metrics that are δ close to g in C^k topology for some k, then the conformal class of g can be constructed from $\{\Lambda_{g'} : g' \in \mathcal{B}_\delta(g)\}$.

Clearly, one actually recovers the conformal class of all the metrics in $\mathcal{B}_\delta(g)$.

5.1.1 History

Both the Riemannian Calderón problem and the Lorentzian Calderón problem are widely open for dimension $n \geq 3$.

The Calderón problem originated from the Electrical Impedance Tomography (EIT) problem considered by Calderón in [22]. EIT asks the following: given a bounded domain $\Omega \subset \mathbb{R}^n$ with an unknown (possibly anisotropic) conductivity tensor $\gamma(x)$, can one recover γ from the Dirichlet-to-Neumann map which sends imposed voltages on $\partial\Omega$ to the resulting boundary currents on $\partial\Omega$? Equivalently, the DN map encodes exactly the voltage-to-current measurements at the boundary. In the anisotropic setting and for $n \geq 3$, this inverse boundary value problem is precisely Calderón's original formulation of recovering the conductivity tensor from boundary data (up to the natural obstruction of boundary fixing diffeomorphisms). For the 2-dimensional Calderón problem, besides the boundary fixing isometry, a boundary fixing conformal change is also a gauge equivalence. The dimension 2 case has been fully solved in [94], but the higher dimension Calderón problem remains one of the central open questions in the theory of inverse problems. For more conductivity results, we refer to [141, 79, 80, 59, 11]; and we refer to [125, 144] for more details on the history.

When $n \geq 3$ and the metric is real analytic, the problem is solved in [93, 94, 98]. In [87], Lassas, Liimatainen and Salo proved the same result using a different approach. They used Poisson embedding to identify interior points with distributions on the boundary. When the metric is only C^∞, in [38] and [48], it was shown that a metric in a fixed conformal class is uniquely determined by boundary measurements if the metric is conformally transversally anisotropic.

The Lorentzian Calderón problem is relatively new comparing to the Riemannian version, but there has been some important progress in the past decade. One of the most important results is the recovery of the potential term in [5] and [6]. In [5], Alexakis, Feizmohammadi and Oksanen proved a type of unique continuation principle when the metric is given under some geometric assumptions, and used it to determine the potential term. When $A = 0$ and the conformal class of g is given, determining g reduces to determining the potential term q. As a result, they recovered the conformal factor under some a priori conditions on the metric. In [6], they proved the same result for metrics sufficiently close to the Minkowski metric. For other results related to the recovery of potential term, see [129, 45, 74, 120]. Note that both of Theorem 5.1.3 and Theorem 5.1.5 recover the conformal class of the metric, one can then combine with results from [5] and [6] to recover the conformal factor.

In [42, 43], Eskin proved the Lorentzian Calderón problem when the metric and potential are both real analytic in time. The main tool used there is Boundary Control method, which was introduced by Belishev in [17] and is closely related to the Unique Continuation Principle, see [19, 18, 142, 122] for details.

As for the recovery of conformal class of the metric, one important result comes from when the equation has non-linearity. Specifically, in [83], Kurylev, Lassas and Uhlmann proved that the conformal class of the metric can be determined when the wave equation is non-linear and of the form

$$\Box_g u + a(x)u^2 = 0.$$

They pioneered the use of higher order linearization and microlocal analysis to generate earliest light observation sets, which recovers the conformal class of the metric. The earliest light observation set is essentially the shape of the light cone intersected with the observation set. The method highly relies on the non-linearity, as it allows the waves to interact with each other and create a collision point in the interior. The same idea also helps recover the non-linearity itself, and the study of non-linear hyperbolic inverse problems has been an active research area ever since, for example see [152, 68, 25, 26, 112, 96, 156, 124, 151] and the surveys [86, 150]. This approach, however, does not work for linear equations, since two waves do not interact with each other.

Finally, there has been some boundary determination results. In [140], Stefanov and Yang proved that the local DN map is a pseudodifferential operator, which stably recovers the jet bundle of g, A and q on the boundary. They also proved that away from the diagonal, the DN map is a Fourier Integral Operator with canonical relation being the lens relation (see Section 5.3.1). Given the metric, they proved stable recovery of lens relation and light ray transform of A and q on the boundary. In [158], the author and Zhang proved boundary determination under partial data assumption using the Fourier Integral Operator part of the DN map. Specifically, the Dirichlet initial data is supported on a fixed region U, while the observation is made on a fixed subset of the boundary V, and U and V are disjoint from each other. In both [140] and [158], there is no interior recovery of the metric.

To the author's knowledge, metric recovery from a family of DN maps indexed by the perturbation of parameters appears to be new in both Riemannian and Lorentzian settings. Classical Calderón results typically assume a single DN map and focus on boundary determination or single-map interior uniqueness. Almost all of the existing results are obtained under additional structural assumptions of the metric (for example, analyticity [94], CTA structures [38, 48], or curvature bounds [45]), and many of them require the metrics to be in the same conformal class. In contrast, we add no such assumptions on the metric structure, but instead work with an active, coefficient-parametric data model: an open family of magnetic or electromagnetic perturbations and the associated family of DN maps.

5.1.2 Main ideas

We now briefly discuss the ideas behind the proofs and make comparison between the Riemannian and Lorentzian versions. As mentioned before, one of the key points in recovering the interior metric is the ability to focus on an interior point. Non-linearity and microlocal analysis in the Lorentzian setting provide this ability via wave interactions and propagation of singularities. However, in a linear equation the solutions do not interact, and in the Riemannian case there is no propagation.

To solve the first problem, note that even though the operator is linear, its dependence on the parameters may be non-linear. Specifically, in our case, both $\Delta_{g,A}$ and $\Box_{g,A,q}$ depend non-linearly on A and on g itself, which provides the possibility of obtaining focused information around interior points via local perturbation. For the Lorentzian case, we again use microlocal analysis, which allows us to focus on a single null-geodesic and analyze the propagation of singularities. If the null-geodesic γ does not travel through some interior point x, then a smooth perturbation in a sufficiently small neighborhood of x should not affect the propagation of the singularity along γ. The main work lies in showing that if γ does pass through x, then some particular perturbation around x would affect the propagation of the singularity, and that this change can be picked up from the boundary observation. We actually show that such perturbations are generic. Together, they provide a criterion for determining whether γ passes through a point, thereby recovering the trajectory of the null-geodesic.

In Theorem 5.1.3, we perturb the electromagnetic potential to probe the propagation of particles along null-geodesics. We show in the proof that this only affects the spin of the particles, and hence the trajectory can be recovered by detecting changes in particle spin. In Theorem 5.1.5, we use the perturbation of the metric itself. The trajectory of the chosen null-geodesic may change in this case. One needs to show that this indeed happens for some perturbation, and this change can be detected from boundary measurements.

The same method obviously does not work for the Riemannian case, since propagation is a feature specific to hyperbolic equations. In order to solve this problem for the Riemannian setting, we turn to the Runge Approximation Theorem, which is only enjoyed by elliptic and parabolic equations. Again, even though there is no propagation, we can use the non-linear dependence on A and local perturbation to focus around an interior point. We show in Theorem 5.1.1 that this focusing gives us information of the gradient of the solution, and turns the problem into determining whether the following transformation gives a gauge equivalence.

Suppose B is an isomorphism on T^*M, and up to a diffeomorphism we may assume $B : T^*_xM \to T^*_xM$ for all $x \in M$. Suppose B is the identity map on the boundary, and the boundary is connected. Consider some Riemannian metric g, and suppose

$$\Delta_g v = 0 \implies B(dv) \text{ is exact.} \tag{5.1.14}$$

Consider the new metric given in local coordinates by

$$\tilde{g}^{ij} = \det(B)^{\frac{1}{n-2}} (B^{-1})^j_k g^{ik}. \tag{5.1.15}$$

For every v such that $\Delta_g v = 0$, $B(dv) = du$ for some u, and $dv = du$ on boundary. By adding a constant, we may assume $u|_{\partial M} = v|_{\partial M}$. Since by construction g and $\tilde{g}$ also agree on the boundary, they have the same unit outward normal vector ν, so $\partial_\nu u|_{\partial M} = \partial_\nu v|_{\partial M}$. Finally, by construction,

$$\Delta_{\tilde{g}} u = |\tilde{g}|^{-\frac{1}{2}} \partial_j (|\tilde{g}|^{\frac{1}{2}} \tilde{g}^{jk} \partial_k u) = |\tilde{g}|^{-\frac{1}{2}} \partial_j (|\tilde{g}|^{\frac{1}{2}} \tilde{g}^{jk} B^i_k \partial_i v) = \frac{|g|^{\frac{1}{2}}}{|\tilde{g}|^{\frac{1}{2}}} \Delta_g v = 0. \tag{5.1.16}$$

Hence B gives a gauge equivalence $\Lambda_g = \Lambda_{\tilde{g}}$. We use a Runge Approximation type of result to show that the only such transformation in the Riemannian setting is the identity map, which proves Theorem 5.1.1.

5.1.3 Outline of the chapter

The structure of the chapter is as follows.

The Riemannian version is studied in Section 5.2. In Section 5.2.1, we study the aforementioned transformation; in Section 5.2.2, we prove Theorem 5.1.1.

The Lorentzian version is studied in Section 5.3. In Section 5.3.1, we include the necessary tools and results; in Section 5.3.2, we prove Theorem 5.1.3; in Section 5.3.3, we prove Theorem 5.1.5.

5.2 Riemannian Calderón problem

5.2.1 Preliminaries

We first include a Runge Approximation type of result which will be used later, for a proof see for example the Appendix of [87].

Proposition 5.2.1 ([87, Proposition A.5])**.** *Let Γ be a non-empty open subset of ∂M. Let $p \in M^\circ \cup \Gamma$, let $a_0 \in \mathbb{R}$, let $\xi_0 \in T_p^*M$, and let H_0 be a symmetric 2-tensor at p satisfying $tr_g(H_0) = 0$. There exists $f \in C_c^\infty(\Gamma)$ such that the solution of*

$$-\Delta_g u = 0 \quad \textit{in } M, \quad u|_{\partial M} = f \tag{5.2.1}$$

satisfies $u(p) = a_0$, $du|_p = \xi_0$ and $Hess_g(u)|_p = H_0$.

It is easy to show that if an isomorphism B on T^*M maps all exact 1-forms to closed 1-forms, then up to a diffeomorphism on M, B has to be a locally constant scaling. By composing with a diffeomorphism we may assume B maps each T_x^*M to T_x^*M. If u is a smooth function on M, then both du and $udu = \frac{1}{2}d(u^2)$ are exact 1-forms. By the assumption, we have

$$0 = d(B(udu)) = d(uB(du)) = du \wedge B(du). \tag{5.2.2}$$

Since this holds for any u, this shows $B = fI$ where I is the identity map and f is a smooth function. Since $B(du) = fdu$ is closed for any u, this shows f must be locally constant.

With the help of Runge Approximation Theorem, one can show that the same result holds if the isomorphism maps all divergence free exact 1-forms to closed 1-forms.

Lemma 5.2.2. *Let M be a smooth connected manifold of dimension $n \geq 3$, B is a fiber-wise isomorphism on T^*M such that $B(T_x^*M) = T_x^*M$ for all $x \in M$. Suppose for every smooth ψ satisfying*

$$\Delta_g \psi = 0 \tag{5.2.3}$$

the image $B(d\psi)$ is a closed 1-form, then $B = CI$ for some constant $C \neq 0$ and I is the identity map.

Proof. Fix some interior point p, let $x = (x^1, \ldots, x^n)$ be the normal coordinate at p. That is, the metric is the Kronecker delta at p and the Christoffel symbol vanishes at p. Let H be any symmetric matrix such that $\mathrm{tr}(H) = 0$, note that at p this is the same as $\mathrm{tr}_g(H) = 0$ in normal coordinates. by Proposition 5.2.1, there exists some solution ψ such that $d\psi|_p = 0$ and $\mathrm{Hess}_g(\psi)|_p = H$. Since we choose the normal coordinate, this simply means $\partial^2_{jk}\psi|_p = H_{jk}$. Use the assumption that $B(d\psi)$ is closed, at p we have

$$0 = d(B(d\psi))|_p = d(B^j_k \partial_j \psi dx^k)|_p = B^j_k H_{jl} dx^l \wedge dx^k. \tag{5.2.4}$$

Thus BH is symmetric for any symmetric H with zero trace. For all pairs of $i \neq j$, test B with the following matrices:

1. $H_{ii} = 1$, $H_{jj} = -1$ and 0 everywhere else;
2. $H_{ij} = H_{ji} = 1$ and 0 everywhere else.

Together we obtain B must be of the form $B = fI$ for some smooth function f. On the other hand, if we choose ψ to be such that $\mathrm{Hess}_g(\psi)|_p = 0$ and $d\psi|_p = dx^i$ for $i = 1, \ldots, n$, then at p we have

$$0 = d(f\partial_j\psi dx^j)|_p = \partial_l f dx^i \wedge dx^l. \tag{5.2.5}$$

As a result, we obtain $f \equiv C$ for some constant $C \neq 0$. ☐

5.2.2 Perturbation of the Magnetic Potential

We first recall the boundary determination result from [98]. The proposition has a specific construction formula, but we shall only state it as a deterministic result since it is all we need, for details we refer to the original paper.

Proposition 5.2.3 ([98, Proposition 1.3])**.** *Suppose M has dimension $n \geq 3$, consider any local coordinate $(x^1, \cdots, x^{n-1})$ on the boundary around some $p \in \partial M$. Then the Taylor series of g at p in boundary normal coordinate is determined by Λ_g.*

Proof of Theorem 5.1.1: We first work with arbitrary metric g. Let h be a smooth 1-form supported in the interior that will be determined later. Consider the perturbed Laplace equation

$$\Delta_{g,\varepsilon h} u_\varepsilon = 0, \quad u_\varepsilon|_{\partial M} = f. \tag{5.2.6}$$

Then $u_\varepsilon = u + i\varepsilon v^h + O(\varepsilon^2)$, where u and v^h solves

$$\Delta_g u = 0, \quad u|_{\partial U} = f; \tag{5.2.7}$$

$$\Delta_g v = h(\nabla_g u) + \div_g(uh^\sharp), \quad v|_{\partial M} = 0, \tag{5.2.8}$$

where $\nabla_g u$ is the gradient of u with respect to g. For notation simplicity, denote derivative of DN map as

$$N(f,h) = -i\partial_\varepsilon|_0 \Lambda_{g,\varepsilon h}(f) = \partial_\nu v|_{\partial M}. \tag{5.2.9}$$

Note that we do not have the $\langle h, \nu\rangle$ term because h supports in the interior. Denote V_g the volume form on M with respect to g, and $V_{g,\partial}$ the volume form on ∂M with respect to $g|_{T\partial M}$. Using divergence theorem and the fact that h supports in the interior, we have

$$\int_{\partial M} N(f,h) V_{g,\partial} = \int_{\partial M} \partial_\nu v V_{g,\partial} = \int_M \Delta_g v V_g = \int_M h(\nabla_g u) V_g. \tag{5.2.10}$$

From Proposition 5.2.3, the metric on the boundary is known, in particular this means the left hand side of the above equation is fully computable.

We now choose h. For every fixed point x_0 in the interior let $x = (x^1, \cdots, x^n)$ be a local coordinate in the interior. Let

$$\varphi^\delta(x) = \delta^{-n} \varphi\left(\frac{x - x_0}{\delta}\right)$$

be a compactly supported approximate of identity at x_0 as $\delta \to 0$. Then for any smooth 1-form ω, we have

$$\int_M \varphi^\delta \omega(\nabla_g u) |g|_x^{\frac{1}{2}} dx \xrightarrow{\delta \to 0} |g(x_0)|^{\frac{1}{2}} \omega(\nabla_g u)|_{x_0}, \tag{5.2.11}$$

where $|g|_x$ is the determinant of g in coordinate x. This means for every initial data f and every point x_0 in the interior, we can explicitly compute

$$\lim_{\delta \to 0} \int_{\partial M} N(f, \varphi^{\delta}_{x_0} \omega) V_{g,\partial} = |g(x_0)|^{\frac{1}{2}} \omega(\nabla_g u)|_{x_0}. \tag{5.2.12}$$

Invariantly speaking, we can thus compute the coupled term $\nabla_g u V_g$. If $\Lambda_{g_1,A} = \Lambda_{g_2,A}$ for all small A, this means

$$\nabla_{g_1} u_1 V_{g_1} = \nabla_{g_2} u_2 V_{g_2} \tag{5.2.13}$$

where u_1 and u_2 solves

$$\Delta_{g_j} u_j = 0, \quad u_j|_{\partial M} = f, \quad j = 1, 2. \tag{5.2.14}$$

This gives a coordinate invariant relationship between du_1 and du_2, via

$$du_1 = \frac{|g_2|^{\frac{1}{2}}}{|g_1|^{\frac{1}{2}}} g_1 g_2^{-1} du_2, \tag{5.2.15}$$

where $g_2^{-1} du_2$ refers to the lift of du_2 to a vector field via g_2, and $g_1 g_2^{-1} du_2$ is the lift of $g_2^{-1} du_2$ to a 1-form again via g_1. The equation is clearly coordinate independent, and in local coordinate we have

$$\partial_j u_1 = \frac{|g_2|^{\frac{1}{2}}}{|g_1|^{\frac{1}{2}}} (g_1)_{jl} (g_2)^{lk} \partial_k u_2 := B^k_j \partial_k u_2. \tag{5.2.16}$$

The operator B is thus a fiber-wise isomorphism on the cotangent bundle, mapping the differential of the solution of the Laplace equation to an exact one form.

By Lemma 5.2.2, $B = CI$ for some constant $C \neq 0$ and I is the identity map. That is,

$$\frac{|g_2|^{\frac{1}{2}}}{|g_1|^{\frac{1}{2}}} g_1 g_2^{-1} = CI. \tag{5.2.17}$$

Since we assumed the dimension $n \geq 3$, take determinant both sides and simplify, we obtain

$$\frac{|g_2|^{\frac{1}{2}}}{|g_1|^{\frac{1}{2}}} = C^{\frac{n}{n-2}}. \tag{5.2.18}$$

Hence

$$g_1 g_2^{-1} = C^{-\frac{2}{n-2}} I. \tag{5.2.19}$$

Use the fact that g_1 and g_2 agree on boundary tangential directions by Proposition 5.2.3, $C = 1$. That is, $g_1 = g_2$. □

5.3 Lorentzian Calderón problem

5.3.1 Preliminaries

We will be working on admissible Lorentzian manifolds, the definition is a slight modification of [66, Definition 1.1]. (The definition has made its appearance in Chapter 4, we restate it here for readers' convenience.)

Definition 5.3.1. *Let $n \geq 3$. Let (M, g) be a smooth connected n-dimensional Lorentzian manifold with non-empty boundary; thus, g has signature $(-, +, \dots, +)$. We call (M, g)* admissible *if*

1. *there exists a smooth, proper, surjective function $\tau \colon M \to \mathbb{R}$, such that $d\tau$ is everywhere timelike;*

2. *the boundary ∂M is timelike, i.e. the induced metric $\bar{g} := g|_{T\partial M \times T\partial M}$ is Lorentzian;*

3. *∂M is* strictly null-convex*: if ν denotes the outward pointing unit normal vector field on ∂M, then*

$$II(V, V) = g(\nabla_V \nu, V) \geq 0 \tag{1.1}$$

for all null vectors $V \in T_p \partial M$, where II is the second fundamental form.

We briefly introduce concepts in microlocal analysis, for details we refer to [69, 40, 70, 71, 39, 65]. Given a smooth submanifold K, we denote N^*K its conormal bundle. A distribution u is called a conormal distribution if locally it can be written as an oscillatory integral of the form

$$u(x) = \int e^{i\phi(x,\theta)} a(x,\theta) d\theta$$

where ϕ is the phase function locally parameterizing N^*K, and a is a smooth symbol, see [69]. More generally, when N^*K is replaced with some Lagrangian submanifold of T^*M, the distribution is called a Lagrangian distribution. An operator mapping functions on X to functions on Y is called a Fourier Integral Operator (FIO) if the corresponding Schwartz kernel is a Lagrangian submanifold of $T^*Y \times T^*X$ with respect to the twisted symplectic form. In this case the Lagrangian submanifold is also referred to as a canonical relation. When $X = Y$ and the Lagrangian submanifold is the diagonal of $T^*X \times T^*X$, we call the operator a pseudodifferential operator. The wavefront set of any Lagrangian distribution is a subset of the Lagrangian submanifold, see [70, 71].

For any operator P acting on distributions on a Lorentzian or Riemannian manifold (M, g), there exists a natural way to turn it into an operator acting on half-density valued distributions (for definition of half-density, see [65]). For any half-density valued distribution ω, define

$$\tilde{P}\omega = |g|^{\frac{1}{4}} P(|g|^{-\frac{1}{4}}\omega).$$

Here $|g|^{\frac{1}{4}} = |g|_x^{\frac{1}{4}} |dx|^{\frac{1}{2}}$ where $|g|_x$ is the determinant of g in local coordinate x, and $|dx|^{\frac{1}{2}}$ is the standard half-density in coordinate x. Note that this conjugation obviously depends on the metric, one can also conjugate P with other half-densities to avoid this. The benefit of $\tilde{P}$ as an operator on half-density valued distribution is that if P is an FIO, then the principal symbol of $\tilde{P}$, denoted by $\sigma(\tilde{P})$, can be invariantly defined modulo lower order terms as a half-density on the Lagrangian submanifold (for simplicity we ignore the Maslov bundle), see [69] for details.

When the Dirichlet data is a conormal distribution corresponding to some conormal bundle N^*K, then the solution is a Lagrangian distribution corresponding to the flowout Lagrangian of N^*K. The flowout Lagrangian refers to the union of the null-bicharacteristics originated from $N^*K \cap \mathcal{H}$ where $\mathcal{H}$ is the hyperbolic set (for definition of hyperbolic set and null-bicharacteristic, we refer to [70, Section 24]). We also refer to

[140] for an optics construction of the solution.

The DN map $\Lambda_{g,A,q}$ has been proved to contain both pseudodifferential part and Fourier Integral Operator part, see for example [140] and [158]. The corresponding canonical relation for the FIO part is given by the powers of lens relation. Denote $\mathcal{H}$ the hyperbolic set and $\pi : T^*M|_{\partial M} \to T^*\partial M$. We can define the lens relation L on a non-trapping admissible Lorentzian manifold as follows.

Definition 5.3.2. *Let (M, g) be a non-trapping admissible Lorentzian manifold. For any $(x', \xi') \in \mathcal{H}$, there exists a unique inward pointing lightlike direction ξ such that $\pi(x', \xi) = (x', \xi')$. Let Γ be the unique future pointing null-bicharacteristic from (x', ξ), and (y', η) is the exit point of Γ. The lens relation is the map*

$$L : \mathcal{H} \to \mathcal{H}, \quad (x', \xi') \to \pi(y', \xi').$$

The projection of Γ on M is a null-geodesic, and we will refer to this null-geodesic as the unique geodesic corresponding to $(y', \eta'; x', \xi')$.

For more details of lens relation and DN map on a Lorentzian manifold, see for example [140] and [158]. Let $U, V \subset \partial M$ be disjoint, denote $\Lambda^{U,V}_{g,A,q}$ the DN map restricted to $V \times U$, that is the Dirichlet data f in (5.1.6) is supported in U and

$$\Lambda^{U,V}_{g,A,q}(f) = \partial_\nu u|_V.$$

In [158], the principal symbol of $\Lambda^{U,V}_{g,A,q}$ is computed.

Proposition 5.3.3 ([158, Proposition 3.5])**.** *View $\Lambda^{U_0,U_k}_{g,A,q}$ as an operator on half-density valued distributions via*

$$\Lambda^{U_0,U_k}_{g,A,q}(f|\bar{g}|^{1/4}) := (\Lambda^{U_0,U_k}_{g,A,q} f)|\bar{g}|^{1/4}.$$

Suppose $L^k(x_0', \xi^{0\prime}) = (x_k', \xi^{k\prime})$ for some $k \geq 1$, $x_0' \in U_0$, $x_k' \in U_k$, and L is the lens relation. Then microlocally in a conic neighborhood of $(x_k', \xi^{k\prime}; x_0', \xi^{0\prime}) \in T^*U_k \times T^*U_0$, the operator $\Lambda^{U_0,U_k}_{g,A,q}$ is an elliptic Fourier integral operator of order 1 whose associated canonical relation is given by

$$\{(L^k(x', \xi'); x', \xi') : (x', \xi') \in T^*U_0 \cap \mathcal{H}\}.$$

Its principal symbol is given by

$$\sigma(\Lambda_{g,A,q}^{U_0;U_k})|_{(x_k',\xi^{k\prime};x_0',\xi^{0\prime})} = \pm 2i(-1)^k \exp\left(i\int_{\gamma^b} A(\dot{\gamma}^b)\right) |\xi_n^k|^{1/2}|\xi_n^0|^{1/2}(L^{-k})^*,$$

where

- γ^b *is the future pointing broken null-geodesic that is the projection of the future pointing broken bicharacteristic connecting*

  $$(x_0', \xi^{0\prime}) \to (x_1', \xi^{1\prime}) \to \cdots \to (x_k', \xi^{k\prime})$$

  (note that $\int_{\gamma^b} A(\dot{\gamma}^b)$ *is independent of parametrization of* γ^b*);*

- *the sign is* $+/-$ *if* $(x_0', \xi^{0\prime})$ *is a past/future pointing covector.*

Finally, we record the well-known equation for computing principal symbols of wave equation which will be used in the proof of perturbation of the metric. If $\tilde{u}$ and $\tilde{v}$ are a half-density valued Lagrangian distribution, $P_g = |g|^{\frac{1}{4}}(\Box_g + q)|g|^{-\frac{1}{4}}$, and $P_g\tilde{u} = \tilde{v}$, then

$$\mathcal{L}_{H_p}\sigma(\tilde{u}) = i\sigma(\tilde{v}),$$

where H_p is the Hamiltonian vector field with respect to the principal symbol of P_g and $\mathcal{L}_{H_p}$ is the Lie derivative along the flowout Lagrangian. We refer to [40] for more details.

5.3.2 Perturbation of The Electromagnetic Potential

In this section, we prove Theorem 5.1.3.

Proof of Theorem 5.1.3. By [140, Theorem 4.3], the lens relation L can be recovered from the DN map. Let $L(x', \xi') = (y', \eta')$, then they are connected by a unique null-bicharacteristic, whose projection γ is a null-geodesic. Note that perturbing A does not change the trajectory of the null-geodesic, but only adds spin to it. In particular, by Proposition 5.3.3 this means for any $A \in \mathcal{B}_\delta(\mathcal{A})$,

$$\left.\frac{\sigma(\Lambda_{g,A,q})}{\sigma(\Lambda_{g,\mathcal{A},q})}\right|_{(y',\eta';x',\xi')} = \exp\left(i\int_\gamma (A-\mathcal{A})(\dot\gamma)\right).$$

Suppose γ does not pass through point x, then it is clear that for all A such that $A-\mathcal{A}$ is supported in a sufficiently small neighborhood of x,

$$\left.\frac{\sigma(\Lambda_{g,A,q})}{\sigma(\Lambda_{g,\mathcal{A},q})}\right|_{(y',\eta';x',\xi')} = 1. \tag{5.3.1}$$

On the other hand, suppose $x = \gamma(t_0)$ for some t_0, and let U be some neighborhood of x, then $\gamma(t) \in U$ for $t \in [t_0-\varepsilon, t_0+\varepsilon]$ and some $\varepsilon > 0$. Consider some $\phi \in C_c^\infty(\mathbb{R})$ such that $\mathrm{supp}(\phi) \subset (-\varepsilon,\varepsilon)$ and $\int \phi = 1$. Let A' be such that

$$A'(\dot\gamma(t)) = \phi(t-t_0), \quad t \in (t_0-\varepsilon, t_0+\varepsilon).$$

One can easily extend A' to be a smooth 1-form supported in U, and divide by a sufficiently large number C so that it is δ close to 0 in C^k topology. In particular, this shows that $A = \mathcal{A} + A' \in \mathcal{B}_\delta(\mathcal{A}; U)$ satisfies

$$\left.\frac{\sigma(\Lambda_{g,A,q})}{\sigma(\Lambda_{g,\mathcal{A},q})}\right|_{(y',\eta';x',\xi')} = \exp\left(\frac{i}{C}\int_{-\varepsilon}^{\varepsilon}\phi\right) \neq 1.$$

The two cases give the criterion of whether γ passes through a point x, hence we proved (5.1.10).

To show the set of such A is generic in $\mathcal{B}_\delta(\mathcal{A}; U)$ when $x \in \gamma$ and U is a neighborhood of x, we need to show it is open and dense. Suppose $A \in \mathcal{B}_\delta(\mathcal{A}; U)$ satisfies

$$\left.\frac{\sigma(\Lambda_{g,A,q})}{\sigma(\Lambda_{g,\mathcal{A},q})}\right|_{(y',\eta';x',\xi')} \neq 1. \tag{5.3.2}$$

then it is clear that it holds in a sufficiently small neighborhood of A in $\tilde{B}_\delta(\mathcal{A}; U)$ with respect to C^k topology. On the other hand, to show density, if A satisfies (5.3.1), same argument shows that there exists some B

arbitrarily close to A such that $\mathrm{supp}(B - A) \subset U$ and

$$\left.\frac{\sigma(\Lambda_{g,B,q})}{\sigma(\Lambda_{g,A,q})}\right|_{(y',\eta';x',\xi')} \neq 1.$$

Hence one can find $B \in \mathcal{B}_\delta(\mathcal{A}; U)$ arbitrarily close to A and (5.3.2) holds for B and $\mathcal{A}$.

Finally, the conformal class can be constructed because every non-trapped null-geodesic can be constructed and the manifold is non-trapping.

□

Remark 5.3.4. *It is easy to see from the proof that when constructing the perturbation, we don't want $A - \mathcal{A}$ to be an exact 1-form supported in the interior, otherwise the integral along the geodesic would still give 0. Another way to see this is that if $A - \mathcal{A} = d\phi$ for some compactly supported ϕ in the interior, then the DN map is the same for A and $\mathcal{A}$ by the gauge equivalence (see [140, Lemma 2.2]). A change by closed form, on the other hand may change the DN map, and one can use them to detect trajectory. This aligns with the Aharonov-Bohm effect in physics (see [1, 2]) , which roughly speaking states that the particle may experience a phase shift caused by the electromagnetic potential even when the induced electromagnetic field is 0. Mathematically, A provides a connection on the principal $U(1)$-bundle, and $F = dA$ is the curvature of the connection. The Aharonov-Bohm effect is simply saying that the holonomy depends not just on the curvature, but also the topology of the path.*

5.3.3 Perturbation of The Metric

In this section, we prove Theorem 5.1.5.

Proof of Theorem 5.1.5. Denote $g_\epsilon = g - \epsilon h$ for h some compactly supported smooth symmetric 2-tensor that will be determined later. Let u_ϵ and u be the solution to (5.1.11) with respect to g_ϵ and g, respectively. Then

$$u_\epsilon = u + \epsilon v + O(\epsilon^2)$$

with v solves

$$\begin{cases} (\Box_g + q_g)v = -|g|^{-\frac{1}{2}}\partial_j(|g|^{\frac{1}{2}}h_g^{jk}\partial_k u) + \frac{1}{2}\partial_j(\mathrm{tr}_g(h))g^{jk}\partial_k u - q^1 u \\ v|_{\partial M} = 0, \quad \partial_\nu v|_{\partial M} = \partial_\epsilon|_0 \Lambda_{g_\epsilon}(f), \end{cases} \tag{5.3.3}$$

where $q_{g_\epsilon} = q_g + \epsilon q^1 + O(\epsilon^2)$, $\mathrm{tr}_g(h) = \mathrm{tr}(g^{-1}h)$, and h_g^{jk} is the jk-entry of the matrix $g^{-1}hg^{-1}$. Note that q^1 depends on g and h, and since the potential depends locally on the metric, $\mathrm{supp}(q^1) \subset \mathrm{supp}(h)$. Let $P_g = |g|^{\frac{1}{4}}\Box_g|g|^{-\frac{1}{4}}$, $\tilde{u} = u|g|^{\frac{1}{4}}$ and $\tilde{v} = v|g|^{\frac{1}{4}}$, the equation for the half-densities is

$$(P_g + q_g)\tilde{v} = -h_g^{jk}\partial_k\partial_j u|g|^{\frac{1}{4}} + l.o.t. \tag{5.3.4}$$

where $l.o.t.$ denotes the lower order terms, that is the terms that only contains u or ∇u.

Let $L(x', \xi') = (y', \eta')$ and γ be the unique null-geodesic corresponds to $(y', \eta'; x', \xi')$. Let $K \subset \partial M$ be such that the conormal bundle N^*K contains (x', ξ'). If $f = u|_{\partial M}$ is a conormal distribution with respect to N^*K, then u is a Lagrangian distribution with canonical relation being the flowout Lagrangian from the hyperbolic part of N^*K along the Hamiltonian vector fields (see for example [68], or the optics construction in [140]). As a result, v is a Lagrangian distribution on the same flowout Lagrangian (one can show this by a symbolic construction as in [101], see also [158]). In particular, the principal symbol satisfies the following transport equation

$$\mathcal{L}_{H_p}\sigma(\tilde{v}) = ih_g^{jk}\xi_j\xi_k\sigma(\tilde{u})$$

where recall that H_p is the Hamiltonian vector field with respect to g and $\mathcal{L}_{H_p}$ is the Lie derivative along the flowout Lagrangian. Let $w_1, \ldots, w_n$ be linearly independent vector fields on the flowout Lagrangian obtained via pushforward along the Hamiltonian flow φ_t, that is

$$w_j|_{\varphi_t(x,\xi)} = \varphi_{t*}(w_j|_{(x,\xi)}), \quad \mathcal{L}_{H_p}w_j = 0.$$

Then we have

$$H_p(\sigma(\tilde{v})(w_1, \cdots, w_n)) = (\mathcal{L}_{H_p}\sigma(\tilde{v}))(w_1, \cdots, w_n) = ih_g^{jk}\xi_j\xi_k\sigma(\tilde{u})(w_1, \cdots, w_n).$$

Denote $\Gamma : [0,T] \to T^*M$ a null-bicharacteristic in the flowout Lagrangian, and let γ be its projection, the principal symbol of $\tilde{v}$ satisfies

$$\sigma(\tilde{v})|_{\Gamma(t)}(w_1, \cdots, w_n) = iC \int_\gamma (g^{-1}hg^{-1})(\dot{\gamma}^\flat, \dot{\gamma}^\flat) = iC \int_\gamma h(\dot{\gamma}, \dot{\gamma})$$

where we used the fact that

$$\sigma(\tilde{v})|_{\Gamma(0)}(w_1, \cdots, w_n) = 0, \quad \sigma(\tilde{u})|_{\Gamma(t)}(w_1, \cdots, w_n) = C \neq 0 \quad \forall t \in [0,T].$$

Now suppose $x \in \gamma$, U is some neighborhood of x and N^*K contains (x', ξ'). It is clear that a generic choice of smooth symmetric 2-tensor h supported in U would satisfy that

$$\int_\gamma h(\dot{\gamma}, \dot{\gamma}) \neq 0.$$

That is to say, for any Dirichlet data f which is a conormal distribution singular at (x', ξ'), a generic choice of smooth symmetric 2-tensor h supported in U gives that $\partial_\epsilon|_0 \Lambda_{g-\epsilon h}(f)$ is not smooth at (y', η').

On the other hand, if $x \notin \gamma$, then for sufficiently small neighborhood U of x and sufficiently small K, the projection of the flowout Lagrangian from the hyperbolic part of N^*K is disjoint from U. From (5.3.3) and the assumption that potential term depends locally on g,

$$(\Box_g + q_g)v \in C^\infty(M^\circ), \quad v|_{\partial M} = 0.$$

As a result, $\partial_\epsilon|_0 \Lambda_{g-\epsilon h}(f)$ is smooth for any smooth symmetric 2-tensor h supported in U.

Finally, the conformal class can be constructed because every non-trapped null-geodesic can be constructed and the manifold is non-trapping.

$\square$

Remark 5.3.5. *An obvious choice of h such that*

$$\int_\gamma h(\dot{\gamma}, \dot{\gamma}) = 0$$

is when $h = fg$ where f is any compactly supported function. This choice of h corresponds to perturbation within the conformally equivalence class, which does not change the DN map when working with Yamabe equation (5.1.12)*. Note that it does not mean we can not use this h for other types of potential terms, since for many of them conformal equivalence is not a gauge equivalence. This simply suggests that one should check lower order terms of* (5.3.3) *in this case. In general, a simple choice can be*

$$h = g \begin{pmatrix} \phi & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} = \begin{pmatrix} -\phi & 0 \\ 0 & 0 \end{pmatrix}$$

for some generic compactly supported ϕ, in any semi-geodesic normal coordinate (see for example [140, Lemma 2.3] or [118])

$$g = -(dx^1)^2 + g_{\alpha\beta}(x) dx^\alpha \otimes dx^\beta, \quad \alpha, \beta \geq 2.$$

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Part II

Geometric inverse problems for Lorentzian manifolds

Chapter 6

Rigidity problems for analytic Lorentzian manifolds

The content of this chapter corresponds a joint work with Yang Zhang [159]. The main results are Theorem 6.1.1, Theorem 6.1.2, Theorem 6.1.4 and Theorem 6.1.4.

6.1 Introduction

Let (M, g) be a Lorentzian manifold with timelike boundary ∂M, where g has the signature $(-, +, \cdots, +)$. A geodesic is said to be *non-trapping* if it exits M in finite time, so that no information can remain confined in the interior. In this chapter, we study several rigidity problems for Lorentzian manifolds whose lightlike geodesics are non-trapping, in the analytic category.

Boundary rigidity concerns the determination of metric from its boundary distance function. However, these distance functions behave quite differently for Riemannian manifolds and Lorentzian manifolds. In a Riemannian manifold, the distance between two arbitrary points x and y is defined as the infimum of the lengths of all piecewise smooth curves connecting x and y. It endows the manifold a metric structure that is compatible with its topology. In a Lorentzian manifold, by contrast, since it is not physically meaningful for signals to travel faster than the speed of light, the distance function is typically defined only for causal curves. Moreover, because in physical applications Lorentzian models are usually time-orientable, we assume throughout that the manifold is time-oriented when discussing the distance function. A connected, time-oriented smooth Lorentzian manifold is also called a *spacetime*, see [16, Definition 3.1]. More explicitly, let

$$LM = \{(x, v) \in TM \setminus 0 : g(v, v) = 0\}$$

be the light cone bundle. Time-orientablility provides a global and continuous distinction between the future and past-directed causal vectors. For $x \in M$, let $J^+(x)$ denote its causal future, see Section 6.2.1. The *Lorentzian distance function*, or more commonly known as the *time separation function*, is defined by

$$d(x, y) = \sup\{L(\alpha) : \alpha \text{ future pointing piecewise smooth causal curve from } x \text{ to } y\},$$

whenever $y \in J^+(x)$, and $d(x, y) = 0$ otherwise. Here L denotes the Lorentzian arc length, see Section 6.2.2. Then, we consider the following boundary rigidity problem.

Problem 4 (Lorentzian boundary rigidity)**.** *For $j = 1, 2$, let (M_j, g_j) be spacetimes with timelike boundaries. Suppose there exists a boundary diffeomorphism $\varphi_0 : \partial M_1 \to \partial M_2$ such that*

$$d_1(x, y) = d_2(\varphi_0(x), \varphi_0(y)), \quad \forall x, y \in \partial M_1.$$

Can φ_0 be extended to a global isometry?

Another commonly studied rigidity problem is the scattering rigidity problem. Since many definitions coincide with those for Riemannian manifolds, here we may regard (M, g) either as a compact Riemannian manifold with boundary, or as a Lorentzian manifold with timelike boundary. We denote the inward $(-)$ and outward $(+)$ pointing vectors on the boundary by

$$\partial_\pm TM = \{(x, v) \in \partial TM : \pm g(v, \nu) > 0\},$$

where ν is the unit outward pointing normal vector field on ∂M. Clearly,

$$\partial TM = \partial_- TM \cup \partial_+ TM \cup T\partial M, \quad \overline{\partial_\pm TM} = \partial_\pm TM \cup T\partial M.$$

There are two similar but different definitions of scattering information in the literature: one records the information when the geodesic first exits M°; the other records the information when the geodesic completely leaves M. See Figure 6.1.

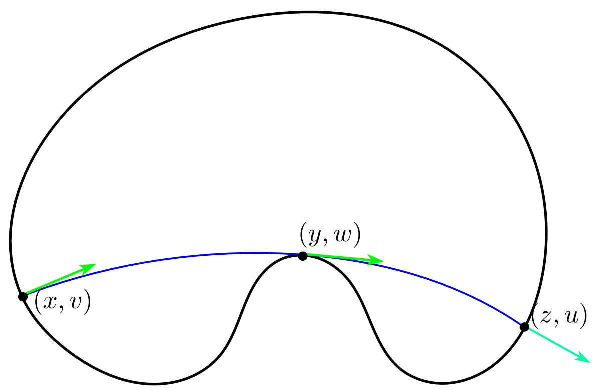

Figure 6.1: For $(x, v) \in \partial_- TM$, denote by γ the corresponding geodesic Let (y, w) be the point and direction at which γ exits M° for the first time (equivalently the first time γ reaches ∂M). Let (z, u) be the point and direction at which γ completely exits M. The complete scattering relation records both transitions $(x, v) \to (z, u)$ and $(y, w) \to (z, u)$. The interior scattering relation, in contrast, records only the first intersection with the boundary, i.e., $(x, v) \to (y, w)$. The second transition $(y, w) \to (z, u)$ is not recorded by the interior scattering relation, as at (y, w) the geodesic is tangential to the boundary.

For $(x, v) \in \partial_- TM$, denote by γ the corresponding geodesic. We define the *interior travel time* as

$$\tau^{in}(x, v) = \sup\{t > 0 : \gamma((0, t)) \subset M^\circ\}.$$

If $\tau^{in}(x, v)$ is finite, we define the *interior scattering relation* and the *interior length function* as

$$S^{in}(x, v) = (\gamma(\tau^{in}(x, v)), \dot{\gamma}(\tau^{in}(x, v))),$$

$$\ell^{in}(x, v) = \tau^{in}(x, v) \cdot |g(v, v)|^{1/2}.$$

The pair (S^{in}, ℓ^{in}) is called the *interior lens data*, and the pair (S^{in}, τ^{in}) is called the *interior travel time data*. Note that the domain of the interior scattering information is a subset of $\partial_- TM$, excluding trapping and boundary tangential directions.

Now let $(\tilde{M}, \tilde{g})$ be an extension of (M, g). For any $(x, v) \in \overline{\partial_- TM}$, as before, let γ be the corresponding geodesic in $\tilde{M}$. We define the *complete travel time* as

$$\tau(x, v) = \sup\{t \geq 0 : \gamma([0, t]) \subset M\}.$$

If $\tau(x, v)$ is finite, we define the *complete scattering relation* and the *complete length function* as

$$S(x, v) = (\gamma(\tau(x, v)), \dot{\gamma}(\tau(x, v))),$$
$$\ell(x, v) = \tau(x, v) \cdot |g(v, v)|^{1/2}.$$

The pair (S, ℓ) is called the *complete lens data*, and the pair (S, τ) is called the *complete travel time data*. Note that the domain of the complete scattering information is a subset of $\overline{\partial_- TM}$, which may include boundary tangential directions when they are non-trapping. Moreover, they do not depend on the choice of extension.

For both Riemannian manifolds with boundaries and Lorentzian manifolds with timelike boundaries, the unit outward normal vector field ν on the boundary is well-defined. We say that $(x, v) \in T\partial M$ is a *strictly convex direction*, if the second fundamental form is positive, i.e.,

$$\mathrm{I\!I}(v, v) := g(\nabla_v \nu, v) > 0. \tag{6.1.1}$$

In particular, if all boundary tangential vectors are strictly convex directions, we say the boundary is strictly convex.[1] In this case, the interior and complete scattering information coincide, since any geodesic in M intersects the boundary transversally. Moreover, in a Lorentzian manifold with timelike boundary, if all boundary tangential lightlike vectors are strictly convex directions, then we say the manifold is strictly null-convex (see [66]). In this case, the interior and complete scattering relations for lightlike geodesics also

[1] More commonly, for a Riemannian manifold with boundary, strict convexity refers to the property that any two sufficiently close boundary points are connected by a distance minimizing geodesic that stays in the interior except at its endpoints. This property implies $\mathrm{I\!I}$ is positive semi-definite. Conversely, a positive definite $\mathrm{I\!I}$ ensures this property. See, for example, [56, Section 3.2]. In this chapter, we will only use the notion of a strictly convex direction.

coincide.

In general, the interior and complete scattering information are different, and recovering one from the other is not obvious. Complete scattering information appears, for example, in [153, 57, 137]. In applications, it models through-transmission setups where rays are detected only after they fully exit the body, such as medical or industrial CT, cross-well seismics, and through-thickness ultrasound. In these setups, the recorded data are the entry and final exit states (possibly with travel time). Meanwhile, interior scattering information has also made its appearance, for example, in [135, 27, 155, 138, 33]. Interior scattering relation matches first-arrival measurements made on the boundary, as in reflection seismology, ground-penetrating radar, or surface-mounted acoustic sensing, where a signal is registered as soon as it first returns to the boundary even if the ray later glides along it or exits elsewhere. We study how the two definitions are related to each other in Section 6.6.

For $j = 1, 2$, suppose (M_j, g_j) are either both compact Riemannian manifolds with boundaries or both Lorentzian manifolds with timelike boundaries. Furthermore, suppose there exists a boundary isometry $\varphi_0 : \partial M_1 \to \partial M_2$, that is,

$$\bar{g}_1 = \varphi_0^* \bar{g}_2, \qquad \text{with } \bar{g}_j = g_j|_{T\partial M_j \times T\partial M_j},$$

where $\bar{g}_j$ is the boundary metric. In boundary normal coordinates, one may write (x, v) as (x, v', v^n), where the boundary is locally given by $x^n = 0$ and the interior corresponds to $x^n > 0$. By abuse of notation, we view $(\varphi_0)_*$ as

$$(\varphi_0)_* : \partial TM_1 \to \partial TM_2, \quad (x, v', v^n) \mapsto (\varphi_0(x), (\varphi_0)_* v', v^n),$$

which maps unit normal directions to unit normal directions. In particular, as φ_0 is a boundary isometry, under these notations we have

$$g_1|_{\partial TM_1 \times \partial TM_1} = \varphi_0^*(g_2|_{\partial TM_2 \times \partial TM_2}). \tag{6.1.2}$$

Then we have the following two scattering rigidity problems.

Problem 5 (Interior scattering rigidity)**.** *For $j = 1, 2$, let (M_j, g_j) be either both compact Riemannian* *manifolds with boundaries or both Lorentzian manifolds with timelike boundaries. Suppose there exists a boundary isometry* $\varphi_0 : \partial M_1 \to \partial M_2$ *such that*

$$(\varphi_0)_* \circ S_1^{in} = S_2^{in} \circ (\varphi_0)_*$$

in boundary normal coordinates and on the domain of S_1^{in} *(which, under the non-trapping assumption, is typically all of* $\partial_- TM_1$ *in the Riemannian case). Can* φ_0 *be extended to a global isometry?*

Problem 6 (Complete scattering rigidity)**.** *For* $j = 1, 2$*, let* (M_j, g_j) *be either both compact Riemannian manifolds with boundaries or both Lorentzian manifolds with timelike boundaries. Suppose there exists a boundary isometry* $\varphi_0 : \partial M_1 \to \partial M_2$ *such that*

$$(\varphi_0)_* \circ S_1 = S_2 \circ (\varphi_0)_*$$

in boundary normal coordinates and on the domain of S_1 *(which, under the non-trapping assumption, is typically all of* $\overline{\partial_- TM_1}$ *in the Riemannian case). Can* φ_0 *be extended to a global isometry?*

For inverse problems, one often omits the subindex j and states the result in a *deterministic* way, without reference to a particular manifold. Thus, Problem 2 (or 3) asks whether the metric g on the entire M can be determined by the given data

$$g|_{\partial TM \times \partial TM} \quad \text{and} \quad S^{in} (\text{or } S).$$

We emphasize that the metric is given on ∂TM instead of $T\partial M$ because $(\varphi_0)_*$ preserves the length of any vector in ∂TM_j, see (6.1.2).

In this chapter, we focus on Lorentzian manifolds with timelike boundaries. As a result, it is natural to focus on scattering information associated with causal geodesics. In particular, we will assume that lightlike geodesics are non-trapping, and work with both the interior and complete scattering information for causal geodesics that are sufficiently close to the light cone.

Finally, in much of the literature for Riemannian manifolds, scattering rigidity is also studied through the *projected scattering relation* (for example, see [135, 72]). Here we demonstrate this using the complete scattering relation; the interior version is similar. Let (M, g) be a Riemannian manifold with boundary, and SM is the spherical bundle. The scattering relation is thus defined from $\overline{\partial_- SM}$ to $\overline{\partial_+ SM}$. Let $\kappa_\pm : \overline{\partial_\pm SM} \to \overline{B\partial M}$ be the projection maps with respect to the metric, where $\overline{B_x \partial M}$ is the closed unit ball in $T_x \partial M$. Note that $\kappa_\pm$ are diffeomorphisms. The projected scattering relation is given by $S^\kappa = \kappa_+ \circ S \circ \kappa_-^{-1}$. For simplicity, suppose (M_1, g_1) and (M_2, g_2) share the same boundary $\partial M = \partial M_1 = \partial M_2$, and $g_1|_{T\partial M \times T\partial M} = g_2|_{T\partial M \times T\partial M}$. Then $S_1^{\kappa_1} = S_2^{\kappa_2}$ is equivalent to the equivalence of S_1 and S_2 under their own boundary normal coordinates.

One can similarly define the projected scattering relation for Lorentzian manifolds. However, it is slightly more complicated as there are two natural projections: the projection for unit timelike vectors $\kappa_\pm$, and the projection for lightlike vectors $\mu_\pm$. Both of them are diffeomorphic onto the image but their images overlap. Specifically, for $v' \in T\partial M$ such that $g(v', v') < -1$, in boundary normal coordinates the timelike and lightlike preimages are

$$\kappa_\pm^{-1}(v') = v' \mp \sqrt{-1 - g(v', v')}\partial_n,$$
$$\mu_\pm^{-1}(v') = v' \mp \sqrt{-g(v', v')}\partial_n.$$

Then using these notation, the timelike projected scattering relation can be defined as $S^\kappa = \kappa_+ \circ S \circ \kappa_-^{-1}$ and the lightlike one can be defined as $S^\mu = \mu_+ \circ S \circ \mu_-^{-1}$. Similar to the Riemannian version, the scattering rigidity problem for Lorentzian manifolds can also be equivalently stated using the projected scattering relations. We emphasize that we do not use the projected version to avoid causing unnecessary confusion.

6.1.1 Main results

We now describe the three main rigidity results for analytic Lorentzian manifolds, concerning the determination of an unknown region from exterior time separation data, the boundary rigidity problem in the presence of timelike boundary, and the scattering rigidity problem from information near the light cone.

We first study the case in which the unknown region is embedded in a larger analytic globally hyperbolic Lorentzian manifold, and we assume that lightlike geodesics do not have cut points there. We show that, given the time separation function on a thin exterior layer of the unknown region, the unknown region can be determined up to isometry.

Theorem 6.1.1. *For $j = 1, 2$, let (N_j, g_j) be an analytic globally hyperbolic Lorentzian manifold of dimension $n \geq 3$, and let d_j be the corresponding time separation function. Let $K_j \subset\subset \tilde{M}_j \subset N_j$ be such that K_j is a compact subset and $\tilde{M}_j$ is an open neighborhood of K_j. Denote by $K_j^c = \tilde{M}_j \setminus K_j$. Suppose there exists a diffeomorphism $\varphi_0 : K_1^c \to K_2^c$ such that*

$$d_1(x, y) = d_2(\varphi_0(x), \varphi_0(y)), \quad \forall x, y \in K_1^c,$$

and such that φ_0 extends continuously to a bijection between ∂K_1 and ∂K_2. Suppose further every lightlike geodesic segment in K_j has no cut point. Then there exists an analytic isometry $\varphi : \tilde{M}_1 \to \tilde{M}_2$ such that $\varphi|_{K_1^c} = \varphi_0$.

We emphasize that no regularity assumption is imposed on the unknown region K_j, for $j = 1, 2$; it may be any compact subset. The exterior region K_j^c may also be an arbitrarily thin layer. To prove this theorem, we use the complete travel time data for lightlike geodesics with respect to a smaller neighborhood of K_j, for $j = 1, 2$. This smaller neighborhood is more convenient to work with, since the boundary of K_j may have low regularity. We reconstruct the lightlike complete travel time data from the exterior time separation function, see Section 6.3.2. Then in Proposition 6.3.1, inspired by [153], we show that this lightlike information establishes a correspondence between the two neighborhoods, which preserves the geometric structure and can be extended to an analytic isometry that matches the given exterior identification. For more details, see Section 6.3.

Next, we study the boundary rigidity problem for analytic globally hyperbolic spacetimes with timelike boundaries. For the definition of globally hyperbolic spacetimes with timelike boundaries, see Section 6.2.1. Here by an analytic spacetime with timelike boundary, we mean a smooth manifold with boundary whose transition maps are analytic, the metric is analytic, and the boundary is timelike. Certainly, the boundary and boundary metric are analytic with respect to the induced analytic structure. We require that after passing to a small analytic collar neighborhood, see Section 6.2.3, the extension is still globally hyperbolic and contains no lightlike cut points. We also require the existence of a strictly convex direction on the boundary, for the recovery of the jet of boundary metric.

Theorem 6.1.2. *For $j = 1, 2$, let (M_j, g_j) be an analytic globally hyperbolic spacetime with timelike boundary of dimension $n \geq 3$, and suppose lightlike geodesics are non-trapping. Assume there exists an analytic collar neighborhood extension (see Section 6.2.3), which is globally hyperbolic and contains no lightlike cut points. Suppose there exists an analytic diffeomorphism $\varphi_0 : \partial M_1 \to \partial M_2$ such that*

$$d_1(x, y) = d_2(\varphi_0(x), \varphi_0(y)), \quad \forall x, y \in \partial M_1.$$

Moreover, assume for any connected component C of ∂M_1, there exists a causal direction $(x, v) \in TC$ such that (x, v) and $(\varphi_0(x), (\varphi_0)_* v)$ are strictly convex directions in (M_1, g_1) and (M_2, g_2), respectively. Then there exists an analytic isometry $\varphi : M_1 \to M_2$ such that $\varphi|_{\partial M} = \varphi_0$.

Comparing to the exterior rigidity problem in Theorem 6.1.1, boundary rigidity is more subtle due to the presence of the boundary. Indeed, since we do not assume the timelike boundary to be strictly convex, the distance maximizing curve may no longer be a geodesic. To prove this theorem, by Proposition 6.3.1, it suffices to reconstruct the lightlike complete travel time data with respect to M_j, for $j = 1, 2$, from the boundary time separation function d_j. For this purpose, we first determine the jet of the metric on the boundary, allowing for an analytic extension. The exterior time separation function $\tilde{d}_j$ can be constructed in this extension, from the knowledge of d_j. As the maximal distance is not necessarily realized by a causal geodesic, we use both $\tilde{d}_j$ and d_j to recover the lightlike complete travel time data, for more details see Section 6.4. Note that we require no lightlike cut points in the extension to avoid the case that a boundary point of M becomes a cut point in the extended manifold.

In [90], the authors show that the universal covering of an analytic Lorentzian spacetime (M, g) is determined by the time separation function restricted to a small timelike submanifold contained in M, assuming that (M, g) is geodesically complete modulo scalar curvature singularities. As only local information is assumed, one can only expect uniqueness up to an isometry of the universal covering space, not of the manifold itself. In contrast, in this chapter, we study the boundary rigidity problem for globally hyperbolic Lorentzian manifolds with timelike boundary. As a manifold with boundary, it is always geodesically incomplete. Moreover, since we have global information, we can completely determine the metric up to isometry. In [131], the local lightlike scattering rigidity near the boundary for analytic metrics is established. For a more comprehensive overview of previous results, see Section 6.1.2.

Finally, we study the scattering rigidity problem for analytic Lorentzian manifolds with timelike boundaries. In this setting, we do not impose any causality conditions. To state the result, we first introduce a mild non-conjugacy condition, which comes from [135].

Definition 6.1.3. *Let (M, g) be a semi-Riemannian manifold with boundary. Consider a boundary tangential direction $(x, v) \in T\partial M$. Let γ be the corresponding geodesic. The direction (x, v) is said to satisfy the* non-conjugacy condition*, if γ is non-trapping and x is not conjugate to any point in $\gamma \cap \partial M$.*

Note that in a Lorentzian manifold with strictly null-convex timelike boundary, any boundary tangential timelike vector sufficiently close to the light cone will satisfy this condition. Recall that, by abuse of notation, we can view the differential of a boundary isometry φ_0 as a map from ∂TM_1 to ∂TM_2, in boundary normal coordinates. For simplicity, we denote by

$$JM = \{(x, v) \in TM \setminus 0 : g(v, v) \leq 0\}$$

the set of causal vectors in M. Let $\partial_\pm JM$ denote the inward $(-)$ and outward $(+)$ pointing causal vectors on the boundary. We show that, under proper assumptions, both the interior and complete scattering rigidity hold.

Theorem 6.1.4. *For $j = 1, 2$, let (M_j, g_j) be an analytic Lorentzian manifold of dimension $n \geq 3$ with analytic timelike boundary. Assume all lightlike geodesics are non-trapping. Then, for sufficiently small conic neighborhoods $\mathcal{U}_j \subset \partial TM_j$ containing ∂LM_j, the interior scattering relation S_j^{in} is well-defined in $\mathcal{V}_j^{in} :=$ $\mathcal{U}_j \cap \partial_- JM_j$. Moreover, assume for each connected component C of ∂M_1, there exists $(x, v) \in TC \cap \overline{\mathcal{V}_1^{in}}$*

such that the non-conjugacy condition holds. If there exists a boundary isometry $\varphi_0 : \partial M_1 \to \partial M_2$ such that

$$(\varphi_0)_*(\mathcal{V}_1^{in}) = \mathcal{V}_2^{in},$$

$$(\varphi_0)_* \circ S_1^{in}|_{\mathcal{V}_1^{in}} = S_2^{in}|_{\mathcal{V}_2^{in}} \circ (\varphi_0)_*,$$

then there exists an analytic isometry $\varphi : M_1 \to M_2$ such that $\varphi|_{\partial M_1} = \varphi_0$.

Theorem 6.1.5. *For $j = 1, 2$, let (M_j, g_j) be an analytic Lorentzian manifold of dimension $n \geq 3$ with analytic timelike boundary. Assume all lightlike geodesics are non-trapping. Then, for sufficiently small conic neighborhoods $\mathcal{U}_j \subset \partial TM_j$ containing ∂LM_j, the complete scattering relation S_j is well-defined in $\mathcal{V}_j \coloneqq \mathcal{U}_j \cap \overline{\partial_- JM_j}$. Moreover, assume for each connected component C of ∂M_1, there exists $(x, v) \in TC \cap \mathcal{V}_1$ such that the non-conjugacy condition holds. If there exists a boundary isometry $\varphi_0 : \partial M_1 \to \partial M_2$ such that*

$$(\varphi_0)_*(\mathcal{V}_1) = \mathcal{V}_2,$$

$$(\varphi_0)_* \circ S_1|_{\mathcal{V}_1} = S_2|_{\mathcal{V}_2} \circ (\varphi_0)_*,$$

then there exists an analytic isometry $\varphi : M_1 \to M_2$ such that $\varphi|_{\partial M_1} = \varphi_0$.

The proof of Theorem 6.1.4 and Theorem 6.1.5 relies on certain boundary determination results, which are developed in Section 6.5 and are inspired by the ideas in [135]. To establish rigidity from both the interior and complete scattering relations, we study their connections in Section 6.6. In particular, in Section 6.6.3, we prove that both the lightlike interior and complete travel time data can be recovered from either the interior or the complete scattering relation for timelike vectors sufficiently close to the light cone, see Lemma 6.6.4 - 6.6.6. All results in Section 6.6 do not rely on any causality or analyticity assumptions on the manifold, or on any convexity assumptions on the timelike boundary. Instead, we derive these results using geometric arguments and the first variation formula for the travel time in Section 6.6.2. Then, by applying Proposition 6.3.1 again, we complete the proof of these two theorems in Section 6.7.

6.1.2 Previous literature

For Riemannian manifolds, a geometric formulation of boundary distance rigidity goes back to Michel [102]. In [117], it is proved that two dimensional compact simple Riemannian manifolds are boundary distance rigid. Besides, this result is known to hold for simple subdomains in Euclidean spaces, in an open hemisphere in two dimensions, in symmetric spaces of constant negative curvature, and for two dimensional spaces of negative curvature, see [54, 102, 20, 29]. When one metric is sufficiently close to the Euclidean metric, boundary rigidity was established in [92] and later improved in [21]. In particular, for smooth metrics in dimensions $n \geq 3$, local and global boundary rigidity results under foliation conditions are established, see [145, 136, 137]. Lens and scattering data provide alternative boundary measurements beyond the boundary distance. Local lens rigidity with incomplete data for classes of non-simple manifolds was proved in [135]. In the presence of trapped geodesics, scattering rigidity and lens rigidity were obtained under different geometric assumptions, see [27, 32, 55]. In the analytic category, lens rigidity was established for non-trapping Riemannian manifolds in [153], and rigidity problems for such manifolds with an analytic magnetic field are considered as well, see [34, 64]. For other related works and generalizations, see [108, 109, 10, 91, 72, 35, 114, 30, 104, 20], and we refer the readers to surveys [138, 73]. These rigidity problems are closely related to the (anisotropic) Calderón problem for the Laplace–Beltrami operator, for example, see [93, 94, 98] for real-analytic Riemannian metrics and surveys [144, 125] for more results.

In Lorentzian manifolds, boundary distance rigidity is stated in terms of the time separation function. The recovery of stationary metrics from the time separation function is studied in [9] for two-dimensional product Lorentzian manifolds, and in [148] for dimension three and higher that are close to the Minkowski. In [90] the Lorentzian universal covering spaces of real-analytic Lorentzian manifolds can be recovered from local time separation function on a timelike hypersurface, with some convexity and geodesic completeness assumptions. The recovery of stationary metrics from the lens relation is studied in [130]. Then it is considered in [107] via timelike geodesics and MP-systems. Analogously, these Lorentzian rigidity problems are closely related to the Lorentzian Calderón problems, where we consider the recovery of the Lorentzian manifold (M, g) from the Dirichlet-to-Neumann map associated with a wave equation. Specifically, the Dirichlet-to-Neumann map determines the lightlike scattering relation in the Lorentzian setting. For ultrastatic manifold with stationary metrics or metrics real-analytic in t, this problem has been extensively studied in the literature, including [17, 42, 76, 84, 89]. The most recent progress is in [5, 6], where the assumption of real-analyticity in t is replaced by bounds on the Lorentzian curvature.

In both Riemannian and Lorentzian settings, boundary determination problems are considered usually as the first step towards interior recovery. In Riemannian geometry, boundary determination has been studied, with well-established results on uniqueness, stability, and constructive reconstruction methods, see [92, 133, 146] for cases with strictly convex boundaries, and [135, 160] for cases with concave points. In Lorentzian geometry, the determination of the jet of the metric on a timelike submanifold from the time separation function is considered in [90]. Boundary determination of the metric, the magnetic field, and the potential from the knowledge of the DN map has been studied in [140], and later discussed in [158] using partial data in disjoint boundary sets.

In contrast to the Lorentzian boundary and scattering rigidity results discussed above, we treat general analytic globally hyperbolic spacetimes and recover the metric globally from exterior time separation data, boundary time separation data, or from either the interior or complete scattering relation near the light cone. In the boundary data cases, we do not impose any global convexity assumption on the boundary, and our results remain valid even when the timelike boundary admits non-convex regions. The accompanying results between interior and complete scattering data seem to be new even for Riemannian manifolds.

6.1.3 Outline

In Section 6.2, we include some preliminary results and some useful lemmas. In Section 6.3, we prove Theorem 6.1.1. In Section 6.4, we prove Theorem 6.1.2. Section 6.5 to Section 6.7 are related to the scattering rigidity results. Specifically, in Section 6.5, we study the determination of the jet of metric on the boundary for both Riemannian manifold and Lorentzian manifold with timelike boundary. In Section 6.6, we study how the interior and complete scattering information relates to each other. The results from these two sections will be used in Section 6.7 to prove Theorem 6.1.4 and Theorem 6.1.5. Finally, Section 6.8 includes some transversal intersection results, and Section 6.9 includes the constructive recovery of the jet of metric on the boundary near strictly convex directions.

6.2 Preliminaries

In this section, we briefly recall some basic definitions and properties of Lorentzian manifolds that will be used (see also Chapter 2 for preliminaries of Lorentzian geometry).

6.2.1 Lorentzian geometry

Recall that a smooth Lorentzian manifold (M, g) is a *spacetime* if it is connected and has a time orientation. We say a smooth path $\mu : (a, b) \to M$ is *timelike,* if $g(\dot{\mu}(s), \dot{\mu}(s)) < 0$ for all $s \in (a, b)$. We say a smooth path $\mu : (a, b) \to M$ is *causal*, if $g(\dot{\mu}(s), \dot{\mu}(s)) \leq 0$ with $\dot{\mu}(s) \neq 0$ for all $s \in (a, b)$. Given $x, y \in M$, we say $x \ll y$ if there exists a future pointing piecewise smooth timelike curve from x to y. We say $x < y$ if there is a future pointing piecewise smooth causal curve from x to y. We write $x \leq y$ if $x = y$ or $x < y$. The *chronological future and past* of $x \in M$ are defined as $I^+(x) = \{y \in M : x \ll y\}$ and $I^-(x) = \{y \in M : y \ll x\}$ respectively. The *causal future and past* of $x \in M$ are defined as $J^+(x) = \{y \in M : x \leq y\}$ and $J^-(x) = \{y \in M : y \leq x\}$ respectively. We denote by $J(x, y) = J^+(x) \cap J^-(y)$ the *causal diamond set*. For a subset $U \subseteq M$, its causal futures (or past) are defined as $J^{\pm}(U) = \bigcup_{x \in U} J^{\pm}(x)$.

In a spacetime (M, g), a *pregeodesic* is a smooth curve whose image is a geodesic, possibly with a non-affine parameterization. An open set $U \subset M$ is called a *geodesically convex neighborhood*, provided U is a normal neighborhood of each of its points, see [113, Definition 5.5]. In particular, for any two points $x, y \in U$ there is a unique geodesic segment $\gamma : [0, 1] \to N$ that lies entirely in U, while there are possibly other geodesics from x to y that do not remain in U.

We say a spacetime is *causal* if it does not have any closed causal curves. For a spacetime (M, g) without boundary, we say it is *globally hyperbolic*, if it is strongly causal (see [113, Definition 14.11]) and every causal diamond is compact. In fact, it suffices to assume causal instead of strongly causal. This provides one possible definition for a subset to be globally hyperbolic (see [113, Definition 14.20]), i.e., $H \subset M$ is said to be a *globally hyperbolic subset*, if it satisfies the strong causality condition, and every causal diamond is compact in H. With this definition, one can then study subsets of M with timelike boundaries, and refer to them as globally hyperbolic Lorentzian manifolds with timelike boundaries. However, this definition is not intrinsic, since the strong causality depends on the ambient manifold. Throughout this chapter, we instead use the following intrinsic definition of globally hyperbolicity for manifolds with timelike boundaries.

Definition 6.2.1. *[3, Definition 2.15] Let (M, g) be a Lorentzian manifold with timelike boundary. It is globally hyperbolic if it is causal and the causal diamond set $J(x, y)$ is compact for any $x, y \in M$.*

The definition is intrinsic as we only consider causal curves in M. By [3, Corollary 5.8], every globally hyperbolic Lorentzian manifold with timelike boundary can be isometrically embedded into a globally hyperbolic Lorentzian manifold without boundary of the same dimension. Moreover, globally hyperbolic Lorentzian manifolds with or without boundaries are always time-orientable, so we may refer to them as globally hyperbolic spacetime.

6.2.2 Null and timelike cut locus

In a spacetime (M, g) without boundary, for $x < y$, we define the Lorentzian arc length

$$L(\alpha) = \int_0^1 \sqrt{-g(\dot{\alpha}(s), \dot{\alpha}(s))}\, \mathrm{d}s,$$

for a piecewise smooth causal path $\alpha : [0, 1] \to M$ from x to y. Note that this definition is independent of the parameterization of α. In addition, by [16, Remark 3.35], the Lorentzian arc length functional L is upper semicontinuous.

Recall the time separation function d for a Lorentzian manifold is the supremum of the arc length of all piecewise smooth causal curves from x to y, if $x \leq y$. It satisfies the reverse triangle inequality

$$d(x, y) + d(y, z) \leq d(x, z), \qquad \text{for } x \leq y \leq z.$$

For $(x, v) \in L^+M$, recall the *null cut locus function*

$$\rho(x, v) = \sup\{\, s \in [0, \mathcal{T}(x, v)) : d(x, \gamma_{x,v}(s)) = 0 \,\},$$

where $\gamma_{x,v}(s)$ is the unique lightlike geodesic starting from x in the direction v, and $\mathcal{T}(x, v)$ is supremum of the parameter value for which $\gamma_{x,v}(s) \in M$ is defined. For a timelike vector $(x, v) \in TM$, recall the *timelike cut locus function*

$$\rho(x, v) = \sup\{\, s \in [0, \mathcal{T}(x, v)) : d(x, \gamma_{x,v}(s)) = s|v|_g \,\},$$

where $\gamma_{x,v}(s)$ is the unique timelike geodesic starting from x in the direction v, see [16, Definition 9.3]. In both cases, when $\rho(x, v) < \mathcal{T}(x, v)$, we call $\gamma_{x,v}(\rho(x, v))$ the first cut point of x along the null or timelike geodesic $\gamma_{x,v}$. By [16, Theorem 9.12] and [16, Theorem 9.15], in a globally hyperbolic Lorentzian manifold without boundaries, the first cut point $\gamma_{x,v}(\rho(x, v))$ is either the first conjugate point to x along $\gamma_{x,v}$, or the first point on $\gamma_{x,v}$ at which there exists another distinct null or maximal timelike geodesic from x to $\gamma_{x,v}(\rho(x, v))$.

Moreover, we include the following lemma about the boundary of the causal future, in a globally hyperbolic Lorentzian manifold without boundary.

Lemma 6.2.2. *Let (M, g) be a globally hyperbolic Lorentzian manifold. Let $x, y \in M$ and $\partial J^+(x) := J^+(x) \setminus I^+(x)$ be the boundary of the causal future of x. The following statements are equivalent.*

1. *$y \in \partial J^+(x)$;*
2. *$y \in J^+(x)$ and $d(x, y) = 0$;*
3. *there exists $y_j \to y$ such that $d(x, y_j) > 0$ and $d(x, y_j) \to 0$;*
4. *there exists a future pointing lightlike geodesic from x to y entirely lying on $\partial J^+(x)$, with no cut points strictly before y.*

In particular, when there are no lightlike cut points, one has $\partial J^+(x) = \mathcal{L}^+_x$, where $\mathcal{L}^+_x$ is the forward light cone defined as the union of all lightlike geodesics starting from x.

Proof. First, we prove the statement (1) is equivalent to (2). Indeed, with $y \in J^+(x)$, one has $d(x, y) \geq 0$. Recall $y \in I^+(y)$ exactly when $d(x, y) > 0$. Then $y \in J^+(x)$ with $d(x, y) = 0$ if and only if $y \in J^+(x) \setminus I^+(x)$.

Next, we prove (2) is equivalent to (3). Indeed, as (M, g) is globally hyperbolic, the causal future $J^+(x)$ is closed and is the closure of the open set $I^+(x)$. Moreover, the time separation function $d(x, y)$ is continuous in $M \times M$. Suppose $y \in J^+(x)$ with $d(x, y) = 0$. Then we can find a sequence $y_j \in I^+(x)$ such that $y_j \to y$. By continuity, one has $d(x, y_j) \to d(x, y) = 0$. On the other hand, if (3) holds, then $y_j \in I^+(x)$ and therefore we must have $y \in J^+(x)$ with $d(x, y) = 0$ by continuity again.

Finally, using [4, Theorem 1], the statement (1) is equivalent to (4), see also [83, Lemma 2.3]. $\square$

In a Lorentzian manifold with timelike boundary, we use a slightly different definition for the cut locus function ρ, see (6.4.1) and (6.4.2). Roughly speaking, we define the cut locus in such a manifold (M, g) as the infimum time when the geodesic fails to be maximal. Now suppose (M, g) is embedded in a globally hyperbolic Lorentzian manifold (N, g) with or without timelike boundary, such that $M \subset N^\circ$. Let d_N be the time separation function and ρ_N be the cut locus in N. Then one has

$$d(x, y) \leq d_N(x, y), \quad \rho(x, v) \geq \rho_N(x, v),$$

for any $x, y \in N$ and for any lightlike or timelike $(x, v) \in TM$. For more details about cut points in a Lorentzian manifold with timelike boundary, see Section 6.4.1.

6.2.3 Normal exponential extension

Next, we briefly talk about the extension of a Lorentzian manifold with boundary by a collar neighborhood. This is usually a crucial first step for the recovery of interior metric. Given an analytic Lorentzian manifold (M, g) with timelike boundary, one can always extend the smooth manifold slightly to a larger smooth manifold $\tilde{M}$. When the extension is sufficiently small, one can analytically extend the metric g to $\tilde{g}$ on $\tilde{M}$. For every point $x \in \partial M$, let U_x be a small neighborhood of x, where the metric can be written in the boundary normal coordinates (see [140, Lemma 2.3]). Choose a subset of $\{U_x : x \in \partial M\}$ that is locally finite and covers ∂M. Note such subset can be made countable and we denote it by $\{U_j\}_{j=1}^{\infty}$. Take $V = \cup_j U_j$, and we shrink V if necessary. For any $y \in V$, there exists a unique $x \in \partial M$ such that y is on the normal geodesic from x in V. This is possible by shrinking V, since the cover is locally finite. Then the following map is well-defined:

$$\exp_\nu^{-1} : V \to \partial M \times (-\epsilon, \epsilon), \quad y = \exp_x(s\nu) \mapsto (x, s),$$

where ν is the unit normal vector field on ∂M. We refer to its inverse, $\exp_\nu$, as the *normal exponential map*, whose domain is the range of $\exp_\nu^{-1}$. Comparing to the compact Riemannian manifold case in [153, Section 2], the domain of $\exp_\nu$ in our setting may not be uniform in size, since M is not compact. Without loss of generality, we may assume $\tilde{M} = V \cup M$ in the first place. In this chapter, we refer to such an extension as the *analytic collar neighborhood extension*. Finally, we can always make sure that after the extension, the outer boundary is still timelike.

6.2.4 Limit curves

In a globally hyperbolic spacetime M with timelike boundary, we recall the following concept of limit curves, which are only used in Section 6.4.

Definition 6.2.3 ([16, Definition 3.28]). *A curve γ is a limit curve of the sequence $\{\gamma_n\}$ if there is a subsequence $\{\gamma_m\}$ such that for all p in the image of γ, each neighborhood of p intersects all but a finite number of curves of the subsequence $\{\gamma_m\}$.*

Note that a sequence of curves $\{\gamma_n\}$ may not have limit curves or may have many limit curves. Moreover, we consider the convergence of curves in C^0 topology as below.

Definition 6.2.4 ([16, Definition 3.33]). *Let γ and all sequence curves γ_n be defined on the closed interval $[a, b]$. Then $\{\gamma_n\}$ is said to converge to γ in the C^0 topology on curves if $\gamma_n(a) \to \gamma(a)$, $\gamma_n(b) \to \gamma(b)$, and given any open set Γ containing γ, we have $\gamma_n \subseteq \Gamma$ for sufficiently large n.*

As is explained in [16], in any spacetime, there exists a sequence $\{\gamma_n\}$ that has a limit curve γ, while it does not converge to γ in the C^0 topology. However, if the spacetime without boundary is strongly causal, then two types of convergence are almost equivalent for sequences of causal curves. Additionally, in a strongly causal spacetime (M, g) with timelike boundary, the same results hold, see Lemma 6.2.6.

Although the definitions of the causal structure and the time separation function are usually defined using piecewise smooth curve, the limit curve of a sequence of piecewise smooth curves may have lower regularity. Thus, we need the following definition about H^1-curves, when considering limit curves.

Definition 6.2.5 ([3, Definition 2.16]). *Let (M, g) be a spacetime with timelike boundary and $I \subset \mathbb{R}$ be any* *interval. A continuous curve $\gamma : I \to M$ is an H^1-causal curve if*

(1) in any local coordinates, γ restricted to any compact interval J belongs to $H^1(J;\mathbb{R}^n)$;

(2) and its almost everywhere derivative is causal.

Here the Sobolev space $H^1(J;\mathbb{R}^n)$ is the set of curves that are absolutely continuous and whose derivatives are L^2 integrable, in the compact interval.

One has the following lemmas about the limit curve of a sequence of H^1-causal curves, see [3].[2]

Lemma 6.2.6 ([3, Proposition 2.19 (1)])**.** *Suppose that $\{\gamma_n\}$ is a sequence of H^1-causal curves defined on $[a,b]$ such that $\gamma_n(a) \to p$, $\gamma_n(b) \to q$. A causal curve $\gamma\colon [a,b] \to M$, with $\gamma(a) = p$ and $\gamma(b) = q$, is a limit curve of $\{\gamma_n\}$ if and only if there is a subsequence $\{\gamma_m\} \subset \{\gamma_n\}$, which converges to γ in the C^0 topology.*

Lemma 6.2.7. *[3, Proposition 2.19 (2)] Let (M,g) be a globally hyperbolic spacetime with timelike boundary. Suppose that $\{p_n\}$ and $\{q_n\}$ are sequences in M converging to p and q in M, respectively, with $p \neq q$, and $p_n \leq q_n$ for each n. Let γ_n be a future pointing H^1-causal curve from p_n to q_n for each n. Then there exists a future pointing H^1-causal limit curve γ which joins p to q.*

Recall in a spacetime (M,g) with timelike boundary, the causal future $J^+(p)$ and past $J^-(p)$, for some $p \in M$, is defined using piecewise smooth causal curves. One may also consider their definitions using H^1-causal curves, i.e., $J^+_{H^1}(p)$ and $J^-_{H^1}(p)$.

Lemma 6.2.8 ([3, Proposition 2.22])**.** *Let (M,g) be a globally hyperbolic Lorentzian manifold with timelike boundary (in our conventions, always defined by causal futures and pasts using piecewise smooth casual curves). Then $J^\pm(p) = J^\pm_{H^1}(p)$ for all $p \in M$.*

This leads to the following lemma.

Lemma 6.2.9. *Let (M,g) a globally hyperbolic spacetime with timelike boundary. Let $x, y \in M$ and $\mu : [0,1] \to M$ be an H^1-causal curve from x to y. Then there exists a piecewise smooth casual curve connecting x and y.*

[2]Note that [3, Proposition 2.19] works for H^1-causal curves, see the remark before it and also [3, Convention 2.23].

Proof. Indeed, with the H^1-causal curve μ, we know $y \in J^+_{H^1}(x)$. It follows that $y \in J^+(x)$ and therefore there is a piecewise smooth causal curve from x to y.

□

6.3 The exterior rigidity problem

We prove Theorem 6.1.1 in this section. Recall that the ambient manifold N_j is globally hyperbolic, K_j is the unknown compact region, $\tilde{M}_j$ is an open neighborhood of K_j, and $K_j^c = \tilde{M}_j \setminus K_j$ is the exterior region. The proof proceeds through the following steps:

1. First we determine the metric in K_j^c, and find a middle layer M_j such that $K_j \subset\subset M_j \subset\subset \tilde{M}_j$ and M_j has piecewise analytic boundary.

2. Then using the exterior time separation function, we recover the complete travel time data (S, τ) for lightlike geodesics with respect to M_j.

3. Using the complete travel time data, we construct a map from L^+M_1 and M_2. We show that this map is actually fiber-preserving and therefore upgrade to a bijection between M_1 and M_2.

4. Finally, we extend this bijection and upgrade it to an analytic isometry between $\tilde{M}_1$ and $\tilde{M}_2$, which agrees with φ_0 in the exterior region.

We summarize Step (3) and Step (4) in the proposition below, as these arguments will also be used in the proof of Theorem 6.1.2, Theorem 6.1.4 and Theorem 6.1.5. For this proposition, we do not require M_j to be contained in a globally hyperbolic Lorentzian manifold, in contrast with the setting of Theorem 1.1. Instead, it suffices to assume that all lightlike geodesics in M_j are non-trapping. In the globally hyperbolic setting of Theorem 6.1.1, this non-trapping property follows automatically from the global hyperbolicity and compactness of M_j.

Proposition 6.3.1. *For $j = 1, 2$, let $(\tilde{M}_j, g_j)$ be analytic Lorentzian manifolds of dimension $n \geq 3$. Let $M_j \subset \tilde{M}_j$ be a closed subset in the interior with piecewise analytic boundary. Assume the lightlike geodesics* *are non-trapping in M_j. Suppose there exists an isometry $\varphi_0 : M_1^c \sqcup \partial M_1 \to M_2^c \sqcup \partial M_2$, where $M_j^c = \tilde{M}_j \setminus M_j$. Assume the lightlike complete travel time data are equivalent, that is*

$$(\varphi_0)_* \circ S_1 = S_2 \circ (\varphi_0)_*, \quad \tau_1 = \tau_2 \circ (\varphi_0)_*, \quad \text{on } \overline{\partial_- LM_1}.$$

Then there exists an analytic isometry $\varphi : \tilde{M}_1 \to \tilde{M}_2$ such that $\varphi|_{M_1^c} = \varphi_0$.

In this section, we will only work with complete travel time data, so we sometimes omit the complete and simply refer to it as the travel time data. Complete travel time data is more natural in the exterior case, because measurements are made outside the unknown region, which requires the information to fully pass through.

6.3.1 Setup

First, we prove the metric in the exterior region is determined in the following lemma.

Lemma 6.3.2. *Let φ_0 be defined as in Theorem 6.1.1. Then φ_0 is an analytic isometry on K_j^c.*

Proof. We first claim that the for any geodesically convex open set U, the time separation function d_U in U determines the metric there. Indeed, for any $x \in U$, the intersection $I^-(x) \cap U = \{y \in U : d_U(y,x) > 0\}$ is a non-empty open set. Now we pick some y in it. By [90, Lemma 5], $d_U(y,\cdot)$ is smooth and $\nabla_g d_U(y,\cdot)|_x$ gives a unit timelike vector at x. Computing this for any $y \in I^-(x) \cap U$, we recover all future pointing timelike vectors at x and therefore the metric at x.

Now consider some $x \in K_1^c$. Let $U \subset K_1^c$ be a geodesically convex neighborhood of x. By strong causality, there exists a smaller neighborhood $V \ni x$ such that for any $y \in V$, any causal curve connecting x and y are fully contained in U. As a result, $d_1(y,x) = d_U(y,x)$ for any $y \in V$, where recall d_1 is the time separation function restricted to K_1^c. Since φ_0 is a diffeomorphism on K_j^c and the strong causality holds on K_2^c as well, we may assume that $d_2(\varphi_0(y),\varphi_0(x)) = d_{\varphi_0(U)}(\varphi_0(y),\varphi_0(x))$ for any $y \in V$ by shrinking the size of V. Use the fact that $d_1 = \varphi^* d_2$, one has $I^-(\varphi_0(x)) \cap \varphi_0(V) = \varphi_0(I^-(x) \cap V)$. Then the set of (past pointing) unit timelike vector at $\varphi_0(x)$ is given by

$$
\begin{aligned}
&\{\nabla_{g_2} d_2(z,\cdot)|_{\varphi_0(x)} : z \in I^-(\varphi_0(x)) \cap \varphi_0(V)\} \\
=&\{\nabla_{g_2} d_2(\varphi_0(y),\cdot)|_{\varphi_0(x)} : y \in I^-(x) \cap V\} \\
=&(\varphi_0)_*\{\nabla_{g_1} d_1(y,\cdot)|_x : y \in I^-(x) \cap V\}.
\end{aligned}
$$

That is, φ_0 preserves the length of timelike vectors. As a result, $g_1|_{K_1^c} = \varphi_0^*(g_2|_{K_2^c})$. Since metrics are analytic and φ_0 is an isometry, it is automatically analytic. □

Since K_j may have very rough boundary, we now find a middle layer M_j with piecewise analytic boundary such that $K_j \subset\subset M_j \subset\subset \tilde{M}_j$.

Lemma 6.3.3. *There exists $M_j \subset\subset \tilde{M}_j$ such that $K_j \subset\subset M_j$ and M_j has piecewise analytic boundary. In particular, we can write $\partial M_j \subset \bigcup_{k=1}^N \partial B_k^j$, where $B_k^j \subset M_j^c$ are compact sets with analytic boundary. Moreover, denoting by $M_j^c = \tilde{M}_j \setminus M_j$, we have $\varphi_0(M_1^c) = M_2^c$ and $\varphi_0(\partial M_1) = \partial M_2$.*

Proof. We start by considering some open neighborhood U_1 of K_1 such that $K_1 \subset\subset U_1 \subset\subset \tilde{M}_1$. For every $x \in U_1$, we may choose local coordinates around x whose domain is contained entirely in K_1^c. Denote the closed unit ball in that coordinate by B_x. By our construction, each B_x has an analytic boundary. By the compactness, there are finitely many such B_{x_j} covering ∂U_1, and we take $M_1 = U_1 \cup (\bigcup_{j=1}^N B_{x_j})$. Now M_1 satisfies the desired property. In addition, since φ_0 is an analytic isometry on K_j^c, the set $\varphi_0(B_{x_j})$ also has an analytic boundary. In particular, $M_2 = \tilde{M}_2 \setminus \varphi_0(\tilde{M}_1 \setminus M_1)$ also satisfies the desired property. □

As M_j has piecewise analytic boundary (here piecewise refers to finitely many pieces), the intersection between the boundary and lightlike geodesics become much simpler.

Lemma 6.3.4. *Let M_j be given by Lemma 6.3.3. Then any causal geodesic only intersects ∂M_j finitely many times. In particular, for any $(x,v) \in \partial L^+ M_j$, there exists some $\epsilon > 0$ such that $\gamma_{x,v}((-\epsilon,\epsilon)) \cap \partial M_j = \{x\}$.*

Proof. By Lemma 6.3.3, we may assume $\partial M_j \subset \cup_{k=1}^N B_k^j$, where each $B_k^j \subset M_j^c$ is a compact set with analytic boundary. Consider a causal geodesic γ. By the global hyperbolicity, it eventually leaves any compact set. Suppose γ intersects with ∂M_j infinitely many times. Then by the pigeonhole principle, it intersects B_k^j

infinitely many times for some k. By the compactness, their intersection has accumulation point, which forces the geodesic to lie completely inside of ∂B_k^j. However, this would contradict γ leaving any compact set. The second statement immediately follows from finite intersection points. $\square$

6.3.2 Lightlike complete travel time data

In this subsection, for convenience we ignore the subscript j and work with $K, M, \tilde{M}, N, g$. The goal of this subsection is to explicitly compute the lightlike complete travel time data (S, τ) for M, using d and g on $K^c \sqcup \partial K$. Recall that even though $d(x, y)$ is only given for all $x, y \in K^c$, the time separation function $d(x, y)$ for all $x, y \in K^c \sqcup \partial K$ is actually known via continuity by [113, Lemma 14.21]. (Here we used the assumption that φ_0 extends continuously to ∂K_j.)

Lemma 6.3.5. *For any $x \in K^c$, we have*

$$\partial(\{y \in K^c : d(x, y) > 0\}) \cap K^c = \partial J^+(x) \cap K^c.$$

Proof. On a globally hyperbolic spacetime we have

$$\begin{aligned} \partial(\{y \in K^c : d(x, y) > 0\}) \cap K^c &= \partial(I^+(x) \cap K^c) \cap K^c \\ &= \Big((\partial I^+(x) \cap \overline{K^c}) \cup (\partial K^c \cap \overline{I^+(x)})\Big) \cap K^c \\ &= \partial J^+(x) \cap K^c. \end{aligned}$$

$\square$

Lemma 6.3.6. *For every $x \in K^c$ and $v \in L_x^+\tilde{M}$, consider some $x' = \gamma_{x,v}(\epsilon) \in K^c$ in a geodesically convex neighborhood of x. Then*

$$\partial J^+(x) \cap \partial J^+(x') \cap K^c = \gamma_{x,v}([\epsilon, \rho(x, v)]) \cap K^c.$$

Proof. Indeed, for $\epsilon \le t \le \rho(x, v)$, then $d(x, \gamma_{x,v}(t)) = d(x', \gamma_{x,v}(t)) = 0$. As $\gamma_{x,v}(t)$ is contained in $J^+(x)$ and $J^+(x')$, this implies $\gamma_{x,v}(t) \in \partial J^+(x) \cap \partial J^+(x')$, by the global hyperbolicity of the spacetime, see Lemma 6.2.2. On the other hand, if $y \in \partial J^+(x) \cap \partial J^+(x')$, then there exists $v' \in L_{x'}^+\tilde{M}$ such that $\gamma_{x',v'}$ passes through y. If $v' \neq \dot{\gamma}_{x,v}(\epsilon)$ up to rescaling, then x and y are connected by $\gamma_{x,v}([0,\epsilon]) \circ \gamma_{x',v'}$, which is a causal path that is not a pregeodesic. By Lemma 6.2.2, $d(x,y) > 0$, contradicting $y \in \partial J^+(x)$. Thus $y = \gamma_{x,v}(t)$ for some $t \geq \epsilon$. Again use $y \in \partial J^+(x)$ to deduce $d(x,y) = 0$, so $\epsilon \leq t \leq \rho(x,v)$. □

Lemma 6.3.7. *Let $\gamma : [0,T] \to \tilde{M}$ be a lightlike geodesic. Then $\rho(\gamma(t), \dot{\gamma}(t))$ has a uniform positive lower bound for all $t \in [0,T]$.*

Proof. Since $\tilde{M}$ is globally hyperbolic, the null cut locus ρ is lower semi-continuous. Clearly ρ is always positive, then on a compact set it has positive lower bound. □

We now start to recover the lightlike complete travel time data for M. Specifically, for any lightlike geodesic γ in M, $\gamma \cap K^c$ and its travel time data are computed step by step below. We show this procedure terminates after finitely many steps by proving that there is a uniform lower bound for every two steps. See Figure 6.2 for the different cases of the two step recovery appeared in the proof.

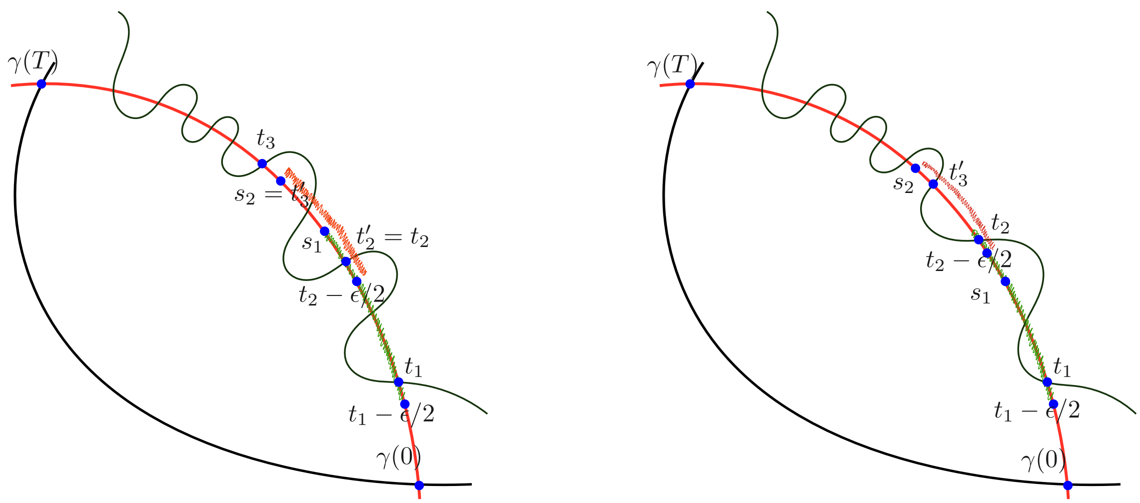

Figure 6.2: The first step goes from t_1 to t_2', where t_1 is the first time γ intersects K. Let s_1 be the time of the first cut point with respect to $\gamma(t_1 - \epsilon)$ for some small $\epsilon > 0$, see (6.3.2). There are two cases based on whether $\gamma(s_1)$ is in K (left graph) or K^c (right graph). The second step goes to t_3', which can be shown is strictly larger than s_1 in both cases. Since $s_1 - t_1$ has a uniform lower bound, this shows for every two steps we gain a uniform lower bound for the recovery.

Proposition 6.3.8. *The lightlike complete travel time data (S,τ) for M can be computed. In particular, this means for any $(x,v) \in \partial L^+ M_1$:*

$$(S_1, \tau_1)(x,v) = (y, w; T) \iff (S_2, \tau_2)(\varphi_0(x), (\varphi_0)_* v) = (\varphi_0(y), (\varphi_0)_* w; T), \tag{6.3.1}$$

where φ_0 is the isometry such that $\varphi_0(M_1^c) = M_2^c$ and $\varphi_0(\partial M_1) = \partial M_2$.

Proof. Pick some $(x, v) \in \partial L^+M$. We denote by γ the corresponding lightlike geodesic. By Lemma 6.3.4, we know γ intersects with ∂M only at finitely many points. Let T be the complete travel time of (x, v) with respect to M, i.e., $\gamma([0, T]) \subset M$ and $\gamma((T, T + \epsilon)) \subset M^c$. Then we write $(S, \tau)(x, v) = (y, w; T)$ where $(y, w) = (\gamma(T), \dot{\gamma}(T))$. The goal is to determine $(y, w; T)$.

Since $g|_{K^c \sqcup \partial K}$ is given, $(y, w; T)$ is directly recovered if γ never enters K° before exiting M. Hence, it suffices to consider the case in which γ intersects K before leaving M. Then there exists $0 < t_1 < T$ such that $\gamma(t_1) \in \partial K$ and $\gamma([0, t_1)) \subset K^c$, i.e, t_1 is the first time γ intersects K. Certainly we can determine t_1, from the knowledge of $g|_{K^c \sqcup \partial K}$. Then we show that $\gamma \cap K^c$ can be recovered step by step, where for each step we can recover the travel time along γ, and T will be reached in finitely many steps.

Denote by i_0 the uniform positive lower bound for $\rho(\gamma(t), \dot{\gamma}(t))$ for $t \in [0, T]$, according to Lemma 6.3.7. We pick sufficiently small $0 < \epsilon < i_0/2$. In the following, let

$$z_1 = \gamma(t_1 - \epsilon), \quad z_1' = \gamma(t_1 - \epsilon/2).$$

We consider the first cut point with respect to z_1 along γ, i.e.,

$$s_1 = \rho(\gamma(t_1 - \epsilon), \dot{\gamma}(t_1 - \epsilon)) + (t_1 - \epsilon). \tag{6.3.2}$$

By Lemma 6.3.5 and Lemma 6.3.6, the geodesic segments $\gamma([t_1 - \epsilon/2, s_1]) \cap K^c$ can be determined. Indeed, $\gamma([t_1 - \epsilon/2, s_1]) \cap K^c$ is union of line segments, so $\gamma(s_1) \in K^c$ if and only if one of the segments has an endpoint in K^c other than $\gamma(t_1 - \epsilon/2)$. In other words, we can determine the subset $\gamma([t_1 - \epsilon/2, s_1)) \cap K^c$. The benefit of removing $\gamma(s_1)$ is that, for any $\gamma(t) \in \gamma([t_1 - \epsilon/2, s_1)) \cap K^c$, the first cut point for z_1 comes after $\gamma(t)$. As a result, γ is the unique causal geodesic connecting z_1 and $\gamma(t)$.

We now recover t and $\dot{\gamma}(t)$ for any $\gamma(t) \in \gamma([t_1 - \epsilon/2, s_1)) \cap K^c$. Since $\gamma(t) \in K^c$, we can pick a sequence of $y_j \in I^+(\gamma(t)) \cap K^c$ converging to $\gamma(t)$. Then $d(z_1, y_j) > 0$. We claim that $d(z_1, \cdot) = |\exp_{z_1}^{-1}(\cdot)|_g$ locally around y_j and $d(\cdot, y_j) = |\exp_{y_j}^{-1}(\cdot)|_g$ locally around z_1, provided j is sufficiently large. Indeed, this comes from the fact that $\gamma(t)$ is before the first cut point for z_1. By lower semi-continuity of ρ and the fact that there is no conjugate point before the first cut point, $\exp_{z_1}$ and $\exp_{y_j}$ are local diffeomorphisms around y_j and z_1, respectively. Moreover, the time separation function is positive around z_1 and y_j because they are connected by a unique distance maximizing timelike geodesic.

A similar computation as [90, Lemma 5] gives

$$v_j := d(z_1, y_j) \cdot \nabla_g d(z_1, \cdot)|_{y_j}, \quad u_j := -d(z_1, y_j) \cdot \nabla_g d(\cdot, y_j)|_{z_1}$$

satisfy $\exp_{z_1}(u_j) = y_j$ and $\mathrm{d}\exp_{z_1}|_{u_j} u_j = v_j$. Since $y_j \to \gamma(t)$, denoting by $u = \lim u_j$ and $v = \lim v_j$, we must have $\gamma(t) = \exp_{z_1}(u)$ and $v = \mathrm{d}\exp_{z_1}|_u u$. Thus, the travel time data from z_1 to $\gamma(t)$ along γ is $(S, \tau)(z_1, u) = (\gamma(t), v; 1)$, which is equivalent to $(S, \tau)(\gamma(t_1 - \epsilon), \dot{\gamma}(t_1 - \epsilon)) = (\gamma(t), \dot{\gamma}(t); t - (t_1 - \epsilon))$ up to scaling. As a result, let $k > 0$ be such that $u = k\dot{\gamma}(t_1 - \epsilon)$, so $t = (t_1 - \epsilon) + k$ and $\dot{\gamma}(t) = \frac{1}{k} v$ are recovered.

As t and $\dot{\gamma}(t)$ can be recovered for any $\gamma(t) \in \gamma([t_1 - \epsilon/2, s_1)) \cap K^c$, we are done if $\gamma([t_1 - \epsilon/2, s_1)) \cap M^c \neq \emptyset$. Otherwise the geodesic is still contained in M and we denote the supremum of all such t by t_2', i.e.,

$$t_2' = \sup\{t \in [t_1 - \epsilon/2, s_1) : \gamma(t) \in K^c\}.$$

Note that $\dot{\gamma}(t_2')$ can be computed by taking the limit. Since we know t_2', $\gamma(t_2')$ and $\dot{\gamma}(t_2')$, we keep extending γ in K^c until one of the following two things happen: either it leaves M, in which case we are done since this means T, $\gamma(T)$ and $\dot{\gamma}(T)$ are recovered; or it hits ∂K again, say at time $t_2 \geq t_2'$. Now we repeat the same argument as above for t_2, which shows that one can recover the travel time data on $\gamma([t_2 - \epsilon/2, s_2)) \cap K^c$, where $s_2 = \rho(\gamma(t_2 - \epsilon), \dot{\gamma}(t_2 - \epsilon)) - (t_2 - \epsilon)$ is the first time when $\gamma(t_2 - \epsilon)$ has the first cut point along γ. Denote by t_3' the supremum of all t such that $\gamma(t) \in \gamma([t_2 - \epsilon/2, s_2)) \cap K^c$. We claim that $t_3' > s_1$. Note that then $t_3' - t_1 > s_1 - (t_1 - \epsilon) - \epsilon > i_0/2$ is bounded below.

To prove the claim above, consider two cases: $\gamma(s_1) \in K^c$ or $\gamma(s_1) \notin K^c$. If $\gamma(s_1) \in K^c$, then $t_2' = s_1$ and $t_2 > t_2'$. Since $\epsilon < i_0/2$, $\gamma(t_2 - \delta) \in \gamma([t_2 - \epsilon/2, s_2)) \cap K^c$ for all sufficiently small δ. In particular, this means $t_3' \geq t_2 > t_2' = s_1$. On the other hand, if $\gamma(s_1) \notin K^c$, then $\gamma([t_2', s_1]) \subset K$, hence $t_2 = t_2'$. Let s' be the first time γ leaves K after t_2', clearly $s' \geq s_1$. By the assumption that lightlike geodesic in M does not have cut point, we know $\rho(\gamma(t_2), \dot{\gamma}(t_2)) > s' - t_2$. In particular, there exists $\delta > 0$ sufficiently small such that $\gamma(s' + \delta) \in K^c$ and $s' + \delta - t_2 < \rho(\gamma(t_2), \dot{\gamma}(t_2))$. Thus $t_3' \geq s' + \delta > s_1$.

Thus we have proved that, after two steps, the amount of recovery of the travel time along γ is $t_3' - t_1$, which is bounded below uniformly by $i_0/2$. We repeat the two-step procedure, by the uniform lower bound of the two-step size, T, $\gamma(T)$ and $\dot{\gamma}(T)$ can be recovered in finitely many steps. As $(x, v) \in \overline{\partial_- L^+ M}$ is arbitrarily chosen, the entire lightlike complete travel time data for M is recovered. $\square$

Thus we have proved M_1 and M_2 admit equivalent lightlike complete travel time data (identified via φ_0). It now suffices to prove Proposition 6.3.1 in Section 6.3.3 and Section 6.3.4.

6.3.3 Construction of bijection

As explained at the beginning of Section 6.3, we now start the construction of the extension. The construction is mostly similar to [153], the difference being we only have lightlike directions and the fact that time separation function is not a true distance. Recall that in the assumption of Theorem 6.1.1, the manifolds are time-orientable, so $L^+ M_j$ is well-defined.

For any $(x, v) \in L^+ M_1$, for convenience, we denote $\gamma^1_{x,-v}$ as the unique past pointing lightlike geodesic, where the superscript 1 indicates it is a geodesic in $\tilde{M}_1$. In the following, let

$$t(x, v) = \sup\{t > 0 : \gamma^1_{x,-v}([0, t]) \subset M_1\}$$

be the first exit time for the past pointing lightlike geodesic $\gamma^1_{x,-v}$. Denote by

$$(y, w) = (\gamma^1_{x,-v}(t(x, v)), -\dot{\gamma}^1_{x,-v}(t(x, v))).$$

We formally define

$$\varphi : L^+ M_1 \to M_2, \quad (x, v) \mapsto \exp^{g_2}_{\varphi_0(y)}(t(x, v)(\varphi_0)_* w).$$

Intuitively, for each future pointing lightlike vector $(x, v) \in L^+ M_1$, we backtrack to find the first entry data (y, w), transport it to M_2 through φ_0, and then follow the corresponding g_2-geodesic issued from $\varphi_0(y)$ in direction $(\varphi_0)_* v$ for the same amount of time $t(x, v)$.

Lemma 6.3.9. *The map φ is well-defined.*

Proof. By the non-trapping property of lightlike geodesics, $t(x,v)$ is finite. By Proposition 6.3.8, there exists travel time data $(S,\tau)(y,w) = (z,u;T)$ on $\tilde{M}_1$ such that

1. $T \geq t(x,v)$;
2. it is the travel time data given by $\gamma^1_{x,v}$;
3. and $(S,\tau)(\varphi_0(y),(\varphi_0)_* w) = (\varphi_0(z),(\varphi_0)_* u;T)$ is a travel time data on $\tilde{M}_2$.

As a result $\exp^{g_2}_{\varphi_0}(t(x,v)(\varphi_0)_* w) \in M_2$. $\square$

Next we show that the map $\varphi(x,\cdot)$ is locally constant for any fixed $x \in M_1$. We first recall a technical lemma for an analytic Riemannian metric.

Lemma 6.3.10 (Lemma 1, [153]). *Let g^R be an analytic Riemannian metric on $\tilde{M}$. If K is a compact subset contained in the subset of $\tilde{M}$, then there is an open $O \subset \tilde{M}$ containing K and a positive number r such that the squared distance function of g^R is analytic on the set*

$$\Delta_{O,r}(K) = \{(x,y) : x \in K, d_R(x,y) < r\}.$$

Proposition 6.3.11. *For any $x \in M_1$, $\varphi(x,\cdot)$ is locally constant. Hence $\varphi(x,\cdot)$ is a constant function.*

Proof. Fix some $(x,v) \in L^+M_1$, for notation simplicity we denote $\gamma^1_{x,v}$ by γ^1, and we set $(\gamma^1(0),\dot{\gamma}^1(0)) = (x,v)$. Recall $t(x,v)$ is the time takes for the first entry point to arrive at (x,v). By Lemma 6.3.4, there exists some $\epsilon > 0$ sufficiently small such that $\gamma^1((-t(x,v)-\epsilon,-t(x,v))) \subset M_1^c$. Denote by

$$s = t(x,v) + \epsilon, \qquad (z,u) := (\gamma^1(-s),\dot{\gamma}^1(-s)).$$

Consider now a Fermi coordinate around γ^1 given by the following procedures. Let $e_0 = u$, $e_i \in T_z\tilde{M}_1$, for $i = 1,\dots,n-1$, be such that $e_0,\dots,e_{n-1}$ are linearly independent, and consider their parallel transport E_i along γ^1. The Fermi coordinate in a neighborhood W_1 of γ^1 is given by

$$(r^0, \cdots, r^{n-1}) \mapsto \exp^{g_1}_{\gamma^1(-s+r^0)}(\sum_{i=1}^{n-1} r^i E_i).$$

One may choose e_i such that the metric g_1 along γ^1 is given by

$$g_1|_{\gamma^1} = 2dr^0 \otimes dr^1 + \sum_{j,k=2}^{n-1} \theta_{jk} dr^j \otimes dr^k,$$

see for example [46, Section 4]. Let us assume that the Fermi coordinate is constructed on $\gamma^1([-s-\delta, \delta])$ for some sufficiently small δ, so that $\gamma^1([-s, 0])$ is in the interior of W_1. See Figure 6.3.

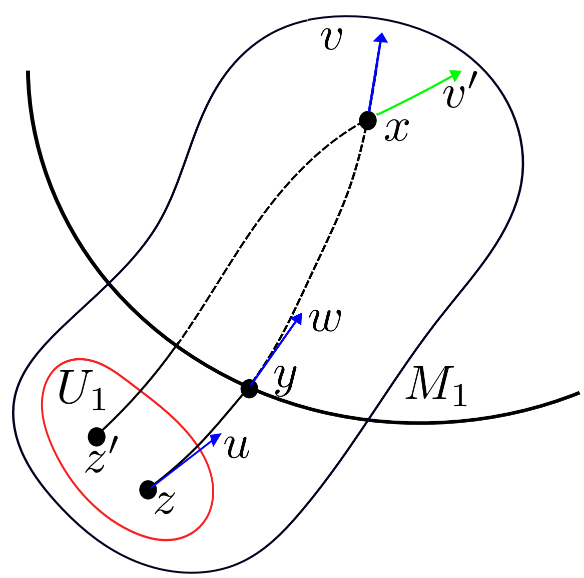

Figure 6.3: Locally we consider Fermi coordinates around a fixed geodesic by parallel transport of $e_0, \ldots, e_{n-1}$, and construct the analytic Riemannian metric g_1^R based on g_1. On M_2 we make the same construction using $(\varphi_0)_* e_k$ to obtain the Fermi coordinate and corresponding Riemannian metric g_2^R. Around z and $\varphi_0(z)$, the two metrics g_1^R and g_2^R agree (preserved by φ_0).

We now proceed to construct a corresponding Fermi coordinate in $\tilde{M}_2$ along γ^2, where

$$\gamma^2(-s+t) = \exp^{g_2}_{\varphi_0(z)}(t(\varphi_0)_* u).$$

First of all, γ^2 is well-defined on $(-s-\delta,\delta)$ for small δ, since

$$\gamma^2([-s,-t(x,v)]) = \varphi_0(\gamma^1([-s,-t(x,v)])) \subset \tilde{M}_2, \tag{6.3.3}$$

$$\gamma^2(0) = \exp^{g_2}_{\varphi_0(z)}(s(\varphi_0)_* u) = \exp^{g_2}_{\varphi_0(y)}(t(x,v)(\varphi_0)_* w) = \varphi(x,v) \in M_2, \tag{6.3.4}$$

where $(y,w) = (\gamma^1(-t(x,v)), -\dot{\gamma}^1(-t(x,v)))$. We then construct the Fermi coordinate using the parallel transport of $(\varphi_0)_* e_i \in T_{\varphi_0(z)}\tilde{M}_2$, denoted by E_i'. Note that φ_0 being isometry on M_j^c implies for all t around $-s$, $\varphi_0(\gamma^1(t)) = \gamma^2(t)$ and $(\varphi_0)_* E_i(t) = E_i'(t)$. The Fermi coordinate in a neighborhood W_2 around $\gamma^2([-s-\delta,\delta])$ is thus given by

$$(r^0,\cdots,r^{n-1}) \mapsto \exp^{g_2}_{\gamma^2(-s+r^0)}(\sum_{i=1}^{n-1} r^i E_i').$$

By the construction of E_i and E_i', we have

$$g_2|_{\gamma^2} = 2dr^0 \otimes dr^1 + \sum_{j,k=2}^{n-1} \theta'_{jk} dr^j \otimes dr^k.$$

For $j = 1,2$, we define $X_j = \partial_{r^0} - \partial_{r^1}$ in the corresponding Fermi coordinates of γ^j. Then by shrinking W_1 and W_2, we may assume X_1 and X_2 are both analytic timelike vector fields. Moreover, there exists $U_1 \subset W_1 \cap M_1^c$ neighborhood of z and $U_2 \subset W_2 \cap M_2^c$ neighborhood of $\varphi_0(z)$ such that $\varphi_0(U_1) = U_2$ and $(\varphi_0)_*(X_1|_{U_1}) = X_2|_{U_2}$. Consider the analytic Riemannian metric on W_j given by

$$g_j^R = g_j - 2(g_j(X_j,X_j))^{-1} X_j^\flat \otimes X_j^\flat,$$

we thus have $g_1^R|_{U_1} = \varphi_0^*(g_2^R|_{U_2})$. In the following, we denote $d_{R,j}$ by the corresponding distance function on W_j.

We are now ready to show $\varphi(x,\cdot)$ is constant around v. By continuity there exists a sufficiently small neighborhood $\mathcal{U}$ of v in $L_x^+ M_1$ such that for all $v' \in \mathcal{U}$,

$$\exp^{g_1}_x(-sv') \in U_1, \quad \{\exp^{g_1}_x(tv') : t \in [-s,0]\} \subset W_1.$$

Denote $(z', u') := (\exp_x^{g_1}(-sv'), d\exp_x^{g_1}|_{-sv'}v')$. Define

$$\rho_1(t) = d_{R,1}^2(\exp_z^{g_1}(tu), \exp_{z'}^{g_1}(tu')),$$

$$\rho_2(t) = d_{R,2}^2(\exp_{\varphi_0(z)}^{g_2}(t(\varphi_0)_* u), \exp_{\varphi_0(z')}^{g_2}(t(\varphi_0)_* u')).$$

By Lemma 6.3.10, ρ_1 and ρ_2 are well-defined analytic function on $[0, s]$, provided that $\mathcal{U}$ is sufficiently small. Since φ_0 preserves both Riemannian and Lorentzian metrics on U_j, $\rho_1(t) = \rho_2(t)$ for all t sufficiently small. Hence $\rho_1 = \rho_2$ on $[0, s]$. By definition of ρ_1, one has $\rho_2(s) = \rho_1(s) = 0$; together with (6.3.4), we have

$$\varphi(x, v) = \exp_{\varphi_0(z)}^{g_2}(s(\varphi_0)_* u) = \exp_{\varphi_0(z')}^{g_2}(s(\varphi_0)_* u').$$

It now suffices to show

$$\exp_{\varphi_0(z')}^{g_2}(s(\varphi_0)_* u') = \varphi(x, v').$$

This follows immediately from the same argument as in [153, Lemma 1], we include here for completeness. By Lemma 6.3.4 every lightlike geodesic only intersects ∂M_j finitely many times, so the set of lightlike travel time data with respect to $\tilde{\gamma}^2(t) = \exp_{\varphi_0(z')}^{g_2}(t(\varphi_0)_* u')$ is given by

$$((\tilde{\gamma}^2(s_k), \dot{\tilde{\gamma}}^2(s_k)); (\tilde{\gamma}^2(s_k + T_k), \dot{\tilde{\gamma}}^2(s_k + T_k)); T_k), \quad k = 1 \dots N$$

with $0 < s_1 \leq s_1 + T_1 < s_2 \leq s_2 + T_2 < \cdots < s_N \leq s \leq s_N + T_N$ and

$$\tilde{\gamma}^2((0, s_1)) \subset M_2^c, \quad \tilde{\gamma}^2([s_k, s_k + T_k]) \subset M_2, \quad \tilde{\gamma}^2((s_k + T_k, s_{k+1})) \subset M_2^c.$$

Denote $\tilde{\gamma}^1(t) = \exp_{z'}^{g_1}(tu')$. By Proposition 6.3.8 and the fact that φ_0 preserves metrics on M_j^c, a simple induction shows that

$$\left(\varphi_0(\tilde{\gamma}^1(t)), (\varphi_0)_*(\dot{\tilde{\gamma}}^1(t))\right) = \left(\tilde{\gamma}^2(t), \dot{\tilde{\gamma}}^2(t)\right), \quad \forall t = s_k, s_k + T_k.$$

Denote $(y', w') = (\tilde{\gamma}^1(s_N), \dot{\tilde{\gamma}}^1(s_N))$ the entering point for (x, v'), we know $t(x, v') = s - s_N$. This implies

$$\begin{aligned}
\exp^{g_2}_{\varphi_0(z')}(s(\varphi_0)_* u') &= \tilde{\gamma}_2(s_N + (s - s_N)) \\
&= \exp^{g_2}_{\varphi_0(\tilde{\gamma}^1(s_N))}((s - s_N)(\varphi_0)_* \dot{\tilde{\gamma}}^1(s_N)) \\
&= \exp^{g_2}_{\varphi_0(y')}(t(x, v')(\varphi_0)_* w') \\
&= \varphi(x, v').
\end{aligned}$$

Thus $\varphi(x, \cdot)$ is locally constant. For Lorentzian manifold of dimension at least 3 the set of future pointing lightlike directions is connected, so we can define $\varphi(x) = \varphi(x, \cdot)$. $\square$

Next we show that it is a bijection.

Proposition 6.3.12. *The map φ is a bijection from M_1 to M_2.*

Proof. Considering the same construction of $\psi : L^+M_2 \to M_1$. Proposition 6.3.11 shows that $\psi : M_2 \to M_1$ is also well-defined. For any $(x, v) \in L^+M_1$, by Proposition 6.3.8, $\exp^{g_2}_{\varphi_0(y)}(t(\varphi_0)_* w) \in M_2$ for all $0 \leq t \leq t(x, v)$ where (y, w) is the first entry point for (x, v). Then

$$\psi(\varphi(x)) = \psi(\exp^{g_2}_{\varphi_0(y)}(t(x, v)(\varphi_0)_* w)) = \exp^{g_1}_{y}(t(x, v)w) = x.$$

So φ is a bijection with $\varphi^{-1} = \psi$. $\square$

6.3.4 Analytic isometry

To extend the analytic identification constructed on M_1 to the entire $\tilde{M}_1$, we glue together the exterior analytic isometry φ_0 and the interior bijection φ, i.e., we set $\tilde{\varphi}(x) = \varphi_0(x)$ if $x \in M_1^c$, and $\tilde{\varphi}(x) = \varphi(x)$ if $x \in M_1$. We prove $\tilde{\varphi}$ is an analytic isometry below.

Proposition 6.3.13. *A curve γ is a future pointing lightlike geodesic in $\tilde{M}_1$ if and only if $\tilde{\varphi}(\gamma)$ is a future pointing lightlike geodesic in $\tilde{M}_2$. In particular, $\tilde{\varphi}$ and $\tilde{\varphi}^{-1}$ preserve causality.*

Proof. Let γ be a (connected) future pointing lightlike geodesic in $\tilde{M}_1$. By extending it we might as well assume $\gamma(0) \notin M_1$. By Lemma 6.3.4 it can be decomposed into finitely many segments: $\gamma((s_k, s_{k+1})) \subset M_1^c$

for k odd, and $\gamma([s_k, s_{k+1}]) \subset M_1$ for k even. Denote by $(y_k, w_k) = (\gamma(s_k), \dot\gamma(s_k))$. Since φ_0 preserves metric on M_j^c, for all k odd and $t \in (0, s_{k+1} - s_k)$, $\tilde\varphi(\gamma(s_k + t)) = \exp^{g_2}_{\varphi_0(y_k)}(t(\varphi_0)_* w_k)$ is a future pointing lightlike geodesic. By the definition of $\varphi(x, v)$, for all k even, one has $\tilde\varphi(\gamma(s_k + t)) = \exp^{g_2}_{\varphi_0(y_k)}(t(\varphi_0)_* w_k)$ where $t \in [0, s_{k+1} - s_k]$. So $\tilde\varphi(\gamma)$ is a piecewise future pointing lightlike geodesic. In particular, Proposition 6.3.8 and a simple induction shows that for all t,

$$\tilde\varphi(\gamma(t)) = \exp^{g_2}_{\varphi_0(\gamma(0))}(t(\varphi_0)_* \dot\gamma(0)). \tag{6.3.5}$$

The proof for $\tilde\varphi^{-1}$ is the same, and so they preserve causality. $\square$

Proposition 6.3.14. *The map $\tilde\varphi : \tilde M_1 \to \tilde M_2$ is an analytic isometry.*

Proof. By Proposition 6.3.13 and [61, Lemma 19], $\tilde\varphi$ is a conformal diffeomorphism. Then $g_1 = f\tilde\varphi^* g_2$ on $\tilde M_1$ with positive $f \in C^\infty(\tilde M_1)$ (note that so far we do not know if f is analytic). We provide several different proofs from here.

Since the dimension is at least 3, $g_1 = f\varphi^* g_2$ provides a system of $\frac{n(n+1)}{2}$ equations for $n+1$ unknowns (namely, $\partial_j \varphi^k$ and f in local coordinates), and g_1 and g_2 are both analytic. By Cauchy-Kowalevski Theorem (see for example [49, Chapter 1D]), the smooth solution φ and f must be analytic. As $f \equiv 1$ on M_1^c, it must be 1 everywhere, and the theorem is proved.

We can also prove $f \equiv 1$ using the special form of $\tilde\varphi$ in (6.3.5), without using any analyticity assumption. Let $\gamma^1(t)$ be a lightlike geodesic in $\tilde M_1$ such that $\gamma^1(0) \in M_1^c$. Then by (6.3.5), $\gamma^1(t) = \tilde\varphi^{-1}(\gamma^2(t))$ for some lightlike geodesic $\gamma^2(t)$ in $\tilde M_2$. On the other hand, $\tilde\varphi^{-1}(\gamma^2(t))$ is also a lightlike geodesic for $\tilde\varphi^* g_2$ on $\tilde M_1$. Since $g_1 = f\tilde\varphi^* g_2$, $\tilde\varphi^{-1}(\gamma^2(s(t)))$ is a lightlike geodesic for g_1 (see for example [131, Section 3.2]) with

$$\dot s = \frac{\mathrm{d}s}{\mathrm{d}t} = \frac{1}{f(\tilde\varphi^{-1}(\gamma^2(s)))}, \quad s(0) = 0.$$

Note that $\gamma^1(t) = \tilde\varphi^{-1}(\gamma^2(t))$ and $\tilde\varphi^{-1}(\gamma^2(s(t)))$ are both geodesics for g_1, and

$$\tilde\varphi^{-1}(\gamma^2(s(0))) = \gamma^1(0), \quad \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \tilde\varphi^{-1}(\gamma^2(s(t))) = \frac{1}{f(\gamma^1(0))}\tilde\varphi_*^{-1}(\dot\gamma^2(0)) = \dot\gamma^1(0),$$

where we used the fact that $\gamma^1(0) \in M_1^c$ and $\tilde{\varphi}$ is an isometry between M_j^c so $f|_{M_1^c} \equiv 1$. This means they must be the same geodesic, so

$$\tilde{\varphi}^{-1}(\gamma^2(t)) = \gamma^1(t) = \tilde{\varphi}^{-1}(\gamma^2(s(t))).$$

Then $s(t) = t$ for all t, so $f \equiv 1$ on γ^1. The set of all lightlike geodesics cover the entire $\tilde{M}_1$, so $f \equiv 1$ on $\tilde{M}_1$, $g_1 = \tilde{\varphi}^* g_2$. Since both g_1 and g_2 are analytic, $\tilde{\varphi}$ is analytic.

Finally, given the special form of $\tilde{\varphi}$ in (6.3.5), we can modify the proof in [61, Lemma 19] to show $\tilde{\varphi}$ is an analytic isometry directly. Consider some $p \in \tilde{M}_1$, locally choose an analytic coordinate system $x^1, \cdots, x^n$, we may assume ∂_{x^1} is timelike and $\partial_{x^2}, \cdots, \partial_{x^n}$ are spacelike. Then for $j = 2, \cdots, n$, define

$$a_j = -g_{1j} + \sqrt{g_{1j}^2 - g_{11}g_{jj}}$$

which is analytic because $g_{1j}^2 - g_{11}g_{jj}$ is uniformly bounded below by a positive number ($g_{11} < 0$ and $g_{jj} > 0$). Then $V_j = a_j \partial_{x^1} + \partial_{x^j}$ are linearly independent analytic lightlike vector fields. Similarly, $V_1 = a_1 \partial_{x^1} - \partial_{x^2}$ where

$$a_1 = g_{12} + \sqrt{g_{12}^2 - g_{11}g_{22}}$$

is lightlike and $V_1, \cdots, V_n$ form an analytic basis for $T\tilde{M}$ around p. By Inverse Function Theorem and analyticity of g_1, locally

$$(t_1, \cdots, t_n) \mapsto \exp^{g_1}(t_n V_n) \circ \cdots \circ \exp^{g_1}(t_1 V_1)(p)$$

is an analytic diffeomorphism from a neighborhood of 0 in $\mathbb{R}^n$ to neighborhood of p in $\tilde{M}_1$, so it is an analytic chart. Let $W_j = \tilde{\varphi}_* V_j$, by (6.3.5) they are linearly independent lightlike directions in $\tilde{M}_2$, so around $\tilde{\varphi}(p)$ we have an analytic chart

$$(t_1, \cdots, t_n) \mapsto \exp^{g_2}(t_n W_n) \circ \cdots \circ \exp^{g_2}(t_1 W_1)(\tilde{\varphi}(p)).$$

With respect to these charts, $\tilde{\varphi}$ is simply the identity map between neighborhoods of 0 by (6.3.5), so it is an analytic map. Thus g_1 and $\tilde{\varphi}^* g_2$ are both analytic metric on $\tilde{M}_1$ that agrees on M_1^c, meaning they are the same everywhere. $\square$

Remark 6.3.15. *We emphasize that, as is shown in the proof, the form of $\tilde{\varphi}$ in* (6.3.5) *directly implies isometry, without referring to analyticity. Hence if two smooth Lorentzian manifolds admit a bijective causal preserving map that also preserves parametrization of lightlike geodesics, then they are isometric.*

Thus the proof of Theorem 6.1.1 can be summarized as follows.

Proof of Theorem 6.1.1: By Lemma 6.3.2, the metric on K^c is determined. By Lemma 6.3.3 and Proposition 6.3.8, one can find a middle layer $K \subset\subset M \subset\subset \tilde{M}$ with piecewise analytic boundary, such that the lightlike complete travel time data for M are determined. By Proposition 6.3.1, $(\tilde{M}, g)$ is determined. $\square$

6.3.5 Non-time-orientable case

Indeed, Section 6.3.3 and Section 6.3.4 have proved Proposition 6.3.1 for spacetime. When the manifold is non-time-orientable, we only need a small modification.

Proof of Proposition 6.3.1: Section 6.3.3 and Section 6.3.4 gave the proof for when the manifolds are time-orientable. Suppose now that they are not time-orientable, then L^+M_j is not well-defined. Instead, since the lightlike complete travel time data are equivalent, we can use the same definition for φ with domain being the entire LM_1. Lemma 6.3.9 and Proposition 6.3.11 still proves φ is well-defined and locally constant, but this time L_xM_1 contains two connected components, so we need to show the two agree.

Consider some $(x, v) \in LM$, again denote

$$(y, w) = (\gamma^1_{x,-v}(t(x, v)), -\dot{\gamma}^1_{x,-v}(t(x, v)))$$

where $t(x, v)$ is the first time the backward geodesic $\gamma_{x,-v}$ leaves M. If we denote

$$(z, u) = (\gamma^1_{x,v}(t(x, -v)), -\dot{\gamma}^1_{x,v}(t(x, -v)))$$

the corresponding one for $(x,-v)$, then by construction

$$(z,-u) = S_1(y,w), \quad t(x,v) + t(x,-v) = \tau_1(y,w).$$

Since the lightlike complete travel time data are equivalent, we immediately have

$$\begin{aligned}
\varphi(x,v) &= \exp^{g_2}_{\varphi_0(y)}(t(x,v)(\varphi_0)_* w) \\
&= \exp^{g_2}_{\varphi_0(z)}((\tau_2(\varphi_0(y),(\varphi_0)_* w) - t(x,v))(\varphi_0)_* u) \\
&= \exp^{g_2}_{\varphi_0(z)}(t(x,-v)(\varphi_0)_* u) \\
&= \varphi(x,-v).
\end{aligned}$$

Thus $\varphi(x,\cdot)$ is a constant function. The rest of the proof is the same as the time-orientable case. $\square$

6.4 Boundary rigidity

We now study the boundary rigidity problem for an analytic globally hyperbolic Lorentzian manifold with timelike boundary. In particular, we prove Theorem 6.1.2 in this section. In the setting of Theorem 6.1.1, since the time separation function is defined in the ambient manifold, then $y \in J^+(x)$ implies the existence of a distance maximizing causal geodesic.

In the contrast, in a manifold with boundary, the situation becomes much more subtle. For example, if the timelike boundary is strictly concave in a small neighborhood, then even locally one can find $x < y$ whose distance maximizing curve is not given by a pregeodesic. This makes identifying the intersection of light cone and boundary more challenging than the exterior case.

6.4.1 Cut locus in Lorentzian manifolds with boundaries

In this subsection, we prove a sequence of lemmas about cut points in a Lorentzian manifold (M,g) with timelike boundary.

For $(x, v) \in L^+M$, we define the *null cut locus function*

$$\rho(x, v) = \inf\{s \in [0, \mathcal{T}(x, v)] : \ d(x, \gamma_{x,v}(s)) > 0\}, \tag{6.4.1}$$

where $\gamma_{x,v}(s)$ is the unique lightlike geodesic starting from x in the direction of v and $\mathcal{T}(x, v)$ is the supremum of the maximal parameter value such that $\gamma_{x,v}(s)$ is defined in M. If there is no s such that $d(x, \gamma_{x,v}(s)) > 0$, then we write $\rho(x, v) = +\infty$. For timelike (x, v), we define the timelike cut locus function

$$\rho(x, v) = \inf\{s \in [0, \mathcal{T}(x, v)] : \ d(x, \gamma_{x,v}(s)) > s|v|_g\}, \tag{6.4.2}$$

where $\gamma_{x,v}(s)$ is the unique timelike geodesic starting from x in the direction of v and $\mathcal{T}(x, v)$ is the supremum of the maximal parameter value such that $\gamma_{x,v}(s)$ is defined in M. Similarly, if there is no such s, then we write $\rho(x, v) = +\infty$. In both cases, when $\rho(x, v) < +\infty$, we call $\gamma_{x,v}(\rho(x, v))$ the first cut point of x along $\gamma_{x,v}$.

In our definition, the cut locus function only detects cut points lying in the interior of M; boundary points are not treated as cut points, even when they are the first conjugate points along the geodesic. This is sufficient for our purposes, because we assume that the extension of the manifold has no lightlike cut points, see Theorem 6.1.2. Later, we will work on the extension, which is a Lorentzian manifold with timelike boundary, and we regard our manifold simply as an interior subset of the extension. Moreover, note that these definitions are slightly different from those for Lorentzian manifolds without boundaries, see Section 6.2.2.

We prove the following lemmas for manifolds with timelike boundaries.

Lemma 6.4.1. *Let (N, g) be a spacetime with timelike boundary. Let $x, y \in N^\circ$ and $\gamma_{x,v} : [0, 1] \to N^\circ$ be a lightlike geodesic segment with $y = \gamma_{x,v}(1)$. Suppose $\rho(x, v) > 1$. Then the first conjugate point to x along $\gamma_{x,v}$ comes after y.*

Proof. Assume for contradiction that there contains a conjugate $s_0 \in (0, 1]$ to x along $\gamma_{x,v}$. By considering the same proper variation as in the proof of [16, Proposition 10.72], there is a timelike path from x to y arbitrarily close to $\gamma_{x,v}([0, 1])$. As $\gamma_{x,v}([0, 1])$ is contained in the interior, such timelike path is there and we can always assume it is a piecewise smooth timelike curve. Thus, we must have $d(x, y) > 0$ which contradicts with $\rho(x, v) > 1$. $\square$

Lemma 6.4.2. *Let (N, g) be a globally hyperbolic spacetime with timelike boundary. Let $x, y \in N^\circ$ and $\gamma : [0, 1] \to N^\circ$ be a lightlike geodesic segment with $y = \gamma_{x,v}(1)$. Suppose there is another H^1-causal curve $\mu : [0, 1] \to N$ from x to y, which cannot be reparameterized to γ. Then y comes on or after the first cut point along $\gamma_{x,v}$, i.e., $\rho(x, v) \leq 1$.*

Proof. First we prove that there exists a piecewise smooth causal curve $\mu_0 : [0, 1] \to N$ from x to y, which cannot be reparameterized to γ. Indeed, with $\gamma([0, 1]) \in N^\circ$, there exist $p \in \mu((0, 1)) \cap N^\circ$ such that $p \notin \gamma([0, 1])$. Note that $p \in J^+_{H^1}(x)$ and $y \in J^+_{H^1}(p)$. By Lemma 6.2.8, there exists a piecewise smooth causal curve from x to p and then p to y. We may refer to it by μ_0 below and reparameterize it such that $\mu_0 : [0, 1] \to N$. Note that μ_0 may touch the boundary. However, with $p \notin \gamma([0, 1])$, there exists $\delta > 0$ such that $y_0 = \mu_0(1 - \delta) \notin \gamma([0, 1])$ and $\mu_0([1 - \delta, 1]) \subset N^\circ$.

By Lemma 6.2.9, We consider $y_j = \gamma(1 + \epsilon_j) \in N$ for a sequence $\epsilon_j \to 0$. With $y \in N^\circ$, we may assume $y_j \in N^\circ$ as well. Note that the causal path $\mu_j := \mu_0([1 - \delta, 1]) \cup \gamma([1, 1 + \epsilon_j])$ from y_0 to y_j is not a null pregeodesic. By [113, Proposition 10.46], there is a timelike curve from y_0 to y_j arbitrarily close to μ_j and therefore $d(y_0, y_j) > 0$. By the reverse inequality, we conclude that $d(x, y_j) \geq d(x, y_0) + d(y_0, y_j) > 0$. As $y_j \to y$, one has $\rho(x, v) \leq 1$. $\square$

Recall that the time separation function is defined by considering all piecewise smooth causal curves. We show that global hyperbolicity guarantees any distance maximizing H^1-causal curve lying in the interior must be a geodesic. Clearly the same result does not hold for points on the boundary, for example the distance maximizing curve in a strictly concave part of the boundary may be realized by boundary geodesics.

Lemma 6.4.3. *Let (N, g) be a globally hyperbolic spacetime with timelike boundary. Let $x, y \in N^\circ$ be such that $y \in I^+(x)$. Let $\alpha([0, 1]) \subset N^\circ$ be an H^1-causal curve from x to y such that $L(\alpha) \geq d(x, y)$. Then α must be a timelike pregeodesic.*

Proof. Assume for contradiction that α is not a timelike pregeodesic. As $\alpha([0, 1])$ is contained in the interior, there exist finitely many open convex neighborhoods U_j, for $j = 0, \ldots N$, covering it. We can find a sequence

$0 = s_0 < s_1 < \ldots < s_N = 1$ such that $\alpha([s_j, s_{j+1}]) \subset U_j$, for $j = 0, \ldots, N$. We set $x_j = \alpha(s_j)$ with $x_N = y$ and note that $x_j < x_{j+1}$. As $L(\alpha) \geq d(x, y) > 0$, there exists $j_0 \in \{0, 1, \ldots, N\}$ such that $x_{j_0} \ll x_{j_0+1}$. This implies one of the following two scenarios can happen:

1. If $\alpha([s_{j_0}, s_{j_0+1}])$ is the unique timelike pregeodesic connecting x_{j_0} and x_{j_0+1}, we take $\alpha_0 = \alpha$. Certainly $L(\alpha_0) = L(\alpha)$.

2. If $\alpha([s_{j_0}, s_{j_0+1}])$ is not the unique timelike pregeodesic connecting x_{j_0} and x_{j_0+1}, we replace this segment of α with the unique timelike geodesic, and denote the new curve as α_0. Note that $L(\alpha_0) > L(\alpha)$ by [116, Proposition 7.2][3].

We now inductively modify α_0, and by abuse of notation we shall keep using α_0 after the modification. The goal is to show that after finitely many steps, α_0 is a piecewise timelike pregeodesic from x to y whose length is strictly larger than α.

We show the first inductive step in details. Let $j = j_0$, by the previous construction, $\alpha_0([s_j, s_{j+1}])$ is a timelike pregeodesic. Pick $\epsilon > 0$ sufficiently small such that $\alpha_0([s_{j+1} - \epsilon, s_{j+2}])$ lies in a geodesically convex neighborhood, denote $x'_{j+1} := \alpha_0(s_{j+1} - \epsilon)$. Since $\alpha_0([s_{j+1} - \epsilon, s_{j+1}])$ is a timelike pregeodesic, $x'_{j+1} \ll x_{j+1} < x_{j+2}$, so one of the following two scenarios can happen:

1. $\alpha([s_{j+1} - \epsilon, s_{j+2}])$ is the unique timelike pregeodesic connecting x'_{j+1} and x_{j+2}. In this case we keep the segment, so α_0 is unchanged.

2. $\alpha([s_{j+1} - \epsilon, s_{j+2}])$ is not the unique timelike pregeodesic connecting x'_{j+1} and x_{j+2}. In this case we replace this segment with the unique timelike geodesic, which strictly increases the total length of α_0 by [116, Proposition 7.2], in particular $L(\alpha_0) > L(\alpha)$. See Figure 6.4.

Now $\alpha_0([s_{j+1}, s_{j+2}])$ is a timelike pregeodesic, and the induction continuous. Certainly it terminates in finitely many steps (0 step if $j_0 + 1 = N$) when $j + 1$ reaches N. Similarly, we inductively modify α_0 in the reverse direction, by checking if $\alpha_0([s_{j-1}, s_j + \epsilon])$ is the unique timelike pregeodesic. Again, we either keep the segment if it is, or replace it with the unique timelike geodesic to increase the total length of α_0. The induction also terminates in finitely many steps when reaching 0.

[3]Indeed, the proof there works for any H^1-causal curve, see [116, Remark 7.3].

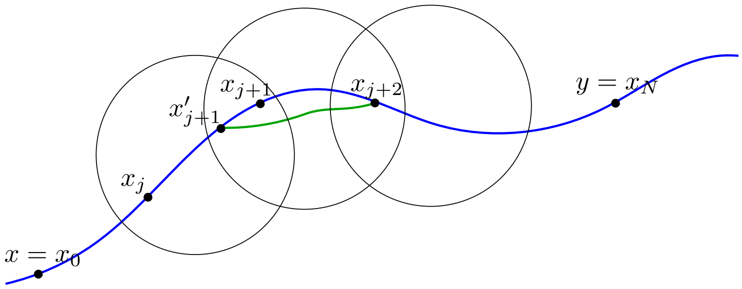

Figure 6.4: The inductive step: we choose x'_{j+1} sufficiently close to x_{j+1} such that it is still contained in the convex neighborhood U_{j+1}. There exists a unique timelike geodesic segment from x'_{j+1} to x_{j+1}.

By the construction, α_0 is now a piecewise timelike pregeodesic. If replacement never happened, then $\alpha_0 = \alpha$, in particular we have $\alpha([s_{j_0}, s_{j_0+1}])$, $\alpha_0([s_k-\epsilon, s_{k+1}])$ and $\alpha_0([s_{l-1}, s_l+\epsilon])$ are all timelike pregeodesic, for all $k > j_0$ and $l < j_0$. Overlapping of the intervals would imply α is a timelike pregeodesic which contradicts our assumption, hence the replacement must happen at least once. In particular, $L(\alpha_0) > L(\alpha)$, contradicting $L(\alpha_0) \le d(x, y) \le L(\alpha)$.

$\square$

By [16, Theorem 9.33], the cut locus function is lower semi-continuous on a globally hyperbolic spacetime without boundary. We show similar results on a globally hyperbolic spacetime with timelike boundary (N, g), around lightlike geodesics lying in the interior. Note that since N has timelike boundary, its interior N° is never globally hyperbolic, because the causal diamond set near the boundary is not compact. As a result, what we proved is not a trivial consequence of the globally hyperbolic without boundary scenario.

Proposition 6.4.4. *Let (N, g) be a globally hyperbolic spacetime with timelike boundary. Let $x, y \in N^\circ$ and $\gamma_{x,v} : [0, 1] \to N^\circ$ be a lightlike geodesic segment with $y = \gamma_{x,v}(1)$. Suppose $\rho(x, v) > 1$, let g_+ be an auxiliary Riemannian metric on N. Then there exists $\epsilon > 0$ such that for any w in the open neighborhood*

$$N(v, \epsilon) = \{w \in T_x M : \|w - v\|_{g^+} < \epsilon,\ g(w, w) < 0\},$$

the cut locus function $\rho(x, w) > 1$.

Proof. It suffices to prove $\rho(x, w) \ge 1$, as $(x, w) \in N(v, \epsilon)$ and y in the interior would imply $(x, (1-\delta)w) \in N(v, \epsilon)$ for sufficiently small δ, then $\rho(x, w) = \frac{1}{1-\delta}\rho(x, (1-\delta)w) \ge \frac{1}{1-\delta}$. By Lemma 6.4.1, the first conjugate point to x along $\gamma_{x,v}$ happens after y. Then there exists $\epsilon > 0$ and a small neighborhood $V \subset N^\circ$ of y such that the exponential map $\exp_x : N(v,\epsilon) \to V$ is a diffeomorphism.

Now assume for contradiction we cannot find $\epsilon > 0$ such that $\rho(x,w) \geq 1$ for any $w \in N(v,\epsilon)$. Then there exists a sequence $w_j \in T_xM$ such that

$$g(w_j, w_j) < 0, \quad \text{and} \quad w_j \to v \text{ as } j \to +\infty,$$

but with $\rho(x,w_j) < 1$. It follows that $y_j = \gamma_{x,w_j}(1) \to y$ is after the first cut point of x along the timelike geodesic γ_{x,w_j}. We may assume $y_j \in V$ by choosing sufficiently small $\epsilon > 0$. By the definition, one has $d(x,y_j) > |w_j|_g$, where d is the time separation function. Then we claim there exists an H^1-causal curve μ_j in N from x to y_j such that $L(\mu_j) \geq d(x,y_j)$, where recall L denotes the Lorentzian arc length. Indeed, by the definition of $d(x,y_j)$, there exists a sequence of future pointing piecewise smooth causal curve $\mu_{j_k} : [0,1] \to N$ from x to y_j, such that $L(\mu_{j_k}) \to d(x,y_j)$. By Lemma 6.2.7, there exists a future pointing H^1-casual limit curve $\mu_j : [0,1] \to N$ from x to y_j. As the length function is upper semi-continuous, we have

$$L(\mu_j) \geq \limsup L(\mu_{j_k}) = d(x,y_j).$$

With y_j being after the first cut point of x along γ_{x,w_j}, we emphasize that μ_j cannot be reparameterized to γ_{x,w_j}.

Next, for the sequence of H^1-casual curve $\{\mu_j\}$ in M from x to y_j, we apply Lemma 6.2.7 again and there exists an H^1-causal limit curve $\mu : [0,1] \to N$ from x to y. If such μ cannot be reparameterized to the lightlike geodesic $\gamma_{x,v}$ from x to y, then by Lemma 6.4.2, we must have y is on or after the first cut point along $\gamma_{x,v}$, which contradicts with $\rho(x,v) > 1$. Thus, any such μ can always be reparameterized to $\gamma_{x,v}$. By Lemma 6.2.6, there is a subsequence of $\{\mu_j\}$ that converges to $\gamma_{x,v}$ in the C^0 topology. We abuse the notation and denote it still by $\{\mu_j\}$. This implies one can find a small tube neighborhood Γ of the lightlike geodesic $\gamma_{x,v}([0,1])$ such that $\mu_j([0,1]) \subset \Gamma$, for sufficiently large j. With $\gamma_{x,v}([0,1]) \subset N^\circ$, we may choose $\Gamma \subset N^\circ$ and therefore $\mu_j \subset N^\circ$ for large j. Then with $L(\mu_j) \geq d(x,y_j)$, by Lemma 6.4.3, such μ_j is a timelike pregeodesic from x to y_j. We may reparameterize and write it as γ_{x,v_j} for some timelike

$v_j \in T_xM$. Recall that μ_j can not be reparameterized to γ_{x,w_j}, this means v_j and w_j are different directions. Since γ_{x,v_j} converges to $\gamma_{x,v}$ in C^0 topology, v_j must converge to v. In particular, $v_j \in N(v,\epsilon)$ for large j. Now for $y_j \in V$, we have found two different timelike geodesic γ_{x,w_j} and γ_{x,v_j} from x to y_j, with both $w_j, v_j \in N(v,\epsilon)$. This contradicts with the fact that $\exp_x$ is a diffeomorphism there. $\square$

Then we can prove the following analog for Lorentzian manifolds with boundaries.

Lemma 6.4.5. *Let (N,g) be a globally hyperbolic spacetime with timelike boundary. Let $x, y \in N^\circ$ and $\gamma([0,1])$ be a lightlike geodesic segment contained in N° with $y = \gamma(1)$ the first cut point. Then at least one of the following hold:*

(1) The point y is the first conjugate point of x along γ.

(2) There exist at least one other piecewise smooth causal curves from x to y that cannot be reparameterized to γ.

Proof. As y is the first cut point of x long γ, there exists $y_j = \gamma(1+\epsilon_j) \to y$ in N, with $\epsilon_j \to 0$ and $d(x, y_j) = \delta_j$ for some $\epsilon_j, \delta_j > 0$. Then using the same arguments as in the proof of Proposition 6.4.4, we claim there exists an H^1-causal curve μ_j in N from x to y_j such that $L(\mu_j) \geq d(x, y_j)$. As (M,g) is globally hyperbolic, by Lemma 6.2.7, there exists a future pointing H^1-causal limit curve $\mu : [0,1] \to N$ of the sequence $\{\mu_j\}$ connecting x and y. If we cannot reparameterize μ to γ, then there exist $p \in \mu((0,1))$ such that $p \notin \gamma([0,1])$. With $p \in J^+_{H^1}(x)$ and $y \in J^+_{H^1}(p)$, by Lemma 6.2.8 again, there exists a piecewise smooth causal curve from x to p and then p to y. This implies case (2). Otherwise, such μ can be reparameterized to γ. By Lemma 6.2.6, we can always find a subsequence of $\{\mu_j\}$ that converges to μ in the C^0 topology. Then the same argument as in the proof of Proposition 6.4.4 shows that y must be a conjugate point of x. $\square$

6.4.2 The proof of the boundary rigidity

We briefly describe the general steps of proving Theorem 6.1.2. The first step is to determine the jet of metric on the boundary. For Riemannian manifolds, it was shown in [135] that near a strictly convex direction, the jet of the metric can be determined from the lens relation. On the other hand, near a strictly convex direction, it is well-known that the knowledge of the distance function implies the lens data, in a geodesically convex neighborhood. Same result holds for spacetime with timelike boundary if the strictly convex direction is timelike. A constructive version is given in [133], we provide a constructive one for Lorentzian manifolds in Section 6.9.

With the jet of the metric at one point on each component of the boundary determined, by analyticity, one can then determine the metric in an analytic collar neighborhood extension, as explained in Section 6.2.3. By the assumption in the theorem, we may assume the extension $(\tilde{M}, \tilde{g})$ is still an analytic globally hyperbolic spacetime with timelike boundary. The goal now is to determine the lightlike complete travel time data of M, then we can use Proposition 6.3.1 to finish the proof. For this purpose, we first determine the time separation function $\tilde{d}$ for points in the exterior region $M^c = \tilde{M} \setminus M$.

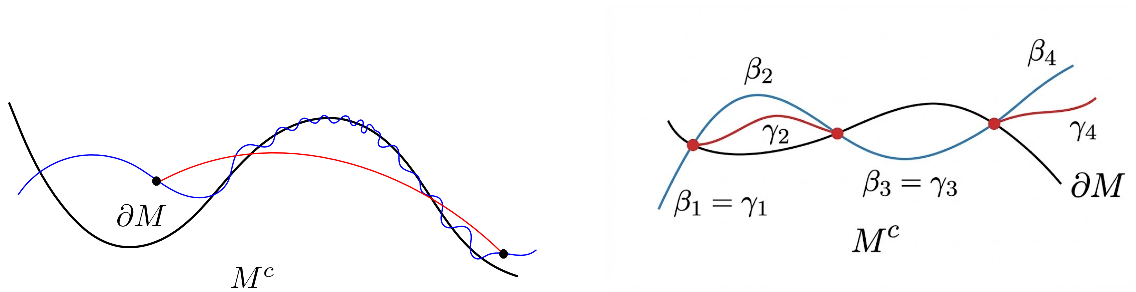

Figure 6.5: In the left picture, if a causal curve intersects with the boundary infinitely many times, then it can be replaced by a piecewise geodesic curve via a finite cover with geodesically convex neighborhoods. By analyticity, the piecewise geodesic curve switches between M and M^c only finitely many times. In the right picture, for each timelike curve $\beta_1 \circ \cdots \circ \beta_N$, we keep the same β_j as γ_j if it is in the exterior region, and find a γ_j that is at least almost the same length as β_j if it is in the interior. This is possible since the boundary time separation functions agree.

Proposition 6.4.6. *Let (M, g) be an analytic Lorentzian manifold with timelike boundary, and d be the boundary time separation function. Let $(\tilde{M}, \tilde{g})$ be an analytic extension and we denote by $M^c = \tilde{M} \setminus M$. Suppose $\tilde{g}|_{M^c}$ is given. Then d uniquely determines the exterior time separation function $\tilde{d}$ with respect to $(\tilde{M}, \tilde{g})$ for all $x, y \in M^c \sqcup \partial M$.*

Proof. For some fixed $x, y \in M^c \sqcup \partial M$, we first show that to compute $\tilde{d}(x, y)$, it suffices to consider all piecewise smooth, future pointing causal curves from x to y that can be decomposed into finitely many segments, with each segment either fully in M or in M^c. Indeed, consider a piecewise causal curve which enters and exits M infinitely many times in a bounded time interval. Denote the curve by α, then the set of points at which this transition occurs contains some accumulation point p. Near p, we may consider a geodesically convex neighborhood W. By analyticity of the boundary, locally a geodesic either fully stays in the boundary or has only discrete intersection point, so the segment of α in W can not be a geodesic. Then, one can replace it with a geodesic segment to increase the length (the geodesic would be causal because α is causal). This geodesic segment now has finitely many intersection points with the boundary. We may perform this for all accumulation points, since the travel time is bounded. Then by compactness we will end up with a curve that only switches finitely many times between M and M^c, whose length is larger than or equal to the original one. We refer to such a curve as a *valid curve* for simplicity. See the left graph in Figure 6.5.

Suppose now there are two metrics $\tilde{g}_j$ on the same $\tilde{M}$ and M, denote d_j the time separation function with respect to $(M, \tilde{g}_j|_M)$ and $\tilde{d}_j$ the time separation function with respect to $(\tilde{M}, \tilde{g}_j)$. Suppose $d_1|_{\partial M \times \partial M} = d_2|_{\partial M \times \partial M}$, and $\tilde{g}_1|_{M^c} = \tilde{g}_2|_{M^c}$. We show that the respective exterior time separation functions $\tilde{d}_j$ must coincide on $M^c = \tilde{M} \setminus M$. Fix x and y in $M^c \sqcup \partial M$ such that $\tilde{d}_1(x,y) > 0$. Suppose α_k is a sequence of valid curves, such that $L_1(\alpha_k) \to \tilde{d}_1(x,y)$, where L_1 denotes the length of the curve with respect to $\tilde{g}_1$. We may assume $\alpha_k \cap M$ is timelike almost everywhere with respect to $\tilde{g}_1$. Indeed, since $\tilde{d}_1(x,y) > 0$ and $\alpha_k \cap M$ lie in the interior of $\tilde{M}$, we can use the same replacement argument as in the proof of Lemma 6.4.3, so that the resulting curve is timelike almost everywhere in M and the length does not decrease. By previous arguments, we may assume each α_k switches between M and M^c finitely many times. For such an α_k, we denote its decomposition by $\beta_1, \cdots, \beta_N$. If $\beta_j \subset M^c$, we let $\gamma_j = \beta_j$ be the same curve, then $L_2(\gamma_j) = L_1(\beta_j)$ since the exterior metrics are the same. If $\beta_j \subset M$, then denote the starting and ending point as $z_1, z_2 \in \partial M$, we must have $d_1(z_1, z_2) \geq L_1(\beta_j) > 0$. As a result $d_2(z_1, z_2) = d_1(z_1, z_2) \geq L_1(\beta_j) > 0$, so there exists a piecewise smooth, future pointing causal curve $\gamma_j \subset M$ with respect to g_2, connecting z_1 and z_2, such that $L_2(\gamma_j) > d_2(z_1, z_2) - \epsilon$ for arbitrarily small ϵ. As a result, $\gamma_1, \cdots, \gamma_N$ form a piecewise smooth, future pointing causal curve with respect to $\tilde{g}_2$ connecting x to y. See the right graph of Figure 6.5. Moreover,

$$L_2(\gamma_1) + \cdots + L_2(\gamma_N) \geq L_1(\beta_1) + \cdots + L_1(\beta_N) - N \cdot \epsilon = L_1(\alpha_k) - N \cdot \epsilon.$$

Since ϵ is arbitrarily small, we have $\tilde{d}_2(x,y) \geq \tilde{d}_1(x,y) > 0$. By symmetry, we also have $\tilde{d}_1(x,y) \geq \tilde{d}_2(x,y)$. As a result, $\tilde{d}_1$ and $\tilde{d}_2$ have the same support in the exterior region, and in their common support, one has

$\tilde{d}_1 = \tilde{d}_2$. ☐

Now we have the information of two different time separation functions, $\tilde{d}$ for $(\tilde{M}, \tilde{g})$ and d for (M, g), for points in $M^c \sqcup \partial M$ and ∂M, respectively. As explained in the beginning of Section 6.4, unlike the globally hyperbolic spacetime without boundary case, x and $y \in \partial J^+(x)$ may not be connected by any lightlike geodesic. To identify those points y that are connected to x by a lightlike geodesic in M, we use a criterion involving both d and $\tilde{d}$. Intuitively, these are the points in $J^+(x) \cap \partial M$ whose distance to x remains 0 after the extension.

Proposition 6.4.7. *Let (M, g) be an analytic globally hyperbolic Lorentzian manifold with timelike boundary. Suppose there exists an analytic extension $(\tilde{M}, \tilde{g})$ such that the lightlike geodesics in $\tilde{M}$ do not have cut points. For any $x \in \partial M$ and $v \in \overline{\partial_- L_x^+ M}$, denote by $\gamma_{x,v}$ the corresponding inextendible geodesic in M. We have*

$$\bigcup_{v \in \overline{\partial_- L_x^+ M}} \gamma_{x,v} \cap \partial M = \partial(\{y \in \partial M : d(x, y) > 0\}) \cap \{y \in \partial M : \tilde{d}(x, y) = 0\}.$$

Proof. Let $y \in \gamma_{x,v} \cap \partial M$ for some $v \in \overline{\partial_- L_x^+ M}$. Since the lightlike geodesics do not have cut points in $\tilde{M}$, we must have $\tilde{d}(x, y) = 0$. Then $d(x, y) \leq \tilde{d}(x, y)$ implies $d(x, y) = 0$. Since the boundary is timelike, pick any future pointing timelike curve α in the boundary starting at y, clearly $d(x, \alpha(s)) > 0$ for any $s > 0$. Thus y belongs to the right hand side.

Now let y be a point in the set on the right-hand side. Let $z_j \in \partial M$ be a sequence of points converging to y such that $d(x, z_j) > 0$. Then we can find a sequence of piecewise smooth causal curves $\gamma_j \subset M$ from x to z_j. By Lemma 6.2.7, there exists a future pointing H^1-causal limit curve $\gamma \subset M$ by the global hyperbolicity of M, going from x to y. Note that γ may not be piecewise smooth, but by Lemma 6.2.9, there exists a piecewise smooth causal curve $\tilde{\gamma}$ in M from x to y. Now assume for contradiction that $\tilde{\gamma}$ is not a lightlike pregeodesic. Then it is a piecewise smooth causal curve in M and therefore contained in the interior of $\tilde{M}$, which cannot be reparameterized to a lightlike geodesic. By [113, Proposition 10.46], there exists a piecewise smooth timelike curve arbitrarily close to $\tilde{\gamma}$ in $\tilde{M}$ from x to y. This contradicts with the assumption that $\tilde{d}(x, y) = 0$. Thus, $\tilde{\gamma}$ must be a lightlike pregeodesic in M. Clearly, the corresponding direction is in $\overline{\partial_- L_x^+ M}$. This finishes the proof. ☐

We are now ready to put these together and prove the main theorem of this section.

Proof of Theorem 6.1.2. Throughout the proof, we omit the subscript j, and say a quantity A is determined if A_1 and A_2 are related by φ_0. We first show the boundary metric can be determined around boundary points with strictly convex directions. Consider the strictly convex direction (x, v) in $T\partial M$ given in the statement of the theorem, then strict convexity holds for all (y, w) sufficiently close to (x, v). Pick a smooth curve α in the boundary such that $(\alpha(0), \dot{\alpha}(0)) = (y, w)$ and $d(y, \alpha(s)) > 0$ for $s \ll 1$. Such (y, w) and α exist because the boundary is timelike and (x, v) is a causal direction. Consider the strictly convex direction $(x, v) \in T\partial M$, as is assumed in the theorem. We may choose $(y, w) \in T\partial M$ sufficiently close to (x, v) such that (y, w) is timelike and is still strictly convex. Then pick a smooth curve α on the boundary with $(\alpha(0), \dot{\alpha}(0)) = (y, w)$ such that $d(y, \alpha(s)) > 0$ for $s \ll 1$. Such (y, w) and α exist because the boundary is timelike and (x, v) is a causal direction. In particular, w is timelike and we need to determine its length. By strict convexity, for $s \ll 1$, the maximal distance $d(y, \alpha(s))$ is obtained by a unique timelike geodesic segment before the first cut point in M, and therefore

$$d(y, \alpha(s)) = |\exp_y^{-1}(\alpha(s))|_g := |w(s)|_g,$$

for $w(s) \in T_y\partial M$. By our construction, $w(0) = 0$ and $\dot{w}(0) = \mathrm{d}\exp_y|_{w(0)}\dot{w}(0) = \dot{\alpha}(0) = w$. In particular,

$$(d(y, \alpha(s)))^2 = -g(\dot{w}(0), \dot{w}(0))s^2 + O(s^3),$$

so $g(w, w)$ is determined from the boundary time separation function. Since this holds for all timelike (y, w) sufficiently close to (x, v), the boundary metric around x is determined.

Next, we determine the scattering relation around strictly convex direction (x, v) from the boundary time separation function. Note that because of the strict convexity, the interior and complete scattering relation for these directions coincide, so we simply use the term scattering relation. Denote by $C \subset \partial M$ the connected component that contains x. Since the boundary metric has been determined, we may assume (x, v) is unit timelike, as the strict convexity is an open condition. Then there exists $V \subset C$, the image of a conic neighborhood of (x, v) in $\overline{\partial_- T_x M}$ under $\exp_x$, such that for any $y \in V$, $d(x, y) > 0$ is the length of the unique timelike geodesic connecting x and y in M. Consider a local extension around x and U a geodesically convex neighborhood of x in the extension. Let $\bar{d}$ be the time separation function for points in U. Then $\bar{d}(x,y) = d(x,y)$ for all $y \in V$, since the unique timelike geodesic from x to y is in $M \cap U$, by shrinking V if necessary. In particular, $d(x,\cdot)$ is smooth around y; similarly, $d(\cdot,y)$ is smooth around x for every $y \in V$. As a result, by [90, Lemma 5], the two differentials

$$\eta := \mathrm{d}(d(x,\cdot))|_y = [\mathrm{d}(\tilde{d}(x,\cdot))|_y]|_{T_yC}, \quad \xi := \mathrm{d}(d(\cdot,y))|_x = [\mathrm{d}(\tilde{d}(\cdot,y))|_x]|_{T_xC}$$

are the projection of unit timelike covectors at x and y respectively, whose corresponding vectors are related by the projected scattering relation. In boundary normal coordinates around x, the scattering relation is thus given by

$$(x, g^{\alpha\beta}\xi_\alpha\partial_\beta + \sqrt{-1-|\xi|_g}\partial_n) \to (y, g^{\alpha\beta}\eta_\alpha\partial_\beta - \sqrt{-1-|\eta|_g}\partial_n).$$

As a result, the scattering relation is determined for unit timelike directions in $\overline{\partial_- T_x M}$ whose projection is sufficiently close to (x,v). As strict convexity is an open condition, the scattering relation is determined for all timelike directions in $\overline{\partial_- TM}$ sufficiently close to (x,v).

Next, we determine the scattering relation in a small neighborhood $\Gamma \subset \overline{\partial_- J_x M}$ of the strictly convex direction $v \in T_x\partial M$, from the boundary time separation function. Since the boundary metric is known, without loss of generosity we may assume (x,v) is unit timelike, as the strict convexity is an open condition. We may also assume Γ contains timelike directions only. For each $w \in \Gamma$, if it is tangential to the boundary, then the strictly convexity implies that the corresponding geodesic $\gamma_{x,w}$ immediately enters the exterior region; if it is transversal, then the strictly convexity implies that, as long as Γ is sufficiently small, $\gamma_{x,w}$ travels in the interior of M for a short period of time and leaves M transversally. Thus, the interior and complete scattering relation for these directions coincide, so we simply use the term scattering relation. Denote by $C \subset \partial M$ the connected component that contains x. Then let $V \subset C$ be the image of Γ under the exponential map $\exp_x$. For sufficiently small Γ, we may assume for any $y \in V$, the time separation $d(x,y) > 0$ is the length of the unique timelike geodesic connecting x and y in M. Indeed, consider a local extension around x and let U be a geodesically convex neighborhood of x in the extension. By the globally hyperbolicity, we can choose U small enough such that no causal curves that leave U never return, for example, see [16]. Let $\bar{d}$ be the time separation function for points in U. Then we have $\bar{d}(x,y) = d(x,y)$ for all $y \in V$ if Γ is sufficiently small, since the unique timelike geodesic from x to y is in $M \cap U$. In particular, $d(x,\cdot)$ is smooth near y; similarly, $d(\cdot,y)$ is smooth around x for every $y \in V$. As a result, by [90, Lemma 5], the two differentials

$$\eta := \mathrm{d}(d(x,\cdot))|_y = [\mathrm{d}(\tilde{d}(x,\cdot))|_y]|_{T_yC}, \quad \xi := \mathrm{d}(d(\cdot,y))|_x = [\mathrm{d}(\tilde{d}(\cdot,y))|_x]|_{T_xC}$$

are the projection of unit timelike covectors at x and y respectively, whose corresponding vectors are related by the projected scattering relation. In boundary normal coordinates around x, the scattering relation is thus given by

$$(x, g^{\alpha\beta}\xi_\alpha\partial_\beta + \sqrt{-1-|\xi|_g}\partial_n) \to (y, g^{\alpha\beta}\eta_\alpha\partial_\beta - \sqrt{-1-|\eta|_g}\partial_n).$$

As a result, the scattering relation is determined for unit timelike directions in $\overline{\partial_- T_x M}$ whose projection is sufficiently close to (x,v). As strict convexity is an open condition, the scattering relation is determined for all timelike directions in $\overline{\partial_- TM}$ sufficiently close to (x,v).

Since the metric and scattering relation are determined around the strictly convex direction (x,v) (recall we can assume (x,v) timelike), we can now apply Theorem 6.9.4, which determines the jet of metric in boundary normal coordinates at x. By analyticity, we determine the jet of metric in boundary normal coordinate of the entire boundary. Consider the extension of a collar neighborhood using normal exponential map, denote the extended manifold as $(\tilde{M},\tilde{g})$. By the same argument as [153, Section 2], the metric $\tilde{g}$ on $M^c = \tilde{M} \backslash M$ is determined, with the only difference being that the collar neighborhood is not uniform in size as the manifold is non-compact. Moreover, by the assumption in the statement of the theorem, we can make the extension sufficiently small so that $(\tilde{M},\tilde{g})$ is a globally hyperbolic spacetime with timelike boundary and lightlike geodesics do not have cut point in $\tilde{M}$.

By Proposition 6.4.6, the time separation function $\tilde{d}$ for $(\tilde{M},\tilde{g})$ is determined for points x, y in the exterior region $M^c \sqcup \partial M$. By Proposition 6.4.7, we can also identify the intersection between lightlike geodesics with the boundary:

$$L_x := \bigcup_{v \in \overline{\partial_- L_x^+ M}} \gamma_{x,v} \cap \partial M, \quad x \in \partial M.$$

Let $y \in L_x$ reached by $\gamma_{x,v}$ at time T, by rescaling we may assume $y = \gamma_{x,v}(1)$. We first recover $w = \mathrm{d}\exp_x|_{Tv} v$.

Since the boundary is timelike and we know the metric in M^c, there exists a sequence of $z_j \to y$ such that $z_j \in I^+(y) \cap M^c$. By the assumption that lightlike geodesics do not have cut points, so they do not have conjugate points by Lemma 6.4.1. Locally, $\exp_x$ is then a diffeomorphism around v. Denote by

$$v_j := \exp_x^{-1}(z_j), \quad u_j := \mathrm{d}\exp_x|_{v_j} v_j.$$

For any v' sufficiently close to v, $\gamma_{x,v'}([0,1])$ are in the interior of $\tilde{M}$, since $\gamma_{x,v}([0,1])$ is in M. By Proposition 6.4.4, for v' sufficiently close to Tv, the cut point on $\gamma_{x,v'}$ comes after $z' := \gamma_{x,v'}(1)$, which implies $\tilde{d}(x,z') = |v'|_{\tilde{g}} = |\exp_x^{-1}(z')|_{\tilde{g}}$. In particular, $|v_j|_{\tilde{g}} = |u_j|_{\tilde{g}} = \tilde{d}(x, z_j)$.

Moreover, $\tilde{d}(x,\cdot)$ is smooth around z_j for sufficiently large j. By the same computation as [90, Lemma 5], $u_j = \tilde{d}(x,z_j) \cdot \nabla_{\tilde{g}}(\tilde{d}(x,\cdot))|_{z_j}$ is determined by $\tilde{d}$, where $\nabla_{\tilde{g}}$ is the gradient. As $z_j \to y$, w is determined via $u_j \to w$.

To determine the corresponding v at x, we perform the same computation around x. Specifically, there exists a sequence of $z'_j \to x$ such that $z'_j \in I^-(x) \cap M^c$. Same argument shows $\tilde{d}(\cdot, y)$ is smooth around z'_j, and $|\tilde{d}(z', y)| = |\exp_y^{-1}(z')|_{\tilde{g}}$ for z' close to x. Then v is recovered as the limit of $-\tilde{d}(z'_j, y) \cdot \nabla_{\tilde{g}}(\tilde{d}(\cdot, y))|_{z'_j}$.

Since this can be computed for all $y \in L_x$, for any $v \in \overline{L_x^+ M}$, we group the (y_j, w_j) whose corresponding (x, v_j) is a scaling of (x, v). The final exit point is thus the (y_j, w_j) whose corresponding v_j is the longest. This gives us the range of the lightlike complete travel time data $(x, v) = (x, \frac{|v|}{|v|_j} v_j) \to (y_j, \frac{|v|}{|v_j|} w_j, \frac{|v_j|}{|v|})$, where $|\cdot|$ is any measure of vector length. Finally, we apply Proposition 6.3.1 to obtain isometry between M_1 and M_2. $\square$

6.5 Jet determination at the boundary

In this section, we study the determination of the boundary jet from both the interior and the complete scattering relation. The main idea is similar to [135, Theorem 1], but under different assumptions. Specifically, whereas [135] used lens data, which under their definition does not require a priori knowledge of the boundary metric, see [135, Section 1], here we work with scattering information. Since we are interested in the scattering rigidity problem, we do not assume the length function or the travel time function. Nevertheless, we prove that the same result can be achieved if one has knowledge of the boundary metric, using the first variation of the length function. We first state the results for Riemannian manifolds, then extend the result to Lorentzian setting. Finally, we refer to Section 6.9 for a constructive proof for a strictly convex direction, which has already been used in Section 6.4.

Theorem 6.5.1. *Let (M, g) be a compact Riemannian manifold with boundary. Let $(x_0, v_0) \in S\partial M$ be such that the maximal geodesic γ_0 through it leaves M in finite time. Suppose x_0 is not conjugate to any point in $\gamma_0 \cap \partial M$. If S^{in} is known on some neighborhood of (x_0, v_0) and the metric restricted to boundary, i.e., $g|_{\partial TM \times \partial TM}$, is known on some neighborhood of $\gamma_0 \cap \partial M$, then the jet of g at x_0 in boundary normal coordinates is determined uniquely.*

Proof. The proof stays close to the proof of [135, Theorem 1]. By the same argument there, this non-conjugacy condition holds for any (x, v) sufficiently close to (x_0, v_0). Now consider $v_\epsilon = v_0 + \epsilon \partial_n$ in boundary normal coordinates, and we write $(y_\epsilon, w_\epsilon) = S^{in}(x_0, v_\epsilon)$. By compactness there exists a subsequence $y_{\epsilon_j} \to y_0^*$ for some y_0^*, and y_0^* lies in $\gamma_0 \cap \partial M$. By the non-conjugacy assumption, y_0^* is not conjugate to x_0.

Denote by γ_j the geodesic corresponding to (x_0, v_{ϵ_j}). The same argument there implies there exists H_j hypersurfaces on ∂M containing x_0 such that a small neighborhood of x_0 on the boundary, denoted by $U_j \subset \partial M$, can be decomposed into $U_j^+ \sqcup (H_j \cap U_j) \sqcup U_j^-$, where U_j^+ is visible from y_{ϵ_j}. Here visible means any point in U_j^+ can be connected to y_{ϵ_j} by a geodesic that is in the interior except endpoints. Since the non-conjugacy condition is an open condition, it holds for $(x, u_j(x))$, where $x \in U_j^+$ and $u_j(x)$ is the unique direction near (x_0, v_0) such that the base point of $S^{in}(x, u_j(x))$ is y_{ϵ_j}. We can thus define $\tau_j(x) := \ell^{in}(x, u_j(x))$ as a function on U_j^+. It is smooth in U_j^+ because the geodesic from y_{ϵ_j} connects to x is transversal at x. Note that so far every step is the same as [135, Theorem 1], with the only difference being we do not have knowledge of τ_j, since we do not have the length function. This is not a problem because we do not need to recover the boundary metric.

Indeed, let $C_j := \tau_j(x_0, v_{\epsilon_j})$ be some positive constant. Although we do not know this constant, it does not affect the future computations. More explicitly, let $x \in U_j^+$ be fixed and $x(s)$, $s \in [0,1]$, be a smooth curve in $U_j^+ \cup \{x_0\}$ connecting x_0 and x. Recall S^{in} maps both (x_0, v_{ϵ_j}) and $(x(s), u_j(s))$ to vectors with the same base point y_{ϵ_j}. By the first variation formula of the length function, for example, see (6.6.5), we obtain

$$\tau_j(x) = C_j - \int_0^1 g(x'(s), u_j(x(s)))ds.$$

Since we have the scattering relation and the boundary metric, this implies τ_j can be determined up to a constant (this fact is also stated in the proof of [135, Theorem 1]). In particular, the derivative of τ_j can be determined from our assumptions. The remaining proof follows exact like the proof of [135, Theorem 1], because the Eikonal equation

$$g^{\alpha\beta}\partial_\alpha \tau_j \partial_\beta \tau_j + (\partial_n \tau_j)^2 = 1$$

only involves the derivatives of τ_j. $\square$

In [135, Theorem 1], the authors used the interior lens data. Next we show that one can also use the complete scattering relation. We emphasize that the existence of U_j^+ requires the geodesic connecting x_0 and y_{ϵ_j} to stay in the interior of M except for endpoints, which can not be guaranteed if such y_{ϵ_j} can be obtained via the complete scattering relation $S(x_0, v_{\epsilon_j})$. On the other hand, a generic perturbation would solve the problem.

Theorem 6.5.2. *Let (M, g) be a compact Riemannian manifold with boundary. Let $(x_0, v_0) \in S\partial M$ be such that the maximal geodesic γ_0 through it leaves M in finite time. Suppose x_0 is not conjugate to any point in $\gamma_0 \cap \partial M$. If S is known on some neighborhood of (x_0, v_0) and the metric restricted to boundary, i.e., $g|_{\partial TM \times \partial TM}$, is known on some neighborhood of $\gamma_0 \cap \partial M$, then the jet of g at x_0 in boundary normal coordinates is determined uniquely.*

Proof. We show that we can still find $\tilde{v}_\epsilon$ arbitrarily close to $v_\epsilon = v + \epsilon\partial_n$ such that the geodesic from $(x_0, \tilde{v}_\epsilon)$

to $S(x_0, \tilde{v}_\epsilon)$ stays in the interior of M, except for two endpoints, whose directions are transversal to the boundary. In particular, this means $S(x_0, \tilde{v}_\epsilon) = S^{in}(x_0, \tilde{v}_\epsilon)$. Indeed, consider an arbitrarily small open set of directions containing v_ϵ in $\partial_- T_x M$. By non-trapping assumption, there exists some T such that all of them leaves M before time T. Let $\tilde{M}$ be any extension of M, by Lemma 6.8.2, the corresponding geodesic for all except a zero measure set of them will intersect with the boundary transversally, whenever it intersects with the boundary. Let $\tilde{v}_\epsilon$ be any such direction, then it is in $\partial_- TM$, and $\gamma_{x_0,\tilde{v}_\epsilon}$ intersects with the boundary only transversally. In particular, this means the first time it leaves the manifold is transversal, and does not touch the boundary in between. This proves what we need. We have shown the existence of such $\tilde{v}_\epsilon$, to explicitly find one, use the fact that a transversal $\tilde{v}_\epsilon$ is such a direction if and only if $S(x_0, \tilde{v}_\epsilon)$ is transversal and it satisfies

$$\{(z, u) : S(z, u) = S(x_0, \tilde{v}_\epsilon)\} = \{(x_0, \tilde{v}_\epsilon)\}.$$

Instead of directly using v_ϵ as in the proof of Theorem 6.5.1, we can now pick $\tilde{v}_\epsilon = v_\epsilon + O(\epsilon^N)$ for any large N so that $\tilde{v}_\epsilon \to v$, the rest of the proof is now the same with $\tilde{v}_\epsilon$ replacing v_ϵ. The higher order error does not affect the computation for lower order jet, and since we may choose N to be arbitrarily large, we may determine all the jet. $\square$

Since we focus on Lorentzian manifolds with timelike boundary, we prove the corresponding boundary determination results in the Lorentzian setting. The ideas are the same when the non-conjugacy condition holds for a tangential timelike direction.

Theorem 6.5.3. *Let (M, g) be a Lorentzian manifold with timelike boundary. Let $(x_0, v_0) \in T\partial M$ be a unit timelike vector tangential to the boundary, such that the maximal geodesic γ_0 through it leaves M in finite time. Suppose x_0 is not conjugate to any point in $\gamma_0 \cap \partial M$. If S^{in} or S is known on some neighborhood of (x_0, v_0) and the metric restricted to boundary, i.e., $g|_{\partial TM \times \partial TM}$, is known on some neighborhood of $\gamma_0 \cap \partial M$, then the jet of g at x_0 in boundary normal coordinates is determined uniquely.*

Proof. The proof is almost identical with the Riemannian case in Theorem 6.5.1 and Theorem 6.5.2, we point out some small adjustments below.

Let $\tilde{M}$ be any extension of M and suppose $\gamma_0(T) \notin M$. Then there exists a small tubular neighborhood U of $\gamma_0([0,T])$ and a sufficiently small open set of directions $\mathcal{U} \subset \partial IM$ containing (x_0, v_0) such that the geodesics from $(x,v) \in \mathcal{U}$ will stay in $\bar{U} \cap M$ before leaving M. Hence we may perform all the steps in this tubular neighborhood, which is compact. Since we are only working with geodesics very close to γ_0, we may assume all the geodesics appear in the proof to be timelike geodesics. For timelike geodesics, one can check that the step involving the existence of visible set still holds, specifically the proof of the statement in [126, Lemma 2.3] works for timelike geodesics as well. For timelike geodesics, the first variation formula for the length function has the opposite sign

$$\tau_j(x) = C_j + \int_0^1 g(x'(s), u_j(x(s)))ds.$$

The Eikonal equation also has the opposite sign

$$g^{\alpha\beta}\partial_\alpha \tau_j \partial_\beta \tau_j + (\partial_n \tau_j)^2 = -1.$$

Both of them do not affect the proof.

The $y_0^* \neq x_0$ case is the same as in [135, Theorem 1], we will rewrite the induction step for the case $y_0^* = x_0$, since timelike geodesics locally maximizing distances instead of minimizing distances. The main idea is exactly the same. Suppose for the two metrics g, h, there exists $k \geq 1$ such that at (x_0, v_0), one has

$$\partial_n^j g^{\alpha\beta} v_\alpha v_\beta = \partial_n^j h^{\alpha\beta} v_\alpha v_\beta, \text{ when } j < k; \quad \partial_n^k g^{\alpha\beta} v_\alpha v_\beta > \partial_n^k h^{\alpha\beta} v_\alpha v_\beta.$$

Then $g^{\alpha\beta} v_\alpha v_\beta > h^{\alpha\beta} v_\alpha v_\beta$ for all (x,v) close to (x_0, v_0), when x is in the interior. This is equivalent to

$$\sqrt{-g^{\alpha\beta} v_\alpha v_\beta} < \sqrt{-h^{\alpha\beta} v_\alpha v_\beta}.$$

Denote L^g, L^h the length of a curve with respect to g, h. Then integrating along γ_j^g, the geodesic connecting x_0 to y_{ϵ_j} (recall $y_{\epsilon_j} \to y^* = x_0$) with respect to g, we obtain

$$L^g(\gamma_j^g) < L^h(\gamma_j^g)$$

since these are short geodesics with initial direction close to (x_0, v_0). Use the fact that they share the same scattering relation near (x_0, v_0), and the boundary metrics agree around x_0, the first variation formula for the length function with initial data $L^g(x_0 \mapsto y_0^*) = L^h(x_0 \mapsto y_0^*) = 0$ gives

$$L^h(\gamma_j^h) = L^g(\gamma_j^g) < L^h(\gamma_j^g).$$

Note that γ_j^g is still a timelike curve with respect to h because $0 > g^{\alpha\beta} v_\alpha v_\beta > h^{\alpha\beta} v_\alpha v_\beta$ along it. This is a contradiction because γ_j^h should be the distance maximizing causal curve between x_0 and y_{ϵ_j} with respect to h, if we consider sufficiently large j so that y_{ϵ_j} is in the geodesic convex neighborhood of x_0. $\square$

6.6 Interior and complete scattering information

In this section, we study the relationship between interior and complete scattering information. As explained in Section 6.1, both definitions have made their appearance in the literature, and neither of them can trivially imply the other one. Even though the work is focusing on Lorentzian manifold, we choose to study the Riemannian setting first. The techniques used in the Riemannian setting will then be used on Lorentzian manifolds. We emphasize that in this section we do not impose analyticity or any causality condition.

6.6.1 Riemannian setting

For a compact Riemannian manifold with boundary, we first show that the interior lens data can be constructed from the complete lens data.

Lemma 6.6.1. *Let (M, g) be a compact Riemannian manifold with boundary. Suppose it is non-trapping. Then $(S^{in}, \ell^{in})|_{\partial_- TM}$ can be recovered from $(S, \ell)|_{\overline{\partial_- TM}}$.*

Proof. Let $(x, v) \in \partial_- SM$. Then $S(x, v) = (y, w)$ is the final exit point. We may view $(S, \ell)(z, u) = (z, u; 0)$ if $(z, u) \in \partial_+ SM$, so that (S, ℓ) is defined on the entire ∂SM. If we denote $\varphi(x, v, \cdot)$ as the complete geodesic flow with respect to (x, v), then

$$\varphi(x, v, \cdot) \cap \partial TM = \{(z, u) : S(z, u) = (y, w)\}.$$

One can order them according to $\ell(z, u)$. We pick

$$(z_0, u_0) = \arg\max\{\ell(z, u) : \ell(z, u) < \ell(x, v) \text{ and } (z, u) \in \varphi(x, v, \cdot) \cap \partial TM\}.$$

Then (z_0, u_0) is the immediate next boundary point, and

$$(S^{in}, \ell^{in})(x, v) = (z_0, u_0; \ell(x, v) - \ell(z_0, u_0)).$$

□

Note that the above proof does not work when one only has the interior scattering relation without the length information. The other direction, however, is more challenging. This is because the interior scattering relation is defined in $\partial_- SM$, which excludes the tangential directions. As a result, if a geodesic tangentially enters M and then tangentially exits M, then it is invisible with respect to the interior scattering information, but should be contained in the complete scattering relation. On the other hand, if one has a prior information of the metric around the boundary, then the tangential geodesics are no longer invisible. Note that we do not claim this to be the optimal result: for example, one may try to approximate the tangential directions using limits of transversal directions without requiring the knowledge of the boundary metric. We prove this weaker version because it suffices for the scattering rigidity result in the analytic setting.

Lemma 6.6.2. *Let (M, g) be a compact Riemannian manifold with boundary, suppose it is non-trapping. Suppose there exists an open neighborhood $U \subset M$ of ∂M such that $g|_U$ is given. Then*

(1) $S|_{\overline{\partial_- TM}}$ can be recovered from $S^{in}|_{\partial_- TM}$.

(2) $(S, \ell)|_{\overline{\partial_- TM}}$ can be recovered from $(S^{in}, \ell^{in})|_{\partial_- TM}$.

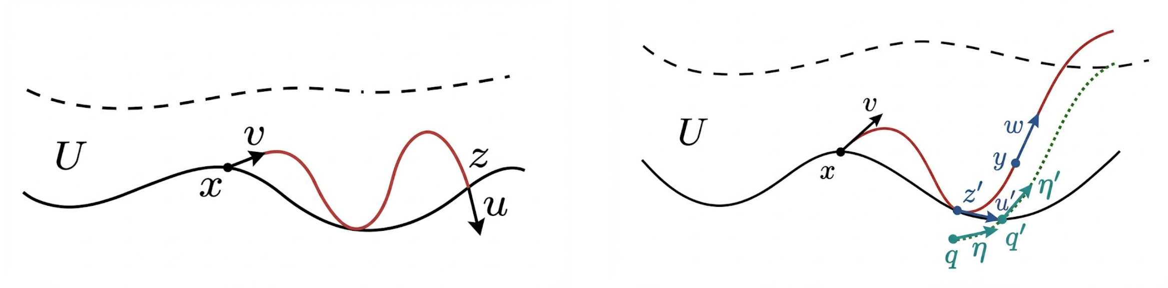

Figure 6.6: There are two possibilities for the geodesic to leave U: either it leaves M or it enters $M \setminus U$. For the second case we use nearby transversal starting points to find a future boundary intersection point of (x, v).

Proof. We start with the first statement. Since we know the metric on U, we may extend M to some $\tilde{M}$. Then we also know the metric on $V := U \cup (\tilde{M} \backslash M)$. For every $(x, v) \in \partial SM$, since we know the metric in U and the geodesics are non-trapping, we can track how the geodesic starting at (x, v) leaves U for the first time. There are two possibilities (see Figure 6.6):

label=(i) it leaves M;

lbbel=(ii) it enters $M \setminus U$.

Denote the time of leaving by T and direction by (z, u). For case (i), we refer to (z, u) as the exiting direction of (x, v).

We now deal with case (ii). Denote by φ the geodesic flow and let $(z', u') = \varphi(x, v, T')$ be such that

$$T' = \inf\{t > 0 : \varphi(x, v, s) \in U \setminus \partial M \text{ for } s \in [t, T)\}.$$

In other words, (z', u') is the last time the geodesic touches the boundary before entering $M \setminus U$. Note that (z', u') may just be (x, v) itself. Now consider a geodesically convex neighborhood $W \subset \tilde{M}$ of z'. Then there exists small δ such that $\varphi(z', u', t)$ stays in $W \cap M$ for all $t \in [0, \delta]$, and we denote by $(y, w) = \varphi(z', u', \delta)$. For $p \in W \setminus M$, the unique geodesic from p to y intersects with ∂M. Let $\xi \in T_pW$ be such that (p, ξ) is sufficiently close to (z', u'). Using Lemma 6.8.1, there exists (q, η) sufficiently close to (p, ξ) such that the geodesic from (q, η) intersects with ∂M at least once before leaving W, and all intersections are transversal. For (q, η) sufficiently close to (p, ξ) and p sufficiently close to z', we have $\varphi(q, \eta, \cdot)$ stays close to $\varphi(z', u', \cdot)$

and eventually enters $M \setminus U$. In particular, we can assume that $\varphi(q,\eta,t)$ stays in the interior of U for $t_W(q,\eta) \leq t \leq t_U(q,\eta)$, where $t_W(q,\eta)$ is the time it leaves W and $t_U(q,\eta)$ is the time it enters $M \setminus U$. Backtracking from $\varphi(q,\eta,t_W(q,\eta))$ until it hits ∂M for the first time, we denote that time by $t'(q,\eta)$ and the vector by $(q',\eta') = \varphi(q,\eta,t'(q,\eta))$. See the right picture in Figure 6.6.

To summarize, we have shown that $\varphi(q,\eta,t) \in U \setminus \partial M$ for $t \in (t'(q,\eta), t_U(q,\eta))$ before entering $M \setminus U$, and is transversal to the boundary at time $t'(q,\eta)$ with direction (q',η'). In particular, $S^{in}(q',\eta')$ is well-defined. We may find a sequence of such (p_j,ξ_j) converging to (z',u'), and a sequence of corresponding $S^{in}(q'_j,\eta'_j) = (y_j,w_j)$. By compactness and passing down to subsequence, we may assume $(y_j,w_j) \to (y_*,w_*) \in \partial SM$. By our construction, (y_*,w_*) will be on the flow of $\varphi(z',u',\cdot)$ because (q_j,η_j) will converge to (z',u'). Moreover, if we denote T_j as the time $\varphi(q_j,\eta_j,T_j) = (y_j,w_j)$, then $T_j > t_U(q_j,\eta_j) > \delta$. Hence the limit of T_j is bounded below by δ, meaning $(y_*,w_*) = \varphi(z',u',t)$ for some $t > \delta$. We emphasize that this means $(y_*,w_*) = \varphi(x,v,t)$ for some $t > \delta$, and δ can be uniformly chosen as the minimum of: half of the injective radius; and minimum times for a boundary direction to enter $M \setminus U$.

We are now ready to prove the lemma. For any $(x,v) \in \overline{\partial_- SM}$, it falls in one of the two categories. If it is case (i), we set $S(x,v)$ to be the exiting direction and we are done. If it is case (ii), then we will find some (y_*,w_*) and we repeat the procedure on (y_*,w_*). Note that we can guarantee (y_*,w_*) is at least δ time after (x,v), so the procedure will terminate in finite steps by the non-trapping assumption. In other words, we will reach case (i) in finitely many steps, and we set $S(x,v)$ to be the final exiting direction (z,u). S is recovered from S^{in}.

To recover (S,ℓ) from (S^{in},ℓ^{in}), it now suffices to show ℓ can be recovered as well. During the above procedure, we can recover $L_j = \ell^{in}(q'_j,\eta'_j) + L((q_j,\eta_j) \to (q'_j,\eta'_j))$. So $\ell^{in}(z',u') = \lim L_j$. We just need to add them up when recursively finding the next (y_*,w_*), which as explained before terminates in finitely many steps. $\square$

6.6.2 The first variation of the travel time

Before considering the Lorentzian setting, we compute the first variation of the travel time function. Even though on Riemannian manifolds, the travel time function is almost the same as the length function, it contains more information in the Lorentzian setting, especially for variations near lightlike geodesics. We will show that this allows us to build stronger relationship between scattering relation and travel time data.

Let (M, g) be a Riemannian or Lorentzian manifold and $H_1, H_2 \subset M$ be smooth hypersurfaces. In the following, we compute the variation of the travel time function of a fixed geodesic between H_1 and H_2. Let $(x_0, v_0) \in T_{H_1}M$ be transversal to H_1. In this part, for simplification, we use $\gamma_0 : [0, \tau_0] \to M$ to denote the unique geodesic segment starting from $x_0 \in H_1$ in the direction of v_0. Suppose γ_0 hits H_2 transversally at $y_0 = \gamma_0(\tau_0)$. For intervals $I_t, I_\lambda \subset \mathbb{R}$ to be specified later, we consider a smooth one-parameter family of geodesics

$$\Gamma : I_\lambda \times I_t \to M$$

near γ_0, such that for each $\lambda \in I_\lambda$, the smooth curve $\gamma_\lambda(\cdot) = \Gamma(\lambda, \cdot)$ is the unique geodesic starting from $x(\lambda) = \Gamma(\lambda, 0) \in H_1$ in the direction of $v(\lambda) = \partial_t \Gamma(\lambda, 0) \in T_{x(\lambda)}M$, with $x(0) = x_0$ and $v(0) = v_0$. Suppose each $\gamma_\lambda(t)$ intersects H_2 at $t = \tau_\lambda$. As γ_0 intersects H_2 transversally, for sufficiently small $|\lambda|$, γ_λ also intersects H_2 transversally. Thus, we may assume the interval I_λ is a small open neighborhood of $\lambda = 0$ such that each γ_λ intersect H_2 transversally and the interval I_t contains $[0, \tau_0]$ such that $\tau_\lambda \in I_t$ for every $\lambda \in I_\lambda$. Note that τ_λ is the solution to $\gamma_\lambda(\tau_\lambda) \in H_2$. By the Implicit Function Theorem, the travel time function τ_λ is smooth for $\lambda \in I_\lambda$.

Now we consider the variation field

$$X_\lambda(t) := \partial_\lambda \Gamma(\lambda, t),$$

which is a Jacobi field along the geodesic $\gamma_\lambda(t)$. We denote by $\dot{\gamma}_\lambda(t) = \partial_t \Gamma(\lambda, t)$ with $\dot{\gamma}_\lambda(0) = v(\lambda)$. Along each geodesic, the quantity $g(\dot{\gamma}_\lambda(t), \dot{\gamma}_\lambda(t))$ is invariant and therefore we denote it by the level set

$$h(\lambda) := g(\dot{\gamma}_\lambda(t), \dot{\gamma}_\lambda(t)). \tag{6.6.1}$$

In particular, we can find $h(\lambda)$ by checking the initial data of this family of geodesics at H_1, i.e., by

$h(\lambda) = g(\dot{\gamma}_\lambda(0), \dot{\gamma}_\lambda(0))$. We compute its derivative as below:

$$h'(\lambda) = \partial_\lambda\big(g(\dot{\gamma}_\lambda(t), \dot{\gamma}_\lambda(t))\big) = 2g(D_\lambda \dot{\gamma}_\lambda(t), \dot{\gamma}_\lambda(t)) = 2g(D_t X_\lambda(t), \dot{\gamma}_\lambda(t)),$$

where D_λ, D_t are covariant derivatives along these curves and the last equality uses the symmetry lemma, for example, see [97, Lemma 6.2]. With $D_t \dot{\gamma}_\lambda(t) = 0$, we have

$$h'(\lambda) = 2g(D_t X_\lambda(t), \dot{\gamma}_\lambda(t)) + 2g(X_\lambda(t), D_t \dot{\gamma}_\lambda(t)) = 2\partial_t\big(g(X_\lambda(t), \dot{\gamma}_\lambda(t))\big)$$

Integrating it with respect to t from 0 to τ_λ, we have

$$h'(\lambda)\tau_\lambda = 2g(X_\lambda(\tau_\lambda), \dot{\gamma}_\lambda(\tau_\lambda)) - 2g(X_\lambda(0), \dot{\gamma}_\lambda(0)). \tag{6.6.2}$$

Moreover, if we denote by

$$y(\lambda) = \gamma_\lambda(\tau_\lambda)$$

the endpoints of the geodesic segment $\gamma_\lambda(t)$ at S_2, then by the chain rules we have

$$y'(\lambda) = \frac{\mathrm{d}}{\mathrm{d}\lambda}\gamma_\lambda(\tau_\lambda) = \partial_\lambda \gamma_\lambda(t)|_{t=\tau_\lambda} + \partial_t \gamma_\lambda(t)\frac{\mathrm{d}\tau_\lambda}{\mathrm{d}\lambda} = X_\lambda(\tau_\lambda) + \dot{\gamma}_\lambda(\tau_\lambda)\frac{\mathrm{d}\tau_\lambda}{\mathrm{d}\lambda}. \tag{6.6.3}$$

Combining (6.6.2) and (6.6.3), one has

$$h'(\lambda)\tau_\lambda = 2g(y'(\lambda), \dot{\gamma}_\lambda(\tau_\lambda)) - 2g(\dot{\gamma}_\lambda(\tau_\lambda), \dot{\gamma}_\lambda(\tau_\lambda))\frac{\mathrm{d}\tau_\lambda}{\mathrm{d}\lambda} - 2g(X_\lambda(0), \dot{\gamma}_\lambda(0)).$$

Using (6.6.1), we prove the following proposition.

Proposition 6.6.3. *Let $H_1, H_2 \subset M$ be smooth hypersurfaces and $\gamma_0 : [0, T_0] \to M$ be the unique geodesic segment starting from $x_0 \in H_1$ in the transversal direction v_0. Suppose γ_0 hits H_2 transversally at $y_0 = \gamma_0(\tau_0)$. Consider a one-parameter family of geodesics $\gamma_\lambda(t)$ near γ_0 with initial data $(x(\lambda), v(\lambda))$ smoothly depending on λ, $x(0) = x_0$ and $v(0) = v_0$. Let τ_λ be the travel time such that $\gamma_\lambda(\tau_\lambda) \in H_2$. Then near $\lambda = 0$,* *one has*

$$2h(\lambda)\frac{\mathrm{d}\tau_\lambda}{\mathrm{d}\lambda} + h'(\lambda)\tau_\lambda = 2g(y'(\lambda), \dot{\gamma}_\lambda(\tau_\lambda)) - 2g(x'(\lambda), \dot{\gamma}_\lambda(0)), \tag{6.6.4}$$

where $h(\lambda) = g(\dot{\gamma}_\lambda(0), \dot{\gamma}_\lambda(0))$ *and* $y(\lambda) = \gamma_\lambda(\tau_\lambda)$.

This proposition relates the change of the travel time with the observations we could get from the scattering relation on H_1 and H_2. Note that $y'(\lambda)$ and $x'(\lambda)$ are always tangent to the hypersurfaces, so the equation can give the first variation of travel time using projected scattering relation as well. In the Riemannian setting or for timelike geodesics in the Lorentzian setting, equation (6.6.4) is the same as the first variation of the length function:

$$l'(\lambda) = \frac{d}{d\lambda}\left(\sqrt{\pm h(\lambda)}\tau(\lambda)\right) = \pm\frac{1}{\sqrt{\pm h}}\left(g(y', \dot{\gamma}_\lambda(\tau_\lambda)) - g(x', \dot{\gamma}_\lambda(0))\right), \tag{6.6.5}$$

with $+$ in the Riemannian case and $-$ in the Lorentzian timelike case. However, if we consider the variation around a lightlike geodesic, the length function fails to be differentiable there and we no longer have the first variation of the length function. Instead, we may use (6.6.4) to recover the travel time of this lightlike geodesic directly, provided we have timelike or spacelike scattering relation so that $h'(0) \neq 0$. Namely, in this case the travel time for the lightlike geodesic is simply

$$\tau_0 = \frac{2}{h'(0)}[g(y'(0), \dot{\gamma}_\lambda(\tau_\lambda)) - g(x'(0), \dot{\gamma}_\lambda(0))]. \tag{6.6.6}$$

However, when both $h'(0) = 0$ and $h(0) = 0$, the equality becomes

$$g(y'(0), \dot{\gamma}_0(\tau_0)) = g(x'(0), \dot{\gamma}_0(0)),$$

or equivalently, $\dot{\gamma}_0(\tau_0)^\flat(y'(0)) = \dot{\gamma}_0(0)^\flat(x'(0))$, which only involves scattering relation for covectors. In particular, this implies the lightlike scattering relation along can not recover the travel time. This is compatible with the fact that a conformal change leaves the scattering relation for covectors unchanged, but the parametrization of the lightlike geodesic is changed, which results in different travel time.

6.6.3 Lorentzian setting

We are now ready for the Lorentzian setting. Let (M, g) be a Lorentzian manifold with timelike boundary, we do not require analyticity here. Suppose the lightlike geodesics are non-trapping. Then there exists a conic open neighborhood $\mathcal{U}'$ of ∂LM such that the corresponding geodesics are also non-trapping. Denote by $\mathcal{U} = \mathcal{U}' \cap \partial JM$ the causal ones. We use $\mathcal{U}_\pm$ to denote inward $(-)$ and outward $(+)$ pointing ones. Then S is well-defined on $\overline{\mathcal{U}}_-$ and S^{in} is well-defined on $\mathcal{U}_-$. We show that interior lightlike travel time can be recovered directly from interior scattering relation. This requires the knowledge of nearby timelike directions, so that we can apply the first variation of travel time formula (6.6.4) in a non-trivial way.

Lemma 6.6.4. *Let (M, g) and $\mathcal{U}_\pm$ be defined as above. Suppose $g|_{\partial TM \times \partial TM}$ is given. Then the lightlike interior travel time data $(S^{in}, \tau^{in})|_{\partial_- LM}$ can be recovered from $S^{in}|_{\mathcal{U}_-}$.*

Proof. Given $(x, v) \in \partial_- LM$, we denote by $(y, w) = S^{in}(x, v)$, where w is either tangential to ∂M or outward pointing. Then the lightlike geodesic corresponding to $(y, -w)$ leaves the manifold at $(x, -v)$ transversally, and stays in the interior of M except for two ends. Pick a smooth one parameter family of outward pointing timelike vectors $w(\lambda)$ for $\lambda \in (0, 1)$ such that $w(\lambda) \to w$ as $\lambda \to 0$. Denoting by $h(\lambda) = g(w(\lambda), w(\lambda))$, we first show that we may choose $w(\lambda)$ such that $h'(0) = -2$.

For this purpose, we consider some normal coordinates at y such that g is Minkowski at y. Then in local coordinates ∂_t is tangent to ∂M and we choose $w = \partial_t + \partial_1$. Moreover, we set

$$w(\lambda) = \partial_t + (1 - \lambda)\partial_1. \tag{6.6.7}$$

If w is outward pointing then this holds for $w(\lambda)$ when $\lambda \ll 1$; if w is tangent to ∂M then so are all the $w(\lambda)$. Then $h'(0) = -2$.

For $\lambda \ll 1$, there exists a unique $(x(\lambda), v(\lambda))$ such that $S^{in}(x(\lambda), v(\lambda)) = (y, w(\lambda))$. Moreover, by the Implicit Function Theorem, $(x(\lambda), v(\lambda))$ is a smooth curve converging to (x, v) when $\lambda \to 0$. By Proposition 6.6.3, see also (6.6.6), the first variation formula for the travel time gives

$$\tau^{in}(x, v) = \frac{-2}{h'(0)} g(x'(0), v) = g(x'(0), v).$$

$\square$

Comparing to Lemma 6.6.1, we show that the lightlike interior travel time data can be recovered from the complete scattering relation defined for timelike vectors near the light cone, without the need of the complete travel time data.

Lemma 6.6.5. *Let (M, g) and $\mathcal{U}_\pm$ be defined as above. Suppose $g|_{\partial TM \times \partial TM}$ is given. Then the lightlike interior travel time data $(S^{in}, \ell^{in})|_{\partial_- LM}$ can be recovered from $S|_{\overline{\mathcal{U}_-}}$.*

Proof. We may view $(S, \ell)(z, u) = (z, u; 0)$ for all $(z, u) \in \partial_+ TM$. Given $(x, v) \in \partial_- LM$, we consider

$$\varphi(x, v, \cdot) \cap \partial TM = \{(z, u) \in \partial LM : S(z, u) = S(x, v)\}.$$

Clearly (x, v) and $S(x, v)$ are in the set. If they are the only elements in the set, then $S^{in}(x, v) = S(x, v)$. If the set has cardinality 2, then besides (x, v) itself, the other element must be $S^{in}(x, v) = S(x, v)$. Otherwise $S^{in}(x, v) \neq S(x, v)$. Now for each

$$(z, u) \in (\varphi(x, v, \cdot) \cap \partial TM) \setminus \{(x, v), S(x, v)\},$$

it must be tangential to ∂M. There are two possibilities:

1. there exists a one parameter family of outward pointing timelike $(z, u(\lambda))$ converging to (z, u) as $\lambda \to 0$, and a one parameter family of inward pointing timelike $(x(\lambda), v(\lambda))$ converging to (x, v) as $\lambda \to 0$, such that $S(x(\lambda), v(\lambda)) = (z, u(\lambda))$;
2. or such one parameter family does not exist.

For case (1), we may construct the same $w(\lambda)$ and then $h(\lambda)$ as in the proof of Lemma 6.6.4, see (6.6.7). Applying Proposition 6.6.3 again, we conclude that the travel time from (x, v) to (z, u) is given by

$$\tau(z, -u) = \frac{-2}{h'(0)} g(x'(0), v).$$

Note that $S^{in}(x, v)$ is in the set and falls in case (1), as is illustrated in the proof of Lemma 6.6.4. In addition, it gives the minimal $\tau(z, -u)$. Thus, we can compute $\tau(z, -u)$ for all (z, u) satisfying case (1) and then set $S^{in}(x, v)$ to be the one that minimizes $\tau(z, -u)$.

$\square$

The recovery of the complete travel time data from the interior scattering relation needs more delicate treatment. If we are given the complete scattering relation for causal directions sufficiently close to the light cone, then we can use Proposition 6.6.3 to recover the travel time. However, the situation is more complicated if we are given the interior scattering relation for causal directions sufficiently close to the light cone. In this case, we first use a similar strategy as Lemma 6.6.2 to recover the complete scattering relation for causal directions sufficiently close to the light cone, and then apply Proposition 6.6.3.

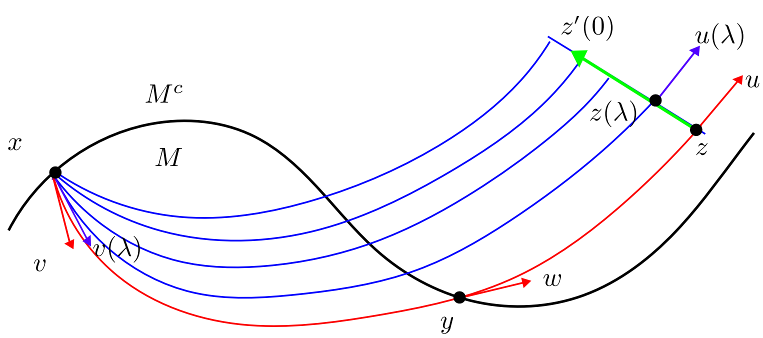

Figure 6.7: For every $(x, v) \in \overline{\partial_- LM}$, one can find a smooth one parameter family of inward pointing timelike directions converging to v. Their flow forms a 2-dimensional band, and in the exterior region we choose a transversal slice $z(\lambda)$ that approaches z as $\lambda \to 0$. The first variation formula of travel time then recovers the travel time of (x, v) to the exterior point z.

Lemma 6.6.6. *Let (M, g) and $\mathcal{U}_\pm$ be defined as above. Suppose there exists $U \subset M$ an open neighborhood of ∂M such that $g|_U$ is given. Then the lightlike complete travel time data $(S, \tau)|_{\overline{\partial_- LM}}$ can be recovered from $S^{in}|_{\mathcal{U}_-}$ or $S|_{\overline{\mathcal{U}}_-}$.*

Proof. We first assume $S^{in}|_{\mathcal{U}_-}$ is given. Again by extending M to some $\tilde{M}$, we know the metric on $V := U \cup (\tilde{M} \setminus M)$. We will prove the statement in two steps:

1. we first show, using the same idea as the Riemannian setting, that we can recover the complete scattering relation for all causal directions sufficiently close to the light cone;

2. we then use the complete scattering relation to compute the complete travel time of lightlike geodesics using first variation formula of travel time.

Step 1. Consider some causal vector $(x, v) \in \overline{\partial_- TM}$ that are close to the light cone. Denote by φ the geodesic flow. By the non-trapping assumption of lightlike geodesics, the geodesic $\varphi(x, v, \cdot)$ is non-trapping as well when (x, v) is sufficiently close to the light cone. As a result, $\varphi(x, v, \cdot)$ is uniformly close to LM. If $\varphi(x, v, t)$ is tangential to the boundary for some t, then we may assume it is in $\overline{U_-}$ by the closeness to LM. In particular, this means for inward pointing causal directions sufficiently close to $\varphi(x, v, t)$, it will be in $\mathcal{U}_-$. We denote $\mathcal{V}$ as the set of such (x, v), and note that it contains $\overline{\partial_- LM}$.

Now pick some $(x, v) \in \mathcal{V}$, since we know the metric in U and $\varphi(x, v, \cdot)$ is non-trapping, we can keep track of how the geodesic from (x, v) leaves U for the first time (see Figure 6.6). There are two possibilities:

label=(i) it leaves M;

lbbel=(ii) it enters $M \setminus U$.

Denote the time of leaving by T and direction by (z, u). For case (i), we refer to (z, u) as the exiting direction of (x, v).

For case (ii), let $(z', u') = \varphi(x, v, T')$ be such that

$$T' = \inf\{t > 0 : \varphi(x, v, s) \in U \setminus \partial M \text{ for } s \in [t, T)\}.$$

In other words, (z', u') is the last time the geodesic touches the boundary before entering $M \setminus U$. Note that (z', u') may just be (x, v) itself. We consider a geodesically convex neighborhood $W \subset \tilde{M}$ of z'. There exists small δ such that $\varphi(z', u', t)$ stays in $W \cap M$ for all $t \in [0, \delta]$. We denote by $(y, w) = \varphi(z', u', \delta)$. Pick $p \in (W \setminus M) \cap I^-(z')$. Now such p exists because the boundary is timelike. Then the unique timelike geodesic from p to y intersects with ∂M. As before, Let $\xi \in T_pW$ be such that (p, ξ) is sufficiently close to (z', u'). Using Lemma 6.8.1, there exists timelike (q, η) sufficiently close to (p, ξ) such that the timelike geodesic from (q, η) intersects with ∂M at least once before leaving W, and all intersections are transversal. For (q, η) sufficiently close to (p, ξ) and p sufficiently close to z', we have $\varphi(q, \eta, \cdot)$ stays close to $\varphi(z', u', \cdot)$ and eventually enters $M \setminus U$. In particular, we can assume that $\varphi(q, \eta, t)$ stays in the interior of U for $t_W(q, \eta) \leq t \leq t_U(q, \eta)$, where $t_W(q, \eta)$ is the time it leaves W and $t_U(q, \eta)$ is the time it enters $M \setminus U$. Backtracking from $\varphi(q, \eta, t_W(q, \eta))$ until it hits ∂M for the first time, we denote that time by $t'(q, \eta)$ and direction by $(q', \eta') = \varphi(q, \eta, t'(q, \eta))$.

To summarize, we have shown that $\varphi(q, \eta, t) \in U \setminus \partial M$ for $t \in (t'(q, \eta), t_U(q, \eta))$ before entering $M \setminus U$, and is transversal to the boundary at time $t'(q, \eta)$ with direction (q', η'). Moreover, (q', η') will be sufficiently close to the light cone if (q, η) is sufficiently close to (z', u'), because $\varphi(x, v, \cdot)$ is uniformly close to light cone. As a result, $(q', \eta') \in \mathcal{U}_-$, and $S^{in}(q', \eta')$ is well-defined. We may find a sequence of such (p_j, ξ_j) converging to (z', u'), and a sequence of the corresponding $S^{in}(q_j', \eta_j') = (y_j, w_j)$. By non-trapping of $\varphi(x, v, \cdot)$, $\varphi(q', \eta', \cdot)$ will be non-trapping as the two are close. Hence we may assume they live in a tubular neighborhood of $\varphi(x, v, \cdot)$, whose closure is thus compact. By compactness and passing down to subsequence, we may assume $(y_j, w_j) \to (y_*, w_*) \in \partial SM$. By our construction, (y_*, w_*) will be on the flow of $\varphi(z', u', \cdot)$ because (q_j, η_j) will converge to (z', u'). Moreover, if we denote T_j as the time $\varphi(q_j, \eta_j, T_j) = (y_j, w_j)$, then $T_j > t_U(q_j, \eta_j) > \delta$. Hence the limit of T_j is bounded below by δ, meaning $(y_*, w_*) = \varphi(z', u', t)$ for some $t > \delta$. We emphasize that this means $(y_*, w_*) = \varphi(x, v, t)$ for some $t > \delta$.

We can finish the first step as the Riemannian setting. For any $(x, v) \in \mathcal{V}$, it falls in one of the two categories. If it is case (i), we set $S(x, v)$ to be the exiting direction and we are done. If it is case (ii), then we will find some (y_*, w_*). By definition of $\mathcal{V}$, (y_*, w_*) will still be in $\mathcal{V}$, and we can repeat the procedure on (y_*, w_*). Note that we can guarantee (y_*, w_*) is at least δ time after (x, v). We claim δ can be uniform in each step. Note that by non-trapping assumption we may assume everything happens in a compact region of the manifold. Then by compactness, the injective radius of the Lorentzian metric (measured with respect to some auxiliary Riemannian metric g_+) is positive. Moreover, there exists a uniform lower bound for boundary unit directions (with respect to g_+) to enter $M \setminus U$. As $|\varphi(x, v, \cdot)|_{g_+}$ is bounded above by non-trappingness, the time it takes for $\varphi(x, v, s)$ to enter $M \setminus U$ every time it touches the boundary is bounded below. We can now take δ to be the minimum of half of the injective radius and the time taken to enter $M \setminus U$ from boundary. Since δ is uniform, the procedure will terminate in finite steps by the non-trapping assumption. In other words, we will reach case (i) in finitely many steps, and we set $S(x, v)$ to be the final exiting direction (z, u). $S|_{\mathcal{V}}$ is recovered from $S^{in}|_{\mathcal{U}_-}$.

Step 1. Now we use $S|_{\mathcal{V}}$ to recover the travel time of lightlike geodesics. For any $(z, u) \in \mathcal{V}$, since we know $S(z, u)$, we may identify the segment of $\varphi(z, u, \cdot) \cap TV$ that contains $S(z, u)$. That is, how the geodesic from (z,u) travels near the boundary around $S(z,u)$. Fix some $(x,v) \in \overline{\partial_- LM}$, denote $(y,w) = S(x,v)$. Let $v(\lambda) \in \partial_- T_x M$ be a smooth one parameter family of timelike directions converging to v as $\lambda \to 0$. Again we may choose the parameterization such that $h(\lambda) := g(v(\lambda), v(\lambda))$ satisfies $h'(0) = -2$. Denote $(z,u) = \varphi(y,w,t_0)$ for some unknown t_0 such that $z \in \tilde{M} \setminus M$. Consider a small hypersurface $H \subset \tilde{M} \setminus M$ containing z such that u is transversal to H. For $\lambda \ll 1$, $\varphi(x, v(\lambda), \cdot)$ will eventually pass through H, in particular it will leave M (not necessarily its first exit) before reaching H.

Consider $S(x, v(\lambda))$, as explained above we can construct the segment of the geodesic containing $S(x, v(\lambda))$ in V. If the segment intersects with H, we denote the intersection as $(z(\lambda), u(\lambda))$. Otherwise, we keep tracking the flow until it reaches $\overline{\partial_- LM}$ again. We repeatedly apply S and check if the segment passes through H, and find the next entry point if not. Eventually it will pass through H for the first time at $(z(\lambda), u(\lambda))$. Note that since $(z(\lambda), u(\lambda))$ is on the flow $\varphi(x, v(\lambda), \cdot)$ with $v(\lambda) \to v$, and (z,u) is the first time $\varphi(x, v, \cdot)$ passes through H, we must have $(z(\lambda), u(\lambda)) \to (z,u)$.

To summarize, we have found $v(\lambda), z(\lambda), u(\lambda)$ such that

- $(z(\lambda), u(\lambda)) \to (z,u)$ as $\lambda \to 0$;
- $z'(0)$ is not scaling of u;
- and $(z(\lambda), u(\lambda)) = \varphi(x, v(\lambda), t(\lambda))$ for some unknown time $t(\lambda)$.

See Figure 6.7. By Proposition 6.6.3, one has

$$t(0) = \frac{-2}{h'(0)} g(z'(0), u(0)) = g(z'(0), u).$$

We gather all such $t(0)$, and take the infimum, denoted by τ. By construction we know $\tau \geq \tau(x,v)$ because each (z,u) is after (y,w). On the other hand, there exists a sequence of such (z_j, u_j) in the exterior region converging to (y,w), so the infimum is precisely $\tau(x,v)$.

We have proved the proposition for S^{in}; if we are given $S|_{\overline{\mathcal{U}}_-}$, then we directly apply Step 2 to recover the lightlike complete travel time. Thus the proof is finished. $\square$

6.7 Scattering rigidity

We can now prove Theorem 6.1.4 and Theorem 6.1.5, using the results from Section 6.5 and Section 6.6.

Proof of Theorem 6.1.4. Since the lightlike geodesics are non-trapping, this holds for all timelike geodesics sufficiently close to the light cone, so $\mathcal{U}_j$ does exist. By the assumptions stated in the theorem, the non-conjugacy condition holds for at least one tangential direction in each connected component of ∂M_1, where S_1^{in} is known for all transversal directions close to it. We can then apply Theorem 6.5.3 to recover the jet of g_1 at that point. By analyticity of the boundary and the metric, this determines the jet of g_1 on each component and thus the entire ∂M_1.

Similar to [153], even though the non-conjugacy condition only holds with respect to g_1, the jet of g_2 can be computed in the same way. This is because the non-conjugacy assumption is only used to find y_{ϵ_j}, y_0^*, and a sequence of diffeomorphisms between U_j^+ and $\partial_- T_{y_{\epsilon_j}} M_1$ for each j in the proof of Theorem 6.5.3. Then, since the scattering data are the same, these diffeomorphisms still hold between $\varphi_0(U_j^+)$ and $(\varphi_0)_*(\partial_- T_{y_{\epsilon_j}} M_1)$ (recall we can view $(\varphi_0)_*$ as a map from ∂TM_1 to ∂TM_2 in boundary normal coordinates). One can then perform the same computation for the Eikonal equation with respect to g_2 and use the same scattering data, which shows that the jet of g_2 agrees with the jet of g_1 (through φ_0 of course). That is, for any $(x, v) \in T\partial M_1$, $k \in \mathbb{N}$, in boundary normal coordinates we have

$$(\partial_n^k (g_1)_{\alpha\beta})(x) v^\alpha v^\beta = (\partial_n^k (g_2)_{\alpha\beta})(\varphi_0(x))((\varphi_0)_* v)^\alpha ((\varphi_0)_* v)^\beta.$$

As a result, one can analytically extend M_j to a slightly larger manifold $\tilde{M}_j$, similar to Section 2 of [153], also see Section 6.2.3. The only difference being that the extension may not be uniform in size. Nevertheless, the determination of jet gives the following: there exists $U_j \subset \tilde{M}_j$ open set of ∂M_j such that φ_0 can be extended to an analytic isometry on U_j via normal exponential maps. We may assume $\tilde{M}_j = U_j \cup M_j$ and by abuse of notation denote φ_0 also as the isometry on U_j. Now we can apply Lemma 6.6.6. Since φ_0 clearly maps $\overline{\partial_- LM_1}$ to $\overline{\partial_- LM_2}$, we obtain from the lemma that the lightlike complete travel time data are equivalent. Finally we apply Proposition 6.3.1 to obtain isometry between M_1 and M_2.

$\square$

Proof of Theorem 6.1.5. The proof is mostly the same as the proof of Theorem 6.1.4. For the recovery of the jet of boundary metric, we still use Theorem 6.5.3. After extending to $U_j \cup M_j$, we again use Lemma 6.6.6 to recover the lightlike complete travel time data. The rest of the proof is the same as Theorem 6.1.4. $\square$

6.8 Transversal geodesics

In this part, we include two lemmas in order to find transversal geodesics.

Lemma 6.8.1. *Let (M, g) be a Riemannian or semi-Riemannian manifold and $H \subset M$ be a smooth hypersurface. Let $(x_0, v_0) \in SM$ and $W \subset M$ be a convex neighborhood of x_0 such that $H \cap W = \{x : \phi(x) = 0\}$, where ϕ is the smooth defining function. Suppose $\phi(x_0) > 0$ and $\phi(y_0) < 0$ with $y_0 = \gamma_{x_0,v_0}(t_0)$ for some $t_0 > 0$. Then there exists a small conic neighborhood $V_0 \subset S_{x_0}M$ of v_0 such that for almost every $v \in V_0$, the geodesic segment $\gamma_{x_0,v}((0, t_0))$ intersects with H and the intersections are always transversal.*

Proof. We prove the statement using Sard's Theorem. With the assumption $\phi(x_0) > 0$ and $\phi(y_0) < 0$, we can find a sufficiently small conic neighborhood $V_0 \subset S_{x_0}M$ of v_0, such that $\phi(\gamma_{x_0,v}(t_0)) < 0$ for any $v \in V_0$. Then we consider the smooth function

$$f(v, t) = \phi(\gamma_{x_0,v}(t)), \quad \text{for } (v, t) \in V_0 \times (0, t_0)$$

and its zero set

$$Z = \{(v, t) \in V_0 \times (0, t_0) : f(v, t) = 0\}.$$

Note that Z is a smooth one-codimensional submanifold of $V_0 \times (0, t_0)$, as we have $\mathrm{d}_{(t,v)} f \neq 0$ provided that V_0 is sufficiently small. Let $\pi : Z \to V_0$ be the projection given by $\pi(v, t) = v$. By our choice of V_0, for each v there, we can find t_v such that $\phi(\gamma_{x_0,v}(t_v)) = 0$, which implies $(v, t_v) \in Z$ and therefore $\pi(Z) = V_0$.

Now consider the differential $\mathrm{d}\pi_{(v,t)} : T_{(v,t)}Z \to T_v V_0$. Let $(\delta v, \delta t) \in T_{(v,t)}Z$, which satisfies

$$\mathrm{d}_v f(\delta v) + \mathrm{d}_t f(\delta_t) = 0.$$

On the one hand, when $\gamma_{x_0,v}$ intersects H transversally at $t = t_v$, the differential $\mathrm{d}\pi$ is surjective with $\mathrm{rank}(\mathrm{d}\pi) = n - 1$. Indeed, in this case, one has $\partial_t f(v,t) = \mathrm{d}\phi(\dot{\gamma}_{x_0,t}(t_v)) \neq 0$. Then we can solve δ_t in terms of δ_v from the condition above. This implies the map $\mathrm{d}\pi_{(v,t)}(\delta v, \delta t) = \delta_v$ is surjective and we have the desired rank. On the other hand, when $\gamma_{x_0,v}$ intersects H tangentially at $t = t_v$, all $(\delta v, \delta t) \in T_{(v,t)}Z$ must satisfy

$$0 = \mathrm{d}_v f(\delta v) = \mathrm{d}\phi \circ \mathrm{d}_v \exp_{x_0}|_{t_v v}(\delta v).$$

With $\gamma_{x_0,v}(t_v) \in W \cap H$, the differential $\mathrm{d}_v \exp_{x_0}|_{t_v v}$ is invertible and therefore

$$\mathrm{d}\phi \circ \mathrm{d}_v \exp_{x_0}|_{t_v v} \neq 0.$$

It follows that the condition above gives restriction for δv and the map $\mathrm{d}\pi_{(v,t)}(\delta v, \delta t) = \delta_v$ has rank less than $n-1$. If we denote the critical set of π by

$$T_0 = \{(t,v) \in Z : \mathrm{rank}(\mathrm{d}\pi) < n-1\},$$

then the arguments above proves

$$T_0 = \{(t,v) \in Z : \partial_t f(t,v) = 0\}.$$

By Sard's Theorem, the image $\pi(T_0)$ has Lebesgue measure zero in $V_0 = \pi(Z)$, where $\pi(T_0)$ is the set of directions of which the geodesics intersect H tangentially before $t = t_0$. Thus, we prove the desired result. $\square$

Lemma 6.8.2. *Let (M,g) be a Riemannian or semi-Riemannain manifold with smooth boundary and $(\tilde{M}, \tilde{g})$ be its extension. Let $(x_0, v_0) \in S\tilde{M}$ with $x_0 \in \tilde{M} \backslash M$. Suppose the geodesic γ_{x_0,v_0} satisfies $\gamma_{x_0,v_0}((0,\delta)) \subset M^\circ$ and then $\gamma(t_0) \in \tilde{M} \setminus M$ for some $t_0 > \delta > 0$. Moreover, suppose x_0 is not conjugate to any point in $\gamma_{x_0,v_0}([0,t_0]) \cap \partial M$. Then there exists a small conic neighborhood $V_0 \subset S_{x_0}M$ of v_0 such that for almost every*

$v \in V_0$, the geodesic segment $\gamma_{x_0,v}((0,t_0))$ intersects with ∂M and the intersections are always transversal.

Proof. We consider the set

$$P_0 = \gamma_{x_0,v_0}([0,t_0]) \cap \partial M.$$

It is compact and we can pick N points $p_1, \ldots, p_N \in P_0$ such that P_0 is covered by the convex neighborhoods of these points, i.e., $P_0 \subset \cup_{j=1}^N W_j$, where W_j is the convex neighborhood of p_j. By choosing sufficiently small $V_0 \subset S_{x_0}M$, we can assume $\gamma_{x_0,v}((0,t_0))$ always intersects ∂M at some points in $\cup_{j=1}^N W_j$.

With the assumption on conjugate points, we may shrink these neighborhood W_j and V_0 such that for any $v \in V_0$, the exponential map $\exp_{x_0} : tv \mapsto y = \exp_{x_0}(tv) \in W_j \cap \partial M$ for some $t > 0$ and $1 \leq j \leq N$ has non-degenerate differential and therefore is a local diffeomorphism. Now for fixed j, let ϕ_j be the smooth boundary defining function for $W_j \cap \partial M$, when W_j is sufficiently small. We perform the same argument as in Lemma 6.8.1. More explicitly, we define

$$f_j(v,t) = \phi_j(\gamma_{x_0,v}(t)), \quad \text{for } (v,t) \in V_0 \times (0,t_0)$$

and its zero set

$$Z_j = \{(v,t) \in V_0 \times (0,t_0) : f_j(v,t) = 0\}.$$

Similarly, Z_j is a smooth one-codimensional submanifold of $V_0 \times (0,t_0)$, as we have $\mathrm{d}_{(t,v)} f_j \neq 0$ by the no conjugate point assumption, provided that V_0 is sufficiently small. Note that $\pi(Z_j)$ might be empty but we always have $\pi(Z_j) \subset V_0$ by its definition. If $\pi(Z_j) \neq \emptyset$, then as we denote the critical set of π by

$$T_{0,j} = \{(t,v) \in Z : \mathrm{rank}(\mathrm{d}\pi) < n-1\}.$$

The same arguments as in the proof of Lemma 6.8.1 proves $T_{0,j} = \{(t,v) \in Z : \partial_t f(t,v) = 0\}$. By the Sard's Theorem, the set $\pi(T_{0,j})$ has Lebesgue measure zero in $\pi(Z_j)$ and therefore has measure zero in V_0. Thus, we consider all elements $v \in V_0 \setminus \bigcup_{j=1}^N \pi(T_{0,j})$. This proves the desired result.

$\square$

6.9 Jet construction with strictly convex directions

We include here a constructive proof of recovering the normal jet of the metric on the boundary near a strictly convex direction. We derive the one-sided Taylor expansion of the travel time, and inductively recover the higher order jet from the expansion. In [133], similar ideas are used to prove the stability of recovery.

6.9.1 Setup

Let (M, g) be a Riemannian manifold with boundary or a Lorentzian manifold with boundary. Suppose the metric around some $p \in \partial M$ and $v \in T_p\partial M$ is strictly convex in the v direction. For the Lorentzian setting, we assume the boundary near p is either timelike or spacelike, and v is either timelike or spacelike. Here we do not treat the case that v is lightlike separately, as the strict convexity is an open condition. Indeed, if v is lightlike and strictly convex, then one can always choose a timelike or spacelike direction that is also strictly convex. Recall we defined strict convexity to be

$$\mathrm{I\!I}(v, v) = g(\nabla_v \nu, v) > 0,$$

where $\mathrm{I\!I}$ is the second fundamental form and ν is the unit outward normal vector (see Section 6.1). Since the manifold is either Riemannian or Lorentzian with timelike or spacelike boundary, we can write the metric locally in semi-geodesic normal coordinate around p as

$$g = g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta + (\mathrm{d}x^n)^2, \quad \alpha, \beta = 1, \dots, n-1,$$

where the interior is $x^n > 0$. Then strict convexity translates to

$$0 < \mathrm{I\!I}(v, v) = g(\nabla_v \nu, v) = -g(\nabla_v \partial_n, v) = -\frac{1}{2}\partial_n g_{\alpha\beta} v^\alpha v^\beta.$$

Note that in this case, for directions near (p,v), the interior and complete scattering information coincide. We will first show that if the scattering relation is known in a neighborhood of $v \in T_p\partial M$, then the jet of g can be constructed from the scattering relation S near (p,v) and the boundary metric $g|_{T\partial M \times T\partial M}$ near p.

Let $v_\varepsilon \in T_p\partial M$ be a one parameter family of directions approaching v, the scattering relation is known for $\varepsilon \in [0,\delta)$ where $\delta \ll 1$. In the Riemannian setting we choose $v_\varepsilon = \sqrt{1-\varepsilon^2}v + \varepsilon\partial_n$; in the Lorentzian setting we choose $v_\varepsilon = \sqrt{1+\varepsilon^2}v + \varepsilon\partial_n$. Then the travel time (exit time) $\tau(\varepsilon) := \tau(x, v_\varepsilon)$ as a function of ε is smooth in $(0,\delta)$ by the Implicit Function Theorem, since the geodesics corresponding to v_ε, denoted by γ_ε, leave the manifold transversally by strict convexity.

For convenience, we use the notation $V_\varepsilon(s) = \dot\gamma_\varepsilon(s)$ to denote the vector field, in which case $V_\varepsilon^j(f)$ means V_ε acting on the function f for j times; while $v_\varepsilon^j(s) = dx^j(V_\varepsilon)(s)$ is used as the j-th component of the geodesic γ_ε at time s. We shall also omit $s=0$ most of the time and simply write $v_\varepsilon^j = v_\varepsilon^j(0)$. Then

$$\begin{aligned} x^n(\gamma_\varepsilon(s)) &= \sum_{j=0}^{N} V_\varepsilon^j(x^n)|_{s=0} \cdot \frac{1}{j!}s^j + O(s^{N+1}) \\ &= \varepsilon s + \frac{1}{4}\partial_n g_{\alpha\beta} v_\varepsilon^\alpha v_\varepsilon^\beta \cdot s^2 + \sum_{j=3}^{N} V_\varepsilon^{j-1}(v_\varepsilon^n)|_{s=0} \cdot \frac{1}{j!}s^j + O(s^{N+1}). \end{aligned}$$

For the last equality, we used the fact that $\Gamma^n_{\alpha\beta} = -\frac{1}{2}\partial_n g_{\alpha\beta}$, and

$$\begin{aligned} 0 &= \mathrm{d}x^\alpha(\nabla_{V_\varepsilon}V_\varepsilon) = V_\varepsilon(v_\varepsilon^\alpha) + \sum_{1\le k,l\le n} \Gamma^\alpha_{kl} v_\varepsilon^k v_\varepsilon^l, \qquad \text{for } 1 \le \alpha \le n-1, \\ 0 &= \mathrm{d}x^n(\nabla_{V_\varepsilon}V_\varepsilon) = V_\varepsilon(v_\varepsilon^n) + \sum_{1\le k,l\le n-1} \Gamma^n_{kl} v_\varepsilon^k v_\varepsilon^l, \end{aligned} \tag{6.9.1}$$

since $\Gamma^n_{nl} = 0$ for $l = 1, \ldots, n$. In particular, by the choice of v_ε, we have

$$V_\varepsilon(0) = v_\varepsilon^\alpha \partial_\alpha + \varepsilon\partial_n.$$

Denote by

$$K_\varepsilon = \frac{1}{2}\partial_n g_{\alpha\beta} v_\varepsilon^\alpha v_\varepsilon^\beta = \frac{1}{2}\partial_n g_{\alpha\beta} v^\alpha v^\beta + O(\varepsilon^2) := K + O(\varepsilon^2). \tag{6.9.2}$$

Note that by strict convexity, we know $K, K_\varepsilon \neq 0$ for $\varepsilon \ll 1$. In fact they are uniformly bounded away from

0. Then at $s = \tau(\varepsilon)$, we have

$$\begin{aligned}\tau(\varepsilon) &= -2K_\varepsilon^{-1}\left[\varepsilon + \sum_{j=2}^{N-1} V_\varepsilon^j(v_\varepsilon^n)|_{s=0} \cdot \frac{1}{(j+1)!}\tau(\varepsilon)^j\right] + O(\tau(\varepsilon)^N) \\ &= -2K_\varepsilon^{-1}\left[\varepsilon + \sum_{j=2}^{N-1} V_\varepsilon^j(v_\varepsilon^n)|_{s=0} \cdot \frac{1}{(j+1)!}\tau(\varepsilon)^j\right] + O(\varepsilon^N) \end{aligned} \tag{6.9.3}$$

by repeatedly substituting the left hand side to the right hand side. Note that the terms are all finite because the only term appearing in the denominator at each level is powers of K_ε which is bounded away from 0.

The construction can be outlined as follow. The above computation shows that the one-sided Taylor expansion of the travel time (exit time) function is well-defined at $\varepsilon = 0^+$. We then show that the m-th normal derivative can be recovered from the ε^{2m-1} term. To do this, we need to show that the term in the ε^{2m-1} level that contains $\partial_n^m g_{\alpha\beta}$ has non-zero coefficient, and that all the other terms contain only $\partial_n^k g_{\alpha\beta}$ with $k < m$. Tangential derivatives do not pose problem: if $\partial_n^k g_{\alpha\beta}$ can be recovered at p, then it can be recovered in a neighborhood of p since strict convexity holds for a neighborhood of $(p, v) \in T\partial M$.

Example 6.9.1. *Consider $B_1(0,1) \subset \mathbb{R}^2$ the unit ball centered at $(0,1)$, equipped with the Euclidean metric. Consider lines of slope k that passes through the origin, we may write the length of the segment inside of $B_1(0,1)$ as a function of k, denote by $l(k)$. Straightforward computation shows that $l(k) = \frac{2|k|}{\sqrt{k^2+1}}$. Even though it is not differentiable at $k = 0$, the coefficients of the Taylor expansion at $k > 0$ will converge to a finite number when $k \to 0$. That is, the one-sided Taylor expansion at 0 is well-defined.*

6.9.2 Analyze contribution of each term

In this subsection, we analyze what terms may contribute to the coefficient of $\varepsilon^{2m-1}\partial_n^m g_{\alpha\beta}v^\alpha v^\beta$ for $m \geq 2$.

First of all, it is straightforward to see that if $j < m$, then $V_\varepsilon^j(v_\varepsilon^n)|_{s=0}$ does not contain any term related to $\partial_n^m g_{\alpha\beta}$. On the other hand, $\tau(\varepsilon)$ is ε level, so if $j > 2m-1$, then $\tau(\varepsilon)^j = O(\varepsilon^{2m})$. Therefore the contribution must come from

$$
\begin{aligned}
&- 2K_\varepsilon^{-1} \sum_{j=m}^{2m-1} V_\varepsilon^j(v_\varepsilon^n)|_{s=0} \cdot \frac{1}{(j+1)!}\tau(\varepsilon)^j \\
&= -2K_\varepsilon^{-1}\tau(\varepsilon)^m \sum_{j=0}^{m-1} V_\varepsilon^{m+j}(v_\varepsilon^n)|_{s=0} \cdot \frac{1}{(m+j+1)!}\tau(\varepsilon)^j.
\end{aligned}
$$

We now make the following observation. The term $V_\varepsilon^k(v_\varepsilon^n)$ contains a finite sum of the product of tangential derivatives of $v_\varepsilon^n, v_\varepsilon^\alpha, \partial_n^l g_{\alpha\beta}$, and

$$
\begin{aligned}
&V_\varepsilon(v_\varepsilon^n) = \frac{1}{2}\partial_n g_{\alpha\beta} v_\varepsilon^\alpha v_\varepsilon^\beta, \quad V_\varepsilon(v_\varepsilon^\alpha) = -\frac{1}{2} g^{\alpha\beta}\partial_n g_{\beta\gamma} v_\varepsilon^n v_\varepsilon^\gamma - \Gamma_{\beta\gamma}^\alpha v_\varepsilon^\beta v_\varepsilon^\gamma, \\
&V_\varepsilon(\partial_n^l g_{\gamma\beta}) = v_\varepsilon^n \partial_n^{l+1} g_{\alpha\beta} + v_\varepsilon^\alpha \partial_\alpha \partial_n^l g_{\alpha\beta}.
\end{aligned}
$$

Since $\tau(\varepsilon)^{m+j} = O(\varepsilon^{m+j})$, to obtain the desired term $\varepsilon^{2m-1}\partial_n^m g_{\alpha\beta} v^\alpha v^\beta$, the contribution from $V_\varepsilon^{m+j}(v_\varepsilon^n)|_{s=0}$ can contain at most ε^{m-j-1}. Meanwhile, (6.9.1) tells us that

$$
V_\varepsilon^{m+j}(v_\varepsilon^n) = \frac{1}{2} V_\varepsilon^{m+j-1}(\partial_n g_{\alpha\beta} v_\varepsilon^\alpha v_\varepsilon^\beta), \tag{6.9.4}
$$

so to obtain $\partial_n^m g_{\alpha\beta}$, we need $v_\varepsilon^n \partial_n$ to repeatedly apply to $\partial_n g_{\alpha\beta}$ exactly $m-1$ times. One byproduct of this action is $(v_\varepsilon^n)^{m-1}$ which is ε^{m-1} when evaluated at $s=0$, and exceeds the ε^{m-j-1} allowance for $j>0$. Fortunately, applying V_ε to $(v_\varepsilon^n)^l$ gives

$$
V_\varepsilon(v_\varepsilon^n)^l = l(v_\varepsilon^n)^{l-1} \cdot \frac{1}{2}\partial_n g_{\alpha\beta} v_\varepsilon^\alpha v_\varepsilon^\beta = lK_\varepsilon (v_\varepsilon^n)^{l-1} = lK(v_\varepsilon^n)^{l-1} + \varepsilon^2 lK(v_\varepsilon^n)^{l-1} \tag{6.9.5}
$$

by (6.9.2) and (6.9.1), which effectively brings down the power of ε from l to $l-1$ when evaluated at $s=0$. To bring it down from $m-1$ to $m-j-1$ we thus need at least j times V_ε acting on powers of v_ε^n. Since we only have $m+j-1$ times V_ε acting on $\partial_n g_{\alpha\beta}$, we need $m-1$ of them to repeatedly act on the derivative of $\partial_n g_{\alpha\beta}$, and j of them to bring down the power of v_ε^n. In other words, every single V_ε must be used to either raise the degree of normal derivative, or bring down the power of v_ε^n, and can not be wasted on any other terms.

Combine the above analysis with the fact that

$$K_\varepsilon = K + O(\varepsilon^2), \quad v_\varepsilon^\alpha v_\varepsilon^\beta = v^\alpha v^\beta + O(\varepsilon^2),$$

in each $V_\varepsilon^{m+j}(v_\varepsilon^n)$, the term that contributes to $\varepsilon^{2m-1}\partial_n^m g_{\alpha\beta} v^\alpha v^\beta$ must only come directly from the zeroth order of the coefficient of $(v_\varepsilon^n)^{m-j-1}\partial_n^m g_{\alpha\beta} v_\varepsilon^\alpha v_\varepsilon^\beta$. We denote the zeroth order of the coefficient by C_j. After expanding K_ε, v_ε^α and use the fact that $\tau(\varepsilon) = -2K^{-1}\varepsilon + O(\varepsilon^2)$, the coefficient of $\varepsilon^{2m-1}\partial_n^m g_{\alpha\beta} v^\alpha v^\beta$ is

$$\sum_{j=0}^{m-1} \frac{C_j}{(m+j+1)!}(-2K^{-1})^{m+j+1}. \tag{6.9.6}$$

6.9.3 Computation of the coefficient

We compute C_j in this subsection. By the analysis from the previous section, we only need to keep track of the coefficient when V_ε acts on either the power of v_ε^n or highest normal derivative of g. To make this more clear, we compute the first several terms to illustrate what we are keeping track of.

When V_ε first acts on v_ε^n we obtain $\frac{1}{2}\partial_n g_{\alpha\beta} v_\varepsilon^\alpha v_\varepsilon^\beta$. Then the second V_ε must directly apply to $\partial_n g_{\alpha\beta}$, specifically only the $v_\varepsilon^n \partial_n$ part of V_ε would matter, which gives $\frac{1}{2} v_\varepsilon^n \partial_n^2 g_{\alpha\beta} v_\varepsilon^\alpha v_\varepsilon^\beta$ plus irrelevant terms. These irrelevant terms will stay irrelevant when applying all future V_ε by the analysis from the previous section. The next V_ε now has a choice, it can either apply to v_ε^n, thus acting as lowering the power; or it can apply to $\partial_n^2 g_{\alpha\beta}$, thus acting as raising the degree of normal derivative. All the other terms would stay irrelevant, and we keep applying V_ε to the relevant terms as such. For a detailed computation, see Section 6.9.5.

Thus, we need to gather the different coefficients caused by lowering the power or increasing the degree of derivative in different orders. Denote by $R^{j,k}$ the relevant term that contains $(v_\varepsilon^n)^j \partial_n^k g_{\alpha\beta}$, then the relevant term in $V_\varepsilon(R^{j,k})$ is thus

$$jK_\varepsilon R^{j-1,k} + R^{j+1,k+1},$$

where the first term is obtained from acting on $(v_\varepsilon^n)^j$ (lowering the power) and use (6.9.1); and the second term is obtained from the $v_\varepsilon^n \partial_n$ part of V_ε acting on $\partial_n^k g_{\alpha\beta}$ (raising the degree of derivative). Ignore the irrelevant term, we obtain the following relation

$$V(R^{j,k}) = jKR^{j-1,k} + R^{j+1,k+1}$$

where for simplicity we omit the lower index in V_ε and use the zeroth order term of K_ε from (6.9.2) as it is the term that would contribute to the final coefficient.

Lemma 6.9.2. *Given the following rule*

$$V(R^{j,k}) = jKR^{j-1,k} + R^{j+1,k+1}.$$

For $l \geq 0$, one has

$$V^l(R^{0,1}) = \sum_{d=0}^{\lfloor l/2 \rfloor} \frac{l!}{(l-2d)!d!2^d} K^d R^{l-2d,1+l-d}.$$

Proof. We prove by induction, $l = 0$ is trivial. Suppose this holds for some l, then

$$\begin{aligned}
V^{l+1}(R^{0,1}) &= \sum_{d=0}^{\lfloor l/2 \rfloor} \frac{l!}{(l-2d)!d!2^d} K^d V(R^{l-2d,1+l-d}) \\
&= \sum_{d=0}^{\lfloor l/2 \rfloor} \frac{l!}{(l-2d)!d!2^d} (l-2d) K^{d+1} R^{l-2d-1,1+l-d} \\
&+ \sum_{d=0}^{\lfloor l/2 \rfloor} \frac{l!}{(l-2d)!d!2^d} K^d R^{l-2d+1,2+l-d} \\
&= \left. \frac{l!}{(l-2d)!d!2^d} (l-2d) K^{d+1} R^{l-2d-1,1+l-d} \right|_{\text{if } l \text{ odd and } d=\frac{l-1}{2}} \\
&+ \sum_{d=0}^{\lfloor l/2 \rfloor -1} \frac{l!}{(l-2d)!d!2^d} (l-2d) K^{d+1} R^{l-2d-1,1+l-d} \\
&+ R^{l+1,2+l} + \sum_{d=1}^{\lfloor l/2 \rfloor} \frac{l!}{(l-2d)!d!2^d} K^d R^{l-2d+1,2+l-d} \\
&= \left. \frac{l!}{(\frac{l-1}{2})!2^{\frac{l-1}{2}}} K^{\frac{l-1}{2}+1} R^{0,1+(l+1)-\frac{l+1}{2}} \right|_{\text{if } l \text{ odd}} + R^{l+1,1+(l+1)} \\
&+ \sum_{d=1}^{\lfloor l/2 \rfloor} \frac{l!}{(l-2d)!(d-1)!2^{d-1}} \left[\frac{1}{l-2d+1} + \frac{1}{2d} \right] K^d R^{l+1-2d,1+(l+1)-d} \\
&= \left. \frac{(l+1)!}{(\frac{l+1}{2})!2^{\frac{l+1}{2}}} K^{\frac{l-1}{2}+1} R^{0,1+(l+1)-\frac{l+1}{2}} \right|_{\text{if } l \text{ odd}}
\end{aligned}$$

$$+\sum_{d=0}^{\lfloor l/2\rfloor}\frac{(l+1)!}{(l+1-2d)!d!2^d}K^dR^{l+1-2d,1+(l+1)-d}.$$

For the last equality, we used the fact that $R^{l+1,1+(l+1)}$ is precisely the $d=0$ term. When l is even, we do not have the first term and $\lfloor\frac{l+1}{2}\rfloor=\lfloor\frac{l}{2}\rfloor$; and when l is odd, the $d=\lfloor\frac{l+1}{2}\rfloor=\frac{l+1}{2}$ term is the first term. $\square$

Since we start from $R^{0,1}$, by (6.9.4) we substitute $l=m+j-1$ and $d=l+1-m=j$. Indeed for this d and l we have $R^{l-2d,1+l-d}=R^{m-j-1,m}$ which corresponds to $(v_\varepsilon^n)^{m-j-1}\partial_n^m g_{\alpha\beta}$ as required. From (6.9.4), we have C_j is the coefficient of the $d=j$ term in $V^{m+j-1}(R^{0,1})$ multiply by 1/2:

$$C_j=\frac{1}{2}\frac{(m+j-1)!}{(m-j-1)!j!2^j}K^j.$$

Plug into (6.9.6), the entire coefficient is

$$\frac{1}{2}(-2K^{-1})^{m+1}\sum_{j=0}^{m-1}\frac{(-1)^j}{(m+j)(m+j+1)(m-j-1)!j!}=\frac{1}{2}(-2K^{-1})^{m+1}\frac{m!}{(2m)!}\tag{6.9.7}$$

by Lemma 6.9.3.

Lemma 6.9.3. *For any integer $m\geq 1$, we have*

$$\sum_{j=0}^{m-1}\frac{(-1)^j}{(m+j)(m+j+1)(m-j-1)!j!}=\frac{m!}{(2m)!}.$$

Proof. We denote the left-hand side by S_m. First, we observe

$$\frac{1}{(m+j)(m+j+1)}=\frac{1}{m+j}-\frac{1}{m+j+1}=\int_0^1 t^{m+j-1}(1-t)dt.$$

This implies

$$\begin{aligned}S_m&=\sum_{j=0}^{m-1}\frac{(-1)^j}{(m-j-1)!j!}\int_0^1 t^{m+j-1}(1-t)dt\\&=\int_0^1 t^{m-1}(1-t)\sum_{j=0}^{m-1}\frac{(-t)^j}{(m-1-j)!j!}dt,\end{aligned}$$

where we swap the finite sum with the integral. On the other hand, using the binomial identity, we have

$$(1-t)^{m-1} = \sum_{j=0}^{m-1} \binom{m-1}{j} (-t)^j = (m-1)! \sum_{j=0}^{m-1} \frac{(-t)^j}{(m-1-j)!j!}.$$

It follows that

$$S_m = \frac{1}{(m-1)!} \int_0^1 t^{m-1}(1-t)(1-t)^{m-1} dt = \frac{1}{(m-1)!} \int_0^1 t^{m-1}(1-t)^m dt.$$

A straightforward computation shows

$$\int_0^1 t^{m-1}(1-t)^m dt = \frac{(m-1)!m!}{(2m)!}.$$

Thus, we have

$$S_m = \frac{1}{(m-1)!} \cdot \frac{(m-1)!m!}{(2m)!} = \frac{m!}{(2m)!}.$$

□

6.9.4 Recover normal jet

In this subsection we recover the normal jet of the metric. From the previous sections, we have computed the expansion of the travel time τ in terms of ε, see (6.9.3). We have also computed that the coefficient for $\varepsilon^{2m-1}\partial_n^m g_{\alpha\beta} v^\alpha v^\beta$ is given by (6.9.7) which is bounded away from 0. Now it suffices to show that all the other terms of ε^{2m-1} level only contains lower order normal derivative of the metric. This can be achieved by using the same argument we used to find contributions.

Certainly contribution from $V_\varepsilon^j(v_\varepsilon^n)$ for $j < m$ is fine since it has at most $m-1$ degree of normal derivative of g. When $j = m$ we have contribution from m-th normal derivative, which is included in (6.9.7) already, all the other terms have lower order normal derivative. When $m < j < 2m-1$, again the contribution from m-th normal derivative term has already been computed; as for higher order normal derivative term, this requires at least $m+1$ times V_ε for raising the degree of normal derivative, but the byproduct is $m-1$ power of v_ε^n, which requires at least $j-m$ times V_ε to bring down to $2m-1-j$ level (because $\tau(\varepsilon)^j = O(\varepsilon^j)$). However, there are only $j < (m+1) + (j-m)$ times V_ε, so the only other terms they can contribute to coefficient of ε^{2m-1} are normal derivatives of the metric with degree strictly smaller than m. Finally, $j > 2m-1$ is at least $O(\varepsilon^{2m})$.

Since K can be read from the coefficient of ε by (6.9.3) and gives $\partial_n g_{\alpha\beta} v^\alpha v^\beta$, the first order normal derivative can be recovered in a neighborhood of p (this is because the boundary is strictly convex with respect to all nearby (x, w)). Inductively, we have that $\partial_n^m g_{\alpha\beta} v^\alpha v^\beta$ can be computed from the coefficient of ε^{2m-1} term for all $m \geq 2$. Thus one can similarly compute $\partial_n^m g_{\alpha\beta}(x) w^\alpha w^\beta$ for an open set of tangential directions (x, w) near (p, v). This is enough information to construct the symmetric two tensor $\partial_n^m g_{\alpha\beta}$ at a neighborhood of p.

To conclude, we have proved the following result.

Theorem 6.9.4. *Let (M, g) be either a Riemannian manifold with boundary, or a Lorentzian manifold with timelike boundary. Suppose the boundary is strictly convex with respect to some $(p, v) \in \partial TM$, with v being timelike in the Lorentzian case. Suppose the scattering relation S around (p, v) and the boundary metric $g|_{T\partial M \times T\partial M}$ around p are given. Then the normal jet of g at p in the boundary normal coordinates can be constructed.*

Remark 6.9.5. *We claim that the same result holds even if $I\!I$ vanishes around p, as long as some higher level of convexity holds. Specifically, suppose there exists $k \in \mathbb{Z}^{\geq 1}$ such that $\partial_n^j g_{\alpha\beta} v^\alpha v^\beta = 0$ for all $j < k$, but there exists some $v \in T_p\partial M$ such that $\partial_n^k g_{\alpha\beta} v^\alpha v^\beta < 0$. What we computed was the $k = 1$ case, when $k > 1$, the expansion of $x^n(\gamma_\varepsilon(s))$ will be*

$$\varepsilon s + \sum_{j=k+1}^{N} V_\varepsilon^j (x^n)|_{s=0} \cdot \frac{1}{j!} s^j + O(s^{N+1}).$$

Then we obtain the expansion of $\tau(\varepsilon)^k$ with the first term being $C_k^{-1}\varepsilon$. Here C_k is a nonzero constant multiple of $\partial_n^k g_{\alpha\beta} v^\alpha v^\beta$, and by the assumption it is nonzero. One can analyze the higher order terms to recover the jet of g in a similar way. We do not prove this here.

6.9.5 Example

As an example, let us compute for $m = 2$. For simplicity we omit the ε subindex, and we use $R^{j,k}$ to denote any term with j-th power of v^n and at most k-th normal derivative of g.

$$
\begin{aligned}
V(v^n) &= \frac{1}{2}\partial_n g_{\alpha\beta} v^\alpha v^\beta \\
V^2(v^n) &= \frac{1}{2} v^n \partial_n^2 g_{\alpha\beta} v^\alpha v^\beta + \frac{1}{2}\partial_\gamma \partial_n g_{\alpha\beta} v^\gamma v^\alpha v^\beta + \partial_n g_{\alpha\beta} V(v^\alpha) v^\beta \\
&= \frac{1}{2} v^n \partial_n^2 g_{\alpha\beta} v^\alpha v^\beta + R^{0,1} + R^{1,1} \\
V^3(v^n) &= \frac{1}{2} K \partial_n^2 g_{\alpha\beta} v^\alpha v^\beta + \frac{1}{2}(v^n)^2 \partial_n^3 g_{\alpha\beta} v^\alpha v^\beta + \frac{1}{2} v^n \partial_\gamma \partial_n^2 g_{\alpha\beta} v^\gamma v^\alpha v^\beta + v^n \partial_n^2 g_{\alpha\beta} V(v^\alpha) v^\beta \\
&+ \frac{1}{2} v^n \partial_\gamma \partial_n^2 g_{\alpha\beta} v^\gamma v^\alpha v^\beta + \frac{1}{2}\partial_\omega \partial_\gamma \partial_n g_{\alpha\beta} v^\omega v^\gamma v^\alpha v^\beta \\
&+ \frac{1}{2}\partial_\gamma \partial_n g_{\alpha\beta} V(v^\gamma) v^\alpha v^\beta + \partial_\gamma \partial_n g_{\alpha\beta} V(v^\alpha) v^\beta \\
&+ v^n \partial_n^2 g_{\alpha\beta} V(v^\alpha) v^\beta + \partial_\gamma \partial_n g_{\alpha\beta} v^\gamma V(v^\alpha) v^\beta + \partial_n g_{\alpha\beta} V^2(v^\alpha) v^\beta + \partial_n g_{\alpha\beta} V(v^\alpha) V(v^\beta) \\
&= \frac{1}{2} K \partial_n^2 g_{\alpha\beta} v^\alpha v^\beta + O(v^n) + R^{0,1}.
\end{aligned}
$$

In the above computation we used the fact that V apply to tangential v^α belongs to $R^{1,1} + R^{0,0}$. Use R^j to denote the term that contains normal derivative of at most j, substitute the above computation into the expansion (6.9.3):

$$
\begin{aligned}
\tau(\varepsilon) &= -2K_\varepsilon^{-1} \Big[\varepsilon + (\frac{1}{2}\varepsilon \partial_n^2 g_{\alpha\beta} v_\varepsilon^\alpha v_\varepsilon^\beta + R^1) \cdot \frac{1}{6}(-2K^{-1}\varepsilon + O(\varepsilon^2))^2 \\
&\qquad + (\frac{1}{2} K_\varepsilon \partial_n^2 g_{\alpha\beta} v_\varepsilon^\alpha v_\varepsilon^\beta + O(\varepsilon) + R^1) \cdot \frac{1}{24}(-2K^{-1}\varepsilon + O(\varepsilon^2))^3 \Big] + O(\varepsilon^4) \\
&= -2K^{-1}\varepsilon + R^1 \varepsilon^2 \\
&+ \left[(-2K^{-1})^3 \frac{1}{12} \partial_n^2 g_{\alpha\beta} v^\alpha v^\beta - (-2K^{-1})^3 \frac{1}{24} \partial_n^2 g_{\alpha\beta} v^\alpha v^\beta + R^1 \right] \varepsilon^3 + O(\varepsilon^4) \\
&= -2K^{-1}\varepsilon + R^1 \varepsilon^2 + (-2K^{-1})^3 \frac{1}{24} \partial_n^2 g_{\alpha\beta} v^\alpha v^\beta \varepsilon^3 + R^1 \varepsilon^3 + O(\varepsilon^4).
\end{aligned}
$$

Plug $m = 2$ into (6.9.7), we obtain

$$\frac{1}{2}(-2K^{-1})^{2+1}\frac{2!}{4!} = (-2K^{-1})^3\frac{1}{24},$$

which agrees with the explicit computation.

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