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Publication Date

2013

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Quantum Gravity in Three Dimensions from Higher-Spin Holography

by
Hai Siong Tan

A dissertation submitted in partial satisfaction of the
requirements for the degree of
Doctor of Philosophy
in
Physics
in the
Graduate Division
of the
University of California, Berkeley

Committee in charge:
Professor Ori Ganor, Chair
Professor Petr Horava
Professor Vaughan Jones

Spring 2013

Abstract

Quantum Gravity in Three Dimensions from Higher-Spin Holography

by

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Doctor of Philosophy in Physics

University of California, Berkeley

Professor Ori Ganor, Chair

In this thesis, I explore various aspects of quantum gravity in three dimensions from the perspective of higher-spin holography in anti-de Sitter spacetime. The bulk theory is a higher-spin Vasiliev gravity theory of which topological sector can be described by a Chern-Simons theory equipped with some suitable Lie (super-)algebra, whereas the boundary conformal field theory is conjectured to be a coset minimal model which contains \mathcal{W} symmetries. I present new black hole solutions and investigate the thermodynamics of these solutions, in particular, I establish a relationship among black hole thermodynamics, asymptotic symmetries and \mathcal{W} algebras. I also construct new conical defect solutions, supersymmetric RG flow solutions in the bulk gravity theories, and present the bulk-boundary propagator for scalar fields interacting with a higher-spin black hole. The main examples used in this thesis are illustrated in the framework of $SL(N)$ and $SL(N|N-1)$ Chern-Simons theories, and I point out how these new solutions can be used to yield some insights into the nature of quantum gravity.

*Dedicated to my Mum,
and in fond memory of my Dad.*

Acknowledgement

It is a pleasure to hereby give thanks to a number of amazing individuals from whom I have received support over the past five years. Foremost, I would like to express my deepest gratitude to my thesis advisor Prof. Ori Ganor for his guidance. Ori is a tremendously creative and knowledgeable string theorist, and it has been an awe-inspiring experience for me to witness his genius at work. He seems to be able to derive anything in physics that is sensible from scratch. On top of his technical prowess, his good-natured and caring personality has always made it easy for me to seek his advice on various issues. From getting research funds to letters for summer schools and post-doc applications, he has always been there. I have learnt that being a graduate student in string theory is a very challenging position to be in, yet Ori has also made my journey much more joyful and memorable than it could have been. I have learnt so many things from him, about physics and beyond physics.

I would like to thank Prof. Petr Hořava for his generous support in many important matters. I am very privileged to have the chance to learn many things from him when he taught the QFT and string theory courses, to have him on my qualifying exam, and to learn from him about his exciting proposal for a new model of quantum gravity back in 2009. His encouragement has played a crucial role in my learning process. I also thank Prof. Vaughan Jones for agreeing to be on my qualifying exam and thesis committee. It is a humbling experience to have him on my committee, and I thank him for his time and kindness. Prof. Sergey Cherkis, who was a former visitor taught me many things about physics, especially about instantons. I had many enjoyable discussions with him when he was at Berkeley. Prof. Neal Synderman, who was my instructor for a quantum mechanics course when I was an exchange undergraduate student at Berkeley many years ago, has inspired me much not only by his depth of physics knowledge, but also by his rich life experiences. I enjoy listening to his numerous interesting encounters with many big characters in physics. I am really glad to know Sergey and Neal in person. Apart from them, I had the chance to benefit from many conversations with a number of physicists, in particular, I thank Mina Aganagic, Jan de Boer, Steve Carlip, Sven Bjarke Gudnason, Yoon Pyo Hong, Bom Soo Kim, Per Kraus, Emanuele Latini, Juan Maldacena, Hitoshi Murayama, Eric Perlmutter and Meng Chwan Tan. Finally, I thank my good friend Phuntsok Tseten for many hours of badminton sparring sessions, and three angels Anne Takizawa, Donna Sakima and Kathy Lee for being so kind in helping me plough through various administrative issues along the way.

Preface

Some of the original materials in this thesis have been published in the following two papers

1. H. S. Tan, “Exploring Three-dimensional Higher-Spin Supergravity based on $sl(N - N - 1)$ Chern-Simons theories,” JHEP **1211**, 063 (2012) [arXiv:1208.2277 [hep-th]].
2. H. S. Tan, “Aspects of Three-dimensional Spin-4 Gravity,” JHEP **1202**, 035 (2012) [arXiv:1111.2834 [hep-th]].

Other papers not directly related to the theme of this thesis and which were published during the course of my study are

3. O. J. Ganor, Y. P. Hong, R. Markov and H. S. Tan, “Static Charges in the Low-Energy Theory of the S-Duality Twist,” JHEP **1204**, 041 (2012) [arXiv:1201.2679 [hep-th]].
4. O. J. Ganor, Y. P. Hong and H. S. Tan, “Ground States of S-duality Twisted N=4 Super Yang-Mills Theory,” JHEP **1103**, 099 (2011) [arXiv:1007.3749 [hep-th]].
5. H. S. Tan, “First order Born-Infeld Hydrodynamics via Gauge/Gravity Duality,” JHEP **0904**, 131 (2009) [arXiv:0903.3424 [hep-th]].

My research journey has led me to a variety of project topics in string theory and quantum gravity. Projects 3 and 4 involve the study of duality-twisted field theories using string-theoretic techniques, and were done during my second to fourth year of study. They gave me many opportunities to learn from my advisor Prof. Ganor and his other graduate students. Purely to keep to a more uniform theme, I have chosen to write this thesis based on projects 1 and 2 which involve an application of the principle of holography to understand quantum gravity in three dimensions. Some of the materials discussed in this thesis were presented in a seminar at University of California at Davis, Department of Mathematics in April 2012.

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 CHAPTER 1

Introduction

This thesis is devoted towards understanding the topic of quantum gravity by the principle of holography which, roughly speaking, states that quantum gravity in $d + 1$ dimensions is somewhat equivalent to a local field theory in d dimensions. There is an obvious analogy of this phenomenon to that of a hologram which stores information of a three-dimensional image in a two-dimensional picture, and hence the name ‘holography’. This idea owes its roots to the study of the thermodynamics of black holes. As was demonstrated beautifully by Bekenstein [1] and Hawking [2] in the 1970s, when the laws of quantum mechanics are applied to a certain approximate extent, a black hole can be understood as a thermodynamical ensemble and in particular carries an entropy proportional to the area of its horizon. This is at first sight rather mysterious because the number of degrees of freedom should scale with the volume of the gravitational theory, unless somehow, $d + 1$ -dimensional quantum gravity is a d -dimensional local field theory in disguise.

In our modern understanding, one of the most concrete realizations of the principle of holography lies in the *AdS/CFT* correspondence. The prototype example, as was first conjectured by Maldacena [3], is type IIB strings compactified on $AdS_5 \times S^5$ being dual to four-dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. Since this seminal work, a plethora of other examples and generalizations of such dualities have been discovered, including those where strings are not on *AdS* spaces (and neither is the dual field theory conformal). As we shall explain more elaborately within the context of *AdS₃/CFT₂* in Section 2.3 in the next chapter, the *AdS/CFT* correspondence is fundamentally a open/closed string duality and in particular, implies a gauge/gravity duality. Apart from the principle of holography, we should mention that this correspondence is intimately related to an idea due to ’t Hooft [4] - that a large N gauge theory (where N is the rank of the gauge group) is equivalent to a string theory. Consider a perturbative expansion of a large N gauge theory in $1/N$ and $g_{YM}^2 N$ of the form $Z = \sum_{g \geq 0} N^{2-2g} Z_g(g_{YM}^2)$ which resembles the loop expansion in string theory which goes as $Z = \sum_{g \geq 0} g_s^{2g-2} Z_g$, with the string coupling being $g_s \sim 1/N$. Roughly speaking, Feynman diagrams of the gauge theory in this limit are turned into surfaces depicting interacting strings.

One of the precursors of the *AdS/CFT* correspondence comes from a study of quantum gravity in three dimensions. Henneaux and Brown [5] discovered that the asymptotic spacetime algebra of *AdS₃* is simply two copies of Virasoro algebras of central charge $c = 3l/2G$ where l is the radius of curvature of *AdS₃* and G the Newtonian gravitational

constant. This is strongly suggestive of the existence of a CFT living on the boundary cylinder of AdS_3 (which is topologically a solid cylinder). This forms a suitable point for us to take a step back to discuss what is special about gravity in three dimensions. In this case, the Einstein's field equations with a negative cosmological constant turn out to dictate all solutions to be locally AdS_3 . Although this implies the absence of gravitational waves, the gravitational theory is not altogether trivial because there are black hole solutions [6] which correspond to taking some suitable global quotients of AdS_3 . Above three dimensions, the presence of gravitational waves implies that it would be highly improbable for us to be able to solve the theory exactly.

Nonetheless, quantum gravity in three dimensions involves many subtleties which are yet to be understood completely. For example, if we simply consider it purely as a theory without any additional field, it was argued by Witten and Maloney [7] that the Euclidean one-loop partition function does not make sense for any putative dual CFT living on the boundary¹, and that unless we embed the pure gravity theory in a more complicated one, such as string theory, it does not seem possible to have a well-defined quantum gravity theory which admits a semi-classical limit and, at the same time, has a dual CFT.

Embedding pure three-dimensional gravity in string theory, we have had some success in AdS_3/CFT_2 . A convenient framework to understand AdS_3/CFT_2 is type IIB string theory. In the near-horizon limit of D1/D5 branes, a factor of AdS_3 geometry appears. Open/closed string duality essentially determines the duality mechanism here. Strominger and Vafa [11] did a stringy computation of the black hole microstates which matches with Bekenstein-Hawking entropy, although it was later appreciated that the entropy formula carries a wider universality and that what is required for matching is the existence of any low-energy CFT that exhibits a Cardy limit at high temperatures (see for example [12]). Closed strings on AdS_3 (with some suitable internal space) can be solved if we add NS-NS flux because one can write the theory as a WZW model. However, it remains challenging to understand many other aspects of quantum gravity, for example, the notion of 'topology-transition' requires us to understand the closed string tachyon propagating on the space, and the vertex operators corresponding to these tachyons have yet to be found. We should also note that it is not easy to solve the dual CFT in the strongly coupled regime. To solve AdS_3 string theory exactly, we had to send all Ramond-Ramond fields to zero, yet this implies that the dual CFT is at a singular point in the moduli space. One can invoke string dualities to understand the picture, such as using M-theory or heterotic strings. In a short review in Chapter 2, we will also mention briefly about the 'Farey tail expansion of the elliptic genus method' [13] later. This is particularly interesting because the elliptic

¹The central point of their argument or computation is that when summing over all $SL(2, \mathbb{Z})$ quotients of the spacetime, the partition does not exhibit holomorphic factorization and predicts negative numbers for degeneracy number of some of the primary fields. Earlier, Witten argued in [8] that any consistent CFT dual to pure quantum gravity in 3D should be an extremal CFT and likely to possess Monster symmetry. The condition of extremal refers to a condition on the mass gap, and the Monster symmetry was motivated mainly via counting of degeneracy of low-lying states to the Hawking entropy of a BTZ black hole. Unfortunately, it was shown by Gaberdiel [9] and Gaiotto [10] later that extremal CFTs do not exist for all k .

genus (see for example [14]) is easier to compute than the partition function, and when expanded in a certain way, admits an interpretation as a sum over all geometries. However, we note that the successes of these stringy programs mentioned rely on supersymmetry-related arguments. Generically speaking, these arguments break down when we wish to embed non-supersymmetric black holes in string theory, and things become difficult to compute.

It is then natural to ask if there exists a self-consistent non-minimal model of quantum gravity in three dimensions, apart from string theory. Two-dimensional conformal field theories are certainly more well-understood than their higher-dimensional counterparts due to the existence of the infinite-dimensional Virasoro symmetry that controls the structure of their Hilbert spaces. In this sense, at least from the perspective of holography, working in three dimensions offers us the best bet to find a model of quantum gravity that we can solve exactly, and ultimately draw useful universal lessons from.

Against the backdrop sketched above, the subject of this thesis revolves around a holographic conjecture, due to Gaberdiel and Gopakumar [15], that relates a gravity theory, known as Vasiliev higher-spin theory [16, 17], to minimal coset conformal field theories which contains an infinite-dimensional \mathcal{W} -symmetry. By this, we mean that there exists conserved currents in these CFTs of which OPEs yield certain classes of \mathcal{W} algebras.

Let us first introduce the bulk gravitational theory. The higher-spin theories contain apart from the graviton, an infinite tower of higher-spin fields. Although there is no known Lagrangian formulation of these theories, the equations of motion (which are rather complex) follow from consistency conditions related to gauge symmetries. If we consider the linearized limit, each higher-spin fields obey the Klein-Gordan equations on AdS_3 space-time. One can also couple external scalar fields to the theory. The general symmetry properties of various fields of this theory descend from an infinite-dimensional Lie-algebra known as $hs[\lambda]$. This group-theoretic parameter λ will also turn out to be the 't Hooft parameter in the conjecture. Just like in the case of ordinary gravity, there is a special bonus that comes in three dimensions for these higher-spin gravity theories. The massless higher spin gauge fields possess no local degrees of freedom and can be regarded as higher spin versions of the graviton which is topological in three dimensions. The global degrees of freedom are those which are associated with boundary excitations of the fields. Further, it is consistent to truncate the infinite tower of higher spin fields to those with spin $s < N$ [18], unlike higher-dimensional Vasiliev theories.

On the other hand, the conjectured boundary theory is a family of conformal field theories which are diagonal cosets of WZW models of the form

$$\frac{SU(N)_k \oplus SU(N)_1}{SU(N)_{k+1}}$$

where k is the level of the affine algebra of one of the sectors of the Hilbert space. There is

a well-defined large N limit for these two-dimensional theories which corresponds to taking

$$k, N \rightarrow \infty, \lambda = \frac{N}{k+N} \text{ being fixed.} \quad (1.1)$$

When the conjecture was first proposed in [15], the duality is between the bulk Vasiliev theory and the boundary CFT in this limit, where λ serves the role of the 't Hooft parameter. As we shall also briefly discuss in Chapter 5, there is another limit which is likely to provide us insights into the workings of the duality, namely the limit

$$c \rightarrow \infty, N \text{ finite} \quad (1.2)$$

This is known as the semi-classical limit. For both the limits (1.1) and (1.2), the bulk theory needs to be equipped with massive complex scalar field(s) of which masses depend on the parameter λ or c and N . This constraint on the mass comes from identification between the scalar field states in the bulk and the coset CFT states in the boundary. We will mention more about these two limits later in Chapter 5.

Since Vasiliev theory is a highly complex gravitational theory, it is natural to ask why then are we considering this model of quantum gravity? The crux is that the minimal coset conformal field theory is exactly solvable for any admissible N and k . For $N = 2$, this family of CFTs is nothing but the minimal series of CFTs well-studied in the context of classification of rational CFTs. Thus, on the boundary theory side, this is a significant advantage as compared to say the sigma model on a symmetric orbifold - the dual CFT in AdS_3/CFT_2 , or other reincarnations of it (which descend from open-string dynamics) when we perform string dualities on D1/D5, or extremal CFTs with monster symmetry in the case of pure gravity for which we are not even sure whether they exist for a generic value of the cosmological constant. It is this solvability that renders this duality conjecture attractive to study in recent years. Nonetheless, the price that comes with it is the complexity of quantum Vasiliev theory in the bulk. At least at the time of writing, quantum Vasiliev theory is yet to be solved and constitutes an important research goal in this area.

In this thesis, I will present ‘evidences’ for this holographic duality. Below is a short list which highlights some of the original contributions that will be explained in greater detail in later chapters:

- **New higher-spin black holes:** we construct new black hole solutions in $SL(4) \oplus SL(4)$ and $SL(N|N-1) \oplus SL(N|N-1)$ Chern-Simons theories which are the topological sectors of Vasiliev theories, and demonstrate that there is a well-defined gravitational thermodynamics for these black holes that appears to be compatible with higher-spin holography.
- **Supersymmetric generalization of the topological sector of the bulk theory:** we initiate a study of a suitable supersymmetrization of the topological sector of the bulk theory, namely $SL(N|N-1) \oplus SL(N|N-1)$ Chern-Simons theories, including the supersymmetry transformation laws, and demonstrate how to recover super-Virasoro

algebras from its asymptotic spacetime symmetries.

- **Relationship between higher-spin black hole thermodynamics and \mathcal{W} algebras:** we furnish a clear evidence that the first law of thermodynamics usually encountered in Euclidean gravitational theories can be realized in a similar fashion for the higher-spin black holes and that this is related to the existence of \mathcal{W} algebras as symmetry algebras at asymptotic infinity. We mainly compute various quantities in a gauge where these black holes are described by a non-gauge invariant metric describing a class of transversable wormholes.
- **Derivation of the bulk-boundary propagator for a scalar field probing a higher-spin black hole:** we present a computation of the bulk-boundary scalar propagator in the background of a higher-spin black hole in $\text{hs}[\lambda]$ Chern-Simons theory. Our result holds for any value of λ between zero and one, and should correspond to an integrated torus three-point function of the boundary coset CFT. The derivation turns out to be rather ‘delicate’ due to the group-theoretic properties of $\text{hs}[\lambda]$.

The organization of the rest of the chapters goes as follows: in Chapter 2, we present a discussion of the role of Wilson loops in three-dimensional gravity and how holonomies characterize classical solutions which correspond to quotients under discrete subgroups of the isometry group. We will also include notes on stringy realizations. In Chapter 3, we present derivation of the bulk-boundary propagator and also include a review of salient properties of Vasiliev gravity and its formulation. In Chapter 4, we will discuss the topological sector of Vasiliev gravity as realized via Chern-Simons theories, in particular, using the cases of $SL(4)$ and $SL(N|N-1)$. In Chapter 5, we have collected some notes on the CFTs which form the background of our understanding of the two different limits of the higher-spin holographic conjecture as discussed above. In Chapter 6, we demonstrate how asymptotic spacetime symmetries are related to \mathcal{W} algebras and super- \mathcal{W} algebras. Finally in Chapter 7, we present new higher-spin black hole solutions various types of higher-spin theories and discuss their gravitational thermodynamics. Other classical solutions such as conical defect solutions and RG flow solutions will be presented too. Then, this thesis ends with a brief discussion on future directions in Chapter 8.

 CHAPTER 2

On 3D Quantum gravity: sketches from different perspectives

This Chapter presents a brief discussion of three different frameworks in understanding quantum gravity in three dimensions, namely Chern-Simons theory in Section 2.1, string theory in Section 2.3 and higher-spin holography in Section 2.4. In Section 2.2, we present an independent formulation of the role of Wilson loops in three-dimensional gravity and how holonomies characterize classical solutions which correspond to quotients under discrete subgroups of the isometry group.

2.1 On pure gravity in three dimensions

In this section, we discuss Chern-Simons formulation of 3D gravity - as was first pointed out by Townsend, Achucarro and Witten [19, 20]. Denote the dreibein by $e^a = e^a_\mu dx^\mu$ and the spin connection by $\omega^a = \frac{1}{2}\epsilon^{abc}\omega_{bc\mu}dx^\mu$. The Einstein-Hilbert action can be re-written in the form of the difference¹ between two Chern-Simons actions as follows (M denotes the 3-manifold):

$$I = \frac{1}{8\pi G} \int_M e^a \wedge \left(d\omega_a + \frac{1}{2}\epsilon_{abc}\omega^b \wedge \omega^c \right) + \frac{\Lambda}{6}\epsilon_{abc}e^a \wedge e^b \wedge e^c = I_{c.s}[A] - I_{c.s}[\bar{A}] \quad (2.1)$$

$$I_{c.s} = \frac{k}{4\pi} \int_M \text{Tr} \left(A \wedge dA + \frac{2}{3}A \wedge A \wedge A \right), \quad A = \left(\omega^a + \frac{1}{l}e^a \right) \tau_a, \quad \bar{A} = \left(\omega^a - \frac{1}{l}e^a \right) \bar{\tau}_a \quad (2.2)$$

where $\Lambda = -1/l^2$, $\tau_a, \bar{\tau}_a$ are 2 independent sets of generators of $SL(2, R)$, A are thus $SL(2, R)$ -valued gauge potentials, and the Chern-Simons level $k \propto l/G$ with the proportionality constant depending on the trace convention. Classically, this nice coincidence is embellished by the fact that general diffeomorphisms (which are smoothly connected to the identity) are related to gauge transformations on-shell, because it can be checked that the Lie derivative of the gauge connection $\mathfrak{L}_\xi A$ is

$$\mathfrak{L}_\xi A = d(\xi.A) + \xi.dA = \xi.F + D_A(\xi.A) \quad (2.3)$$

¹If we take the sum instead of difference, we obtain Chern-Simons gravity.

where the curvature $F = dA + A \wedge A$, D_A is the usual gauge-covariant exterior derivative, and ξ is some vector. Since it can be further checked that the flat connection condition ($F = 0$) is equivalent to the field equations, what (2.3) tells us is that *on-shell*, performing a co-ordinate transformation is the same as an ordinary gauge transformation with parameter $\lambda = \xi^\mu A_\mu$. For completeness, let us summarize the gauge groups of Chern-Simons including the Euclidean and $\Lambda > 0$ cases:

1. AdS_3 : $SL(2, R) \times SL(2, R)$
2. Euclidean AdS_3 : $SL(2, C)$ as a complexification of $SU(2)$
3. Euclidean dS_3 : $SU(2) \times SU(2)$
4. dS_3 : $SL(2, C)$ as a complexification of $SL(2, R)$

It is important to note that when the gauge group is $SL(2, C)$ as a complexification of G , then A, \bar{A} are hermitian conjugates, but when the gauge group is of the form $G \times G$, then the two connections are independent. Now, in each case, the target space geometry is either $SL(2, C)/G$ or $G \times G/G$. In particular, $AdS_3 \sim SL(2, R) \times SL(2, R)/SL(2, R)$ (universal cover of $SL(2, R)$) and Euclideanized $AdS_3 \sim SL(2, C)/SU(2) \sim \mathbb{H}^3$ where \mathbb{H}^3 is the hyperbolic 3-space.

Now, since the equations of motions give flat connections, one may wonder if the excitations of the theory can be classified under ‘pure gauge’ excitations. However, these become physical at the conformal boundary, i.e. physical observables need not be invariant under gauge transformations at the boundary. To see this, consider Chern-Simons theory on a (2+1) manifold $R \times \Sigma$ where Σ is a 2-manifold with boundary. Upon canonical quantization, it can be checked that since A_0 is a lagrange multiplier, its corresponding first-class constraints $G_{(0)}$ generate gauge transformations. But they can be checked to be non-differentiable, and are actually second-class constraints at the boundary. In such a framework, the CFT that describes quantum gravity is obtained by a suitable restriction of the bulk gravitational dynamics to the boundary. Begin by performing a gauge transformation $A^g = g^{-1}Ag + g^{-1}dg$, then it is straightforward to check that the action $I_{c.s}$ transforms as

$$I_{c.s} \rightarrow I_{c.s} - \frac{k}{4\pi} \text{Tr} \int_{\partial M} (dg g^{-1}) \wedge A - \frac{k}{12\pi} \text{Tr} \int_M (g^{-1}dg)^3 \quad (2.4)$$

For a closed manifold, the last term is a topological winding number and upon suitable quantization of the level k , $\exp(iI_{c.s})$ is invariant. However, for a manifold with boundary, this term cannot be discarded and will reside in the eventual WZW theory. Next, consider the variation of $I_{c.s}$ which can be shown to yield $\delta I_{c.s} = \frac{k}{2\pi} \int_M \text{Tr} (\delta A (dA + A \wedge A)) - \frac{k}{4\pi} \int_{\partial M} \text{Tr} (A \wedge \delta A)$. We see that we need to add boundary contribution to the action to cancel the boundary term. To do that, it is convenient to choose a complex structure

$dz \wedge d\bar{z}$ on the boundary, and prescribe the boundary value. We can fix half of the gauge fields, i.e. either $(A_z, \bar{A}_{\bar{z}})$ or $(\bar{A}_{\bar{z}}, A_z)$. This implies that we add the boundary term

$$I_b = \frac{k}{4\pi} \int_{\partial M} \text{Tr} A_z A_{\bar{z}} \quad (2.5)$$

It can be straightforwardly checked that the sum of the $I_{total} = I_{c.s} + I_b$ transforms as $I_{total} \rightarrow I_{total} + kI_{WZW}$ where

$$I_{WZW}[g^{-1}, A_z] = \frac{1}{4\pi} \int_{\partial M} \text{Tr} (g^{-1} \partial_z g g^{-1} \partial_{\bar{z}} g - 2g^{-1} \partial_{\bar{z}} g A_z) + \frac{1}{12\pi} \int_M \text{Tr} (g^{-1} dg)^3 \quad (2.6)$$

This is the chiral WZW action in which the gauge element g is coupled to a background gauge field A_z . When we apply this to $I_{c.s}[A] - I_{c.s}[\bar{A}]$, and evaluate the action on classical solutions (in which $A_z = \bar{A}_{\bar{z}} = 0$), then using Polyakov-Wiegman identities, the two chiral WZW actions combine to form an ‘non-chiral’ WZW action which depends on the combination $g = g\bar{g}^{-1}$. Explicitly,

$$I = kI_{WZW}[g^{-1}, A_z = 0] - kI_{WZW}[\bar{g}^{-1}, \bar{A}_{\bar{z}} = 0] = kI_{WZW}[g = g\bar{g}^{-1}] \quad (2.7)$$

This non-chiral WZW model is thus a sigma model with target space $SL(2, C)/G$ (or $G \times G/G$) [21].

Let us briefly touch on two arguments made against Chern-Simons theory as the correct theory of quantum gravity as summarized in [8]. Perturbatively, if one perform a path integral around the background of a valid classical solution, Chern-Simons theory will not take us out of the region in which the metric is invertible. However, non-perturbatively, the vierbein may cease to be invertible and there is nothing at first sight to constrain the path integral to run over invertible metrics. Also, we expect to sum over different topologies, but there is no natural instruction for us to sum the Chern-Simons action over different 3-manifolds. Recently in [22], Deser published a very brief but interesting note to address the first issue in the classical picture. Recall that the metric’s invertibility is a tacit, basic assumption of classical gravity, and one might ask (even without concerns with quantum gravity) why this metric variable has no zero vacuum. He proposed to append the term

$$L_D = M \left(\sqrt{gh} - \phi^2 \right) \quad (2.8)$$

to the usual gravitational action, where ϕ can be interpreted as a Higgs scalar which has a non-zero VEV, g is the metric determinant, M is some tensor density and h is some tensor anti-density (the tensorial nature of M and h is to ensure diffeomorphism invariance). Varying with respect to M , we obtain $g \neq 0$ (metric invertibility is thus embodied as a field equation). On the other hand, varying with respect to h implies vanishing of M , and thus the usual equations of motion are not disturbed.

In the following, we will attempt to study what such an attempt implies in the language of Chern-Simons and its boundary WZW theory. We consider Lorentzian AdS_3 as a first

example. Now, the metric determinant \mathcal{G} can be expressed as (setting $l = 1$ for simplicity)

$$\sqrt{\mathcal{G}} d^3x = \frac{1}{6} \epsilon_{abc} e_i^a e_j^b e_k^c dx^i \wedge dx^j \wedge dx^k = \frac{1}{48} \epsilon_{abc} \Delta A_i^a \Delta A_j^b \Delta A_k^c \epsilon^{ijk} d^3x, \quad \Delta A \equiv A - \bar{A} \quad (2.9)$$

Recall that the boundary dynamics essentially arises from the way the Chern-Simons action changes under gauge transformations. For us to see the consequence of (2.8) on the boundary CFT, we will use (2.9) as the ansatz for the fields M as well. We will denote this independent gauge field as B . We will assume that the anti-density h can be realized by a function of a suitably defined ‘dual’ gauge field \tilde{A} , and that $h(\tilde{A})$ transforms as the inverse of $1/\sqrt{\mathcal{G}}$, and defer a more serious treatment to future work. Thus, in addition to A, \bar{A} , we have also $B, \bar{B}, \tilde{A}, \tilde{\tilde{A}}$ and a scalar field ϕ of which VEV is defined to be non-zero.² What we want to determine is how their combined behavior under gauge transformation affects the boundary CFT. Just like in topological BF theory, the new gauge fields have to be also valued by two independent $SL(2, R)$ connections since this is the case for the primary gauge field A . As a first step, we consider an infinitesimal gauge transformation $g \approx 1 + u, \bar{g} \approx 1 + \bar{u}$, and keep terms to linear order in u and \bar{u} . Then one can show after some tedious but straightforward algebra that the change δL_D is (note that the contributions due to $\sqrt{g}\sqrt{h}$ is multiplied to M which vanishes on-shell)

$$\frac{\delta L_D}{(1 + \phi^2)} = \frac{1}{16} \epsilon_{abc} d \wedge ((u - \bar{u})^a (B_t)^b \wedge (B_t)^c) + \frac{1}{8} (u - \bar{u})^a \epsilon_k^{ij} ((\bar{B}_j - B_j)_a B_i^e \bar{B}_e^k - \epsilon_{abc} \partial_i (B_t)^b_j (B_t)^c_k) \quad (2.10)$$

$$B_t \equiv B + \bar{B} \quad (2.11)$$

Since there are no kinetic terms for B , and the field equations only demand $B = \bar{B}$, we see that the L_D is only invariant if we further impose the vanishing of curvature $F_{B_t} \equiv dB_t + B_t^2$ of B_t . This can be done by adding a Chern-Simons term for B_t (and perhaps coupled to other fields). Such an on-shell condition then simplifies (2.10) to a boundary term that looks like

$$I_D = (1 + \phi^2) \epsilon_{abc} \int_{\partial M} (u - \bar{u})^a B_t^b B_t^c \quad (2.12)$$

It is interesting to note that $g\bar{g}^{-1} \approx 1 + (u - \bar{u}) + \dots$ and thus it is likely that if we perform full analysis (to all orders), we obtain a non-chiral WZW model with gauge group G_C/G , but ‘deformed’ by auxiliary fields B_t and possibly \tilde{A} .

2.2 Playing with Wilson loops

Apart from the BTZ black hole, a large variety of space-times can be obtained by identifying points in AdS_3 by means of a discrete group of isometries. Begin by considering a killing

²As commented by Deser himself, such an approach may not appear to be completely satisfactory as it does not really give physical insights to this problem. Nonetheless, it does effectively set up a very simple model in which \mathcal{G} is restricted not to vanish by the field equations.

vector ξ that defines a one-parameter subgroup of isometries of $SO(2, 2)$: $P \rightarrow e^{t\xi}P$, the mappings for which t is an integer multiple of some basic step. Since the transformations are isometries, the quotient space obtained by identifying points in a given orbit inherits from AdS_3 a well-defined metric which has constant negative curvature and is a valid solution.³ In [10], a complete classification of the 1-parameter subgroups of $SO(2, 2)$ was done, and we will discuss this shortly in the language of the conjugacy classes of $SL(2, R)$ and $SL(2, C)$.

2.2.1 Worm-holes and black holes from the Möbius transformations of the Poincaré disk

Consider AdS_3 as the quadric surface $X^2 + Y^2 - U^2 - V^2 = -1$ embedded in $ds^2 = dX^2 + dY^2 - dU^2 - dV^2$, where we set $l = 1$ for convenience. We write the defining equation of the quadric as condition on the determinant of a matrix

$$|\mathbf{X}| \equiv \begin{vmatrix} V + X & Y + U \\ Y - U & V - X \end{vmatrix} = 1 \quad (2.13)$$

Such a condition is clearly preserved by a transformation $\mathbf{X} \rightarrow g\mathbf{X}\bar{g}^{-1}$ where g, \bar{g} belong to two $SL(2, R)$. The trace of g is invariant under conjugation and the classes are defined as: (i)elliptic, ($|\text{Tr } g| < 2$) (ii)parabolic, ($|\text{Tr } g| = 2$) (iii)hyperbolic, ($|\text{Tr } g| > 2$). Now, introduce intrinsic coordinates (t, ρ, ϕ) via

$$X = \frac{2\rho}{1 - \rho^2} \cos \phi, \quad Y = \frac{2\rho}{1 - \rho^2} \sin \phi, \quad U = \frac{1 + \rho^2}{1 - \rho^2} \cos t, \quad V = \frac{1 + \rho^2}{1 - \rho^2} \sin t \quad (2.14)$$

The metric becomes⁴

$$ds^2 = - \left(\frac{1 + \rho^2}{1 - \rho^2} \right)^2 dt^2 + \frac{4}{(1 - \rho^2)^2} (d\rho^2 + \rho^2 d\phi^2) \quad (2.15)$$

At constant t , we have the Poincaré disk. For static black hole spacetimes, it turns out to be convenient to focus on the initial surface $t = 0$. (Once the $t = 0$ plane is understood, one just evolves the identification in time to obtain the spacetime.) This surface is preserved by the discrete group Γ which is a subgroup of the diagonal subgroup of $SL(2, R) \times SL(2, R)$. We have mentioned briefly about the conjugacy classes which for the Poincaré disk are described as: (i)elliptic: 1 fixed point within disk, (ii)parabolic: 1 fixed point at boundary, (iii)hyperbolic: 2 fixed points at boundary. The action of Γ on the disk can be understood

³For the quotient space to have an admissible causal structure, closed curves generated by the identification should not be timelike or null, a necessary condition being $\xi^2 > 0$. Thus, the covering space of all good solutions are open regions of AdS_3 such that the flows of all ξ that create the subgroup Γ are spacelike.

⁴The covering space is now the interior of the cylinder $\rho < 1$ and conformal compactification is done by adjoining the surface of the cylinder $\rho = 1$.

by finding the fundamental region.⁵

Let's consider some examples [23]: first, the static BTZ black hole of which Γ consists of a single hyperbolic element γ . The fundamental region is that between two geodesics mapped into one another by generator γ . The distance between the two geodesics measured along their common normal is precisely the horizon circumference. As a second example: consider the group Γ generated by two hyperbolic elements γ_1, γ_2 . The fundamental region to be part of the disc between 4 geodesics which are identified *cross-wise*. The geometry has only one asymptotic region and topology inside the event horizon can be shown to be a torus. As a final example, consider another group Γ of two hyperbolic elements but they identify the geodesics *side-wise* instead. There are now three asymptotic regions and corresponding event horizons⁶.

If we take the group Γ to be more complicated, one can construct an enormous class of spacetimes, with any number of asymptotic regions and any number of handles hidden behind the horizons. We also note that the conjugacy classes have the following physical meanings: hyperbolic elements correspond to general BTZ black holes, parabolic elements correspond to extremal ones (if both $g, \bar{g} \in$ parabolic, then $M = 0$, if one is parabolic and the other hyperbolic, we have the case of $J = Ml$) while elliptic elements correspond to particles creating conical singularities). The main class of wormholes constructed via this method has all elements of the discrete subgroup with $g, \bar{g} \in$ hyperbolic. Here's a practical way [12] of realizing the elements γ and relating them to g, \bar{g} : let $J_{XY} = X\partial_Y - Y\partial_X$, $J_{YV} = Y\partial_V + V\partial_Y$, etc, and let $\gamma_{\{0,1,2\}}$ be the usual gamma matrices in 2+1 dimensions, then we can identify

$$\begin{aligned} -\frac{1}{2}(J_{XU} + J_{YV}) &= J_1 = -\frac{1}{2}\gamma_1, & -\frac{1}{2}(J_{XU} - J_{YV}) &= \tilde{J}_1 \\ -\frac{1}{2}(J_{XV} - J_{YU}) &= J_2 = -\frac{1}{2}\gamma_2, & -\frac{1}{2}(J_{XU} + J_{YV}) &= \tilde{J}_2 \\ -\frac{1}{2}(J_{XY} - J_{UV}) &= J_3 = -\frac{1}{2}\gamma_0, & -\frac{1}{2}(J_{XU} + J_{YV}) &= \tilde{J}_3 \end{aligned} \quad (2.16)$$

Let us briefly relate what we have discussed to the theory of uniformization of Riemann surfaces. The $t = 0$ surface is non-compact and are obtained from the disc by discrete identifications by Fuchsian group, yet these are *not* the classical Fuchsian groups because one gets non-compact surfaces. (If the $t = 0$ slice is compact, then the subgroup generated by $2G$ elements has a fundamental region = $4G$ -sided polyhedron and resultant geometries are compact genus G Riemann surfaces. These were studied as interesting cosmological models in [24].

We now consider analytic continuation to Euclidean AdS_3 which we shall model by the

⁵The fundamental region \mathbf{D} of the unit disc \mathbf{H} for group Γ is such that any point on \mathbf{H} can be obtained as an image of point in \mathbf{D} under Γ but no two points of \mathbf{D} are related.

⁶Let a circle depict the Poincaré disc, and draw 4 semi-circular geodesic arcs at the north, south, east and west portions of the circle. The fundamental regions are visualized by identification of these arcs, and so are the asymptotic regions.

upper half-space \mathbf{H}_3 . The Euclidean BTZ black hole is a solid torus. What about the more exotic black objects considered above? Instead of the usual Wick rotation, let us consider identifying points in \mathbf{H}_3 using the same group Γ . This is of course consistent since $SL(2, R) \subset SL(2, C)$ which is the group of isometries of \mathbf{H}_3 . What is then the relation between the $t = 0$ slice and the conformal boundary of the Euclideanized solution which is just the Riemann sphere? It turns out that the latter is the Schottky double of the former.⁷ Indeed, this is simply *uniformization by Schottky groups* Σ which are discrete subgroups of $SL(2, C)$ freely generated by a number, say q , of loxodromic generators $L_1, \dots, L_q \in SL(2, C)$.⁸ Formally, the Euclidean manifolds are quotients of \mathbf{H}_3 by a certain quasi-Fuchsian group. Consider the toroidal wormhole discussed earlier - if it is of genus g and K asymptotic regions, then it can be shown that the genus of the Euclidean boundary is $g_E = 2g + K - 1$, and the Euclideanized 3-manifold is thus a g_E -genus handle-body.

Practically, here's the recipe: we know that $SL(2, C)$ can be represented as a direct product of two copies of $SU(2)$, and it acts on \mathbf{X} (see 2.13) by $\mathbf{X} \rightarrow \mathbf{SXS}^\dagger$, $S \equiv e^{\frac{i}{2}\sigma_i(a^i - ib^i)}$. We can analytically continue S into an element of $SL(2, R)$ by

$$ib^i \rightarrow b^i, \quad i\sigma_1 \rightarrow \gamma_1, \quad i\sigma_2 \rightarrow -\gamma_0, \quad i\sigma_3 \rightarrow \gamma_2. \quad (2.17)$$

After applying the map (2.17) to both S and S^\dagger , we then obtain the left(g) and right(\bar{g}) parts of the $SL(2, R)$ isometry.

2.2.2 Where the Wilson loops are

Since Wilson loops are all the physical observables in Chern-Simons theories, these loop holonomies should play a crucial role in the gravitational setting. As was first mentioned in [6], these holonomies are actually of the form $e^{n\xi}$, and this correspondence appears to hold under general conditions on the manifold. There is thus a correspondence between the identification matrix γ and the holonomies of the connection $W = e^{i\int A}$.

For a single Euclideanized BTZ black hole, this is *exactly* the same, with the Wilson loop defined along the non-contractible cycle of the solid torus. Let us briefly review how this can be understood. When V of (2.13) is Wick-rotated, we can redefine $\mathbf{X} \rightarrow \begin{pmatrix} V + U & X + iY \\ X - iY & U - V \end{pmatrix}$. Perform a boost along the UX plane with parameter $-b$ and an anti-clockwise rotation in the YV plane with parameter a , then it can be shown that the identification matrix acting on \mathbf{X} is

$$L_1 = \exp \left(\frac{i\sigma_1}{2}(a + ib) \right). \quad (2.18)$$

⁷This arises commonly in boundary CFTs. Briefly speaking, the Schottky double of a surface X with a boundary is made by glueing two copies of X along the boundary to obtain a connected surface.

⁸Note that the classes of parabolic, elliptic and hyperbolic remain as defined in the case of $SL(2, R)$, while a loxodromic L is defined as $0 \leq \text{Tr}^2(L) \in C \setminus [0, 4]$.

We now need explicit coordinates to compute the Wilson loop. The Poincaré coordinates can be obtained from the quadric by defining $z = \frac{1}{U+X}$, $\beta = \frac{y}{U+X}$, $\alpha = \frac{-V}{U+X}$, then the metric reads

$$ds^2 = \frac{1}{z^2} (dz^2 + d\beta^2 + d\alpha^2) = \frac{1}{R^2 \sin^2 \chi} (dR^2 + R^2(d\chi^2 + \cos^2 \chi d\theta^2)) \quad (2.19)$$

where we have further invoked spherical coordinates: $\alpha = R \cos \theta \cos \chi$, $\beta = R \sin \theta \cos \chi$, $z = R \sin \chi$ ⁹. Now, let $r_{\pm}^2 = \frac{1}{2} (M \pm \sqrt{M^2 - J^2})$, where r_{\pm} denote the radius of the event and Cauchy horizon respectively. Note that we define M, J as the ADM mass and angular momentum. When expressed in Poincaré coordinates, the metric is just (2.19) but with the identifications

$$(R, \theta, \chi) \sim \left(R e^{2\pi r_+}, \theta + 2\pi|r_-|, \chi + \frac{\pi}{2} \right), \quad b = 2\pi r_+, \quad a = -2\pi|r_-| \quad (2.20)$$

where the fundamental region is described by $R \in (1, e^{2\pi r_+})$. In the Euclidean picture, the Chern-Simons connection $A = \omega + ie$. Now, $\chi = \text{constant}$ is some non-contractible cycle in the solid torus, with $\chi = 0$ being the boundary T^2 and $\chi = \pi/2$ the degenerate core. We have now all the ingredients to compute the Wilson loop along a constant $\chi = \chi_0$ longitude. After some algebra, we find¹⁰

$$W_{\chi} = \exp i \left(\int_{\chi=\chi_0} A \right) = \begin{pmatrix} \cos \varpi - \cot \chi \sin \varpi & -i \sin \varpi \operatorname{cosec} \chi \\ -i \sin \varpi \operatorname{cosec} \chi & \cos \varpi - \cot \chi \sin \varpi \end{pmatrix} \quad (2.21)$$

where $2\varpi \equiv -a - ib$. A straightforward check ensures that indeed, we have the nice relation $W_{\chi_0=\pi/2} = L_1$. This Wilson loop at the core of the solid torus changes upon gauge transformation, but remains in the same conjugacy class since its trace is gauge-invariant.

Consider now the exotic wormhole of which Euclideanized handle-body can be generated by the discrete group having loxodromic letters

$$\Gamma = \left\{ L_1 = \exp \left(\frac{i\sigma_1}{2} (a + ib) \right), L_2 = \exp \left(\frac{i\sigma_3}{2} (a + ib) \right) \right\} \quad (2.22)$$

We have studied the Wilson loop equivalent of L_1 . What about L_2 ? Clearly, it is a rotated version of L_1 since it belongs to the same conjugacy class. To see this more convincingly, we attempt to *locate* the second Wilson loop relative to the first one. We will see that they are in fact *orthogonal*, but we will have to resort to some geometrical computations. Begin by observing that L_2 is related to L_1 by switching $X \leftrightarrow V$ as it can be checked that L_2 is equivalent to an inverse boost along the UX -plane and rotation in the YV -plane. We also

⁹This helps us check that it is a solid torus.

¹⁰Our result differs from [25], but is probably due to a different choice of representation of $SL(2, C)$ We thank S. Carlip for discussion.

write down the inverse map:

$$z' = \frac{2z}{1 + R^2 - 2\alpha}, \beta' = \frac{2\beta}{1 + R^2 - 2\alpha}, \alpha' = \frac{R^2 - 1}{1 + R^2 - 2\alpha} \quad (2.23)$$

where the primed Poincaré coordinates refer to those of L_1 but with X, V switched. Recall that the curve $\chi = \pi/2$ is simply $\alpha = \beta = 0, z = R$. Let us denote the curve of Wilson loop of L_2 by C_2 . This is defined by $\alpha' = 0, \beta' = 0, z' = R'$. The inverse map (2.23) allows us to picture this curve in the fundamental domain of unprimed coordinates, and this gives:

$$C_2 = (R = 1, \beta = 0, z = R'(1 - \alpha)) \quad (2.24)$$

which is a finite circular arc in the unprimed chart since the length corresponds to the interval that defines the fundamental region $R' \in [1, e^{-b}]$. Another way to see that these two Wilson loops are orthogonal geometrically is to invoke the embedding co-ordinates. In terms of quadric coordinates, the two Killing vectors that generate L_1 and L_2 are $\xi_1 = (bU, 0, 0, -aX)$ and $\xi_2 = (0, 0, -bU, -aV)$ in the basis $\{\partial_V, \partial_U, \partial_X, \partial_Y\}$. Clearly, the orbits lie in orthogonal planes, and thus remain orthogonal within the submanifold \mathbf{H}_3 . More formally, a flat connection is determined by its holonomy, and what we have demonstrated are examples of the homomorphism

$$\rho : \pi_1(M) \rightarrow SL(2, C) = \text{Isom}(\mathbf{H}_3) \quad (2.25)$$

2.3 Notes on string theoretic realizations

In this section, we will very briefly sketch a couple of well-known stringy constructions of AdS_3 geometries, our focus being on understanding how to reproduce the partition function (and thus the spectrum of states) from both the CFT and bulk gravity side as coming from an approximate computation in the wider framework of string/M theory. Our eventual purpose is to furnish a broader context to see the motivation for understanding higher-spin holography.

The two constructions that we will be reviewing in the following (see for example, [12] and references therein) are

- (a) Type IIB string theory on $\mathbb{R}^{4,1} \times S^1 \times \left\{ \begin{array}{c} T^4 \\ K3 \end{array} \right\}$ and D1/D5 branes
- (b) M5 branes wrapped on $AdS_3 \times S^2 \times \left\{ \begin{array}{c} T^6 \\ K3 \times T^2 \\ CY_3 \end{array} \right\}$.

Let us begin with the set-up in (a), where we have Type IIB string theory on the ambient spacetime as shown, with the following non-perturbative objects included (denoting the

internal space simply by M_4 :

- N_5 D5-branes wrapped on $S^1 \times M_4$
- N_1 D1-branes wrapped on S^1

Such a brane configuration preserves eight susy charges (which is $\frac{1}{4}$ of the original thirty-two). Now, consider the regime of weak string coupling, i.e. $g_s \leq 1$, and impose the overall size of M_4 to be much smaller than S^1 . The low-energy dynamics is now contained in the intersection $S^1 \times \mathbb{R}(\text{time})$. When the open strings are quantized, the Chan-Paton degrees of freedom imply the existence of a $U(N_1) \times U(N_5)$ gauge theory. It also turns out that in the infra-red, the theory flows (in the sense of the renormalization group) to a conformal field theory with (4, 4) SUSY, with thus $SU(2)_{\text{left}} \times SU(2)_{\text{right}}$ R-symmetry. For each sector, the R-symmetry currents are anomalous when coupled to external gauge fields. The anomaly arises from a one-loop effect which yields $k = N_1 N_5$, where k is the level of the current algebra. The central charge $c = 6k = 6N_1 N_5$. This summarizes, in a sense, a microscopic derivation of the central charge.

From the IIB supergravity description, the classical solution has a metric which has $AdS_3 \times S^3 \times M_4$ near horizon geometry. In the low-energy effective action, apart from the metric, the dilaton field and the Ramond-Ramond three-form are turned on. To compute the central charge, one gauges the global $SO(4)$ isometry of the S^3 factor as follows: replace the S_3 metric by

$$d\Omega_3^2 \rightarrow (dn^i - A^{ij}n^j)^2 \quad (2.26)$$

where $\sum_i n_i^2 = 1$, $A^{ij} = -A^{ji}$ are the $SO(4)$ Kaluza-Klein gauge fields, and importantly, we note the invariance under the gauge transformation

$$A^{ij} \rightarrow d\alpha^{ij} + [\alpha, A]^{ij}, \quad n^i \rightarrow n^i + \alpha^{ij}n^j \quad (2.27)$$

where α is the gauge parameter. Under (2.27), the IIB supergravity low energy effective action which reads

$$I \sim \int dx \sqrt{G} e^{-2\Phi} \left(R + 4(\partial\Phi)^2 + \frac{1}{2} e^{2\Phi} |G_3|^2 \right) \quad (2.28)$$

where Φ is the dilaton field and G_3 the RR 3-form and G the metric determinant, must remain unchanged up to some boundary term. This turns out to demand a modified G_3 which originally reads for the solution (which has the $AdS_3 \times S^3 \times M_4$ near horizon geometry) as $2gN_5\alpha'(\eta_3 + i \star_6 \eta_3)$ where η is the S^3 volume form and \star_6 is the 6D Hodge dual. An explicit computation shows that after modifying G_3 , the induced boundary term goes as

$$\delta I = -i \frac{N_1 N_5}{4\pi} \int_{\partial AdS} \text{Tr} (\Lambda dA - \bar{A} d\bar{A}) \quad (2.29)$$

from which we can read off the level $k = N_1 N_5$ and the central charge agrees with the IIB

string theory computation, or more precisely, its low-energy CFT living on the intersection between the branes.

The case of wrapped M5 branes in case (b) is similar. Consider M theory on $AdS_3 \times S^2 \times M_6$ where M_6 is one of the following $\{T^6, K3 \times T^2, CY_3\}$, and denote a set of four cycles in M_6 by the volume form Ω_B . The M5 brane wraps the set of four cycles $P = p^B \Omega_B$, and thus is manifest as a string-like object in the remaining $AdS_3 \times S^2$. Compactifying the spatial direction of the string to be S^1 , the low energy theory turns out to be a CFT with $(0, 4)$ supersymmetry. The central charge can be computed by taking into account the massless fluctuations of the M5 brane, and it can be written in the form

$$c = c_{IJK} p^I p^J p^K + (c_2)_{MP} p^M \quad (2.30)$$

where c_{IJK} is the number of triple intersections for the three four-cycles and c_2 is the second Chern-class of M_6 expanded in the basis of the four-cycles. These are topological invariants so the central charge is moduli-independent. How does one reproduce the central charge from the gravitational side then? It is essentially the same as in case (a). We begin with the 11D supergravity action

$$I = \frac{1}{2\kappa_1 l^2} \left[\int d^4x \sqrt{g} \left(R + \frac{1}{2} |F_4|^2 \right) + \frac{i}{6} \int A_3 \wedge F_4 \wedge F_4 \right. \\ \left. + N \int A_3 \wedge \left(\text{Tr} R^4 - \frac{1}{4} (\text{Tr} R^2)^2 \right) \right] \quad (2.31)$$

where N is some normalization constant $\sim \kappa^4/3$. One then further reduces the effective action (2.31) to five dimensions in the presence of the M5 branes. To proceed with the gravitational computation of the central charge, one considers the asymptotically flat solution of the resulting five-dimensional theory and finds that the near-horizon geometry is $AdS_3 \times S^2$. Finally, in a similar spirit as what was done in case (a), one gauges the global $SO(3)$ isometry of the S^2 factor, and compute the induced boundary terms, from which one can read off the central charges.

The matching of central charges in both cases rely on the existence of a low-energy CFT with some supersymmetry that relates the central charges to the R-symmetry anomalies. This reproduces the black hole entropy at the level of Cardy, i.e.

$$S = 2\pi \sqrt{\frac{c}{6} \left(L_0 - \frac{c}{24} \right)} + 2\pi \sqrt{\frac{\bar{c}}{6} \left(L_0 - \frac{\bar{c}}{24} \right)} \quad (2.32)$$

which is valid in the high-temperature limit. It is natural to ask if it is possible to check the validity of

$$Z_{CFT} = Z_{AdS} \quad (2.33)$$

beyond the Cardy limit. In such a case, one needs to include all states of the full string theory which is not a tractable computation. Note that both sides of (2.33) descend from

string theory, the LHS being open strings and the RHS being closed strings. We will mention more about AdS_3/CFT_2 at the end of the section. The previous arguments we have sketched relies on low-energy approximations, and the fact that the central charges are fortunately protected by supersymmetry to be valid in the absence of many other terms. For example, for the M5 brane computations, there are infinitely many higher derivative terms in the action (2.31) which we omitted because they do not contribute to the central charge. Beyond the Cardy limit, we would need to either understand (2.33) as tractable perturbation sum or compute some other quantities less difficult but enough to give us a sound idea of what the gravitational Hilbert space is. An example of such a quantity is the ‘elliptic genus’.

Let us thus finally end this section by briefly discussing the notion of the elliptic genus (see for example, [13, 14] and references therein). Apart from the mass, let us also consider the presence of other conserved Noether charges $\{Q^j\}$ and their associated chemical potentials $\{z_j\}$. Then, instead of the partition function, consider the following trace over the Ramond sector (denoted as the subscript below) for the $(0, 4)$ CFT in case (b).

$$\chi(\tau, z_j) = \text{Tr}_R \left[e^{2\pi i\tau(L_0 - \frac{c}{24})} e^{-2\pi i\bar{\tau}(\bar{L}_0 - \frac{\bar{c}}{24})} e^{2\pi i z_j Q^j} (-1)^F \right] \quad (2.34)$$

where we have denoted F to be the R-charge. In the trace above, the inclusion of the $(-1)^F$, like the fermion number in the Witten index in supersymmetric gauge theories, implies Bose-Fermi cancellation among states except for states which obey $\bar{L}_0 = \frac{\bar{c}}{24}$ - the Ramond ground states. Meanwhile, all left-moving states contribute. The general strategy is to compute this quantity in the approximation that the CFT consists of free fields only, then extrapolate to strong coupling before comparing with the same quantity in the supergravity picture.

There are two symmetry properties of (2.34) that reveal the elliptic genus to be a weak Jacobi form of weight zero. The first is the modular property

$$\chi \left(\frac{a\tau + b}{c\tau + d}, \frac{z_j}{c\tau + d} \right) = e^{2\pi i \frac{cz^2}{c\tau + d}} \chi(\tau, z_j) \quad (2.35)$$

where $z^2 \equiv k^{ij} z_i z_j$, with the metric k^{ij} arising from the relation among the modes of the conserved currents J^i , namely, $[J_m^i, J_n^j] = \frac{m}{2} k^{ij} \delta_{m+n, 0}$. The second is the property

$$\chi(\tau, z_j + l_j \tau + m_j) = e^{-2\pi i(l^2 \tau + 2l^k z_k)} \chi(\tau, z_j) \quad (2.36)$$

It turns out that these two properties imply that there exists a transform, known as a ‘Farey-Tail’ transform, of the elliptic genus of which form is strongly suggestive of a gravitational interpretation that one is computing a sum over different geometries. The Farey-Tail expansion is as follows.

$$\tilde{\chi}(\tau, z_j) = \left(\frac{\partial_\tau}{2\pi i} + \frac{\partial_z^2}{16\pi^2} \right)^{3/2} \chi(\tau, z_j) \quad (2.37)$$

On the gravity side, the contributions to the Farey-tail transformed elliptic genus originate from (a) supergravity states from the KK spectrum in the higher-dimensions, which consist of the vector multiplets, hypermultiplets, gravitino and gravity multiplets in case (b) for instance (b) branes wrapping cycles of the internal space (M_6 in case (b) for instance). This computation is difficult to perform in the general case on both sides, but it is insightful because it can be written as a sum over global quotients of states which correspond to BPS states below the zero mass black hole.

Finally, let us mention a few salient points about AdS_3/CFT_2 from the viewpoint of string theory. A formal derivation of AdS_3/CFT_2 is usually given via strings propagating on $AdS_3 \times S^3 \times M_4$, where M_4 is T^4 or $K3$. The key point is that this geometry arises as the near horizon limit of the D1/D5 brane system. The dual CFT is then the field theory that captures the degrees of freedom of this brane system. In this set-up, the dual CFT arises from open string dynamics on the D1/D5 branes. The bulk is described by closed strings propagating on $AdS_3 \times S^3 \times M_4$. AdS_3/CFT_2 is then manifestly an open/closed string duality.

Let us mention briefly about some key points in this realization. For definiteness, we take M_4 to be T^4 in the following summary. The dual CFT that describes the bound states of N_1 D1 branes and N_5 D5 branes is a $(4, 4)$ sigma model with target space $(T^4)^{N_1 N_5} / S(N_1 N_5)$, where we have orbifolded the product of T^4 by the permutation group $S(N_1 N_5)$. This descends from a description of D1 branes as instantons on the D5 branes, with the degrees of freedom being captured by the instanton moduli space.¹¹ What is the structure of the underlying Hilbert space? It consists of twisted sectors labeled by conjugacy classes of the permutation group $S(N_1 N_5)$ that contains cyclic groups of various lengths, ie. denoting M_n to be the multiplicity of the cycle and n to its length, we have the defining relation $\sum_{n=0}^{N_1 N_5} n M_n = N_1 N_5$ (see for example, a nice discussion in [26].)

On the other hand, the type IIB closed string theory can be solved exactly as a $SL(2, \mathbb{R})$ WZW model if we add in the NS-NS B-field (which corresponds to the winding number term in the WZW), and setting all Ramond-Ramond fields to be zero.¹² It is also useful to note that the bulk theory flows in the low-energy limit to a supergravity theory which contains a sector that is described by Chern-Simons theory equipped with the super-Lie algebra $SU(1, 1|2) \times SU(1, 1|2)$. This fact was utilized to identify conical defects with vertex operators in the dual CFT. The higher temperature thermal partition function of the dual CFT was also computed to agree with that in the bulk. We can also understand the bulk spectrum as follows. BTZ black holes were shown to correspond to density matrices (or equivalently thermal states) of the dual CFT in the Ramond sector. The ground state AdS_3 belongs to the NS sector, and a spectral flow connects it to the massless BTZ.

¹¹There is an alternative description which gives a different dual theory. Open strings stretching between D1 and D5 branes give rise to the sigma model with target space $(T^4)^{N_1 N_5} / S(N_1) \times S(N_5)$ instead. There should however exist an isomorphism between these two theories.

¹²This thus constitutes a special point in the moduli space where in general, the RR fields are non-vanishing. This turns out to be a singular point, i.e. the dual CFT is singular, and in the bulk, this is manifest in the continuum of long strings.

2.4 Higher-spin holography: quantum Vasiliev gravity from coset CFTs

Let us now discuss some generalities concerning higher-spin holography in three dimensions, as first conjectured by Gaberdiel and Gopakumar. We note that since their seminal paper on the duality between the higher-spin Vasiliev theory and coset minimal models, there has been proposals to refine this conjecture. We will mention briefly about these refinements towards the end of this section.

As mentioned in the introduction, in Gaberdiel-Gopakumar conjecture, the boundary CFT can be represented as a diagonal coset of WZW models of the form

$$\frac{SU(N)_k \oplus SU(N)_1}{SU(N)_{k+1}}, \quad (2.38)$$

with the large- N limit corresponding to taking

$$k, N \rightarrow \infty, \quad \lambda = \frac{N}{k+N} \text{ being fixed} \quad (2.39)$$

where λ serves the role of a 't Hooft parameter. On the CFT side, it is known that the \mathcal{W}_N CFT is integrable, and in principle, correlation functions can be computed precisely for all N and k . It relies on the equivalence between the \mathcal{W}_N algebra and Vasiliev-type higher-spin algebra, and how the former dictates the representation theory of the minimal CFT (see [27] for related evidences). In the limit (2.39), the central charge goes as $c = N(1 - \lambda^2)$ and thus behaves like a vector model. Generally for AdS/CFT to work, one needs a large number of degrees of freedom in some large N limit to recover bulk gravity at least parametrically from the finite N region. The higher-spin holography passes this basic consistency check.

The bulk Vasiliev gravitational theory contains an infinite tower of higher-spin fields. To match with states in the CFT, one adds massive complex scalar fields which interact with the higher-spin fields in the bulk. When linearized, these scalar fields satisfy the Klein-Gordon equation on the background AdS_3 spacetime. The topological sector of the theory is $hs[\lambda]$ Chern-Simons theory. The mass of the complex massive scalar fields reads $M^2 = -1 + \lambda^2$, which give rise to two possible conformal weights $h_{\pm} = \frac{1}{2}(1 \pm \lambda)$. In the 't Hooft limit, these two values correspond to the conformal dimensions of two primary fields labelled by the primitive representations of the cosets (we will discuss this in greater detail in Chapter 5), which are labelled as $(\square; 0)$ and $(0; \square)$, where the first label refers to an affine weight of $SU(N)_k$ and the second an affine weight of $SU(N)_{k+1}$.

There is however a problem with the finite N limit. In [29], the three and four point functions in the minimal CFT were computed, and it was argued that there are several additional light states difficult to see in the bulk and which do not decouple (see also [30] for related issues). At the same time that this problem surfaced in the literature, it was realized that there are non-perturbative objects in the bulk which are generalizations of the conical defect spacetimes in ordinary gravity [31], that could be possible bulk candidates

to be identified with the boundary light states of the form $(\Lambda; \Lambda)$. These objects however are manifest in $SL(N, C)$ Chern-Simons theory, so it was argued that for finite N , one should perform an analytic continuation to $\lambda = -N$ to interpret the light states as non-perturbative objects. Only one perturbative scalar field $(\square; 0)$ should be kept, while $(0; \square)$ should be interpreted as a non-perturbative state after an analytic continuation $\lambda \rightarrow -N$. Thus, it was suggested that there is another limit which is likely to provide us insights into the workings of the duality, namely the limit

$$c \rightarrow \infty, \quad N \text{ finite} \tag{2.40}$$

Most recently in [32], a more refined analysis of the structure of null states in the CFT in this limit together with an analysis of the symmetries that govern the perturbative fluctuations about the conical defect spacetimes reveals that it seems more appropriate to interpret the bulk conical defect spacetimes as dual to the state $(0; \square)$, whereas the light states should be interpreted as bound states of the scalar field and the conical defect spacetimes.

The caveat is that taking the semi-classical limit actually brings us to the non-unitary regime of the coset CFTs, because the conformal weights become negative. For example, that of the state $(0; \square)$ goes like $-\frac{c}{2N^2}$. The corresponding bulk theory is also Euclidean if we wish to match the bulk objects and the CFT states. This leaves us with the crucial, open question of what is the correct analytic continuation to take to understand the whole setting in Lorentzian signature, and the analytic continuation from the semi-classical limit to the 't Hooft limit.

In the rest of the chapters, we will be presenting results which would appear to be directly relevant in each of the limits, depending on the topological sector of the theory. If the gauge group is $hs[\lambda]$, then the 't Hooft limit is easier to see, whereas the cases of $SL(N)$ and $SL(N|N-1)$ should be taken as analytically continuing $\lambda = N$, and then taking large c . In both cases, it is apparent that we are in the semi-classical regime.

CHAPTER 3

Adventures in Vasiliev Gravity I: probing spacetime with scalar fields

In this chapter, we will derive the bulk-boundary propagator for a scalar field propagating on the background of a higher-spin black hole. We expect it to serve as an important quantity in future developments of higher-spin holography. We begin with a brief survey of important points in Vasiliev gravity in Section 3.1 and demonstrate that the topological sector of this gravity theory can be described by Chern-Simons theory in Section 3.2. After reviewing the group-theoretic properties of $\text{hs}[\lambda]$ in Section 3.3 and the generalized Klein Gordan equation in Section 3.5, we present the derivation of the bulk-boundary propagator in Section 3.6.

3.1 Some aspects of the master field equations

In this section, we furnish a brief review of some essential aspects of Vasiliev gravity in three dimensions [16, 17]. We discuss its basic formulation and in particular, for the purpose of pedagogical clarity, consider the dynamics of scalar fields in the linearized limit.

This higher-spin gravity does not admit any known Lagrangian, and its self-consistent existence and formulation relies thinly (but firmly) upon a higher-spin gauge symmetry. As we shall see, it has a rather intricate structure. Let us first introduce the matter and gauge field contents of Vasiliev gravity. There exists one higher spin gauge field for each integer spin $s \geq 2$, coupled to matter fields in a manner which we will describe shortly. It turns out that the dynamics is rather complex, and it is more convenient to describe the ingredients of the theory via a set of three classes of ‘master fields’, namely

Spacetime one-form $W_\mu dx^\mu$

Spacetime zero-form B

Auxiliary twistor field S_α , $\alpha = 1, 2$.

We now summarize the physical meanings of each of these ‘master fields’ (see for example [33, 34, 35] for a nice introduction). The spacetime one-form W captures the gauge sector of the theory, and in particular it contains a pair of Chern-Simons gauge field A, \tilde{A} which

parametrizes the topological sector of the theory as a Chern-Simons theory equipped with the infinite-dimensional Lie algebra $hs[\lambda] \oplus hs[\lambda]$ where λ is related to the mass of the scalar matter. W can be interpreted as a generating function for the infinite tower of higher-spin gauge fields (note that the $sl(2)$ subalgebra corresponds to the spin-2 graviton). On the other hand, the spacetime zero-form B is the generating function for the matter fields, and as we shall explain later, it parametrizes the anti-de Sitter vacua of the theory. The auxiliary field S_α is invoked to ensure that the higher-spin theory has the required internal gauge symmetry.

To discuss these three master fields more precisely, we need to explain about the variables of which they are functions of. The fields depend on, apart from spacetime coordinates,

1. Auxiliary bosonic twistor variables z_α, y_α
2. Two pairs of Clifford elements $\psi_{1,2}, \{\kappa, \rho\}$ that satisfy the following algebra

$$\begin{aligned} \{\psi_i, \psi_j\} &= 2\delta_{ij}, \\ \{\kappa, \rho\} &= 0, \quad \kappa^2 = \rho^2 = \psi^2 = 1, \\ \{\kappa, y_\alpha\} &= \{\kappa, z_\alpha\} = [\rho, y_\alpha] = [\rho, z_\alpha] = 0. \end{aligned} \quad (3.1)$$

We remark that the twistor indices are raised and lowered by the rank two Levi-Civita symbol $\epsilon_{\alpha\beta}$ as follows: $z^\alpha = \epsilon^{\alpha\beta} z_\beta, z_\alpha = \epsilon_{\beta\alpha} z^\beta$. Functions of the twistors $\{z, y\}$ are multiplied by the Moyal star product

$$f(z, y) \star g(z, y) = \frac{1}{4\pi^2} \int d^2u \int d^v e^{iu_\beta v^\beta} f(z + u, y + u) g(z - v, y + v) \quad (3.2)$$

from which it is straightforward to show that the following Moyal star-commutation relations hold.

$$[y_\alpha, y_\beta]_\star = -[z_\alpha, z_\beta]_\star = 2i\epsilon_{\alpha\beta}, \quad [y_\alpha, z_\beta]_\star = 0. \quad (3.3)$$

Roughly speaking, the properties of the elements $\{z_\alpha, y_\alpha, \kappa, \rho\}$ realize the internal gauge symmetries and the master fields can be treated as a set of generating functions for the various fields of the theory. In the higher-spin literature, the fields are sometimes written in the form of an expansion in the variables $\{\kappa, \rho, \psi, z, y\}$ as follows.

$$F(z, y; \psi, \kappa, \rho|x) = \sum_{b,c,d,e} \sum_{m,n=0}^{\infty} \frac{1}{m!n!} F_{bcde}^{\alpha_1 \dots \alpha_m \beta_1 \dots \beta_n}(x) \kappa^b \rho^c \psi_1^d \psi_2^e z_{\alpha_1} \dots z_{\alpha_m} y_{\beta_1} \dots y_{\beta_n} \quad (3.4)$$

where F can be any of the master fields. Now, the full non-linear equations of motion of

Vasiliev gravity can be written as follows¹

$$dW = W \wedge \star W \quad (3.5)$$

$$dB = W \star B - B \star W \quad (3.6)$$

$$dS_\alpha = W \star S_\alpha - S_\alpha \star W \quad (3.7)$$

$$S_\alpha \star S^\alpha = -2i(1 + B \star \kappa e^{iz_\alpha y^\alpha}) \quad (3.8)$$

$$S_\alpha \star B = B \star S_\alpha \quad (3.9)$$

These highly non-linear equations of motion are invariant under the infinitesimal higher spin gauge transformations

$$\delta W = d\epsilon + [\epsilon, W]_\star \quad (3.10)$$

$$\delta B = [\epsilon, B]_\star \quad (3.11)$$

$$\delta S_\alpha = [\epsilon, S_\alpha]_\star \quad (3.12)$$

where ϵ is the infinitesimal gauge parameter independent of ρ . As mentioned earlier, W is the generating function for higher spin gauge fields, B is that for the matter fields, and S_α describes auxiliary degrees of freedom. There is a simplification of the equations of motion often alluded to in the modern literature. Note that (3.5) enjoy the symmetry

$$\rho \rightarrow -\rho, \quad S_\alpha \rightarrow -S_\alpha$$

upon which the system can be truncated to a simpler one in which W and B are independent of ρ , while S is linear in ρ .

3.2 On the vacuum/topological sector

As the most basic point in understanding the nonlinear Vasiliev equations, let us consider the vacuum solutions, for which we will simply add a superscript (0) onto the notations for the master fields. We wish to demonstrate below that there is a limit in which Vasiliev theory reduces to a direct sum of Chern-Simons theories equipped with some infinite-dimensional gauge algebra, describing a purely gravitational theory containing an infinite tower of higher-spin versions of the graviton. This limit is simply

$$B^{(0)} = \nu \Rightarrow \text{Chern-Simons with gauge algebra } \text{hs} \left[\frac{1}{2}(1 \pm \nu), \kappa = \mp 1 \right] \quad (3.13)$$

¹The factor $\kappa e^{iz_\alpha y^\alpha}$ that appears in one of the equations of motion is sometimes known as the Kleinian.

In the above limit, the field equations are reduced to

$$dW^{(0)} = W^{(0)} \star \wedge W^{(0)} \quad (3.14)$$

$$dS^{(0)} = [W^{(0)}, S_\alpha^{(0)}]_\star \quad (3.15)$$

$$S_\alpha^{(0)} S^{(0)\alpha} = -2i(1 + \nu\kappa e^{izy}) \quad (3.16)$$

To solve the above equations, it turns out to be convenient to define new twistor variables by using κ and the Kleinian $K \equiv \kappa e^{izy}$ as follows.

$$\tilde{z}_\alpha = z_\alpha + \nu w_\alpha \kappa, \quad \tilde{y}_\alpha = y_\alpha + \nu w_\alpha K, \quad (3.17)$$

$$w \equiv (z + y) \int_0^1 dt t e^{it(zy)} \quad (3.18)$$

These deformed \tilde{y}, \tilde{z} satisfy the commutation relations

$$[\tilde{y}_\alpha, \tilde{y}_\beta]_\star = 2i\epsilon_{\alpha\beta}(1 + \nu\kappa), \quad [\rho\tilde{z}_\alpha, \rho\tilde{z}_\beta]_\star = -2i\epsilon_{\alpha\beta}(1 + \nu K), \quad [\rho\tilde{z}_\alpha, \tilde{y}_\beta]_\star = 0 \quad (3.19)$$

from which it can be verified straightforwardly that the auxiliary fields $S^{(0)}$ satisfy the field equations if we write it as

$$S^{(0)} = \rho\tilde{z} \quad (3.20)$$

The last equation in (3.5) implies that $W^{(0)}$ should commute with $S^{(0)}$ which can be satisfied by taking $W^{(0)}$ to be independent of κ and \tilde{z} . Therefore $W^{(0)}$ depends only on the variables x, ψ, \tilde{y} , and we can expand it as (following the notation in (3.4))

$$W^{(0)}(\tilde{y}; \psi|x) = \sum_{d,e} \sum_{n=0}^{\infty} \frac{1}{n!} W_{00de}^{\beta_1 \dots \beta_n}(x) \psi_1^d \psi_2^e \tilde{y}_{\beta_1} \star \dots \star \tilde{y}_{\beta_n} \quad (3.21)$$

The relation to the Lie algebra then arises as follows. Under the Moyal star product, the auxiliary variables \tilde{y} generate the higher-spin algebra $\text{hs}[\lambda]$. A symmetric product $\tilde{y}_{\alpha_1} \star \dots \star \tilde{y}_{\alpha_n}$ corresponds to a generator of $\text{hs}[\lambda]$ with spin $n + 1$. Projecting onto $\kappa = \pm 1$ gives us the commutation relations of those of $\text{hs}[\lambda]$. The last equation of motion (3.14) to solve can be written as simply the flatness condition of Chern-Simons theory. To see this, we introduce $\text{hs}[\lambda] \oplus \text{hs}[\lambda]$ -valued gauge fields A and \bar{A} by writing

$$W^{(0)} = -\left(\frac{1 + \psi_1}{2}\right) A - \left(\frac{1 - \psi_1}{2}\right) \bar{A} \quad (3.22)$$

where A and \bar{A} are functions of x, \tilde{y} . Then (3.14) can be written as

$$dA + A \wedge \star A = 0, \quad d\bar{A} + \bar{A} \wedge \star \bar{A} = 0 \quad (3.23)$$

which is nothing but the flatness condition for a Chern-Simons theory equipped with a direct sum of two independent $\text{hs}[\lambda]$ algebras.

3.3 About the infinite-dimensional Lie algebra $\mathfrak{hs}[\lambda]$

In this section, we discuss some main points concerning the infinite-dimensional Lie algebra $\mathfrak{hs}[\lambda]$ (see for example, [27] for a nice exposition). This one-parameter family of higher spin Lie algebra has generators

$$V_n^s, \quad s \geq 2, \quad |n| < s. \quad (3.24)$$

of which $V_{0,\pm 1}^2$ form an $\mathfrak{sl}(2)$ subalgebra under which V_n^s has spin $s - 1$, i.e.

$$[V_m^2, V_n^s] = (-n + m(s - 1))V_{m+n}^s. \quad (3.25)$$

The bulk Chern-Simons gauge fields conjugate to the generator V_n^s will have spacetime spin s . The full commutation relations are typically written as

$$[V_m^s, V_n^t] = \sum_{u=2,4,\dots}^{s+t-1} g_u^{st}(m, n; \lambda) V_{m+n}^{s+t-u} \quad (3.26)$$

with the structure constants $g_u^{st}(m, n; \lambda)$ as

$$g_u^{st}(m, n; \lambda) = \frac{q^{u-2}}{2(u-1)!} \phi_u^{st}(\lambda) N_u^{st}(m, n) \quad (3.27)$$

where q is a normalization that can be scaled away by re-defining the generators but which we will take as $1/4$ for definiteness, and

$$\begin{aligned} N_u^{st}(m, n) &= \sum_{k=0}^{u-1} (-1)^k \binom{u-1}{k} [s-1+m]_{u-1-k} [s-1-m]_k [t-1+n]_k [t-1-n]_{u-1-k} \\ \phi_u^{st}(\lambda) &= \left[\begin{array}{c} \frac{1}{2} + \lambda, \frac{1}{2} - \lambda, \frac{2-u}{2}, \frac{1-u}{2} \\ \frac{3}{2} - s, \frac{3}{2} - t, \frac{1}{2} + s + t - u \end{array} \middle| 1 \right] \end{aligned} \quad (3.28)$$

where $[a]_n$ is the descending Pochhammer symbol defined as

$$[a]_n = a(a-1)\dots(a-n+1) \quad (3.29)$$

Before we move on to discuss a deeper way to understand $\mathfrak{hs}[\lambda]$, let us briefly discuss the relation to Moyal star product (3.2). The commutation relation (3.26) is defined with respect to a ‘lone-star’ product that reads

$$V_m^s \star V_n^t = \frac{1}{2} \sum_{u=1,2,\dots}^{s+t-|s-t|-1} g_u^{st}(m, n; \lambda) V_{m+n}^{s+t-u} \quad (3.30)$$

from which one recovers (3.26) via the relation

$$g_u^{st}(m, n; \lambda) = (-1)^{u+1} g_u^{ts}(n, m; \lambda) \quad (3.31)$$

The relevance to the Moyal star product (3.2) is that the lone-star product acting on $\text{hs}[\lambda]$ generators is isomorphic to the Moyal product involving deformed oscillators \tilde{y}, \tilde{z} with an identification between the generators and oscillator polynomials, i.e. there exists an isomorphism between the generator and polynomial bases. It turns out that this comes with the relation $\lambda = (1 - \nu\kappa)/2$. For example, one can identify the $\text{sl}(2)$ subalgebra spanned by the symmetric polynomials $S_{\alpha\beta} = \tilde{y}_{(\alpha}\tilde{y}_{\beta)}$, which can be shown to satisfy $\text{sl}(2)$ subalgebra of $\text{hs}[\lambda]$, spanned by the generators $V_0^2, V_{\pm 1}^2$. More precisely, the identification turns out to be simply

$$V_m^s = \left(\frac{-i}{4}\right)^{s-1} S_m^s \quad (3.32)$$

where S_m^s is the symmetrized product of $2s - 2$ oscillators, with m as half the difference between the number of \tilde{y}_1 and the number of \tilde{y}_2 .

There exists a deeper and at the same time more intuitive description of $\text{hs}[\lambda]$ to which we will allude to now. Consider the quotient of the universal enveloping algebra $U(\text{sl}(2))$ by the ideal generated by $(C_2 - \mu\mathbb{I})$, formally,

$$B[\mu] = \frac{U(\text{sl}(2))}{\langle C_2 - \mu\mathbb{I} \rangle} \quad (3.33)$$

where C_2 is the quadratic Casimir of $\text{sl}(2)$. Let us denote the generators of $\text{sl}(2)$ to be J_0, J_{\pm} satisfying $[J_+, J_-] = 2J_0$ and $[J_{\pm}, J_0] = \pm J_{\pm}$, then $C_2 \equiv J_0^2 - \frac{1}{2}(J_+J_- + J_-J_+)$. Since unitary representations of $\text{sl}(2)$ have $C_2 > -\frac{1}{4}$, C_2 is parametrized by

$$\mu = \frac{1}{4}(\lambda^2 - 1).$$

The Lie algebra $\text{hs}[\lambda]$ is then identified as a subspace of $B[\mu]$ of (3.33) which can be decomposed as $B[\mu] = \text{hs}[\lambda] \oplus \mathbb{C}$, where the vector corresponding to \mathbb{C} is the identity generator which can be formally identified with V_0^1 . We can neatly identify the modes $V_{0,\pm 1}^2$ in the $\text{sl}(2)$ subalgebra with $J_{0,\pm 1}$, while for $n \geq 2$, we write

$$V_n^s = (-1)^{s-1-n} \frac{(n+s-1)!}{(2s-2)!} \left[\underbrace{J_-, \dots, [J_-, [J_-, J_+^{s-1}]]}_{s-1-n \text{ terms}} \right] \quad (3.34)$$

Formally, $B[\mu]$ is an associative algebra whose product is the lone-star product. The Lie algebraic structure of $\text{hs}[\lambda] \oplus \mathbb{C}$ is then defined via the commutator. The identity generator V_0^1 is central. Moreover, if $\lambda = N$ with integer $N \geq 2$, then the following quadratic form degenerates

$$\text{tr}(V_m^s \star V_n^r) = 0 \quad \forall s > N \quad (3.35)$$

implying that an ideal φ_N emerges, consisting of all generators V_n^s , with $s > N$. Factoring out this ideal then truncates $\text{hs}[\lambda]$ algebra to the finite algebra $\text{sl}(N)$. We can write this

fact as

$$\mathfrak{sl}(N) = \mathfrak{hs}[N]/\varphi_N \quad (N \geq 2). \quad (3.36)$$

Finally, let us mention a few special cases of λ . For $\lambda = 1$, this algebra is the wedge subalgebra of a certain \mathcal{W}_∞ algebra considered by Pope, Romans and Shen in [36] (we will discuss in detail the notion of \mathcal{W} -algebras later), and represents the maximal coupling limit in the Gaberdiel-Gopakumar conjecture. For $\lambda = \frac{1}{2}$, this is sometimes known in the literature as $\mathfrak{hs}(1,1)$ or the ‘undeformed’ case. In fact, historically speaking, this was the first case considered for higher-spin gravity in AdS_3 . In the infinite λ limit, after re-scaling, the generators J_0, J_\pm can be considered as co-ordinates on a 2D hyperboloid, and $\mathfrak{hs}[\infty]$ is the area-preserving diffeomorphisms acting on the hyperboloid. It is also worthwhile to note that for all λ , the zero modes V_0^s commute since they are simply polynomials of J_0 after quotienting out the Casimir. Thus, there are infinitely many commuting charges.

3.4 On coupling to the matter fields

Having understood the topological sector, let us introduce propagating matter fields. We shall consider simpler scenario in which we want to understand fluctuations of the field B to linear order around a constant background $B^{(0)} = \nu$. Thus, let us write

$$B = \nu + \mathcal{C} \quad (3.37)$$

For W and S_α , we suppress the fluctuations. Substituting this into the fields equations, it can be shown that we obtain two equations for \mathcal{C} which read

$$\begin{aligned} d\mathcal{C} - W^{(0)} \star \mathcal{C} + \mathcal{C} \star W^{(0)} &= 0 \\ [S_\alpha^{(0)}, \mathcal{C}]_\star &= 0 \end{aligned} \quad (3.38)$$

Recalling that $S_\alpha^{(0)} = \rho \tilde{z}_\alpha$, it is easy to see that (3.38) can be satisfied if \mathcal{C} does not depend on \tilde{z}_α or κ , and hence is only a function of \tilde{y}, ψ, x . Vasiliev and Prokushkin further demonstrated that without loss of generality, one can write \mathcal{C} as

$$\mathcal{C} = C(\tilde{y}|x)\psi_2 + \tilde{C}(\tilde{y}|x)\psi_2 \quad (3.39)$$

where one can check that essentially, we are decomposing \mathcal{C} like in (3.40), since

$$C = \left(\frac{1 + \psi_1}{2}\right) \mathcal{C}, \quad \tilde{C} = \left(\frac{1 - \psi_1}{2}\right) \mathcal{C}. \quad (3.40)$$

Finally, the equations of motion imply that

$$dC + A \star C - C \star \bar{A} = 0, \quad d\tilde{C} + \bar{A} \star \tilde{C} - \tilde{C} \star A = 0 \quad (3.41)$$

which capture the dynamics of the matter interacting with an arbitrary higher spin background (up to linear order) that is specified by the gauge fields A, \bar{A} . The latter can be written in terms of a generalized vielbein e and generalized spin connection ω as follows

$$A = \omega + e, \quad \bar{A} = \omega - e \quad (3.42)$$

We will see explicit examples of these gauge fields later in the framework of $sl(N)$ and $sl(N|N-1)$ Chern-Simons theories.

3.5 Generalized Klein-Gordan equations

Before we end our review on the bulk Vasiliev gravity theory, let us briefly mention two very recent investigations of the field equations when we want to study linearized matter in [33, 34]. The first is the extraction of the correction scalar field equation in the background of some higher-spin fields, and the second is the derivation of the bulk-boundary propagator in the background of a higher-spin black hole. In our modern understanding, these have turned out to be useful starting steps in understanding coupling of matter of higher-spin gravity.

We begin by writing down the gauge fields that correspond to AdS_3 . Since we know that when the gauge algebra is $sl(2)$ and in the absence of all other scalar fields, the theory is ordinary gravity, the gauge connection is valued in just spin-2 generators. Working in the Euclidean signature in Fefferman-Graham gauge, the connection can be written as

$$A = e^\rho V_1^2 dz + V_0^2 d\rho, \quad \bar{A} = e^\rho V_{-1}^2 d\bar{z} - V_0^2 d\rho \quad (3.43)$$

This gives rise to the metric $g_{\mu\nu} \sim \text{Tr}(e_{(\mu} e_{\nu)})$

$$ds^2 = d\rho^2 + e^{2\rho} dz d\bar{z} \quad (3.44)$$

and vanishing higher-spin fields. Note that in the above, ρ refers to the radial co-ordinate, and at the boundary, we have a 2-sphere. (In the Lorentzian signature, the boundary is a cylinder, and the entire spacetime is a solid cylinder, as opposed to a solid ball in the Euclidean case.) Substituting this ansatz into (3.41), and decomposing along both spacetime and internal gauge algebraic space, and using the lone-star product, we find

$$\partial_\rho C_m^s + 2C_m^{s-1} + C_m^{s+1} g_3^{(s+1)2}(m, 0) = 0 \quad (3.45)$$

$$\partial C_m^s + e^\rho \left(C_{m-1}^{s-1} + \frac{1}{2} g_2^{2s}(1, m-1) C_{m-1}^s + \frac{1}{2} g_3^{2(s+1)}(1, m-1) C_{m-1}^{s+1} \right) = 0 \quad (3.46)$$

$$\bar{\partial} C_m^s - e^\rho \left(C_{m+1}^{s-1} - \frac{1}{2} g_2^{2s}(-1, m+1) C_{m+1}^s + \frac{1}{2} g_3^{2(s+1)}(-1, m+1) C_{m+1}^{s+1} \right) = 0 \quad (3.47)$$

where $|m| < s$ and $\partial = \partial_z, \bar{\partial} = \partial_{\bar{z}}$, and we have suppressed the λ -dependence of the

structure constants for notational simplicity.

The generalized Klein-Gordan equation can now be derived using the equations for the set $\{C_0^1, C_0^2, C_0^3, C_1^2\}$. These equations straightforwardly give us

$$(\partial_\rho^2 + 2\partial_\rho + 4e^{-2\rho}(\partial\bar{\partial} - \eta\partial^3) - (\lambda^2 - 1)) C_0^1 = 0 \quad (3.48)$$

which is the Klein-Gordan equation in AdS_3 with the scalar mass-squared $m^2 = \lambda^2 - 1$. Of course, the master field C contain C_0^1 and infinitely many mixed and higher derivatives of it. Using the same approach, we can then derive the scalar field equation for any higher-spin background. For example, consider a general chiral spin- s deformation of AdS_3 for which the connection reads as

$$\begin{aligned} A &= e^\rho V_1^2 dz - \mu e^{(s-1)\rho} V_{s-1}^s d\bar{z} + V_0^2 d\rho \\ \bar{A} &= e^\rho V_{-1}^2 d\bar{z} - V_0^2 d\rho \end{aligned} \quad (3.49)$$

One can show that the generalized Klein-Gordan equation now reads as

$$(\partial_\rho^2 + 2\partial_\rho + 4e^{-2\rho}(\partial\bar{\partial} - \mu(-\partial)^s) - (\lambda^2 - 1)) C_0^1 = 0 \quad (3.50)$$

3.6 Propagators in the background of a higher-spin black hole

In this section, we will derive the bulk-boundary scalar propagator for any general value of λ , in the background of a higher-spin black hole. Our analysis is inspired by the work in [33] which deals with the specific case of the scalar field under alternate quantization and the symmetry algebra being $hs[\lambda = \frac{1}{2}]$. In the following, we will present results for any generic λ and for both methods of quantizations of the scalar field.

We begin by discussing some generalities that concern the computation of the scalar field propagator. Writing Euclidean AdS_3 in the chart

$$ds^2 = d\rho^2 + e^{2\rho} dz d\bar{z}. \quad (3.51)$$

For $0 \leq \lambda \leq 1$, the scalar field admits standard and alternate quantizations which we will denote using the subscripts \pm in the expressions for the bulk-boundary propagator obeying the following boundary conditions at $\rho \rightarrow \infty$.

$$G_\pm(\rho, x; x') \sim [e^{-(1\mp\lambda)\rho} \delta^{(2)}(x - x') + \dots] + [e^{-(1\pm\lambda)\rho} \phi_1(x - x') + \dots] \quad (3.52)$$

Note that $\phi_1(x - x')$ refers to the scalar vacuum expectation value. In the above chart, the bulk-boundary propagator reads

$$G_\pm(\rho, x; x') = \pm \frac{\lambda}{\pi} \left(\frac{e^{-\rho}}{e^{-2\rho} + |z - z'|^2} \right)^{1\pm\lambda} \quad (3.53)$$

We wish to derive (3.53) in the language of $sl(2)$ Chern-Simons theory. First, let us write down the flat connections corresponding to the AdS_3 background, choosing a gauge such that²

$$\begin{aligned} A(\rho, z, \bar{z}) &= a_z(z, \bar{z})dz + a_{\bar{z}}(z, \bar{z})d\bar{z} + V_0^2 d\rho \\ &= b^{-1}ab + b^{-1}db \\ \bar{A}(\rho, z, \bar{z}) &= \bar{a}_z(z, \bar{z})dz + \bar{a}_{\bar{z}}(z, \bar{z})d\bar{z} - V_0^2 d\rho \\ &= b^{-1}\bar{a}b + bdb^{-1}, \end{aligned} \tag{3.54}$$

with $b = e^{\rho V_0^2}$. To compute a, \bar{a} , recall that the generalized vielbein e and spin connection ω are related to the metric via

$$A = \omega + e, \bar{A} = \omega - e, g_{\mu\nu} \sim \text{Tr}(e_\mu e_\nu)$$

which allows us to read off

$$a_{AdS} = V_1^2 dz, \quad \bar{a}_{AdS} = V_{-1}^2 d\bar{z}. \tag{3.55}$$

For the various higher-spin backgrounds that we shall discuss later, both a and \bar{a} are independent of z, \bar{z} . Then, to make it apparent that the connections are pure gauge locally, we can write

$$\Lambda = a_\mu x^\mu, \bar{\Lambda} = \bar{a}_\mu x^\mu, \quad g = e^\Lambda \star b, \bar{g} = e^{\bar{\Lambda}} \star b^{-1} \tag{3.56}$$

and thus

$$A = g^{-1} \star dg, \quad \bar{A} = \bar{g}^{-1} \star d\bar{g}. \tag{3.57}$$

It also turns out to be convenient to introduce the quantity

$$\Lambda_\rho = b^{-1} \star \Lambda \star b, \bar{\Lambda}_\rho = b \star \bar{\Lambda} \star b^{-1}. \tag{3.58}$$

It can be shown straightforwardly that $\Lambda_\rho, \bar{\Lambda}_\rho$ can be obtained from $\Lambda, \bar{\Lambda}$ by the replacements $V_m^s \rightarrow e^{m\rho} V_m^s$ and $V_m^s \rightarrow e^{-m\rho} V_m^s$ respectively. Via the fact that the connections are locally pure gauge, we can proceed to prescribe a way to derive the scalar propagator. We first solve the Klein Gordan equation in the gauge $A = \bar{A} = 0$ and then act with a gauge transformation to obtain the solution in AdS_3 .

3.6.1 Traces of master fields from an infinite-dimensional representation of $sl(2)$

In the following, we shall use \mathfrak{M}_\pm to denote the scalar master field C in this gauge. The AdS_3 master field reads

$$C_\pm = e^{-\Lambda_\rho} \star b^{-1} \star \mathfrak{M}_\pm \star b^{-1} \star e^{\bar{\Lambda}_\rho}. \tag{3.59}$$

²It is straightforward to see that flatness of (A, \bar{A}) implies the flatness of (a, \bar{a}) .

For some generic λ , taking \mathfrak{M}_\pm to be highest weight states of $\mathfrak{hs}[\lambda]$ leads to the appropriate bulk-boundary scalar propagators Φ_\pm with delta-function boundary conditions.

It is natural to ask if we can find a CFT explanation in the spirit of holography. We recall that the bulk scalar is dual to a scalar primary \mathcal{O} in a CFT with $\mathcal{W}_\infty[\lambda]$ symmetry which is the asymptotic symmetry algebra of $\mathfrak{hs}[\lambda]$ gravity in AdS_3 . This algebra contains an infinite set of conserved currents which denote as $J^{(s)}$, where $s = 2, 3, \dots$ refer to the spin. They admit a Laurent expansion of the form

$$J^{(s)}(z) = -\frac{1}{2\pi} \sum_{m \in \mathbb{Z}} \frac{J_m^{(s)}}{z^{s+m}}. \quad (3.60)$$

To relate it to $\mathfrak{hs}[\lambda]$, we note that the modes with $m < |s|$ are the wedge modes which generate $\mathfrak{hs}[\lambda]$, i.e. the modes $J_m^{(s)} = V_m^s$, $|m| < s$. The CFT primary state $|\mathcal{O}_\pm\rangle$ is a highest weight state and an eigenstate of the spin- s zero modes V_0^s , i.e. $V_0^s|\mathcal{O}_\pm\rangle = f_\pm(s)|\mathcal{O}_\pm\rangle$, with $f_\pm(s)$ some spin-dependent coefficients, and $V_{m>0}^s|\mathcal{O}_\pm\rangle = 0$. In the bulk theory, the gauge master fields \mathfrak{M}_\pm lead to the scalar propagators in AdS_3 with delta function boundary conditions are highest weight state of $\mathfrak{hs}[\lambda]$ and obeying $V_0^s \star \mathfrak{M}_\pm = f_\pm(s)\mathfrak{M}_\pm$, $V_{m<0}^s \star \mathfrak{M}_\pm = 0$.

The propagator can be computed in a generic background (A, \bar{A}) , since it is the trace of C_\pm . After algebra, from (3.59), we have

$$\Phi_\pm = e^{(1+\lambda)\rho} \text{Tr} \left(e^{-\Lambda_\rho} \star \mathfrak{M}_\pm \star e^{\bar{\Lambda}_\rho} \right) \quad (3.61)$$

For example, in the Poincare chart of AdS_3 , the expression (3.61) reads

$$\Phi_\pm^{(AdS_3)} = e^{(1+\lambda)\rho} \text{Tr} \left(e^{-ze^\rho V_1^2} \star \mathfrak{M} \star e^{\bar{z}e^\rho V_{-1}^2} \right) \quad (3.62)$$

Note that even in such a case, computing Φ involves an infinite sum. There exists an infinite-dimensional matrix representation of $\mathfrak{hs}[\lambda]$ which is induced from an infinite-dimensional representation of $\mathfrak{sl}(2)$ via the universal enveloping algebra construction. This tends to simplify the actual computation of this sum. For the case of positive quantization, this reads³

$$\begin{aligned} (J_0)_{mm} &= \frac{1-\lambda}{2} - m \\ (J_+)_{m+1,m} &= -\sqrt{(-m-\lambda)m} \\ (J_-)_{m,m+1} &= \sqrt{(-m-\lambda)m} \end{aligned} \quad (3.63)$$

where $m = 1, 2, \dots, \infty$. One should regard this as a representation of $\mathfrak{sl}(2)$ for arbitrary spin, from which one could construct the full $\mathfrak{hs}[\lambda]$ algebra. We can also truncate this representation to the fundamental of $\mathfrak{sl}(N)$ by setting $\lambda = -N$ and restricting to the upper

³Henceforth, we write the modes $V_{0,\pm 1}^2$ in the $\mathfrak{sl}(2)$ subalgebra as $J_{0,\pm 1}$.

left $N \times N$ block. To compute the scalar propagator, one needs an explicit realization of \mathfrak{M} , which reads

$$\mathfrak{M}_{ij} = \delta_{i1}\delta_{j1} \quad (3.64)$$

In the $\mathfrak{sl}(2)$ algebra, we observe that

$$J_0\mathfrak{M} = -\left(\frac{1+\lambda}{2}\right)\mathfrak{M} = \mathfrak{M}J_0, \quad J_-\mathfrak{M} = \mathfrak{M}J_+ = 0 \quad (3.65)$$

with similar equations for all other higher spin generators, following the enveloping algebra construction. One can obtain a similar representation for the alternative quantization by switching $\lambda \rightarrow -\lambda$ above. Substituting (3.63)-(3.63) into (3.61), one obtains

$$\Phi_+ = e^{(1+\lambda)\rho} \left[e^{\bar{\Lambda}_\rho} e^{-\Lambda_\rho} \right]_{11} \quad (3.66)$$

Thus, the computation of the propagator is reduced to computing a single matrix element.

3.6.2 Chiral higher spin deformations of AdS_3

As a warm-up to computing the scalar propagator in the background of a higher-spin black hole, we shall first understand the case of a chiral spin-3 perturbation of the AdS_3 vacuum. The connections read

$$a = J_+ dz - \mu V_2^3 d\bar{z}, \quad \bar{a} = J_- d\bar{z} \quad (3.67)$$

where μ is constant. From the viewpoint of holography, this is the dual to a deformation of the boundary by a dimension (3,0) operator $\mathcal{W}(z)$ with constant coupling, schematically, $\delta S_{CFT} = \mu \int d^2z \mathcal{W}(z)$. The higher-spin black hole solution with chiral charge is a finite temperature version of this solution. To compute the propagator in this background, we first calculate

$$\Lambda_\rho = e^\rho z J_+ - \mu e^{2\rho} \bar{z} J_+^2, \quad \bar{\Lambda}_\rho = e^\rho \bar{z}. \quad (3.68)$$

The propagator can then be computed straightforwardly as

$$\Phi_+ = e^{(1+\lambda)\rho} \left(e^{e^\rho \bar{z} J_-} e^{-e^\rho z J_+ + \mu e^{2\rho} \bar{z} J_+^2} \right)_{11} \quad (3.69)$$

To proceed, we derive a list of identities to aid us in the computation. Firstly, using $[J_\pm, J_0] = \pm J_\pm$, $[J_+, J_-] = 2J_0$, it is straightforward to derive the following ‘braiding’ identities:

$$\begin{aligned} e^{c_+ J_+ + c_- J_-} J_0 e^{-c_+ J_+ - c_- J_-} &= \cos(2\sqrt{c_+ c_-}) J_0 + \frac{\sin(2\sqrt{c_+ c_-})}{2\sqrt{c_+ c_-}} (c_+ J_+ - c_- J_-), \quad (3.70) \\ e^{c_+ J_+ + c_- J_-} J_+ e^{-c_+ J_+ - c_- J_-} &= J_+ - \frac{c_-}{\sqrt{c_+ c_-}} \sin(2\sqrt{c_+ c_-}) J_0 - \frac{1 - \cos(2\sqrt{c_+ c_-})}{2c_+} (c_+ J_+ - c_- J_-), \end{aligned}$$

$$e^{c_+J_++c_-J_-} J_- e^{-c_+J_+-c_-J_-} = J_- + \frac{c_+}{\sqrt{c_+c_-}} \sin(2\sqrt{c_+c_-}) J_0 + \frac{1 - \cos(2\sqrt{c_+c_-})}{2c_-} (c_+J_+ - c_-J_-) \quad (3.71)$$

$$(3.72)$$

We also need

$$[e^{\Lambda_-J_-} e^{-\Lambda_+J_+} J_+^n]_{11} = \frac{\Gamma(\lambda + n + 1)}{\Gamma(\lambda + 1)} \Lambda_-^n (1 + \Lambda_- \Lambda_+)^{-(\lambda+n+1)} \quad (3.73)$$

To see (3.73), consider for example $n = 2$.

$$\begin{aligned} [e^{\Lambda_-J_-} e^{-\Lambda_+J_+} J_+^2]_{11} &= \sum_{q, \tilde{q}} \frac{(-\Lambda_+)^{\tilde{q}} \Lambda_-^q}{q! \tilde{q}!} (J_-^q J_+^{\tilde{q}} J_+^2)_{11} \\ &= \sum_{q, \tilde{q}} \frac{(-\Lambda_+)^{\tilde{q}} \Lambda_-^q}{q! \tilde{q}!} \sqrt{2(\lambda+2)(\lambda+1)} (-J_-^q J_+^{\tilde{q}})_{13} \end{aligned} \quad (3.74)$$

Since

$$\begin{aligned} \frac{(J_-^q)_{11}}{q!} &= (i^q) \delta_{l, 1+q} \sqrt{\frac{(\lambda+q)!}{q! \lambda!}} \\ \frac{(J_+^{\tilde{q}})_{13}}{\tilde{q}!} &= (-i)^{\tilde{q}} \delta_{l-\tilde{q}, 3} \sqrt{\frac{(\lambda+l-1)!}{\tilde{q}! (\lambda+l-1-\tilde{q})!}} \sqrt{\frac{(l-1)!}{\tilde{q}! (l-1-\tilde{q})!}}, \end{aligned} \quad (3.75)$$

simplifying (3.74) then gives us

$$\begin{aligned} [e^{\Lambda_-J_-} e^{-\Lambda_+J_+} J_+^2]_{11} &= \sum_{q=2}^{\infty} (-1)^q \Lambda_-^q \Lambda_+^{q-2} (\lambda+2)(\lambda+1) \frac{(\lambda+q)!}{(q-2)! (\lambda+2)!} \\ &= (\lambda+2)(\lambda+1) \Lambda_-^2 (1 + \Lambda_- \Lambda_+)^{-(\lambda+3)} \end{aligned} \quad (3.76)$$

The proof works similarly for all other values of any positive integral n . Similarly, it is straightforward to derive

$$[J_-^n e^{\Lambda_-J_-} e^{-\Lambda_+J_+}]_{11} = \frac{\Gamma(\lambda + n + 1)}{\Gamma(\lambda + 1)} (-\Lambda_+)^n (1 + \Lambda_- \Lambda_+)^{-(\lambda+n+1)} \quad (3.77)$$

Returning to the computation of the (3.69), after some algebra, we thus have

$$\begin{aligned} \Phi_+ &= e^{(1+\lambda)\rho} \left(\sum_{s=0}^{\infty} e^{e^{\rho} \bar{z} J_-} e^{-e^{\rho} z J_+} \frac{(\mu e^{2\rho} \bar{z} J_+^2)^s}{s!} \right)_{11} \\ &= e^{(1+\lambda)\rho} \sum_0^{\infty} \frac{(\mu \bar{z}^3 e^{4\rho})^s}{s!} \frac{(\lambda+2s)!}{\lambda! (1 + e^{2\rho} \bar{z} z)^{1+\lambda+2s}} \end{aligned} \quad (3.78)$$

A similar derivation holds for the case of alternative quantization, and together with this case, we can state the result for the chiral spin-3 background's propagator as

$$\begin{aligned}\Phi_{\pm} &= \left(\frac{e^{\rho}}{1 + e^{2\rho}|z|^2} \right)^{1\pm\lambda} \sum_{n=0}^{\infty} c_{n,\pm} \left(\frac{\mu \bar{z}^3 e^{4\rho}}{(1 + |z|^2 e^{2\rho})^2} \right)^n, \\ c_{n,\pm} &\equiv \prod_{i=1}^n (i \pm \lambda)\end{aligned}\quad (3.79)$$

3.6.3 Scalar propagator in the BTZ background and physical interpretations

Before we compute the propagator in the background of a higher-spin black hole, it is instructive to use the above procedure to compute the propagator in the background of the BTZ black hole. The Chern-Simons connections in this case read

$$a = \left(J_+ + \frac{1}{4\tau^2} J_- \right) dz, \bar{a} = \left(J_- + \frac{1}{4\bar{\tau}^2} J_+ \right) d\bar{z}, \quad (3.80)$$

This gives rise to the BTZ metric in Euclidean signature

$$ds^2 = d\rho^2 + \frac{2\pi}{k} (\mathcal{L} dz^2 + \bar{\mathcal{L}} d\bar{z}^2) + \left(e^{2\rho} + \left(\frac{2\pi}{k} \right)^2 \mathcal{L} \bar{\mathcal{L}} e^{-2\rho} \right) dz d\bar{z} \quad (3.81)$$

where $\mathcal{L} = -k/(8\pi\tau^2)$, $\bar{\mathcal{L}} = -k/(8\pi\bar{\tau}^2)$ are the left and right moving components of the boundary stress-momentum tensor. Note that τ is the modular parameter of the Euclidean boundary torus and that Euclidean BTZ has the topology of a solid torus. The black hole horizon is located at $\rho_{\text{hori.}} = -\log(4\tau\bar{\tau})/2$. The scalar bulk-boundary propagator is known to be

$$G_{\pm}(\rho, x) = \left(\frac{e^{-\rho}}{e^{-2\rho} \cos(\frac{z}{2\tau}) \cos(\frac{\bar{z}}{2\bar{\tau}}) + 4\tau\bar{\tau} \sin(\frac{z}{2\tau}) \sin(\frac{\bar{z}}{2\bar{\tau}})} \right)^{1\pm\lambda} \quad (3.82)$$

Below, we will re-derive (3.82) as a matrix element. This will serve as a good consistency check of our computational approach. First, we need the following BCH theorem for $SU(2)$

$$e^{(\lambda_+ J_+ - \lambda_- J_-)} = e^{\Lambda_+ J_+} e^{-\ln \Lambda_3 J_0} e^{-\Lambda_- J_-} \quad (3.83)$$

where

$$\Lambda_3 = \text{sech}^2 \sqrt{\lambda_+ \lambda_-}, \quad \Lambda_{\pm} = \frac{\lambda_{\pm}}{\sqrt{\lambda_+ \lambda_-}} \tanh \sqrt{\lambda_+ \lambda_-} \quad (3.84)$$

For the BTZ, the connections imply that we need to compute the matrix element

$$\Phi_{\pm}^{BTZ} = e^{\rho(1\pm\lambda)} \left[e^{a_+ J_+ - a_- J_-} e^{\tilde{a}_+ J_+ - \tilde{a}_- J_-} \right]_{11} \quad (3.85)$$

where

$$a_- = -e^\rho \bar{z}, \quad a_+ = \frac{e^{-\rho} \bar{z}}{4\bar{\tau}^2}, \quad \tilde{a}_+ = -e^\rho z, \quad \tilde{a}_- = \frac{e^{-\rho} z}{4\tau^2}. \quad (3.86)$$

From (3.83), and the relation $(J_0)_{11} = -\frac{1+\lambda}{2}$, we find (3.85) to simplify to

$$\Phi_\pm^{BTZ} = \Upsilon_3^{(1\pm\lambda)/2} \tilde{\Upsilon}_3^{(1\pm\lambda)/2} (1 + \Upsilon_- \tilde{\Upsilon}_+)^{-(1\pm\lambda)}, \quad (3.87)$$

$$\Upsilon_3 = \operatorname{sech}^2 \alpha, \quad \tilde{\Upsilon}_3 = \operatorname{sech}^2 \tilde{\alpha}, \quad \Upsilon_- = \frac{a_-}{\alpha} \tanh \alpha, \quad \tilde{\Upsilon}_+ = \frac{\tilde{a}_+}{\tilde{\alpha}} \tanh \tilde{\alpha}, \quad \alpha = i \frac{\bar{z}}{2\bar{\tau}}, \quad \tilde{\alpha} = i \frac{z}{2\tau} \quad (3.88)$$

After some simplification, one can straightforwardly verify that indeed, (3.87) is the scalar propagator (3.82). Before we extend this result to higher-spin black holes, let us briefly comment on the physical implications of the scalar propagator. Now, the Lorentzian BTZ has a Kruskal extension with two asymptotically AdS_3 regions at $\rho \rightarrow \pm\infty$. In the spirit of holography, the BTZ solution can be interpreted as being dual to an entangled state in a tensor product of two CFTs. In calculating the two-point function of scalar fields, we can take the two operators to be associated with the same CFT ($\rho \rightarrow \infty$) or each of the two ($\rho \rightarrow -\infty$). If both scalar operators are defined within the Hilbert space associated with the boundary at $\rho \rightarrow \infty$, then the two-point function is extracted from the large ρ limit of the propagator, from (3.82) and goes like $(\sin(z/2\tau) \sin(\bar{z}/2\bar{\tau}))^{-(1\pm\lambda)}$. On the other hand, in the $\rho \rightarrow -\infty$ limit, the mixed correlator propagator goes as $(\cos(z/2\tau) \cos(\bar{z}/2\bar{\tau}))^{-(1\pm\lambda)}$. Note that since in the Lorentzian signature we take $z = \phi + t, \bar{z} = \phi - t$, the single sided two point function is singular on the light cone $\phi = \pm t$ where the mixed correlator is not. This reflects the causal disconnection between the two asymptotic boundaries of the BTZ black hole solution.

3.6.4 A derivation of the scalar propagator on a higher-spin black hole

The connection for the higher-spin black hole with spin-3 charge reads

$$\begin{aligned} a_z &= J_+ - \frac{2\pi\mathcal{L}}{k} J_- - \frac{\pi\mathcal{W}}{2k} J_+^2 \\ a_{\bar{z}} &= -\mu(a_z \star a_z - \text{trace}). \end{aligned} \quad (3.89)$$

and similarly for the anti-holomorphic connection,

$$\begin{aligned} \bar{a}_z &= J_- - \frac{2\pi\bar{\mathcal{L}}}{k} J_+ - \frac{\pi\bar{\mathcal{W}}}{2k} J_+^2 \\ \bar{a}_{\bar{z}} &= -\bar{\mu}(\bar{a}_{\bar{z}} \star \bar{a}_{\bar{z}} - \text{trace}). \end{aligned} \quad (3.90)$$

Defining the spin-3 chemical potential $\alpha = \mu\bar{\tau}$, $\bar{\alpha} = \bar{\mu}\tau$, the charges, up to first order in α read

$$\mathcal{L} = -\frac{k}{8\pi\tau^2} + \mathcal{O}(\alpha^2), \bar{\mathcal{L}} = -\frac{k}{8\pi\bar{\tau}^2} + \mathcal{O}(\bar{\alpha}^2), \mathcal{W} = -\frac{k}{3\pi\tau^5}\alpha + \mathcal{O}(\alpha^3), \bar{\mathcal{W}} = -\frac{k}{3\pi\bar{\tau}^5}\bar{\alpha} + \mathcal{O}(\bar{\alpha}^3), \quad (3.91)$$

To proceed, we have, from (3.58) and up to first order in α ,

$$\begin{aligned} \Lambda_\rho &= e^\rho z J_+ + \frac{e^{-\rho} z}{4\tau^2} J_- + \alpha \left(\left(\frac{e^{-2\rho} z}{6\tau^5} - \frac{e^{-2\rho} \bar{z}}{16\tau^4 \bar{\tau}} \right) J_-^2 - \frac{e^{2\rho} \bar{z}}{\bar{\tau}} J_+^2 - \frac{\bar{z}}{2\bar{\tau}\tau^2} \left(\frac{1}{3} J_0^2 + \frac{1}{6} \{J_+, J_-\} \right) \right) + \dots \\ \bar{\Lambda}_\rho &= e^\rho \bar{z} J_- + \frac{e^{-\rho} \bar{z}}{4\bar{\tau}^2} J_+ + \bar{\alpha} \left(\left(\frac{e^{-2\rho} \bar{z}}{6\bar{\tau}^5} - \frac{e^{-2\rho} z}{16\bar{\tau}^4 \tau} \right) J_+^2 - \frac{e^{2\rho} z}{\tau} J_-^2 - \frac{z}{2\tau\bar{\tau}^2} \left(\frac{1}{3} J_0^2 + \frac{1}{6} \{J_+, J_-\} \right) \right) + \dots \end{aligned} \quad (3.92)$$

where we have taken the trace of $\{J_+, J_-\}$ to be $-\frac{1}{3}(\lambda^2 - 1)$, and the relation $\{J_+, J_-\} = 2J_0^2 - \frac{1}{2}(\lambda^2 - 1)$. It is also important to note that this turns out also to be the unique combination of J_0^2 and $\{J_+, J_-\}$ that will preserve periodicity in the eventual expression we have for the scalar propagator.

In the limit of vanishing $\alpha, \bar{\alpha}$, we recover the BTZ black hole, and thus we regard the propagator Φ_\pm as a perturbation sum around the BTZ result $\Phi_\pm^{BTZ} \equiv \Phi_\pm^{(0)}$. Thus,

$$\Phi_\pm = \Phi_\pm^{(0)} + \sum_{n=1}^{\infty} \Phi_\pm^{(n)}, \quad \Lambda_\rho = \Lambda_\rho^{(0)} + \sum_{n=1}^{\infty} \alpha^n \Lambda_\rho^{(n)}, \quad \bar{\Lambda}_\rho = \bar{\Lambda}_\rho^{(0)} + \sum_{n=1}^{\infty} \bar{\alpha}^n \bar{\Lambda}_\rho^{(n)}, \quad (3.93)$$

In the following, we will focus on computing $\Phi_\pm^{(1)}$, by expanding about $\alpha = 0$, and for simplicity, setting $\bar{\alpha} = 0$. We begin with

$$\Phi_\pm^{(1)} = e^{(1\pm\lambda)\rho} \int_0^1 ds \operatorname{Tr} \left(\alpha \left(e^{-s\Lambda_\rho^{(0)}} \star (-\Lambda_\rho^{(1)}) \star e^{-(1-s)\Lambda_\rho^{(0)}} \star \mathfrak{M}_\pm \star e^{\bar{\Lambda}_\rho^{(0)}} \right) \right) \quad (3.94)$$

Using the infinite-dimensional matrix realization of $\mathfrak{sl}(2)$ discussed earlier, we can write (3.94) as

$$\Phi_\pm^{(1)} = -\alpha e^{(1\pm\lambda)\rho} \left[\int_0^1 ds e^{\bar{\Lambda}_\rho^{(0)}} e^{-s\Lambda_\rho^{(0)}} \Lambda_\rho^{(1)} e^{s\Lambda_\rho^{(0)}} e^{-\Lambda_\rho^{(0)}} \right]_{11} \quad (3.95)$$

Introducing the notations

$$Z \equiv \frac{z}{2\tau}, \quad \bar{Z} \equiv \frac{\bar{z}}{2\bar{\tau}}, \quad T \equiv \tau e^\rho, \quad \bar{T} \equiv \bar{\tau} e^\rho,$$

and after invoking the braiding formulae (3.71), we obtain for each term that appears in $\Lambda_\rho^{(1)}$,

$$e^{-s\Lambda_\rho^{(0)}} J_+^2 e^{s\Lambda_\rho^{(0)}} = (\cos^4(sZ)) J_+^2 \left(\frac{\sin^2(2sZ)}{4T^2} \right) J_0^2 + \left(\frac{\sin^4(sZ)}{16T^4} \right) J_-^2$$

$$\begin{aligned}
& + \left(\frac{\cos^2(sZ) \sin(2sZ)}{2T} \right) \{J_+, J_0\} + \left(\frac{\cos^2(sZ) \sin^2(sZ)}{4T^2} \right) \{J_+, J_-\} \\
& + \left(\frac{\sin(2sZ) \sin^2(sZ)}{8T^3} \right) \{J_0, J_-\}
\end{aligned} \tag{3.96}$$

$$\begin{aligned}
e^{-s\Lambda_\rho^{(0)}} J_-^2 e^{s\Lambda_\rho^{(0)}} & = (\cos^4(sZ)) J_+^2 (4T^2 \sin^2(2sZ)) J_0^2 + (16T^4 \sin^4(sZ)) J_-^2 \\
& - (2T \cos^2(sZ) \sin(2sZ)) \{J_-, J_0\} + (4T^2 \cos^2(sZ) \sin^2(sZ)) \{J_+, J_-\} \\
& - (8T^3 \sin(2sZ) \sin^2(sZ)) \{J_0, J_+\}
\end{aligned} \tag{3.97}$$

$$\begin{aligned}
e^{-s\Lambda_\rho^{(0)}} J_0^2 e^{s\Lambda_\rho^{(0)}} & = (\cos^2(2sZ)) J_0^2 (T^2 \sin^2(2sZ)) J_+^2 + \left(\frac{1}{16T^2} \sin^2(2sZ) \right) J_-^2 \\
& - (T \cos(2sZ) \sin(2sZ)) \{J_+, J_0\} - \left(\frac{\sin^2(2sZ)}{4} \right) \{J_+, J_-\} \\
& + \left(\frac{\sin(2sZ) \cos(2sZ)}{4T} \right) \{J_0, J_-\}
\end{aligned} \tag{3.98}$$

$$\begin{aligned}
e^{-s\Lambda_\rho^{(0)}} \frac{1}{2} \{J_+, J_-\} e^{s\Lambda_\rho^{(0)}} & = (T^2 \sin^2(2sZ)) J_+^2 - (\sin^2(2sZ)) J_0^2 + \left(\frac{\sin^2(2sZ)}{16T^2} \right) J_-^2 \\
& - \left(\frac{T \sin(4sZ)}{2} \right) \{J_+, J_0\} + \left(\frac{\cos^4(sZ) + \sin^4(sZ)}{2} \right) \{J_+, J_-\} \\
& + \left(\frac{\sin(4sZ)}{8T} \right) \{J_0, J_-\}
\end{aligned} \tag{3.99}$$

Then, integrating over s , and summing up the coefficients for each term, we obtain

$$\int_0^1 ds e^{-s\Lambda_\rho^{(0)}} \Lambda_\rho^{(1)} e^{s\Lambda_\rho^{(0)}} = \left(F_{++} J_+^2 + F_{00} J_0^2 + F_{--} J_-^2 + F_{+0} \{J_+, J_0\} + F_{+-} \{J_+, J_-\} + F_{0-} \{J_0, J_-\} \right) \tag{3.100}$$

where

$$\begin{aligned}
F_{++} & = \left(-\frac{\bar{Z}}{8} + \frac{Z}{3} \right) \left(\frac{12Z - 8 \sin(2Z) + \sin(4Z)}{2Z} \right) - \frac{\bar{Z}(12Z + 8 \sin(2Z) + \sin(4Z))}{16Z} \\
& \quad - \bar{Z} \left(\frac{1}{2} - \frac{\sin(4Z)}{8Z} \right),
\end{aligned} \tag{3.101}$$

$$F_{+-} = \frac{\bar{Z}(-4Z + \sin(4Z))}{64T^2 Z} - \frac{\left(-\frac{\bar{Z}}{8} + \frac{Z}{3} \right) (-4Z + \sin(4Z))}{8T^2 Z} - \frac{\bar{Z} \left(-\frac{1}{2} + \frac{\sin(4Z)}{8Z} \right)}{4T^2} \tag{3.102}$$

$$F_{0-} = \frac{\left(-\frac{\bar{Z}}{8} + \frac{Z}{3} \right) (-1 + \cos^4(Z))}{T^3 Z} - \frac{\bar{Z} \cos^2(Z) \sin^2(Z)}{4T^3 Z} - \frac{\bar{Z} \sin^4(Z)}{8T^3 Z} \tag{3.103}$$

$$F_{+0} = \frac{4 \left(\frac{\bar{Z}}{8} - \frac{Z}{3} \right) \sin^4(Z)}{TZ} + \frac{\bar{Z} \cos^2(Z) \sin^2(Z)}{TZ} - \frac{\bar{Z} (-1 + \cos^4(Z))}{2TZ} \quad (3.104)$$

and $F_{--} = \frac{F_{++}}{16T^4}$, $F_{00} = 4F_{+-}$. Note that at this stage, we have computed (3.95) to read as

$$\begin{aligned} \Phi_{\pm}^{(1)} = & -\alpha e^{(1\pm\lambda)\rho} \left[e^{\bar{\Lambda}_{\rho}^{(0)}} e^{-\Lambda_{\rho}^{(0)}} e^{\Lambda_{\rho}^{(0)}} \left(F_{++} J_+^2 + F_{00} J_0^2 + F_{--} J_-^2 + F_{+0} \{J_+, J_0\} \right. \right. \\ & \left. \left. + F_{+-} \{J_+, J_-\} + F_{0-} \{J_0, J_-\} \right) e^{-\Lambda_{\rho}^{(0)}} \right]_{11} \end{aligned} \quad (3.105)$$

We can now invoke the braiding formulae again to simplify (3.105) to be

$$\Phi_{\pm}^{(1)} = -\alpha e^{(1\pm\lambda)\rho} \left[e^{\bar{\Lambda}_{\rho}^{(0)}} e^{-\Lambda_{\rho}^{(0)}} \left(\mathfrak{F}_{++} J_+^2 + \mathfrak{F}_{00} J_0^2 + \mathfrak{F}_{+0} \{J_+, J_0\} + \mathfrak{F}_{+-} \{J_+, J_-\} \right) \right]_{11} \quad (3.106)$$

where we have discarded the terms in J_-^2 and $\{J_-, J_0\}$ since these vanish upon computing the matrix element, and

$$\begin{aligned} \mathfrak{F}_{++} &= F_{++} \cos^4(Z) + F_{00} T^2 \sin^2(2Z) + 2F_{+-} T^2 \sin^2(2Z) + 16F_{--} T^4 \sin^4(Z) \\ &\quad + 2F_{+0} T \sin(2Z) \cos^2(Z) + 8F_{0-} T^3 \sin(2Z) \sin^2(Z) \\ &= \frac{1}{6} (12(Z - \bar{Z}) - 8 \sin(2Z) + \sin(4Z)) \\ \mathfrak{F}_{+0} &= F_{++} \frac{\sin^2(2Z)}{4T^2} + F_{00} \cos^2(2Z) - 2F_{+-} \sin^2(2Z) + 4F_{--} T^2 \sin^2(2Z) \\ &\quad - F_{+0} \frac{\cos(2Z) \sin(2Z)}{T} + 2F_{0-} T \sin(4Z) \\ &= -\frac{1}{6T^2} (4(\bar{Z} - Z) + \sin(4Z)) \\ \mathfrak{F}_{+-} &= F_{++} \frac{\cos^2(Z) \sin^2(Z)}{4T^2} + \frac{1}{4} F_{00} \sin^2(2Z) + F_{+-} (\cos^4(Z) + \sin^4(Z)) + F_{--} T^2 \sin^2(2Z) \\ &\quad - F_{+0} \frac{\sin(2Z) \cos(2Z)}{4T} + 2F_{0-} T \sin(4Z) \\ &= \frac{1}{24T^2} (\cos(4Z) - 2) (4(\bar{Z} - Z) + \sin(4Z)) \\ \mathfrak{F}_{00} &= -F_{++} \left(\frac{1}{2T} \sin(2Z) \cos^2(Z) \right) + F_{00} (T \sin(2Z) \cos(2Z)) + F_{+-} (T \sin(4Z)) \\ &\quad + 8F_{--} T^3 \sin^2(Z) \sin(2Z) + F_{+0} \left(\cos(2Z) \cos^2(Z) - \frac{1}{2} \sin^2(2Z) \right) \\ &\quad + F_{0-} (2T^2 \sin^2(2Z) + 4T^2 \sin^2(Z) \cos(2Z)) \\ &= \frac{4 \sin^4(Z)}{3T} \end{aligned} \quad (3.107)$$

Further, one can use (3.83) to simplify (3.106) to read

$$\begin{aligned} \Phi_{\pm}^{(1)} &= -\alpha e^{(1\pm\lambda)\rho} e^{-\log(\sec^2(\bar{Z}))(J_0)_{11}} \left[e^{(2\bar{T}\tan(\bar{Z}))J_-} e^{(-2T\tan(Z))J_+} e^{-\log(\sec^2(Z))(J_0)} e^{-\left(\frac{\tan(Z)}{2T}\right)J_-} \right. \\ &\quad \left. \times \left(\mathfrak{F}_{++}J_+^2 + \mathfrak{F}_{00}J_0^2 + \mathfrak{F}_{+0}\{J_+, J_0\} + \mathfrak{F}_{+-}\{J_+, J_-\} \right) e^{\frac{\tan(Z)}{2T}J_-} \right]_{11} \end{aligned} \quad (3.108)$$

We can then use the braiding relations (3.70)-(3.71) to write (3.108) as

$$\begin{aligned} \Phi_{\pm}^{(1)} &= -\alpha e^{(1\pm\lambda)\rho} e^{-\log(\sec^2(\bar{Z}))(J_0)_{11}} \left(\mathfrak{F}_{++} [\mathbf{N}J_+^2]_{11} + \left(\mathfrak{F}_{+0} + \frac{\tan(Z)}{T}\mathfrak{F}_{++} \right) [\mathbf{N}\{J_+, J_0\}]_{11} \right. \\ &\quad + \left(\frac{2\tan(Z)}{T}\mathfrak{F}_{+0} + \frac{\tan^2(Z)}{T^2}\mathfrak{F}_{++} + \mathfrak{F}_{00} \right) [\mathbf{N}J_0^2]_{11} \\ &\quad \left. + \left(\frac{\tan(Z)}{2T}\mathfrak{F}_{+0} + \frac{\tan^2(Z)}{4T^2}\mathfrak{F}_{++} + \mathfrak{F}_{+-} \right) [\mathbf{N}\{J_+, J_-\}]_{11} \right) \end{aligned} \quad (3.109)$$

where we have defined the infinite-dimensional matrix \mathbf{N} to be

$$\mathbf{N} \equiv e^{2\bar{T}\tan(\bar{Z})J_-} e^{-2T\tan(Z)J_+} e^{-(\log \sec^2(Z))J_0}.$$

Finally, we invoke (3.73) and (3.77) to obtain, after some algebra,

$$[\mathbf{N}J_+^2]_{11} = \frac{\sec^{(\pm\lambda+5)}(Z)(\pm\lambda+2)(\pm\lambda+1)(2\bar{T}\tan(\bar{Z}))^2}{(1+4T\bar{T}\tan(Z)\tan(\bar{Z}))^{\pm\lambda+3}} \quad (3.110)$$

$$[\mathbf{N}\{J_+, J_0\}]_{11} = -\frac{\sec^{(\pm\lambda+3)}(Z)(\pm\lambda+2)(\pm\lambda+1)2\bar{T}\tan(\bar{Z})}{(1+4T\bar{T}\tan(Z)\tan(\bar{Z}))^{\pm\lambda+2}} \quad (3.111)$$

$$[\mathbf{N}J_0^2]_{11} = \frac{\sec^{(\pm\lambda+1)}(Z)(\pm\lambda+1)^2}{4(1+4T\bar{T}\tan(Z)\tan(\bar{Z}))^{\pm\lambda+1}} \quad (3.112)$$

$$[\mathbf{N}J_0]_{11} = -\frac{\sec^{(\pm\lambda+1)}(Z)(\pm\lambda+1)}{2(1+4T\bar{T}\tan(Z)\tan(\bar{Z}))^{\pm\lambda+1}} \quad (3.113)$$

Defining $\Theta \equiv 4T\bar{T}\tan(Z)\tan(\bar{Z})$, the final expression for $\Phi_{\pm}^{(1)}$ reads

$$\begin{aligned} \Phi_{\pm}^{(1)} &= -\alpha \frac{e^{(3\pm\lambda)\rho} \sec^{(1\pm\lambda)}(\bar{Z}) \sec^{(1\pm\lambda)}(Z)}{(1+\Theta)^{1\pm\lambda}T^2} \left((1\pm\lambda)\mathfrak{G}_{+-} + \frac{(1\pm\lambda)^2}{4}\mathfrak{G}_{00} \right. \\ &\quad \left. - \frac{\mathfrak{G}_{+0} \sec^2(Z)(2\pm\lambda)(1\pm\lambda)(2T\bar{T}\tan(\bar{Z}))}{1+\Theta} + \frac{\mathfrak{G}_{++} \sec^4(Z)(2\pm\lambda)(1\pm\lambda)(2T\bar{T}\tan(\bar{Z}))^2}{(1+\Theta)^2} \right) \end{aligned} \quad (3.114)$$

where we have defined

$$\mathfrak{G}_{+-} = \left(\frac{\sec^2(Z)}{192} \right) \left(-80(\bar{Z} - Z) + 24(\bar{Z} - Z) \cos(2Z) + 16(\bar{Z} - Z) \cos(4Z) + 8\bar{Z} \cos(6Z) \right. \\ \left. -8z \cos(6Z) - 18 \sin(2Z) - 4 \sin(4Z) - \sin(6Z) + 2 \sin(8Z) + \sin(10Z) \right) \quad (3.115)$$

$$\mathfrak{G}_{++} = (12(-\bar{Z} + Z) - 8 \sin(2Z) + \sin(4Z))/6 \quad (3.116)$$

$$\mathfrak{G}_{+0} = -2 \tan(Z)(\bar{Z} - Z + \cos(Z) \sin(Z)) \quad (3.117)$$

$$\mathfrak{G}_{00} = -\sec(Z)^2(4(\bar{Z} - Z) - 2(\bar{Z} - Z) \cos(2Z) + \sin(2Z))/3 \quad (3.118)$$

One can perform a simple consistency check for this result by considering the limit

$$\tau \rightarrow \infty, \bar{\tau} \rightarrow \infty, Z \rightarrow 0, \bar{Z} \rightarrow 0. \quad (3.119)$$

which should take the solution to the $\mathcal{O}(\mu)$ part of the propagator in the background of the chiral spin-3 deformation (3.79). Indeed, in this limit, it is straightforward to check that the propagator becomes

$$\Phi_{\pm} \rightarrow \Phi_{\pm}^{BTZ} \times \left(\frac{\mu \bar{Z}^3 e^{4\rho} (2 \pm \lambda)(1 \pm \lambda)}{(1 + |Z|^2 e^{2\rho})^2} \right) \quad (3.120)$$

which agrees with (3.79). There is also another limit that one can take to elucidate the physical interpretation of the scalar propagator we found. The boundary CFT correlators capture singularity structure of the spacetime geometry, and these can be computed by taking the large $|\rho|$ limits.

At $\rho \rightarrow \infty$, $T, \bar{T} \rightarrow \infty$ and the term in \mathfrak{G}_{++} dominates and we have

$$\Phi_{\pm}^{(1)}|_{\rho \rightarrow \infty} \approx \frac{-\alpha e^{-(1 \pm \lambda)\rho} (2 \pm \lambda)(1 \pm \lambda) \mathfrak{G}_{++}}{(4 \sin(\bar{z}) \sin(z) \tau \bar{\tau})^{1 \pm \lambda} \tau^2 \sin^2(2z)} \quad (3.121)$$

This yields the boundary correlator in which both scalar operators are on the same boundary. There is an absence of singularities away from the origin (and its thermal images obtained by summing over $z \sim z + 2\pi\mathbb{Z}$ and $\bar{z} \sim \bar{z} + 2\pi\mathbb{Z}$). Near the origin, we have

$$\Phi_{\pm}^{(1)} \approx \Phi_{\pm}^{BTZ} \times \left(\frac{(2 \pm \lambda)(1 \pm \lambda) \bar{z} \mu}{z^2} + \dots \right) \quad (3.122)$$

where near the origin, we have $\Phi^{BTZ} \approx (e^{-\rho}/|z|^2)^{1 \pm \lambda}$ since we took the infinite ρ limit first. The spin-3 deformation of the CFT sharpens the UV singularity at the boundary near the origin. Similarly, one can take the limit $\rho \rightarrow -\infty$, in which case we would have the mixed

boundary correlator. In this limit, we have

$$\Phi_{\pm}^{(1)}|_{\rho \rightarrow -\infty} \approx -\alpha \frac{e^{(1\pm\lambda)\rho}(1\pm\lambda)(4\mathfrak{G}_{+-} + (1\pm\lambda)\mathfrak{G}_{00})}{4\tau^2 \cos^{(1\pm\lambda)} \bar{Z} \cos^{(1\pm\lambda)} Z} \quad (3.123)$$

This gives us some valuable insights into the causal structure of the higher-spin black hole solution. If one computes the metric from the black hole's Chern-Simons connections, it is one in which a traversable wormhole connects two asymptotic regions. Naively, the two boundaries (at ρ being plus and negative infinities) appears to be causally connected. Typically it implies that a minimally coupled scalar field's correlator would exhibit some singularities in the mixed boundary correlator. In this case, the higher-spin symmetry, via the group theoretic properties of $\text{hs}[\lambda]$, control the coupling of spacetime to the set of higher spin fields. Hence, the physical causal structure is determined not purely by the apparent metric, but by the full set of higher spin fields. Evaluated in the Lorentzian signature, (3.123) is nonsingular. This gives us the physical picture that the two boundaries should be taken as being causally disconnected, just like how the two boundaries of the Lorentzian BTZ are spacelike separated.

Finally, let us comment that the scalar propagator is manifestly periodic under the thermal identification of (z, \bar{z})

$$z \sim z + 2\pi\tau, \quad \bar{z} \sim \bar{z} + 2\pi\bar{\tau} \quad (3.124)$$

This characteristic feature of the propagator acts as a good consistency check that we have taken the trace factor in the expression of the Chern-Simons connections correctly, since it is straightforward to check that other values would actually give rise to a non-periodic expression, by virtue of producing non-periodic terms in Z and \bar{Z} which are not of the form $Z - \bar{Z}$.

There exists an argument to see why the thermal periodicity is a property of all the other fields of the Vasiliev master field C_{\pm} . As we will elaborate more in later chapters, the higher-spin black hole is constructed by imposing the condition that the Wilson holonomies⁴

$$H = \mathcal{P}e^{\oint A}, \quad \bar{H} = \mathcal{P}e^{\oint \bar{A}} \quad (3.125)$$

are gauge equivalent and continuously deformable to the BTZ. The higher-spin black holes have flat Chern-Simons connections that generalize those of the BTZ to include higher-spin chemical potentials and charges. The requirement of being gauge-equivalent to the BTZ can be imposed as follows. Denoting $\oint A = \omega$, $\oint \bar{A} = \bar{\omega}$, being gauge equivalent to the BTZ implies the condition

$$\text{Tr}(\omega^n) = \text{Tr}(\omega_{BTZ}^n) \quad (3.126)$$

These conditions also ensure that the higher-spin black holes admit sensible thermodynamics properties that match those of a putative boundary CFT. The BTZ's holonomy belongs to the center of the group $\text{HS}[\lambda]$ and thus, so is H, \bar{H} . Now recall that the Vasiliev master

⁴Please note that in (3.125), \mathcal{P} denotes path-ordering being imposed.

field read

$$C_{\pm} = e^{(1\pm\lambda)\rho} \left[e^{-\Lambda_{\rho}} \star \mathfrak{M}_{\pm} \star e^{\bar{\Lambda}_{\rho}} \right],$$

where

$$\Lambda_{\rho} = A_z(\rho)z + A_{\bar{z}}(\rho)\bar{z}, \bar{\Lambda}_{\rho} = \bar{A}_z(\rho)z + \bar{A}_{\bar{z}}(\rho)\bar{z}.$$

Using the flatness condition and the fact that the holonomy belongs to the center of the group, it is straightforward to show that

$$C_{\pm}(\rho, z + 2\pi\tau, \bar{z} + 2\pi\bar{\tau}) = C_{\pm}(\rho, z, \bar{z}). \quad (3.127)$$

This furnishes a supporting evidence that higher-spin black hole background admits a thermal nature.

CHAPTER 4

Adventures in Vasiliev Gravity II: the topological sector

In this chapter, we present a detailed formulation of the $SL(4, R)$ Chern-Simons theory as the topological sector Vasiliev gravity when λ is analytically continued to 4, including various non-principal embeddings. In Section 4.3, we supersymmetrize the framework to study $SL(N|N - 1)$ Chern-Simons theories, including a derivation of the supersymmetry transformation laws. In particular, we noted that global AdS_3 preserves eight supercharges, the massless BTZ preserving four and the extremal BTZ preserving two. By performing an analytic continuation on N , we have also checked that this classification holds in the infinite-dimensional $shs[\lambda]$ gauge algebra case. The consistency of some aspects of our computations was verified by checking that, upon truncating some of the gauge fields, we obtain ordinary SUGRA defined via Chern-Simons with $osp(2|2)$ and $osp(1|2)$ gauge algebras which are sub-superalgebras of $SL(N|N - 1)$. From a partial analysis of the asymptotic symmetry algebra, we recovered the $\mathcal{N} = 2$ superconformal algebra (for each chiral sector), and computed explicitly the Sugawara redefinition of the energy-momentum tensor. The $u(1)$ field generates spectral flows of bulk solutions, and is quantized to ensure either the ϕ -periodicity of the Killing spinor or smooth holonomies. Various results of this section form the backdrop for our discussion of spacetime asymptotic symmetry in Chapter 6 and higher-spin black holes in Chapter 7, and they could be found in our papers [39, 40].

4.1 Chern-Simons theories as the topological sector of the bulk

Let us begin by reviewing the basic formulation of higher-spin gravity in the framework of Chern-Simons theory. As was briefly discussed earlier in Chapter 2, recall that the Chern-Simons action reads

$$S_{CS}[A] = \frac{k}{4\pi} \int \text{tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right). \quad (4.1)$$

and that the combination (with the same Chern-Simons level k)

$$S = S_{CS}[A] - S_{CS}[\tilde{A}] \quad (4.2)$$

where A and \tilde{A} are independent Chern-Simons connections labelled in $SL(2, \mathbb{R})$, reduces to the Einstein-Hilbert action¹ if we identify

$$A = \left(\omega^a + \frac{e^a}{l} \right) J_a, \quad \tilde{A} = \left(\omega^a - \frac{e^a}{l} \right) J_a \quad (4.3)$$

where the one-forms e^a, ω^a are the vielbeins and spin connection, and J_a are the $SL(2, \mathbb{R})$ generators. This identification is up to boundary terms, and, in particular, is made with the normalization $\text{Tr}(J_a J_b) = \frac{1}{2} \eta_{ab}$, and the identification $k = \frac{l}{4G}$. This can be generalized to an $SL(N, \mathbb{R}) \times SL(N, \mathbb{R})$ Chern-Simons action, with the vector potential expressed as

$$A = \left(\omega^a + \frac{e^a}{l} \right) J_a + \sum_{i=2}^{N-1} \left(\omega^{a_1 a_2 \dots a_i} + \frac{e^{a_1 a_2 \dots a_i}}{l} \right) T_{a_1 a_2 \dots a_i} \quad (4.4)$$

where $e^{a_1 a_2 \dots a_i}, \omega^{a_1 a_2 \dots a_i}$ are the analogous gauge potentials for the higher-spin fields, and $T_{a_1 a_2 \dots a_m}$ are the spin- m generators which are completely symmetric and traceless in their indices (i.e. $T_{ba_3 \dots a_i}^b = 0$). Like (4.3), the expression for \tilde{A} is similar but with $e \rightarrow -e$.

The higher-spin generators satisfy

$$[J_a, J_b] = \epsilon_{abc} J^c, \quad [J_b, T_{a_1 a_2 \dots a_{s-1}}] = \epsilon^m_{b(a_1} T_{a_2 \dots a_{s-1})m}, \quad (4.5)$$

and they clearly transform as $SL(2, \mathbb{R})$ tensors. For a general N , when the vector potentials are valued as in (4.4), we have a consistent description of a ‘gravitational’ sector. More precisely, when the equations of motion are linearized, we obtain the physics of a spin- N field propagating on an AdS_3 background (see, for example, Section 2 of [37] for a brief review). Let us now write down the full non-linear action and the field equations for spin-4 gravity. In terms of the vielbein-like and the spin connection-like fields, the action (4.2) in the case of $N = 4$ reads, after some algebra,

$$\begin{aligned} S = & \frac{1}{8\pi G} \int e^a \wedge d\omega_a + 2e^{abc} \wedge d\omega_{abc} + \frac{1}{6l^2} \epsilon_{abc} e^a \wedge e^b \wedge e^c + \frac{1}{2} \epsilon_{abc} e^a \wedge \omega^b \wedge \omega^c \\ & + 2e^{ab} \wedge d\omega_{ab} + \frac{2}{l^2} \epsilon_{abc} e^a \wedge e^{b\alpha} \wedge e^c_{\alpha} + 2\epsilon_{abc} e^a \wedge \omega^{b\alpha} \wedge \omega^c_{\alpha} + 4\epsilon_{abc} \omega^a \wedge \omega^{b\alpha} \wedge e^c_{\alpha} \\ & + 2e^{abc} \wedge d\omega_{abc} + \frac{2}{l^2} \epsilon_{abc} e^a \wedge e^{b\alpha\beta} \wedge e^c_{\alpha\beta} + 2\epsilon_{abc} e^a \wedge \omega^{b\alpha\beta} \wedge \omega^c_{\alpha\beta} + 4\epsilon_{abc} \omega^a \wedge \omega^{b\alpha\beta} \wedge e^c_{\alpha\beta} \\ & \epsilon_{abc} \left\{ 2\omega^{a\alpha\beta} \wedge \omega^b_{\alpha\gamma} \wedge e^{c\gamma}_{\beta} + \frac{2}{3l^2} e^{a\alpha\beta} \wedge e^b_{\alpha\gamma} \wedge e^{c\gamma}_{\beta} + 2\omega^{a\alpha} \wedge \omega^{b\beta} \wedge e^c_{\alpha\beta} \right. \\ & \left. + \frac{2}{l^2} e^{a\alpha} \wedge e^{b\beta} \wedge e^c_{\alpha\beta} + 4\omega^{a\alpha} \wedge e^{b\beta} \wedge \omega^c_{\alpha\beta} \right\} \end{aligned} \quad (4.6)$$

¹We note in passing that if the Chern-Simons levels are allowed to be different, we then have topologically massive gravity[41], of which a higher-spin analogue was considered most recently in [42].

The equations of motion for the gravitational fields are

$$\begin{aligned} de^a + \epsilon^{abc} \omega_b \wedge e_c + 4\epsilon^{abc} (e_{bd} \wedge \omega_c^d + \omega_{b\alpha\beta} \wedge e_c^{\alpha\beta}) &= 0, \\ d\omega^a + \epsilon^{abc} \left(\frac{1}{2} \omega_b \wedge \omega_c + \frac{e_b \wedge e_c}{2l^2} + 2\omega_{bd} \wedge \omega_c^d + 2\frac{e_{bd} \wedge e_c^d}{l^2} \right. \\ &\quad \left. + 2\omega_{b\alpha\beta} \wedge \omega_c^{\alpha\beta} + \frac{2}{l^2} e_{b\alpha\beta} \wedge e_c^{\alpha\beta} \right) = 0. \end{aligned} \quad (4.7)$$

from which we note that the higher-spin fields, like torsion fields, destroy the metric compatibility condition. The field equations for the spin-3 fields are

$$\begin{aligned} de^{ab} + \epsilon^{cd(a} (\omega_c \wedge e_d^{b)}) + e_c \wedge \omega_d^{b)}) + \epsilon^{cd(a} (\omega_{c\beta} \wedge e_d^{b)\beta} + e_{c\beta} \wedge \omega_d^{b)\beta}) &= 0, \\ d\omega^{ab} + \epsilon^{cd(a} \left(\omega_c \wedge \omega_d^{b)} + \frac{1}{l^2} e_c \wedge e_d^{b)} + \omega_{c\beta} \wedge \omega_d^{b)\beta} + \frac{1}{l^2} e_{c\beta} \wedge e_d^{b)\beta} \right) &= 0. \end{aligned} \quad (4.8)$$

Finally, the field equations for the spin-4 fields are

$$\begin{aligned} de^{abc} &= -\frac{2}{3} \epsilon^{ed(a} (\omega_{bc)e} \wedge e_d + e_{bc)e} \wedge \omega_d) - \frac{1}{3} \epsilon^{ef(a} (\omega_{b|ek} \wedge e^{c)fk} + \omega_{b|e} \wedge e^{c)f}), \\ d\omega^{abc} &= \frac{2}{3} \epsilon^{ef(a} \left(\frac{1}{l^2} e^{bc)f} \wedge e_e + \omega^{bc)f} \wedge \omega_e + \frac{1}{4} \left(\omega_{ek}^{b|} \wedge \omega^{c)fk} \right. \right. \\ &\quad \left. \left. + \omega_e^b \wedge \omega^{c)f} + e_{ek}^{b|} \wedge e^{c)fk} + e_e^b \wedge e^{c)f} \right) \right) \end{aligned} \quad (4.9)$$

From these explicit expressions, we can write down precisely the entire set of gauge transformations acting on the higher-spin fields. Besides the usual diffeomorphism, the spin-2 fields acquire new gauge transformations proportional to the spin-3 gauge parameters ξ^{ab} and Λ^{ab} and spin-4 gauge parameters ξ^{abc} and Λ^{abc} :

$$\begin{aligned} \delta e^a &= 4\epsilon^{abc} \omega_{bd} \xi_c^d + 4\epsilon^{abc} e_{bd} \Lambda_c^d, \\ \delta \omega^a &= 4\epsilon^{abc} \omega_{bd} \Lambda_c^d + 4\frac{1}{l^2} \epsilon^{abc} e_{bd} \xi_c^d. \end{aligned} \quad (4.10)$$

For the spin-3 and spin-4 fields, we have the gauge transformations

$$\begin{aligned} \delta e^{ab} &= d\xi^{ab} + \epsilon^{cd(a} \omega_c \xi_d^{b)} + \epsilon^{cd(a} e_c \Lambda_d^{b)} + \epsilon^{cd(a} \omega^b)_c \xi_d + \epsilon^{cd(a} e^b)_c \Lambda_d, \\ \delta \omega^{ab} &= d\Lambda^{ab} + \epsilon^{cd(a} \omega_c \Lambda_d^{b)} + \frac{1}{l^2} \epsilon^{cd(a} e_c \xi_d^{b)} + \epsilon^{cd(a} \omega^b)_c \Lambda_d + \frac{1}{l^2} \epsilon^{cd(a} e^b)_c \xi_d. \end{aligned} \quad (4.11)$$

Thus, both spin-3 and spin-4 fields transform under spin-3 and spin-4 gauge transformations and diffeomorphisms.²

²In our derivations above, we have used various trace relations like $\text{tr}(J_a J_b J_c) = 1/4 \epsilon_{abc}$, $\text{tr}(J_a J_c T_{ef}) = \eta_{ac} \eta_{fe} + \eta_{af} \eta_{ec}$, $\text{tr}(J_a T_{ef} T_{ij}) = \epsilon_{aei} \eta_{fj}$, etc. For the general spin- N , the trace relations for the various J^a

Another useful basis which we will rely on in later sections is one in which one does not have to take into account the trace constraints on the generators. One general expression for such a basis for higher-spin fields (see, for example, [38]) is

$$\begin{aligned} [L_+, L_-] &= 2L_0, \quad [L_\pm, L_0] = \pm L_\pm, \\ [L_i, W_m^l] &= (il - m) W_{i+m}^l, \\ W_m^l &= (-1)^{l-m} \frac{(l+m)!}{(2l)!} ad_{L_-}^{l-m} (L_+^l). \end{aligned} \quad (4.12)$$

In the above notations, the spin is $(l+1)$, $i = 0, \pm 1$, $-l \leq m \leq l$, and $ad_L(f) = [L, f]$ refers to the adjoint action of L on f . This is easily motivated by letting $W_i^l = L_+^l$ in the fundamental, and then deriving the rest of the generators by the lowering operator L_- . In [37], the isomorphism in the case of spin-3 between the spin generators $T_{a_1 a_2 \dots a_{s-1}}$ and the W_m^l generators is computed. Similarly, we derive the isomorphism in the spin-4 case. Up to one constant scaling factor, we find that for the ten spin-4 generators, the mapping goes as

$$\begin{aligned} T_{222} &= U_0, \quad T_{220} = \frac{1}{2}(U_1 + U_{-1}), \quad T_{221} = \frac{1}{2}(U_1 - U_{-1}), \quad T_{200} = \frac{1}{4}(U_2 + U_{-2}) + \frac{1}{2}U_0, \\ T_{012} &= \frac{1}{4}(U_2 - U_{-2}), \quad T_{211} = \frac{1}{4}(U_2 + U_{-2}) - \frac{1}{2}U_0, \quad T_{000} = \frac{1}{8}(U_3 + U_{-3} + 3(U_1 + U_{-1})), \\ T_{001} &= \frac{1}{8}(U_3 - U_{-3} + U_1 - U_{-1}), \quad T_{011} = \frac{1}{8}(U_3 + U_{-3} - U_1 - U_{-1}), \\ T_{111} &= \frac{1}{8}(U_3 - U_{-3} + 3(U_{-1} - U_1)) \end{aligned} \quad (4.13)$$

where we have denoted $\mathcal{U}_m = \mathcal{W}_m^3$. For the general spin- N case, we can derive this isomorphism straightforwardly by starting with the ‘highest-weight’ generator $T_{22\dots 2}$.

The physical interpretation of these Chern-Simons theories begins after we identify metric-like fields. They are identified by demanding invariance under local Lorentz invariance parametrized by the gauge parameters Λ^a , Λ^{ab} , and Λ^{abc} . This condition is only sufficient for the spin-2 (metric) and spin-3 fields. In [38], it was explained that for higher-spin fields, the choice of identification is unique for spin-4 and spin-5 fields if we further demand that in the linearized regime, rewriting them in terms of vielbeins reproduces the definition in the free theory. Then, it can be shown (see [37, 38]) that this leads to the following definitions of metric-like fields:

$$g \sim \text{tr}(e \cdot e), \quad \psi_3 \sim \text{tr}(e \cdot e \cdot e), \quad \psi_4 \sim \text{tr}(e^4) - \frac{3\lambda^2 - 7}{10} (\text{tr}e^2)^2 \quad (4.14)$$

and $T^{a_1 a_2 \dots a_{s-1}}$ are determined by all possible contraction of indices of the generators in the product via the tensors η and ϵ .

where λ is related to the quadratic Casimir by

$$L_0^2 - \frac{1}{2}(L_+L_- + L_-L_+) = \frac{\lambda^2 - 1}{4}. \quad (4.15)$$

In our case, $\lambda = 4$, but is not necessarily an integer if we consider the larger framework of $hs[\lambda]$ algebra. Consider building an infinite tower of higher-spin $(l + 1)$ fields in the basis of generators W_m^l , each appearing once, so l in (4.12) runs from 1 to ∞ . In general this algebra $hs[\lambda] = \bigoplus_{l=1}^{\infty} \mathfrak{g}^{(l)}$ (where λ is as defined in (4.15)) is distinct for different values of λ , and when $\lambda = N \in \mathbb{Z}$, all higher-spin generators for spin $> N$ can be truncated and the algebra reduces to $SL(N, \mathbb{R})$ algebra. An interesting fact is that the commutator between even- l W 's yields a sum of odd- l W 's, whereas the commutator between even- l W 's and odd- l W 's yields a sum of even- l W 's. A related implication of this is that it is possible for the algebra to be truncated of all even- l generators (which correspond to odd spins). In four dimensions, the well-known counterpart is the minimal Vasiliev model.

4.2 Non-principal Embeddings

Before we proceed to discuss the spacetime interpretation of the ansatz (6.10), let us mention that there are other Drinfeld-Sokolov procedures with which one can construct \mathcal{W} -algebras. What we have done above corresponds to the choice of the ‘‘principal embedding’’ of $SL(2, \mathbb{R})$ in $SL(4, \mathbb{R})$. Let us first quickly review the meaning of having different embeddings of $SL(2, \mathbb{R})$ in the general $SL(N, \mathbb{R})$ theory.

As explained in for example [43], that one considers $SL(2, \mathbb{R})$ embeddings is closely related to the requirement that one wants the algebra to be an extended conformal algebra, i.e. containing the Virasoro as a subalgebra and other generators to be primary fields with respect to this Virasoro algebra. To each $SL(2, \mathbb{R})$ embedding within the simple Lie algebra that underlies the affine algebra, one can associate a generalized classical Drinfeld-Sokolov reduction of the affine algebra to obtain a \mathcal{W} -algebra.

For $SL(N, \mathbb{R})$, the number of inequivalent $SL(2, \mathbb{R})$ embeddings is equal to the number of partitions of N , and the standard reduction leading to \mathcal{W}_N algebras is associated with the so-called principal embedding. Also, the inequivalent $SL(2, \mathbb{R})$ embeddings are completely characterized by the branching rules of the fundamental representation. In our discussion below where we will give explicit examples of the statements above, we will parametrize the branching by various $SL(2, \mathbb{R})$ -multiplets, and ‘‘spin’’ in this context refers to the dimensionality of the representation. The conformal weight of each field is obtained from the $SL(2, \mathbb{R})$ spin by adding one.

What is the relevance of these non-principal embeddings in a three-dimensional $SL(N, \mathbb{R}) \times SL(N, \mathbb{R})$ higher-spin theory? A nice discussion was first made in [50], where it was pointed out that non-principal embeddings describe AdS_3 vacua of possibly different radii, with the corresponding \mathcal{W} -algebra as the asymptotic symmetry algebra. Specifically in the case

of $N = 3$, the Polyakov-Bershadsky algebra $\mathcal{W}_3^{(2)}$ is the only non-principal embedding in $SL(3, \mathbb{R})$. A solution that represents an interpolation between the $\mathcal{W}_3^{(2)}$ (in the UV) and \mathcal{W}_3 vacua (in the IR) was constructed, and an elegant linearized analysis of the RG flow background was presented in [50].

Let us very briefly review the $N = 3$ case as presented in [50]. For the principal embedding (\mathcal{W}_3), we have one spin-1 multiplet generated by $(L_0, L_{\pm 1})$ and one spin-2 multiplet generated by $(W_0, W_{\pm 1}, W_{\pm 2})$. For $\mathcal{W}_3^{(2)}$, the branching reads as: (i) one spin-1 multiplet $(\frac{1}{4}W_2, \frac{1}{2}L_0, -\frac{1}{4}W_{-2})$, (ii) one spin-0 multiplet (W_0) , (iii) two spin-1/2 multiplets: $(W_1, L_{-1}), (L_1, W_{-1})$, where the $\mathcal{W}_3^{(2)}$'s generators have been expressed in terms of the \mathcal{W}_3 's. An analysis similar to that done for \mathcal{W}_3 can be done to obtain the classical $\mathcal{W}_3^{(2)}$ algebra.

In terms of the Chern-Simons connections, we have

$$\begin{aligned} A_{AdS_3} &= e^\rho \left(\frac{1}{4}W_2 \right) dx^+ + \left(\frac{1}{2}L_0 \right) d\rho, \\ \bar{A}_{AdS_3} &= -e^\rho \left(\frac{1}{4}W_{-2} \right) dx^- - \left(\frac{1}{2}L_0 \right) d\rho \end{aligned} \quad (4.16)$$

which translates to the metric (the higher spin field $\psi_{abc} = 0$)

$$ds^2 = \frac{l^2}{4} (d\rho^2 - e^{2\rho} dx^+ dx^-) \quad (4.17)$$

which is AdS_3 with radius $= \frac{l}{2}$, if we assume the same metric normalization as that for the principal embedding.

The difference in the AdS radius can be traced to the trace relations of the new $SL(2, \mathbb{R})$ generators, in particular that of L_0 which is also known as the defining vector of the embedding. For $\mathcal{W}_3^{(2)}$, we note that $\text{Tr} \left(\left(\frac{1}{2}L_0 \right)^2 \right) = \frac{1}{4} \text{Tr} (L_0^2)$, giving rise to an AdS_3 of half the radius of that of the principal embedding. This implies that the overall normalization of the Chern-Simons action restricted to the $SL(2, \mathbb{R})$ subalgebra must be reduced by an overall factor of 1/4, and thus both the Chern-Simons level k and central charge of the $\mathcal{W}_3^{(2)}$ would be reduced by a similar factor.

It is straightforward to make similar statements in the general N case. What one needs to do is to find an explicit representation of the generators of various $SL(2, \mathbb{R})$ -multiplets, in particular that of the three gravitational spin-one generators in terms of the original ones associated with the principal embedding. Retaining the original metric normalization factor, and denoting the defining vector of the non-principally embedded $SL(2, \mathbb{R})$ algebra by \tilde{L}_0 , we then have the metric describing an AdS_3 of radius R_{AdS_3} and vacuum's central

charge c , with

$$\frac{R_{AdS_3}^2}{l^2} = \frac{\text{Tr}(\tilde{L}_0^2)}{\text{Tr}(L_0^2)} = \frac{c}{6k}, \quad (4.18)$$

while noting that l is the radius of the AdS_3 vacuum of the principal $SL(2, \mathbb{R})$ embedding.

In [44, 43], the general embedding of $SL(2, \mathbb{R})$ in $SL(N, \mathbb{R})$ was discussed very nicely, and this gives one the basic tools for analyzing this aspect of these higher-spin gravity theories. Following [43], let (n_1, n_2, \dots) be a partition of N , with $n_1 \geq n_2 \geq \dots$, then define a different partition (m_1, m_2, \dots) of n , with m_k equal to the number of i for which $n_i \geq k$, and let $s_t = \sum_i^t m_i$. The embedded $SL(2, \mathbb{R})$'s generators $(\tilde{L}_0, \tilde{L}_\pm)$ can then be expressed explicitly as [43]

$$\begin{aligned} \tilde{L}_+ &= \sum_{l \geq 1} \sum_{k=1}^{n_l-1} E_{l+s_{k-1}, l+s_k}, \\ \tilde{L}_0 &= \sum_{l \geq 1} \sum_{k=1}^{n_l} \left(\frac{n_l+1}{2} - k \right) E_{l+s_{k-1}, l+s_{k-1}}, \\ \tilde{L}_- &= \sum_{l \geq 1} \sum_{k=1}^{n_l-1} k(n_l - k) E_{l+s_k, l+s_{k-1}}, \end{aligned} \quad (4.19)$$

where E_{ij} denotes the matrix with a one in its (i, j) entry and zeroes everywhere else. Applying (4.19) to the $SL(4, \mathbb{R})$ case where there are 15 generators, we can describe the branching of each non-principal embedding as a direct sum of $(2j+1)$ -dimensional irreducible $SL(2, \mathbb{R})$ representations $\bigoplus_{j \in \frac{1}{2}\mathbb{N}} n_j \cdot \underline{2j+1}$, where n_j is the degeneracy of the spin- j representation. The branching of the $SL(4, \mathbb{R})$ fundamental representation $\underline{15}^{(n_1, n_2, \dots)}$ goes as

(1) $\underline{15}^{(2,2)} \sim 4 \cdot \underline{3} + 3 \cdot \underline{1}$. Apart from $(\tilde{L}_0, \tilde{L}_\pm)$, this representation consists of three spin-1 multiplets and three singlets. We compute $\text{Tr}(\tilde{L}_0^2) = 1$, $R_{AdS_3} = \sqrt{\frac{1}{5}}$.

(2) $\underline{15}^{(3,1)} \sim \underline{3} + \underline{5} + 2 \cdot \underline{3} + \underline{1}$. Apart from $(\tilde{L}_0, \tilde{L}_\pm)$, this representation consists of two spin-1 multiplets, one spin-2 multiplet and one singlet. We compute $\text{Tr}(\tilde{L}_0^2) = 2$, $R_{AdS_3} = \sqrt{\frac{2}{5}}$.

(3) $\underline{15}^{(2,1,1)} \sim \underline{3} + 4 \cdot \underline{2} + 4 \cdot \underline{1}$. Apart from $(\tilde{L}_0, \tilde{L}_\pm)$, this representation consists of four spin-1/2 multiplets, four singlets. We compute $\text{Tr}(\tilde{L}_0^2) = \frac{1}{2}$, $R_{AdS_3} = \sqrt{\frac{1}{10}}$.

We note that in the notations above, the principal embedding is $\underline{15}^{(4)} \sim \underline{3} + \underline{5} + \underline{7}$, and we have set the radius $l = 1$. We will use the above results in understanding a subtle aspect of the asymptotic behavior of the black hole solutions in Chapter 7.

4.3 Supersymmetric generalizations and a class of higher-spin supergravity theories

4.3.1 General remarks on higher-spin AdS_3 SUGRA as a Chern-Simons theory

Below, we review some basics of ordinary and higher-spin anti-de Sitter SUGRA based on Chern-Simons theory, mainly following the introduction in [69]. There are two basic requirements of the superalgebra associated with an AdS_3 SUGRA: (i) it contains an $sl(2, \mathbb{R}) \oplus sl(2, \mathbb{R})$ as a sub-algebra, and (ii) the fermionic generators transform in the $\mathbf{2}$ of the $sl(2)$. This condition is satisfied by seven classes of superalgebras (see table 1 of [69]). In this section, our main interest lies in a higher-spin counterpart of $osp(2|2, \mathbb{R}) \oplus osp(2|2, \mathbb{R})$, which belongs to the general class of $osp(N|2, \mathbb{R}) \oplus osp(M|2, \mathbb{R})$ Chern-Simons theory (these are SUGRA theories which have $\mathcal{N} = (N, M)$ supersymmetry).

In the non-supersymmetric case, a consistent higher spin gravity can be defined as a $sl(N, \mathbb{R})$ Chern-Simons theory which includes higher spin gauge fields up to spin $\leq N$. In a particular $N \rightarrow \infty$ limit, the gauge algebra becomes an infinite-dimensional $hs(2, \mathbb{R})$ algebra. Supersymmetrization of this construction was first discussed in [69] in which the infinite-dimensional superalgebra $shs(N|2, \mathbb{R}) \oplus shs(M|2, \mathbb{R})$ is studied as the higher-spin extension of $osp(N|2, \mathbb{R}) \oplus osp(M|2, \mathbb{R})$. The first can be represented as the quotient of the universal enveloping algebra of the latter by a certain ideal.

In the framework of Chern-Simons theory, supersymmetrization implies among other things, that the Lie algebra is replaced by suitable superalgebras, along with the supertrace in place of the ordinary one. In the following, we will study the particular case of $sl(N|N-1)$ superalgebras as the higher-spin gauge algebras, with $N = 3$ as our main example. We should remark that $sl(N|N-1)$ is not a consistent truncation of $shs[\lambda]$ in the sense that it is *not* a subalgebra of the latter. Indeed, the only non-trivial sub-superalgebra of $shs[\lambda]$ is $osp(2|2)$. Nonetheless, $sl(N|N-1)$ contains $sl(2|1) \simeq osp(2|2)$ as a sub-superalgebra, and further there is a well-defined analytic continuation procedure to send it to $shs[\lambda]$. This situation appears identically in the $sl(N)$ case. As noted in [48], $sl(3)$ cannot be obtained as an algebraic truncation of $hs[\lambda]$. However, if we force terms valued in higher spin fields of spin greater than two to be zero by hand, then the truncated algebra is isomorphic to $sl(3)$. We are interested in $sl(N|N-1)$ as a higher-spin SUGRA theory which is a natural supersymmetric generalization of the $sl(N)$ higher-spin theories.

Another important relation which we will discuss further is that to super- \mathcal{W} algebras. As is well-known, the Drinfeld-Sokolov Hamiltonian reduction procedure, when applied to WZW models, takes affine $sl(2)$ current algebra to Virasoro algebra. Generalization of this method has been recently used to explain how \mathcal{W}_N and $\mathcal{W}_\infty[\lambda]$ can be obtained from affine $sl(N)$ and $hs[\lambda]$ algebras respectively, in the context of higher-spin gravitational theories. Supersymmetrization of this computation was done in [69]. Although we do not explicitly carry out the full computation to obtain the classical $\mathcal{N} = 2 \mathcal{W}_3$ algebra (see [72] for the full quantum $\mathcal{N} = 2 \mathcal{W}_3$ algebra and [73] for the classical algebra), later we shall truncate some of the gauge fields to recover the $\mathcal{N} = 2$ superconformal algebra, and in the process,

compute the Sugawara redefinition of the energy-momentum tensor. We should note that for non-higher spin supergravity theories, the relationship between superconformal algebras and asymptotic dynamics has been explained elegantly in the nice work of [74] for the finite-dimensional gauge algebras, and the Hamiltonian reduction procedure explained in [75].

In the ordinary case of $osp(N|2)$ gauge algebra, the odd-graded generators are fundamental spinors of $sl(2)$ and vectors of $so(N)$, while the even-graded ones consist of the sum of $sl(2)$ and $so(N)$ generators. In particular, the $N = 2$ case is important for us due to the isomorphism $osp(2|2) \simeq sl(2|1)$, the latter being a sub-superalgebra of $sl(N|N - 1)$. Henceforth, we will pay attention to the $N = 3$ case. The Killing spinor equations were solved for a number of classical backgrounds, and global AdS_3 and the massless BTZ black hole [6] arise as the Neveu-Schwarz and Ramond vacua of the theory. The supersymmetric higher-spin theories based on $sl(N|N - 1)$ gauge algebra will have $\mathcal{N} = 2$ supersymmetry in either/both chiral sectors of the Chern-Simons theory, due to the $osp(2|2)$ sub-superalgebra.

4.3.2 About $sl(N|N - 1)$ in the Racah basis

Now, the general $sl(N|N - 1)$ element can be decomposed as [76]

$$sl(N|N - 1) = sl(2) \oplus \left(\bigoplus_{s=2}^{N-1} g^{(s)} \right) \oplus \left(\bigoplus_{s=0}^{N-2} g^{(s)} \right) \oplus 2 \times \left(\bigoplus_{s=0}^{N-2} g^{(s+\frac{1}{2})} \right) \quad (4.20)$$

where $g^{(s)}$ is defined as a spin- s multiplet of $sl(2)$. This is one less than the conformal or spacetime spin³. Hence, for example, for $sl(3|2)$, the even-graded sector will consist of three $sl(2)$ generators, five spin-2 generators, one abelian gauge field, one spin-1 field, whereas the odd-graded part consists of two copies of a spin-1/2 multiplet and a spin-3/2 multiplet.

Let us begin with the action which is the difference between two super-Chern Simons action at level k ,

$$S_{CS}[\Gamma, \tilde{\Gamma}] = \frac{k}{4\pi} \int \text{str} \left(\Gamma \wedge d\Gamma + \frac{2}{3} \Gamma \wedge \Gamma \wedge \Gamma \right) - \frac{k}{4\pi} \int \text{str} \left(\tilde{\Gamma} \wedge d\tilde{\Gamma} + \frac{2}{3} \tilde{\Gamma} \wedge \tilde{\Gamma} \wedge \tilde{\Gamma} \right) \quad (4.21)$$

where ‘str’ stands for the super-trace, and Γ (chiral sector) and $\tilde{\Gamma}$ (anti-chiral sector) are the connection one-forms valued in the elements of the superalgebra. When the higher-spin theory is cast as a Chern-Simons theory, it is essential to specify how the gravitational $sl(2)$ sector is embedded in the gauge algebra. For example, in the $sl(N)$ case, Chern-Simons theory based on the gauge algebra $sl(N)$ can realize physically distinct higher-spin theories arising from inequivalent embeddings of $sl(2)$.⁴ Demanding that the gravitational $sl(2)$ is part of the $osp(2|2)$ superalgebra, we can work in the so-called ‘Racah’ basis to write down the commutation relations that are more suited for us to identify the physical interpretation

³From now on, by ‘spin’ we refer to the $sl(2)$ -spin unless otherwise stated.

⁴There can however be problems with unitarity for the non-principal embeddings as discussed in [77].

of various fields. We will leave explicit details of the matrix realization of the superalgebra to the Appendix B, but now, let us summarize and review some essential points about the way we describe the $sl(N|N-1)$ in the Racah basis following [76].

It is convenient to start from $gl(N|N-1)$ and obtain $sl(N|N-1)$ by quotienting out its center. The generators are \mathbb{Z}_2 -graded by the usual Grassmann parity function: the bosonic ones we denote by T, U which generate $gl(N)$ and $gl(N-1)$ in (4.20), and the fermionic ones by Q, \bar{Q} which generate the half-integer spin multiplets.⁵ The supercommutation relations read schematically (the structure constants are multiples of Wigner $6j$ -symbols (see Appendix B)):

$$[T, T] \sim T, \quad [U, U] \sim U, \quad \{Q, \bar{Q}\} \sim T + U \quad (4.22)$$

$$[T, \bar{Q}] \sim \bar{Q}, \quad [U, \bar{Q}] \sim \bar{Q}, \quad [T, Q] \sim Q, \quad [U, Q] \sim Q. \quad (4.23)$$

In the notation of (4.20), we denote $T_m^s, -s \leq m \leq s$, to generate each $g^{(s)}$ multiplet, and similarly for U, Q, \bar{Q} . The identity matrix $\mathbf{1} = \sqrt{N}T_0^0 + \sqrt{N-1}U_0^0$ is the center, and after modding it out, we have $sl(N|N-1) \sim gl(N|N-1)/\mathbf{1}$.

Further for our purpose, we should re-define the generators such that we can form a basis for the $sl(2)$ in (4.20) with all the other generators transforming under its irreducible representations. This is the gravitational $sl(2)$ sub-algebra when $sl(2)$ is embedded principally in $sl(N|N-1)$. Henceforth, we will allude to the specific case of $N=3$ as a concrete example. First, let us consider linear combinations of the T_m^s, U_m^s as follows

$$\begin{aligned} L_0 &= \frac{1}{\sqrt{2}} (2T_0^1 + U_0^1), \quad L_{\pm 1} = 2T_{\pm 1}^1 + U_{\pm 1}^1, \\ A_0 &= \frac{1}{\sqrt{2}} (2T_0^1 - U_0^1), \quad A_{\pm 1} = 2T_{\pm 1}^1 - U_{\pm 1}^1. \\ W_{\pm 2} &= 4T_{\pm 2}^2, \quad W_{\pm 1} = 2T_{\pm 1}^2, \quad W_0 = \sqrt{\frac{8}{3}}T_0^2 \\ U_0 &= \frac{1}{\sqrt{3}}T_0^0 + \frac{1}{\sqrt{2}}U_0^0 \end{aligned} \quad (4.24)$$

which realize the even part of $sl(3|2)$ as

$$\begin{aligned} [L_i, L_j] &= (i-j)L_{i+j}, \quad [A_i, A_j] = (i-j)L_{i+j}, \quad [L_i, A_j] = (i-j)A_{i+j} \\ [W_i, W_j] &= \frac{(j-i)}{3} (2j^2 + 2i^2 - ij - 8) \times \frac{1}{2} (L_{i+j} + A_{i+j}) \\ [L_i, W_j] &= (2i-j)W_{i+j}, \quad [A_i, W_j] = (2i-j)W_{i+j}. \end{aligned} \quad (4.25)$$

The generators L_i form the basis for the total $sl(2)$. Note that the commutator between the W 's is the same as that in $sl(3)$ but with $A \rightarrow L$. Indeed, the subset of generators

⁵Formally, let the Grassmann parity function be $P(T) = P(U) = 1 = -P(Q) = -P(\bar{Q})$, and define the supercommutator by $[A, B] = AB - (-1)^{P(A)P(B)}BA$.

$\{(L_i + A_i)/2, W_i\}$ forms the $sl(3)$ algebra. Also, some commutation relations involving Q, \bar{Q}, U_0 and J_m can be written as

$$\{Q_r^{(\frac{1}{2})}, \bar{Q}_s^{(\frac{1}{2})}\} = \frac{1}{3} J^a (\gamma_a \gamma_0)_{rs} + (\gamma_0)_{rs} \mathcal{U}_0, \quad (4.26)$$

$$[U_0, Q_r^{(\frac{1}{2})}] = \frac{1}{6} Q_r^{(\frac{1}{2})}, \quad [U_0, \bar{Q}_r^{(\frac{1}{2})}] = -\frac{1}{6} \bar{Q}_r^{(\frac{1}{2})}, \quad (4.27)$$

$$[J_m, Q_r^{(\frac{1}{2})}] = -\frac{1}{2} (\gamma_m)_{rs} Q_s^{(\frac{1}{2})}, \quad [J_m, Q_r^{(\frac{3}{2})}] = \frac{1}{2} (\gamma_m^{(\frac{3}{2})})_{rs} Q_s^{(\frac{3}{2})}, \quad (4.28)$$

where $r, s = -\frac{1}{2}, \frac{1}{2}$ and we define

$$J_0 = \frac{1}{2} (L_1 + L_{-1}), \quad J_1 = \frac{1}{2} (L_1 - L_{-1}), \quad J_2 = L_0. \quad [J_a, J_b] = \epsilon_{abc} J^c$$

$$\gamma_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \quad [\gamma_a, \gamma_b] = 2\epsilon_{abc} \gamma^c.$$

$$\gamma_0^{(\frac{3}{2})} = \begin{pmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -\sqrt{4} & 0 \\ 0 & \sqrt{4} & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}, \quad \gamma_1^{(\frac{3}{2})} = - \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & \sqrt{4} & 0 \\ 0 & \sqrt{4} & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}, \quad \gamma_2^{(\frac{3}{2})} = 2[\gamma_0, \gamma_1]$$

where $\gamma_m^{(\frac{3}{2})}$ are spin- $\frac{3}{2}$ matrix realizations of $sl(2)$. Identical relations hold for the \bar{Q} generators. Thus, the fermionic generators Q, \bar{Q} transform as irreducible $sl(2)$ tensors. With the identification of the gravitational $sl(2)$, we checked that the Chern-Simons level k can be identified as

$$k = \frac{l}{4G} \quad (4.29)$$

provided we normalize the supertrace 'str' = $\frac{1}{3}(\sum_{i=1}^3 - \sum_{i=4}^5)M_{ii}$ where M is any supermatrix.

4.3.3 A class of gauge transformations

We now turn to the subject of deriving the suitable supersymmetry transformation laws for the $sl(3|2)$ Chern-Simons theory. Our consideration below lies in the chiral sector, but applies equally to the anti-chiral sector. Now, the gauge connection Γ is $sl(3|2)$ -valued and parametrized as

$$\Gamma = (e^a + \omega^a) J_a + \Upsilon^i K_i + \mathcal{U} U_0 + \mathcal{W}^m W_m + \psi_r^{(\frac{1}{2})} Q_r^{(\frac{1}{2})} + \bar{\psi}_r^{(\frac{1}{2})} \bar{Q}_r^{(\frac{1}{2})} + \psi_r^{(\frac{3}{2})} Q_r^{(\frac{3}{2})} + \bar{\psi}_r^{(\frac{3}{2})} \bar{Q}_r^{(\frac{3}{2})} \quad (4.30)$$

with

$$K_2 = A_0, \quad K_1 = \frac{1}{2} (A_1 - A_{-1}), \quad K_0 = \frac{1}{2} (A_1 + A_{-1}).$$

Also, we should relate the gravitational vielbeins e^a and spin connection $\omega^a = \frac{1}{2}\epsilon^{abc}\omega_{bc}$ to the relevant fields in the other copy of Chern-Simons by setting

$$\tilde{\Gamma} = (-e^a + \omega^a)J_a + \dots \quad (4.31)$$

Now, it is useful to begin by considering the invariance of the action under a gauge transformation that is valued in $Q^{(\frac{1}{2})}, \bar{Q}^{(\frac{1}{2})}$ (in the notation introduced in the previous section), i.e.

$$\epsilon_{susy} = \epsilon_r Q_r^{(\frac{1}{2})} + \bar{\epsilon}_s \bar{Q}_s^{(\frac{1}{2})} \quad (4.32)$$

Then, the invariance of the action under $\delta\Gamma$ which reads

$$\delta\Gamma = d\epsilon_{susy} + [\Gamma, \epsilon_{susy}] \quad (4.33)$$

is guaranteed up to total derivative terms. From (4.33), we can then compute the supersymmetry transformation laws purely from the superalgebra. This method relies on the coincidence that we can write the gravity theory as a Chern-Simons theory of which bulk action is gauge invariant, modulo total derivatives which may not vanish at the boundary. Equation (4.33) can then be used to derive the supersymmetry transformation laws, based on the $sl(2|1) \simeq osp(2|2)$ subalgebra, albeit a subtlety that involves the reality conditions of the fermionic fields. We note that when written in components, the kinetic terms read schematically as

$$\begin{aligned} \mathcal{L}_{kin.} = & (e^a + \omega^a) \wedge d(e_a + \omega_a) + 2(e^a + \omega^a) \wedge d\Upsilon_a + \Upsilon^a \wedge d\Upsilon_a + \mathcal{U} \wedge d\mathcal{U} + \mathcal{W}^m \wedge d\mathcal{W}_m \\ & - \bar{\psi}^{(\frac{1}{2})}\gamma_0 \wedge d\psi^{(\frac{1}{2})} - \psi^{(\frac{3}{2})}\tilde{\gamma}_0 \wedge d\bar{\psi}^{(\frac{3}{2})}, \end{aligned} \quad (4.34)$$

where the indices are contracted via the Killing metric of $sl(3|2)$, and

$$\tilde{\gamma}_0 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Now, there is a crucial yet subtle point that affects how one derives the supersymmetry transformation laws. We note that if $\bar{\psi}$ and ψ are independent, real Grassmann fields, then their kinetic terms are not real. As in the case of $osp(N|M)$ Chern-Simons theories, one inserts extra factors of ‘ i ’ in the Lagrangian appropriately.

In the following, we will keep track of these insertions by imposing suitable reality conditions on the fermionic fields ψ as follows.

$$\bar{\psi}_r^{(\frac{1}{2})} = i\psi_r^{(\frac{1}{2})\dagger}, \quad \bar{\psi}_r^{(\frac{3}{2})} = -i\psi_r^{(\frac{3}{2})\dagger}. \quad (4.35)$$

We should note that the number of degrees of freedom involving ψ remains the same, and that the sub-superalgebra generated by the generators $L_i, U_0, Q_r^{(\frac{1}{2})}, \bar{Q}_r^{(\frac{1}{2})}$ closes, and can be

identified as $sl(2|1) \simeq osp(2|2)$. More precisely, we find that these algebras are isomorphic upon the identifications (please see Appendix A for the generators of $osp(2|2)$ displayed below):

$$E \sim L_-, \quad F \sim -L_+, \quad H \sim 2L_0,$$

$$J_{12} \sim i6U_0, \quad Q_{\pm\frac{1}{2}}^{(\frac{1}{2})} \sim -\frac{i}{2\sqrt{3}} (R_1^\pm + iR_2^\pm), \quad \bar{Q}_{\pm\frac{1}{2}}^{(\frac{1}{2})} \sim \frac{1}{2\sqrt{3}} (R_1^\pm - iR_2^\pm). \quad (4.36)$$

We note from (4.36) that the matching between $sl(2|1)$ and $osp(2|2)$ involves changing the reality conditions of the generators U_0 and $Q_{\pm\frac{1}{2}}^{(\frac{1}{2})}$. These conditions ensure the reality of the supersymmetric Lagrangian, since they generate appropriate factors of ‘ i ’ in the coefficients of the fermionic kinetic terms. Later, we shall use both (4.35) and (4.33) to derive the supersymmetry transformation laws, but first let us consider the cases where the fields are truncated to those in $osp(1|2)$ and $osp(2|2)$ Chern-Simons supergravity theories. The $sl(3|2)$ field content differs from that of the $osp(2|2)$ case by the additional $\mathcal{N} = 2$ multiplet $\{\Upsilon, \mathcal{W}, \psi^{(3/2)}\}$. The supersymmetry transformation laws for these theories have been derived some time ago, and we want to re-derive them by using (4.33) and (4.35), after setting the irrelevant fields in $sl(3|2)$ Chern-Simons theory to vanish. This should serve as a good consistency check of our approach.

4.3.4 On the supersymmetry of $osp(1|2)$ and $osp(2|2)$ Chern-Simons theories

We begin with a comparison to $osp(1|2)$ Chern-Simons theory [20]. This theory can be recovered after a truncation of the $sl(3|2)$ theory, by setting

$$\mathcal{W} = \Upsilon = \mathcal{U} = \psi^{(\frac{3}{2})} = \bar{\psi}^{(\frac{3}{2})} = 0, \quad (4.37)$$

$$\bar{\psi}^{(\frac{1}{2})} = -i\psi^{(\frac{1}{2})}. \quad (4.38)$$

To motivate (4.38), we set the fields conjugate to the generator R_2^\pm (see (4.36) and Appendix A) to be zero and the fields conjugate to R_1^\pm to be real. After taking into account the overall trace normalization factor, we compute the action to be ⁶

$$S = \frac{1}{8\pi G} \int d^3x e^a \wedge \left(d\omega_a + \frac{1}{2} \epsilon_{abc} \omega^b \wedge \omega^c \right) + \frac{1}{6} \epsilon_{abc} (e^a \wedge e^b \wedge e^c) \\ + i\psi^{(\frac{1}{2})} \gamma_0 \wedge \left(d - \frac{1}{2} (e^a + \omega^a) \gamma_a \right) \wedge \psi^{(\frac{1}{2})} \quad (4.39)$$

where the part consisting only of the vielbein and spin-connection one-forms is the usual Einstein-Hilbert action with a negative cosmological constant, and the fields $\psi^{(\frac{1}{2})}$ are real.

⁶We take the cosmological constant to be unity, and henceforth, rescaled all the fermionic fields, including the gauge parameters ϵ and $\bar{\epsilon}$ by a factor of $\sqrt{\frac{3}{2}}$.

This, including the factor of i , is the action for $osp(1|2) \oplus sl(2)$ Chern-Simons supergravity [20]. From (4.38) and (4.33), we obtain⁷

$$\delta e^a = \frac{1}{2} \epsilon \gamma_0 \gamma^a \psi^{(\frac{1}{2})} \quad (4.40)$$

$$\delta \psi^{(\frac{1}{2})} = \left(d - \frac{1}{2} (e^a + \omega^a) \gamma_a \right) \epsilon \quad (4.41)$$

where we have also taken $\bar{\epsilon} = -i\epsilon$. We checked that (4.40) agrees with the supersymmetry transformation laws for $osp(1|2)$ Chern-Simons supergravity as stated in the literature. Similarly, we can consider the $osp(2|2) \oplus sl(2)$ Chern-Simons theory where the $u(1)$ gauge field \mathcal{U} plays a role. Instead of (4.37), we set

$$\mathcal{W} = \Upsilon = \psi^{(\frac{3}{2})} = \bar{\psi}^{(\frac{3}{2})} = 0, \quad (4.42)$$

$$\mathcal{U} = i6\mathcal{B}, \quad \bar{\psi}^{(\frac{1}{2})} = i\psi^{(\frac{1}{2})\dagger}, \quad (4.43)$$

which is motivated by constraining the fields conjugate to $R_{1,2}^\pm$ and J_{12} to be real. This leads to the action

$$\begin{aligned} S = & \frac{1}{8\pi G} \int d^3x e^a \wedge \left(d\omega_a + \frac{1}{2} \epsilon_{abc} \omega^b \wedge \omega^c \right) + \frac{1}{6} \epsilon_{abc} (e^a \wedge e^b \wedge e^c) \\ & - i\psi^{(\frac{1}{2})\dagger} \gamma_0 \wedge \left(d - \frac{1}{2} (e^a + \omega^a) \gamma_a + i\mathcal{B} \right) \wedge \psi^{(\frac{1}{2})} + 2\mathcal{B} \wedge d\mathcal{B}, \end{aligned} \quad (4.44)$$

where the fermionic fields are now complex, and thus possess twice the degrees of freedom as in the $osp(1|2)$ case. We checked that this is indeed the action for $osp(2|2) \oplus sl(2)$ Chern-Simons supergravity [79] as stated in the literature. From (4.43) and (4.33), we obtain the supersymmetry transformation laws to be

$$\delta e^a = -\frac{i}{4} \left(\epsilon^\dagger \gamma_0 \gamma^a \psi^{(\frac{1}{2})} + \psi^{(\frac{1}{2})\dagger} \gamma_0 \gamma^a \epsilon \right) \quad (4.45)$$

$$\delta \psi^{(\frac{1}{2})} = \left(d - \frac{1}{2} (e^a + \omega^a) \gamma_a + i\mathcal{B} \right) \epsilon \quad (4.46)$$

$$\delta \mathcal{B} = \frac{1}{4} \left(\epsilon^\dagger \gamma_0 \psi^{(\frac{1}{2})} + \psi^{(\frac{1}{2})\dagger} \gamma_0 \epsilon \right). \quad (4.47)$$

We checked that (4.45) is indeed the transformation laws for $osp(2|2)$ Chern-Simons supergravity as stated in the literature [79]. We now proceed to derive the supersymmetry transformation laws for $sl(3|2)$ Chern-Simons supergravity theory, based on (4.35) and (4.33). The results of this subsection vindicated the consistency of such an approach when the fields are truncated to yield $osp(1|2)$ and $osp(2|2)$ theories.

We should remark that the equations for $\psi^{(\frac{3}{2})}$ in (4.35) cannot be derived by seeking

⁷Since we are supersymmetrizing only a copy of Chern-Simons theory, $\delta e = \frac{1}{2} \delta A$. The spin-connection one-forms are treated as auxiliary forms.

consistent truncations to the $osp(N|2)$ cases since in the first place, they are absent in these theories. But, as we shall observe next, this relation can be easily motivated by the form of some of the equations that we obtain from (4.33). Finally, let us summarize our approach in the following. One starts with $sl(3|2; \mathbb{R})$ superalgebra in deriving the Chern-Simons action, but then, insert factors of ‘ i ’ into appropriate parts of the Lagrangian by hand to ensure reality. Starting with (4.35) can be viewed as a way to keep track of these changes. The results of this subsection showed that in each sector, when the multiplet $(\Upsilon, \mathcal{W}, \psi^{(\frac{3}{2})})$ is truncated, this procedure reduces the $sl(3|2)$ theory to the $osp(2|2)$ Chern-Simons theory.

4.3.5 $\mathcal{N} = 2$ supersymmetry and Killing spinors

Upon substituting (4.30) and (4.35) into (4.33), excluding the equations for $\psi^{(\frac{3}{2})}$ in (4.35), and inserting factors of ‘ i ’ in the generators U_0 and \mathcal{W}^m , we find the following supersymmetry transformation laws

$$\delta\psi^{(\frac{1}{2})} = \left(d + i\mathcal{U} - \frac{1}{2} \left((e^a + \omega^a) + \frac{5}{3}\Upsilon^a \right) \gamma_a \right) \epsilon \quad (4.48)$$

$$\delta\bar{\psi}^{(\frac{1}{2})} = \left(d - i\mathcal{U} - \frac{1}{2} \left((e^a + \omega^a) + \frac{5}{3}\Upsilon^a \right) \gamma_a \right) \bar{\epsilon} \quad (4.49)$$

$$\delta\psi^{(3/2)} = (i\mathcal{W}^m \eta_m + \Upsilon^a \lambda_a) \epsilon \quad (4.50)$$

$$\delta\bar{\psi}^{(3/2)} = (i\mathcal{W}^m \eta_m - \Upsilon^a \lambda_a) \bar{\epsilon} \quad (4.51)$$

$$\delta\mathcal{U} = \frac{1}{4} \left(\epsilon^\dagger \gamma_0 \psi^{(\frac{1}{2})} + \psi^{(\frac{1}{2})\dagger} \gamma_0 \epsilon \right) \quad (4.52)$$

$$\delta\mathcal{W}^m = \frac{i}{\sqrt{8}} \left(\bar{\psi}^{(\frac{3}{2})} \alpha^m \epsilon + \psi^{(\frac{3}{2})} \alpha^m \bar{\epsilon} \right) \quad (4.53)$$

$$\delta e^a = -\frac{5}{6} (\delta\Upsilon^a) - \frac{i}{4} \left(\epsilon^\dagger \gamma_0 \gamma^a \psi^{(\frac{1}{2})} + \psi^{(\frac{1}{2})\dagger} \gamma_0 \gamma^a \epsilon \right) \quad (4.54)$$

$$\delta\Upsilon^a = \bar{\psi}^{(\frac{3}{2})} \beta^a \epsilon - \psi^{(\frac{3}{2})} \beta^a \bar{\epsilon}, \quad (4.55)$$

where the field \mathcal{U} is multiplied by a factor of 6, and the field $\psi^{(\frac{1}{2})}$ by a factor of $\sqrt{\frac{3}{2}}$ as in Section 4.3.4 for convenience. The non-vanishing elements of the 4×2 matrices $\beta, \eta, \lambda, \alpha$ are

$$\alpha_{(\frac{1}{2}, -\frac{1}{2})}^0 = \alpha_{(-\frac{1}{2}, \frac{1}{2})}^0 = -\alpha_{(\frac{1}{2}, \frac{1}{2})}^{-1} = -\alpha_{(-\frac{1}{2}, -\frac{1}{2})}^1 = 1.$$

$$\alpha_{(\frac{3}{2}, \frac{1}{2})}^{-2} = \alpha_{(-\frac{3}{2}, -\frac{1}{2})}^2 = -\alpha_{(\frac{3}{2}, -\frac{1}{2})}^{-1} = -\alpha_{(-\frac{3}{2}, \frac{1}{2})}^1 = \sqrt{\frac{1}{3}}.$$

$$\lambda_{(\frac{1}{2}, \frac{1}{2})}^2 = \lambda_{(-\frac{1}{2}, -\frac{1}{2})}^2 = -\frac{2\sqrt{2}}{3}, \quad -\lambda_{(\frac{3}{2}, \frac{1}{2})}^1 = \lambda_{(-\frac{3}{2}, -\frac{1}{2})}^1 = \sqrt{\frac{2}{3}}, \quad -\lambda_{(\frac{1}{2}, -\frac{1}{2})}^1 = \lambda_{(-\frac{1}{2}, \frac{1}{2})}^1 = \frac{\sqrt{2}}{3}.$$

$$\lambda_{\left(\frac{3}{2}, \frac{1}{2}\right)}^0 = \lambda_{\left(-\frac{3}{2}, -\frac{1}{2}\right)}^0 = \sqrt{\frac{2}{3}}, \quad \lambda_{\left(\frac{1}{2}, -\frac{1}{2}\right)}^0 = \lambda_{\left(-\frac{1}{2}, \frac{1}{2}\right)}^0 = \frac{\sqrt{2}}{3}.$$

$$\eta_m = \frac{1}{\sqrt{8}} \tilde{\gamma}_0 \operatorname{str}(W_m W_n) \alpha^n, \quad \beta_a = -i \frac{3}{8} \tilde{\gamma}_0 \lambda^a.$$

Equation (4.51) suggests we should look for a reality condition relating $\bar{\psi}^{(\frac{3}{2})}$ and $\psi^{(\frac{3}{2})}$ since $\bar{\epsilon} = i\epsilon^\dagger$. A simple observation tells us that imposing

$$\bar{\psi}_r^{(\frac{3}{2})} = -i\psi_r^{(\frac{3}{2})\dagger} \quad (4.56)$$

throughout is consistent with (4.50) and (4.51). This then completes the derivation of (4.35). We should recall again that this analytic continuation is accompanied by inserting ‘ i ’ in the generators U_0 and W^m . Now, the Killing spinor equations can then be directly read off from (4.48)–(4.51) to be

$$\left(d + i\mathcal{U} - \frac{1}{2} \left((e^a + \omega^a) + \frac{5}{3} \Upsilon^a \right) \gamma_a \right) \epsilon = 0, \quad (4.57)$$

$$(i\mathcal{W}^m \eta_m + \Upsilon^a \lambda_a) \epsilon = 0. \quad (4.58)$$

As we shall observe later, if we demand all fields to be real-valued, (4.58) presents a rather stringent condition on the higher-spin fields that are allowed for a non-vanishing two-component complex spinor ϵ . These results apply equally to the anti-chiral sector.

4.3.6 Solving the Killing spinor equations

As a warm-up, we begin with an ansatz for a class of flat connections with the vielbein and spin connection as the only non-vanishing fields,

$$\Gamma = (e^\rho L_1 - \mathcal{L} e^{-\rho} L_{-1}) dx^+ + L_0 d\rho \quad (4.59)$$

$$\tilde{\Gamma} = - \left(e^\rho L_{-1} - \tilde{\mathcal{L}} e^{-\rho} L_1 \right) dx^- - L_0 d\rho. \quad (4.60)$$

In ordinary gravity where the gauge group is $sl(2) \oplus sl(2)$, the parameters $\mathcal{L}, \tilde{\mathcal{L}}$ are related to the ADM mass M and angular momentum J via the following equations

$$\mathcal{L} = \frac{M - J}{2k}, \quad \tilde{\mathcal{L}} = \frac{M + J}{2k}. \quad (4.61)$$

In particular, global AdS_3 corresponds to taking $\mathcal{L} = \tilde{\mathcal{L}} = -\frac{1}{4}$. Extremal black holes with positive J correspond to taking $\mathcal{L} = 0, \tilde{\mathcal{L}} > 0$, while those with negative J correspond to taking $\tilde{\mathcal{L}} = 0, \mathcal{L} > 0$.

Consider the copy of Chern-Simons theory parametrized by flat connections Γ . The

corresponding Killing spinor equations read

$$(\partial_+ - e^\rho \gamma_+ + \mathcal{L} e^{-\rho} \gamma_-) \epsilon = 0, \quad \left(\partial_\rho - \frac{1}{2} \gamma_2 \right) \epsilon = 0, \quad \partial_- \epsilon = 0. \quad (4.62)$$

where we have defined

$$\gamma_\pm \equiv \frac{1}{2} (\gamma_0 \pm \gamma_1). \quad (4.63)$$

If we let the spinor ϵ to be variable separable in ρ, x^\pm , then (4.62) gives us

$$\begin{pmatrix} e^{-\frac{\rho}{2}} \partial_+ K(x^+) \\ e^{\frac{\rho}{2}} K(x^+) \end{pmatrix}, \quad \partial_+^2 K = \mathcal{L} K. \quad (4.64)$$

This implies the following classification: for $\mathcal{L} > 0$, since K is a sum of hyperbolic functions which are not periodic in x^+ , there is no admissible Killing spinors. For $\mathcal{L} = 0$, $K \sim c_1 x^+ + c_2$ where $c_{1,2}$ are constants. We have to set $c_1 = 0$ to preserve the periodicity, and hence the Killing spinor preserved is of the form

$$\epsilon_{\mathcal{L}=0} \sim \begin{pmatrix} 0 \\ e^{\frac{\rho}{2}} \end{pmatrix}. \quad (4.65)$$

Finally, for negative \mathcal{L} , we have

$$K = c_1 \sin(\sqrt{|\mathcal{L}|} x^+) + c_2 \cos(\sqrt{|\mathcal{L}|} x^+). \quad (4.66)$$

These are periodic for $|\mathcal{L}| = N^2, N \in \mathbb{Z}^+$ and anti-periodic if $|\mathcal{L}| = \left(\frac{2N+1}{2}\right)^2$. In particular, if we restrict ourselves to $\mathcal{L} \geq -\frac{1}{4}$, then the only supersymmetric case corresponds to global AdS_3 which will have two linearly independent ϵ 's. Identical results hold for $\bar{\epsilon}$, and thus the number of real Killing spinors preserved in each case is to be doubled.

We can also supersymmetrize the other copy of Chern-Simons theory, and obtain similar results depending on the sign of $\bar{\mathcal{L}}$. We found that substituting (4.60) into (4.57) amounts to switching

$$x^\pm \rightarrow x^\mp, \quad \gamma_2 \rightarrow -\gamma_2, \quad \gamma_\pm \rightarrow -\gamma_\mp. \quad (4.67)$$

Instead of (4.65), the Killing spinors are

$$\begin{pmatrix} e^{\frac{\rho}{2}} \tilde{K}(x^-) \\ e^{-\frac{\rho}{2}} \partial_- \tilde{K}(x^-) \end{pmatrix}, \quad \partial_-^2 \tilde{K} = \tilde{\mathcal{L}} \tilde{K}. \quad (4.68)$$

This implies that in the $\mathcal{N} = (2, 2)$ theory based on $sl(3|2) \oplus sl(3|2)$, global AdS_3 ($\mathcal{L} = \tilde{\mathcal{L}} = -\frac{1}{4}$) preserves 8 real supercharges, the massless BTZ preserves 4, and extremal black holes with non-zero angular momentum preserve 2. The generic BTZ with $J \neq M$ will break all supersymmetries. These results agree with those belonging to the case of $osp(2|2) \oplus osp(2|2)$ Chern-Simons supergravity theories [80].

To be more general, given any x^\pm -component of the gauge connection Γ , it is straightforward to solve for the form of any admissible Killing spinors as:

$$\epsilon = e^{(-i\mathcal{U}+m_0)x} \left[\alpha e^{A_+x} \begin{pmatrix} A_+ \\ m_+ \end{pmatrix} + \beta e^{A_-x} \begin{pmatrix} A_- \\ m_+ \end{pmatrix} \right] \quad (4.69)$$

$$A_\pm = -m_+m_0 \pm \sqrt{m_+^2m_0^2 + m_+m_-}, \quad m_i \equiv \left(e^i + \omega^i + \frac{5}{3}\Upsilon^i \right), \quad x \equiv x^\pm, \quad (4.70)$$

with α, β being arbitrary complex constants. The remaining spinor equation (4.58) constrains the Killing spinors to lie within the null-space of the following matrix of one-forms \mathcal{W}^m and Υ^m :

$$\begin{pmatrix} -i\mathcal{W}^{-1} + 2\Upsilon^{-1} & -4i\mathcal{W}^{-2} \\ i\mathcal{W}^0 - \Upsilon^0 & \frac{3}{2}i\mathcal{W}^{-1} + \Upsilon^{-1} \\ -\frac{3}{2}i\mathcal{W}^1 + \Upsilon^1 & -i\mathcal{W}^0 - \Upsilon^0 \\ 4i\mathcal{W}^2 & i\mathcal{W}^1 + 2\Upsilon^1 \end{pmatrix}. \quad (4.71)$$

This turns out to be highly restrictive on the fields \mathcal{W} and Υ in the gauge connection, apart from the periodicity conditions that one should further impose on the spinors in (4.69).

4.3.7 On the $u(1)$ gauge field

As mentioned earlier, it is well-known that $\mathcal{N} = 2$ superconformal algebra admits an automorphism, with the spectral flow generated by the zero mode of the $u(1)$ field. We expect to find this automorphism symmetry in the bulk Chern-Simons theory. Let us first consider the chiral sector with a generic gauge connection Γ that contains only even-graded generators. It is easy to observe that a constant shift of the $u(1)$ field component in the x^+ direction

$$\mathcal{U} \rightarrow \mathcal{U} + \delta\mathcal{U}$$

can be realized by performing a large gauge transformation

$$\Gamma \rightarrow \Gamma + e^{-i\delta\mathcal{U}U_0x^+} de^{i\delta\mathcal{U}U_0x^+}, \quad (4.72)$$

since U_0 commutes with all the even-graded generators. Also, as observed earlier, any Killing spinor attains a phase factor

$$\epsilon \rightarrow e^{-i\delta\mathcal{U}x^+} \epsilon. \quad (4.73)$$

Similarly, for the anti-chiral sector, we have

$$\tilde{\Gamma} \rightarrow \tilde{\Gamma} + e^{i\delta\tilde{\mathcal{U}}U_0x^-} de^{-i\delta\tilde{\mathcal{U}}U_0x^-}, \quad \tilde{\epsilon} \rightarrow e^{i\delta\tilde{\mathcal{U}}x^-} \tilde{\epsilon}. \quad (4.74)$$

From (6.53), we can compute the changes to the energy-momentum tensor

$$T_{++} \rightarrow T_{++} + 2\delta\mathcal{U} + \delta\mathcal{U}^2, \quad \tilde{T}_{--} \rightarrow \tilde{T}_{--} + 2\delta\tilde{\mathcal{U}} + \delta\tilde{\mathcal{U}}^2. \quad (4.75)$$

Starting from any generic solution with zero $u(1)$ charge, one can generate a family of solutions. For supersymmetric ones which preserve Killing spinors, the $u(1)$ charge is thus quantized as, in our choice of normalization,

$$\mathcal{U} \in \mathbb{Z}/2, \quad \tilde{\mathcal{U}} \in \mathbb{Z}/2. \quad (4.76)$$

It turns out that this quantization is also consistent with the requirement of a smooth holonomy. As an explicit example, consider the ansatz (4.59) and (4.60), and recall from earlier discussion that for supersymmetry to be preserved, we require $\mathcal{L} \leq 0$. This implies the existence of a family of supersymmetric solutions satisfying

$$\sqrt{|\mathcal{L}|} + \mathcal{U} \in \mathbb{Z}/2, \quad \sqrt{|\tilde{\mathcal{L}}|} + \tilde{\mathcal{U}} \in \mathbb{Z}/2. \quad (4.77)$$

It is straightforward to check that imposing a trivial holonomy along the ϕ -direction for this class of solutions yields both (4.77) and (4.76). Also, generalizing (4.61), the physical charges of mass (M) and angular momentum (J) read

$$M = k \left(\mathcal{L} + \tilde{\mathcal{L}} + \mathcal{U}^2 + \tilde{\mathcal{U}}^2 \right) \quad (4.78)$$

$$J = k \left(-\mathcal{L} + \tilde{\mathcal{L}} - \mathcal{U}^2 + \tilde{\mathcal{U}}^2 \right). \quad (4.79)$$

In supergravity theories that arise from type IIB string theory compactified on some internal space, the conical defect spacetimes in ordinary 3d gravity, with masses interpolating between AdS_3 and the massless BTZ (i.e. in our notation, $-\frac{1}{4} < \mathcal{L} < 0$), can be embedded as solutions and made supersymmetric by turning on the $U(1)$ charge [26, 81, 82]. Here, they are also supersymmetric solutions in the higher-spin supergravity theories, but we note that the trivial holonomy condition will be lost. The solutions in (4.77) correspond to conical surpluses instead, with $\mathcal{L} < -\frac{1}{4}$. We will discuss more about conical defect spacetimes later in Section 7.6.3.

CHAPTER 5

Notes on the boundary CFT

This is a rather short chapter that furnishes a brief summary of some important points about the coset CFTs (see [45, 46] for an excellent introduction). We begin with a review of the structure of Hilbert space of the coset minimal models, and then in Section 5.2, we discuss the two different limits of this family of CFTs which are of direct relevance to many results presented in this thesis.

5.1 A review of important points about the coset models

The family of boundary theories that are conjectured to be dual to the Vasiliev theory in the bulk are coset CFTs with the following Lie-algebraic symmetry

$$\frac{SU(N)_k \oplus SU(N)_1}{SU(N)_{k+1}} \quad (5.1)$$

This is a special case of the general G/H coset construction. In our case, we are considering a WZW theory based on the algebra $SU(N) \oplus SU(N)$ and gauge the diagonal subalgebra $H = SU(N)$. More generally, for diagonal coset models of the form $(g_1 \oplus g_2)/g$, the generators of the diagonal are simply the sum of the generators of each copy of g . We can write

$$J_{diag}^a = J_{(1)}^a + J_{(2)}^a \quad (5.2)$$

with $[J_{(1)}^a, J_{(2)}^b] = 0$. The level of the diagonal algebra is the sum of the other two. The stress-energy tensor reads $T_{coset} = T_{(g_1 \oplus g_2)} - T_g$, with the Virasoro modes thus satisfying

$$L_m^{coset} = L_m^{(g_1 \oplus g_2)} - L_m^g \quad (5.3)$$

leading to the commutation relation

$$[L_m^{coset}, L_n^{coset}] = (m - n)L_{m+n}^{coset} - (c^{(g_1 \oplus g_2)} - c^g) \frac{m^3 - m}{12} \delta_{m+n,0}. \quad (5.4)$$

which is the Virasoro algebra with the central charge

$$c = \dim(g) \left(\frac{k_1}{k_1 + \mathfrak{g}} + \frac{k_2}{k_2 + \mathfrak{g}} - \frac{k_1 + k_2}{k_1 + k_2 + \mathfrak{g}} \right) \quad (5.5)$$

where we have defined \mathfrak{g} to be the dual Coxeter of g . Since for our case (5.1), $\dim(g) = N-1$, $\mathfrak{g} = N$, $k_1 = k$, $k_2 = 1$, from (5.5), it is straightforward to derive the central charge to be

$$c = (N-1) \left[1 - \frac{N(N+1)}{(N+k)(N+k+1)} \right] \quad (5.6)$$

From (5.6), it is easy to see that it is bounded from the above by $N-1$. Some comments on special values of various parameters: (i) for $N=2$, this family of cosets forms the well-studied unitary series of the Virasoro minimal models with $c = 1 - 6/((k+2)(k+3))$ (ii) for general N , the coset theory with the minimum $k=1$ has central charge $c = 2(N-1)/(N+2)$ can be realized by parafermions. (iii) for a finite N but infinite k , $c = N-1$ and the algebra is the Casimir algebra of the $SU(N)$ affine algebra at level one and in this limit, the CFT simplifies to a free boson theory.

Let us now briefly discuss the structure of the Hilbert space. It can be understood primarily via the decomposition ($G \equiv SU(N)_k \oplus SU(N)_1$, $H = SU(N)_{k+1}$)

$$\mathcal{H}_G^{(\Lambda)} \bigoplus_{\Lambda'} \left(\mathcal{H}_{G/H}^{(\Lambda, \Lambda')} \otimes \mathcal{H}_H^{(\Lambda')} \right) \quad (5.7)$$

where Λ, Λ' are the highest weight representations. They are related by a simple selection rule as follows. We can parametrize Λ as

$$\Lambda = \rho \oplus \mu \quad (5.8)$$

where ρ is a highest weight representation of $SU(N)_k$, μ is a highest weight representation of $SU(N)_1$ and letting $\Lambda' = \nu$ be a highest weight representation of $SU(N)_{k+1}$, the selection rule reads

$$\rho + \mu - \nu \in \Lambda_{root} \quad (5.9)$$

where omitting the affine Dynkin label for the moment, we can treat ρ, μ, ν as weights of the finite-dimensional algebra $SU(N)$ with the root lattice Λ_{root} . To be more precise, we need to add the affine Dynkin label. We can write the highest weight representations of each of the affine algebra as

$$\left\{ [l_0; l_1, \dots, l_{N-1}] : l_j \in \mathbb{Z}_0^+, \sum_{j=0}^{N-1} l_j = k \right\}$$

Since we have the selection rule, the highest weight representation of the coset theory can be labelled by simply $(\rho; \nu)$. They are Young diagrams of at most N rows, and at most k and $k+1$ columns respectively. Note that from (5.9), μ can be uniquely determined by

ρ and ν . Further, from a Drinfeld-Sokolov description of the coset CFTs, the conformal weights can be computed to be

$$h(\rho, \nu) = \frac{1}{2(N+k+1)(N+k+2)} (|(N+k+1)(\rho + \hat{w}) - (N+k)(\nu + \hat{w})|^2 - \hat{w}^2) \quad (5.10)$$

where we have denoted \hat{w} as the Weyl vector of $SU(N)$. A convenient representation is to regard Λ as a N -dimensional vector with components

$$\Lambda_i = r_i - \frac{B}{N}, \quad i = 1, 2, \dots, N \quad (5.11)$$

where B stands for the number of boxes in the Young diagram, and r_i denote the number of Young diagram boxes in the i th row. In this notation, the Weyl vector reads

$$\rho_i = \frac{N+1}{2} - i \quad (5.12)$$

As a simple example, let us consider $N = 2$, and denote the number of boxes in ρ be α and that in ν be β . Then, we have

$$h(\alpha, \beta) = \frac{1}{4(k+2)(k+3)} (((k+3)(\alpha+1) - (k+2)(\beta+1))^2 - 1) \quad (5.13)$$

where the constraints on the sizes of the Young Tableau read $1 \leq \alpha \leq k, 1 \leq \beta \leq k+1$. For the case of ρ being trivial and ν being the fundamental representation (and thus so is μ), we have

$$h(0; \square) = \frac{N-1}{2N} \left(1 - \frac{N+1}{N+k+1} \right) \quad (5.14)$$

whereas for the case of ρ being the fundamental representation, ν being trivial (and thus μ is the anti-fundamental representation), we have

$$h(\square; 0) = \frac{N-1}{2N} \left(1 + \frac{N+1}{N+k+1} \right) \quad (5.15)$$

These two cases will each play an important role in the higher-spin holography conjecture, as we shall discuss shortly. We should also mention that there are also field identifications to be made. By this, we mean taking the equivalence of the following two representations of the coset algebra.

$$(\rho, \mu; \nu) \sim (A\rho, A\mu; A\nu) \quad (5.16)$$

where $A = \mathbb{Z}_N$ is an outer automorphism of the affine $SU(N)_k$ and generated by cyclic permutations of the Dynkin labels, i.e. $[l_0; l_1, l_2, \dots, l_{N-1}] \rightarrow [l_1; l_2, \dots, l_{N-1}, l_0]$.

Next, let us discuss the higher-spin currents. We begin by considering the following

cubic combination of currents

$$W^{(3)}(z) = d_{abc} \left(c_1 (J_{(1)}^a J_{(1)}^b J_{(1)}^c)(z) + c_2 (J_{(2)}^a J_{(1)}^b J_{(1)}^c)(z) + c_3 (J_{(2)}^a J_{(2)}^b J_{(1)}^c)(z) \right. \\ \left. + c_4 (J_{(2)}^a J_{(2)}^b J_{(2)}^c)(z) \right) \quad (5.17)$$

where d_{abc} is the totally symmetric cubic invariant of $SU(N)$, the c_i are some parameters which one can set to ensure that W^3 has a regular OPE with $J_{(1)}^a + J_{(2)}^a$. This defines the chiral spin-3 current of weight 3 in the coset theory. For a useful record, let us write down the precise expression for the spin-3 current $W^{(3)}(z)$.

$$W^{(3)}(z) \sim \sum_{i+j=3} (-1)^j \prod_{p=i+1}^3 \left(k + (p-1) \frac{N}{2} \right) \prod_{q=j+1}^3 \left(1 + (q-1) \frac{N}{2} \right) \tilde{T}^{(i,j)}(z), \\ \tilde{T}^{(i,j)}(z) \equiv \frac{1}{i!j!} d_{a_1 a_2 a_3} \left(J_{(1)}^{a_1} \dots J_{(1)}^{a_i} J_{(2)}^{a_{i+1}} \dots J_{(2)}^{a_3} \right)(z) \quad (5.18)$$

Similarly, since $SU(N)$ has independent invariant symmetric tensors for each rank $s \leq N$, which are just the independent Casimirs of $SU(N)$, we can construct similarly the other higher spin currents, each of spin $s \leq N$. This procedure actually works for general cosets, though when one of the levels is one, it can be shown that the OPEs of the higher-spin currents close among themselves. The other fields become null and decouple, leaving us with the \mathcal{W}_N algebras generated by the higher-spin currents $W^{(s)}(z)$. Another quantity which turns out to be useful are the higher-spin charges, i.e. the eigenvalue of the zero modes $W_0^{(s)}$. As a concrete example, the spin-3 charge $w^{(3)}$ of the primary in the (ρ, ν) representation can be computed to be

$$w^{(3)} = -\sqrt{\frac{2N(N^2-1)}{(N-2)(3N^2+(c-1)(N+2))}} \frac{1}{3} \sum_i (\theta_i)^3, \quad (5.19)$$

where θ is the vector

$$\theta = \sqrt{\frac{N+k+1}{N+k}} (\rho + \hat{w}) - \sqrt{\frac{N+k}{N+k+1}} (\nu + \hat{w}) \quad (5.20)$$

5.2 Two different limits for the holography conjecture

5.2.1 The 't Hooft limit

For AdS/CFT to work, one generally requires a large number of degrees of freedom in the boundary CFT to recover a classical gravitational theory in the bulk, for instance in a large

N vector-like or gauge-like model, where N is the rank of some underlying gauge group.¹

When the higher-spin holography conjecture was first proposed by Gopakumar and Gaberdiel, the following 't Hooft limit of the coset minimal model was first considered.

$$N, k \rightarrow \infty, \quad (5.21)$$

while keeping the parameter

$$\lambda = \frac{N}{N+k} \quad (5.22)$$

fixed. In this limit, it is straightforward to check that the central charge behaves as

$$c = N(1 - \lambda^2). \quad (5.23)$$

Hence, in this 't Hooft-like limit, the coset minimal model has the degrees of freedom scaling in the same way as a vector-like theory. Due to modular invariance, one expects the Hilbert space of this coset model to consist not merely of the vacuum representation (where both ρ and ν are trivial). All admissible non-trivial representations can however be regarded as being built up from tensor products of two basic representations and their complex conjugates, i.e. (ρ, ν) are direct products of the following four elements.

$$(\square; 0), \quad (\square; 0), \quad (0; \square), \quad (0; \square) \quad (5.24)$$

Thus, it is natural to consider adding to the bulk higher-spin theory a scalar field of suitable mass for the correspondence to hold. Unlike in the case of four-dimensions, the scalar field is not part of the higher-spin multiplet and can be added by hand. Denote the complex scalar field by Φ , and its mass by M . From our understanding of *AdS/CFT*, the relation between mass and conformal dimension (Δ) in three dimensions reads

$$M^2 = \Delta(\Delta - 2) \quad (5.25)$$

Taking the 't Hooft limit of (5.14) and (5.15), the L_0 weights of the representations $(\square; 0), (0; \square)$ read

$$h(\square; 0) \equiv h_+ = \frac{1}{2}(1 + \lambda), \quad h(0; \square) \equiv h_- = \frac{1}{2}(1 - \lambda). \quad (5.26)$$

and similarly for the anti-holomorphic sector. Thus, if we let $M^2 = \lambda^2 - 1$, then $\Delta = 1 \pm \lambda = (h_{\pm} + \bar{h}_{\pm})$. It is then a possibility that we can identify the scalar fields with the primary states as labelled as $(\square; 0)$ and $(0; \square)$. We note that since $0 \leq \lambda \leq 1$, there are two possible quantization conditions² for the scalar fields which we will label by Φ_+, Φ_- each corresponding to the two distinct values of Δ and thus h, \bar{h} . It was first proposed that one

¹The number of degrees of freedom of the vector-like and gauge-like model would scale as N , and N^2 respectively.

²This stems from two possible boundary conditions for Φ .

could match

$$\begin{aligned} (\square; 0)_{CFT} &\leftrightarrow \Phi_+ \text{ (Bulk Scalar with standard quantization)} \\ (0; \square)_{CFT} &\leftrightarrow \Phi_- \text{ (Bulk Scalar with alternate quantization)} \end{aligned} \quad (5.27)$$

One of the first checks of the above identification is the computation of the one-loop thermal partition function. For the bulk gravitational theory, this includes the contributions of both the higher-spin fields (only the kinetic terms in the equations of motion are needed at one-loop) and the scalar field of either quantization (only the conformal dimensions enter into the computation since at one-loop, we can consider the non-interacting limit). For the boundary theory, one computes the vacuum character of the \mathcal{W} algebra (which corresponds to the higher-spin fields' contributions), and also other representations via the branching functions that correspond to either representation (and which corresponds to the bulk scalar fields' perturbative spectrum). The partition functions were computed to agree on both sides, and can be written as

$$Z^{pert} = (q\bar{q})^{-c/24} \left| \prod_{s=2} \prod_{n=2} |1 - q^n|^{-2} \right|^2 \left| \prod_{j,j'=0}^{\infty} \frac{1}{1 - q^{h+j}\bar{q}^{h+j'}} \right|^2 \quad (5.28)$$

where $q = e^{2i\pi\tau}$, and we have noted that this partition function captures the dynamics of perturbative fluctuations about the vacuum. To relate this result to the complete CFT partition function, one needs to ensure that the null states that appear in the 't Hooft limit are decoupled and can be taken out of the sum. In the $N \rightarrow \infty$ limit, one can show that indeed, they are. But this turns out not to be true for any finite N . The so-called light states are of the form $(\rho, \nu) = (\lambda, \lambda)$ for some non-trivial representation λ . For these states, the conformal dimension read (from (5.10))

$$h(\Lambda, \Lambda) = \frac{C_2(\Lambda)}{(N+k)(N+k+1)} \quad (5.29)$$

where $C_2(\Lambda)$ is the eigenvalue of the quadratic Casimir in the Λ representation. In the 't Hooft limit (yet one has to take strictly $N \rightarrow \infty$), these dimensions vanish thus forming a continuum of states near vacuum. At finite N , these states do not decouple [29], and one must work in a different limit to understand the physical meaning of these states, or at least what they correspond to in the bulk gravity theory. This limit involves sending the central charge to infinity while fixing a possibly finite N .

Before we leave our discussion on the 't Hooft limit, let us mention briefly that the consistency of taking a large N limit also passes a couple of consistency tests on the matching of correlation functions. The three-point function involving two scalar primaries with one spin- s current $\langle \mathcal{O}\bar{\mathcal{O}}J^s \rangle$ was computed in the bulk [34] assuming that the bulk theory has $\mathcal{W}_\infty[\lambda]$ symmetry and the corresponds with the expected CFT result. This is relevant to the 't Hooft limit in the sense that such a limit of \mathcal{W}_N models have \mathcal{W}_∞ algebra. Apart from the above three-point functions, the four-point functions of primary fields built

from tensor products of $(\square; 0)$ and $(0; \square)$ factorize at large N , have a well-defined large N limit, and behaves as multi-particle states [35].

5.2.2 The semiclassical limit

Let us now treat the primary field $(0; \square)$ more carefully, and shortly we shall review a modification of the identification in (5.27). The limit that we will now consider is as follows.

$$c \rightarrow \infty, \quad N \text{ fixed.} \quad (5.30)$$

From the exact formula for the central charge, we can compute k to be

$$k = -N - \frac{1}{2} \pm \frac{1}{2} \left[1 - \frac{4N(N^2 - 1)}{c + 1 - N} \right]^{1/2} \quad (5.31)$$

In the following we shall adopt the negative branch in (5.31). In the limit of (5.30), we find

$$k \approx -N - 1 + \frac{N(N^2 - 1)}{c} + \mathcal{O}\left(\frac{1}{c^2}\right) \quad (5.32)$$

Note that in this limit, $\lambda \rightarrow -N$.³ This implies that the bulk theory undergoes an analytic continuation in λ . Recall that for $\text{hs}[\lambda]$, an ideal forms when $\lambda = -N$, and upon factoring out this ideal, we obtain higher-spin gravity theory equipped with $sl(N)$ Lie algebra. Later, we will match the states with objects in this bulk theory. But first, let us remark the important fact that in this limit, the conformal dimensions for the primaries (apart from the trivial representation) become negative and the coset CFT is non-unitary. In particular, let us revisit the conformal dimensions of the fields $(\square; 0)$, $(0; \square)$, (Λ, Λ) which in this limit read

$$h(\square; 0) \approx -\frac{N-1}{2}, \quad h(0; \square) \approx -\frac{c}{2N^2}, \quad h(\Lambda; \Lambda) \approx -\frac{c}{N(N^2-1)} C_2(\Lambda) \quad (5.33)$$

Hence we see that since the conformal dimension of $(0; \square)$ scales as c in this limit, the large number of degrees of freedom suggest strongly that we should identify this state with some non-perturbative objects in the bulk gravity theory (instead of a perturbative scalar field of alternate quantization in the 't Hooft limit). These objects [31] turn out to be generalizations of conical defects in ordinary three-dimensional gravity. They are generalizations because although one can find a gauge in which the metric has explicitly conical defects/surpluses, the metric is not gauge invariant, and one uses the notion of holonomies to characterize these classical solutions.

There is a one-to-one correspondence between these spacetimes and $sl(N)$ Young tableau, which makes it easy to match between boundary and bulk states. Essentially, recall that the Chern-Simons connection associated with each conical defect spacetime can be expressed

³Choosing the other branch gives us an infinite λ .

as

$$A = e^{-\rho L_0} a(z) e^{\rho L_0} dz + e^{-\rho L_0} d(e^{\rho L_0}), \quad (5.34)$$

where

$$a(z) = L_+ + \sum_{s=2}^N N_s W_0^s V_{1-s}^s \quad (5.35)$$

where W_0^s are the spin- s charges and N_s are some normalization constants. Imposing a trivial holonomy along the ϕ -circle, trivial in the sense of $sl(N, \mathbb{Z})$ and thus $e^{\oint ad\phi} = e^{\frac{2\pi im}{N}} \neq 1$, where m is some integer, we find that $a(z)$ has the eigenvalues $-in_j$ which turn out to read

$$n_j = r_j - \frac{\sum_k r_k}{N} + \frac{N+1}{2} - j, \quad r_1 \geq r_2 \geq r_3 \dots \geq r_N = 0 \quad (5.36)$$

where r_j , $j = 1, 2, \dots, N-1$ are positive integers, and we have imposed the connection to satisfy the condition that at infinity we have AdS_3 geometry as the induced metric. We observe that we can take r_j to denote the number of boxes in the j th row of a Young diagram, and that (5.36) then reads as

$$n_j = \Lambda_j + \hat{w}_j \quad (5.37)$$

where \hat{w} is the Weyl vector as before. Equation (5.37) spells out the matching between the bulk and boundary states $(0, \Lambda)$ explicitly. One may wonder if this matching is unique. Given a weight Λ , apart from $(0, \Lambda)$, one can also consider (Λ, Λ) . To probe this issue, let us first consider the conformal dimension in the semi-classical limit of the general case of (Λ^+, Λ^-) . Denoting $n^\pm = \Lambda^\pm + \hat{w}$, it reads

$$h(\Lambda^+, \Lambda^-) = c \left[-\frac{C_2(n)}{N(N_1^2)} + \frac{1}{24} \right] + \left[\frac{C_2(n^-)}{N(N+1)} - \frac{N-1}{24} + \sum_i \Lambda_i^- n_i^- - \sum_i \Lambda_i^+ n_i^- \right] + \mathcal{O}(1/c) \quad (5.38)$$

where we have included the zeroth order piece as well, and ignored higher orders in $1/c$. Thus, we see that the leading order term is independent of Λ^+ . If we write (5.38) more suggestively as

$$h(\Lambda^+, \Lambda^-) = h(0, \Lambda^-) - \sum_i \Lambda_i^+ \Lambda_i^- + (\mathcal{O})(1/c) \quad (5.39)$$

and observe that the inner product in (5.39) is always positive, the energy of the state is mainly due to $(0, \Lambda^-)$ and there is a continuum of states of energies below that of $(0, \Lambda^-)$ which one may interpret as bound states between multi-particle states $(\Lambda^+, 0)$ of the scalar field $\Phi_+ \sim (\square, 0)$ and the non-perturbative state $(0, \Lambda^-)$. Indeed, a similar statement holds for the higher-spin charges which read

$$w^s(\Lambda^+, \Lambda^-) = c f_{\text{leading}}(N, s, \Lambda^-) + f_{\text{sub-leading}}(N, s, \Lambda^+, \Lambda^-) + \mathcal{O}(1/c) \quad (5.40)$$

where f_{leading} and $f_{\text{sub-leading}}$ are some functions. An analysis of $f_{\text{sub-leading}}$ reveals that they originate from zero-mode eigenvalues of scalar fluctuations about the conical surplus

spacetime identified to be $(0, \Lambda^-)$. Also, the null states of $(0, \Lambda^-)$ were found to furnish a non-unitary representation of the symmetries of the conical surplus' backgrounds. These are supporting evidences of how to interpret $(0, \square)$, $(0, \Lambda^-)$ in the higher-spin holography. In general then, (Λ^+, Λ^-) can be interpreted as the bound state between some multi-particle excited states of the perturbative scalar fields (represented by $(\Lambda^+, 0)$), and a conical surplus background (non-perturbative) represented by $(0, \Lambda^-)$. A preliminary computation of the bulk and boundary partition functions in this limit supports this picture [32] (in this case, the conical surplus background is the saddle point in the path integral). The caveat is that right from the outset, we are considering a non-unitary coset CFT.

CHAPTER 6

Asymptotic Spacetime Symmetries and \mathcal{W} algebras

In this chapter, we demonstrate that the asymptotic spacetime algebras of $SL(N)$ Chern-Simons theories are \mathcal{W} algebras, using the case of $SL(4)$ as an explicit example. This is essentially achieved by showing that bulk field equations with suitable boundary conditions yield Ward identities on the boundary CFT side, and then invoking Noether's theorem to read off the OPEs among the Virasoro primary fields of the \mathcal{W}_N algebras. This entire process is actually equivalent to performing a Drinfeld-Sokolov reduction of the affine $SL(N)$ algebras. We begin with a review of the relevant \mathcal{W} algebras following [45, 46], and then consider the $SL(4)$ case in Section 6.2. Finally, in Section 6.4, we consider the super-algebras $SL(N|N-1)$, and showed that the same procedure allows us to recover $\mathcal{N} = 2$ super-Virasoro algebra in each sector of the theory.

6.1 On \mathcal{W} algebras

\mathcal{W} algebras are extensions of Virasoro and Kac-Moody algebras with their defining feature being that they are non-linear. The non-linearity refers to the presence of polynomial terms in the OPEs (or equivalently, in the commutation relations among their Laurent modes) that depend on the fields and their derivatives. In the following, we will be mainly using a field-theoretic language to describe \mathcal{W} algebras. They are then defined as meromorphic CFTs of which Hilbert space consists of the entire space of fields/states (equivalent via the state-operator correspondence) are normal-ordered products of fields $W^{(s)}(z)$ and their derivatives. The fields $W^{(s)}(z)$ are quasi-primary fields, $s = 2$ being the energy-momentum tensor, and $s = 3, 4, \dots$ are generalized conserved currents which could be interpreted as higher-spin currents. They have integral conformal dimensions s . The OPEs among the fields could be determined by imposing the above definition and demanding crossing symmetry which is a consistency condition on four-point functions (or conformal blocks when the four-point functions are expressed in terms of the blocks). Nonetheless, such a direct construction turns out to be rather tedious except for a few \mathcal{W} algebras, and in physics applications, and more common constructions are the coset constructions and Drinfeld-Sokolov reduction.

We should mention that there is a large family of \mathcal{W} algebras, and very likely, many more remain to be discovered. In the present context, \mathcal{W} algebras constitute the symmetry

of the boundary CFT, and they are manifest in the asymptotic spacetime symmetry of the bulk gravitational theory. The class of \mathcal{W} algebras that we are specifically interested in is usually denoted as $\mathcal{W}[\mu]$ algebras. For each value of μ , the algebra is parametrized by the central charge. As we have seen earlier in our exposition of coset minimal models, for the coset CFT which contains N conserved currents $W^{(s)}$ (N can be infinite), the central charge is bounded by $N - 1$. We shall now turn to the cases where $\mu \in \mathbb{Z}_0^+$.

- (a) $\mu = 1$. For this value, the \mathcal{W} algebra is a Lie algebra because there are no non-linear terms arising in the OPEs of the fields! One can realize this algebra via free bosonic fields as follows. Consider M free and independent complex bosons $\phi^i, i = 1, 2, \dots, M$ satisfying the standard OPE relation

$$\partial\bar{\phi}^i(z)\partial\phi_j(0) \sim -\frac{\delta_j^i}{z^2}$$

One finds that there exist M higher-spin currents which read simply as

$$W_{boson}^s(z) \sim \sum_{k=0}^{s-2} f_{k,s} \partial^{s-k-1} \bar{\phi} \partial^{k+1} \phi \quad (6.1)$$

where s runs from 2 to ∞ , $f_{k,s}$ are constant coefficients fixed by demanding that the currents are quasi-primary. The OPEs can be computed straightforwardly and the linearity of the algebra follows nicely from the quadratic nature of (6.1). The currents are the singlets under the global $SU(M)$, and one can check that in this case, the central charge $c = 2M$.

- (b) $\mu = 0$. For this value, the algebra can be realized by M free Dirac fermions φ^i satisfying the OPE

$$\bar{\varphi}^i(z)\varphi_j(0) \sim \frac{\delta_j^i}{z}$$

and the higher-spin currents

$$W_{fermion}^s(z) \sim \sum_{k=0}^{s-1} g_{k,s} \partial^{s-k-1} \bar{\varphi} \partial^k \varphi \quad (6.2)$$

The OPEs can be computed straightforwardly. There is a small caveat that is absent in the case of $\mu = 1$, namely that the spin-3 current is not quasi-primary unless we project out the spin-1 current by hand, and thus s runs from 2 to ∞ . This could be seen from the OPE $T(z)W^3(0)$ where the energy momentum tensor $T(z) = -\frac{1}{2}(\bar{\varphi}^i\partial\varphi_i - \varphi_i\partial\bar{\varphi}^i)$. After projecting out $W^1(z)$, the spin-3 current is quasi-primary, and the resulting \mathcal{W} -algebra is a Lie algebra that is sometimes denoted as $w_{1+\infty}$ in the literature. The central charge in this case reads $c = M$.

- (c) $\mu = 2, 3, \dots$ For integral values of μ , the algebra truncates consistently to one with currents of maximum spin $s = \mu = N$ for some finite N . The resulting algebra is

then the \mathcal{W}_N algebra. This algebra, as explained in the previous chapter, could be realized via a coset minimal model - this is of course the boundary CFT of interest in the higher spin holography. From the viewpoint of the coset minimal CFT described previously, it enjoys a $\mathcal{W}_\infty[\lambda]$ symmetry, for a generic N and k , where we recall that in this realization, $\lambda = N/(N+k)$. Since the same CFT also realizes \mathcal{W}_N algebras, this implies that there should be an isomorphism between the two. It is not so obvious why this is expected since in the former case $0 \leq \lambda \leq 1$, and this isomorphism is related to an analytic continuation of λ to be N . More generally, it turns out that this isomorphism is part of a ‘trianality’ relation in the quantum $\mathcal{W}_\infty[\mu]$ algebras which read

$$\mathcal{W}_\infty[N] \approx \mathcal{W}_\infty \left[\frac{N}{N+k} \right] \approx \mathcal{W}_\infty \left[-\frac{N}{N+k+1} \right] \quad (6.3)$$

The way \mathcal{W} algebras would emerge in our current context is related to the asymptotic spacetime symmetry. This Brown-Henneaux like analysis determines the Poisson brackets of the \mathcal{W} algebra generators and thus the derived OPEs or commutation relations are only valid at large c . These are classical \mathcal{W}_N or $\mathcal{W}_\infty[N]$ algebras. In this thesis, we will not attempt to derive the quantum \mathcal{W} algebras from higher-spin holography, but only the classical versions. We note that the quantum algebras are significant deformations of their classical counterparts and in principle can be derived from the latter by replacing the Poisson brackets by commutators and then imposing Jacobi identities. On the other hand, to obtain the classical versions from the quantum ones, one should take the large central charge limit.

6.2 Asymptotic Spacetime Symmetries

We first summarize how \mathcal{W}_N -algebras emerge from the asymptotic symmetries of AdS_3 , and certain holographic aspects of the boundary CFT with \mathcal{W}_N symmetry. Let us begin with the metric that describes the entire set of asymptotically AdS_3 solutions with a flat boundary metric (see [53]):

$$ds^2 = l^2 \left\{ d\rho^2 - \frac{8\pi G}{l} \left(\mathcal{L}(dx^+)^2 + \tilde{\mathcal{L}}(dx^-)^2 \right) - \left(e^{2\rho} + \frac{64\pi^2 G^2}{l^2} \mathcal{L} \tilde{\mathcal{L}} e^{-2\rho} \right) dx^+ dx^- \right\} \quad (6.4)$$

where $(\rho, x^\pm \equiv t \pm \phi)$ describes the solid cylinder, and $\mathcal{L} = \mathcal{L}(x^+)$, $\tilde{\mathcal{L}} = \tilde{\mathcal{L}}(x^-)$ are arbitrary functions of $x^{\pm 1}$. In terms of the Chern-Simons connections, denoting $b = e^{\rho L_0}$,

$$\begin{aligned} A &= b^{-1} a(x^+) b + b^{-1} db, \quad \bar{A} = b \bar{a}(x^-) b^{-1} + b db^{-1}, \\ a &= \left(L_1 - \frac{2\pi}{k} \mathcal{L} L_{-1} \right) dx^+, \quad \bar{a} = \left(-L_{-1} + \frac{2\pi}{k} \tilde{\mathcal{L}} L_1 \right) dx^- \end{aligned} \quad (6.5)$$

¹For example, $\mathcal{L} = \tilde{\mathcal{L}} = -\frac{M}{4\pi}$ for the static BTZ of mass M , with the global AdS_3 vacuum corresponding to $M = -1/8G$. If $\mathcal{L} = \tilde{\mathcal{L}} = 0$, we recover the Poincaré patch of AdS_3 .

It was argued in [37] that in the context of this higher-spin theory, a more appropriate definition of an asymptotically AdS_3 solution is the set of conditions:

$$(A - A_{AdS_3}) \Big|_{\text{boundary}} = \mathcal{O}(1), \quad A_\rho = L_0, \quad A_- = 0 \quad (6.6)$$

where the first equation in (6.6) refers to a finite difference at the boundary (as $\rho \rightarrow \infty$). Similar expressions hold for the anti-holomorphic sector. It was then explained in [37, 38] that (6.6) translates into the Drinfeld-Sokolov condition on A , and if we consider the branching of \mathfrak{g} according to the sign of the eigenvalues of the adjoint action of L_0 , this implies, by (4.12) that we can set terms in W_m^l where m is positive to vanish. These are first-class constraints which generate gauge transformations, and we can fix the residual gauge freedom by letting only terms in W_{-l}^l to survive.² A similar procedure works for the anti-holomorphic \bar{A} . Altogether, we end up with the ansatz for the connections:

$$a = \left(L_1 + \sum_l \mathcal{W}_{-l}^l W_{-l}^l \right) dx^+, \quad \bar{a} = - \left(L_{-1} + \sum_l \bar{\mathcal{W}}_l^l W_l^l \right) dx^- \quad (6.7)$$

with the $\mathcal{W}, \bar{\mathcal{W}}$'s being general functions of ϕ , and A, \bar{A} obtained by gauge transforming via b as in (6.5). The global symmetries of the space of solutions described by a are described by the gauge transformations

$$\lambda(\phi) = \sum_i \xi^i(\phi) L_i + \sum_{l,m} \chi_m^l(\phi) W_m^l. \quad (6.8)$$

Identifying those that leave the structure of (6.7) invariant, we can express each gauge parameter $\chi_m^l, m < l$ as functions of the fields $\mathcal{W}_{-l}^l, \chi_l^l$ and their derivatives. Finally, we can write down the gauge transformations $\delta_\chi \mathcal{W}$ of \mathcal{W}_{-l}^l with respect to the parameters χ_l^l . In [37, 38], from this point, the asymptotic symmetry algebra is then obtained from the Poisson brackets of the charges that generate these transformations, and we obtain the two copies of \mathcal{W}_N -algebra.

Now in [49, 50], a slightly different approach was adopted to elucidate both the emergence of the \mathcal{W}_N -algebras and some holographic aspects at the same time. It was shown, explicitly in the case of spin-3, that the bulk field equations evaluated on a more general ansatz than (6.7) (which corresponds to generalized boundary conditions) yields the Ward identities in the CFT in the presence of spin-3 sources. This means that certain terms in the connection can be related precisely to extra source terms in the boundary CFT lagrangian, making feasible the existence of an AdS/CFT dictionary for the higher-spin sources. This was argued to be important in demonstrating that we have a consistent holographical dictionary for computing correlation functions of the stress tensor and spin-3 currents. By

²This procedure is the Drinfeld-Sokolov reduction in the highest-weight gauge. See, for example, [43] and [45].

invoking Noether's theorem

$$\delta\mathcal{O} = 2\pi\text{Res}_{z\rightarrow 0} \left[\sum_l \chi_l^l(z) \mathcal{W}_{-l}^l(z) \mathcal{O}(0) \right], \quad (6.9)$$

upon obtaining the various $\delta\mathcal{W}$, one can also read off the OPEs between the Virasoro primary fields of the \mathcal{W}_N algebra conveniently. In [50], this was done for the $\mathcal{W}_3^{(2)}$ algebra.

6.3 Ward Identities and the case of $SL(4)$ Chern-Simons theory

We now proceed to study explicitly if the procedures developed for the spin-3 case in [49, 50] generalize neatly to the spin-4 case as well. In this Section, we shall perform an analogous calculation for the spin-4 case below.

We begin with the ansatz

$$\begin{aligned} a &= (L_1 + \alpha\mathcal{L}L_{-1} + \beta\mathcal{W}W_{-2} + \gamma\mathcal{U}U_{-3}) dx^+ + \left(\sum_{m=-2}^2 \chi_m W_m + \sum_{m=-3}^3 f_m U_m + \nu L_{-1} \right) dx^- \\ \bar{a} &= - (L_{-1} + \alpha\bar{\mathcal{L}}L_1 + \beta\bar{\mathcal{W}}W_2 + \gamma\bar{\mathcal{U}}U_3) dx^- - \left(\sum_{m=-2}^2 \bar{\chi}_m W_{-m} + \sum_{m=-3}^3 \bar{f}_m U_{-m} + \bar{\nu} L_1 \right) dx^+, \end{aligned} \quad (6.10)$$

where W 's are the spin-3 generators, U 's are the spin-4 ones, all fields are functions of x^\pm , and $\{\alpha, \beta, \gamma\}$ are scaling parameters to be specified later.

In the language of Chern-Simons theory, the bulk field equations are conditions for flat connections. From the viewpoint of the boundary CFT with \mathcal{W}_4 symmetry, the fields \mathcal{W} and \mathcal{U} are dimension $(3, 0)$ and $(4, 0)$ primary fields, the anti-holomorphic ones being $(0, 3)$ and $(0, 4)$ primaries. The ansatz (6.10) generalizes the one in (6.7) and the modified boundary conditions can be interpreted as adding to the boundary CFT sources terms

$$I \rightarrow I - \int d^2x \left(\chi_2(x) \mathcal{W}(x) + \bar{\chi}_2(x) \bar{\mathcal{W}}(x) + f_3(x) \mathcal{U}(x) + \bar{f}_3(x) \bar{\mathcal{U}}(x) \right) \quad (6.11)$$

It turns out that the functions $\chi_2, \bar{\chi}_2, f_3, \bar{f}_3$ can be identified with those in the connections (6.10) by imposing the bulk field equations to yield the Ward identities of the CFT in the presence of these spin-3 and spin-4 sources. As explained in [49], the higher-spin operators are irrelevant in the RG sense and adding them changes the UV structure of the CFT. Correspondingly, the bulk geometry will asymptote to a different AdS_3 geometry.

Below, we analyze the holomorphic connection a which is gauge-equivalent to A .³ The flat connection condition $da + a \wedge a = 0$ gives rise to the following system of differential equations for sixteen unspecified functions of x^\pm . There are fourteen equations with two free parameters χ_2, f_3 that can be chosen freely. For notational simplicity, we denote

³The corresponding computation for \bar{A} is similar.

$\chi_2 \equiv \chi$, $f_3 \equiv f$, and superscripted primes denote ∂_{x^+} . For the χ 's:

$$\begin{aligned}
 \chi_1 &= -\chi' \\
 \chi_0 &= 2\alpha\mathcal{L}\chi - 6\beta f\mathcal{W} + \frac{1}{2}\chi'' \\
 \chi_{-1} &= 4\beta\mathcal{W}f' - \frac{2}{3}\alpha\chi\mathcal{L}' + 2\beta f\mathcal{W}' - \frac{5}{3}\alpha\mathcal{L}\chi' - \frac{1}{6}\chi''' \\
 \chi_{-2} &= \alpha^2\mathcal{L}^2\chi - 3\gamma\mathcal{U}\chi - \frac{3}{2}\beta f'\mathcal{W}' + \frac{7}{12}\alpha\mathcal{L}'\chi' \\
 &\quad - \frac{13}{10}\beta\mathcal{W}f'' + \frac{1}{6}\alpha\chi\mathcal{L}'' - \frac{1}{10}f(48\alpha\beta\mathcal{L}\mathcal{W} + 5\beta\mathcal{W}'') + \frac{2}{3}\alpha\mathcal{L}\chi'' + \frac{1}{24}\chi^{(4)}
 \end{aligned} \tag{6.12}$$

while for the f 's, we have

$$\begin{aligned}
 f_2 &= -f' \\
 f_1 &= \frac{1}{2}(6\alpha f\mathcal{L} + f'') \\
 f_0 &= -\frac{8}{3}\alpha\mathcal{L}f' - \alpha f\mathcal{L}' - \frac{1}{6}f''' \\
 f_{-1} &= \frac{1}{24}(48\beta\mathcal{W}\chi + 22\alpha f'\mathcal{L}' + 28\alpha\mathcal{L}f'' + 6f(12\alpha^2\mathcal{L}^2 + 12\gamma\mathcal{U} + \alpha\mathcal{L}'') + f^{(4)}) \\
 f_{-2} &= \frac{1}{120} \left(-264\alpha^2\mathcal{L}^2f' - 216\gamma\mathcal{U}f' - 72\gamma f\mathcal{U}' - 48\beta\chi\mathcal{W}' - 192\beta\mathcal{W}\chi' \right. \\
 &\quad \left. - 50\alpha\mathcal{L}'f'' - 28\alpha f'\mathcal{L}'' - 8\alpha\mathcal{L}(27\alpha f\mathcal{L}' + 5f''') - 6\alpha f\mathcal{L}''' - f^{(5)} \right) \\
 f_{-3} &= \frac{1}{720} \left(288\gamma f'\mathcal{U}' + 240\beta\mathcal{W}'\chi' + 544\alpha^2\mathcal{L}^2f'' + 360\gamma\mathcal{U}f'' + 78\alpha f''\mathcal{L}'' + 48\beta\chi\mathcal{W}'' \right. \\
 &\quad + 432\beta\mathcal{W}\chi'' + 90\alpha\mathcal{L}'f''' + 34\alpha f'\mathcal{L}''' + 2\alpha\mathcal{L}(720\beta\mathcal{W}\chi + 482\alpha f'\mathcal{L}' + 25f'''' + 6f(120\alpha^3\mathcal{L}^3 \\
 &\quad \left. - 480\beta^2\mathcal{W}^2 + 36\alpha^2\mathcal{L}'^2 + \alpha\mathcal{L}(264\gamma\mathcal{U} + 46\alpha\mathcal{L}'') + 12\gamma\mathcal{U}'' + \alpha\mathcal{L}^{(4)} + f^{(6)} \right) \tag{6.13}
 \end{aligned}$$

Finally, the higher spin-fields and ν are subject to the following equations:

$$\nu = \frac{6}{5}(9\gamma\mathcal{U}f - 4\beta\mathcal{W}\chi) \quad (6.14)$$

$$\alpha\partial_-\mathcal{L} = \nu' + \frac{12}{5}\beta\mathcal{W}\chi' - \frac{18}{5}\gamma\mathcal{U}f' \quad (6.15)$$

$$\begin{aligned} \beta\partial_-\mathcal{W} = & -3\gamma[\mathcal{U}'\chi + 2\mathcal{U}\chi'] + \frac{\alpha}{12}[2\mathcal{L}'''\chi + 9\mathcal{L}''\chi' + 15\mathcal{L}'\chi'' + 10\mathcal{L}\chi'''] \\ & + \frac{8\alpha^2}{3}[\mathcal{L}\mathcal{L}'\chi + \mathcal{L}^2\chi'] + \frac{1}{24}\chi^{(5)} - \left(\frac{24}{5}\alpha\beta\mathcal{L}'\mathcal{W} + \frac{34}{5}\alpha\beta\mathcal{L}\mathcal{W}' + \frac{1}{2}\beta\mathcal{W}'''\right)f \\ & - \left(\frac{44}{5}\alpha\beta\mathcal{L}\mathcal{W} + 2\beta\mathcal{W}''\right)f' - \frac{14}{5}\beta\mathcal{W}'f'' - \frac{13}{10}\beta\mathcal{W}f''' \end{aligned} \quad (6.16)$$

$$\begin{aligned} \gamma\partial_-\mathcal{U} = & \frac{\beta}{15}[\mathcal{W}'''\chi + 6\mathcal{W}''\chi' + 14\mathcal{W}'\chi'' + 14\mathcal{W}\chi'''] \\ & + \frac{2\alpha\beta}{15}[25\mathcal{L}'\mathcal{W}\chi + 18\mathcal{L}\mathcal{W}'\chi + 52\mathcal{L}\mathcal{W}\chi'] \\ & + \frac{\gamma}{10}[\mathcal{U}'''f + 5\mathcal{U}''f' + 9\mathcal{U}f'' + 6\mathcal{U}f'''] \\ & + \frac{\alpha}{360}[3\mathcal{L}^{(5)}f + 20\mathcal{L}^{(4)}f' + 56\mathcal{L}'''f'' + 84\mathcal{L}''f''' + 70\mathcal{L}'f^{(4)} + 28\mathcal{L}f^{(5)}] \\ & - 12\beta^2[\mathcal{W}\mathcal{W}'f + \mathcal{W}^2f'] + \frac{14}{5}[\alpha\mathcal{L}'\gamma\mathcal{U}f + \alpha\mathcal{L}\gamma\mathcal{U}'f + 2\alpha\mathcal{L}\gamma\mathcal{U}f'] \\ & + \frac{\alpha^2}{180}[177\mathcal{L}'\mathcal{L}''f + 78\mathcal{L}\mathcal{L}'''f + 295\mathcal{L}'^2f' + 352\mathcal{L}\mathcal{L}''f' + 588\mathcal{L}\mathcal{L}'f'' + 196\mathcal{L}^2f'''] \\ & + \frac{8\alpha^3}{5}[3\mathcal{L}^2\mathcal{L}'f + 2\mathcal{L}^3f'] + \frac{1}{720}f^{(7)}. \end{aligned} \quad (6.17)$$

We now come to an important point(the logic here parallels the spin-3 analysis in [49]): adding the source terms to the CFT action causes the stress tensor to attain \bar{z} dependence. Upon inserting

$$e^{\int d^2x(\chi_2(x)\mathcal{W}(x)+\bar{\chi}_2(x)\bar{\mathcal{W}}(x)+f_3(x)\mathcal{U}(x)+\bar{f}_3(x)\bar{\mathcal{U}}(x))} \quad (6.18)$$

within the expectation value of the stress-energy tensor, and invoking the OPEs between the stress-energy tensor and higher spin operators which read

$$T(z)\mathcal{W}(0) \sim \frac{3}{z^2}\mathcal{W}(0) + \frac{1}{z}\partial\mathcal{W}(0) + \mathcal{O}(1), \quad T(z)\mathcal{U}(0) \sim \frac{4}{z^2}\mathcal{U}(0) + \frac{1}{z}\partial\mathcal{U}(0) + \mathcal{O}(1), \quad (6.19)$$

we obtain

$$\frac{1}{2\pi}\partial_{\bar{z}}\langle T(z, \bar{z}) \rangle_{\chi, f} = 2\mathcal{W}'\chi + 3\mathcal{W}\chi' + 3\mathcal{U}'f + 4\mathcal{U}f' \quad (6.20)$$

where $\langle \dots \rangle$ denotes inserting (6.18) within the expectation value. Note that in obtaining (6.20), we have expanded in powers of χ , f and invoke the useful formula $\partial_{\bar{z}}(1/z) =$

$2\pi\delta^{(2)}(z, \bar{z})$. Now, observe that if we set

$$\beta = -\frac{5}{12}\alpha, \quad \gamma = \frac{5}{18}\alpha \quad (6.21)$$

then upon setting

$$2\pi\mathcal{L} = T \quad (6.22)$$

we find that (6.20) is nothing but (6.15). Thus, the stress-energy tensor corresponds to one single term in the connection. This was observed for the spin-3 case in [49], and it is nice to see explicitly that it is true for the spin-4 case as well. To fix α , we note that in the absence of all the higher-spin charges and conjugate potentials, if we demand the solution to reduce to BTZ in the chart (6.4), then

$$\alpha = \frac{2\pi}{k}, \quad (6.23)$$

which can be checked to yield precisely the Brown-Henneaux central charge $c = 6k$. The normalization of the parameters in (6.21) will enter into the geometries of the classical solutions.

Apart from the stress-energy tensor Ward identity, we can also use the bulk equations and (6.9) to read off the higher-spin OPEs or Ward identities. As a concrete example, we can easily read off the OPEs between two spin-3 and two spin-4 currents in our normalization. From (6.16), (6.17) and (6.9) (letting $\mathcal{O} = \mathcal{W}$ and \mathcal{U}), we have (suppressing the scaling constants to compare with existing results in literature, see for example [38])⁴

$$2\pi\langle\mathcal{W}(z)\mathcal{W}(0)\rangle = \frac{5}{z^6} + \frac{5\mathcal{L}}{z^4} + \frac{5\mathcal{L}'}{2z^3} + \frac{(\frac{8}{3}\mathcal{L}^2 + \frac{3}{4}\mathcal{L}'' - 6\mathcal{U})}{z^2} + \frac{(-3\mathcal{U}' + \frac{1}{6}\mathcal{L}''' + \frac{8}{3}\mathcal{L}\mathcal{L}')}{z} \quad (6.24)$$

$$\begin{aligned} 2\pi\langle\mathcal{U}(z)\mathcal{U}(0)\rangle &= \frac{1}{z} \left(\frac{1}{120}\mathcal{L}^{(5)} - 12\mathcal{W}\mathcal{W}' + \frac{14}{5}(\mathcal{U}\mathcal{L})' + \frac{1}{10}\mathcal{U}''' + \frac{177}{180}\mathcal{L}'\mathcal{L}'' + \frac{13}{30}\mathcal{L}\mathcal{L}''' + \frac{24}{5}\mathcal{L}^2\mathcal{L}' \right) \\ &+ \frac{1}{z^2} \left(\frac{1}{18}\mathcal{L}^{(4)} + \frac{1}{2}\mathcal{U}'' - 12\mathcal{W}^2 + \frac{28}{5}\mathcal{L}\mathcal{U} + \frac{59}{36}\mathcal{L}'^2 + \frac{88}{45}\mathcal{L}\mathcal{L}'' + \frac{16}{5}\mathcal{L}^3 \right) \\ &+ \frac{1}{5z^3} \left(9\mathcal{U} + \frac{14}{9}\mathcal{L}''' + \frac{98}{3}\mathcal{L}\mathcal{L}' \right) + \frac{6}{5z^4} \left(3\mathcal{U} + \frac{7}{6}\mathcal{L}'' + \frac{49}{9}\mathcal{L}^2 \right) \\ &+ \frac{14\mathcal{L}'}{3z^5} + \frac{28\mathcal{L}}{3z^6} + \frac{7}{z^8} \end{aligned} \quad (6.25)$$

The above results provide a foothold for understanding the holography dictionary in the presence of spin-3 and spin-4 sources, and is essentially, the spin-4 generalization of what was achieved in [49].

⁴Observe that if instead of (6.21), we have set $\beta = \frac{\alpha}{4}$, then (6.24) agrees exactly with equation (4.8) of [49].

6.4 Super- \mathcal{W} algebras and $SL(N|N-1)$ Chern-Simons theories

6.4.1 On super- \mathcal{W} symmetries as asymptotic spacetime symmetry

In the following, we shall briefly discuss the asymptotic spacetime symmetry of $sl(3|2)$ higher-spin SUGRA. This was initiated briefly in [67]. Our results generalize straightforwardly to the general $sl(N|N-1)$ cases. We begin by taking the manifold for the Chern-Simons theory (both copies) to be $\mathbb{R} \times D^2$, with co-ordinates (ρ, ϕ) and $t \in \mathbb{R}$. The radial co-ordinate is ρ and the boundary cylinder at infinite ρ is parametrized by $x^\pm \equiv t \pm \phi$.

Consider the chiral sector (our analysis that follows generalizes straightforwardly to the anti-chiral sector). We can fix some of the gauge freedom by choosing

$$\Gamma_- = 0, \quad \Gamma_+ = e^{-\rho L_0} a(x^+) e^{\rho L_0}, \quad \Gamma_\rho = L_0 \quad (6.26)$$

where $\Gamma_\pm = \Gamma_t \pm \Gamma_\phi$. By imposing the boundary condition that we obtain an asymptotically AdS_3 space, one can write, in a ‘highest-weight’ gauge,

$$a(x^+) = L_1 + \mathcal{L}L_{-1} + \mathcal{W}W_{-2} + \mathcal{U}U_0 + \Upsilon A_{-1} + \varphi_+ Q_{\frac{1}{2}}^{\frac{1}{2}} + \bar{\varphi}_+ \bar{Q}_{\frac{1}{2}}^{\frac{1}{2}} + \Phi_+ Q_{\frac{3}{2}}^{\frac{3}{2}} + \bar{\Phi}_+ \bar{Q}_{\frac{3}{2}}^{\frac{3}{2}}. \quad (6.27)$$

It can be shown straightforwardly that such a gauge choice is still preserved by gauge transformations of the form

$$\Lambda(x^+) = e^{-\rho L_0} \lambda(x^+) e^{\rho L_0} \quad (6.28)$$

where the gauge parameter λ is valued (with x^+ dependence in the components suppressed in notation) in the basis generators as follows:

$$\begin{aligned} \lambda = & \xi^n L_n + \chi^n W_n + \alpha^n A_n + \eta U_0 + \sum_{i=-\frac{1}{2}, \frac{1}{2}} \nu_i Q_i^{\frac{1}{2}} + \sum_{i=-\frac{1}{2}, \frac{1}{2}} \sigma_i Q_i^{\frac{3}{2}} + \sum_{i=-\frac{3}{2}, \frac{3}{2}} \zeta_i Q_i^{\frac{3}{2}} \\ & + \sum_{i=-\frac{1}{2}, \frac{1}{2}} \bar{\nu}_i \bar{Q}_i^{\frac{1}{2}} + \sum_{i=-\frac{1}{2}, \frac{1}{2}} \bar{\sigma}_i \bar{Q}_i^{\frac{3}{2}} + \sum_{i=-\frac{3}{2}, \frac{3}{2}} \bar{\zeta}_i \bar{Q}_i^{\frac{3}{2}}. \end{aligned} \quad (6.29)$$

Performing this gauge transformation at fixed time, the gauge connection a changes as

$$\delta a = d_\phi \lambda + [a, \lambda] \quad (6.30)$$

upon which we require that the form of the ansatz (6.27) be preserved. This will lead to a set of constraint equations for the variations of various fields. At the boundary, the gauge transformations may generate a physically inequivalent state. These physical symmetries are generated by boundary charges. From the Chern-Simons action, one can compute the classical Poisson brackets among these charges.

In the non-supersymmetric case, this procedure yields the \mathcal{W}_N algebras[37, 38]. Essentially, imposing the condition that we have AdS_3 boundary conditions at infinity turns out to be equivalent to the Drinfeld-Sokolov reduction of the current algebra. In an identical

fashion, we can perform the same analysis in the supersymmetric case. We solve the constraint equations due to (6.30) and compute the variation of various fields. The various parameters are reduced to the following set of independent ones which we denote as

$$\eta, \xi \equiv \xi^{(1)}, \chi \equiv \chi^{(2)}, \alpha \equiv \alpha_1, \nu_- \equiv \nu_{-\frac{1}{2}}, \bar{\nu}_- \equiv \bar{\nu}_{-\frac{1}{2}}, \zeta_- \equiv \zeta_{-\frac{3}{2}}, \bar{\zeta}_- \equiv \bar{\zeta}_{-\frac{3}{2}}. \quad (6.31)$$

The variation of various fields can be computed straightforwardly, but it is very cumbersome even for the case of $sl(3|2)$. The OPEs can be computed straightforwardly because by the Ward identities, the fields' variations are identical to those generated via

$$\delta\mathcal{O} = 2\pi\text{Res}(J(\phi)\mathcal{O}(0)) \quad (6.32)$$

where the Noether current J taking the form of

$$J = \frac{1}{2\pi} (\xi\mathcal{L} + \eta\mathcal{U} + \alpha\Upsilon + \chi\mathcal{W} + \nu_-\bar{\varphi}_+ + \bar{\nu}_-\varphi_+ + \zeta_-\bar{\Phi}_+ + \bar{\zeta}_-\Phi_+). \quad (6.33)$$

The OPEs depend on the choice of basis and are sensitive to field/parameter redefinitions. To adopt the appropriate convention, it is natural to do so such that the $\mathcal{N} = 2$ super-Virasoro algebra can be obtained after a truncation. We will not explicitly carry out the full computation to obtain the classical $\mathcal{N} = 2$ \mathcal{W}_3 algebra (see [73] for the classical and [72] for the quantum $\mathcal{N} = 2$ \mathcal{W}_3 algebra), but in the next section, after truncating some fields, we recover the $\mathcal{N} = 2$ superconformal algebra, and in the process, compute explicitly the Sugawara redefinition of the energy-momentum tensor.

6.4.2 Recovering the $\mathcal{N} = 2$ super-Virasoro algebra

As explained earlier, $sl(3|2)$ contains as a sub-algebra, $sl(2|1) \simeq osp(2|2)$. Below, we demonstrate explicitly, as a consistency check, that restricting to fields valued in this sub-algebra, we recover precisely the OPE relations pertaining to the well-known $\mathcal{N} = 2$ super-Virasoro algebra. This is the symmetry algebra that dictates the boundary degrees of freedom corresponding to bulk Chern-Simons fields valued in the $sl(2|1)$ sector, as we shall shortly verify.

In the following, we first present the relevant variations of the relevant fields $\varphi_+, \bar{\varphi}_+, \mathcal{L}$

and \mathcal{U} .

$$\delta\varphi_+ = \frac{3}{2}\varphi_+\xi' + \varphi'_+\xi + \frac{1}{6}\mathcal{U}\varphi_+\xi - \frac{1}{6}\varphi_+\eta + \left(\frac{5}{3}\Upsilon + \mathcal{L} + \frac{1}{6}\mathcal{U}' + \frac{1}{36}\mathcal{U}^2\right)\nu_- + \frac{1}{3}\mathcal{U}\nu'_- + \nu''_- + \dots, \quad (6.34)$$

$$\delta\bar{\varphi}_+ = \frac{3}{2}\varphi_+\xi' + \varphi'_+\xi - \frac{1}{6}\mathcal{U}\varphi_+\xi + \frac{1}{6}\varphi_+\eta + \left(\frac{5}{3}\Upsilon + \mathcal{L} - \frac{1}{6}\mathcal{U}' + \frac{1}{36}\mathcal{U}^2\right)\nu_- + \frac{1}{3}\mathcal{U}\nu'_- + \nu''_- + \dots, \quad (6.35)$$

$$\delta\mathcal{L} = \frac{1}{2}\xi''' + 2\mathcal{L}\xi' + \mathcal{L}'\xi - \left(\frac{1}{18}\mathcal{U}\bar{\varphi}_+ + \frac{1}{6}\bar{\varphi}'_+ + \frac{5}{4\sqrt{6}}\bar{\Psi}_+\right)\nu_- - \frac{1}{2}\bar{\varphi}_+\nu'_- + \dots \quad (6.36)$$

$$\delta\Upsilon = \Upsilon'\xi + 2\Upsilon\xi' + \sqrt{\frac{3}{32}}(\bar{\Psi}_+\nu_- - \Psi_+\bar{\nu}_-) + \dots \quad (6.37)$$

$$\delta\mathcal{U} = \eta' + \bar{\varphi}_+\nu_- - \varphi_+\bar{\nu}_- + \dots \quad (6.38)$$

where we refer the reader to Appendix B for the ellipses which are not important for the following discussion. After some algebra, we found that the suitable field/parameter redefinitions are as follows (hatted variables are the new ones):

$$\begin{aligned} \hat{\varphi}_+ &= \frac{c}{3}\varphi_+, & \hat{\varphi}_- &= \frac{c}{3}\bar{\varphi}_+, \\ \hat{U} &= -\frac{c}{18}\mathcal{U}, \\ \hat{T} &= \frac{c}{6}\left(\mathcal{L} + \frac{5}{3}\Upsilon + \frac{1}{36}\mathcal{U}^2\right) \\ \hat{\eta} &= -\frac{1}{6}\left(\eta + \frac{18}{c}(\mathcal{U}\xi)'\right) \end{aligned} \quad (6.39)$$

with c a constant which, as we shall see shortly, has the meaning of the central charge. Upon the above redefinitions, we found that (6.34)-(6.38) can be written as (the ellipses refer to other terms unimportant for this particular computation)

$$\delta\hat{U} = \frac{c}{3}\hat{\eta}' + \hat{U}'\xi + \hat{U}\xi' + \dots \quad (6.40)$$

$$\delta\hat{T} = \frac{c}{12}\xi''' + 2\hat{T}\xi' + \hat{T}'\xi + \dots \quad (6.41)$$

$$\delta\hat{\varphi}_+ = \frac{3}{2}\hat{\varphi}_+\xi' + \hat{\varphi}'_+\xi + \hat{\varphi}_+\hat{\eta} + \frac{c}{3}\nu''_- - 2\hat{U}\nu'_- + (2\hat{T} - \hat{U}')\nu_- + \dots \quad (6.42)$$

$$\delta\hat{\varphi}_- = \frac{3}{2}\hat{\varphi}_-\xi' + \hat{\varphi}'_-\xi - \hat{\varphi}_-\hat{\eta} + \frac{c}{3}\bar{\nu}''_- - 2\hat{U}\bar{\nu}'_- + (2\hat{T} + \hat{U}')\bar{\nu}_- + \dots \quad (6.43)$$

The relevant Noether current which we denote as J_s reads

$$J_s = \frac{1}{2\pi}\left(\xi\hat{T} + \hat{\eta}\hat{U} + \nu_-\hat{\varphi}_- + \bar{\nu}_-\hat{\varphi}_+\right) \quad (6.44)$$

and it generates the variations (6.40)-(6.43). Invoking Cauchy's residue theorem and (6.32)

yields the following OPEs:

$$\hat{T}(z)\hat{T}(0) \sim \frac{c}{2z^4} + \frac{2\hat{T}}{z^2} + \frac{\partial\hat{T}}{z} \quad (6.45)$$

$$\hat{T}(z)\hat{U}(0) \sim \frac{\hat{U}}{z^2} + \frac{\partial\hat{U}}{z} \quad (6.46)$$

$$\hat{T}(z)\hat{\varphi}_{\pm}(0) \sim \frac{3\hat{\varphi}_{\pm}}{2z^2} + \frac{\partial\hat{\varphi}_{\pm}}{z} \quad (6.47)$$

$$\hat{\varphi}_{\pm}(z)\hat{\varphi}_{\mp}(0) \sim \frac{2c}{3z^3} + \frac{2\hat{T}}{z} \pm \frac{2\hat{U}}{z^2} \pm \frac{\partial\hat{U}}{z} \quad (6.48)$$

$$\hat{U}(z)\hat{\varphi}_{\pm}(0) \sim \pm \frac{\hat{\varphi}_{\pm}(0)}{z} \quad (6.49)$$

$$\hat{U}(z)\hat{U}(0) \sim \frac{c}{3z^2}. \quad (6.50)$$

From the OPEs, we see that the constant c can be identified as the central charge. One can expand the fields in terms of their Laurent modes

$$\hat{T} = \sum_n \frac{\hat{L}_n}{z^{n+2}}, \quad \hat{\varphi}^{\pm} = \sum_r \frac{\hat{\varphi}_r^{\pm}}{z^{r+3/2}}, \quad \hat{U} = \sum_n \frac{\hat{U}_n}{z^{n+1}}, \quad (6.51)$$

upon which the OPEs lead to the following commutation relations displayed below for completeness.

$$\begin{aligned} [\hat{L}_m, \hat{\varphi}_n^{\pm}] &= \left(\frac{1}{2}m - n\right) \hat{\varphi}_{m+n}^{\pm} \\ [\hat{L}_m, \hat{U}_n] &= -n\hat{U}_{m+n} \\ [\hat{U}_m, \hat{U}_n] &= \frac{1}{3}cm\delta_{m+n,0} \\ [\hat{U}_m, \hat{\varphi}_r^{\pm}] &= \pm\hat{\varphi}_{m+r}^{\pm} \\ \{\hat{\varphi}_r^{\pm}, \hat{\varphi}_s^{\mp}\} &= 2\hat{L}_{r+s} \pm (r-s)\hat{U}_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0} \end{aligned} \quad (6.52)$$

We can immediately recognize (6.52) as the $\mathcal{N} = 2$ super-Virasoro algebra. Apart from being a good consistency check on our computations, the above exercise demonstrates that the Sugawara redefinition of the energy-momentum tensor should read as

$$\hat{T} = \frac{c}{6}\left(\mathcal{L} + \frac{5}{3}\Upsilon + \frac{1}{36}\mathcal{U}^2\right) = \frac{c}{36}\text{str}(a^2) \quad (6.53)$$

where the gauge connection a takes the form in (6.27).⁵ Finally, we recall that the $\mathcal{N} = 2$

⁵The second equality in (6.53) relies on inserting a factor of ‘ i ’ in the generator U_0 . As explained later, it turns out this is also required by a consistent reduction of this theory to $osp(N|2)$ theories.

superconformal algebra enjoys the following spectral flow as an automorphism [78]

$$\begin{aligned}\hat{U}_m &\rightarrow \hat{U}_m + \frac{1}{3}\alpha c\delta_{m,0}, \\ \hat{\varphi}_r^\pm &\rightarrow \hat{\varphi}_{r\pm\alpha}^\pm, \\ \hat{L}_m &\rightarrow \hat{L}_m + \alpha\hat{U}_m + \frac{1}{6}\alpha^2 c\delta_{m,0}.\end{aligned}\tag{6.54}$$

Later, we will see this automorphism manifest when we discuss some flat connections in the bulk Chern-Simons theory. A constant shift of the field conjugate to the generator U_0 induces a phase shift in the Killing spinor, and modifies the energy-momentum tensor following (6.53).

CHAPTER 7

Higher-spin black holes

In this chapter, we present new higher-spin black hole solutions in the framework of $SL(4)$ and $SL(N|N-1)$ theories. We first discuss the role of Wilson holonomies in $SL(N)$ theories and their relations to black holes and conical defect spacetimes. This is followed by an explicit construction of the $SL(4)$ black hole, and how this class of black holes exhibits consistent gravitational thermodynamics deeply related to the emergence of \mathcal{W}_4 algebra at the boundary. In Section 7.4, we present the higher-spin black holes in $SL(3|2)$ theory and demonstrated how their gravitational thermodynamics is related to $\mathcal{N} = 2$ super- \mathcal{W}_3 algebras. In Section 7.5, we present RG flow solutions carrying an abelian charge. They are a class of supersymmetric solutions (preserving four supercharges) which are massless BTZ solutions carrying higher-spin fields. Finally in Section 7.6, we extend our results to $SL(N|N-1)$ theories, and also present new conical defect solutions.

7.1 The role of Wilson loops

As we have discussed earlier in Chapter 2, the 3D Hilbert action is well-known to be equivalent to a $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ Chern-Simons theory. In the framework of $SL(N, \mathbb{R}) \times SL(N, \mathbb{R})$ Chern-Simons, where the $SL(2, \mathbb{R})$'s can always be embedded, one can, of course, also recover pure gravity, after setting all higher-spin fields to be zero.

The $SL(2, \mathbb{R})$'s are the isometries of AdS_3 , and apart from the vacuum, one can generate a rich class of non-trivial spacetimes by orbifolding AdS_3 by a pair of suitable generators. One particular example is the BTZ solution which derives from the action of two hyperbolic generators. Realizing $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ in terms of their left- and right- action on the embedding hyperboloid $X^2 + Y^2 - U^2 - V^2 = -1$, we can write

$$\begin{aligned}
 J_1 &= -\frac{1}{2}(J_{XU} + J_{YV}), & \tilde{J}_1 &= -\frac{1}{2}(J_{XU} - J_{YV}) \\
 J_2 &= -\frac{1}{2}(J_{XV} - J_{YU}), & \tilde{J}_2 &= -\frac{1}{2}(J_{XV} + J_{YU}) \\
 J_3 &= -\frac{1}{2}(J_{XY} - J_{UV}), & \tilde{J}_3 &= -\frac{1}{2}(J_{UV} + J_{XY})
 \end{aligned} \tag{7.1}$$

where $J_{ab} \equiv x_b \partial_a - x_a \partial_b$, and J_i, \tilde{J}_i are the generators of the left and right $SL(2, \mathbb{R})$ groups.

As explained in [6], the identification group generated by the Killing vector $\xi = -r_+(J_1 + \tilde{J}_1)$ yields a quotient space which is the BTZ black hole of radius r_+ .

We can choose to represent the quotient space construction via matrices. Let us write the defining equation of the AdS_3 quadric as the condition on the determinant of a matrix \mathbf{X} as follows

$$\mathbf{X} = \begin{pmatrix} V + X & Y + U \\ Y - U & V - X \end{pmatrix}, \quad \det|\mathbf{X}| = 1 \quad (7.2)$$

This condition is preserved by a transformation

$$\mathbf{X} \rightarrow g_l \mathbf{X} g_r^{-1} \quad (7.3)$$

where $(g_l, g_r) \in SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$. The trace is invariant under conjugation of which classes determine different spacetime solutions. For the BTZ, (g_l, g_r) are hyperbolic generators, and in general, it is natural to ask how these matrices can be mapped via a homomorphism to (7.1)-(7.1) (i.e. in this context, we want to represent $e^{t\xi}$ as an action on \mathbf{X}). Relating this to our conventions for $SL(2, \mathbb{R})$, we find

$$L_{-1} = J_3 - J_2, L_1 = J_3 + J_2, L_0 = J_1 \quad (7.4)$$

and similarly for the anti-holomorphic quantities. As an explicit example, consider the static BTZ. One can first parametrize the $SO(2, 2)$ group element as

$$\mathbf{X} = \begin{pmatrix} r e^\phi & e^t \sqrt{r^2 - 1} \\ e^{-t} \sqrt{r^2 - 1} & r e^{-\phi} \end{pmatrix}, \quad ds^2 = -(r^2 - 1) dt^2 + \frac{dr^2}{r^2 - 1} + r^2 d\phi^2 \quad (7.5)$$

The hyperbolic quotient by $e^{t\xi_\phi}$ is realized by taking¹

$$g_l = g_r^{-1} = \begin{pmatrix} e^{tr_+} & 0 \\ 0 & e^{-tr_+} \end{pmatrix}, \quad (7.6)$$

from which we see that since $t = 2\pi\mathbb{Z}$, making a BTZ is equivalent to identifying ϕ as a periodic coordinate. Rescaling $r \rightarrow r/r_+$, $\phi \sim \phi + 2\pi$ and we can interpret the spacetime as having a horizon at r_+ , and with ADM mass $\sim r_+^2$. Going once around the non-contractible cycle along ϕ , we have $e^{\oint_C \xi_\phi}$ acting on \mathbf{X} via (g_l, g_r) as demonstrated explicitly.

Let us now view things from the Chern-Simons perspective. In the $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ Chern-Simons theory, the solutions are flat connections and thus locally can always be expressed as pure gauges, i.e. $A = g^{-1} dg$. Globally, when the spacetime has non-trivial topology, then the gauge function g is not single-valued. When the spacetime has a non-contractible cycle C , as we go around the cycle once, g attains a factor of the holonomy $\mathcal{P}\exp(\oint_C A)$. Up to an overall gauge transformation, the flat connections are thus uniquely specified by their holonomies around non-contractible cycles of the manifold.

¹We can explicitly realize the $SL(2, \mathbb{R})$ generators as $L_0 = -\sigma_z/2$, $L_\pm = (i\sigma_y \pm \sigma_x)/2$.

Indeed, the holonomies (of (A, \tilde{A})) are precisely the (g_l, g_r) described above.² For the BTZ, the ϕ -cycle is non-contractible, and will have a non-trivial holonomy. From (6.5), we have checked that the eigenvalues of $\oint a_\phi d\phi = 2\pi a_+$ are precisely those of $e^{-2\pi r_+ L_0}$, and similarly $\oint \bar{a}_\phi d\phi = -2\pi \bar{a}_-$ shares identical eigenvalues with $e^{-2\pi r_+ \tilde{L}_0}$. Now, there is another time-like Killing vector $\xi_t \sim -(J_1 - \tilde{J}_1)$. After a Wick rotation, the thermodynamics of the black hole can be obtained by demanding that the $SL(2, \mathbb{R})$ connection be single-valued along Euclidean time direction which is periodic. This cycle is contractible and vanishes at the horizon. The periodic identification of the thermal time can be represented via a pair of (g_l, g_r) acting on \mathbf{X} . From the metric in (7.5), we compute the period to be 2π . This means $g_l = e^{2\pi L_0}, g_r^{-1} = e^{-2\pi \tilde{L}_0}$. Like in the case of the ϕ direction, this is equivalent to the action of $e^{\oint \xi_\tau}$ if we take $\xi_\tau = -r_+ (L_0 - \tilde{L}_0)$ since $\tau = 1/r_+$. The eigenvalues are $\pm\pi$ which we check to be equivalent to that of $\oint a_\tau d\tau = 2\pi\tau a_+$ and $\oint \bar{a}_\tau d\tau = 2\pi\tau \bar{a}_-$.

For our purpose, it is natural to ponder about whether a similar interpretation for Wilson loops holds for the gravitational thermodynamics in the context of higher-spin space-time geometries? We can begin to answer this question by first embedding pure gravity, and thus the ordinary BTZ, in the $SL(N, \mathbb{R}) \times SL(N, \mathbb{R})$ theory. After a straightforward computation, we find that the N -dimensional fundamental representation of $SL(2, \mathbb{R})$ yields the following set of N eigenvalues for the holonomies for any even $N > 2$:

$$(\pm(N-1)\pi, \pm(N-3)\pi, \pm(N-5)\pi, \dots, \pm\pi) \quad (7.7)$$

with a similar result for any odd N but with the last pair of values replaced by 0. In higher-spin theories, the connection is now also valued in other higher-spin generators, and thus the holonomies no longer correspond neatly to the quotienting action by subgroups of isometries. Nonetheless, we can still classify various solutions according to the holonomies' eigenvalues, as explained nicely in [31] and [51]. Apart from higher-spin black holes, as shown recently in [31], when the ϕ -direction is a contractible spatial cycle, demanding the holonomies to be trivial elements of the $SL(N, \mathbb{R})$ gives us a discrete set of solutions which are higher-spin generalizations of conical defects in the pure gravity case.

Before leaving this brief discussion, let us remark that the above argument does not hold for all gauge groups. For example, when $\Lambda = 0$, and the group is $ISO(2, 1)$, the identification holonomies are no longer the Wilson loops of the theory, but simply $\oint \omega^a$. This is because the connection ω and the vielbein ϵ are valued differently, i.e. in this case (see, for example, [54] for a nice review),

$$A = e^a J_a + \omega^a P_a, \text{ with } [J^a, P^b] = \epsilon^{abc} P_c, [J^a, J^b] = \epsilon^{abc} J_c, [P^a, P^b] = 0. \quad (7.8)$$

In our case, they are both valued in the same set of generators and the holonomies are precisely the Wilson loops.

²We note that this relationship between identifications and holonomies is emphasized and discussed nicely in Section 1.3 of [54].

7.2 Higher-spin black holes and spacetime geometry

7.2.1 The case of $SL(N)$ black holes

In [49], the bulk field equations for $SL(3, \mathbb{R}) \times SL(3, \mathbb{R})$ Chern-Simons were solved to yield an ansatz that was interpreted to be a generalized BTZ solution which carries a spin-3 charge. These ‘black-hole’ solutions are defined not by a manifest event horizon (which would be inappropriate since the metric changes under a general spin-3 gauge transformation), but by demanding that the Wilson holonomy along the time-like direction of (6.4) has eigenvalues identical to that of the ordinary BTZ black hole. This places constraints on the functions \mathcal{L} and \mathcal{W} which are, in principle, functions of the Euclidean time τ and the chemical potential $\varrho = -\tau\chi$, where χ is the spin-3 charge (similar relations hold for the anti-holomorphic entities).

Apart from satisfying gauge invariance, such a prescription was argued to be tenable based on the following resulting conditions: (i) the variables \mathcal{L} and \mathcal{W} in the ansatz (6.10) satisfy a nice integrability condition:

$$\frac{\partial \mathcal{L}}{\partial \rho} = \frac{\partial \mathcal{W}}{\partial \tau} \quad (7.9)$$

(ii) in the limit when the spin-3 field vanishes, the BTZ is recovered smoothly, and (iii) that there exists a gauge in which the solution exhibits a regular event horizon and spin-3 field, both of which are smooth in the Euclidean (τ, ρ) plane. Such a black hole is then argued to be a saddle point contribution to a partition function of the form

$$Z = \text{tr} \left(e^{4\pi^2 i(\tau \mathcal{L} + \rho \mathcal{W} - \bar{\tau} \bar{\mathcal{L}} - \bar{\rho} \bar{\mathcal{W}})} \right) \quad (7.10)$$

By performing a Legendre transform, we can compute the entropy from this partition function. This generalizes the usual notion of the area law. On general grounds of symmetry, it may be natural to expect that if the above procedure works fine for the spin-3 case, it is likely to be valid as well for all higher spins, since \mathcal{W}_N symmetry arises as the asymptotic symmetry of these higher-spin theories as demonstrated in [38].

But we think that the consistency of such an approach cannot be guaranteed by asymptotic symmetry arguments alone, and it would be important to analyze some manageable cases in the absence of a general and rigorous proof.

In this Section, we will study the case of spin-4 carefully, and demonstrate that the Wilson-loop-defined black holes can be understood precisely in the same elegant fashion as described above. Let us begin by reviewing the general algorithm of constructing higher-spin black holes as instructed in [52]: (i) begin by computing the flat connection for the ordinary BTZ in the $SL(N, \mathbb{R}) \times SL(N, \mathbb{R})$ theory (ii) extend this connection by including higher-spin charges together with their conjugate chemical potentials (iii) compute the Wilson loop in the time-like direction of the Fefferman-Graham chart and constrain the holonomy’s eigenvalues to be identical to those of the BTZ.

Following [52], the eigenvalue constraint equation can be formulated via a system of trace equations:

$$\mathrm{Tr}(\omega^n) = \mathrm{Tr}(\omega_{BTZ}^n), \quad n = 2, 3, \dots \quad (7.11)$$

Note that we computed these eigenvalues earlier in (7.7). In the case of spin-3, (7.11) terminates at $n = 3$, and can be solved via a quartic equation. In the general case, a perturbative approach is easier. For example, as briefly mentioned earlier, in a certain $N \rightarrow \infty$ limit, one can lift the algebra of $SL(N, \mathbb{R}) \times SL(N, \mathbb{R})$ to $hs[\lambda] \oplus hs[\lambda]$, and an illuminating perturbative analysis of (7.11) was performed in [52] for this case.

For the spin-4 black holes, from the bulk equations, we can straightforwardly compute the connection to be

$$\begin{aligned} A &= (e^\rho L_1 + \mathcal{L}e^{-\rho}L_{-1} + \mathcal{W}e^{-2\rho}W_{-2} + \mathcal{U}e^{-3\rho}U_{-3}) dx^+ + L_0 d\rho \\ &+ \left(\chi \left(e^{2\rho}W_2 + 2\mathcal{L}W_0 + 2\mathcal{W}e^{-\rho}U_{-1} - \frac{24}{5}\mathcal{W}e^{-\rho}L_{-1} + (\mathcal{L}^2 - 3\mathcal{U})e^{-2\rho}W_{-2} + 2\mathcal{L}\mathcal{W}e^{-3\rho}U_{-3} \right) \right. \\ &+ \left. f \left(e^{3\rho}U_3 + 3\mathcal{L}e^\rho U_1 - 6\mathcal{W}W_0 + (3\mathcal{L}^2 + 3\mathcal{U})e^{-\rho}U_{-1} + \frac{54}{5}\mathcal{U}e^{-\rho}L_{-1} - \frac{24}{5}\mathcal{W}\mathcal{L}e^{-2\rho}W_{-2} \right. \right. \\ &\quad \left. \left. \left(\mathcal{L}^3 + \frac{11}{5}\mathcal{L}\mathcal{U} - 4\mathcal{W}^2 \right) e^{-3\rho}U_{-3} \right) \right) dx^- \end{aligned} \quad (7.12)$$

where the factors of $e^{n\rho}$ for $n = -3 \dots 3$ originate from conjugating a ρ -independent a with $b = e^{\rho L_0}$. Please note that we have rescaled the variables $\alpha\mathcal{L} \rightarrow \mathcal{L}, \beta\mathcal{W} \rightarrow \mathcal{W}, \gamma\mathcal{U} \rightarrow \mathcal{U}$ in this Section to avoid cluttering the notations, but will emphasize at appropriate moments later on, when the constants α, β, γ are needed to be restored.

The other anti-holomorphic $SL(4, \mathbb{R})$ connection reads

$$\begin{aligned} \bar{A} &= -(e^\rho L_{-1} + \bar{\mathcal{L}}e^{-\rho}L_1 + \bar{\mathcal{W}}e^{-2\rho}W_2 + \bar{\mathcal{U}}e^{-3\rho}U_3) dx^- - L_0 d\rho \\ &- \left(\bar{\chi} \left(e^{2\rho}W_{-2} + 2\bar{\mathcal{L}}W_0 + 2\bar{\mathcal{W}}e^{-\rho}U_1 - \frac{24}{5}\bar{\mathcal{W}}e^{-\rho}L_1 + (\bar{\mathcal{L}}^2 - 3\bar{\mathcal{U}})e^{-2\rho}W_2 + 2\bar{\mathcal{L}}\bar{\mathcal{W}}e^{-3\rho}U_3 \right) \right. \\ &- \left. \bar{f} \left(e^{3\rho}U_{-3} + 3\bar{\mathcal{L}}e^\rho U_{-1} - 6\bar{\mathcal{W}}W_0 + (3\bar{\mathcal{L}}^2 + 3\bar{\mathcal{U}})e^{-\rho}U_1 + \frac{54}{5}\bar{\mathcal{U}}e^{-\rho}L_1 - \frac{24}{5}\bar{\mathcal{W}}\bar{\mathcal{L}}e^{-2\rho}W_2 \right. \right. \\ &\quad \left. \left. \left(\bar{\mathcal{L}}^3 + \frac{11}{5}\bar{\mathcal{L}}\bar{\mathcal{U}} - 4\bar{\mathcal{W}}^2 \right) e^{-3\rho}U_3 \right) \right) dx^+ \end{aligned} \quad (7.13)$$

From these connections, we can derive the metric as explained in the discussion surrounding

(4.14) with the normalization

$$g_{\mu\nu} = \frac{1}{\text{Tr}(L_0^2)} \text{Tr}(e_\mu e_\nu) \quad (7.14)$$

Note that the above is consistent with the choice made in the particular $SL(3, \mathbb{R})$ case presented in [49]. We have checked that this normalization condition corresponds to AdS_3 spacetime with unit radius in the limit of vanishing higher-spin charges. Since the difference between the connections is proportional to the vielbein, we can already read off the asymptotic behavior from (7.13).

For completeness sake, let us display the metric explicitly³:

$$\begin{aligned} ds^2 = & d\rho^2 + \frac{1}{5} (2(-3f\mathcal{W} + \chi\mathcal{L})dx^- + 2(-3\bar{f}\bar{\mathcal{W}} + \bar{\chi}\bar{\mathcal{L}})dx^+)^2 \\ & + \frac{12}{5} \left| (e^{2\rho}\chi + e^{-2\rho}\bar{\mathcal{W}})dx^- + \left(-\frac{24}{5}\bar{\mathcal{W}}\bar{\mathcal{L}}e^{-2\rho}\bar{f} + \bar{\chi}e^{-2\rho}(\bar{\mathcal{L}}^2 - 3\bar{\mathcal{U}})dx^+\right) \right|^2 \\ & - \frac{6}{25} \left| 3f\mathcal{L}e^\rho dx^- + (2\bar{\chi}\bar{\mathcal{W}}e^{-\rho} + 3\bar{f}e^{-\rho}(\bar{\mathcal{L}}^2 + \bar{\mathcal{U}})) dx^+ \right|^2 \\ & - \frac{18}{5} \left| (e^{3\rho}f + e^{-3\rho}\bar{\mathcal{U}})dx^- + (2\bar{\chi}\bar{\mathcal{L}}\bar{\mathcal{W}} + \bar{f}\bar{\mathcal{L}}^3 + \frac{11}{5}\bar{f}\bar{\mathcal{L}}\bar{\mathcal{U}} - 4\bar{f}\bar{\mathcal{W}}^2)e^{-3\rho}dx^+ \right|^2 \\ & - \left| \bar{\mathcal{L}}e^{-\rho}dx^- + \left(e^\rho - e^{-\rho}\frac{24}{5}\bar{\chi}\bar{\mathcal{W}} + e^{-\rho}\frac{54}{5}\bar{f}\bar{\mathcal{U}}\right)dx^+ \right|^2, \end{aligned} \quad (7.15)$$

where the notation $|\dots|^2$ refers to multiplying the enclosed expression with its conjugate, i.e. all barred quantities become unbarred and vice-versa, and $dx^\pm \leftrightarrow dx^\mp$. It is straightforward to check that when all higher-spin charges and their conjugate potentials vanish, the metric (7.15) reduces to that of the BTZ with its Noether charges proportional to $\mathcal{L}, \bar{\mathcal{L}}$. From (7.15), we see that as $\rho \rightarrow \infty$, the terms $\sim e^{3\rho}$ dominate, and the metric asymptotes to

$$ds^2 = d\rho^2 - \left(\frac{18}{5} \bar{f} f e^{6\rho} \right) dx^+ dx^- \quad (7.16)$$

After re-scaling the boundary cylinder, (7.16) is just global AdS_3 with radius $= \frac{1}{3}$. From a holographical perspective, we can say that the addition of spin-4 potentials in the bulk corresponds to adding to the boundary CFT an irrelevant dimension 4 operator, thereby changing its UV behavior. Indeed, from the general algorithm of constructing this class of higher-spin black holes, it is clear that the metric asymptotes to an AdS_3 with radius $= \frac{1}{N-1}$, where N is the largest spin added. This is independent of the rank of the Chern-Simons gauge group that embeds the solution.

Consider switching off the spin-4 field, and we would expect to find the $SL(3, \mathbb{R})$ solutions of [49] embedded in our $SL(4, \mathbb{R})$ framework. This is *almost* the case, as can be

³Please refer to the Appendix A for our choice of basis for the $SL(4, \mathbb{R})$ generators.

seen easily from (7.15). The trace of the spin-4 generators affects the scaling constants of each line of (7.15) and we find that it is not possible, via simply a re-scaling of coordinates, to obtain exactly the $SL(3, \mathbb{R})$ solutions of [49]. We choose to interpret this difference by looking at the asymptotic behavior $\rho \rightarrow -\infty$. If the spin-4 potential is kept, this limit takes us again to AdS_3 with radius $= \frac{1}{3}$. When the spin-4 potential is discarded and the spin-3 potential retained, this limit still remains. In conventional GR language, this traversable wormhole⁴ asymptotes between two distinct AdS_3 . On the other hand, for the $SL(3, \mathbb{R})$ solutions in [49], we have, instead, a wormhole with two asymptotic AdS_3 with identical radius $= \frac{1}{2}$.

Incidentally, another related point is that for the spin-3 solution in the principal embedding, since the metric asymptotes to an AdS_3 with radius $= \frac{1}{2}$, it was interpreted in [50] as being the $\mathcal{W}_3^{(2)}$ vacuum. In this case, we observe that clearly, this is not the case for the spin-4 solution in the principal embedding, i.e. (7.15). The asymptotic AdS_3 vacua of solution (7.15) *cannot* be identified with any of those that belong to non-principal \mathcal{W}_4 's, after the metric is normalized to yield an AdS_3 of unit radius in the limit of vanishing higher-spin charges. We conclude that this is the case in general for higher-spin black holes of this type, as can be checked from the expression for the defining vector \tilde{L}_0 in (4.19). Thus, the identification/interpretation made in this particular aspect for the spin-3 case is due to more of a coincidence.

In [49], the spin-3 ‘black-hole’ solutions are defined by a manifest event horizon by demanding that the Wilson holonomy along the time-like Killing direction of (6.4) has eigenvalues identical to that of the ordinary BTZ black hole. Such a procedure was shown to be equivalent to (7.64) which is a necessary condition for the consistency of the gravitational thermodynamics of the solutions. As a bonus, it was checked that the resulting constraints placed on the variables \mathcal{L} and \mathcal{W} lead to the solution approaching the BTZ (and its thermodynamical behavior) in the limit of vanishing higher spin and potential. A caveat is that when the holonomy constraint is imposed, the solution does not exhibit an event horizon, but, in the conventional sense, is instead a transversable wormhole connecting two AdS_3 vacua. It was further shown in [50] that there exists a gauge transformation that takes the metric to one in which g_{tt} has a double zero relative to the radial direction, and thus there could be an event horizon. This is reasonable as higher-spin gauge transformations make the usual notions of Riemannian geometry non gauge-invariant, and one is naturally led to proposing gauge-invariant entities like Wilson holonomies to discuss the physics of these solutions.

In such an approach, that the thermodynamical consistency has been invoked to be the fundamental definition of a ‘black hole’ is clearly motivated by holography. At this point, it is pertinent to recall that in a certain $N \rightarrow \infty$ limit, one can lift the algebra of $SL(N, \mathbb{R}) \times SL(N, \mathbb{R})$ to $hs[\lambda] \oplus hs[\lambda]$. In this limit, and if we add two additional complex scalar fields in the bulk (to be interpreted in the ‘t Hooft limit), we arrive at the bulk picture of the Gaberdiel-Gopakumar conjecture.

⁴We will explain later why the event horizon is absent in this solution in the next Section.

Next, we will mainly be concerned about checking the validity of the integrability condition and the interpretation of the gravitational thermodynamics of the spin-4 solutions. We will see that our results demonstrate clearly that the spin-4 generalization of the framework discussed in [49, 50] is valid, and thus if this program is to work for all N , it has, at least, passed an explicit non-trivial check.

7.3 On the first law of higher-spin black hole thermodynamics

Let us begin this subsection by performing an elementary review of some basic thermodynamical relations like (7.10). Mainly, this spells out various sign conventions, scaling factors and, of course, to set up the basic interpretative framework.

Since we have two higher-spin fields, (7.10) should now be

$$Z(\tau, \varrho, \mu, \bar{\tau}, \bar{\varrho}, \bar{\mu}) = \text{tr} \left(e^{4\pi^2 i(-\tau \mathcal{L} + \varrho \mathcal{W} + \mu \mathcal{U} + \bar{\tau} \bar{\mathcal{L}} - \bar{\varrho} \bar{\mathcal{W}} - \bar{\mu} \bar{\mathcal{U}})} \right) \quad (7.17)$$

where τ is the inverse Euclidean temperature of the BTZ in the limit of vanishing higher-spin, and $\varrho = -\tau \chi$, $\mu = -\tau f$ are the chemical potentials⁵. For a static BTZ limit, we can relate the holomorphic and anti-holomorphic quantities to be

$$\tau = -\bar{\tau}, \quad \varrho = \bar{\varrho}, \quad \mu = \bar{\mu}, \quad \bar{\mathcal{L}} = \mathcal{L}, \quad \bar{\mathcal{W}} = -\mathcal{W}, \quad \bar{\mathcal{U}} = -\mathcal{U} \quad (7.18)$$

The spin-4 solutions contribute to the above generalized partition function which includes chemical potentials ϱ, μ conjugate to the spin-3 and spin-4 currents \mathcal{W}, \mathcal{U} respectively. From the CFT perspective, assigning the dependence of $\mathcal{L}, \mathcal{W}, \mathcal{U}$ on τ, ϱ, μ (and hence generalizing the BTZ) amounts to the following equations for the expectation values:

$$\langle \mathcal{L} \rangle = \frac{i}{4\pi^2} \frac{\partial \ln Z}{\partial \tau}, \quad \langle \mathcal{W} \rangle = -\frac{i}{4\pi^2} \frac{\partial \ln Z}{\partial \varrho}, \quad \langle \mathcal{U} \rangle = -\frac{i}{4\pi^2} \frac{\partial \ln Z}{\partial \mu}. \quad (7.19)$$

This gives us, as necessary conditions, the integrability constraints:

$$\frac{\partial \mathcal{L}}{\partial \varrho} = -\frac{\partial \mathcal{W}}{\partial \tau}, \quad \frac{\partial \mathcal{L}}{\partial \mu} = -\frac{\partial \mathcal{U}}{\partial \tau}, \quad \frac{\partial \mathcal{W}}{\partial \mu} = \frac{\partial \mathcal{U}}{\partial \varrho}. \quad (7.20)$$

We note that in the spin-3 case, only the first two relations of (7.20) are relevant in ensuring the validity of the Wilson loop prescription. In the following, we will see that all the three equations of (7.20) are satisfied, and thus this constitutes an important and non-trivial evidence for the program to hold for the spin-4 case as well.

In the absence of a geometric area law for the entropy, one can still define it via a Legendre transform of the free energy. Such a definition implies that the first law of thermodynamics will be automatically satisfied, when \mathcal{L} is identified as the energy-momentum

⁵The rest of this Section will demonstrate why such an interpretation is viable.

tensor divided by 2π . The entropy S reads

$$S = \ln Z - 4i\pi^2 (-\tau\mathcal{L} + \varrho\mathcal{W} + \mu\mathcal{U} + \bar{\tau}\bar{\mathcal{L}} - \bar{\varrho}\bar{\mathcal{W}} - \bar{\mu}\bar{\mathcal{U}}) \quad (7.21)$$

which can be computed once $\mathcal{L}, \mathcal{W}, \mathcal{U}$ are determined. In the spin-3 case, determining these relations from the Wilson loop prescription is still feasible analytically, but it proves to be difficult in spin > 3 cases. Inspired by the treatment for the $hs[\lambda] \oplus hs[\lambda]$ case [52], we shall perform the analysis perturbatively.

As a warm-up, let us re-visit the spin-3 case [49]. Absorbing various scaling constants in \mathcal{L}, \mathcal{W} , the Chern-Simons connection is $A = b^{-1}ab + L_0 d\rho$, with a being

$$a_+ = L_1 - \mathcal{L}L_{-1} - \mathcal{W}W_{-2}, \quad a_- = \chi(W_2 - 2\mathcal{L}W_0 + \mathcal{L}^2W_{-2} + 8\mathcal{W}L_{-1}), \quad (7.22)$$

for which, upon taking $n = 2, 3$, (7.11) reads

$$\begin{aligned} \frac{-2}{\tau^2} &= 8\mathcal{L} - 96\chi\mathcal{W} + \frac{128}{3}\chi^2\mathcal{L}^2, \\ 0 &= -\mathcal{W} + \frac{8}{3}\chi\mathcal{L}^2 - 16\chi^2\mathcal{L}\mathcal{W} + 64\chi^3 \left(\mathcal{W}^2 - \frac{2}{27}\mathcal{L}^3 \right) \end{aligned} \quad (7.23)$$

We note that when the higher-spin fields are set to zero, the BTZ Euclidean temperature τ can be computed by demanding that the Euclidean section is smooth and is related to \mathcal{L} by $\mathcal{L} = -1/4\tau^2$. Now, we can expand the variables as

$$\mathcal{L} = \sum_n c_n \chi^n \tau^{-n-2}, \quad \mathcal{W} = \sum_n d_n \chi^n \tau^{-n-3} \quad (7.24)$$

where $n \in \mathbb{Z}_0^+$. Substituting (7.24) into (7.23), it can be checked that the following recursion relations are solutions:

$$c_n = \frac{16(2n+1)}{3(n-1)} \sum_{i+j=n-2} c_i c_j, \quad d_{n-1} = \frac{4n}{3(n-1)} \sum_{i+j=n-2} c_i c_j \quad (7.25)$$

From (7.25), we can see that

$$nc_n = 4(2n+1)d_{n-1} \quad (7.26)$$

Upon restoring the scaling constants $\mathcal{L} \rightarrow \alpha\mathcal{L} = \frac{2\pi}{k}\mathcal{L}$, $\mathcal{W} \rightarrow \beta\mathcal{W} = \frac{\alpha}{4}\mathcal{W}$, then (7.26) immediately implies the integrability condition (7.64). We note that in the original work of [49], the integrability condition was verified by differentiating (7.23) instead.

We can seek the analogue of (7.26) in the case of spin-4, and see whether the integrability conditions (7.20) hold. The gauge connection was previously displayed in (7.13), and by setting $\rho = 0$, we can obtain a as the simpler yet also valid variable to compute the

holonomy's eigenvalues. The three holonomy equations are

$$\mathrm{Tr}(a_+ + a_-)^2 = -\frac{5}{\tau^2}, \quad \mathrm{Tr}(a_+ + a_-)^3 = 0, \quad \mathrm{Tr}(a_+ + a_-)^4 = \frac{41}{4\tau^4}. \quad (7.27)$$

Substituting (7.13) into (7.27), and expanding the variables as

$$\mathcal{L} = \sum_{n,m} c_{nm} \frac{\chi^n f^m}{\tau^{n+2+2m}}, \quad \mathcal{W} = \sum_{n,m} d_{nm} \frac{\chi^n f^m}{\tau^{n+3+2m}}, \quad \mathcal{U} = \sum_{n,m} b_{nm} \frac{\chi^n f^m}{\tau^{n+4+2m}}, \quad (7.28)$$

we can express the holonomy equations (7.27) as polynomial equations in the coefficients. For example, the first one (which is the simplest) reads

$$\begin{aligned} 0 = & -20c_{nm} + 144d_{n-1,m} - 288b_{n,m-1} - 144b_{n-2,m} + 64 \sum_{i+j=n-2} \sum_{k+l=m} c_{ik}c_{jl} \\ & + 432 \sum_{i+j=m-2} \sum_{k+l=n} d_{k,i}d_{l,j} - \frac{2496}{5} \sum_{i+q=n-1} \sum_{j+r=m-1} c_{ij}d_{qr} \\ & - \frac{576}{5} \sum_{i+j+k=m-2} \sum_{p+q+r=n} c_{pi}c_{qj}c_{rk} - \frac{1008}{5} \sum_{i+j=m-2} \sum_{p+q=n} c_{pi}b_{qj}. \end{aligned} \quad (7.29)$$

We verify, up to the 5th order, that the following recursive relations analogous to (7.26) solve (7.29) and the other two holonomy equations in (7.27)

$$-\frac{5}{18}mc_{nm} = (2n+3m+1)b_{n,m-1}, \quad \frac{5}{12}nc_{nm} = (2n+3m+1)d_{n-1,m}, \quad 3nb_{nm} = -2(m+1)d_{n-1,m+1} \quad (7.30)$$

It is remarkable to note the appearance of the scaling constants β, γ defined earlier in (6.21) in (7.30). Indeed, the integrability conditions (7.20) are precisely equal to (7.30) only upon rescaling

$$\mathcal{L} \rightarrow \alpha \mathcal{L} = \frac{2\pi}{k} \mathcal{L}, \quad \mathcal{W} \rightarrow \beta \mathcal{W} = -\frac{5\alpha}{12} \mathcal{W}, \quad \mathcal{U} \rightarrow \frac{5\alpha}{18} \mathcal{U}. \quad (7.31)$$

Now recall that earlier, we mentioned that (7.51) is necessary if we demand that the bulk equations (6.14), (6.15) are equivalent to the OPE between the stress-energy tensor and the higher spin operators. In this aspect, it is nice to see that the consistency condition for holography is precisely the one that surfaces when we equate the holonomy condition to the integrability condition. With this compelling evidence, on top of the spin-3 case in [49], it is natural to expect this to hold generally for this class of higher-spin gravitational theories.

This strongly suggests that in fact, writing down a valid gravitational thermodynamics for these geometries can be understood as demanding a consistent holography dictionary, in particular, that the boundary \mathcal{W}_N symmetry's Ward identities emerge from the bulk equations.

Let us close this Section by displaying, to some finite order, all the relevant parameters

in terms of τ and the higher-spin chemical potentials. The partition function and entropy can be calculated using (7.29), (7.30), (7.21), (7.19) to be

$$S = 4\pi^2 i \left(\frac{1}{2\alpha\tau} - \frac{4\varrho^2}{5\alpha\tau^5} + \frac{27\mu^2}{50\alpha\tau^7} + \frac{63\mu^3}{125\alpha\tau^{10}} + \frac{21\mu\varrho^2}{5\alpha\tau^8} \right) + \mathcal{O}(4) \quad (7.32)$$

$$\ln Z = 4\pi^2 i \left(\frac{1}{4\alpha\tau} - \frac{\varrho^2}{5\alpha\tau^5} + \frac{7\mu\varrho^2}{10\alpha\tau^8} + \frac{\varrho^4}{\alpha\tau^9} - \frac{91\mu^2\varrho^2}{50\alpha\tau^{11}} + \frac{9\mu^2}{100\alpha\tau^7} + \frac{63\mu^3}{1000\alpha\tau^{10}} \right) + \mathcal{O}(5) \quad (7.33)$$

We expect (7.32) to furnish the leading order approximation of the partition function of any candidate boundary CFT with \mathcal{W}_4 symmetry. From (7.32), we observe that when the higher-spin currents vanish, the BTZ entropy, and thus the familiar area law, is recovered. Finally, the currents $\mathcal{L}, \mathcal{W}, \mathcal{U}$ read as

$$\mathcal{L} = \mathcal{L}_0 - \chi^2 (16\alpha\mathcal{L}_0^2) + f^2 \left(\frac{1008}{25}\alpha^2\mathcal{L}_0^3 \right) - f^3 \left(\frac{4032}{25}\alpha^3\mathcal{L}_0^4 \right) - \chi^2 f \left(\frac{1792}{5}\alpha^2\mathcal{L}_0^3 \right) + \mathcal{O}(4) \quad (7.34)$$

$$\mathcal{W} = -\chi \left(\frac{8}{3\beta}\alpha^2\mathcal{L}_0^2 \right) - \chi f \left(\frac{112}{3\beta}\alpha^3\mathcal{L}_0^3 \right) + \chi^3 \left(\frac{320}{3\beta}\alpha^3\mathcal{L}_0^3 \right) - \chi f^2 \left(\frac{5824}{15\beta}\alpha^4\mathcal{L}_0^4 \right) + \mathcal{O}(4) \quad (7.35)$$

$$\begin{aligned} \mathcal{U} = & -f \left(\frac{16}{5\gamma}\alpha^3\mathcal{L}_0^3 \right) + f^2 \left(\frac{336}{25\gamma}\alpha^4\mathcal{L}_0^4 \right) + \chi^2 \left(\frac{112}{9\gamma}\alpha^3\mathcal{L}_0^3 \right) + \chi^2 f \left(\frac{11648}{45\gamma}\alpha^4\mathcal{L}_0^4 \right) \\ & - f^3 \left(\frac{38528}{125\gamma}\alpha^5\mathcal{L}_0^5 \right) + \mathcal{O}(4), \end{aligned} \quad (7.36)$$

where $\mathcal{L}_0 = \frac{1}{4\alpha\tau^2}$ gives us the relation between the τ and \mathcal{L} in the BTZ when all higher-spin fields are zero. We note that the perturbative expansions are governed by \mathcal{L}, \mathcal{U} having even χ -parity, and \mathcal{W} having odd χ -parity.

With the higher-spin parameters $\mathcal{L}, \mathcal{W}, \mathcal{U}$ computed to be (7.34), (7.35) and (7.36) via the holonomy prescription, it is straightforward to check that the resulting solution does not possess an event horizon. As mentioned earlier, the higher-spin gauge transformations imply that notions like horizons are not gauge-invariant quantities, and the definition of a higher-spin black hole has been based on the holonomy prescription. As explained in [49] and nicely demonstrated in [50], along the gauge orbit of these solutions, one can find a unique geometry which exhibits a smooth horizon.

Although in the general case, even for the simplest spin-3 solution, it is tediously challenging ([50]) to write down the precise gauge that unveils the event horizon explicitly, an easy linearized analysis can already be useful in helping us understand related aspects. Below, we will allude to the linearized spin-4 solution, and furnish some preliminary evidence that the holonomy prescription yields in some gauge, an Euclidean geometry carrying higher-spin fields, all of which are free of conical singularity at the Lorentzian event horizon.

If we keep only terms that are linear in the chemical potentials and the higher-spin currents in (7.12) and (7.13), then explicitly, the holomorphic connection reads

$$\begin{aligned}
 A_{linear} &= L_0 d\rho + (e^\rho L_1 + \mathcal{L}e^{-\rho} L_{-1} + \mathcal{W}e^{-2\rho} W_{-2} + \mathcal{U}e^{-3\rho} U_{-3}) dx^+ + \\
 &+ f (e^{3\rho} U_3 + 3\mathcal{L}e^\rho U_1 + 3\mathcal{L}^2 e^{-\rho} U_{-1} + \mathcal{L}^3 e^{-3\rho} U_{-3}) dx^- \\
 &+ \chi (e^{2\rho} W_2 + 2\mathcal{L}W_0 + \mathcal{L}^2 e^{-2\rho} W_{-2}) dx^-,
 \end{aligned} \tag{7.37}$$

with a similar-looking expression for \bar{A} . Keeping to first order, this gives us the BTZ metric since the corrections to the metric are second order and above, with the horizon located at $\rho_+ = \left(\text{Log}(\sqrt{2\pi|\mathcal{L}|/k})\right)/2$. Also, imposing smoothness of the Euclidean geometry in the (τ, ρ) plane gives us a relation for \mathcal{W}, \mathcal{U} in terms of χ and τ which correspond to the first terms in (7.35) and (7.36), as expected.

For the higher-spin fields defined in (4.14) (setting $\lambda = 4$ since we are working in $SL(4, \mathbb{R})$ Chern-Simons theory), we define smoothness at the horizon by demanding relations among the field components identical to those determined when we impose the regularity of the (τ, ρ) plane of the Euclidean BTZ. Explicitly, we demand

$$\left. \frac{\partial_\rho^2 \psi_{\phi\phi tt}}{\psi_{\phi\phi\rho\rho}} \right|_{\rho_+} = \left. \frac{\partial_\rho^2 \psi_{\phi tt}}{\psi_{\phi\rho\rho}} \right|_{\rho_+} = \frac{16\pi\mathcal{L}}{k} = \left. \frac{\partial_\rho^2 g_{tt}}{g_{\rho\rho}} \right|_{\rho_+} \tag{7.38}$$

where ϕ is the spectator direction. Implementing this procedure by expanding around the horizon, we find that at the first order, the spin-4 field naturally satisfies the condition (7.38) because⁶

$$\psi_{\phi\phi\rho\rho} = \frac{\pi\mathcal{L}}{k} (1 + (\rho - \rho_+)^2) + \dots, \quad \psi_{\phi\phi tt} = \frac{8\mathcal{L}^2\pi^2}{k^2} (\rho - \rho_+)^2 + \dots \tag{7.39}$$

However, the spin-3 fields are not smooth even at the first order, so a natural question is whether, analogous to the spin-3 case [49], one can perform a gauge transformation to let it develop a double zero at the Lorentzian horizon.

After some experimentation, we simply find that the ansatz that was used for the spin-3 case in [49] does the job here as well. The spin-3 fields can be computed to be

$$\psi_{\phi\rho\rho} = \frac{16\pi\mathcal{L}\chi}{k} + \dots, \quad \psi_{\phi tt} = \frac{40\pi\mathcal{W}}{k} (\rho - \rho_+) - \left(\frac{60\pi\mathcal{W}}{k} + \frac{896\pi^2\mathcal{L}^2\chi}{k^2} \right) (\rho - \rho_+)^2 + \dots \tag{7.40}$$

We then gauge transform on the background connection (that gives purely the BTZ) via a gauge parameter $F(\rho)$, i.e.

$$\delta A = d\lambda + [A, \lambda], \quad \lambda = F(\rho) (W_1 - W_{-1}) = -\bar{\lambda} \tag{7.41}$$

⁶We adopt the normalization for components ψ_{abb} and ψ_{aabb} such that their definitions in (4.14) is divided by the number of distinct permutations of the indices.

to obtain

$$\delta\psi_{\phi tt} = -192\sqrt{\frac{2\pi^3|\mathcal{L}^3|}{k^3}} (F(\rho_+) (\rho - \rho_+) + F'(\rho_+) (\rho - \rho_+)^2) \quad (7.42)$$

$$\delta\psi_{\phi\rho\rho} = 24\sqrt{\frac{2\pi|\mathcal{L}|}{k}} F'(\rho_+) \quad (7.43)$$

Comparing (7.42) and (7.40), a straightforward calculation shows that if we set

$$F(\rho_+) = \frac{5\mathcal{W}}{24} \sqrt{\frac{k}{2\pi|\mathcal{L}^3|}}, \quad (7.44)$$

then the term in $(\rho - \rho_+)$ vanishes, removing the previous singularity. Also, upon demanding (7.38), we obtain the first term of (7.35), and thus showing a nice consistency with the holonomy condition.

7.4 The case of $SL(N|N-1)$ black holes

7.4.1 Holonomy conditions and higher-spin black holes

In the $sl(N)$ Chern-Simons theory, higher-spin black holes and conical defects are defined via holonomy conditions. For black holes, the holonomy along the Euclidean time direction is trivial whereas for conical defects, the holonomy along the angular direction ϕ is trivial. In the $sl(3|2) \oplus sl(3|2)$ theory, consider first the case where only the fields $\mathcal{L}, \tilde{\mathcal{L}}$ survive. As shown in 4.3.5, global AdS_3 and extremal BTZ are shown to be supersymmetric solutions in the theory. As reviewed earlier, their corresponding gauge connection Γ can be parametrized as $(L_1 - \mathcal{L}L_{-1})$ of which eigenvalues of $\oint \Gamma_\phi d\phi$ read as $2\pi(0, \sqrt{\mathcal{L}}, -\sqrt{\mathcal{L}}, 2\sqrt{\mathcal{L}}, -2\sqrt{\mathcal{L}})$, with $\mathcal{L} = -1/4$ for global AdS_3 .

If we wick-rotate the time-like direction $t \rightarrow -i\tau$, and let the Euclidean time period $(\Delta\tau)$ be such that the Euclidean manifold is smooth, then for the BTZ black hole, we have $\Delta\tau = \frac{\pi}{\sqrt{\mathcal{L}}}$. For thermal AdS_3 or the Euclidean BTZ, the holonomy along τ -direction reads

$$e^{i\oint \Gamma d\tau} \sim \left[\begin{array}{c|c} \mathbf{1}_{3 \times 3} & 0 \\ \hline 0 & -\mathbf{1}_{2 \times 2} \end{array} \right]. \quad (7.45)$$

We should note that although (7.45) is not the center of $sl(3|2)$, it is a central element among all group elements derived from exponentiating even-graded generators. To see this, we note that (7.45) is a linear combination of the identity and U_0 . On the other hand the holonomy, $e^{\oint \Gamma d\phi}$ possesses the eigenvalues $(1, e^{2\pi\sqrt{\mathcal{L}}}, e^{-2\pi\sqrt{\mathcal{L}}}, e^{4\pi\sqrt{\mathcal{L}}}, e^{-4\pi\sqrt{\mathcal{L}}})$. In the limit $\mathcal{L} \rightarrow 0^+$, the holonomy along the ϕ -direction becomes precisely the identity (the Euclidean-time direction becomes non-compact in this limit). This is the holonomy condition for the massless BTZ. An important result of [49] is that for $sl(N)$ higher-spin black holes constructed in this

manner, the holonomy conditions coincide with the integrability conditions (which we shall review in a moment) that point towards the existence of a partition function containing the boundary source terms.

To find appropriate generalizations of the higher-spin $sl(3)$ black holes in the $sl(3|2)$ theory, it seems natural to begin by considering an ansatz where Γ_+ is in the highest-weight gauge⁷. Paralleling the approach in [49], we begin with

$$\Gamma = b^{-1}a(x^+)b + b^{-1}db, \quad \bar{\Gamma} = b\bar{a}(x^-)b^{-1} + bdb^{-1} \quad (7.46)$$

where $b = e^{\rho L_0}$, and

$$\begin{aligned} a &= (L_1 - \mathcal{L}L_{-1} + \mathcal{W}W_{-2} + \Upsilon A_{-1}) dx^+ + \left(\sum_{i=-2}^2 \chi_i W_i + \sum_{i=-1}^1 \xi_i L_i + \sum_{i=-1}^1 \mathcal{A}_i A_i \right) dx^-, \\ \bar{a} &= -(L_{-1} - \bar{\mathcal{L}}L_1 + \bar{\mathcal{W}}W_2 + \bar{\Upsilon}A_1) dx^- - \left(\sum_{i=-2}^2 \bar{\chi}_i W_{-i} + \sum_{i=-1}^1 \bar{\xi}_i L_{-i} + \sum_{i=-1}^1 \bar{\mathcal{A}}_i A_{-i} \right) dx^-. \end{aligned} \quad (7.47)$$

In the above, the unspecified functions are functions of the boundary co-ordinates (x^\pm). To furnish a prescription for bulk computations in the presence of source terms, the underlying principle is that the sources are associated with generalized boundary conditions for the various bulk higher-spin fields. Similar to the $sl(N)$ story, we wish to propose that functions $(\xi_1, \chi_2, \mathcal{A}_1)$ are proportional to the sources in the putative CFT, and below, we will support this claim at the level of perturbation theory in the sources. This is done simply by comparing bulk field equations to the CFT's Ward identities, which will be performed explicitly below for the chiral sector.

Now, the bulk field equations are just the conditions for a flat connection. For the ansatz (7.46), it is straightforward to solve for the various functions in terms of products of derivatives of the following set $(\mathcal{L}, \Upsilon, \mathcal{W}, \chi \equiv \chi_2, \xi \equiv \xi_1, \mathcal{A} \equiv \mathcal{A}_1)$, in particular, with the fields $(\mathcal{L}, \Upsilon, \mathcal{W})$ obeying the following differential equations.

$$\begin{aligned} \partial_- \mathcal{L} &= \frac{1}{2} \partial_+^3 \xi + 2\mathcal{L} \partial_+ \xi + \xi \partial_+ \mathcal{L} + \mathcal{A} \partial_+ \Upsilon + 2\Upsilon \partial_+ \mathcal{A} - 6\mathcal{W} \partial_+ \chi - 4\chi \partial_+ \mathcal{W}, \\ \partial_- \Upsilon &= \frac{1}{2} \partial_+^3 \mathcal{A} + 2\mathcal{L} \partial_+ \mathcal{A} + \mathcal{A} \partial_+ \mathcal{L} + \xi \partial_+ \Upsilon + 2\Upsilon \partial_+ \xi - 6\mathcal{W} \partial_+ \chi - 4\chi \partial_+ \mathcal{W}, \\ \partial_- \mathcal{W} &= 3\mathcal{W}(\partial_+ \xi + \partial_+ \mathcal{A}) + (\xi + \mathcal{A}) \partial_+ \mathcal{W} + \frac{8}{3} ((\mathcal{L} + \Upsilon)^2 \partial_+ \chi + \chi(\mathcal{L} + \Upsilon) \partial_+(\mathcal{L} + \Upsilon)) \\ &\quad + \frac{1}{24} \partial_+^5 \chi + \frac{5}{6} (\mathcal{L} + \Upsilon) \partial_+^3 \chi + \frac{5}{4} \chi \partial_+(\mathcal{L} + \Upsilon) + \frac{3}{4} \partial_+ \chi (\partial_+^2 \mathcal{L} + \partial_+^2 \Upsilon) + \frac{1}{6} \chi (\partial_+^3 \mathcal{L} + \partial_+^3 \Upsilon). \end{aligned} \quad (7.48)$$

$$(7.49)$$

On the other hand, from the perspective of the boundary CFT, upon the insertion of the

⁷We set the abelian field $\mathcal{U} = 0$ for simplicity in this Section.

source terms in the Lagrangian, in the form

$$\int d^2x (\chi(x)\mathcal{W}(x) + \mathcal{A}(x)\Upsilon(x) + \xi(x)\mathcal{L}(x) + \bar{\chi}(x)\bar{\mathcal{W}}(x) + \bar{\mathcal{A}}(x)\bar{\Upsilon}(x) + \bar{\xi}(x)\bar{\mathcal{L}}(x)) \quad (7.50)$$

the various expectation values of $(\mathcal{L}, \Upsilon, \mathcal{W})$ will pick up the source terms due to the singular terms in the OPEs, as explained in [49]. We have assumed that the source terms are precisely the set of (ξ, \mathcal{A}, χ) in the bulk ansatz. To validate this assumption, we need to ensure that the OPEs among the fields belong to that of the $\mathcal{N} = 2 \mathcal{W}_3$ algebra. To conveniently compare with the bulk equations (7.48), we first switch to Euclidean coordinates $(z = it + \phi, \bar{z} = -it + \phi)$, and perform the re-scaling for the fields in the bulk ansatz

$$\mathcal{L} \rightarrow \alpha\mathcal{L}, \Upsilon \rightarrow \beta\Upsilon, \mathcal{W} \rightarrow \gamma\mathcal{W}. \quad (7.51)$$

After some algebra, we find that (7.48) lead to the OPEs

$$\mathcal{L}(z)\mathcal{L}(0) \sim \frac{3}{\alpha z^4} + \frac{2\mathcal{L}}{z^2} + \frac{\partial\mathcal{L}}{z}, \quad (7.52)$$

$$\Upsilon(z)\Upsilon(0) \sim \frac{3}{\beta z^4} + \left(\frac{\alpha}{\beta}\right) \left(\frac{2\mathcal{L}}{z^2} + \frac{\partial\mathcal{L}}{z}\right), \quad (7.53)$$

$$\mathcal{L}(z)\mathcal{W}(0) \sim \frac{3\mathcal{W}}{z^2} + \frac{\partial\mathcal{W}}{z}, \quad (7.54)$$

$$\Upsilon(z)\mathcal{W}(0) \sim \frac{3\mathcal{W}}{z^2} + \frac{\partial\mathcal{W}}{z}, \quad (7.55)$$

$$\mathcal{W}(z)\Upsilon(0) \sim \left(-\frac{\gamma}{\beta}\right) \left(\frac{6\mathcal{W}}{z^2} + \frac{4\mathcal{W}'}{z}\right), \quad (7.56)$$

$$\mathcal{W}(z)\mathcal{L}(0) \sim \left(-\frac{\gamma}{\alpha}\right) \left(\frac{6\mathcal{W}}{z^2} + \frac{4\mathcal{W}'}{z}\right) \quad (7.57)$$

$$\mathcal{L}(z)\Upsilon(0) \sim \frac{2\Upsilon}{z^2} + \frac{\partial\Upsilon}{z}, \quad (7.58)$$

$$\Upsilon(z)\mathcal{L}(0) \sim \left(\frac{\beta}{\alpha}\right) \left(\frac{2\Upsilon}{z^2} + \frac{\partial\Upsilon}{z}\right), \quad (7.59)$$

$$\begin{aligned} \mathcal{W}(z)\mathcal{W}(0) \sim & \frac{5}{\gamma z^6} + \frac{1}{6\gamma} \left(\frac{\partial^3(\alpha\mathcal{L} + \beta\Upsilon)}{z} + \frac{9\partial^2(\alpha\mathcal{L} + \beta\Upsilon)}{2z^2} + \frac{15\partial(\alpha\mathcal{L} + \beta\Upsilon)}{z^3} + \frac{30(\alpha\mathcal{L} + \beta\Upsilon)}{z^4} \right) \\ & + \frac{4}{3\gamma} \left(\frac{2(\alpha\mathcal{L} + \beta\Upsilon)^2}{z^2} + \frac{\partial(\alpha\mathcal{L} + \beta\Upsilon)^2}{z} \right), \end{aligned} \quad (7.60)$$

where in particular, we note the non-linear terms arising in the $\mathcal{W}(z)\mathcal{W}(0)$ OPE. Indeed, we find that we can choose appropriate scaling parameters (α, β, γ) such that (7.52) are actually those of the $\mathcal{N} = 2 \mathcal{W}_3$ algebra restricted to the bosonic fields (excluding the $U(1)$ field omitted for simplicity of discussion). For a precise comparison, let us adopt the basis

of the super \mathcal{W}_3 algebra as constructed in reference [73]. We find that upon choosing

$$\alpha = \beta = -2\gamma = \frac{3}{c} \quad (7.61)$$

and re-defining

$$T = \frac{1}{2}(\mathcal{L} + \Upsilon), \quad \bar{T} = \frac{1}{2}(\mathcal{L} - \Upsilon), \quad \tilde{W} = 2W, \quad (7.62)$$

the fields (T, \bar{T}, \tilde{W}) generate the relevant OPEs in eqn. 19 of reference [73]. In particular, we have established that χ acts as the source for the spin-3 field \mathcal{W} .

We now turn to the subject of constructing higher-spin black hole solutions, and investigating if the integrability of the higher-spin charges are related to the holonomy condition (7.45). Consider the same ansatz (7.47) but now with all the functions set to be constants. The connection now reads as⁸

$$\begin{aligned} a_+ &= L_1 - \mathcal{L}L_{-1} - \frac{1}{2}\mathcal{W}W_{-2} + \Upsilon A_{-1}, \\ a_- &= \left[(2\mathcal{W}\chi + \Upsilon\mathcal{A})L_{-1} + \left((\mathcal{L} - \Upsilon)^2\chi - \frac{1}{2}\mathcal{A}\mathcal{W} \right) W_{-2} + \chi W_2 + 2\chi(-\mathcal{L} + \Upsilon)W_0 \right. \\ &\quad \left. - (-2\mathcal{W}\chi + \mathcal{L}\mathcal{A})A_{-1} + \mathcal{A}A_1 \right] \end{aligned} \quad (7.63)$$

where, for simplicity of discussion, we have set ξ - the source for \mathcal{L} - to be zero. Apart from \mathcal{L} , the parameter space of the ansatz consists of the fields \mathcal{W}, Υ and their conjugate potentials χ, \mathcal{A} .

What are the ‘integrability conditions’? Now, in the context of the solution as described in (7.63), the chemical potentials for the fields \mathcal{W} and Υ read as

$$\mu_{\mathcal{W}} \equiv -\tau\chi, \quad \mu_{\Upsilon} \equiv -\tau\mathcal{A}$$

respectively, up to some normalization. The integrability conditions refer to the relations

$$\frac{\partial \mathcal{L}}{\partial \mu_{\mathcal{W}}} = \frac{\partial \mathcal{W}}{\partial \tau}, \quad \frac{\partial \mathcal{L}}{\partial \mu_{\Upsilon}} = \frac{\partial \Upsilon}{\partial \tau}, \quad \frac{\partial \mathcal{W}}{\partial \mu_{\Upsilon}} = \frac{\partial \Upsilon}{\partial \mu_{\mathcal{W}}}, \quad (7.64)$$

which is motivated by the existence of a partition function at the boundary that reads

$$Z(\tau, \mu_{\mathcal{W}}, \mu_{\Upsilon}) = \text{Tr} \left(e^{k(\tau\mathcal{L} + \mu_{\mathcal{W}}\mathcal{W} + \mu_{\Upsilon}\Upsilon - \bar{\tau}\bar{\mathcal{L}} + \bar{\mu}_{\mathcal{W}}\bar{\mathcal{W}} + \bar{\mu}_{\Upsilon}\bar{\Upsilon})} \right) \quad (7.65)$$

where, for a static BTZ limit, the chiral and anti-chiral quantities are related as

$$\tau = -\bar{\tau}, \mu_{\mathcal{W}} = -\bar{\mu}_{\mathcal{W}}, \mu_{\Upsilon} = -\bar{\mu}_{\Upsilon}, \mathcal{L} = \bar{\mathcal{L}}, \mathcal{W} = -\bar{\mathcal{W}}, \Upsilon = -\bar{\Upsilon}. \quad (7.66)$$

⁸The factor of $-1/2$ was inserted in accordance with (7.61).

Let us now demand the holonomy to be that of (7.45) and study its compatibility with (7.64). For the solution (7.63), after some algebra, we find that the five eigenvalues can be reduced nicely to the three roots of one cubic equation and the remaining two taking a rather nice form

$$\text{Eigenvalues} = \{\pm(\mathcal{A} - 1)\sqrt{\mathcal{L} + \Upsilon}, \text{ Roots of } x^3 - Bx + C = 0\}. \quad (7.67)$$

where B, C are polynomial functions in $\{\mathcal{L}, \mathcal{W}, \Upsilon, \chi, \mathcal{A}\}$, which read as

$$B \equiv -\frac{4}{3} (16\mathcal{L}^2\chi^2 - 3(1 + \mathcal{A})^2\Upsilon + 16\chi^2\Upsilon^2 + \mathcal{L}(3 + 6\mathcal{A} + 3\mathcal{A}^2 - 32\chi^2\Upsilon) + 18(1 + 2\mathcal{A})\chi\mathcal{W}) \quad (7.68)$$

$$C \equiv 4 \left[256\mathcal{L}^3\chi^3 - 144(1 + \mathcal{A})^2\chi\Upsilon^2 - 48\mathcal{L}^2\chi(3 + 6\mathcal{A} + 3\mathcal{A}^2 + 16\chi^2\Upsilon) - 27(1 + \mathcal{A})^2(-1 + 2\mathcal{A})\mathcal{W} \right. \\ \left. + 432(3 + 2\mathcal{A})\chi^2\Upsilon\mathcal{W} + 48\mathcal{L}\chi(6(1 + \mathcal{A})^2\Upsilon + 16\chi^2\Upsilon^2 - 9(3 + 2\mathcal{A})\chi\mathcal{W}) - 128\chi^3(2\Upsilon^3 + \mathcal{W}^2) \right] \quad (7.69)$$

We now impose the BTZ holonomy condition by solving for the fields $(\mathcal{L}, \mathcal{W}, \Upsilon)$ in terms of inverse temperature τ , and potentials \mathcal{A}, χ . The ‘ $\mathbf{1}_{2 \times 2}$ ’ factor in (7.45) gives us conveniently, from (7.67),

$$\Upsilon = \frac{\pi^2}{\tau^2(1 - \mathcal{A})^2} - \mathcal{L}(\tau, \mathcal{A}, \chi). \quad (7.70)$$

For the ‘ $\mathbf{1}_{3 \times 3}$ ’ factor in (7.45), the zero eigenvalue demands setting $C = 0$, which in turn gives us a quadratic equation for \mathcal{W} . The other two eigenvalues of $(\pm 2\pi i)$ are then the roots of the cubic equation in (7.67), i.e.

$$B = -\frac{4\pi^2}{\tau^2}. \quad (7.71)$$

Substituting the expressions for Υ and \mathcal{W} into B in (7.71), we can re-arrange the various terms to obtain a quartic equation for \mathcal{L} . Although an analytic, closed form for \mathcal{L} is not manageable, we can perform a perturbative analysis to any desired order rather easily, and obtain \mathcal{L} (and thus also Υ and \mathcal{W}) as a Taylor series in (\mathcal{A}, χ) . Below, we present the expressions for the various fields and the free energy up to some order.

$$\begin{aligned} \mathcal{L} &= \frac{\pi^2}{\tau^2} + \frac{40\pi^4\chi^2}{3\tau^4} + \frac{3\pi^2\mathcal{A}^2}{\tau^2} - \frac{80\pi^4\mathcal{A}\chi^2}{\tau^4} + \frac{1280\pi^6\chi^4}{3\tau^6} + \frac{280\pi^4\chi^2\mathcal{A}^2}{\tau^4} + \frac{5\pi^2\mathcal{A}^4}{\tau^2} + \mathcal{O}(5), \\ \mathcal{W} &= \frac{16\pi^4\chi}{3\tau^4} - \frac{80\pi^4\chi\mathcal{A}}{3\tau^4} + \frac{80\pi^4\chi\mathcal{A}^2}{\tau^4} + \frac{5120\pi^6\chi^3}{27\tau^6} - \frac{5120\pi^6\chi^3\mathcal{A}}{3\tau^6} - \frac{560\pi^4\chi\mathcal{A}^3}{3\tau^4} + \mathcal{O}(5) \\ \Upsilon &= \frac{2\pi^2\mathcal{A}}{\tau^2} - \frac{40\pi^4\chi^2}{3\tau^4} + \frac{4\pi^2\mathcal{A}^3}{\tau^2} + \frac{80\pi^4\chi^2\mathcal{A}}{\tau^4} - \frac{1280\pi^6\chi^4}{3\tau^6} - \frac{280\pi^4\chi^2\mathcal{A}^2}{\tau^4} + \mathcal{O}(5). \end{aligned} \quad (7.72)$$

It can then be checked that the integrability conditions of (7.64) are indeed satisfied, and that free energy in (7.65) is integrable. Indeed, the free energy can be computed from (7.65) and (7.72) after taking into account the contributions from the anti-chiral sector, and we obtain

$$\ln Z = -2k \left(\frac{\pi^2}{\tau} + \frac{8\pi^4 \mu_{\mathcal{W}}^2}{3\tau^5} + \frac{\pi^2 \mu_{\Upsilon}^2}{\tau^3} + \frac{40\pi^4 \mu_{\mathcal{W}}^2 \mu_{\Upsilon}}{3\tau^6} + \frac{40\pi^4 \mu_{\mathcal{W}}^2 \mu_{\Upsilon}^2}{\tau^7} + \frac{\pi^2 \mu_{\Upsilon}^4}{\tau^5} \right) + \mathcal{O}(5) \quad (7.73)$$

If further, as first proposed in [49], the entropy is defined via a Legendre transform of the free energy, then the first law of black hole thermodynamics is naturally satisfied. Our black hole solutions thus provide a concrete evidence that defining higher-spin black holes via the BTZ holonomy condition, as originally proposed in the $sl(N)$ case [49, 50], generalizes consistently to the $sl(N|N-1)$ theories. On the other hand, as briefly pointed out earlier, there exists a subtle difference with the $sl(N)$ theories, since (7.45) is not the center of the group. One may ask if imposing the supermatrix identity as the holonomy condition can also be compatible with the integrability conditions, for example, by taking $\mathcal{L} = -\Upsilon$ in (7.67), and whether there exists a gauge in which we can find a smooth horizon. We leave these questions for future investigations.

Finally, we point out that these solutions do not preserve supersymmetries due to the higher-spin fields they carry. In the next section, we will study a class of solutions that do preserve some amount of supersymmetry. It turns out that this requires us to strip off all the higher-spin fields except for the one conjugate to the lowest weight generator.

7.5 Other classical solutions: RG flow solutions

7.5.1 Supersymmetric solutions with higher-spin fields

If one demands the non-vanishing of any of the fields $\mathcal{W} = \Upsilon = \chi = \mathcal{A} = 0$, then for the ansatz (7.63), (4.71) implies $\Upsilon^{-1} \equiv \Upsilon = 0$, which forces either $\mathcal{W}^{-2} \equiv \mathcal{W} = 0$ or the spinor component $\epsilon_2 = 0$, the latter condition fixing $\mathcal{L} = 0$. Solving the other two spinor equations and adding the $U(1)$ charge yields the following supersymmetric class of connections⁹

$$\Gamma = (e^\rho L_1 + 6\mathcal{U} U_0) dx^+ + \mu e^{2\rho} W_2 dx^- + L_0 d\rho, \quad (7.74)$$

$$\tilde{\Gamma} = \left(-e^\rho L_{-1} - 6\tilde{\mathcal{U}} U_0 \right) dx^- - \tilde{\mu} e^{2\rho} W_{-2} dx^+ - L_0 d\rho. \quad (7.75)$$

Just as in the case of a vanishing μ , demanding the holonomy $e^{\oint \Gamma d\phi}$ to be trivial turns out to be equivalent to (4.76). There is no additional constraint arising from the parameter μ , but as in the non-supersymmetric theory, its non-vanishing implies that the metric grows as $e^{4\rho}$ and is thus asymptotic to AdS_3 with half its original ($\mu = 0$) radius. Invoking the

⁹We note that this class of solutions is the RG flow solution of [50], carrying some $U(1)$ charge.

relation

$$g_{\mu\nu} = \frac{1}{\text{str}(L_0^2)} \text{str}(e_\mu \cdot e_\nu), \quad e = \frac{1}{2} (\Gamma - \tilde{\Gamma}), \quad (7.76)$$

we can compute the line element to be

$$ds^2 = d\rho^2 - \left(e^{2\rho} + \frac{16}{3} \mu \tilde{\mu} e^{4\rho} + 2\mathcal{U}\tilde{\mathcal{U}} \right) dx^+ dx^- + \left(\mathcal{U}^2 dx_+^2 + \tilde{\mathcal{U}}^2 dx_-^2 \right). \quad (7.77)$$

To ensure the correct signature at infinity, we impose the parameter constraint

$$\mu \tilde{\mu} \geq 0.$$

Further, one can compute the spin-3 field $Q \sim \text{str}(e \cdot e \cdot e)$ (in the absence of \mathcal{U} and $\tilde{\mathcal{U}}$) to read

$$Q \sim e^{4\rho} (\tilde{\mu} dx_+^3 + \mu dx_-^3). \quad (7.78)$$

As explained earlier, this class of solutions preserves two real supercharges in each chiral sector. We should also remark that (7.74) and (7.75) can be further generalized to include a term $\sim W_2$ in Γ_+ (and correspondingly another term $\sim W_{-2}$ in $\bar{\Gamma}_-$), but including these terms destroy the asymptotically AdS_3 condition.

7.6 Extension of results to $sl(N|N-1)$ for a general finite N .

7.6.1 On the Killing spinor equations

Recall from equation (4.20) how the general $sl(N|N-1)$ element can be decomposed as a direct sum of $sl(2)$ multiplets. In Appendix B, we present explicit expressions for the generators and commutator relations for $sl(N|N-1)$. Below, we shall briefly present some important points following [76]. The gravitational $sl(2)$ is generated by L_m^1 , of which the other generators are irreducible representations of. They can be expressed as

$$L_0 = \sqrt{\frac{N(N+1)}{12}} \left(\sqrt{N+2} T_0^1 + \sqrt{N-1} U_0^1 \right), \quad (7.79)$$

$$L_{\pm 1} = \sqrt{\frac{N(N+1)}{6}} \left(\sqrt{N+2} T_{\pm 1}^1 + \sqrt{N-1} U_{\pm 1}^1 \right), \quad (7.80)$$

while V_m^1 generates $h^{(1)}$ (another spin-1 multiplet of $sl(2)$). The spin-0 sector (i.e. $h^{(0)}$) is generated by

$$U = \sqrt{N} T_0^0 + \sqrt{N+1} U_0^0 \quad (7.81)$$

which commutes with all even-graded generators. Together with $L_0, L_{\pm 1}$, they generate the even part of the $sl(2|1)$ sub-superalgebra, whereas $Q_r^{(\frac{1}{2})}$ and $\bar{Q}_r^{(\frac{1}{2})}$ generate the odd part of

it. To describe the other sectors, it is convenient to define

$$\begin{aligned} L_m^s &\equiv \sqrt{\frac{(N+s+1)!}{(2s+1)!(N-s)!}} T_m^s + \sqrt{\frac{(N+s)!}{(2s-1)!(N-s-1)!}} U_m^s, \\ V_m^s &\equiv \sqrt{\frac{(N+s+1)!}{(2s+1)!(N-s)!}} T_m^s - \sqrt{\frac{(N+s)!}{(2s-1)!(N-s-1)!}} U_m^s, \end{aligned} \quad (7.82)$$

which generate $g^{(s)}$ and $h^{(s)}$, $s = 3, 4 \dots N-2$ respectively. Finally, T_m^{N-1} generate $g^{(N-1)}$.

In Appendix B, we collect the structure constants, which can be derived via intertwining properties of the Clebsch-Gordan coefficients. Just as we have demonstrated for the $sl(3|2)$ case, one can invoke (D-1) to derive the supersymmetry transformation laws. In particular, we wish to write down the Killing spinor equation to classify the classical solutions. This relies on the commutation relations between $\{T_m^{s=0,1,2}, U_m^{s=0,1,2}\}$ and $\{Q_r^{(\frac{1}{2})}, \bar{Q}_r^{(\frac{1}{2})}\}$, which we display below to be explicit. Consider first the commutation relations between the spin-0 and spin-1 fields with the Q 's. Keeping the same notation as for the $sl(3|2)$ case, we denote the generators of the spin-1 multiplet $h^{(1)}$ by $A \equiv V_m^1$, we have¹⁰

$$\begin{aligned} [A_0, Q_r^{(\frac{1}{2})}] &= r \left(\frac{2N+1}{3} \right) Q_r^{(\frac{1}{2})} - \frac{\sqrt{2(N-1)(N+2)}}{3} Q_r^{(\frac{3}{2})}, \\ [A_{\pm 1}, Q_{\mp \frac{1}{2}}^{(\frac{1}{2})}] &= 2\sqrt{\frac{(N+2)(N-1)}{6}} Q_{\mp \frac{3}{2}}^{(\frac{3}{2})}, \\ [A_{\pm 1}, Q_{\pm \frac{1}{2}}^{(\frac{1}{2})}] &= \mp \left(\frac{2N+1}{3} \right) Q_{\mp \frac{1}{2}}^{(\frac{1}{2})} + \frac{2}{3} \sqrt{\frac{(N+2)(N-1)}{2}} Q_{\mp \frac{3}{2}}^{(\frac{3}{2})}, \\ [U, Q_r^{(\frac{1}{2})}] &= \frac{1}{\sqrt{N(N+1)}} Q_r^{(\frac{1}{2})}, \quad [U, \bar{Q}_r^{(\frac{1}{2})}] = -\frac{1}{\sqrt{N(N+1)}} \bar{Q}_r^{(\frac{1}{2})}, \end{aligned} \quad (7.83)$$

noting that the gravitational $sl(2)$ subalgebra is generated by L_m^1 . This serves as a consistency check that the $sl(2|1)$ sub-superalgebra is embedded in the same manner. Using (7.83), we can write down the Killing spinor equation generalizing (4.57) as

$$\left(d + i \frac{1}{\sqrt{N(N+1)}} \mathcal{U} - \frac{1}{2} \left((e^a + \omega^a) + \left(\frac{2N+1}{3} \right) \Upsilon^a \right) \gamma_a \right) \epsilon = 0. \quad (7.84)$$

We observe that it is essentially the same except for N -dependent scaling constants which can be absorbed into the various fields. The other fields do not play any role here, basically due to a simple constraint imposed by the Wigner-6j symbols appearing in the structure

¹⁰The commutation relations between A and \bar{Q} are identical, and thus omitted.

constants in the relevant commutation relations. Recall that the Wigner-6j symbol

$$\left\{ \begin{array}{ccc} s & s' & s'' \\ a & b & c \end{array} \right\} = 0 \quad \text{unless } s = |s' - s''|, \dots, s + s''. \quad (7.85)$$

For the supersymmetry transformation laws, the index s' in (7.85) takes the value of $\frac{1}{2}$. Since (7.84) is derived from the vanishing of $\delta\psi^{(\frac{1}{2})}$, the index s'' takes the value of $\frac{1}{2}$ as well, and the relevant values for s are restricted to $\{0, 1\}$. Thus, (7.84) takes on essentially the same form as (4.57).

Similarly, for the analogues of (4.58) in the $sl(N|N-1)$ higher-spin theory, for each $\frac{3}{2} \leq s'' \leq \frac{2N-3}{2}$, the vanishing of $\delta\psi^{(s'')}$ induces a constraint equation that involves even-graded fields of spin index $s = s'' \pm \frac{1}{2}$. Hence, the equation (4.58) generalizes to a series of linear constraint equations (each labelled by s and r) of the following form

$$\sum_{l=-\frac{1}{2}, \frac{1}{2}} \left(\zeta_{rl}^{(s-\frac{1}{2})} \mathcal{V}_{l-r}^{(s-\frac{1}{2})} + \zeta_{rl}^{(s+\frac{1}{2})} \mathcal{V}_{l-r}^{(s+\frac{1}{2})} + \eta_{rl}^{(s-\frac{1}{2})} \mathcal{L}_{l-r}^{(s-\frac{1}{2})} + \eta_{rl}^{(s+\frac{1}{2})} \mathcal{L}_{l-r}^{(s+\frac{1}{2})} \right) \epsilon_l = 0 \quad (7.86)$$

which ensures the vanishing of the rest of the fermionic fields other than $\psi^{(\frac{1}{2})}$, i.e.

$$\delta\psi_r^{(s)} = 0, \quad s = \frac{3}{2}, \frac{5}{2}, \dots, \frac{2N-3}{2}, \quad |r| \leq s,$$

where η, ζ are constant matrices that can be computed straightforwardly from the structure constants. We note that

$$\eta^{(1)} = 0, \quad \zeta^{(N+\frac{1}{2})} = 0, \quad (7.87)$$

since $[L_m^1, \psi^{(\frac{1}{2})}] \sim \psi^{(\frac{1}{2})}$, and the multiplets $h^{(s)}$ terminate at $s = N - 2$. Similar relations hold for the barred variables.

We note that the above conclusions extend to the case of the infinite-dimensional algebra $shs[\lambda]$, since as explained in [76], the generators (7.82) are those of $shs[\lambda]$, provided we analytically continue the integer N to a positive real value λ and abolish the restrictions to the index s . Further, there is a certain limit that involves taking $N \rightarrow \infty$, in which case we have a higher-spin gravity theory based on the infinite-dimensional algebra $shs[\infty]$.¹¹ Essentially, we perform a redefinition of the generators as follows

$$\tilde{L}_m^s = N^{-s+1} L_m^s, \quad \tilde{V}_m^s = N^{-s} V_m^s, \quad \tilde{Q}_m^s = N^{1/2-s} Q_m^s, \quad \tilde{\bar{Q}}_m^s = N^{1/2-s} \bar{Q}_m^s, \quad (7.88)$$

Then, we take the limit $N \rightarrow \infty$ and further perform another step of redefinition

$$L_m^s = \tilde{L}_m^s - \frac{(s-1)}{2} \tilde{V}_m^s, \quad (7.89)$$

¹¹As explained in [76], this is the supersymmetric analogue of $hs[\infty]$ which can be physically understood as the algebra of area-preserving diffeomorphisms of 2D hyperboloids.

noting that the gravitational $sl(2)$ is generated by L_m^1 . In this limit, we can choose the generator U_0 to be normalized as

$$U_0 = -L_0^0 - 2V_0^0$$

and we check, using the new commutation relations obtained after this limiting procedure, that the Killing spinor equation (7.84) is preserved and reads

$$\left(d + i\mathcal{U} - \frac{1}{2} \left((e^a + \omega^a) + \frac{2}{3} \Upsilon^a \right) \gamma_a \right) \epsilon = 0. \quad (7.90)$$

Thus, the supersymmetry classification of ordinary solutions like the BTZ remains the same. Like in the finite-dimensional gauge algebra case, one can derive straightforwardly the rest of the supersymmetry transformation laws based on the structure constants - which we review in the Appendix B.

7.6.2 On solutions with higher spin fields

In the general $sl(N|N-1) \oplus sl(N|N-1)$ theory, the supersymmetry properties of solutions without higher-spin charges are thus identical as described earlier in Section 4.3.6. In the following, we will briefly present the supersymmetric solutions with non-zero higher spin fields that we studied earlier in Section 7.4.1 for the $sl(3|2)$ theory.

We begin with the ‘highest-weight’ ansatz

$$\Gamma = b^{-1} \left(L_1 - \mathcal{L}L_{-1} + \sum_{s=2}^{N-1} \mathcal{L}_{-s}^{(s)} L_{-s}^{(s)} + \sum_{r=1}^{N-2} \mathcal{V}_{-r}^{(r)} V_{-r}^{(r)} + \mathcal{U}U_0 \right) b dx^+ + \Gamma_- dx^- + b^{-1} db, \quad b \equiv e^{\rho L_0}, \quad (7.91)$$

where we write

$$\Gamma_- = \sum_{s=2}^{N-1} \sum_{m=-s}^s \mu_m^{(s)} L_m^{(s)} + \sum_{r=1}^{N-2} \sum_{m=-r}^r \Phi_m^{(r)} V_m^{(r)}. \quad (7.92)$$

Here we are interested in solutions of the form (7.91) which preserve some amount of supersymmetry. Applying the constraints (7.86) in the x^+ direction, the higher-spin fields $\mathcal{L}_{-s}^{(s)}$ and $V_{-r}^{(r)}$ in (7.91) must vanish, upon which we can compute Γ_- rather simply. The various generators transform under the embedded gravitational $sl(2)$ as

$$[L_m, V_r^{(s)}] = (sm - r) V_{m+r}^{(s)}, \quad [L_m, L_r^{(s)}] = (sm - r) L_{m+r}^{(s)}. \quad (7.93)$$

Then, it is straightforward to show that for each multiplet in (7.92), the various fields can be solved in terms of \mathcal{L} and $\{\mu_s^{(s)}, \Phi_s^{(s)}\}$. Explicitly, we have

$$\mu_{s-2m}^{(s)} = {}^s C_m (-\mathcal{L})^m \mu_s^{(s)}, \quad m = 1, 2, \dots, s, \quad (7.94)$$

$$\mu_{s-1}^{(s)} = \mu_{s-3}^{(s)} = \dots = \mu_{-s+1}^{(s)} = 0, \quad (7.95)$$

and identically for the fields $\Phi_r^{(s)}$. Applying the same constraint equations in the x^- direction further kills off all fields, including \mathcal{L} , except for $\mu_{N-1}^{(N-1)}$. Thus, the supersymmetric solution reads simply as

$$\Gamma = (e^\rho L_1 + \mathcal{U}U_0) dx^+ + \left(e^{(N-1)\rho} \mu_{N-1}^{(N-1)} L_{N-1}^{(N-1)} \right) dx^- + L_0 d\rho, \quad (7.96)$$

with a similar expression for the anti-chiral sector. We observe that no component fields of the entire set of multiplets $h^{(s)}$ in (4.20) survive in a supersymmetric ansatz of the form (7.91). Apart from the highest-spin field $\mu_{N-1}^{(N-1)}$, this class of solutions is parametrized by the fields \mathcal{U} and $\tilde{\mathcal{U}}$ which satisfy the quantization condition (4.76). Just like the $\mathcal{L} = 0$ solutions embedded in $osp(2|2)$ supergravity, in this case, we note that two real supercharges are preserved in each sector.

7.6.3 On holonomy conditions and conical defects solutions

As discussed earlier, holonomy conditions play an important role in the study of $sl(N) \oplus sl(N)$ Chern-Simons theory. In Section 4.3.7, we have briefly studied the $sl(3|2)$ case, and it is straightforward to state some results for the general case of $sl(N|N-1)$. For global AdS_3 , the gauge connection shares identical eigenvalues as iL_0 , and the eigenvalues of $\oint \Gamma_\phi d\phi$ are those of $sl(N)$ and $sl(N-1)$. When exponentiated, the holonomy reads as

$$e^{\oint \Gamma_\phi d\phi} \sim \left[\begin{array}{c|c} \pm \mathbf{1}_{N \times N} & 0 \\ \hline 0 & \mp \mathbf{1}_{(N-1) \times (N-1)} \end{array} \right], \quad (7.97)$$

with the sign depending on whether N is odd or even. We note that (7.97) is, for any N , a linear combination of the identity and the $u(1)$ generator. We take both the supermatrix identity and (7.97) to be the defining conditions for a smooth holonomy. We also note that it is straightforward to show that, apart from unimportant normalization constants which can be absorbed via a field redefinition, imposing the holonomy condition on (7.96) yields also (4.76), which is also the (anti-)periodicity condition to be imposed on the Killing spinors.

In the non-supersymmetric $sl(N)$ Chern-Simons theory, there is an interesting class of solutions which has been argued to be conical defects (and surpluses) spacetimes [31]. They play a critical role in the holography duality conjecture, and we would like to investigate if there are natural generalizations of them in the supersymmetric theory. First, we begin with a brief review of some elementary aspects of these solutions following [31] but in the context of $sl(N|N-1) \oplus sl(N|N-1)$ theory.

Consider the ansatz

$$\Gamma = b^{-1} \left(\sum_{k=1}^{2N-1} B_k(a_k, b_k) \right) b dx^+ + L_0 d\rho, \quad \tilde{\Gamma} = -b \left(\sum_{k=1}^{2N-1} B_k(c_k, d_k) \right) b^{-1} dx^- - L_0 d\rho, \quad b \equiv e^{\rho L_0}, \quad (7.98)$$

with

$$[B_k(x, y)]_{ij} = x\delta_{i,k}\delta_{j,k+1} - y\delta_{i,k+1}\delta_{j,k}.$$

The constant matrices B_k are linear combination of weight-one generators (i.e. $T_{\pm 1}, U_{\pm 1}$). They are diagonalizable with imaginary eigenvalues. The interesting solutions are found by imposing three essential conditions, namely (i) the metric induced by the connection is locally AdS_3 , (ii) the holonomy along the ϕ -direction is trivial and (iii) the stress-energy tensor is negative and bounded from below by its value for global AdS_3 .

To conveniently list the equivalence classes of solutions, we can restrict the ansatz to consist of the following maximal commuting set as parametrized by B_k .

$$\begin{aligned} &\text{For even } N, k = 1, 3, \dots, 2N - 3. \\ &\text{For odd } N, k = 1, 3, \dots, N - 2, N + 1, \dots, 2N - 2. \end{aligned} \quad (7.99)$$

with $a_k = b_k = c_k = d_k$ for all cases. Our choice is slightly different from the $sl(N) \oplus sl(N)$ theories considered in [31] for the cases of odd N , due to the fact that we are now taking a supertrace instead of the ordinary matrix trace. Then, from (7.98), we find the line element

$$ds^2 = d\rho^2 - \frac{1}{\text{str}(L_0^2)} (e^\rho + \Lambda e^{-\rho})^2 dt^2 + \frac{1}{\text{str}(L_0^2)} (e^\rho - \Lambda e^{-\rho})^2 d\phi^2, \quad (7.100)$$

$$\Lambda \equiv \frac{1}{2} \left(\sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} a_{2k-1}^2 - \sum_{k=1}^{N-1-\lfloor \frac{N}{2} \rfloor} a_{2k+N-1}^2 \right) = -\frac{1}{4} \text{str}(\Gamma_+^2), \quad (7.101)$$

where we have shifted $\rho \rightarrow \rho + \log(\sqrt{\Lambda})$. Note that the negative sign in (7.101) is due to the supertrace convention. The metric is locally AdS_3 and we can interpret the higher-spin fields as topological matter, with Λ capturing its global properties. Since $\delta\phi = 2\pi$, one can compute the conical deficit δ_c of the spacetime (7.100) to be

$$\delta_c = 2\pi \left(1 - \sqrt{\frac{4\Lambda}{\text{str}(L_0^2)}} \right), \quad \text{str}(L_0^2) = \frac{N(N-1)}{4}. \quad (7.102)$$

Further, the stress tensor for this class of solutions (in units of l_{AdS}/G) can be expressed as

$$M = -\frac{8k\Lambda}{N(N-1)}.$$

In the case of global AdS_3 , the gauge connection Γ_+ has identical eigenvalues as that of the complex generator iL_0 , and a similarity transformation brings it to the form above, with $\Lambda = \frac{1}{4}\text{str}(L_0^2)$, yielding $\delta_c = 0$, $M = -k/2$ expectedly. This yields an upper bound for Λ as set by global AdS_3 (and thus lower bound for the mass M) for conical defect spacetimes

$$0 < \Lambda < \frac{N(N-1)}{16}. \quad (7.103)$$

Using both the holonomy condition (7.97) and the trivial one, we can determine the set of parameters $\{a_k\}$ that defines this class of spacetime. As the simplest example, for the $sl(3|2) \oplus sl(3|2)$ theory, the holonomy condition (7.97) translates into requiring a_1 to be an integer and a_4 to be a half-integer, whereas the trivial holonomy condition imposes both parameters to be integral. The valid interval (7.103) reads as

$$0 < a_1^2 - a_4^2 < \frac{3}{4}, \quad (7.104)$$

where the possible values of the 2Λ are thus just either $1/4$ or $1/2$. These Diophantine equations can be solved to show that for both cases, there are no solutions. Thus, analogous to the $sl(3) \oplus sl(3)$ case, the $sl(3|2) \oplus sl(3|2)$ theory contains no conical defect spacetimes.

Let us move on to consider the $sl(4|3) \oplus sl(4|3)$ theory for which the bound (7.103) reads as

$$0 < a_1^2 + a_3^2 - a_5^2 < \frac{3}{2}, \quad (7.105)$$

with the holonomy conditions translating into solutions of trivariate Diophantine equations:

(i) For the holonomy (7.97), both a_1, a_3 half-integers and a_5 is integral, with all possible solutions taking the mass $M = -\frac{k}{6}$. There is an infinite set of solutions. An example for which it is easy to obtain a closed form is the subset defined by $a_1 = a_3 \equiv 2y + 1$ which reduces to a Pell-like equation which we find to have the solution $8y = -4 + 2((3 - 2\sqrt{2})^r + (3 + 2\sqrt{2})^r)$, $r \in \mathbb{Z}^+$.

(ii) For the trivial holonomy, all parameters are integral with the mass $M = -\frac{k}{3}$. There is also an infinite set of solutions. A rather simple example is the family of solutions defined by $a_1 = a_5, a_3 = 1$.

Thus, the solution space is notably distinct from the $sl(4) \oplus sl(4)$ theory, where there are only three distinct solutions labelled by (a_1, a_3) as follows: $(\frac{1}{2}, \frac{1}{2})$, $(1, 0)$ and $(1, 1)$. One can attribute the infinitely-degenerate spectrum in the case of the $sl(4|3) \oplus sl(4|3)$ theory to the fact that Λ is the difference between two separate sets of a_k^2 (see (7.101)) in this case as we are computing supertraces. But in the $sl(4) \oplus sl(4)$ case, Λ is simply the sum of all a_k^2 and thus for any given finite mass, the degeneracy is finite. It is straightforward to carry out a similar analysis for other values of N . These solutions generically break all supersymmetries. It would be interesting to understand these solutions from the dual CFT point of view, just as in the case of the $sl(4)$ theory in [31]. This would give us a clearer picture of their physical significance and interpretation. We leave these investigations to future work.

CHAPTER 8

Conclusion and future directions

In this thesis, we have explored a plethora of aspects of quantum gravity in three dimensions in the framework of higher-spin holography in anti-de Sitter spacetime. The bulk theory is a higher-spin Vasiliev gravity theory of which topological sector can be described by a Chern-Simons theory equipped with some suitable Lie (super-)algebra, whereas the boundary conformal field theory is a coset minimal model which contains \mathcal{W} symmetries. We have presented new black hole solutions and established a clear relationship among the gravitational thermodynamics of these solutions, asymptotic spacetime symmetries and \mathcal{W} algebras. We also constructed a few classes of new conical defect solutions and supersymmetric RG flow solutions in the bulk gravity theories. The main examples used in the topological sector are illustrated in the framework of $SL(N)$ and $SL(N|N-1)$ Chern-Simons theories. For the latter, we derived the supersymmetry transformation laws and recovered $\mathcal{N} = 2$ super-Virasoro algebra when the higher-spin fields are truncated.

When the scalar field is taken into account, Vasiliev gravity imposes rather complex interactions between the scalar and the massless higher-spin fields. Towards understanding this class of problem, we have derived the bulk-boundary propagator for scalar fields interacting with a higher-spin black hole which is a classical solution in $hs[\lambda]$ Chern-Simons theory having the same Wilson holonomy as the BTZ. This result should correspond to an integrated torus three-point function of the boundary coset CFT (at least in the 't Hooft limit), and thus furnishes an interesting quantity for further investigations of the higher-spin holography conjecture.

Let us end off by commenting on some future directions. We will begin with a few straightforward and immediate generalizations of our results presented in the thesis, and then discuss a couple of bigger issues.

In [52], the spin-3 black holes were lifted to solutions in $hs[\lambda]$ Chern-Simons theory. This was basically achieved by adding an infinite series of higher-spin charges and appropriately inserting normalization factors $N(\lambda)$ such that upon truncation of all spins $s > 3$, we have the $SL(3, \mathbb{R})$ solution with identical generator normalizations. The Wilson holonomy prescription implies that we have to first compute the Wilson loop's eigenvalues for the BTZ via the infinite collection of traces $\text{Tr}(\omega_{BTZ}^n) \forall n \geq 2$, after choosing a trace convention for the lone-star product that reduces correctly to the $SL(3, \mathbb{R})$ conventions. The holonomy constraints are then imposed similarly to the ansatz with spin-3 chemical potential. It

was demonstrated remarkably in [52] that this $hs[\lambda]$ solution yields a high-temperature¹ partition function that agrees with that of the boundary CFT at $\lambda = 0, 1$ with spin-3 chemical potential inserted. A natural direction, thus, would be to turn on other higher-spin potentials and if feasible, this may shed further light on the working principles of the duality. For example, for $\lambda = 1$, this can be described by a theory of D free complex bosons with central charge $c = 2D$, and the spin-4 current reads $U \sim (\partial\phi\partial^3\bar{\phi} - 3\partial^2\phi\partial^2\bar{\phi} + \partial^3\phi\partial\bar{\phi})$ [57], and it would be useful to check if this agrees with the gravity result. Our spin-4 black hole would serve as a useful limit in such a computation.

We have seen that within the topological sector of Vasiliev theory, Wilson holonomies play a critical role in defining higher-spin black holes. Interestingly, in a different scenario, holonomies also seem to play a crucial role when one discusses the entropy function in AdS_2/CFT_1 [58] which computes the entropy of extremal black holes with a near horizon geometry of the form $AdS_2 \times K$, where K is a compact space, and with $U(1)$ gauge fields A^i and charges q^i . The well-known formula of Sen, $d_{micro}(\vec{q}) = \langle \exp[-iq_i \oint d\tau A_{\tau}^i] \rangle_{AdS_2}$ ², computes the microstates in the presence of an inserted Wilson loop lying along the boundary of AdS_2 . Although these two settings are rather different, their geometrical aspects invite a comparison. Note that at constant ϕ , the BTZ metric reduces to AdS_2 , and the Wilson loop which we have used to define the black hole also bounds this AdS_2 by definition. More remarkably, we note that in both cases, the role played by Legendre transformation is critical in defining entropy in the absence of a bifurcate horizon. For these reasons, it may be interesting to explore this parallel on a deeper level.

Another natural avenue for further study is to investigate the properties, both from the bulk and boundary perspectives, of the higher-spin black holes and conical defect spacetimes discussed in this thesis. For example, in the arena of higher-spin black hole thermodynamics, it was recently shown in [84]³ that the relation between integrability conditions and smooth holonomy condition can be understood fundamentally as arising from a careful computation of the on-shell Chern-Simons action. It may be interesting to re-visit this computation in the context of $SL(N|N-1)$ theories.

A bigger question is whether we can extend the holography conjecture in a meaningful way to the case of a positive cosmological constant. In the case of four-dimensions, it was recently argued in [86] that Vasiliev higher-spin gravity in four-dimensional de Sitter space is dual to a three-dimensional CFT living on the spacelike boundary at future timelike infinity. The CFT is the Euclidean $Sp(N)$ vector model with anticommuting scalars. This conjecture is based on some preliminary results on analytic continuations of bulk and boundary correlators in the proposed duality relating the $O(N)$ vector model with Vasiliev gravity in AdS_4 (see for example [87, 88]). It is then natural to ask if one could study an analogous proposal in the three-dimensional case. This was first discussed in [89], where an analytic

¹The limit taken in [52] was $\tau, \varrho \rightarrow 0, \varrho/\tau^2$ fixed.

²The expectation value refers to the path integral over various fields on Euclidean global AdS_2 associated with the attractor geometry for charge q^i . See, for example, [58].

³See also [85] for a clarifying discussion.

continuation of the radius of curvature

$$l_{AdS_3} \rightarrow il_{dS_3} \tag{8.1}$$

reveals immediately that a naive generalization of Gaberdiel-Gopakumar conjecture seems to yield a dual CFT that is an exotic non-unitary WZW coset of the form

$$\frac{SL(N)_k \oplus SL(N)_1}{SL(N)_{k+1}} \tag{8.2}$$

It would be interesting to check in greater detail this proposal and explore other alternatives. Note that although dS_3 and AdS_3 are related by an analytic continuation in l , there are many physical quantities which admit vastly different interpretations. For example, in dS/CFT , the CFT partition function now computes the Hartle-Hawking wavefunction of the universe. Non-unitary WZW cosets are not well-understood and it is not clear whether the admissible representations of the current algebra could be matched to entities in the bulk de Sitter spacetime.

It would be very interesting to pursue higher-spin dS_3/CFT_2 further, perhaps with a focus on how to perform the correct analytic continuation implied by (8.1) so as to seek alternatives to (8.2) or to interpret it correctly. The work of Witten in [90] would be relevant not only for this purpose but also for the AdS case where we still do not know how to perform an analytic continuation from the 't Hooft to the semi-classical limit of both the coset CFT and the bulk Vasiliev theory. Finally, let us briefly remark that string theory includes many higher-spin excitations whose detailed behavior is still largely unknown. (See, for example, [91, 92, 93] for a review of recent, exciting work along these lines.) It would be very interesting to see if the higher-spin fields descending from tensionless strings/branes could be related to Vasiliev higher-spin gravity.

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APPENDIX A

 $SL(4, \mathbb{R})$ generators

Below, we collect the fifteen $SL(4, \mathbb{R})$ generators which were used to derive the spin-4 black hole solution in the principal embedding.

$$\begin{aligned}
L_0 &= \frac{1}{2} \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}, & L_1 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & L_{-1} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -3 & 0 \end{pmatrix}, \\
W_0 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & W_1 &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & W_{-1} &= 3 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \\
W_2 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & W_{-2} &= 12 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, & U_0 &= \frac{3}{10} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
U_1 &= \frac{1}{5} \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & U_2 &= \frac{1}{2} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & U_3 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
U_{-1} &= \frac{6}{5} \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, & U_{-2} &= 6 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, & U_{-3} &= -36 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.
\end{aligned} \tag{A-1}$$

These matrices were constructed by starting with $W_2 = L_+^2$, $U_3 = L_+^3$ and then deriving the rest by the lowering operator L_- . The $SL(2, \mathbb{R})$ generators (L_0, L_\pm) can be realized via (4.19).

APPENDIX B

On the isomorphism $sl(2|1) \simeq osp(2|2)$

We begin by displaying the commutation relations for $osp(2|2)$ as reviewed in the Appendix of [69]. $R_i^\pm, i = 1, 2$ are the four fermionic generators; E, F, H generate the $sl(2)$ while $J_{12} = -J_{21}$ is the $u(1)$ generator.

$$\begin{aligned}
i\{R_i^+, R_j^-\} &= J_{ij} - \delta_{ij} H, & i\{R_i^-, R_j^-\} &= -2\delta_{ij} F, & i\{R_i^+, R_j^+\} &= 2\delta_{ij} E. \\
[H, R_i^+] &= R_i^+, & [F, R_i^+] &= R_i^-, & [H, R_i^-] &= -R_i^-, & [E, R_i^-] &= R_i^+, & [F, R_i^-] &= 0. \\
[H, E] &= 2E, & [H, F] &= -2F, & [E, F] &= H, & [J_{ij}, R_k^\pm] &= \delta_{jk} R_i^\pm - \delta_{ik} R_j^\pm
\end{aligned} \tag{B-1}$$

We verify that this algebra is isomorphic to $sl(2|1)$ via the following identifications between the generators of the latter (written in our choice of basis) and those in (B-1) :

$$\begin{aligned}
E &\sim L_-, & F &\sim -L_+, & H &\sim 2L_0, \\
J_{12} &\sim i6U_0, & Q_{\pm\frac{1}{2}} &\sim \frac{-i}{2\sqrt{3}} (R_1^\pm + iR_2^\pm), & \bar{Q}_{\pm\frac{1}{2}} &\sim \frac{1}{2\sqrt{3}} (R_1^\pm - iR_2^\pm),
\end{aligned} \tag{B-2}$$

We note from (B-2) that the matching between $sl(2|1)$ and $osp(2|2)$ involves changing the reality conditions of the generators U_0 and $Q_{\pm\frac{1}{2}}$. In our paper, the classification of solutions based on the number of real supersymmetries preserved is performed after continuing $U_0 \rightarrow iU_0$ and treating $\psi, \bar{\psi}$ as complex Grassmann variables. Although $sl(2|1)$ and $osp(2|2)$ are isomorphic algebras, at the level of representation, $sl(2|1)$ and $osp(2|2)$ are distinct. The latter admits only real representations whereas the former admits both real and complex ones.

APPENDIX C

Variations of the $osp(2|2)$ gauge fields

In Section 6.4.2, we recovered the $\mathcal{N} = 2$ superconformal algebra by considering the variations of the fields φ_+ , $\bar{\varphi}_+$, \mathcal{L} and \mathcal{U} under a gauge transformation that preserves the form of the highest-weight ansatz. Although we do not need the full expressions for these variations in our computation, we collect them here for completeness and verification.

$$\begin{aligned}
\delta\varphi_+ &= \frac{3}{2}\varphi_+\xi' + \varphi'_+\xi + \frac{1}{6}\mathcal{U}\varphi_+\xi + \frac{5}{6}\varphi_+\alpha' + \sqrt{\frac{8}{3}}\Psi_+\alpha - \frac{1}{6}\varphi_+\eta + \left(\frac{5}{3}\Upsilon + \mathcal{L} + \frac{1}{6}\mathcal{U}' + \frac{1}{36}\mathcal{U}^2\right)\nu_- \\
&\quad + \frac{1}{3}\mathcal{U}\nu'_- + \nu''_- + 5\sqrt{\frac{2}{3}}\Psi_+\chi' + \left(4\sqrt{\frac{2}{3}}\Psi'_+ + \frac{16}{9}\Upsilon\varphi_+ + \sqrt{\frac{8}{27}}\mathcal{U}\Psi_+\right)\chi \\
&\quad + \left(-\sqrt{\frac{8}{3}}\Upsilon' + 4\sqrt{\frac{2}{3}}\mathcal{W} - \frac{4}{9}\sqrt{\frac{2}{3}}\Upsilon\mathcal{U}\right)\zeta_- - \left(\frac{8}{3}\sqrt{\frac{2}{3}}\Upsilon\right)\zeta'_- \\
\delta\bar{\varphi}_+ &= \frac{3}{2}\bar{\varphi}_+\xi' + \bar{\varphi}'_+\xi - \frac{1}{6}\mathcal{U}\bar{\varphi}_+\xi + \frac{5}{6}\bar{\varphi}_+\alpha' - \sqrt{\frac{8}{3}}\bar{\Psi}_+\alpha + \frac{1}{6}\bar{\varphi}_+\eta + \left(\frac{5}{3}\Upsilon + \mathcal{L} + \frac{1}{6}\mathcal{U}' + \frac{1}{36}\mathcal{U}^2\right)\bar{\nu}_- \\
&\quad + \frac{1}{3}\mathcal{U}\bar{\nu}'_- + \bar{\nu}''_- + 5\sqrt{\frac{2}{3}}\bar{\Psi}_+\chi' + \left(4\sqrt{\frac{2}{3}}\bar{\Psi}'_+ - \frac{16}{9}\Upsilon\bar{\varphi}_+ - \sqrt{\frac{8}{27}}\mathcal{U}\bar{\Psi}_+\right)\chi \\
&\quad + \left(\sqrt{\frac{8}{3}}\Upsilon' + 4\sqrt{\frac{2}{3}}\mathcal{W} - \frac{4}{9}\sqrt{\frac{2}{3}}\Upsilon\mathcal{U}\right)\bar{\zeta}_- + \left(\frac{8}{3}\sqrt{\frac{2}{3}}\Upsilon\right)\bar{\zeta}'_- \\
\delta\mathcal{U} &= \eta' - \bar{\Psi}_+\zeta_- + \Psi_+\bar{\zeta}_- + \bar{\varphi}_+\nu_- - \varphi_+\bar{\nu}_-
\end{aligned}$$

$$\begin{aligned}
\delta\mathcal{L} &= \frac{1}{2}\xi''' + 2\mathcal{L}\xi' + \mathcal{L}'\xi + \Upsilon'\alpha + 2\Upsilon\alpha' + \left(\frac{17}{12}\sqrt{\frac{2}{3}}(\Psi_+\bar{\varphi}_+ + \bar{\Psi}_+\varphi_+)\chi - 4\mathcal{W}' - \frac{5}{6}(\bar{\varphi}_+\varphi_+)'\right)\chi \\
&\quad + \left(\frac{5}{36}(\varphi_+\bar{\varphi}_+) - 6\mathcal{W}\right)\chi' - \left(\frac{1}{18}\mathcal{U}\bar{\varphi}_+ + \frac{1}{6}\bar{\varphi}'_+ + \frac{5}{4\sqrt{6}}\bar{\Psi}_+\right)\nu_- - \frac{1}{2}\bar{\varphi}_+\nu'_- \\
&\quad + \left(\frac{5}{8}\sqrt{\frac{3}{2}}(\Upsilon - \mathcal{L}) + \frac{5\mathcal{U}'}{144\sqrt{6}} - \frac{5\mathcal{U}^2}{864\sqrt{6}}\right)\bar{\varphi}_+\zeta_- + \left(\frac{5}{8\sqrt{6}}\bar{\varphi}''_+ + \frac{5\mathcal{U}}{72\sqrt{6}}\bar{\varphi}'_+ - \frac{1}{8}\bar{\Psi}'_+ - \frac{1}{72}\mathcal{U}\bar{\Psi}_+\right)\zeta'_- \\
&\quad - \left(\frac{5}{6\sqrt{6}}\bar{\varphi}'_+ + \frac{5}{24}\bar{\Psi}_+\right)\zeta'_- - \frac{5\bar{\varphi}_+}{12\sqrt{6}}\zeta''_- + \left(\frac{5}{8}\sqrt{\frac{3}{2}}(-\Upsilon + \mathcal{L}) + \frac{5\mathcal{U}'}{144\sqrt{6}} + \frac{5\mathcal{U}_0^2}{864\sqrt{6}}\right)\varphi_+\bar{\zeta}_-
\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{5}{8\sqrt{6}}\varphi''_+ + \frac{5U_0}{72\sqrt{6}}\varphi'_+ - \frac{1}{8}\Psi'_+ + \frac{1}{72}\mathcal{U}\Psi_+ \right) \bar{\zeta}'_- + \left(\frac{5}{6\sqrt{6}}\varphi'_+ - \frac{5}{24}\Psi_+ \right) \bar{\zeta}'_- + \frac{5\varphi_+}{12\sqrt{6}}\bar{\zeta}''_- \\
\delta\Upsilon = & \Upsilon'\xi + 2\Upsilon\xi' + 2\mathcal{L}\alpha' + \mathcal{L}'\alpha + \frac{1}{2}\alpha'' + \sqrt{\frac{3}{32}}(\bar{\Psi}_+\nu_- - \Psi_+\bar{\nu}_-) \\
& + \left(\frac{5}{4}\varphi_+\bar{\varphi}_+ - 6\mathcal{W} \right) \chi' + \left(\frac{5}{6}(\varphi_+\bar{\varphi}_+)' + \frac{25}{24}(\bar{\varphi}_+\Psi_+ + \varphi_+\bar{\Psi}_+) - 4\mathcal{W}' \right) \chi \\
& + \left(\sqrt{\frac{3}{128}}\bar{\varphi}''_+ + \frac{\mathcal{U}}{24\sqrt{6}}\bar{\varphi}'_+ + \left(\frac{\mathcal{U}'}{16\sqrt{6}} + \sqrt{\frac{27}{128}}(\mathcal{L} - \Upsilon) + \frac{\mathcal{U}^2}{288\sqrt{6}} \right) \bar{\varphi}_+ + \frac{3}{4}\bar{\Psi}'_+ + \frac{\mathcal{U}}{24}\bar{\Psi}_+ \right) \zeta_- \\
& + \frac{5}{8\sqrt{6}}\bar{\varphi}_+\zeta''_- + \left(\bar{\Psi}_+ + \frac{1}{\sqrt{6}}\bar{\varphi}'_+ + \frac{\mathcal{U}}{12\sqrt{6}}\bar{\varphi}_+ \right) \zeta'_-
\end{aligned}$$

where the superscripted primes refer to derivatives with respect to ϕ .

APPENDIX D

Structure constants

Following Racah [76], introduce a basis as follows, where E_{ij} refers to a matrix with unity in the i^{th} row and j^{th} column (our convention for the even-graded generators differs slightly from [76], but is presented in full here for clarity)

$$\begin{aligned}
T_m^s &= \sqrt{\frac{2s+1}{N+1}} \sum_{r,q} C_{r m q}^{\frac{N}{2} s \frac{N}{2}} E_{\frac{N}{2}+1-q, \frac{N}{2}+1-r}, \quad (s = 0, 1, \dots, N) \\
U_m^s &= \sqrt{\frac{2s+1}{N}} \sum_{r,q} C_{r m q}^{\frac{N-1}{2} s \frac{N-1}{2}} E_{\frac{3(N+1)}{2}-q, \frac{3(N+1)}{2}+1-r}, \quad (s = 0, 1, \dots, N-1) \\
Q_m^s &= \sqrt{\frac{2s+1}{N}} \sum_{r,q} C_{r m q}^{\frac{N}{2} s \frac{N-1}{2}} E_{\frac{3(N+1)}{2}-q, \frac{N}{2}+1-r}, \quad (s = \frac{1}{2}, \frac{3}{2}, \dots, N - \frac{1}{2}) \\
\bar{Q}_m^s &= \sqrt{\frac{2s+1}{N+1}} \sum_{r,q} C_{r m q}^{\frac{N-1}{2} s \frac{N}{2}} E_{\frac{N}{2}+1-q, \frac{3(N+1)}{2}-r}, \quad (s = \frac{1}{2}, \frac{3}{2}, \dots, N - \frac{1}{2})
\end{aligned}$$

The structure constants read

$$\begin{aligned}
[T_m^s, Q_{m'}^{s'}] &= \sum_{s'', m''} (-1)^{(s+s'-\frac{3}{2}+N+m)} \sqrt{(2s+1)(2s'+1)} \left\{ \begin{matrix} s & s' & s'' \\ \frac{N-1}{2} & \frac{N}{2} & \frac{N}{2} \end{matrix} \right\} C_{-m m' m''}^{s s' s''} Q_{m''}^{s''} \\
[T_m^s, \bar{Q}_{m'}^{s'}] &= \sum_{s'', m''} (-1)^{(s''-\frac{1}{2}+N+m)} \sqrt{(2s+1)(2s'+1)} \left\{ \begin{matrix} s & s' & s'' \\ \frac{N-1}{2} & \frac{N}{2} & \frac{N}{2} \end{matrix} \right\} \\
&\quad \times C_{-m m' m''}^{s s' s''} \bar{Q}_{m''}^{s''} \\
[U_m^s, Q_{m'}^{s'}] &= \sum_{s'', m''} (-1)^{(2(s+s')-\frac{1}{2}+N+m-s'')} \sqrt{(2s+1)(2s'+1)} \left\{ \begin{matrix} s & s' & s'' \\ \frac{N}{2} & \frac{N-1}{2} & \frac{N-1}{2} \end{matrix} \right\} \\
&\quad \times C_{-m m' m''}^{s s' s''} Q_{m''}^{s''} \\
[U_m^s, \bar{Q}_{m'}^{s'}] &= \sum_{s'', m''} (-1)^{(s+s'+\frac{1}{2}+N+m)} \sqrt{(2s+1)(2s'+1)} \left\{ \begin{matrix} s & s' & s'' \\ \frac{N}{2} & \frac{N-1}{2} & \frac{N-1}{2} \end{matrix} \right\} \\
&\quad \times C_{-m m' m''}^{s s' s''} \bar{Q}_{m''}^{s''} \\
[T_m^s, T_{m'}^{s'}] &= \sum_{s'', m''} (-1)^{(s''+N)} \left(1 - (-1)^{s-s'-s''} \right) \sqrt{(2s+1)(2s'+1)} \left\{ \begin{matrix} s & s' & s'' \\ \frac{N}{2} & \frac{N}{2} & \frac{N}{2} \end{matrix} \right\} \\
&\quad \times C_{m m' m''}^{s s' s''} T_{m''}^{s''}
\end{aligned}$$

$$\begin{aligned}
[U_m^s, U_{m'}^{s'}] &= - \sum_{s'', m''} (-1)^{(s''+N)} \left(1 - (-1)^{s-s'-s''}\right) \sqrt{(2s+1)(2s'+1)} \left\{ \begin{matrix} s & s' & s'' \\ \frac{N-1}{2} & \frac{N-1}{2} & \frac{N-1}{2} \end{matrix} \right\} \\
&\quad \times C_{m m' m''}^{s s' s''} U_{m''}^{s''} \\
\{Q_m^s, \bar{Q}_{m'}^{s'}\} &= \sum_{s'', m''} C_{-m m' m''}^{s s' s''} \sqrt{(2s+1)(2s'+1)} \left((-1)^{(s+s'-1+N)} \left\{ \begin{matrix} s & s' & s'' \\ \frac{N}{2} & \frac{N}{2} & \frac{N-1}{2} \end{matrix} \right\} T_{-m''}^{s''} \right. \\
&\quad \left. + (-1)^{(s''+N)} \left\{ \begin{matrix} s & s' & s'' \\ \frac{N-1}{2} & \frac{N-1}{2} & \frac{N}{2} \end{matrix} \right\} U_{-m''}^{s''} \right)
\end{aligned}$$

The above formulae are used in explicit computations in the $sl(3|2)$ Chern-Simons theory. Please note that $\left\{ \begin{matrix} s & s' & s'' \\ a & b & c \end{matrix} \right\}$ are the Wigner $6j$ symbols, while $C_{m m' m''}^{s s' s''}$ denote Clebsch-Gordan coefficients.