Title
APPLICATION OF GENERAL THEORETICAL PRINCIPLES TO EXPERIMENTS - LECTURE 7 APPLICATION OF CAUSALITY TO SCATTERING

Author
Capps, R.H.

Publication Date
1956-02-21

UCRL 3290

UNIVERSITY OF CALIFORNIA

Radiation Laboratory

BERKELEY, CALIFORNIA

DISCLAIMER

UCRL-3290

UNIVERSITY OF CALIFORNIA

Radiation Laboratory
Berkeley, California

Contract No. W-7405-eng-48

APPLICATION OF GENERAL THEORETICAL PRINCIPLES TO EXPERIMENTS - LECTURE 7

APPLICATION OF CAUSALITY TO SCATTERING

R. H. Capps

February 21, 1956

Printed for the U. S. Atomic Energy Commission

APPLICATION OF CAUSALITY TO SCATTERING

R. H. Capps

February 21, 1956

I. Basic Connection between Causality and Dispersion

The causality principle states that no signal may propagate with a velocity greater than the speed of light. In several branches of physics this principle may be used to derive a dispersion relation, which is a simple integral relation between a dispersive and an absorptive process. We will consider two examples of dispersion relations, the propagation of light through a homogeneous, refractive medium, and a quantum-mechanical scattering problem.

The connection between causality and dispersion relations arises from the fact that the momentum and energy of a particle (or wave) are variables canonically conjugate to the position of the particle in space and time. A wave packet may be expressed in either of the sets of variables $(\vec{x}, t)$ or $(\vec{p}, E)$, and these two representations are related by Fourier transforms. Thus the causality principle, which restricts the motion of a wave packet in space and time, may also place limitations on the behavior of a reaction process as a function of energy and momentum.

Dispersion relations may apply in processes in which an input wave and an output wave are defined and the output wave is a linear function of the input wave. For example, in a scattering problem there is the relation $\psi^+ = S\psi^-$, where S is the scattering matrix. In order to simply illustrate the connection between causality and dispersion, we first consider some general reaction process. We limit our attention to a fixed point in space, and assume that the input wave $\psi^-(t)$ and output wave $\psi^+(t)$ may in some way be separated. Either wave may be expressed in terms of the time variable or the frequency variable,

$$\psi^{\pm}(t) \approx \int_{-\infty}^{\infty} e^{-i\omega t} \psi^{\pm}(\omega) d\omega , \tag{1}$$

$$\psi^{\pm}(\omega) \approx \int_{-\infty}^{\infty} e^{i\omega t} \psi^{\pm}(t) dt , \tag{2}$$

where the constants of proportionality have been omitted. If $\psi^{\pm}(t)$ is taken to be a real variable, such as the magnitude of the electromagnetic field at some point, then the transform $\psi^{\pm}(\omega)$ must satisfy the condition

$$\psi^{\pm}(-\omega) = \psi^{\pm *}(\omega), \tag{3}$$

where * indicates complex conjugate.

We assume that the output wave of a particular frequency is related to the input wave of the same frequency by the relation

$$\psi^{+}(\omega) = S(\omega)\, \psi^{-}(\omega) , \tag{4}$$

where $S(\omega)$ is a complex number, of absolute magnitude no greater than one. From Eqs. (3) and (4), $S(\omega)$ is defined for negative frequencies by

$$S(-\omega) = S^{*}(\omega) . \tag{5}$$

Let the Fourier transform of $S(\omega)$ be $S(t)$; thus we have

$$S(\omega) \approx \int_{-\infty}^{\infty} e^{i\omega t} S(t) dt . \tag{6}$$

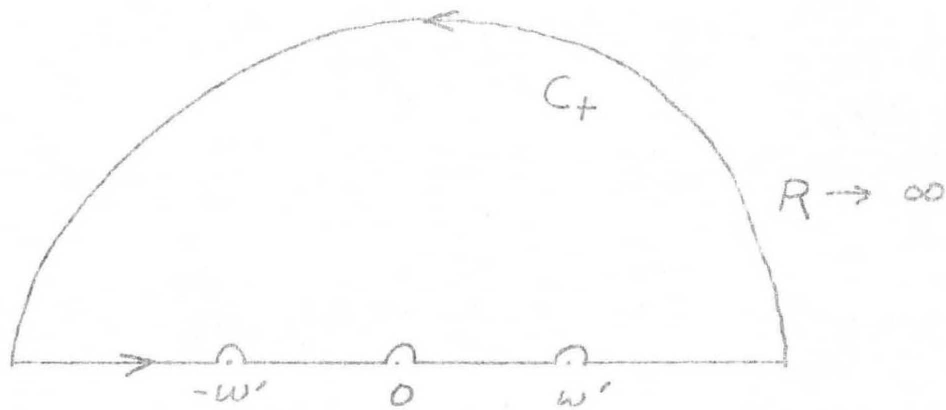
Then, taking the Fourier transform of the quantities in Eq. (4), we find

$$\psi^{+}(t_2) \approx \int d\, t_1 \;\; S(t_2 - t_1)\, \psi^{-}(t_1) . \tag{7}$$

In this case the causality principle states that the output wave, as a function of time, can depend on the input wave at previous times only, where a is a positive number. Since $f(\omega)$ is analytic, we may apply Cauchy's theorem to evaluate such an integral as

$$I = \int_{C^+} \frac{d\omega}{\omega} \frac{f(\omega)}{(\omega^2 - \omega'^2)} \tag{10}$$

along the following contour in the complex ω-plane.



If we assume from some physical argument that $f(\omega)$ on the real axis increases no faster than linearly with ω as $|\omega| \to \infty$, then the theorem of Titchmarsh tells us that the contribution from the semicircle vanishes as the radius R approaches infinity. Let $f(\omega)$ on the real axis be given by

$$f(\omega) = a(\omega) + ib(\omega) \quad , \tag{11}$$

where $a(\omega)$ and $b(\omega)$ are real. Then if $f(\omega)$ satisfies Eq. (5),

$$\begin{aligned} a(\omega) &= a(-\omega) \quad , \\ b(\omega) &= -b(-\omega) \quad . \end{aligned} \tag{12}$$

If we assume $f(\omega)$ has no poles on the real axis, then by carrying out the integral, Eq. (10), and using the symmetry relations, Eq. (12), we find or

$$S(t) = 0 \qquad \text{when} \quad t < 0 \ . \tag{8}$$

Thus the reaction function $S(\omega)$ is the Fourier transform of a function that vanishes for negative values of the argument. We call such a function a causal function.

II. Dispersion Relation for a Causal Function

Let a function of frequency $f(\omega)$ be given by

$$f(\omega) = \int_0^\infty e^{i\omega t} f(t)dt \ . \tag{9}$$

Though $f(\omega)$ has been defined only for real values of the frequency, Eq. (9) may be used to define $f(\omega)$ for complex values of ω. In the region above the real axis in the complex ω plane, the exponential $e^{i\omega t}$ will have magnitude smaller than one, since t is positive. Therefore it is reasonable to expect that $f(\omega)$ will have some sort of convergence property as the imaginary part of ω becomes large. In fact a theorem given in Titchmarsh[1] states that if the integral on the real axis,

[1] E. C. Titchmarsh, Theory of Fourier Integrals, Oxford, Clarendon Press, 2nd ed., 1948, p. 128.

$$\int_{-\infty}^{\infty} d\omega \left| f(\omega) \right|^2 = M \ ,$$

is finite, then $f(\omega)$ given by Eq. (9) is analytic in the entire region above the real axis, and the line integral

$$\int_{ia-\infty}^{ia+\infty} d\omega \left| f(\omega) \right|^2 \leq M \ ,$$

$$a(\omega') - a(0) = \frac{2\,\omega'^{2}}{\pi} \; P \int_{0}^{\infty} \frac{d\omega \, b(\omega)}{\omega(\omega^2 - \omega'^2)} \, , \tag{13}$$

where P indicates the principal part of the integral. This is an example of a dispersion relation.

III. Causality Restrictions on the Complex Refractive Index of a Homogeneous Isotropic Medium

We shall illustrate with the physical example of light traveling through a homogeneous isotropic medium of complex index of refraction. For simplicity we neglect the quantum nature and the polarization of the light, and treat the radiation as a classical neutral scalar field.

In actual practice, in order to separate the input wave from the output wave, we must consider more than one point in space. If the light wave consists of a plane wave propagating in the positive $\vec{x}$ direction, then the amplitude of this wave measured at two different points, $x = 0$ and $x = x_0 > 0$, may be considered the input and output, respectively,

Let the amplitude at $x = 0$ (input wave) be given by

$$\psi_0(t_1) = \psi^-(t_1) = \int_{-\infty}^{\infty} \varphi^-(\omega) e^{-i\omega t_1} \, d(\omega) \, . \tag{14}$$

Then the amplitude at $x = x_0$ (output wave) is given by

$$\psi_{x_o}(t) = \psi^{+}(t_2) = \int_{-\infty}^{\infty} \psi^{-}(\omega) e^{-i\omega t_2 + ik(\omega)x_o} d\omega , \tag{15}$$

where $k(\omega)$ is a complex propagation vector. If we define a complex dispersion function $f(\omega)$ by

$$f(\omega) = k(\omega) - \omega/c , \tag{16}$$

then $f(\omega)$ is related to the real refractive index $n(\omega)$ and real linear absorption coefficient $\alpha(\omega)$ of the medium by

$$f(\omega) = \omega \left[n(\omega) - 1 \right] /c + i\alpha(\omega)/2 . \tag{17}$$

We define a new variable T by

$$T = t_2 - x_o/c . \tag{18}$$

Equation (15), written in terms of T and $f(\omega)$, becomes

$$\psi^{+}(T) = \int_{-\infty}^{\infty} \psi^{-}(\omega) e^{-i\omega T + if(\omega)x_o} d\omega . \tag{19}$$

Thus the Fourier transform of $\psi^{+}(T)$ is equal to

$$\psi^{+}(\omega) = \psi^{-}(\omega) e^{if(\omega)x_o} , \tag{20}$$

so that the reaction function for this example is

$$S(\omega) = e^{if(\omega)x_o} . \tag{21}$$

From the Fourier transform of $S(\omega)$ and $\psi^{-}(\omega)$, Eq. (19) becomes

$$\psi^{+}(T) = \int_{-\infty}^{\infty} dt_1 \, \psi^{-}(t_1) \, S(T - t_1) . \tag{22}$$

The causal principle states that the wave cannot travel from 0 to x_o faster than the velocity c, or

$$S(T - t_1) = S(t_2 - x_0/c - t_1) = 0 \qquad \text{when} \quad T - t_1 < 0 . \tag{23}$$

Therefore $S(\omega)$ is a causal function.

Since $S(\omega) = e^{if(\omega)x_0}$ depends on the distance, the useful dispersion relation in this case is an integral relation involving real and imaginary parts of $f(\omega)$. If $S(\omega)$ satisfies Eq. (5), then $f(\omega)$ must satisfy

$$f(\omega) = -f^*(-\omega) . \tag{24}$$

By using this relation, and the boundedness of $S(\omega)$ in the upper half plane, one may show from complex variable theory that $n(\omega)$ is given by

$$n(\omega') = 1 \quad \frac{c}{\pi} \quad P \int_0^\infty \frac{\alpha(\omega) d\omega}{\omega^2 - \omega'^2} , \tag{25}$$

which is the familiar relation between the real index of refraction and the absorption coefficient.

IV. Dispersion Relations for a Forward-Scattering Amplitude

In this section we sketch the connection between causality and dispersion for forward scattering.

The scattering of a neutral particle in the forward direction is somewhat similar to the problem of the plane wave in the homogeneous refracting medium. In the scattering problem that we consider, however, the interaction is limited to a finite region of space, and the observations are made at distances large compared with the size of the interaction region. For an elastic process we may write schematically

$$\psi^+(E) = S(E)\, \psi^-(E) , \tag{26}$$

$$D(\omega') - D(0) = \frac{\omega'}{2\pi^2} \; P \int_0^\infty d\omega \; \frac{\sigma_t(\omega)}{(\omega^2 - \omega'^2)} \; . \tag{32}$$

This equation applies to the coherent forward-scattering amplitude for scattering of photons from protons or electrons.

where S(E) is the element of the scattering matrix corresponding to coherent, elastic scattering in the forward direction. (Here, the initial and final states are described by the same sets of quantum numbers.) We are interested in the corresponding element of the reaction matrix defined by

$$T(E) = \frac{S(E) - 1}{2ik} . \tag{27}$$

The quantity S(E) may usually be defined for negative frequencies in such a way that S(E) is a causal function. Let T(E) be written in terms of its real and imaginary parts,

$$T(E) = D(E) + iA(E) . \tag{28}$$

Then, for the scattering of neutral particles, it may usually be shown that

$$D(E) = D(-E) ,$$
$$A(E) = -A(-E) . \tag{29}$$

The imaginary part of the amplitude, A(E), is related to the total cross section for all final states by the relation [Eq. (14) of Lecture 5]

$$A(E) = \sigma_T k/(4\pi) . \tag{30}$$

Then, if σ_T and D(E)/k are bounded as functions of real energy, causality may be used to prove the analyticity and boundedness of T(E) in the upper half plane, and a dispersion relation may be derived. If the mass of the scattered particle is zero the dispersion relation may be written in the form similar to Eq. (13), i.e.,

$$D(\omega') - D(0) = \frac{2\omega'^2}{\pi} P \int_0^\infty d\omega \frac{A(\omega)}{\omega(\omega^2 - \omega'^2)} , \tag{31}$$

or, by Eq. (30),