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A Non-renormalization Theorem for the Wilsonian Gauge Couplings in Supersymmetric Theories*

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Abstract

We show that the holomorphic Wilsonian *beta*-function of a renormalizable asymptotically free supersymmetric gauge theory with an arbitrary semi-simple gauge group, matter content, and renormalizable superpotential is exhausted at 1-loop with no higher loops and no non-perturbative contributions.

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1 Introduction

In this Letter we prove that the holomorphic Wilsonian *beta*-function of an arbitrary renormalizable asymptotically-free supersymmetric gauge theory with matter is exhausted at 1-loop with no higher loops and no non-perturbative contributions.

The technique we employ was introduced by Seiberg [1] and it is briefly reviewed here. To obtain the *beta*-function we compare two versions of the theory with different cutoffs and coupling constants and the same low energy physics. The couplings of the theory with the lower cutoff can be expressed in terms of the couplings of the theory with the higher cutoff and the ratio of the two cutoffs. We can restrict their functional dependence on the high cutoff couplings using *holomorphy* of the superpotential and gauge kinetic terms and *selection rules*. Holomorphy is a consequence of supersymmetry. To see this, elevate the couplings to background chiral superfields. They must appear holomorphically in the superpotential in order to preserve supersymmetry. Selection rules generalize global symmetries in the sense that we allow the couplings in the superpotential to transform under these symmetries. Non-zero vacuum values of these couplings then spontaneously break these symmetries. Here we only consider $U(1)$ and $U(1)_R$ symmetries. In the quantum theory they are generally anomalous, but we can use the same technique we used for the coupling in the superpotential. We assume that the θ-angle is a background field and transform it non-linearly to make the full quantum effective action invariant.

Then, following a method used in [2] we translate these conditions on the functional relations between the couplings of the theories at different cutoffs into restrictions of the functional form of the gauge *beta*-function. We can show that the gauge *beta*-function is a function of the holomorphic invariants allowed by selection rules. Then we can restrict further the functional dependence of the *beta*-function by varying the couplings while keeping the invariants fixed. This allows us to relate the *beta*-function of the original

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theory to the *beta*-function of a theory with vanishing superpotential. In addition, we also obtain a strong restriction of the functional dependence of the *beta*-function on the gauge coupling. It has exactly the form of a one-loop *beta*-function. The only ambiguity left is a numerical coefficient which can be calculated in perturbation theory.

Next we make a short detour to explain what we mean by the Wilsonian *beta*-function [3]. The Wilsonian *beta*-function describes the renormalization group flow of the *bare* couplings of the theory so that the low energy theory is cutoff invariant. Additionally, we do not renormalize the vector and chiral superfields, i.e. we do not require canonical normalization of the kinetic terms. The usual convention in particle physics is to canonically normalize the kinetic term. It is obtained by using the covariant derivative $\partial + gA$. Instead, here we allow non-canonical normalization of the kinetic term. The normalization of the gauge fields is such that the covariant derivative has the form $\partial + A$. The gauge coupling only appears in front of the gauge kinetic term. In this case it is convenient to combine the θ-angle and gauge coupling constant g into the complex variable $\tau = \theta/2\pi + 4\pi i/g^2$. In supersymmetric gauge theories the *beta*-function is holomorphic in the bare couplings only if we do not renormalize the fields. Even if we start with canonical normalization at a higher cutoff, the Kahler potential will not be canonical at the lower cutoff. The rescaling of the chiral or gauge superfields is an anomalous transformation [2] that destroys the holomorphy of the superpotential and the *beta*-function [a]. The relation between the *beta*-functions in the two normalizations for the case of a pure supersymmetric Yang-Mills was discussed in [2, 4]. The *beta*-function for canonically normalized fields receives contributions to all orders in perturbation theory. Again we emphasize that here we are only concerned with the holomorphic Wilsonian *beta*-function.

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[a]For some special theories like $N = 2$ SUSY-YM the rescaling anomaly of the chiral superfields cancels the rescaling anomaly of the vector superfield [2]. For these theories we can make stronger statements since the canonical and holomorphic Wilsonian couplings coincide.

We should also clearly state that the theorem is not valid if any one of the one-loop gauge *beta*-functions is not asymptotically free. This includes the case when the one-loop *beta*-function vanishes. As we will see, exactly in this case the $U(1)_R$ symmetry is non-anomalous. This makes it difficult to control the dependence of the *beta*-function on the gauge coupling.

Various partial versions of this result already exist. The case of a simple gauge group with a vanishing superpotential was discussed in [2, 4]. It is also known that the *beta*-function is independent of the gauge coupling in the case of a simple gauge group with only Yukawa interactions present in the superpotential [5].

Finally, we note that the theorem is valid in theories where no mass terms are allowed by the symmetries of the theory. This is of phenomenological interest as many supersymmetric extensions of the Standard Model share this characteristic.

2   Simple Gauge Group

We will consider first the case of a simple gauge group G. Let the generalized superpotential $\widetilde{W}$ be defined to include the kinetic term for the gauge fields[b]

$$\widetilde{W} = \frac{\tau}{64\pi it_R}\mathrm{tr}_R(W_\alpha W^\alpha) + W, \tag{1}$$

where

$$W = \sum \lambda_{ijk}\Phi_i\Phi_j\Phi_k + M\sum m_{ij}\Phi_i\Phi_j + M^2\sum c_i\Phi_i \tag{2}$$

is the usual superpotential and $\mathrm{tr}_R T^a T^b = t_R\delta^{ab}$. Here M is the cutoff mass and was factored out so that all the couplings are dimensionless. The gauge coupling g and θ-angle are combined in the complex variable

$$\tau \equiv \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}. \tag{3}$$

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[b]The normalization of the gauge fields is that of Reference [2].

Note that unitarity requires τ to be valued in the upper half plane. Since θ is a periodic variable it is convenient to introduce a new variable $q \equiv e^{2\pi i \tau}$. It is valued in the complex plane and transforms linearly under the anomalous transformations to be discussed below. Weak coupling is at $q = 0$.

Consider now a theory with a different cutoff M' and with the same low energy physics. The Lagrangian at the new cutoff is

$$\mathcal{L} = \sum_i \int d^2\theta d^2\bar{\theta} Z_i \Phi_i^\dagger e^{2V_h} \Phi_i + \left(\int d^2\theta \widetilde{W}(\tau', \lambda'_{ijk}, m'_{ij}, c'_i, M') + \text{h.c.} \right) \quad (4)$$

where in particular, the fields Φ_i are not renormalized to canonical normalization. The Z_i depends non-holomorphically on the couplings, so renormalizing the chiral superfields would destroy the holomorphic form of $\widetilde{W}$. The new coupling τ' is a function of the old dimensionless couplings and the ratio M/M'. For later convenience we write this as

$$\tau' = \tau'(\tau, \lambda_{ijk}, m_{ij}, c_i; \ln(M/M')). \quad (5)$$

Supersymmetry requires a holomorphic dependence of τ on the first four arguments. To see this, note that the couplings in the generalized superpotential can be considered as vacuum values of background chiral superfields. Invariance of the action under supersymmetry transformations requires holomorphy of the superpotential.

To prove the non-renormalization theorem we will use selection rules. These are global symmetries of the superpotential with all couplings considered as chiral superfields. We assign them non-trivial transformation properties under the symmetry group. These symmetries will be spontaneously broken by non-zero vacuum values of the couplings. In general they are also anomalous. We will make them non-anomalous by assigning a charge to q, i.e. transforming θ to compensate for the anomaly. Consider the $U(1)_R \times U(1)$ global symmetry with the following charge assignment:

	W_α	Φ_i	λ_{ijk}	m_{ij}	c_i	q
$U(1)_R$	1	2/3	0	2/3	4/3	$2b_0/3$
$U(1)$	0	1	−3	−2	−1	$2\sum_i t(R_i)$

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The quantity b_0 is given by $b_0 = 3t_{adj} - \sum_i t(R_i)$, where $t(R_i)$ is the normalization of the generators for the representation of the chiral superfield Φ_i. For example, $t = 1/2$ for a fundamental of $SU(N)$. Define the gauge β-function by

$$\beta_{2\pi i \tau} = \frac{d}{d \ln(M/M')} 2\pi i \tau'|_{M'=M} = \beta(\tau, \lambda_{ijk}, m_{ij}, c_i). \tag{6}$$

The holomorphy of τ in (5) translates into holomorphy of the *beta*-function.

Since $\tau \to \tau + 1$ is a symmetry of the theory, β is a single valued function of q

$$\beta_{2\pi i \tau} = f(q, \lambda_{ijk}, m_{ij}, c_i). \tag{7}$$

First, consider the case when at least one mass term, let us call it m_*, can be non-zero. If any c_i could be non-zero, then there is a gauge singlet field which could be given a Majorana mass, so this is the same case as above.

The gauge *beta*-function is $U(1)_R \times U(1)$ invariant. This statement is non-trivial and requires some explanation. Consider some arbitrary coupling λ that transforms linearly under some $U(1)$ or $U(1)_R$ symmetry. Its *beta*-function β_λ must also transform linearly with the same charge as λ

$$e^{iQ_\lambda \alpha} \beta_\lambda(\lambda, \ldots) = \beta_\lambda(e^{iQ_\lambda \alpha} \lambda, \ldots) \tag{8}$$

where Q_λ is the charge of λ. This is true in particular for the *beta*-function of q. However when we go to the τ variable we have

$$\beta_{2\pi i \tau} = \frac{d}{d \ln(M/M')} 2\pi i \tau' = \frac{d}{dq} 2\pi i \tau' \, \beta_q = q^{-1} \beta_q. \tag{9}$$

The additional q factor makes the τ *beta*-function invariant. In what follows we only consider the gauge *beta*-function since all the others are trivial, i.e. there are no perturbative [6] or non-perturbative [1] corrections to the usual superpotential. We will drop the subscript and denote it β.

First consider $U(1)_R$ invariance. It requires that

$$\beta = \tilde{f}\left(\frac{q}{m_*^{b_0}}, \frac{m_{ij}}{m_*}, \frac{c_i}{m_*^2}, \lambda_{ijk}\right). \tag{10}$$

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However, the variables of $\tilde{f}$ are not $U(1)$ invariant. They have charges $6T_{adj}, 0, 3, -3$, respectively. Invariance under $U(1)_R \times U(1)$ requires that β is a yet another function

$$\beta = F\left(q\, \frac{\lambda_{ijk}^{2t_{adj}}}{m_*^{b_0}}, q^{-1}\, \frac{c_i^{2t_{adj}}}{m_*^{t_{adj}+\sum t_i}}, \frac{m_{ij}}{m_*}\right). \tag{11}$$

We next take the limit $m_* \to 0$ keeping q and all the arguments of F constant. If $b_0 > 0$, this corresponds to taking all couplings except τ to zero. Assuming that β is continuous we see that $\beta(q, \lambda_{ijk}, m_{ij}, c_i) = \beta(q, \lambda_{ijk} = m_{ij} = c_i = 0)$ and thus it is independent of all the couplings in the superpotential. In fact when the superpotential vanishes it is known [2, 4] that the *beta*-function is a constant and the gauge coupling only runs at 1-loop[c]. This just reflects the fact that no $U(1)_R \times U(1)$ holomorphic invariant can be constructed solely in terms of q. Note the importance of holomorphy in these arguments. For example, if we do not require holomorphy $q\bar{q}$ is invariant under an arbitrary $U(1)$ and $U(1)_R$ symmetry. No higher loops or non-perturbative corrections are present and we conclude that

$$\beta = b_0. \tag{12}$$

An exception to the previous argument occurs when the gauge and global symmetries of the theory allow only Yukawa couplings to be present in the superpotential. For these theories

$$\beta = f(q, \lambda_{ijk}). \tag{13}$$

The *beta*-function must be $U(1)_R$ invariant. This requires

$$f(e^{2b_0\alpha i/3}q, \lambda_{ijk}) = f(q, \lambda_{ijk}). \tag{14}$$

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[c]Note that this result can also be written as $\frac{d}{dt}g = -\frac{b_0}{16\pi^2}g^3$ which is just the standard 1-loop *beta*-function.

Then by holomorphy β is independent of q. Further, invariance of β under the $U(1)$ symmetry requires that f is a function of ratios of λ_{ijk} only. We may choose one of the non-zero λ_{ijk}, λ_* say, and divide through by λ_*. Then

$$\beta = f(\lambda_{ijk}) = F\left(\frac{\lambda_{ijk}}{\lambda_*}\right). \tag{15}$$

Consider the limit $\lambda_{ijk} \to 0$ while keeping the ratios λ_{ijk}/λ_* constant. We know that in this limit β reduces to the one-loop result. So assuming that β is continuous, we find $\beta(\lambda_{ijk}) = \beta(\lambda_{ijk} = 0) = b_0$, i.e. it is independent of the Yukawa couplings.

To conclude this Section, we note that our discussion of the proof of the theorem was divided into two cases requiring separate proofs. Here we present a short argument that extends the proof of the theorem, valid when at least one mass term is allowed, to theories which do not admit any bare mass terms. Consider a theory with Lagrangian $\mathcal{L}$ for which the symmetries of the theory forbid the presence of any mass terms. To this theory, *add* a non-interacting gauge-singlet field with mass m_*. More concretely, the new theory defined at M is described by the Lagrangian

$$\mathcal{L}_{new} = \mathcal{L} + \int d^2\theta d^2\bar{\theta}\Phi_0^\dagger\Phi_0 + \left(\int d^2\theta M m_*\Phi_0^2 + \text{h.c.}\right). \tag{16}$$

This new theory satisfies the conditions of the theorem proven when at least one mass term is allowed, so the *beta*-function of the new theory, β_{new}, is exhausted at one-loop. But we can conclude on physical grounds that β_{new} is identical to β, the *beta*-function of the original theory, since in integrating over momentum modes M to M' the contribution from the gauge singlet completely factors out since it is non-interacting. So by this argument the proof of the theorem for theories with mass terms can be extended to theories for which mass terms are forbidden by the symmetries of the model.

The results of this section are also valid for a semisimple gauge group. We shall sketch the proof in the next section.

3   Extension to a semi-simple gauge group

Assume that the gauge group is $G = \Pi_A G_A$ with each G_A a simple group. Also assume that the superpotential has the form given in Section 2. Then if all the simple gauge groups are asymptotically-free the Wilsonian *beta*-functions of all the gauge couplings are one-loop exact.

For each simple gauge group G_A define

$$\tau_A = \frac{\theta_A}{2\pi} + \frac{4\pi i}{g_A^2} \tag{17}$$

and introduce $q_A \equiv e^{2\pi i \tau_A}$ as in Section 2. We extend the $U(1)_R \times U(1)$ selection rules of Section 2 by assigning all gauge chiral multiplets $W_{\alpha,A}$ charge $(1,0)$. Then q_A has charge $(2b_0^A/3, 2\sum_i t_A(R_i))$. It will be conveinent to define $\kappa_A \equiv (q_A)^{\frac{1}{b_0^A}}$. Then κ_A has charge $(2/3, 2\sum_i t_A(R_i)/b_0^A)$. Weak coupling is at $\kappa_A = 0$ since b_0^A is positive.

The *beta*-functions for each simple gauge group are defined as in Section 2, so that

$$\beta_A = f_A(q_B, \lambda_{ijk}, m_{ij}, c_i) \tag{18}$$

is a function of holomorphic invariants and invariant under the $U(1)_R \times U(1)$ symmetry.

We do the proof for two cases:

1. Only Yukawa couplings are allowed.

2. At least one $m_{ij} \neq 0$ is allowed.

In the first case invariance of β_A under $U(1)_R$ requires that β_A is a function of ratios of κ_B only. That is,

$$\beta_A = F_A(\kappa_B/\kappa_{B_*}, \lambda_{ijk}). \tag{19}$$

We have divided through by an arbitrarily chosen κ_{B_*}, so that each κ_B other than κ_{B_*} appears in the argument of F only once. Now consider the weak

coupling limit $\kappa_B \to 0$ for all the gauge couplings. The argument of the *beta*-functions is

$$\kappa_B / \kappa_{B_*} = \exp 2\pi i (\tau_B / b_0^B - \tau_{B_*} / b_0^{B_*}). \tag{20}$$

Since by assumption the one-loop *beta*-functions all have the same sign it is possible to take this limit while keeping the ratios κ_B / κ_{B_*} fixed. In this limit the *beta*-function is a function of the Yukawa couplings only. So assuming that the *beta*-functions are continuous in this limit, we find that $\beta_A(\kappa_B, \lambda_{ijk}) = \beta_A(\kappa_B = 0, \lambda_{ijk}) = F_A(\lambda_{ijk})$. But now we may use the $U(1)$ symmetry to conclude that β_A is a function of $\lambda_{ijk} / \lambda_*$. The argument of Section 2 may now be repeated and we conclude that $\beta_A(q_B, \lambda_{ijk}) = $ constant.

For the second case a straightforward generalization of the argument of Section 2 may be repeated and we conclude that

$$\beta_A = F_A \left(\frac{\kappa_B}{\kappa_{B_*}} \right). \tag{21}$$

Then the argument used in the first case of this Section is used to conclude that F_A is independent of all of the q_B and superpotential couplings.

Acknowledgements

Note

The statement of this theorem for the case of a simple gauge group was also made in the lecture notes [7]. In that proof the author considers a superpotential containing no composite operators, i.e. only operators linear in the fundamental fields. Of course such superpotential is not gauge invariant. However it is is only used in an intermediate step to simplify the study the charge assignment for the couplings in the physical gauge invariant superpotential. The $U(1)$ charge of the coupling of a composite operator equals the sum of the charges of the couplings of the fundamental fields entering the composite. However in [7] it is also assumed that the $U(1)_R$ charge of the couplings of composite gauge invariant operators in the superpotential equals the sum of the charges of the couplings of fundamental fields forming the composite. While this is true for usual $U(1)$ symmetries since the superpotential has charge zero and the sum of charges of the couplings must equal minus the sum of charges of the fields entering the composite, for $U(1)_R$ symmetries the superpotential has charge two and the arithmetic is more complicated. Because of this, the proof in [7] only works for a superpotential linear in matter fields, i.e. when only gauge singlet chiral superfields are present. We also generalized the theorem to a semi-simple gauge group.

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