Title

TWO STUDIES IN POTENTIAL SCATTERING THEORY I. A GENERALIZATION OF LEVINSON'S THEOREM TO THREE-BODY SYSTEMS II. A VARIATIONAL METHOD AND EIGENFUNCTION EXPANSION FOR TWO-AND THREE-BODY SYSTEMS.

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(Ph.D. Thesis)

January 6, 1965

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ABSTRACT

In Part One Levinson's theorem is generalized to systems of three particles. The usual two-body result relates the number of bound states of given angular momentum to the corresponding eigenphase shifts of the S matrix. Because of disconnected diagrams the three-body S matrix has continuous eigenphase shifts in addition to any discrete ones; however it is possible to define a unitary connected matrix that has only discrete eigenphase shifts. Levinson's theorem is given in terms of these phase shifts, and it is the same as the usual multichannel result, except that there are an infinite number of eigenphase shifts to be summed over for each value of the total angular momentum. The proof is carried out within the framework of the Faddeev equations by generalizing Jauch's proof for two-body systems.

In Part Two we develop a variational principle for finding approximate eigenvalues and eigenfunctions of the kernel of the Lippmann-Schwinger equation. Regge trajectories are then easily found from the solution to the eigenvalue equation. We apply the variational principle to other potential theory calculations with very good results in both accuracy and simplicity. Finally we extend the variational principle to the three-body Faddeev equations.

I. INTRODUCTION

One of the important problems in the theory of elementary particles is the determination of whether or not a particle is elementary or composite. In a Lagrangian theory an elementary particle must be put in the Lagrangian. In a model based on dispersion theory there is the well known ambiguity of Castillejo, Dalitz and Dyson.[1] They showed that an infinite number of solutions exist in the charged scalar theory without recoil. In both kinds of theories it has been suggested that Levinson's theorem[2] could be used as a means of selecting the proper Lagrangian or the proper solution to the dispersion relations. In its simplest form as first given by Levinson the theorem says that in the scattering of a particle from a spherically symmetric central potential, the number of bound states of the particle in a given angular momentum state is related to the phase shift by

$$N\pi = \delta(0) - \delta(\infty) \quad . \tag{1.1}$$

Jauch[3] generalized the proof to a larger class of potentials than that treated by Levinson, and also he showed that the relation (1.1) is a result of the completeness of the eigenfunctions of two operators H and H_o . H is the full Hamiltonian for the system and H_o is the Hamiltonian in the absence of interactions. The result has been generalized to the case where H_o also has a discrete spectrum,[4,5]

$$(N_H - N_{H_o})\pi = \delta(0) - \delta(\infty) \quad . \tag{1.2}$$

N_H and N_{H_o} are the number of bound states of H and H_o respectively. Since H_o is the Hamiltonian operator for a noninteracting system, all points in its discrete spectrum represent elementary particles. Levinson's theorem has been further generalized to many channel systems by Kazes.[6] In view of the possible application of Levinson's theorem to determining which equations and which solutions to them nature actually selects, it seems important to extend the theorem to systems of more than two particles.

We propose to prove the theorem in potential theory for three particle systems within the framework of a set of equations developed by Faddeev.[7,8,9,10] One important difference between two and three body systems is the connectness structure, that is there exist interactions between two particles with the third one always beyond the range of the forces. As a result of this disconnectedness, the kernel of the Lippmann-Schwinger equation has a continuous spectrum.[11] Similarly the S-matrix will have a continuous spectrum, that is it will not have discrete eigenphase shifts which can be summed to give an equation such as (1.1) . However, due to the simple origin of the continuous spectrum, it is possible to define a unitary operator closely related to the S-matrix and having only a discrete spectrum. We derive a Levinson theorem in terms of the eigenphase shifts of this unitary operator.

The two body Lippmann-Schwinger equation[12] can be reduced to a one dimensional integral equation by projecting out the angular momentum variables, and for a given total angular momentum and energy the S-matrix is a constant. This is not the case in three body scattering, as a

continuous subenergy variable is needed in addition to the total angular momentum. This of course makes the problem much harder. Fortunately Faddeev[10] has discussed the three body problem in great detail and has derived a set of equations which is in principle soluble. More recently Lovelace[13] and Weinberg[11] have suggested similar sets of equations; we will however use Faddeev's throughout this paper.

Another new feature of three body scattering which doesn't exist in two body systems is the possibility of breakup and rearrangement collisions which can occur if two particles can form a bound state. This greatly increases the complexity of the algebra so it will be treated separately. First it will be assumed that there exist no two body bound states.

This article is organized into several parts, many of which can be omitted for the casual reader. In Section II we discuss Jauch's[3] proof of the two body result. In Section III we introduce Faddeev's[10] equations and the projection operator onto the three particle bound states assuming there are no two body bound states. In Section IV we derive the three body Levinson's theorem. The important ideas are contained in this section and the reader who is only interested in the essential result need go no further.

In Section V we allow for the possibility of one bound state between any two pairs of particles and we review the properties of the various operators that Faddeev introduces to handle these bound states. In Section VI Levinson's theorem is extended to allow for the above possibility. Finally the more tedious calculations can be found in the appendices.

II. LEVINSON'S THEOREM FOR TWO PARTICLE SYSTEMS

Since the three particle equations are so complicated, it is easy to get lost in the proof of Levinson's theorem. We therefore review the method of proof due to Jauch[3] for two particle systems, since it is much simpler and illustrates the important ideas. First the Hamiltonian is split into two parts,

$$H = H_o + V \quad , \tag{2.1}$$

where H_o is the free particle Hamiltonian and V is the interaction term. We assume that all of the eigenstates, ϕ_E of H_o are continuum states with energy $E > 0$,

$$H_o \phi_E = E \phi_E \quad , \tag{2.2}$$

and that H has N points in the discrete spectrum with $E_n < 0$ $(n = 1, 2, \cdots N)$. H is assumed to have the same continuous spectrum as H_o.

$$\begin{aligned} H \psi_E &= E \psi_E & E > 0 \\ H \psi_n &= E_n \psi_n & E_n < 0 \end{aligned} \tag{2.3}$$

All the calculations are carried out for fixed angular momentum, so the functions ϕ and ψ only depend upon the variable E. The isometric operator that maps the continuum eigenstates of H_o onto the continuum

eigenstates of H is called the Møller wave operator[14] and is given by

$$\Omega = \int_0^\infty dE \; |\psi_E><\phi_E| \quad . \tag{2.4}$$

The completeness of the eigenstates of H and H_o gives the relationships

$$\Omega^\dagger \Omega = \int_0^\infty dE \; |\phi_E><\phi_E| = I \tag{2.5}$$

and

$$\Omega \Omega^\dagger = \int_0^\infty dE \; |\psi_E><\psi_E| = I - P_d \quad . \tag{2.6}$$

I is the identity operator and P_d is the projection operator on the discrete spectrum of H. Combining Eqs. (2.5) and (2.6) we have,

$$P_d = \Omega^\dagger \Omega - \Omega \Omega^\dagger \quad . \tag{2.7}$$

Since the trace of a projection operator is the dimension of the space it projects onto, we have

$$N = \text{trace } P_d = \text{trace } (\Omega^\dagger \Omega - \Omega \Omega^\dagger) \quad . \tag{2.8}$$

The matrix elements of Ω are given by

$$\langle \phi_{E'} \mid \Omega \mid \phi_E \rangle = \langle \phi_{E'} \mid \psi_E \rangle \quad , \tag{2.9}$$

and ψ_E can be found from the Lippmann-Schwinger equation,

$$\psi_E = \phi_E - \frac{1}{H_o - E - i\epsilon} V \psi_E \quad . \tag{2.10}$$

The result is

$$\langle \phi_{E'} \, |\Omega| \, \phi_E \rangle = \delta(E - E') - \frac{\langle \phi_{E'} \, |V| \, \psi_E \rangle}{E' - E - i\epsilon} \quad , \tag{2.11}$$

which can be rewritten in terms of the t-matrix as

$$\langle \phi_{E'} \, |\Omega| \, \phi_E \rangle = \delta(E - E') - \frac{\langle \phi_{E'} \, |t| \, \phi_E \rangle}{E' - E - i\epsilon} \quad . \tag{2.12}$$

The S-matrix is given by

$$\langle \phi_{E'} \, |S| \, \phi_E \rangle = \delta(E - E') \left[1 - 2\pi i \, \langle \phi_E \, |t| \, \phi_E \rangle \right] \quad . \tag{2.13}$$

Using (2.12), the projection operator P_d becomes

$$P_d = \int_0^\infty dE'' \left\{ \frac{\langle E \, |t^\dagger| \, E'' \rangle \langle E'' \, |t| \, E' \rangle - \langle E \, |t| \, E'' \rangle \, \langle E'' \, |t^\dagger| \, E' \rangle}{(E - E'' - i\epsilon)\,(E' - E'' + i\epsilon)} \right\} \quad . \tag{2.14}$$

The procedure for evaluating trace P_d is to use the idendity,

$$\frac{1}{E - E' - i\epsilon} = \frac{P}{E - E'} + i\pi \, \delta(E - E') \tag{2.15}$$

when this identity is substituted into (2.14) and then into (2.8) , we obtain three different terms. One, which will be called A , comes from the product of both delta functions. The second, called C comes form the product of principle parts. The third, B , contains the cross terms. The assumption is made that all orders of integration can be interchanged except where the denominators are singular. Hence if there were not singularities in the denominator N would be zero since then trace $\Omega\,\Omega^{+}$ = trace $\Omega^{+}\,\Omega$. We now evaluate the three contributions A , B and C . A is easily evaluated giving

$$A = \pi^2 \text{ trace } \delta(E - E') \left\{ \langle E |T| E\rangle\langle E' |T^{\dagger}| E'\rangle - \langle E |T^{\dagger}| E\rangle\langle E' |T| E'\rangle \right\} . \tag{2.16}$$

The coefficient of the delta function vanishes at $E = E'$ and hence A is identically zero. For C we have

$$C = P \int_0^{\infty} dE \int_0^{\infty} dE' \left\{ \frac{\langle E |T| E'\rangle\langle E' |T^{\dagger}| E\rangle - \langle E |T^{\dagger}| E'\rangle\langle E' |T| E\rangle}{(E - E')^2} \right\} . \tag{2.17}$$

The integral is well defined through the principle part since the numerator vanishes there, and we assume that the integral converges absolutely and uniformly at infinity so that the orders of integration may be interchanged. Since the integrand is antisymmetric in E and E' the double integral vanishes identically upon interchange of the order of integration. The contribution from the cross terms is given by

$$B = -i\pi \text{ trace } \frac{1}{E - E'} \Big\{ \langle E |t| E\rangle\langle E |t^{\dagger}| E'\rangle - \langle E' |t| E'\rangle\langle E |t^{\dagger}| E'\rangle + \langle E |t| E'\rangle\langle E' |t^{\dagger}| E'\rangle - \langle E |t| E'\rangle\langle E |t^{\dagger}| E\rangle \Big\} \quad . \tag{2.18}$$

This has the form

$$\lim_{E' \to E} g(E) \left[\frac{f(E) - F(E')}{E - E'} \right] = g(E) \frac{df}{dE} \quad ,$$

therefore B becomes

$$B = i\pi \int_0^{\infty} \left\{ t^{+}(E) \frac{\partial}{\partial E} t(E) - t(E) \frac{\partial t^{+}}{\partial E}(E) \right\} dE \quad . \tag{2.19}$$

The t-matrix is given in terms of phase-shifts by

$$t(E) = \frac{e^{i\delta(E)} \sin \delta(E)}{\pi} \quad , \tag{2.20}$$

And B becomes

$$B = -\frac{2}{\pi} \int_0^{\infty} dE \sin^2 \delta(E) \frac{d\delta}{dE} \quad . \tag{2.21}$$

Finally we have for N

$$N = \frac{1}{\pi} \left[\delta(0) - \delta(\infty) - \frac{\sin 2\delta(0)}{2} + \frac{\sin 2\delta(\infty)}{2} \right] \quad . \tag{2.22}$$

To see that this is the same as the usual form let

$$\delta(0) - \delta(\infty) = N\pi + \alpha \quad ,$$

and use the relation

$$\frac{1}{2}\left[\sin 2\delta(0) - \sin 2\delta(\infty)\right] = \sin\left[\delta(0) - \delta(\infty)\right] \cos\left[\delta(0) + \delta(\infty)\right] .$$

Substituting in (2.22) we obtain

$$\alpha = \sin\alpha \cos\left[\delta(0) + \delta(\infty)\right]$$

for which the only solution is $\alpha = 0$. Hence we have the desired result,

$$N = \frac{1}{\pi}\left[\delta(0) - \delta(\infty)\right] \tag{2.23}$$

The result obtained here can be modified if we assume that H_o also has a discrete spectrum. Then (2.7) will read

$$P_H - P_{H_o} = \Omega^\dagger \Omega - \Omega \Omega^\dagger$$

and by taking the trace of both sides we obtain[4,5,6]

$$N_H - N_{H_o} = \frac{1}{\pi}\left[\delta(0) - \delta(\infty)\right] \tag{2.24}$$

III. THREE-BODY WAVE MATRICES

In this section we will introduce the Faddeev[7,8,9,10] equations and the isometric operators which are the generalization of the Møller[14] wave matrices to three particle systems. A complete account of the operators and their properties can be found in Reference 10 .

It will be convenient to use two sets of variables in our work. The final answer is given in terms of a trace and is independent of the variables used, but the proofs are often simpler for a particular choice of variables. One set is the same as that used by Omnes,[15] which consists of the individual kinetic energies $(\omega_1, \omega_2, \omega_3)$ in the overall center of mass system, a total angular momentum J and its projections M on a space fixed axis and M' on a body fixed axis.

The second set of variables is essentially an angular momentum decomposition of Faddeev's. A pair of particles is denoted by the symbol α, for example the $2 - 3$ pair is denoted by $\alpha = 1$. In the center of mass of pair α we introduce the kinetic energy ν_α and the relative angular momentum variables ℓ_α and m_α . These variables refer only to pair α . In the total center of mass system we let ω_α be the translational energy of the center of mass of pair α and the third particle. A third total energy variable $E = \omega_\alpha + \nu_\alpha$ will often be used instead of ω_α . For simplicity we denote the angular variables ℓ_α and m_α by λ_α ; sometimes λ_α will be omitted entirely as it is inessential to the calculations. Obviously there are three sets of variables as there are three distinct pairs of particles, and we will often change from one description to another.

The total angular momentum J and its projection M on a space fixed axis complete the set of variables. We will always work in a system with J and M fixed so they will be omitted.

Before discussing the three body problem it is necessary to have the solution to the two-body Lippmann-Schwinger equation[12] for the t-matrix,

$$t_\alpha(\nu_\alpha;\ \nu'_\alpha;\ \lambda_\alpha;\ s) = V_\alpha(\nu_\alpha;\ \nu'_\alpha;\ \lambda_\alpha) - \int_0^\infty d\nu''_\alpha \frac{V_\alpha(\nu_\alpha;\ \nu''_\alpha;\ \lambda_\alpha)\ t_\alpha(\nu''_\alpha;\ \nu'_\alpha;\ \lambda_\alpha;\ s)}{\nu''_\alpha - s} \tag{3.1}$$

We have assumed that the potential is of the form $V_{12}(|\vec{r}_1 - \vec{r}_2|)$ in coordinate space so that V_α and t_α are diagonal in λ_α. The kernel of the three body equations involves the operator $T_\alpha(s)$,

$$\langle \nu,\lambda,\omega\ |T_\alpha(s)|\ \nu',\lambda',\omega' \rangle = \delta(\omega_\alpha - \omega'_\alpha)\ \delta(\lambda_\alpha,\ \lambda'_\alpha)\ t_\alpha(\nu_\alpha;\ \nu'_\alpha;\ \lambda_\alpha;\ s - \omega_\alpha) \tag{3.2}$$

Although the three-body transition operator satisfies an integral equation like (3.1), the kernel is not compact due to the disconnected graphs. However it is possible to define operators which satisfy a set of coupled integral equations in which the disconnected terms are explicitly summed. An iterate of the kernel of these equations has been shown to be compact by Faddeev.

Let $M_{\alpha\beta}(s)$ be the amplitude for an interaction where pair α is the first to interact and pair β is the last. These operators satisfy the equations

$$M_{\alpha\beta}(s) = \delta_{\alpha\beta}\, T_\alpha(s) - T_\alpha(s) \frac{1}{H_o - s} \sum_{\gamma \neq \alpha} M_{\gamma\beta}(s) \quad . \tag{3.3}$$

H_o is the energy operator for all particles free and noninteracting. In our representation it is just multiplication by $E = \nu_\alpha + \omega_\alpha$. The kernel of the operator will be written

$$\langle \nu, \omega, \lambda \; | M_{\alpha\beta}(s) | \; \nu', \omega', \lambda' \rangle = M_{\alpha\beta}(\omega, \nu, \lambda; \omega', \nu', \lambda'; s)$$

or

$$\langle \omega_1, \omega_2, \omega_3, M, \; | M_{\alpha\beta}(s) | \; \omega_1', \omega_2', \omega_3', M' \rangle = M_{\alpha\beta}(\omega_1, \omega_2, \omega_3, M; \omega_1', \omega_2', \omega_3', M'; s) \tag{3.4}$$

depending upon which variables we are using.

The generalization of the wave matrix is given by

$$\Omega_o = \delta(\omega - \omega')\, \delta(\nu - \nu')\, \delta(\lambda, \lambda') - \sum_{\alpha,\beta} \frac{M_{\alpha\beta}(\omega, \nu, \lambda; \omega', \nu', \lambda'; s = \omega' + \nu' + i\epsilon)}{\omega + \nu - \omega' - \nu' - i\epsilon} \tag{3.5}$$

If there are no two particle bound states, the projection operator on the three particle bound states is

$$\Lambda = \Omega_0^+ \Omega_0 - \Omega_0 \Omega_0^+ \quad . \tag{3.6}$$

The operator Ω_0 is a sum of several terms which we write as

$$\Omega_0 = 1 - W_1 - W_2 - W_3 - W_0 \tag{3.7}$$

with W_1, W_2, W_3 being the disconnected terms,

$$\langle \omega,\nu,\lambda \,|W_\alpha|\, \omega',\nu',\lambda' \rangle = \frac{\delta(\omega_\alpha - \omega'_\alpha)\,\delta(\lambda_\alpha,\lambda'_\alpha)\,t_\alpha(\nu_\alpha;\nu'_\alpha;\lambda_\alpha;\ \nu_\alpha + i\epsilon)}{(\nu_\alpha - \nu'_\alpha - i\epsilon)} \quad . \tag{3.8}$$

The term W_0 is that part of (3.5) with no delta functions, that is the connected part. Using Eq. (3.7) we have for Λ,

$$\Lambda = \sum_{\substack{\alpha,\beta=0 \\ \alpha \neq \beta}}^{3} \left[W_\alpha^\dagger, W_\beta\right] + \left[W_0^\dagger, W_0\right] + \sum_{\alpha=1} \left[W_\alpha^\dagger, W_\alpha\right] \quad . \tag{3.9}$$

Using (2.14) the last term can be rewritten,

$$\left[W_\alpha^\dagger, W_\alpha\right] = \delta(\omega_\alpha - \omega'_\alpha)\, P_\alpha \tag{3.10}$$

where P_α is a projection operator on the two-particle bound states of pair α. Since we assume there are no two particle bound states, $P_\alpha = 0$. Later we will include the possibility of these bound states.

Because $P_\alpha = 0$, we need only take the trace of the terms in (3.9) that don't have an overall delta function. The answer will be given in terms of the three-to-three S-matrix which is defined by

$$S_{00} = \delta(\omega - \omega')\,\delta(\nu - \nu')\,\delta(\lambda,\lambda') - 2\pi i\,\delta(\omega + \nu - \omega' - \nu')\,T_{00} \tag{3.11}$$

with

$$T_{00} = \sum_{\alpha,\beta} M_{\alpha\beta}(\omega,\nu,\lambda;\omega',\nu',\lambda';s = \nu' + \omega' + i\epsilon) \quad . \tag{3.12}$$

The trace of Λ is evaluated in Appendix I and the separation of eigenphase shifts is discussed in the next Section.

IV. THREE-BODY LEVINSON THEOREM

The number of three body bound states of the system can now be obtained by taking the trace of both sides of (3.6) . The result as given in Appendix I is

$$N = -i\pi \int_0^{\infty/} dE \text{ trace } \left\{ T_{00}^{\dagger} \frac{\partial T_{00}}{\partial E} - T_{00} \frac{\partial T_{00}^{\dagger}}{\partial E} \right\} . \tag{4.1}$$

The prime on the integral means that terms with an overall delta function are to be omitted from the trace. To obtain a result in terms of a sum over eigenphase shifts, it is necessary to have a compact operator. A connected T-matrix is defined by

$$S_c = 1 - 2\pi i \, \delta(E - E') \, T_c \tag{4.2}$$

with

$$S_c = S_1^{\dagger} S_2^{\dagger} S_3^{\dagger} S_{00} . \tag{4.3}$$

S_1 is the two body S-matrix multiplied by $\delta(\omega_1 - \omega_1')$,

$$S_1 = 1 - 2\pi i \, \delta(E - E') \, T_1 . \tag{4.4}$$

S_c is a unitary operator and it is easily verified that T_c has no delta functions in it. For fixed total energy, T_c is a square integrable operator since its kernel is bounded for all values of the variables and the integration is over a finite range; that is

$$\text{trace } T_c \; T_c^\dagger < \infty \quad . \tag{4.5}$$

Because of unitarity, T_c is also a normal operator,

$$T_c^\dagger T_c = T_c T_c^\dagger \tag{4.6}$$

and therefore it has a spectral expansion of the form

$$T_c = -\frac{1}{\pi} \sum_n e^{i\delta_n} \sin \delta_n \; |e_n\rangle\langle e_n| \quad . \tag{4.7}$$

The eigenvalues depend upon the particular order of the S_α in (4.3), but the final result will not. For the total energy $E = 0$, trace $(T_c^+ T_c^-) \equiv 0$ since the subenergy integrations are over an interval of zero length. Therefore the eigenvalues $\sin^2 \delta(E = 0)$ all vanish identically.

We now write (4.1) in terms of S_{00},

$$N = i\pi \int_0^{\infty/} dE \; \text{trace} \left\{ \frac{1}{4\pi^2} \left[S_{00}^\dagger \frac{\partial S_{00}}{\partial E} - S_{00} \frac{\partial S_{00}^\dagger}{\partial E} \right] - \frac{1}{2\pi i} \frac{\partial}{\partial E} (T_{00} + T_{00}^\dagger) \right\} \tag{4.8}$$

then we use the fact that trace T_{00} vanishes at zero and infinite energy to eliminate all but the S-matrix. Substituting (4.3) for S_{00} we have

$$N = \frac{i\pi}{4\pi^2} \int_0^{\infty/} dE \; \text{trace} \left[(S_c^\dagger S_1 S_2 S_3) \frac{\partial}{\partial E} (S_3 S_2 S_1 S_c) - (S_3 S_2 S_1 S_c) \frac{\partial}{\partial E} (S_c^\dagger S_1 S_2 S_3) \right] \quad . \tag{4.9}$$

Using the unitarity of the S-matrices and the identity trace A B = trace B A, we have

$$N = \frac{i\pi}{4\pi^2} \int_0^{\infty/} dE \text{ trace} \left[S_c^\dagger \frac{\partial}{\partial E} S_c - S_c \frac{\partial}{\partial E} S_c^\dagger + S_1^\dagger \frac{\partial}{\partial E} S_1 + S_2^\dagger \frac{\partial}{\partial E} S_2 \right.$$

$$\left. + S_3^\dagger \frac{\partial}{\partial E} S_3 - S_1 \frac{\partial}{\partial E} S_1^\dagger - S_2 \frac{\partial}{\partial E} S_2^\dagger - S_3 \frac{\partial}{\partial E} S_3^\dagger \right] . \tag{4.10}$$

The prime on the integral reminds us that all the terms with an overall delta function are to be omitted. Finally then we have

$$N = \frac{i\pi}{4\pi^2} \int_0^{\infty} dE \text{ trace} \left[S_c^\dagger \frac{\partial}{\partial E} S_c - S_c \frac{\partial}{\partial E} S_c^\dagger \right] . \tag{4.11}$$

This can be rewritten with T_c rather than S_c,

$$N = i\pi \int_0^{\infty} dE \text{ trace} \left\{ T_c^\dagger \frac{\partial}{\partial E} T_c - T_c \frac{\partial}{\partial E} T_c^\dagger \right\} . \tag{4.12}$$

To compute the trace, we use the eigenfunctions of T_c as a basis. The diagonal elements are easily computed to give

$$\langle e_n | T_c^\dagger \frac{\partial T_c}{\partial E} - T_c \frac{\partial}{\partial E} T_c^\dagger | e_n \rangle = e^{i\delta_n} \sin \delta_n \langle e_n | \frac{\partial T_c}{\partial E} | e_n \rangle$$

$$- e^{i\delta_n} \sin \delta_n \langle e_n | \frac{\partial T_c^\dagger}{\partial E} | e_n \rangle . \tag{4.13}$$

The eigenvalues of T_c are given as a functional which is stationary with respect to variations of the wave functions,

$$-\frac{e^{i\delta_n}}{\pi} \sin \delta_n = \frac{\langle e_n | T_c | e_n \rangle}{\langle e_n | e_n \rangle} \quad . \tag{4.14}$$

Taking the derivative of both sides with respect to E we have

$$\frac{\langle e_n | \frac{\partial T_c}{\partial E} | e_n \rangle}{\langle e_n | e_n \rangle} = \frac{-\partial(e^{i\delta_n} \sin \delta_n)}{\pi \partial E} \quad , \tag{4.15}$$

since the derivative of the eigenfunction gives zero because of the stationary property. Finally then (4.13) becomes

$$\langle e_n | T_c^\dagger \frac{\partial T_c}{\partial E} - T_c \frac{\partial T_c^\dagger}{\partial E} | e_n \rangle = \frac{2i}{\pi^2} \sin^2 \delta_n \frac{d\delta_n}{dE} \quad . \tag{4.16}$$

To obtain the trace, the above expression is summed over n to yield

$$N = \frac{-2}{\pi} \int_0^\infty dE \sum_n \sin^2 \delta_n \frac{d\delta_n}{dE} \quad . \tag{4.17}$$

Interchanging integration and summation we have,

$$N = \frac{1}{\pi} \sum_n \left\{ \delta_n(0) - \delta_n(\infty) - \frac{\sin 2\delta_n(0)}{2} + \frac{\sin 2\delta_n(\infty)}{2} \right\} \tag{4.18}$$

The integration and summation can be interchanged if the partial sums are bounded by an integrable function. The partial sums will be bounded if only

a finite number of phase shifts have arbitrarily large derivatives. We assume that this is the case. The bound will be integrable provided that the T-matrix falls to zero sufficiently rapidly as $E \to \infty$.

Since the amplitude vanished at infinite energy, $\sin \delta(\infty) = 0$, and we have already shown $\sin \delta(0) = 0$, therefore we have

$$N = \frac{1}{\pi} \sum \left\{ \delta_n(0) - \delta_n(\infty) \right\} \tag{4.19}$$

V. THREE-BODY WAVE MATRICES IN THE PRESENCE OF TWO-BODY BOUND STATES

In this section we extend the discussion of Section III to allow for the presence of two-body bound states. In that case it is necessary to know the bound state wave function. ψ_α,

$$\psi_\alpha(\nu_\alpha,\lambda_\alpha) = \frac{1}{\nu_\alpha + B_\alpha} \int_0^\infty d\nu'_\alpha \; V_\alpha(\nu_\alpha;\nu'_\alpha;\lambda_\alpha)\, \psi_\alpha(\nu_\alpha;\lambda_\alpha) \quad , \tag{5.1}$$

where the binding energy is $-B_\alpha$. The wave functions are normalized to one,

$$\int_0^\infty d\nu_\alpha \; |\psi_\alpha(\nu_\alpha;\lambda_\alpha)|^2 = 1 \quad . \tag{5.2}$$

We will assume that there is one s-wave bound state in each two-body system. This is not essential, but it simplifies the algebra considerably. In this case $\lambda_\alpha = \{\ell,m\}_\alpha \equiv 0$ for the bound state pair α.

The bound state causes the two-body t-matrix to have a pole at $s = -B_\alpha$. The three-body amplitude $M_{\alpha\beta}$ will then have a pole at $s = \omega_\alpha - B_\alpha$. Similarly $M_{\alpha\beta}$ has a pole at $s = \omega'_\beta - B_\beta$. The residue at these poles and at the double pole $s = \omega'_\beta - B_\beta = \omega_\alpha - B_\alpha$ are closely related to the S-matrices for bound state scattering. To be more precise, it is not the residue of $M_{\alpha\beta}$ but rather the residue of $M_{\alpha\beta}$ with the two-body wave function projected out. We list these residues and their relationship to the S-matrices and the Møller wave matrices in Equations (5.3) to (5.12). For a complete discussion of their properties the reader may consult Reference 10. The residue at $s = \omega'_\beta - B_\beta$ with

the wave function projected out is given by

$$L_{\alpha\beta}(\omega,\nu,\lambda;\omega'_\beta;s) = (s + B_\beta - \omega'_\beta) \int_0^\infty \frac{M_{\alpha\beta}(\omega,\nu,\lambda;\omega'_\beta,\nu'_\beta,0;s)}{\nu'_\beta + \omega'_\beta - s} \psi_\beta(\nu'_\beta)\, d\nu'_\beta \quad , \tag{5.3}$$

and the residue at $s = \omega_\alpha - B_\alpha$ by

$$\tilde{L}_{\alpha\beta}(\omega_\alpha;\omega',\nu',\lambda';s) = (s + B_\alpha - \omega_\alpha) \int_0^\infty \sum_{\lambda_\alpha} \frac{d\nu_\alpha M_{\alpha\beta}(\omega_\alpha,\nu_\alpha,0;\omega,\nu,\lambda;s)\psi^*(\nu_\alpha)}{\nu_\alpha + \omega_\alpha - s} \quad . \tag{5.4}$$

The $M_{\alpha\beta}$ satisfy the relation,

$$M_{\alpha\beta}(\omega,\nu,\lambda;\omega',\nu',\lambda';s) = M^*_{\beta\alpha}(\omega',\nu',\lambda';\omega,\nu,\lambda;s^*) \quad , \tag{5.5}$$

and the $L_{\alpha\beta}$ satisfy

$$L^*_{\alpha\beta}(\omega,\nu,\lambda;\omega'_\beta;s^*) = \tilde{L}_{\beta\alpha}(\omega'_\beta;\omega,\nu,\lambda;s) \quad . \tag{5.6}$$

The $L_{\alpha\beta}$ operator has a unity term in it coming from the projection of the term $T_\alpha \delta_{\alpha\beta}$ in Eq. (5.3). Separating this term out we define an operator $K_{\alpha\beta}$ and the corresponding operator $\tilde{K}_{\alpha\beta}$ by

$$L_{\alpha\beta} = (\nu_\alpha + B_\alpha)\, \psi_\alpha(\nu_\alpha)\, \delta(\omega_\alpha - \omega'_\alpha)\, \delta_{\alpha\beta} + K_{\alpha\beta}(\omega,\nu,\lambda;\omega'_\beta;s) \quad . \tag{5.7}$$

The residue of $M_{\alpha\beta}$ at the double pole with both wave functions removed is denoted $F_{\alpha\beta}$, and it is obtained by separating $K_{\alpha\beta}$ into a term regular at $s = -B_\alpha + \omega_\alpha$ and a pole term.

$$K_{\alpha\beta} = G_{\alpha\beta} + \frac{(\nu_\alpha + B_\alpha)\psi_\alpha(\nu_\alpha)}{s + B_\alpha - \omega_\alpha} F_{\alpha\beta}(\omega_\alpha;\omega'_\beta;s) \tag{5.8}$$

$$\tilde{K}_{\alpha\beta} = \tilde{G}_{\alpha\beta} + F_{\alpha\beta}(\omega_\alpha;\omega'_\beta;s) \frac{(\nu'_\beta + B_\beta)\psi_\beta(\nu'_\beta)}{s + B_\beta - \omega'_\beta} \tag{5.9}$$

We define three isometric operators by

$$\Omega_\alpha = \sum_\beta L_{\beta\alpha} \frac{(\omega,\nu,\lambda;\omega'_\alpha;s = -B_\alpha + \omega'_\alpha + i\epsilon)}{(\omega + \nu + B_\alpha - \omega'_\alpha - i\epsilon)} \tag{5.10}$$

The S-matrix is given by

$$S_{\alpha\beta} = \delta_{\alpha\beta}\,\delta(\omega_\alpha - \omega'_\alpha) - 2\pi i\,\delta(\omega_\alpha - B_\alpha - \omega'_\beta + B_\beta)\,T_{\alpha\beta}(\omega_\alpha;\omega'_\beta) \tag{5.11}$$

and

$$S_{0\alpha} = -2\pi i\,\delta(\omega + \nu + B_\alpha - \omega'_\alpha)\,T_{0\alpha}(\omega,\nu,\lambda;\omega'_\alpha) \quad , \tag{5.12}$$

where

$$T_{\alpha\beta}(\omega_\alpha;\omega'_\beta) \equiv F_{\alpha\beta}(\omega_\alpha;\omega'_\beta;s = \omega'_\beta - B_\beta + i\epsilon)$$

and

$$T_{0\alpha}(\omega,\nu,\lambda;\omega'_\alpha) \equiv \sum_\beta K_{\beta\alpha}(\omega,\nu,\lambda;\omega'_\alpha;s = -B_\alpha + \omega'_\alpha + i\epsilon)$$

The subscript zero denotes a state with all particles free, for example S_{01} is the S-matrix for particle one scattering on a bound state of particles two and three with all final particles free. S_{12} is the S-matrix for a rearrangement collision with particle two free initially and particle one free in the final state.

The Ω operators are formally defined by Faddeev[10] to be a mapping of one Hilbert space onto another. Define the space $\hat{h}$ by the orthogonal sum

$$\hat{h} = h_0 \oplus h_1 \oplus h_2 \oplus h_3 \quad . \tag{5.13}$$

h_0 is the space of functions of the variables ω, ν, λ that satisfy

$$\sum_{\lambda} \int_0^{\infty} d\omega \int_0^{\infty} d\nu \; |f_0(\omega,\nu,\lambda)|^2 < \infty \quad . \tag{5.14}$$

h_α is the space of square integrable functions of ω_α,

$$\int_0^{\infty} d\omega_\alpha \; |f_\alpha(\omega_\alpha)|^2 < \infty \quad . \tag{5.15}$$

The subspaces h_0, h_α reduce the total energy operator $\hat{H}$ defined on $\hat{h}$ as follows:

$$\begin{aligned} &\text{if } f_0 \in h_0, \text{ then } \hat{H} f_0 = (\omega + \nu) f_0 \quad ; \\ &\text{if } f_\alpha \in h_\alpha, \text{ then } \hat{H} f_\alpha = (-B_\alpha + \omega_\alpha) f_\alpha \quad . \end{aligned} \tag{5.16}$$

$\hat{H}$ is the total energy of a free or "asymptotic" system, either a bound state plus a free particle or all particles free. The total Hamiltonian H acts on a space h which is formally identical to h_0. We now define an isometric operator $\hat{\Omega}$ which maps $\hat{h}$ onto h. It is reduced by the subspaces h_0, h_α with

$$\hat{\Omega} f_0 = \Omega_0 f_0$$

and

$$\hat{\Omega} f_\alpha = \Omega_\alpha f_\alpha \quad . \tag{5.17}$$

The states f_0 and f_α are continuum states and they are only mapped onto continuum states of H in h. Hence if f_d is a discrete eigenstate of H, $\hat{\Omega}^\dagger f_d = 0$. The orthogonality relations

$$\Omega_\beta^\dagger \Omega_\alpha = I_\alpha \delta_{\alpha\beta} = \delta(\omega_\alpha - \omega_\alpha') \delta_{\alpha\beta} \quad ,$$

$$\Omega_0^\dagger \Omega_\alpha = 0 \quad ,$$

and

$$\Omega_0^\dagger \Omega_0 = I_0 = \delta(\omega - \omega') \delta(\nu - \nu') \delta(\lambda,\lambda') \tag{5.18}$$

also hold where I_0 and I_α are the identity operators on h_0 and h_α respectively. Finally then we have

$$\Omega_0 \Omega_0^\dagger + \sum_\alpha \Omega_\alpha \Omega_\alpha^\dagger = I - P_d \quad , \tag{5.19}$$

where I is the identity on h and P_d is the projection operator on

the space spanned by the discrete eigenstates of H. Since h is formally the same space as h_0, I is the same as I_0 and (5.19) becomes

$$\Lambda_d = \Omega_0^\dagger \Omega_0 - \Omega_0 \Omega_0^\dagger - \sum_\alpha \Omega_\alpha \Omega_\alpha^\dagger \quad . \tag{5.20}$$

By taking the trace of (5.20) we have an expression for the number of bound states of H.

VI. THREE-BODY LEVINSON'S THEOREM IN THE PRESENCE OF TWO-BODY BOUND STATES

The trace of the first two terms of the above equation has already been evaluated with the exception of the parts having an overall delta function. That part was given in Eq. (3.10),

$$\left[W_\alpha^\dagger, W_\alpha\right] = \delta(\omega_\alpha - \omega'_\alpha)\, P_\alpha = I_\alpha\, P_\alpha \quad .$$

When there were no two body bound states P_α was zero, but now it must be included. The identity operator is replaced by $\Omega_\alpha^\dagger\, \Omega_\alpha$ since they are equal and then Λ_d becomes

$$\Lambda_d = \sum_{\substack{\alpha,\beta=0 \\ \alpha\neq\beta}}^{3} \left[W_\alpha^\dagger, W_\beta\right] + \left[W_0^\dagger, W_0\right] + \sum_{\alpha=1}^{3} \left\{P_\alpha\, \Omega_\alpha^\dagger\, \Omega_\alpha - \Omega_\alpha\, \Omega_\alpha^\dagger\right\} \quad . \tag{6.1}$$

The W operators are given in Eqs. (3.8) and (3.9), and the actual calculation of the traces is done in Appendices A and B. The number of three body bound states is given by

$$N = i\pi \left\{ \int_0^{\infty} dE\, \text{trace} \left[T_{00}^\dagger \frac{\partial T_{00}}{\partial E} - T_{00} \frac{\partial T_{00}^\dagger}{\partial E} \right] + \sum_\alpha \int_0^\infty dE \right.$$

$$\left. \text{trace} \left[T_{0\alpha}^\dagger \frac{\partial}{\partial E} T_{0\alpha} - T_{0\alpha} \frac{\partial}{\partial E} T_{0\alpha}^\dagger \right] + \sum_{\alpha,\beta} \int_{-\min(B_\alpha, B_\beta)}^{\infty} dE \left[T_{\alpha\beta}^\dagger \frac{\partial}{\partial E} T_{\alpha\beta} - T_{\alpha\beta} \frac{\partial}{\partial E} T_{\alpha\beta}^\dagger \right] \right\}. \tag{6.2}$$

If we write the T-matrix in block form

$$T = \begin{pmatrix} T_{00} & T_{01} & T_{02} & T_{03} \\ T_{10} & T_{11} & T_{12} & T_{13} \\ T_{20} & T_{21} & T_{22} & T_{23} \\ T_{30} & T_{31} & T_{32} & T_{33} \end{pmatrix} , \tag{6.3}$$

then

$$N = i\pi \int' dE \; \text{trace} \left[T^{+} \frac{\partial}{\partial E} T - T \frac{\partial}{\partial E} T^{+} \right] . \tag{6.4}$$

The S-matrix can also be written in block form,

$$S = \hat{I} - 2\pi i \, \delta(E - E') \, T , \tag{6.5}$$

with

$$\hat{I} = \begin{pmatrix} I_0 & 0 & 0 & 0 \\ 0 & I_1 & 0 & 0 \\ 0 & 0 & I_2 & 0 \\ 0 & 0 & 0 & I_3 \end{pmatrix}$$

and I_0 , I_α are defined in Eq. (5.18) . With the use of the above relation, the expression for N can be rewritten in a form similar to Eq. (4.8),

$$N = \frac{i}{4\pi} \int' dE \text{ trace} \left[S^\dagger \frac{\partial S}{\partial E} - S \frac{\partial S^\dagger}{\partial E} + \frac{\partial}{\partial E} (S^\dagger - S) \right] . \tag{6.6}$$

Since the trace of each amplitude $T_{\alpha\beta}$ is assumed to vanish at its threshold and at infinite energy, and since the amplitudes are continuous through other thresholds, the term

$$\int' dE \text{ trace} \left[\frac{\partial}{\partial E} (S^+ - S) \right] \equiv 0 .$$

Define a unitary operator U, by

$$U = \begin{pmatrix} S_1^+ S_2^+ S_3^+ & 0 & 0 & 0 \\ 0 & I_1 & 0 & 0 \\ 0 & 0 & I_2 & 0 \\ 0 & 0 & 0 & I_3 \end{pmatrix} . \tag{6.7}$$

The operator $S_1^+ S_2^+ S_3^+$ was discussed in Section IV. A unitary connected S-matrix can now be defined by

$$S_c = U S , \tag{6.8}$$

and a connected T-matrix by

$$S_c = \hat{I} - 2\pi i \, \delta(E - E') \, T_c . \tag{6.9}$$

Substituting $U^\dagger S_c$ for S in Eq. (6.6), we obtain

$$N = \frac{i}{4\pi} \int' dE \text{ trace } \left[(S_c^\dagger U) \frac{\partial}{\partial E} (U^\dagger S_c) - U^\dagger S_c \frac{\partial}{\partial E} (S_c^\dagger U) \right] .$$

Using $S_c S_c^\dagger = \hat{I}$, $U^\dagger U = \hat{I}$ and trace $(AB - BA) = 0$, the above expression simplifies to

$$N = \frac{i}{4\pi} \int' dE \text{ trace } \left[U \frac{\partial U^\dagger}{\partial E} - U^\dagger \frac{\partial U}{\partial E} + S_c^\dagger \frac{\partial S_c}{\partial E} - S_c \frac{\partial S_c^\dagger}{\partial E} \right] . \tag{6.10}$$

The prime on the integral requires that the terms with an overall delta function be omitted, that is the U terms. Finally then we have

$$N = \frac{i}{4\pi} \int dE \text{ trace } \left[S_c^\dagger \frac{\partial S_c}{\partial E} - S_c \frac{\partial S_c^\dagger}{\partial E} \right] . \tag{6.11}$$

The eigenfunctions of T_c are used to compute the trace. For fixed energy T_c is a normal operator since unitarity requires

$$T_c^\dagger T_c = T_c T_c^\dagger ,$$

and it is square integrable since all integrations are over a finite range and there are no singularities in T_c. Hence it has a spectral decomposition,

$$T_c = -\frac{1}{\pi} \sum_n e^{i\delta_n} \sin \delta_n \; | \phi_n \rangle \langle \phi_n | , \tag{6.12}$$

where ϕ_n form an orthonormal set not necessarily complete. To make the set complete an orthonormal set of functions spanning the null space of T_c is added. The trace in Eq. (6.11) is computed using this basis. The diagonal elements are given by

$$\langle \phi_n | S_c^\dagger \frac{\partial S_c}{\partial E} - S_c \frac{\partial S_c^\dagger}{\partial E} | \phi_n \rangle = 4i \frac{d\delta_n}{dE} \quad . \tag{6.13}$$

Suppose the thresholds are ordered in the following way:

$$0 > -B_1 > -B_2 > -B_3 \quad .$$

Then the answer for the number of three body bound states is

$$N\pi = \sum_n \left[\delta_n(0) - \delta_n(\infty)\right] + \sum_n \left[\delta_n(-B_1) - \delta_n(0)\right] + \sum_n \left[\delta_n(-B_2) - \delta_n(-B_1)\right]$$

$$+ \sum_n \left[\delta_n(-B_3) - \delta_n(-B_2)\right] \quad . \tag{6.14}$$

The phase shifts are only determined modulo π, and since they must be a multiple of π at infinite energy, we are free to choose them to be zero. We can further require them to be continuous across the thresholds of newly opening channels. Rather than require the phase shifts at infinite energy to be zero, we will specify that only a finite number can be non zero. The sum of the phase shifts will converge at any energy and the only contribution will be from the elastic phase shifts at their thresholds,

$$N\pi = \sum_\alpha \sum_{n_\alpha} \delta_{n_\alpha}^{el}(-B_\alpha) + \sum_n \delta_n^{el}(0) - \sum_n \delta_n(\infty) \quad . \tag{6.15}$$

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APPENDIX A

In this appendix we will evaluate the trace of the right hand side of equation (3.9),

$$\Lambda = \sum_{\substack{\alpha,\beta=0 \\ \alpha\neq\beta}}^{3} \left[W_\alpha^\dagger, W_\beta\right] + \left[W_0^\dagger, W_0\right] \quad .$$

We have left out the term $\left[W_\alpha^\dagger, W_\alpha\right]$ since it is given by Eq. (3.10). Although there are a great many terms to evaluate, only three of them are different, so it is sufficient to calculate the following:

$$\Lambda_{12} = \text{trace} \left[W_1^\dagger, W_2\right] \quad ,$$

$$\Lambda_{01} = \text{trace} \left[W_1^\dagger, W_0\right] \quad ,$$

and

$$\Lambda_{00} = \text{trace} \left[W_0^\dagger, W_0\right] \quad . \tag{A.1}$$

It is convenient to use the set of variables used by Omnes[15] and discussed in Section III. We add one redundant variable, the total energy $E = \omega_1 + \omega_2 + \omega_3$. With this choice of variables the operators W_α become

$$\langle \omega_1, \omega_2, \omega_3, M \,|W_\alpha|\, \omega_1', \omega_2', \omega_3', M' \rangle =$$

$$= \delta(\omega_\alpha - \omega_\alpha') \frac{t_\alpha(\omega_1, \omega_2, \omega_3; \omega_1', \omega_2', \omega_3'; s = \omega_1' + \omega_2' + \omega_3' - \omega_\alpha' + i\epsilon; M, M')}{E - E' - i\epsilon} \quad , \tag{A.2}$$

$$\langle \omega_1, \omega_2, \omega_3, M \,|W_0|\, \omega_1, \omega_2, \omega_3, M' \rangle =$$

$$= T_0(\omega_1, \omega_2, \omega_3; \omega_1', \omega_2', \omega_3'; s = \omega_1' + \omega_2' + \omega_3' + i\epsilon; M, M') \quad . \tag{A.3}$$

The total T-matrix as given in Eq. (3.12) is just the sum

$$T_{00} = \sum_\alpha \delta(\omega_\alpha - \omega_\alpha')t_\alpha + T_0 \quad . \tag{A.4}$$

The variables M, M' and s will be omitted as the M, M' variables are always involved in finite sums which present no problem. The arguments ω_α are always positive, so if one of them is replaced by $E - \omega_1 - \omega_2$ for example, the entire expression is to be multiplied by a step function $\theta(E - \omega_1 - \omega_2)$. This will also be omitted, but implicitly understood to be present. To further simplify the notation, the set of variables ω_1, ω_2, ω_3 will be denoted by ω whenever there can be no misunderstanding.

In this notation, the expression for Λ_{12} becomes

$$\Lambda_{12} = \text{trace} \int_0^\infty d\omega'' \left\{ \frac{t_1^\dagger(\omega, \omega'')t_2(\omega'', \omega')\delta(\omega_1 - \omega_1'')\delta(\omega_2'' - \omega_2')}{(E - E'' - i\epsilon)(E' - E'' + i\epsilon)} - \right.$$

$$\left. \frac{t_2(\omega, \omega'')t_1^\dagger(\omega'', \omega')\delta(\omega_1'' - \omega_1')\delta(\omega_2 - \omega_2'')}{(E - E'' - i\epsilon)(E' - E + i\epsilon)} \right\} \quad . \tag{A.5}$$

To evaluate this expression we separate the singular denominators into principle parts and delta functions. We assume that all integrals

converge absolutely and uniformly at infinity so that it is permissible to interchange orders of integration except at the point where the denominators both vanish. For simplicity of notation we let A be the contribution from the product of the two delta functions, C be from the product of the principle parts and B be from the cross terms. A and C will be zero. A_{12} is easy to evaluate because of the delta functions,

$$A_{12} = \pi^2 \text{ trace } \delta(E - E') \Big\{ t_1^\dagger(\omega_1, \omega_2, E - \omega_1 - \omega_2; \omega_1, \omega_2', E - \omega_1 - \omega_2') \times$$

$$\times t_2(\omega_1, \omega_2', E' - \omega_1 - \omega_2'; \omega_1', \omega_2', E' - \omega_1' - \omega_2') -$$

$$t_1^\dagger(\omega_1', \omega_2, E - \omega_1' - \omega_2; \omega_1', \omega_2', E' - \omega_1' - \omega_2') \times$$

$$\times t_2(\omega_1, \omega_2, E - \omega_1 - \omega_2; \omega_1', \omega_2; E - \omega_1' - \omega_2) \Big\} . \tag{A.6}$$

The diagonal elements of the term in the brackets vanish identically and since $x\delta(x) = 0$, $A_{12} \equiv 0$.

The trace in C_{12} is written out explicitly,

$$C_{12} = P \int_0^\infty dE \int_0^\infty dE' \int_0^\infty d\omega_1 \int_0^\infty \frac{d\omega_2}{(E - E')^2}$$

$$\Big\{ t_1^\dagger(\omega_1, \omega_2, E - \omega_1 - \omega_2; \omega_1, \omega_2, E' - \omega_1 - \omega_2) \times$$

$$\times t_2(\omega_1, \omega_2, E' - \omega_1 - \omega_2; \omega_1, \omega_2; E - \omega_1 - \omega_2) - t_1^\dagger(\omega_1, \omega_2, E' - \omega_1 - \omega_2; \omega_1, \omega_2, E - \omega_1 - \omega_2) \times$$

$$\times t_2(\omega_1, \omega_2, E - \omega_1 - \omega_2; \omega_1, \omega_2, E' - \omega_1 - \omega_2) \Big\} . \tag{A.7}$$

Since the numerator vanishes at $E = E'$ and the integrals converge absolutely and uniformly at infinity, it is permissible to interchange the orders of integration. Since the integrand is antisymmetric in E and E', $C_{12} \equiv 0$. The only nonvanishing contribution to Λ_{12} is B_{12},

$$B_{12} = i\pi \,\mathrm{trace}\, \frac{P}{E' - E} \int d\omega_1\, d\omega_2 \Big\{ t_1^\dagger(\omega_1,\omega_2,E-\omega_1-\omega_2;\omega_1,\omega_2,E-\omega_1-\omega_2)\times$$

$$t_2(\omega_1,\omega_2,E-\omega_1-\omega_2;\omega_1,\omega_2,E'-\omega_1-\omega_2) - t_1^\dagger(\omega_1,\omega_2,E'-\omega_1-\omega_2;\omega_1,\omega_2,E'-\omega_1-\omega_2)\times$$

$$t_2(\omega_1,\omega_2,E-\omega_1-\omega_2;\omega_1,\omega_2,E'-\omega_1-\omega_2) + t_1^\dagger(\omega_1,\omega_2,E-\omega_1-\omega_2;\omega_1,\omega_2,E'-\omega_1-\omega_2)\times$$

$$t_2(\omega_1,\omega_2,E'-\omega_1-\omega_2;\omega_1,\omega_2,E'-\omega_1-\omega_2) - t_1^\dagger(\omega_1,\omega_2,E-\omega_1-\omega_2;\omega_1,\omega_2,E'-\omega_1-\omega_2)$$

$$t_2(\omega_1,\omega_2,E-\omega_1-\omega_2;\omega_1,\omega_2,E-\omega_1-\omega_2)\Big\} \quad . \tag{A.8}$$

This is of the form

$$\mathrm{trace}\, \frac{1}{E - E'} \Big\{ f(E,E')\, g(E,E) - f(E,E')\, g(E',E') \Big\}$$

which upon taking the limit $E \to E'$ and integrating becomes

$$\int dE\, f(E,E)\, \frac{\partial g(E,E)}{\partial E} \quad .$$

Finally then we have

$$B_{12} = i\pi \int_0^\infty dE \int_0^\infty d\omega_1\, d\omega_2 \left\{ t_1^\dagger(\omega_1,\omega_2,E-\omega_1-\omega_2;\omega_1,\omega_2,E-\omega_1-\omega_2) \times \right.$$

$$\frac{\partial}{\partial E} t_2(\omega_1,\omega_2,E-\omega_1-\omega_2;\omega_1,\omega_2,E-\omega_1-\omega_2) - t_2(\omega_1,\omega_2,E-\omega_1-\omega_2;\omega_1,\omega_2,E-\omega_1-\omega_2) \times$$

$$\left. \frac{\partial}{\partial E} t_1^\dagger(\omega_1,\omega_2,E-\omega_1-\omega_2;\omega_1,\omega_2,E-\omega_1-\omega_2) \right\} . \tag{A.9}$$

This can be put in a compact form by using T_1 and T_2 (see Eq. (3.2)) where T_1 is now an on the energy shell T-matrix,

$$B_{12} = i\pi \int_0^\infty dE \ \text{trace} \left\{ T_1^\dagger \frac{\partial T_2}{\partial E} - T_2 \frac{\partial}{\partial E} T_1^\dagger \right\} . \tag{A.10}$$

The analysis for Λ_{01} is quite similar,

$$\Lambda_{01} = \text{trace} \int_0^\infty d\omega'' \left\{ \frac{t_1^\dagger(\omega,\omega'')T_0(\omega'',\omega')\delta(\omega_1-\omega_1'') - T_0(\omega,\omega'')t_1^\dagger(\omega'',\omega')\delta(\omega_1'-\omega_1'')}{(E-E''-i\epsilon)(E'-E''+i\epsilon)} \right\} \tag{A.11}$$

Proceeding as before and doing all the traces except the E trace we have

$$A_{01} = \pi^2 \ \text{trace}\ \delta(E-E') \int d\omega_2''\, d\omega_1\, d\omega_2 \left\{ t_1^\dagger(\omega_1,\omega_2,E-\omega_1-\omega_2;\omega_1,\omega_2'',E-\omega_1-\omega_2'') \times \right.$$

$$T_0(\omega_1,\omega_2'',E'-\omega_1-\omega_2'';\omega_1,\omega_2,E'-\omega_1-\omega_2) - t_1^\dagger(\omega_1,\omega_2'',E'-\omega_1-\omega_2'';\omega_1,\omega_2,E'-\omega_1-\omega_2)$$

$$\left. \times\ T_0(\omega_1,\omega_2,E-\omega_1-\omega_2;\omega_1,\omega_2'',E-\omega_1-\omega_2'') \right\} . \tag{A.12}$$

Interchange ω_2 and ω_2'' in the second term and then the expression is explicitly antisymmetric in E and E' and hence vanishes. In the expression for C_{01} we do all the traces explicitly,

$$C_{01} = \int_0^\infty dE \int_0^\infty dE' \int_0^\infty \frac{d\omega_2' d\omega_1 d\omega_2}{(E - E')^2} \Big\{ t_1^\dagger(\omega_1, \omega_2, E-\omega_1-\omega_2; \omega_1, \omega_2', E'-\omega_1-\omega_2') \times$$

$$T_0(\omega_1, \omega_2', E' - \omega_1 - \omega_2'; \omega_1, \omega_2, E - \omega_1 - \omega_2) -$$

$$t_1^\dagger(\omega_1, \omega_2', E'-\omega_1-\omega_2'; \omega_1-\omega_2', E-\omega_1-\omega_2) T_0(\omega_1, \omega_2, E-\omega_1-\omega_2; \omega_1, \omega_2', E'-\omega_1-\omega_2') \Big\} \tag{A.13}$$

Again we interchange ω_2 and ω_2' in the second term. The expression is then explicitly antisymmetric in E and E' and the integral therefore vanishes. As before the entire contribution comes from B.

$$B_{01} = i\pi \text{ trace } \frac{P}{E' - E} \int d\omega_1 \, d\omega_2 \, d\omega_2' \Big\{ t_1^\dagger(\omega_1, \omega_2, E-\omega_1-\omega_2; \omega_1, \omega_2', E-\omega_1-\omega_2') \times$$

$$T_0(\omega_1, \omega_2', E-\omega_1-\omega_2'; \omega_1, \omega_2, E'-\omega_1-\omega_2) - t_1^\dagger(\omega_1, \omega_2', E'-\omega_1-\omega_2'; \omega_1, \omega_2, E'-\omega_1-\omega_2) \times$$

$$T_0(\omega_1, \omega_2, E-\omega_1-\omega_2; \omega_1, \omega_2', E'-\omega_1-\omega_2') + t_1^\dagger(\omega_1, \omega_2, E-\omega_1-\omega_2; \omega_1, \omega_2', E'-\omega_1-\omega_2') \times$$

$$T_0(\omega_1, \omega_2', E'-\omega_1-\omega_2'; \omega_1, \omega_2, E'-\omega_1-\omega_2) - t_1^\dagger(\omega_1, \omega_2', E-\omega_1-\omega_2'; \omega_1, \omega_2, E'-\omega_1-\omega_2') \times$$

$$T_0(\omega_1, \omega_2, E - \omega_1 - \omega_2; \omega_1, \omega_2', E - \omega_1 - \omega_2') \Big\} . \tag{A.14}$$

After interchanging ω_2 and ω_2' in the appropriate places and taking the limit $E \to E'$, we obtain

$$B_{01} = i\pi \int_0^\infty dE \ \text{trace} \left\{ T_1^\dagger \frac{\partial T_0}{\partial E} - \frac{T_0 \partial T_1^\dagger}{\partial E} \right\} . \tag{A.15}$$

The final term we have to calculate is Λ_{00} .

$$\Lambda_{00} = \text{trace} \int d\omega'' \left\{ \frac{T_0^\dagger(\omega, \omega'') T_0(\omega'', \omega') - T_0(\omega, \omega'') T_0^\dagger(\omega'', \omega')}{(E - E'' - i\epsilon)(E' - E'' + i\epsilon)} \right\} \tag{A.16}$$

$$A_{00} = \pi^2 \ \text{trace} \ \delta(E - E') \int_0^\infty d\omega_1 \, d\omega_2 \, d\omega_1' \, d\omega_2' \left\{ T_0^\dagger(\omega_1, \omega_2, E-\omega_1-\omega_2; \omega_1', \omega_2', E-\omega_1'-\omega_2') \right.$$

$$T_0(\omega_1', \omega_2', E-\omega_1'-\omega_2'; \omega_1, \omega_2, E-\omega_1-\omega_2) - T_0(\omega_1, \omega_2, E-\omega_1-\omega_2; \omega_1', \omega_2', E-\omega_1'-\omega_2') \times$$

$$\left. T_0^\dagger(\omega_1', \omega_2', E-\omega_1'-\omega_2'; \omega_1, \omega_2, E-\omega_1-\omega_2) \right\} . \tag{A.17}$$

Upon interchange of ω_1', ω_2' and ω_1, ω_2 , the expression inside the brackets vanishes identically.

$$C_{00} = \text{trace} \int_0^\infty dE \int_0^\infty dE' \int_0^\infty \frac{d\omega_1 \, d\omega_2 \, d\omega_1' d\omega_2'}{(E - E')^2} \left\{ T_0^\dagger(\omega_1, \omega_2, E-\omega_1-\omega_2; \omega_1', \omega_2', E'-\omega_1'-\omega_2') \times \right.$$

$$T_0(\omega_1', \omega_2', E'-\omega_1'-\omega_2'; \omega_1, \omega_2, E-\omega_1-\omega_2) - T_0^\dagger(\omega_1', \omega_2', E'-\omega_1'-\omega_2'; \omega_1, \omega_2, E-\omega_1-\omega_2) \times$$

$$\left. T_0(\omega_1, \omega_2, E-\omega_1-\omega_2; \omega_1', \omega_2', E'-\omega_1'-\omega_2') \right\} . \tag{A.18}$$

The integrand is antisymmetric upon interchange of all variables and hence the integral vanishes.

$$B_{00} = i\pi \text{ trace } \frac{P}{E' - E} \int d\omega_1 d\omega_2 d\omega_1' d\omega_2' \Big\langle T_0^\dagger(\omega_1, \omega_2, E-\omega_1-\omega_2; \omega_1', \omega_2', E-\omega_1'-\omega_2')$$

$$\times T_0(\omega_1', \omega_2', E-\omega_1'-\omega_2'; \omega_1, \omega_2, E'-\omega_1-\omega_2) - T_0^\dagger(\omega_1', \omega_2', E'-\omega_1'-\omega_2'; \omega_1, \omega_2, E'-\omega_1-\omega_2)$$

$$\times T_0(\omega_1, \omega_2, E-\omega_1-\omega_2; \omega_1', \omega_2', E'-\omega_1'-\omega_2') + T_0^\dagger(\omega_1, \omega_2, E-\omega_1-\omega_2; \omega_1', \omega_2', E'-\omega_1'-\omega_2')$$

$$\times T_0(\omega_1', \omega_2', E'-\omega_1'-\omega_2'; \omega_1, \omega_2, E'-\omega_1-\omega_2) - T_0^\dagger(\omega_1', \omega_2', E-\omega_1'-\omega_2'; \omega_1, \omega_2, E'-\omega_1-\omega_2)$$

$$\times T_0(\omega_1, \omega_2, E-\omega_1-\omega_2; \omega_1', \omega_2', E-\omega_1'-\omega_2') \Big\rangle \quad . \tag{A.19}$$

This reduces in the usual way to

$$B_{00} = i\pi \int_0^\infty dE \text{ trace } \left\{ T_0 \frac{\partial T_0^\dagger}{\partial E} - T_0^+ \frac{\partial T_0}{\partial E} \right\} \tag{A.20}$$

From Eqs. (A.4) and (3.12) the three body T-matrix associated with S_{00} is given by

$$T_{00} = T_1 + T_2 + T_3 + T_0$$

$$S_{00} = 1 - 2\pi i \, \delta(E - E') \, T_{00} \quad . \tag{A.21}$$

Combining all the results of this appendix, we have

$$\Lambda_0 = i\pi \int_0^{\infty\prime} dE \text{ trace } \left\{ T_{00}^{\dagger} \frac{\partial T_{00}}{\partial E} - T_{00} \frac{\partial T_{00}^{\dagger}}{\partial E} \right\} , \tag{A.22}$$

where the prime on the integral means that the disconnected parts - that is the terms with an overall delta function - are to be left out.

APPENDIX B

In this appendix we will derive in detail the trace of the third term in Eq. (6.1), which we will call Λ_B.

$$\Lambda_B = \sum_{\alpha=1}^{3} \text{trace} \left\{ P_\alpha \Omega_\alpha^\dagger \Omega_\alpha - \Omega_\alpha \Omega_\alpha^\dagger \right\} \quad . \tag{B.1}$$

The Ω_α operators were defined in (5.10)

$$\Omega_\alpha = \sum_\beta \frac{L_{\alpha\beta}(\omega, \nu, \lambda; \omega'_\alpha)}{\omega + \nu - B'_\alpha + B_\alpha - i\epsilon} \quad ,$$

and the $L_{\alpha\beta}$ and $K_{\alpha\beta}$ in (5.7) and (5.8),

$$L_{\alpha\beta} = (\nu_\alpha + B_\alpha)\, \psi_\alpha(\nu_\alpha)\, \delta(\omega_\alpha - \omega'_\alpha)\, \delta_{\alpha\beta} + K_{\alpha\beta} \quad ,$$

$$K_{\alpha\beta} = G_{\alpha\beta} + \frac{(\nu_\alpha + B_\alpha)\, \psi_\alpha(\nu_\alpha)\, F_{\alpha\beta}(\omega_\alpha; \omega'_\beta)}{\omega_\alpha - B_\alpha - \omega'_\beta + B_\beta - i\epsilon} \quad .$$

P_α is a projection operator on the two body bound state,

$$P_\alpha = |\psi_\alpha(\nu_\alpha)\rangle\langle\psi_\alpha(\nu'_\alpha)| \tag{B.2}$$

The first term factors into $P_\alpha(\nu_\alpha, \nu'_\alpha)\, \delta(\omega_\alpha - \omega'_\alpha)$, so the trace over ν_α can be taken giving unity. $L_{\alpha\beta}$ has a term which is essentially a unit operator and commutes with the other terms to give zero. Therefore Λ_B can be written in terms of K,

$$\Lambda_B = \sum_{\alpha,\beta,\gamma} \text{trace} \left\{ \sum_\lambda \int_0^\infty d\omega'' \, d\nu'' \, \frac{\tilde{K}_{\alpha\gamma}(\omega_\alpha;\omega'',\nu'',\lambda)\, K_{\alpha\beta}(\omega'',\nu'',\lambda;\omega'_\alpha)}{(\omega_\alpha - B_\alpha - \omega'' - \nu'' - i\epsilon)(\omega'_\alpha - B_\alpha - \omega'' - \nu'' + i\epsilon)} \right.$$

$$\left. - \int_0^\infty d\omega''_\alpha \, \frac{K_{\beta\alpha}(\omega,\nu,\lambda;\omega''_\alpha)\, \tilde{K}_{\alpha\gamma}(\omega''_\alpha;\omega',\nu',\lambda')}{(\omega + \nu - \omega''_\alpha + B_\alpha - i\epsilon)(\omega' + \nu' - \omega''_\alpha + B_\alpha + i\epsilon)} \right\} . \qquad (B.3)$$

The usual assumption is made that the orders of integration could be interchanged except for the singularities from the denominators. There are two sources of singularities which occur when $\omega'_\alpha = \omega_\alpha$ and $\omega' + \nu' = \omega + \nu$. The first is exhibited explicitly above and is located at $\omega'' + \nu'' = \omega_\alpha - B_\alpha$. The second is hidden in the K term itself, and can be seen in Eq. (5.8) above. It is located at $\omega_\alpha - B_\alpha = \omega''_\beta - B_\beta$ and will occur only when $\beta = \gamma$ in Eq. (B.3). The two singularities occur at different points so they can be discussed separately.

In the proofs there will be many changes of variable of an essentially trivial nature. As in Appendix A, we will omit all explicit reference to changes in the integration region. If an argument of a K or an F function is negative, the function will be taken to be zero; that is a step function of all arguments is implied. With this restriction, the integration on all variables is taken over the region of positive arguments of the functions K and F. The variable E is used for the total energy, either $\omega + \nu$ or $\omega_\alpha - B_\alpha$. Hereafter the operation "trace" will refer only to E. All other traces will be done explicitly.

First the singularity at $\omega'' + \nu'' = \omega_\alpha - B_\alpha$ is discussed as though the one from the F term didn't exist. Then presuming that the

first singularity is absent we treat the F term. The evaluation follows the by now familiar procedure of splitting the denominators into principle parts and delta functions. The term from the product of delta functions is called A_K or A_F depending upon which singularity is being discussed. The term from the product of principle parts is called C_K or C_F. The cross terms are B_K and B_F. The first contribution to be evaluated is A_K.

$$A_K = \sum_{\alpha,\beta,\gamma} \text{trace } \delta(E - E'') \int \sum_{\lambda} d\nu \left[K_{\beta\alpha}(E-\nu,\nu,\lambda;E + B_\alpha) \tilde{K}_{\alpha\gamma}(E + B_\alpha;E''-\nu,\nu,\lambda) \right.$$

$$\left. - K_{\beta\alpha}(E - \nu,\nu,\lambda;E'' + B_\alpha) \tilde{K}_{\alpha\gamma}(E + B_\alpha;E - \nu,\nu,\lambda) \right] . \tag{B.4}$$

The term in brackets vanishes at $E = E''$ so $A_K = 0$. Hereafter the variable λ will be omitted as it adds nothing to the proof. The evaluation of C_K is straight forward.

$$C_K = \sum_{\alpha,\beta,\gamma} P \int_0^\infty dE \int_0^\infty \frac{dE'}{(E-E')^2} \int_0^\infty d\nu \left[\tilde{K}_{\alpha\gamma}(E-B_\alpha;E'-\nu,\nu) K_{\beta\alpha}(E'-\nu,\nu;E-B_\alpha) \right.$$

$$\left. - \tilde{K}_{\alpha\gamma}(E'-B_\alpha;E-\nu,\nu) K_{\beta\alpha}(E-\nu,\nu;E'-B_\alpha) \right] . \tag{B.5}$$

The term in brackets vanishes at $E = E'$, so the principle part integration is well defined. Therefore the orders of integration may be interchanged and $C_K = 0$ due to the antisymmetry of the integrand.

We now consider the contribution from the term involving only one delta function,

$$B_K = i\pi \sum_{\alpha,\beta,\gamma} \text{trace} \frac{1}{E'-E} \int_0^\infty d\nu \left\{ \tilde{K}_{\alpha\gamma}(E+B_\alpha;E-\nu,\nu)\, K_{\beta\alpha}(E-\nu,\nu;E'+B_\alpha) \right.$$

$$- \tilde{K}_{\alpha\gamma}(E+B_\alpha;E'-\nu,\nu)\, K_{\beta\alpha}(E'-\nu,\nu;E'+B_\alpha) + \tilde{K}_{\alpha\gamma}(E-B_\alpha;E'-\nu,\nu)\, K_{\beta\alpha}(E-\nu,\nu;E+B_\alpha)$$

$$\left. - \tilde{K}_{\alpha\gamma}(E'+B_\alpha;E'-\nu,\nu)\, K_{\beta\alpha}(E-\nu,\nu;E'+B_\alpha) \right\} . \tag{B.6}$$

To evaluate B_K take the limit $E \to E'$ which gives a derivative and then integrate over E. The final result including the angular variable λ is

$$B_K = i\pi \sum_{\alpha,\beta,\gamma} \sum_{\gamma} \int_0^\infty dE \int_0^\infty d\nu \left\{ \tilde{K}_{\alpha\gamma}(E+B_\alpha;E-\nu,\nu,\lambda) \frac{\partial}{\partial E} K_{\beta\alpha}(E-\nu,\nu,\lambda;E+B_\alpha) \right.$$

$$\left. - K_{B\alpha}(E-\nu,\nu,\lambda;E+B_\alpha) \frac{\partial}{\partial E} \tilde{K}_{\alpha\gamma}(E+B_\alpha;E-\nu,\nu,\lambda) \right\} . \tag{B.7}$$

In terms of the transition operators $T_{0\alpha}$ defined in Eq. (5.12), the result is

$$B_K = i\pi \sum_{\alpha} \int_0^\infty dE \, \text{trace} \left\{ T_{0\alpha}^\dagger \frac{\partial T_{0\alpha}}{\partial E} - T_{0\alpha} \frac{\partial T_{0\alpha}^\dagger}{\partial E} \right\} . \tag{B.8}$$

In addition to the singularity from the three to two amplitude, there is a term from the two to two amplitude. It comes from the singularity of the F term implicit in (B.3). Referring to Eq. (5.8) and substituting the F term for the K term we see that the only new singularity will come when $\beta = \gamma$. Therefore we have

$$\Lambda_F = \sum_{\alpha,\beta,\gamma} \delta_{\beta\gamma} \text{ trace } \int_0^\infty d\nu_\beta \Bigg\{ \int_0^\infty \frac{d\omega_\beta'' \, \tilde{F}_{\alpha\beta}(\omega_\alpha;\omega_\beta'') \, F_{\beta\alpha}(\omega_\beta'';\omega_\alpha') \, (\nu_\beta+B_\beta)^2 \, |\psi_\beta(\nu_\beta)|^2}{\left[(\omega_\alpha-B_\alpha-\omega''-\nu'')(\omega_\alpha'-B_\alpha-\omega''-\nu'')\right]\left[(\omega_\beta''-B_\beta-\omega_\alpha'+B_\alpha+i\epsilon)(\omega_\gamma''-B_\gamma-\omega_\alpha-B_\alpha+i\epsilon)\right]}$$

$$- \int_0^\infty \frac{d\omega_\alpha'' \, F_{\beta\alpha}(\omega_\beta;\omega_\alpha'') \, \tilde{F}_{\alpha\beta}(\omega_\alpha'';\omega_\beta') \, (\nu_\beta+B_\beta)^2 \, |\psi_\beta(\nu_\beta)|^2}{\left[(\omega+\nu-\omega_\alpha''+B_\alpha)(\omega'+\nu'-\omega_\alpha''+B_\alpha)\right]\left[(\omega_\beta-B_\beta+B_\alpha-\omega_\alpha''-i\epsilon)(\omega_\gamma'-B_\gamma-\omega_\alpha''+B_\alpha+i\epsilon)\right]} \Bigg\} \tag{B.9}$$

The $i\epsilon$ has been left out of the denominators already treated, as they are presumed to be nonsingular. The γ dependence has been indicated in the denominators to make it clear that only $\beta = \gamma$ terms are singular. The expression is now evaluated in the usual way in terms of principle parts and delta functions. For A_F we have,

$$A_F = -\pi^2 \sum_{\alpha,\beta} \text{ trace } \int_0^\infty d\nu \, |\psi_\beta(\nu)|^2 \Big\{ \delta(\omega_\alpha-\omega_\alpha') \, \tilde{F}_{\alpha\beta}(\omega_\alpha;B_\beta+\omega_\alpha-B_\alpha) \, F_{\beta\alpha}(B_\beta+\omega_\alpha-B_\alpha;\omega_\alpha)$$

$$- \delta(\omega_\beta-\omega_\beta') \, \tilde{F}_{\alpha\beta}(\omega_\beta-B_\beta+B_\alpha;\omega_\beta) \, F_{\beta\alpha}(\omega_\beta;\omega_\beta-B_\beta+B_\alpha) \Big\} . \tag{B.10}$$

A simple change of variables,

$$\omega'_\alpha,\ \omega_\alpha \rightarrow \omega' + B_\alpha,\ \omega + B_\alpha$$

and

$$\omega'_\beta,\ \omega_\beta \rightarrow \omega' + B_\beta,\ \omega + B_\beta$$

puts the expression in a form in which it is explicitly antisymmetric in ω and ω'. It therefore vanishes since it multiplies $\delta(\omega - \omega')$. The calculation for C_F proceeds along similar lines. We change variables to $E = \omega_\gamma - B_\gamma$ and obtain

$$C_F = \sum_{\alpha,\beta} P \int_{-\infty}^{\infty} dE \int_{-\infty}^{\infty} dE' \int_0^{\infty} d\nu \ \frac{(\nu + B_\beta)^2\ |\psi_\beta(\nu)|^2}{(E - E')^2}$$

$$\times \left\{ \frac{\tilde{F}_{\alpha\beta}(E+B_\alpha;E'+B_\beta)\ F_{\beta\alpha}(E'+B_\beta;E+B_\alpha)}{(E-\nu-E'-B_\beta)^2} - \frac{F_{\beta\alpha}(E+B_\beta;E'+B_\alpha)\ \tilde{F}_{\alpha\beta}(E'+B_\alpha,E+B_\beta)}{(E + B_\beta + \nu - E')^2} \right\} . \tag{B.11}$$

The integrand vanishes at $E = E'$ and therefore the principle part integration is well defined. Interchanging orders of integration and using the antisymmetry of the integrand we obtain $C_F = 0$.

The final term to be evaluated is B_F,

$$B_F = i\pi \sum_{\alpha,\beta} \text{trace} \int_0^{\infty} d\nu \ \frac{(\nu + B_\beta)^2\ |\psi(\nu)|^2}{(E' - E)}$$

$$\times \left\{ \frac{\tilde{F}_{\alpha\beta}(E+B_\alpha;E'+B_\beta)\ F_{\beta\alpha}(E'+B_\beta;E'+B_\alpha)}{(E' + \nu + B_\beta - E)(\nu + B_\beta)} - \frac{\tilde{F}_{\alpha\beta}(E+B_\alpha;E'+B_\beta)\ F_{\beta\alpha}(E+B_\beta;E+B_\alpha)}{(\nu + B_\beta)(E' + B_\beta + \nu - E)} \right.$$

$$\left. + \frac{\tilde{F}_{\alpha\beta}(E+B_\alpha;E+B_\beta)\ F_{\beta\alpha}(E+B_\beta;E'+B_\alpha)}{(B_\beta + \nu)^2\ (E + \nu + B_\beta - E')} - \frac{\tilde{F}_{\alpha\beta}(E'+B_\alpha;E'+B_\beta)\ F_{\beta\alpha}(E+B_\beta;E'+B_\alpha)}{(E + \nu + B_\beta - E')(\nu + B_\beta)} \right\} . \tag{B.12}$$

By expanding the term $(E' + \nu + B_\beta - E)^{-1}$ in powers of $(E - E')$, it is easily seen that the only term that need be kept in the above equation is the constant term as all others cancel in the limit $E \to E'$. From Eq. (5.2) we have the normalization integral,

$$\int_0^\infty d\nu \, |\psi_\beta(\nu)|^2 = 1 .$$

In the limit $E \to E'$ the remaining terms give a derivate to yield

$$B_F = i\pi \sum_{\alpha,\beta} \int_{-\min(B_\alpha, B_\beta)}^{\infty} dE \left\{ \tilde{F}_{\alpha\beta}(E + B_\alpha; E + B_\beta) \frac{\partial F_{\beta\alpha}}{\partial E} (E + B_\beta; E + B_\alpha) \right.$$
$$\left. - F_{\beta\alpha}(E + B_\beta; E + B_\alpha) \frac{\partial}{\partial E} \tilde{F}_{\alpha\beta}(E + B_\alpha; E + B_\beta) \right\} . \qquad \text{(B-13)}$$

This expression can be rewritten in terms of $T_{\alpha\beta}$ defined in Eq. (5.11),

$$B_F = i\pi \sum_{\alpha,\beta} \int_{-\min(B_\alpha, B_\beta)}^{\infty} dE \left\{ T^\dagger_{\beta\alpha} \frac{\partial T_{\beta\alpha}}{\partial E} - T_{\beta\alpha} \frac{\partial}{\partial E} T^\dagger_{\beta\alpha} \right\} . \qquad \text{(B-14)}$$

Combining Eqs. (B-14) and (B-8) we have

$$\Lambda_B = i\pi \left[\sum_\alpha \int_0^\infty dE \, \text{trace} \left\{ T^\dagger_{\alpha\alpha} \frac{\partial}{\partial E} T_{\alpha\alpha} - T_{\alpha\alpha} \frac{\partial}{\partial E} T^\dagger_{\alpha\alpha} \right\} \right.$$
$$\left. + \sum_{\alpha,\beta} \int_{-\min(B_\alpha, B_\beta)}^{\infty} dE \left\{ T^\dagger_{\alpha\beta} \frac{\partial}{\partial E} T_{\alpha\beta} - T_{\alpha\beta} \frac{\partial}{\partial E} T^\dagger_{\alpha\beta} \right\} \right] .$$
$$\qquad \text{(B-15)}$$

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I. INTRODUCTION

In this part we develop a method of simplifying and improving some potential theory calculations. The basic idea is to expand the scattering matrix T in eigenfunctions of the kernel of the Lippmann-Schwinger equation.[1] In general these eigenfunctions can not be found exactly, but we have developed a variational principle which enables us to make a reasonably accurate approximation to them.

Most of the existing methods for finding approximate eigenfunctions and eigenvalues are only useful for Hermitian operators. Unfortunately the kernel of the Lippmann-Schwinger equation is not Hermitian; however it has similar properties and the variational principle for the eigenvalues takes a form analogous to the one for Hermitian operators. In Section II we discuss this relationship and prove this variational principle.

In Section III we discuss the form the approximate trial wave functions should take, and we extend the results of Section II to partial waves. In Section IV we discuss Regge poles and their residues and present the results of a calculation of Regge trajectories from the eigenvalues of the Lippmann-Schwinger kernel. The poles of the T-matrix occur when one of these eigenvalues is equal to one, so it is a simple matter to find a Regge pole once an eigenvalue is known as a function of the complex angular momentum and the complex energy.

In Section V we discuss the relationship of the eigenfunction expansion to Weinberg's[2] quasiparticle method. They are quite closely

related, and our techniques can be used to improve calculations by that method. We calculate some low energy parameters and compare them with some presumably accurate calculations and also with those of Scadron and Weinberg.[3]

In Section VI we consider the possibility of extending these methods to the three-body problem. A more complicated variational principle is proved for the Faddeev[4] equations. It is suggested that these equations can be simplified by using the eigenfunction expansion for the two-body T-matrices in these equations.

II. EIGENFUNCTION EXPANSION

In this section we introduce a variational principle for finding approximate eigenfunctions and eigenvalues of the kernel for the Lippmann-Schwinger equation

$$T(s) = V + T(s)\, G_o(s) V \,. \tag{2.1}$$

The formal solution to this equation can be written

$$T = V[1 - G_o V]^{-1} \,. \tag{2.2}$$

For well-behaved potentials the kernel has been shown to be compact[4,5,6,7] for all complex and real s, and therefore the only singularities of the kernel occur when an eigenvalue of $G_o V$ is equal to one. The eigenvalue equation is

$$G_o(s) V \,|\psi(s)\rangle = \lambda(s) \,|\psi(s)\rangle \,, \tag{2.3}$$

which is just the Schroedinger equation

$$V \,|\psi(s)\rangle = \lambda(s)(s - H_o) \,|\psi(s)\rangle \tag{2.4}$$

for a complex potential V/λ. Weinberg has shown that the eigenvalues are real analytic functions of s, and that the phases of ψ may be chosen so that it is also real analytic; that is

$$\langle \vec{r} \mid \psi(s) \rangle = \langle \vec{r} \mid \psi(s^*) \rangle^* , \tag{2.5}$$

and

$$\lambda^*(s^*) = \lambda(s) \,. \tag{2.6}$$

For spherically symmetric potentials $V(|\vec{r}|)$, the momentum space wave

functions can also be chosen such that

$$\langle \vec{p} \,|\psi(s)\rangle = \langle \vec{p} \,|\psi(s^*)\rangle^* . \tag{2.7}$$

To have a practical stationary principle it is necessary that there exist a simple relationship between the solutions of (2.4) and its adjoint,

$$V\,|\phi(s)\rangle = \lambda^*(s)\,(s^* - H_o)\,|\phi(s)\rangle . \tag{2.8}$$

Change s to s^* in (2.8) and use (2.6) to see that $|\phi(s^*)\rangle$ is a solution of (2.4). We choose the normalization and phase such that

$$|\psi(s^*)\rangle = |\phi(s)\rangle . \tag{2.9}$$

With this simple relationship we have the following useful theorem,

Theorem

The functional form

$$\lambda[\psi(s)] = \frac{\langle\psi(s^*)\,|V|\,\psi(s)\rangle}{\langle\psi(s^*)\,|s - H_o|\,\psi(s)\rangle} \tag{2.10}$$

is stationary, i.e., $\delta\lambda = 0$, if and only if the state $|\psi(s)\rangle$ is a solution of (2.4)

Proof

The variation of λ is given by

$$\langle\psi(s^*)|s - H_o|\psi(s)\rangle\delta\lambda = \langle\delta\psi(s^*)|V - \lambda(s)(s - H_o)|\psi(s)\rangle$$

$$+ \langle\psi(s^*)|V - \lambda(s)(s - H_o)|\delta\psi(s)\rangle .$$

Let $\delta\lambda = 0$. Then using (2.5) we have

$$[V - \lambda(s)(s - H_o)] \, |\psi(s)\rangle = 0 \, .$$

Obviously if (2.4) is satisfied, $\delta\lambda \neq 0$.

Special care must be taken for s on the positive real axis. The proper method to handle the singularities is to evaluate $\langle\psi(s^*)|s - H_o|\psi(s)\rangle$ in momentum space for s complex with a small positive imaginary part. It is often more convenient to compute $\langle\psi(s^*)|V|\psi(s)\rangle$ in coordinate space.

In principle the kernel G_oV can be expanded in its eigenfunctions,[8]

$$G_oV = \sum_n \frac{|\psi_n(s)\rangle\langle\psi_n(s^*)|V}{\langle\psi_n(s^*)|s - H_o|\psi_n(s)\rangle} \, . \tag{2.11}$$

Using this expansion we formally solve (2.2) to obtain

$$T_1 = V + \sum_n \frac{V|\psi_n(s)\rangle \, \lambda_n(s) \, \langle\psi_n(s^*)|V}{\langle\psi_n(s^*)|V|\psi_n(s)\rangle[1 - \lambda_n(s)]} \, . \tag{2.12}$$

If only one of the eigenvalues is large and all the others are much smaller than one, the T matrix can be approximated quite accurately by the potential plus one term in the expansion. Bound states occur when $\lambda(s) = 1$ for $s < 0$, and the resonance poles occur when $\text{Im}\,\lambda(s) \ll 1$ and $\text{Re}\,\lambda(s) \cong 1$ for $s > 0$. In the next section we discuss the partial wave problem and some criteria for selecting approximate eigenfunctions.

III. PARTIAL-WAVE EIGENFUNCTIONS

When we consider spherically symmetric potentials, the eigenvalues λ_n and the eigenfunctions $|\psi_n(s)\rangle$ become parameterized by the angular momentum ℓ

$$\lambda_n(s) \to \lambda_\nu(\ell;s) \qquad \{n\} \to \{\nu, \ell\} \tag{3.1}$$

$$\langle \vec{r}|\psi_n(s)\rangle = \frac{1}{r}\langle r|\psi_{\ell,\nu}(s)\rangle\, Y_\ell^m(\hat{r}) . \tag{3.2}$$

For simplicity we will omit the index ν. The partial-wave Schroedinger equation with the complex potential $\frac{V}{\lambda_\ell(s)}$ is

$$\left[\frac{d^2}{dr^2} + k^2 - \frac{\ell(\ell+1)}{r^2} - \frac{V(r)}{\lambda_\ell(s)}\right]\langle r|\psi_\ell(s)\rangle = 0 \tag{3.3}$$

where

$$k^2 = s , \qquad \text{Im } k > 0 \tag{3.4}$$

and

$$k^*(s^*) = -k(s) . \tag{3.5}$$

We shall use the Riccatti Bessel functions

$$u_\ell(z) = \sqrt{\frac{\pi z}{2}}\, J_{\ell+\frac{1}{2}}(z) \tag{3.6}$$

where

$$\frac{2}{\pi}\int_0^\infty u_\ell(qr)\, u_\ell(pr)dr = \delta(q-p) \tag{3.7}$$

so that the partial-wave Green's function is

$$G_\ell(s;\, r,r') \;=\; \frac{2}{\pi}\int_0^\infty \frac{dq}{s - q^2}\; u_\ell(qr)\; u_\ell(qr'), \qquad \text{Im } s > 0\ . \tag{3.8}$$

The partial-wave momentum-space wave functions can be found from the solutions of (3.3) by

$$\langle\, q|\psi_\ell(s)\,\rangle \;=\; \sqrt{\frac{2}{\pi}}\int_0^\infty dr\; u_\ell(qr)\,\langle\, r|\psi_\ell(s)\,\rangle \tag{3.9}$$

so that the functional (2.10) becomes

$$\lambda_\ell[\psi] \;=\; \frac{\displaystyle\int_0^\infty V(r)\left\{\langle\, r\,|\psi_\ell(s)\,\rangle\right\}^2}{\displaystyle\int_0^\infty dq\,\langle\,\psi_\ell(s^*)|q\,\rangle(s - q^2)\langle q|\psi_\ell(s)\rangle}\ . \tag{3.10}$$

At this point we restrict ourselves to the Yukawa potential

$$V(r) \;=\; -g\,\frac{e^{-r}}{r}\ , \tag{3.11}$$

for which

$$\langle\, p|V|q\,\rangle \;=\; -\frac{g}{\pi}\; Q_\ell\left(\frac{q^2 + p^2 + 1}{2qp}\right)\ . \tag{3.12}$$

Our trial wave functions are chosen in as simple a form as possible consistent with the boundary conditions they must satisfy. These can be found directly from the defining equation,

$$\lambda(\ell,s)\,\langle\, q|\psi_\ell(s)\,\rangle \;=\; \frac{-1}{s - q^2}\int_0^\infty \frac{g}{\pi}\, Q_\ell\left(\frac{q^2 + p^2 + 1}{2qp}\right)\langle\, p|\psi_\ell(s)\,\rangle dp \tag{3.13}$$

from which we see that

$$\langle\, q|\psi_\ell(s)\,\rangle \xrightarrow[q \to 0]{} q^{\ell+1}\ , \tag{3.14}$$

$$\langle\, q|\psi_\ell(s)\,\rangle \xrightarrow[q \to \infty]{} q^{-\ell-3}\ , \tag{3.15}$$

and

$$\langle q|\psi_\ell(s)\rangle \propto \frac{1}{s - q^2} \quad . \tag{3.16}$$

We now construct a trial wave function corresponding to the lowest eigenfunction, with one arbitrary parameter, μ .

$$\langle q|\psi_\ell(s)\rangle = \frac{q^{\ell+1}}{(q^2 - s)\left[q^2 + (\mu - ik)^2\right]^{\ell+1}} \quad , \tag{3.17}$$

with $k^2 = s$. In the limit $\mu \to 0$, this wave function goes over to the exact lowest coulomb wave functions,

$$\langle q|\psi_\ell(s)\rangle_{\text{coulomb}} = \frac{q^{\ell+1}}{(q^2 - s)^{\ell+2}} \quad . \tag{3.18}$$

With these trial wave functions we can use the variational principle to find good approximate eigenvalues; we do so in the next two sections to find Regge trajectories and some other properties of scattering by a Yukawa potential.

IV. EIGENVALUE REGGE TRAJECTORIES

We now let ℓ become complex and examine the expansion (2.12) near a pole. The poles occur at those points where

$$\lambda(\ell,s) = 1 . \tag{4.1}$$

This solution is then inverted to give a Regge trajectory,

$$\ell = \alpha(s) . \tag{4.2}$$

The reflection properties (2.5, (2.6) and (2.7) become

$$\lambda^*(\ell^*,s^*) = \lambda(\ell,s)$$

$$\langle\, p \,|\psi(\ell,s)\,\rangle = \langle\, p \,|\psi(\ell^*,s^*)\,\rangle^*$$

$$\langle\, r \,|\psi(\ell,s)\,\rangle = \langle\, r \,|\psi(\ell^*,s^*)\,\rangle^* . \tag{4.3}$$

The residues of the poles are also easily determined from (2.12), since near a pole we have

$$T(\ell,s) \approx \frac{\left|\langle\, p^2 = s \,|V|\psi(\ell,s)\,\rangle\right|^2}{[1 - \lambda(\ell,s)]\,\langle\, \psi(\ell^*,s^*) \,|V|\psi(\ell,s)\,\rangle} \tag{4.4}$$

The factor $[1 - \lambda(\ell,s)]$ can be further simplified to give

$$1 - \lambda(\ell,s) \approx -[\ell - \alpha(s)] \left[\frac{\partial\lambda(\ell,s)}{\partial\ell}\right]_{\ell=\alpha(s)} , \tag{4.5}$$

with

$$\left[\frac{\partial\lambda(\ell,s)}{\partial\ell}\right] = -\left[\frac{d\alpha}{ds}\right]^{-1} \left[\frac{\partial\lambda(\ell,s)}{\partial s}\right]_{\ell=\alpha(s)} . \tag{4.6}$$

It is a very simple matter to calculate $\frac{\partial\lambda}{\partial s}$ by using the stationary principle (2.10). The derivative of the wave function with respect to s must vanish because $\delta\lambda = 0$, and the entire contribution comes from the derivative of the operators $(s - H_o)$ and V. Since we assume energy independent potentials the only contribution is from

$$\frac{d}{ds}(s - H_o) = 1 .$$

Therefore we have

$$\left[\frac{\partial\lambda(\ell,s)}{\partial s}\right]_{\ell=\alpha(s)} = -\left[\frac{\left(\langle\, \psi(\ell^*,s^*) \mid \psi(\ell,s)\,\rangle\right)}{\langle\, \psi(\ell^*,s^*) \,|V|\, \psi(\ell,s)\,\rangle}\right]_{\ell=\alpha(s)} \tag{4.7}$$

and the residue of the pole in T (4.4) is

$$R(s) = \frac{-\left(\langle\, p^2 = s \,|V|\, \psi[\alpha(s),s]\,\rangle\right)^2 \frac{d\alpha(s)}{ds}}{\langle\, \psi[\alpha(s^*),s^*] \mid \psi[\alpha(s),s]\,\rangle} . \tag{4.8}$$

Using (4.8) it is easy to prove that the reduced residues don't change sign for $s < 0$. The reduced residues are defined by factoring out the threshold behavior. There is also an additional factor of $-\frac{\pi}{2s}$ since our amplitude is not the usual partial wave amplitude, and this gives for the usual reduced residue

$$\gamma(s) = \frac{\frac{\pi}{2}\left\{\int_0^\infty \sqrt{r}\; J_{\alpha+\frac{1}{2}}(\sqrt{s}\; r)\,\langle\, r \,|\psi[\alpha(s),s]\,\rangle\, dr\right\}^2 \frac{d\alpha(s)}{ds}}{s^{\alpha+\frac{1}{2}}\int_0^\infty \left\{\langle\, r \,|\psi[\alpha(s),s]\,\rangle\right\}^2 dr}$$

The Bessel functions $J_p(ix)$ are related to $I_p(x)$ by

$$J_p(ix) = i^p I_p(x) ,$$

and $I_p(x)$ is a real function for x real. Therefore $\gamma(s)$ is real and positive for $s < 0$ since $\frac{d\alpha}{ds}$ is positive.

As an illustration of the use of (4.1) in calculating Regge trajectories, we consider the coulomb potential $V(r) = -\frac{g}{r}$. If the wave function (3.18) is inserted in (3.10), the result is of course the exact lowest coulomb eigenvalue

$$\lambda(\ell, s) = \frac{ig}{2k(\ell + 1)} , \tag{4.10}$$

which gives the exact leading trajectory,[9]

$$\alpha(s) = -1 + \frac{ig}{2k} . \tag{4.11}$$

For the Yukawa potential (4.11) we were forced to do the integrals numerically. Using the wave functions (4.17), we have the result,

$$\lambda(\ell, s) = \frac{\dfrac{2g}{\pi}\displaystyle\int_0^\infty dr\, \frac{e^{-r}}{r} \left\{ \int_0^\infty dq\, \frac{q^{\ell+1} u_\ell(qr)}{(q^2 - s)[q^2 + (\mu - ik)^2]^{\ell+1}} \right\}}{\displaystyle\int_0^\infty \frac{q^{2\ell+2}\, dq}{(q^2 - s)[q^2 + (\mu - ik)^2]^{2\ell+2}}} . \tag{4.12}$$

Combining the denominators in (4.12) by Feynman's method, we obtain the simplification

$$\lambda(\ell,s) = g\, 4^{\ell}\, \frac{I_2}{I_1} , \tag{4.13}$$

where

$$I_1 = \int_0^1 \frac{x^{2\ell+1}\, dx}{[b(x)]^{2\ell+3}} \tag{4.14}$$

and

$$I_2 = \int_0^1 \frac{x^{\ell}\, dx}{b(x)} \int_0^1 \frac{y^{\ell}\, dy}{b(y)\,[b(x) + b(y) + 1]^{2\ell+2}} \tag{4.15}$$

with

$$b(x) = [\, x(\mu^2 - 2i\mu k) - k^2 \,]^{\frac{1}{2}} . \tag{4.16}$$

In the above form the integrals are easily evaluated numerically.

In Figures 1 and 2 we compare our results with the Regge trajectories calculated by Lovelace and Masson.[10] We used the same value of μ for an entire trajectory, with μ being chosen at some average value.[11] In a more accurate calculation μ would be determined at each point on the trajectory by finding stationary points of (3.10).

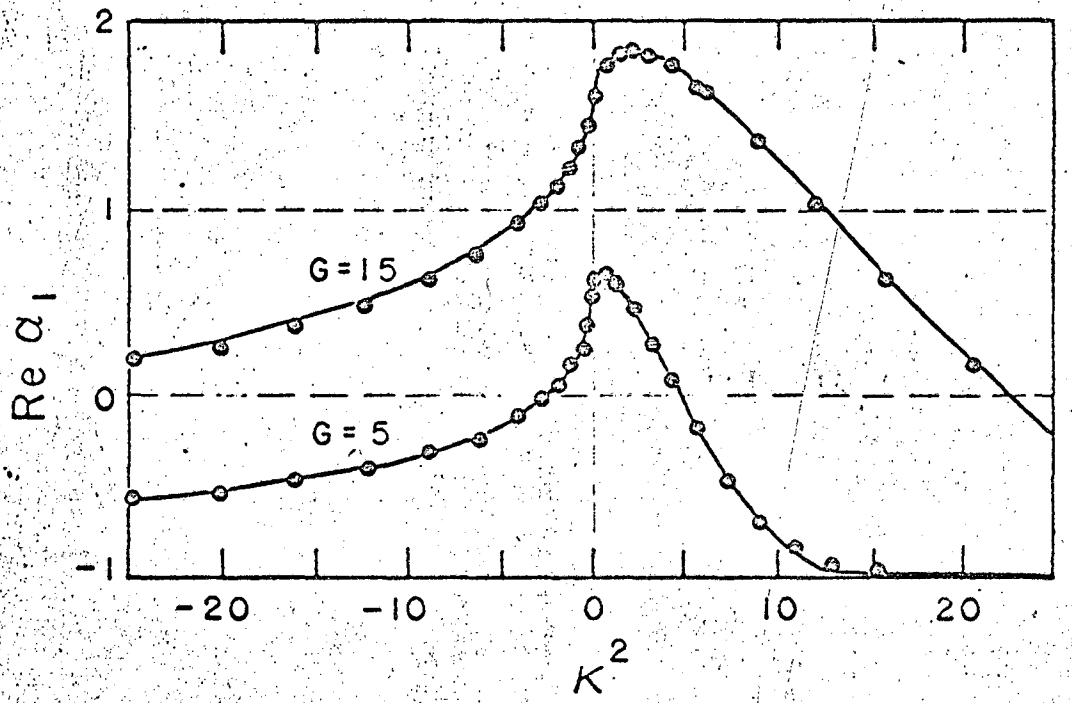

Fig. 1. First Regge trajectory of a Yukawa potential $V(r) = -\frac{G\, e^{-r}}{r}$.

Real part of $\alpha(s)$ as a function of s.

—— C. Lovelace and D. Masson, Ref. 10.

● Our results.

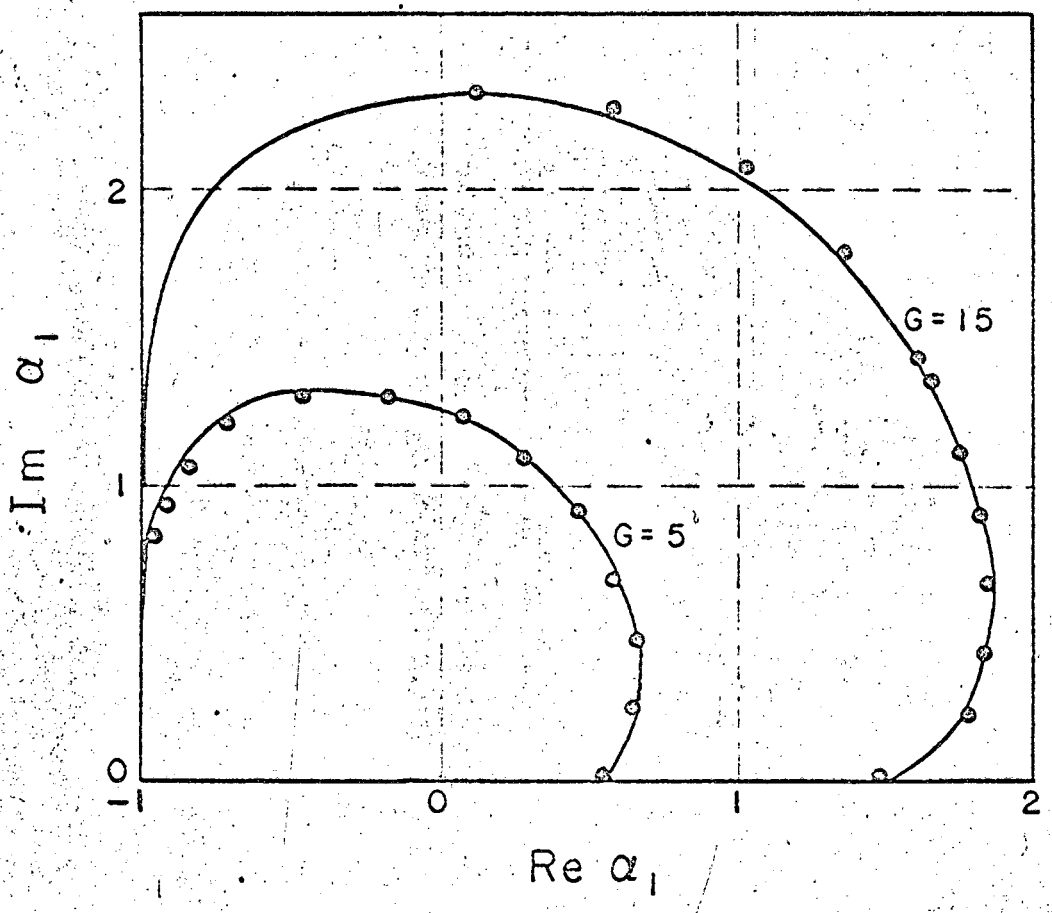

Fig. 2. First Regge trajectory of a Yukawa potential $V(r) = -\frac{G\, e^{-r}}{r}$, plotted in the complex ℓ plane.

——— C. Lovelace and D. Masson, Ref. 10.

• Our results.

V. THE VARIATIONAL METHOD AND QUASIPARTICLES

The methods developed in Section II are ideally suited to the quasiparticle[2] method of calculating scattering amplitudes. The two improvements we have to offer are the use of the variational principle to improve the choice of the subtraction dyad and the use of a simpler expression than Weinberg's[2] to calculate the eigenvalues. To illustrate the difference in computational labor and the increase in accuracy obtained by the suggested modifications, we perform two simple calculations.

The introduction of a quasiparticle into the theory is just the separation from $G_o V$ of a dyad term, which we take to be formed from our approximate eigenfunctions.

$$G_o V = K_Q + \frac{|\psi(s)\rangle\langle\psi(s^*)|V}{\langle\psi(s^*)\,|s - H_o|\,\psi(s)\rangle} \,. \tag{5.1}$$

A little algebra[12] yields the result for the T matrix,

$$T(s) = T_Q(s) + T_Q(s)\,|\psi(s)\rangle\,\Delta(s)\,\langle\,\psi(s^*)|T_Q(s) \tag{5.2}$$

where

$$\Delta(s) = \frac{1}{\langle\,\psi(s^*)\,|s - H_o|\,\psi(s)\,\rangle - \langle\,\psi(s^*)\,|T_Q|\,\psi(s)\,\rangle} \,, \tag{5.3}$$

and

$$T_Q(s) = V[1 - K_Q]^{-1} \,. \tag{5.4}$$

For potentials weak enough so that

$$T_Q \approx V ,$$

Eq. (5.2) simplifies to

$$T(s) \approx V + \frac{V|\psi(s)\rangle\, \lambda(s)\, \langle \psi(s^*)|V}{\langle \psi(s^*)\,|V|\,\psi(s)\rangle\,(1-\lambda)} , \tag{5.5}$$

which is the same as keeping one term in (2.12). Weinberg uses a much more complicated expression for $\lambda(s)$ than (2.10), namely

$$\lambda(s) = \frac{\langle \psi(s^*)\,|\,V\,G_o\,V\,|\,\psi(s)\rangle}{\langle \psi(s^*)\,|V|\,\psi(s)\rangle} . \tag{5.6}$$

If $|\psi\rangle$ is the exact eigenfunctions the two are the same, but in practice it is harder to calculate with (5.6).

In our calculations we treat the Yukawa potential $V = -\frac{g\,e^{-r}}{r}$ for $\ell = 0$. The trial wave function (3.17) becomes

$$\langle q|\psi(s)\rangle = \frac{q}{(q^2 - s)[q^2 + (\mu - ik)^2]} , \tag{5.7}$$

and

$$\langle r|\psi(s)\rangle = \sqrt{\frac{\pi}{2}}\ \frac{e^{ikr}\,(1 - e^{-\mu r})}{\mu(\mu - 2ik)} . \tag{5.8}$$

Using (3.10) we obtain the eigenvalue

$$\lambda(s) = \frac{2(g - ik)}{\mu^2}\ \log\left[\frac{(1 + \mu - 2ik)^2}{(1 + 2\mu - 2ik)(1 - 2ik)}\right] . \tag{5.9}$$

Using (5.6), Scadron and Weinberg[3] obtained a very complicated expression involving several di-logarithms with complex argument. For sample calculations to compare with known results, we choose first to obtain an upper limit on the radius of convergence of the Born series at zero energy. This gives the coupling strength necessary to yield a zero-energy bound state and is obtained by setting $\lambda = 1$ in (5.9) for $s = 0$ and solving for g.

Without benefit of a variational principle Scadron and Weinberg[3] chose $\mu = 1$ in (5.7) and obtained $g = 1.693$. We obtain with less effort $g = 1.6804$ (with $\mu = 1.5350$). The best value in the literature is $g = 1.6798$.[13]

For our second example we perform a scattering length calculation and compare results with those obtained from Blatt and Jackson's[14] interpolation formula. The scattering length is given by

$$a_s = -g\left\{1 + \frac{\lambda(0)\left(\frac{\mu}{1+\mu}\right)^2}{[1-\lambda(0)]\,\ell n\left[\frac{(1+\mu)^2}{1+2\mu}\right]}\right\}. \tag{5.10}$$

We calculate the value of g necessary to give a scattering length of $a_g = 5$ obtaining

$$g = 2.234 \tag{5.11}$$

as compared with Scadron and Weinberg's[3] value of 2.342 and Blatt and Jackson's value of 2.222.

In both examples our results are more accurate and were obtained with less effort. Calculations at $s > 0$ will show up the difference even more, as the di-logarithms of complex argument obtained in Reference 3 are very hard to handle.

VI. THREE-BODY VARIATIONAL PRINCIPLE

The methods developed in Section II for finding eigenfunctions and eigenvalues of the kernel of the Lippmann-Schwinger equation can be generalized to the three-body Faddeev[4] equations. Formally the equations can be written

$$\begin{pmatrix} \tau_1(s) \\ \tau_2(s) \\ \tau_3(s) \end{pmatrix} = \begin{pmatrix} T_1(s) \\ T_2(s) \\ T_3(s) \end{pmatrix} + \begin{pmatrix} 0 & T_1(s) & T_1(s) \\ T_2(s) & 0 & T_2(s) \\ T_3(s) & T_3(s) & 0 \end{pmatrix} \frac{1}{s - H_o} \begin{pmatrix} \tau_1(s) \\ \tau_2(s) \\ \tau_3(s) \end{pmatrix} . \tag{6.1}$$

Th S matrix is given by

$$S_{ab} = \delta_{ab} - 2\pi i\, \delta(E_a - E_b)\, T_{ab}(s) \tag{6.2}$$

with

$$T(s) = \tau_1(s) + \tau_2(s) + \tau_3(s) . \tag{6.3}$$

The kernel of Eq. (6.1) involves the two-body t matrices. $T_1(s)$ is the scattering amplitude for particles two and three with particle one noninteracting. If we describe the relative motion of particle two and three in their center-of-mass system by $\vec{k}_1$, and the motion of particle one in the total center-of-mass system by $\vec{p}_1$, then

$$\langle \vec{k}, \vec{p} \,|T_1(s)|\, \vec{k}', \vec{p}' \rangle = t_1(\vec{k}_1, \vec{k}_1', s - p_1^2)\, \delta(p_1 - p_1') \tag{6.4}$$

where t_1 is the solution to Eq. (2.1). The two-body t matrices and the free Green's function $G_o = \frac{1}{s - H_o}$ satisfy $t^\dagger(s) = t(s^*)$ and

$G_o^{\dagger}(s) = G_o(s^*)$. Using these relationships it is possible to derive a variational principle similar to (2.10). Define

$$K(s) = \begin{pmatrix} 0 & T_1(s) & T_1(s) \\ T_2(s) & 0 & T_2(s) \\ T_3(s) & T_3(s) & 0 \end{pmatrix} . \tag{6.5}$$

We are interested in the kernel $K\,G_o$, but it is simpler to derive the equations for $G_o\,K$ after which it is easy to obtain $K\,G_o$.

Consider the two eigenvalue equations

$$\lambda(s)\, G_o^{-1}\, \psi(s) = K(s)\, \psi(s) \tag{6.6}$$

$$\mu(s)\, G_o^{-1}\, \phi(s) = K^T(s^*)\, \phi(s) , \tag{6.7}$$

where ψ and ϕ have three components,

$$\psi = (\psi_1, \psi_2, \psi_3) .$$

If we take the scalar product of Eq. (6.6) with ϕ and (6.7) with ψ and subtract the two equations, we obtain

$$[\lambda(s) - \mu^*(s)] \langle\, \phi(s) \,|G_o^{-1}(s)|\, \psi(s) \,\rangle = 0 . \tag{6.8}$$

ϕ and ψ are right and left eigenvectors of (6.6) and they satisfy the orthogonality relation (6.8). It is important to identify ϕ_i with some linear combination of the ψ_i's . This is easily done by writing

out the three equations implied in (6.6) and (6.7).

$$\lambda(s)\, G_o^{-1}(s)\, \psi_1(s) = T_1(s)\, [\psi_2(s) + \psi_3(s)]$$

$$\lambda(s)\, G_o^{-1}(s)\, \psi_2(s) = T_2(s)\, [\psi_1(s) + \psi_3(s)]$$

$$\lambda(s)\, G_o^{-1}(s)\, \psi_3(s) = T_3(s)\, [\psi_1(s) + \psi_2(s)]$$

and

$$\lambda(s)\, G_o^{-1}(s)\, \phi_1^*(s) = T_2(s)\, \phi_2^*(s) + T_3(s)\, \phi_3^*(s)$$

$$\lambda(s)\, G_o^{-1}(s)\, \phi_2^*(s) = T_1(s)\, \phi_1^*(s) + T_3(s)\, \phi_3^*(s)$$

$$\lambda(s)\, G_o^{-1}(s)\, \phi_3^*(s) = T_1(s)\, \phi_1^*(s) + T_2(s)\, \phi_2^*(s) \quad .$$

By taking appropriate linear combinations of (6.7) we obtain

$$\lambda(s)\, G_o^{-1}(s)\, [\phi_2^*(s) + \phi_3^*(s) - \phi_1^*(s)] = 2\, T_1(s)\, \phi_1^*(s) \quad . \tag{6.9}$$

Comparing (6.9) and (6.6) we can make the identification,

$$\begin{aligned} 2\, \phi_1^*(s) &= \psi_2(s) + \psi_3(s) \\ \psi_1^*(s) &= \phi_2(s) + \phi_3(s) - \phi_1(s) \quad . \end{aligned} \tag{6.10}$$

The kernel $G_o K$ will have a dyad term in it equal to

$$G_o K \propto \frac{\lambda(s)\, |\, \psi(s)\, \rangle\, \langle\, \phi(s)\, |\, K(s)}{\langle\, \phi(s)\, |\, K(s)\, |\, \psi(s)\, \rangle} \tag{6.11}$$

where

$$\lambda(s) = \frac{\langle \phi(s) \,|K(s)|\, \psi(s) \rangle}{\langle \phi(s) \,|G_o^{-1}(s)|\, \psi(s) \rangle} \,. \tag{6.12}$$

As in the two-body problem, the expression for λ can be shown to be stationary. The proof is formally identical to that given in Section II. Equation (6.11) is easily rewritten in terms of $K G_o$,

$$K G_o \;\propto\; \lambda(s) \, \frac{|G_o^{-1}(s)\, \psi(s) \rangle \langle \phi(s)|}{\langle \phi(s) \,|G_o^{-1}(s)|\, \psi(s) \rangle} \,. \tag{6.13}$$

Using (6.13) and (6.3) it is easy to isolate the effect of the eigenfunction $|\psi\rangle$ on the T matrix.

$$T(s) \;\propto\; \frac{1}{1 - \lambda(s)} \, \frac{|G_o^{-1}(s) \sum_i \psi_i(s)\rangle \langle \sum_i T_i^{\dagger}(s)\, \phi_i(s)|}{\langle \phi(s) \,|G_o^{-1}(s)|\, \psi(s) \rangle} \,. \tag{6.14}$$

With the aid of (6.10), this becomes

$$T(s) \;\propto\; \frac{\lambda(s)}{1 - \lambda(s)} \left\{ \frac{|G_o^{-1}(s) \sum_i \psi_i(s)\rangle \langle G_o^{-1}(s^*) \sum_i \psi_i^*(s)|}{\langle \phi(s) \,|G_o^{-1}(s)|\, \psi(s) \rangle} \right\} . \tag{6.15}$$

The poles of the T matrix occur when $\lambda = 1$, and by using the expression (6.12), the positions of bound states and resonances can be calculated.

If it were desired to expand the T matrix in the eigenfunctions ψ, the square of the kernel would have to be used since the kernel is not compact.

The equations simplify somewhat if a separable expansion such as (2.12) is used for the two-body T matrix. Lovelace[5] has suggested such an approximation; however his method must always be approximate, whereas if (2.12) is used the resulting equations are in principle exact.

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