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Authors
Li, Shaofan
Morgan, Elise

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by

Shaofan Li

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On Permeable Cracks in A Piezoelectric Ceramic. I. Global Energy Release Rate

Shaofan Li

Department of Civil and Environmental Engineering,
University of California, Berkeley, California, USA

Abstract

A permeable crack model is crafted and is used to analyze permeable crack growth in a piezoelectric ceramic. The permeable crack model takes into account of the effect of surface charge distribution on crack surfaces of the piezoelectric ceramic, which may be caused by the discharge of the dielectric medium inside the crack, or charge separation due to the surface separation of the poled pezoelectric matrix. In this study, a permeable crack is modeled as a vanishing thin, finite dimension, rectangular slit with surface charge deposited along the crack surface. A first order approximation theory is developed with respect to slit height, h_0, to analyze electrical and mechanical fields in the vicinity of a permeable crack tip.

A closed form solution is obtained for the permeable crack perpendicular to the poling direction under both mechanical as well electrical loads. Both local and global energy release rates are calculated based on the permeable crack solution. It is found that the global energy release rate for a permeable crack has a remarkable expression,

$$J_c^g = \Big(\frac{\pi b}{2M}\Big)\Big(\sigma_\infty^2 + \frac{4e}{\pi}\sigma_\infty E_\infty\Big)$$

which is in broad agreement with the known experimental observations and may be served as the fracture criterion for piezoelectric materials. This contribution reconsiles the discrepancy between experimental observation and theoretic analysis without invoking any nonlinear theory, and it elucidates, via rigorous analysis, how an applied electric field affects crack growth in piezoelectric ceramic through its interaction with the permeable environment surrounding the crack.

1 Introduction

During past decades, an impressive amount of literature has been documented on the subject of fracture mechanics of piezoelectric materials. Early contribution can be traced back to Parton [33], Dee [5], Parton and Kudryatvsev [34], and in addition with Pak and Herrmann [28,29] and McMeeking [21,20,22], if one considers the closely related topic on elastic dielectrics. Since 1990s, there has been a flux of research papers published dealing with linear fracture mechanics of piezoelectric solids in specific, e.g. Pak [26,27], Li et al [13], Sosa [36–38], Suo et al [41,40], Wang [44], Pak and Tobin [30], Dunn [6], Maugin [19], Dascalu and Maugin [1994a],[1995], Park and Sun [32,31], Gao and Barnett [11], and Gao et al [12], Lynch et al [18,17], Zhang and Hack [48], Kumar and Singh [1997ab], Fulton and Gao [8], Ru [1998][1999][2000], Yang and Zhu [46,47,53], Zhang et al [50,49], McMeeking [23,24], Yang [46,47] and many others. A recent review by Zhang et al [51] may provide adequate information on current status of the research.

The impetus for such intense research activities is largely due to the emerging technology in smart materials and structures. Another reason for such enthusiasm might be, this author privately speculated, because using impermeable approximation one can easily link a crack problem in a piezoelectric solid to an available crack solution of a purely elastic solid. Thus, it seems to be mathematically trivial to solve a crack problem in a piezoelectric media. The best part is: one gets credit for solving it! at least for the analogy.

However, a setback, or to certain extent a tragedy for such theoretical movement, is that the analytical results are not supported by experimental observations ! In a landmark experimental work [32], Park and Sun showed that the experimental observations contradict some of basic conclusions of linear piezoelectric fracture mechanics theory.

For instance, the experimental results obtained by Park and Sun [32] show that there is a decrease in the critical stress of a cracked piezoelectric body if the electric field is applied along the poling axis, and there is an increase in critical stress if the electric field is applied to the opposite direction, whereas according to linear fracture mechanics theory, the electric field does not induce any non-zero stress intensity factor (e.g. Pak [26,27], Suo et al [41]). In addition, in high-cycle fatigue test (Tobin and Park [43]) and low-cycle fatigue test (Cao and Evans [1]; Lynch et al [18,17]), stable crack growths have been observed in both cases when crack growth is perpendicular to the poling direction under purely electric loading, for which the linear piezoelectric fracture mechanics predicts a negative definite energy release rate, which implies that the crack growth should have never occurred.

Currently, there are three remedies that have been proposed to explain the discrepancy between linear fracture mechanics theory and experimental observations.

- Yang and Zhu's transformation toughness theory (Yang and Zhu [46,47,53,52,54])

- Gao et al's strip saturation model (Gao et al [11,12] or equivalently, the electric dipole model proposed by Fulton and Gao [8]) and
- McMeeking's discharge-charge separation model (McMeeking [24]).

From micro-mechanics view point, the so-called *ferroelectric domain switching phenomenon* is the source of nonlinear piezoelectricity (e.g. Sun and Jiang [39] and Yang and Zhu [53,46,47]). The explanation is as follows: the concentrated stress field at a crack tip of piezoelectric ceramic will cause depoling, or ferroelectric domain switching and cause a hysteresis loop between the electric displacement and the electric field, and a butterfly loop between the strain and electric field. This will in turn induce toughness variations in ferroelectricity, and alter the constitutive relations at the crack tip region (e.g Pisarenko et al [1985], Mehta and Vitkar [25], Sun and Jiang [39], Zhu and Yang [53], Yang and Zhu [46,47]). The overall constitutive relation at the crack tip region becomes nonlinear. Thus, the fracture toughness of ferroelectric ceramics is controlled by domain switching.

Zhu and Yang [53] and Yang and Zhu [47] adopted micromechanics based technique to treat the domain switching induced toughness variation as transformation toughening. At phenomenological level, Gao et al [12] and Fulton and Gao [8] proposed a strip-saturation model or equivalently an electric dipole distribution model to estimate the nonlinearity possibly due to the overall effect of domain switching, or polarization. The piezoelectric saturation model is the direct analogous of Dugdale crack in cohesive elastic medium of purely mechanical fracture mechanics. Gao and his co-workers [11,12] showed that the local energy release rate criterion derived from strip saturation model is in close agreement with the experimental observation, which has become the first theoretical result in this research area that is actually useful.

Apparently, the dissipative nature of strip-saturation model seems to be a nuisance, though the solution of "an electrically yielded crack" provides a satisfactory explanation. Recently, McMeeking [24] proposed a discharge-charge separation model for a permeable crack in another attempt to provide a possible reconciliation between the theories and experiments.

Some researchers have been very cautious on adopting impermeable approximation, suspecting that there may be some fundamental differences between an impermeable crack and a permeable crack due to the presence of the permeable environment surrounding the crack, e.g. McMeeking [20], Sosa [37], Dunn [6], Li and Mataga [14,15], Gao and his co-workers [9,10], and Yang and Kao [45] and others. Among them, the permeable crack solution degenerated from an elliptic cavity (Zhang et al [50,49]) is the most complete and detailed. Nevertheless in all these studies, above speculation has never been substantiated, because there is lacking of rigorous treatment of the permeable crack.

In the present work, a permeable crack model is carefully crafted to render an analytical tractable solution while capturing all the major features of a permeable crack. By using the permeable crack model, we are re-examining the linear fracture mechanics theory of piezoelectric materials. One of the novelties of the present treatment is that it combines the discharge-charge separation model with some essential technical ingredients of the strip-saturation model.

The paper is organized in six sections. In section 2, the simplified opening crack model proposed by Gao et al [12] is briefly outlined within the framework of permeable crack. The complete solution procedure is provided in section 3, and asymptotic fields of electrical and mechanical variables are documented in section 4. The main results are presented in section 5 focusing on the energy release rate of a permeable crack. Few remarks are made in section 6.

2 Formulation of the Problem

2.1 Simplified Constitutive Model

The notation of Tiersten [42] is adopted to write the governing equations for linear piezoelectric materials as follows:

- equations of motion

$$\sigma_{ij,i} = 0 \quad ; \tag{1}$$

- electrostatic charge conservation

$$D_{i,i} = 0 \quad ; \tag{2}$$

- strain-displacement relations

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \quad ; \tag{3}$$

- electric field-electric potential relations

$$E_k = -\phi_{,k} \quad ; \tag{4}$$

- linear, piezoelectric constitutive relations

$$\sigma_{ij} = c^E_{ijkl}\varepsilon_{kl} - e_{kij}E_k \quad , \tag{5}$$

$$D_i = e_{ikl}\varepsilon_{kl} + \epsilon^S_{ik}E_k \quad ; \tag{6}$$

where c^E_{ijkl} are the elastic moduli, e_{kij} are the piezoelectricity coefficients, and ϵ^S_{ij} are the dielectric permittivities (with the superscript E or S indicating material constants measured under conditions of constant electric field or constant strain, respectively).

Using Voigt notation, the constitutive relations (5) and (6) of the type of piezoelectric materials we are interested in can be put into the following form

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} c^E_{11} & c^E_{12} & c^E_{13} & 0 & 0 & 0 \\ c^E_{12} & c^E_{11} & c^E_{13} & 0 & 0 & 0 \\ c^E_{13} & c^E_{13} & c^E_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c^E_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c^E_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & (c^E_{11} - c^E_{12})/2 \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{bmatrix} - \begin{bmatrix} 0 & 0 & e_{31} \\ 0 & 0 & e_{31} \\ 0 & 0 & e_{33} \\ 0 & e_{15} & 0 \\ e_{15} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} \tag{7}$$

and

$$\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & e_{15} & 0 \\ 0 & 0 & 0 & e_{15} & 0 & 0 \\ e_{31} & e_{31} & e_{31} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{bmatrix} + \begin{bmatrix} \epsilon^s_{11} & 0 & 0 \\ 0 & \epsilon^s_{11} & 0 \\ 0 & 0 & \epsilon^s_{33} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} \tag{8}$$

Using the precise anisotropic constitutive relation in an analysis often increase the degree of difficulty in finding the closed form solution, or has the danger to obscure the essential physical element of problem.

Considering the material constants of PZT-4 piezoelectric ceramics used by Park and Sun [1995] in their experiment, one may gain a sense of the magnitude of primary aspect of the physical problem.

Elastic Constants (N/m^2)

$c_{11}^E = 13.9 \times 10^{10}$,	$c_{12}^E = 7.78 \times 10^{10}$,	$c_{13}^E = 7.43 \times 10^{10}$
$c_{33}^E = 11.3 \times 10^{10}$,	$c_{44}^E = 2.56 \times 10^{10}$	

Piezoelectric Constants (C/m^2)

$e_{31} = -6.98$,	$e_{33} = 13.84$,	$e_{15} = 13.44$

Dielectric Constants (C/Vm)

$\epsilon_{11}^S = 6.0 \times 10^{-9}$,	$\epsilon_{33}^S = 5.47 \times 10^{-9}$

Neglecting anisotropic effect and taking into account the magnitude of each material constant, Gao, Zhang, and Tong [1997] proposed to use the following simplified constitutive relations to study Mode-I fracture

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = M \begin{bmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&0 \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{bmatrix} - e \begin{bmatrix} 0&0&-1 \\ 0&0&-1 \\ 0&0&1 \\ 0&1&0 \\ 1&0&0 \\ 0&0&0 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} \tag{9}$$

and

$$\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = e \begin{bmatrix} 0&0&0&0&1&0 \\ 0&0&0&1&0&0 \\ -1&-1&1&0&0&0 \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{bmatrix} + \epsilon \begin{bmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} \tag{10}$$

because they capture the essential part of electrical-mechanical behaviors. In the rest of the paper, we refer this simplified model as the GZT model.

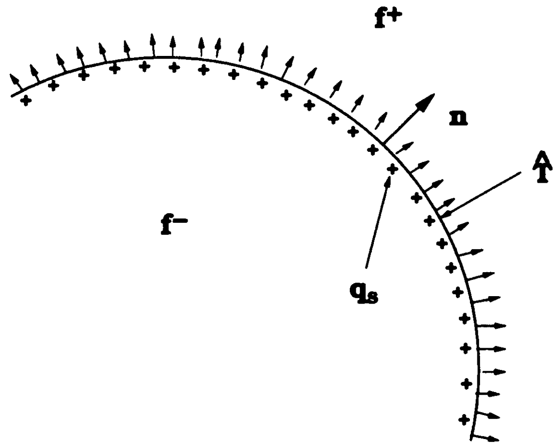

Fig. 1. Convention for boundary conditions

2.2 Boundary Conditions

The boundary conditions or interface conditions for two different dielectric media are

- mechanical boundary conditions

$$\mathbf{n}\cdot[|\boldsymbol{\sigma}|] = -\hat{\mathbf{T}} \quad \text{on } S_\sigma; \qquad \mathbf{u} = \hat{\mathbf{u}} \quad \text{on } S_u \quad ; \tag{11}$$

- electrical boundary conditions

$$\mathbf{n}\cdot[|\mathbf{D}|] = q_s \quad \text{and} \quad \mathbf{n}\times[|\mathbf{E}|] = 0 \quad \text{on } S \ . \tag{12}$$

where S_σ, S_u identify appropriate subsets of the domain boundary. and $S = S_\sigma \cup S_u$. Note that the notation $[|f|] := f^+ - f^-$, and the normal vector $\mathbf{n}$ is pointing from medium $-$ to medium $+$ as shown in Fig. 1.

2.3 Crack models

2.3.1 Slit Geometry

In this paper, a planar crack is modeled as a vanishing thin, finite dimension, rectangular-shaped slit with height $2h_0$ and width $2b$ as shown in Fig. 2

As $h_0 \to 0$, the permeable crack becomes a conventional mathematical crack. One may write crack height as the function of abssica,

$$h(X) = \begin{cases} h_0, & |X| < b \\ 0, & |X| > b \end{cases} \tag{13}$$

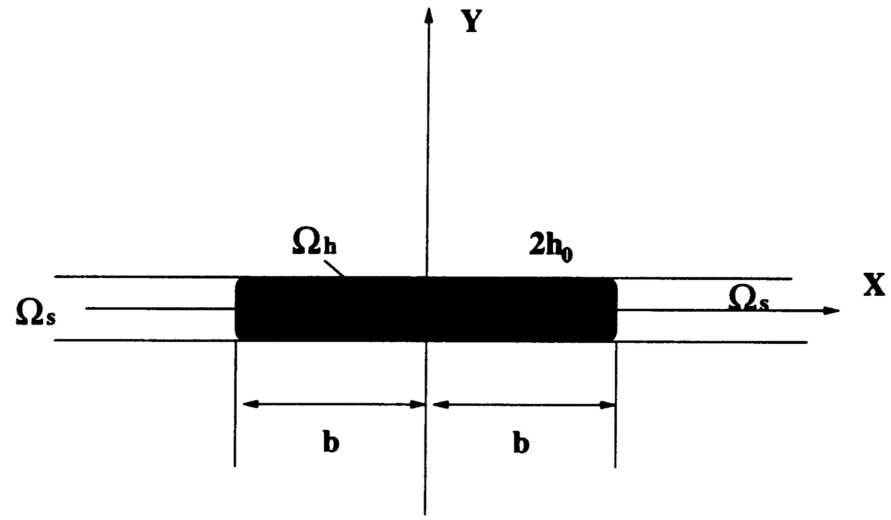

Fig. 2. Geometrical configuration of the permeable crack.

The interior region of the crack is denoted as the set Ω_h,

$$\Omega_h := \Big\{(X,Y)\;\Big|\; -b < X < b, \text{ and } -h_0 < Y < h_0\Big\} \tag{14}$$

For convenience, the adjacent regions to the slit is denoted as Ω_s, the two semi-infinite strips,

$$\Omega_s := \Big\{(X,Y)\;\Big|\; b < |X|, \text{ and } -h_0 < Y < h_0\Big\} \tag{15}$$

Following Gao et al [12], we consider a simplified crack model: a permeable crack that is perpendicular to the poling direction, which is termed as GZT crack.

2.3.2 GZT crack

Let $X = x_1$, $Y = x_3$, and $Z = x_2$ denote regular Cartesian coordinates, where the Y-axis orients in the poling direction. Gao et al [12] made the following simplifications:

$$u_x := u_1 = 0,\; u_z := u_3 = 0, \text{ and } u_y = u_3(X,Y) = u(X,Y) \tag{16}$$

$$E_2 = 0,\; E_x = E_1 = -\frac{\partial \phi}{\partial X},\; E_y = E_3 = -\frac{\partial \phi}{\partial Y}, \tag{17}$$

Consequently, the governing equations simplify considerably. The constitutive relations (9) and (10) become

$$\sigma_{xy} = M\frac{\partial u}{\partial X} + e\frac{\partial \phi}{\partial X} \tag{18}$$

$$\sigma_{yy} = M\frac{\partial u}{\partial Y} + e\frac{\partial \phi}{\partial Y} \tag{19}$$

$$D_x = e\frac{\partial u}{\partial X} - \epsilon\frac{\partial \phi}{\partial X} \tag{20}$$

$$D_y = e\frac{\partial u}{\partial Y} - \epsilon\frac{\partial \phi}{\partial Y} \tag{21}$$

The non-trivial equilibrium equation

$$\frac{\partial \sigma_{xy}}{\partial X} + \frac{\partial \sigma_{yy}}{\partial Y} = 0 \tag{22}$$

and electrostatic charge equation

$$\frac{\partial D_x}{\partial X} + \frac{\partial D_y}{\partial Y} = 0 \tag{23}$$

yield the following simplified governing equations

$$M\nabla^2 u + e\nabla^2 \phi = 0 \tag{24}$$

$$e\nabla^2 u - \epsilon\nabla^2 \phi = 0 \tag{25}$$

where ∇^2 is the two-dimensional Laplacian operator, $\nabla^2 := \dfrac{\partial^2}{\partial X^2} + \dfrac{\partial^2}{\partial Y^2}$;

Since $M\epsilon + e^2 \neq 0$, both u and ϕ are harmonic functions

$$\nabla^2 u = 0\ , \quad \nabla^2 \phi = 0\ , \quad \forall\ (X, Y) \in \mathbb{R}^2/\Omega_h \tag{26}$$

In the interior of the crack,

$$\nabla^2 \phi^a = 0\ , \quad \forall\ (X, Y) \in \Omega_h \tag{27}$$

where ϕ_a is the electric potential in the dielectric medium inside the permeable crack.

3 Solution for cracks perpendicular to the poling axis

Consider a permeable crack that is perpendicular to poling direction, and it is subjected to remote traction and charge distribution at remote boundary (see Fig. 3).

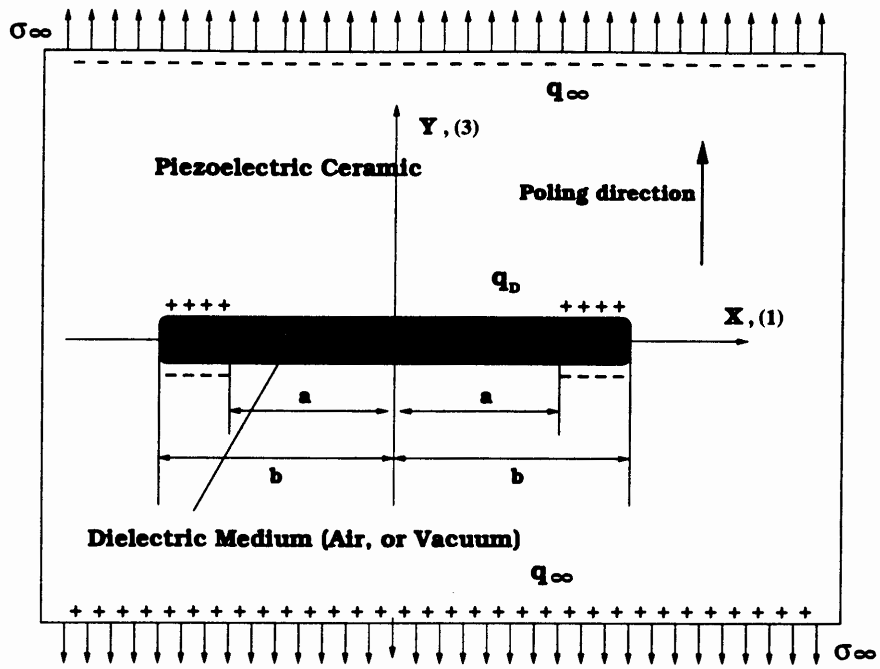

Fig. 3. A permeable crack with remote traction and charge distribution and surface charge distribution at the corner of the crack

Let $\hat{\mathbf{T}} = \sigma_\infty \mathbf{e}_y$ and $q_s = -q_\infty$.

$$\mathbf{n} \cdot [|\boldsymbol{\sigma}|] = -\hat{\mathbf{T}} \quad \rightarrow \quad \sigma_{yy} = \sigma_\infty, \ \forall\, y \rightarrow \infty \tag{28}$$
$$\mathbf{n} \cdot [|\mathbf{D}|] = q_s \quad \rightarrow \quad D_y = q_\infty\ , \ \forall\, y \rightarrow \infty \tag{29}$$

where $q_s = -q_\infty$.

Assume that due to either the discharge of dielectric medium inside the crack, or charge separation due to the newly formed crack surfaces, there is a charge distribution at the corner of the crack (see Fig. 3), which can be described as $q_S = q_D H(|X| - a)$.

The boundary conditions on the crack surfaces,

$$\mathbf{n} \cdot [|\boldsymbol{\sigma}|] = 0\ , \quad \forall\, Y = \pm h_0 \ \text{ and } \ |X| \leq b \tag{30}$$
$$\mathbf{n} \cdot [|\mathbf{D}|] = q_s\ , \quad \forall\, Y = \pm h_0 \ \text{ and } \ |X| \leq b \tag{31}$$
$$\mathbf{n} \times [|\mathbf{E}|] = 0\ , \quad \forall\, Y = \pm h_0 \ \text{ and } \ |X| \leq b \tag{32}$$

take the form

$$\sigma_{yy}(X, \pm h_0) = 0, \qquad \forall\, |X| \leq b \tag{33}$$
$$D_Y(X, \pm h_0) - D_y^a(X, \pm h_0) = \pm q_D H(|X| - a), \quad \forall\, |X| \leq b \tag{34}$$
$$E_x(X, \pm h_0) - E_x^a(X, \pm) = 0, \qquad \forall\, |X| \leq b \tag{35}$$

And symmetry conditions

$$
\begin{aligned}
u(X,0) &= 0, \qquad && \forall\ |X| > b && (36)\\
\phi(X,0) &= 0, && \forall\ |X| > b && (37)\\
\phi^a(X,0) &= 0, && \forall\ 0 < |X| < b && (38)\\
\text{or}\quad E_x(X,0) &= 0, && \forall\ |X| > b && (39)\\
E_x^a(X,0) &= 0, && \forall\ 0 < |X| < b && (40)
\end{aligned}
$$

In the dielectric medium inside the crack, $D_i^a = \epsilon_0 E_i^a$ and $E_i^a = -\phi_{,i}^a$, $i = X, Y$.

Consider the general fields consisting two parts: a uniform part and disturbance part due to the presence of the crack.

$$u = u_0 + \tilde{u} \tag{41}$$

$$\phi = \phi_0 + \tilde{\phi} \tag{42}$$

and choose

$$u_0 = \mathcal{E}_\infty Y, \qquad \phi_0 = -E_\infty Y \tag{43}$$

and

$$\sigma_\infty = M\mathcal{E}_\infty - eE_\infty \tag{44}$$

$$q_\infty = e\mathcal{E}_\infty + \epsilon E_\infty \tag{45}$$

such that $\tilde{u}, \tilde{\phi} \to 0$ as $Y \to \infty$.

It is convenient to write the inverse relationship among key physical variables on the remote boundary,

$$\mathcal{E}_\infty = \frac{1}{\Delta_i}(\epsilon\sigma_\infty + eq_\infty) \tag{46}$$

$$E_\infty = \frac{1}{\Delta_i}(-e\sigma_\infty + Mq_\infty) \tag{47}$$

where $\Delta_i := M\epsilon + e^2$.

Extend the definition domain of ϕ^a into $\Omega_h \cup \Omega_s$ and let

$$\tilde{\phi}^a = \begin{cases} \phi^a - \phi_0^a, & \forall\ (X,Y) \in \Omega_h \\ 0, & \forall\ (X,Y) \in \Omega_s \end{cases} \tag{48}$$

where the uniform part of the electric potential is chosen $\phi_0^a := -\frac{q_\infty}{\epsilon_0} Y$.

Introduce Fourier cosine transform

$$\begin{cases} F^*(\zeta, Y) = \sqrt{\frac{2}{\pi}} \int\limits_0^\infty F(X,Y)\cos(\zeta X) d\zeta \\ \\ F(X,Y) = \sqrt{\frac{2}{\pi}} \int\limits_0^\infty F^*(\zeta,Y)\cos(\zeta X) d\zeta \end{cases} \tag{49}$$

where $F(X,Y) = \tilde{u}(X,Y)$, $\tilde{\phi}(X,Y)$, and $\tilde{\phi}^a(X,Y)$, and $F^*(\zeta,Y) = \tilde{u}^*(\zeta,Y)$, $\tilde{\phi}^*(\zeta,Y)$, and $\tilde{\phi}^{a*}(\zeta,Y)$. and then

$$\frac{d^2}{dY^2} F^* + \zeta^2 F^* = 0 \tag{50}$$

Within the piezoelectric ceramic,

$$\tilde{u}^*(\zeta,Y) = A(\zeta)\exp(-\zeta Y) \ , \quad \forall\, Y > 0 \tag{51}$$
$$\tilde{\phi}^*(\zeta,Y) = B(\zeta)\exp(-\zeta Y) \ , \quad \forall\, Y > 0 \tag{52}$$

Inside the permeable crack,

$$\tilde{\phi}^{a*}(\zeta,Y) = C(\zeta)\sinh(\zeta Y) \ , \quad \forall\, Y > 0 \tag{53}$$

which satisfies the symmetry condition $\tilde{\phi}^a(X,0) = 0$.

Consider the boundary condition,

$$E_x(X, \pm h_0) - E_x^a(X, \pm h_0) = 0, \quad |X| < b \tag{54}$$

and the symmetry condition

$$E_x(X,0) = 0 \ , \quad |X| > b \tag{55}$$

On the other hand, in extended domain

$$\tilde{E}_x^a(X,0) = 0 \ , \quad |X| > b \tag{56}$$

Combining Eqs. (54)–(56), one has

$$\tilde{E}_x(X, \pm h(X)) - \tilde{E}_x^a(X, \pm h(X)) = 0, \quad \forall\, -\infty < X < +\infty \tag{57}$$

where function $h(X)$ is defined in Eq. (14).

In transformed space (ζ, Y), the condition (57) reads as

$$\tilde{E}_x^*(\zeta, \pm h^*(\zeta)) - \tilde{E}_x^{a*}(\zeta, \pm h^*(\zeta)) = 0, \quad \forall\, 0 < \zeta < +\infty \tag{58}$$

where

$$h^*(\zeta) = h_0 \frac{\sin(b\zeta)}{\zeta} \tag{59}$$

Considering Eqs. (52) and (53), one has

$$\begin{aligned} B(\zeta) &= C(\zeta)\frac{1}{2}\Big(\exp(2\zeta h^*(\zeta)) - 1\Big) \\ &= C(\zeta)\left(h_0 \sin(b\zeta) + h_0^2 \sin^2(b\zeta) + \frac{2}{3} h_0^3 \sin^3(b\zeta) + \cdots\cdots\right) \end{aligned} \tag{60}$$

Let

$$A(\zeta) = A_1(\zeta) + h_0 A_2(\zeta) + h_0^2 A_3(\zeta) + \cdots\cdots \tag{61}$$

$$B(\zeta) = B_1(\zeta) + h_0 B_2(\zeta) + h_0^2 B_3(\zeta) + \cdots\cdots \tag{62}$$

By virtue of Eq. (60),

$$B_1(\zeta) = C(\zeta) h_0 \sin(b\zeta) \tag{63}$$

$$B_2(\zeta) = C(\zeta) h_0 \sin^2(b\zeta) \tag{64}$$

$$B_3(\zeta) = C(\zeta) \frac{2h_0}{3} \sin^3(b\zeta) \tag{65}$$

$$\cdots\cdots \tag{66}$$

After Fourier transform, the boundary condition (34) becomes

$$\begin{aligned} &\sqrt{\frac{2}{\pi}} \int_0^{\infty} \zeta\Big([eA(\zeta) - \epsilon B(\zeta)] \exp(-h_0\zeta) - \epsilon_0 C(\zeta) \cosh(\zeta h_0)\Big) \cos(\zeta X) d\zeta \\ &= -q_D H(X - a)\,, \quad \forall\, 0 < X < b \end{aligned} \tag{67}$$

Note the subtlety in terms of crack surface position between Eq. (58) and Eq. (67). For $|X| < b$ the upper crack surface is at $Y = h_0$ in the physical plane.

Consider the series expansion

$$\Big[eA(\zeta) - eB(\zeta)\Big] = \Big[eA_1(\zeta) - eB_1(\zeta)\Big] + h_0\Big[eA_2(\zeta) - eB_2(\zeta)\Big]$$
$$+h_0^2\Big[eA_3(\zeta) - eB_3(\zeta)\Big] + \cdots\cdots \tag{68}$$
$$\exp(-h_0\zeta) = 1 - h_0\zeta + \frac{(h_0\zeta)^2}{2!} - \frac{(h_0\zeta)^3}{3!} + \cdots\cdots \tag{69}$$
$$\cosh(h_0\zeta) = 1 + \frac{(h_0\zeta)^2}{2!} + \frac{(h_0\zeta)^4}{4!} + \cdots \tag{70}$$

Note that the permitivity constant is very small and comparable to h_0. It may be sensible to write $\epsilon_0 = \epsilon_0(h_0)$ to mark its relation (not dependence !) to the height of the slit. The following asymptotic series integral equations may then be derived,

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \zeta\left\{eA_1(\zeta) - \Big(\epsilon B_1(\zeta) + \frac{\epsilon_0(h_0)}{h_0\sin(b\zeta)}\Big)B_1(\zeta)\right\}\cos(\zeta X)d\zeta$$
$$= -q_D H(X-a)\ , \quad \forall\, 0 < X < b \tag{71}$$

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \Big\{-\zeta^2\Big([eA_1(\zeta) - \epsilon B_1(\zeta)] + \zeta[eA_2(\zeta) - \epsilon B_2(\zeta)]\Big\}\cos(\zeta X)d\zeta$$
$$= 0\ , \quad \forall\, 0 < X < b \tag{72}$$

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \left\{\frac{\zeta^3}{2}\Big(eA_1(\zeta) - \Big[\epsilon + \frac{\epsilon_0(h_0)}{h_0\sin(b\zeta)}\Big]B_1(\zeta)\Big) - \zeta^2\Big(eA_2(\zeta) - \epsilon B_2(\zeta)\Big)\right.$$
$$\left.-\zeta\Big(eA_3(\zeta) - \epsilon B_3(\zeta)\Big)\right\}\cos(\zeta X)d\zeta = 0\ , \quad \forall\, 0 < X < b \tag{73}$$
$$\cdots\cdots \tag{74}$$

Considering the symmetry conditions, one may find that

$$\sqrt{\frac{2}{\pi}}\int_0^\infty A_i(\zeta)\cos(\zeta X)d\zeta = 0, \quad \forall\, |X| > b \tag{75}$$
$$\sqrt{\frac{2}{\pi}}\int_0^\infty B_i(\zeta)\cos(\zeta X)d\zeta = 0, \quad \forall\, |X| > b \tag{76}$$

for $i = 1, 2, \cdots$.

Combining Eqs. (71)–(73) and Eqs. (75)–(76), one may derive a set of recursive dual integral equations, which may be considered solvable in principle.

In the remainder of the paper, only the first order approximation is considered. Moreover, since $h_0 \to 0$, $\sin(b\zeta)$ is always bounded, we adopt the following average approximation to render a tractable solution,

$$h_0 \sin(b\zeta) \approx h_0 \overline{\sin(b\zeta)} \to 0 \tag{77}$$

where

$$\overline{\sin(b\zeta)} := \sqrt{\frac{\pi}{2}} \int_0^\infty \sin(b\zeta) d\zeta = \sqrt{\frac{\pi}{2}} \frac{1}{b} \tag{78}$$

The identity (78) is in the sense of generalized function (see : Erdélyi et al [7] or Lighthill [16] page 33).

Let

$$r := \sqrt{\frac{2}{\pi}} \frac{b}{h_0} \tag{79}$$

Eq. (67) becomes

$$\sqrt{\frac{2}{\pi}} \int_0^\infty \zeta \Big(eA_1(\zeta) - (\epsilon + \epsilon_0 r) B_1(\zeta) \Big) \cos(\zeta X) d\zeta = -q_D H(X - a) \ , \ \forall \ 0 < X < b \tag{80}$$

The first order approximation of boundary condition (33) provides the integral equation

$$\sqrt{\frac{2}{\pi}} \int_0^\infty \zeta \Big(MA_1(\zeta) + eB_1(\zeta) \Big) \cos(\zeta X) d\zeta = \sigma_\infty, \quad \forall \ 0 < X < b \tag{81}$$

Considering the symmetry conditions $u(X, 0) = \phi(X, 0) = 0$, $\forall \ |X| > b$. Two sets of standard dual integral equations may be derived,

$$\begin{cases} \sqrt{\frac{2}{\pi}} \int_0^\infty \zeta A_1(\zeta) \cos(\zeta X) d\zeta = \sigma(X) & |X| < b \\ \\ \int_0^\infty A_1(\zeta) \cos(\zeta X) d\zeta = 0 & |X| > b \end{cases} \tag{82}$$

and

$$\begin{cases} \sqrt{\frac{2}{\pi}} \int\limits_0^\infty \zeta B_1(\zeta) \cos(\zeta X) d\zeta = q(X) & |X| < b \\ \\ \int\limits_0^\infty B_1(\zeta) \cos(\zeta X) d\zeta = 0 & |X| > b \end{cases} \tag{83}$$

in which

$$\sigma(X) = \begin{cases} S, & 0 < |X| < a \\ S - p_0, & a \leq |X| < b \end{cases} \tag{84}$$

$$q(X) = \begin{cases} T, & 0 < |X| < a \\ T + q_0, & a \leq |X| < b \end{cases} \tag{85}$$

where

$$S := \frac{(\epsilon + \epsilon_0 r)\sigma_\infty}{\Delta} \tag{86}$$

$$T := \frac{e\sigma_\infty}{\Delta} \tag{87}$$

$$p_0 := \frac{e}{\Delta} q_D \tag{88}$$

$$q_0 := \frac{M}{\Delta} q_D \tag{89}$$

and $\Delta = M(\epsilon + \epsilon_0 r) + e^2$.

Let

$$A_1(\zeta) = \sqrt{\frac{\pi}{2}} \int\limits_0^b f(t) t J_0(\zeta t) dt \tag{90}$$

$$B_1(\zeta) = \sqrt{\frac{\pi}{2}} \int\limits_0^b g(t) t J_0(\zeta t) dt \tag{91}$$

where $f(t)$ and $g(t)$ are unknown functions, and

$$\sqrt{\frac{2}{\pi}}\int_0^\infty A_1(\zeta)\cos(\zeta X)d\zeta = \begin{cases} \int_x^b \frac{f(t)t}{\sqrt{t^2-X^2}}dt, & |X|<b \\ \\ 0, & |X|>b \end{cases} \tag{92}$$

$$\sqrt{\frac{2}{\pi}}\int_0^\infty B_1(\zeta)\cos(\zeta X)d\zeta = \begin{cases} \int_x^b \frac{g(t)t}{\sqrt{t^2-X^2}}dt, & |X|<b \\ \\ 0, & |X|>b \end{cases} \tag{93}$$

The dual integral equations (82)-(83) are then reduced to a set of Abel integral equations

$$\frac{d}{dX}\int_0^X \begin{pmatrix} f(t) \\ g(t) \end{pmatrix} \frac{tdt}{\sqrt{X^2-t^2}} = \begin{pmatrix} \sigma(X) \\ q(X) \end{pmatrix} \tag{94}$$

Following the standard procedure (e.g. Sneddon and Lowengrub [35]), one may find that

$$f(t) = \frac{2}{\pi}\int_0^t \frac{\sigma(X)}{\sqrt{t^2-X^2}}dX = \begin{cases} S, & 0<t<a \\ \\ S - p_0\left(\frac{2}{\pi}\right)\cos^{-1}\left(\frac{a}{t}\right), & a<t<b \end{cases} \tag{95}$$

$$g(t) = \frac{2}{\pi}\int_0^t \frac{q(X)}{\sqrt{t^2-X^2}}dX = \begin{cases} T, & 0<t<a \\ \\ T + q_0\left(\frac{2}{\pi}\right)\cos^{-1}\left(\frac{a}{t}\right), & a<t<b \end{cases} \tag{96}$$

Consequently, one may find that $\forall\ |X| < b$,

$$u(X,0) = \int_x^b \frac{f(t)t}{\sqrt{t^2-X^2}}dt, \tag{97}$$

$$\phi(X,0) = \int_x^b \frac{g(t)t}{\sqrt{t^2-X^2}}dt. \tag{98}$$

and $\forall\ |X| > b$,

$$\frac{\partial u}{\partial Y} = \frac{X}{\sqrt{X^2-b^2}} f(b) + (\mathcal{E}_\infty - f(0)) - X \int_0^b \frac{f'(t)}{\sqrt{X^2-t^2}} dt, \tag{99}$$

$$\frac{\partial \phi}{\partial Y} = \frac{X}{\sqrt{X^2-b^2}} g(b) - (E_\infty + g(0)) - X \int_0^b \frac{g'(t)}{\sqrt{X^2-t^2}} dt, \tag{100}$$

where

$$\begin{cases} f(0) = S, \\ f(b) = S - p_0\left(\frac{2}{\pi}\right)\cos^{-1}\left(\frac{a}{b}\right) \end{cases} \tag{101}$$

$$\begin{cases} g(0) = T, \\ g(b) = T + q_0\left(\frac{2}{\pi}\right)\cos^{-1}\left(\frac{a}{b}\right) \end{cases} \tag{102}$$

4 Asymptotic fields

Let $Y = 0$. The general forms of asymptotic fields of both mechanical and electric variables in front of the crack tip can be found as follows

$$\begin{aligned} \epsilon_{yy} =& \frac{1}{\Delta}\left\{(\epsilon + \epsilon_0 r)\sigma_\infty - e q_D\left(\frac{2}{\pi}\right)\cos^{-1}\left(\frac{a}{b}\right)\right\}\frac{X}{\sqrt{X^2-b^2}} \\ &+ \left(\mathcal{E}_\infty - \frac{(\epsilon+\epsilon_0 r)\sigma_\infty}{\Delta}\right) + \text{higher order terms} \end{aligned} \tag{103}$$

$$\begin{aligned} E_y =& -\frac{1}{\Delta}\left\{e\sigma_\infty + M q_D\left(\frac{2}{\pi}\right)\cos^{-1}\left(\frac{a}{b}\right)\right\}\frac{X}{\sqrt{X^2-b^2}} \\ &+ \left(E_\infty + \frac{e\sigma_\infty}{\Delta}\right) + \text{higher order terms} \end{aligned} \tag{104}$$

$$\sigma_{yy} = \frac{\sigma_\infty X}{\sqrt{X^2-b^2}} + \text{higher order terms} \tag{105}$$

$$\begin{aligned} D_y =& \frac{1}{\Delta}\left\{e\epsilon_0 r\sigma_\infty - (M\epsilon + e^2) q_D\left(\frac{2}{\pi}\right)\cos^{-1}\left(\frac{a}{b}\right)\right\}\frac{X}{\sqrt{X^2-b^2}} \\ &+ \left(q_\infty - \frac{e\epsilon_0 r\sigma_\infty}{\Delta}\right) + \text{higher order terms} \end{aligned} \tag{106}$$

The relevant field intensity factors can be found as follows

$$K_I^S = \lim_{X \to b^+} \sqrt{2\pi(X-b)}\,\epsilon_{yy}(X,0)$$

$$= \left((\epsilon + \epsilon_0 r)\sigma_\infty - e q_D \left(\frac{2}{\pi}\right) \cos^{-1}\left(\frac{a}{b}\right) \right) \frac{\sqrt{\pi b}}{\Delta} \tag{107}$$

$$K_I^E = \lim_{X \to b^+} \sqrt{2\pi(X-b)} E_y(X,0)$$

$$= - \left(e\sigma_\infty + M q_D \left(\frac{2}{\pi}\right) \cos^{-1}\left(\frac{a}{b}\right) \right) \frac{\sqrt{\pi b}}{\Delta} \tag{108}$$

$$K_I^T = \lim_{X \to b^+} \sqrt{2\pi(X-b)} \sigma_{yy}(X,0) = \sigma_\infty \sqrt{\pi b} \tag{109}$$

$$K_I^D = \lim_{X \to b^+} \sqrt{2\pi(X-b)} D_y(X,0)$$

$$= \left(e\epsilon_0 r \sigma_\infty - \Delta_i q_D \left(\frac{2}{\pi}\right) \cos^{-1}\left(\frac{a}{b}\right) \right) \frac{\sqrt{\pi b}}{\Delta} \tag{110}$$

A few particular cases are deserved special attention.

4.1 Without surface charge

Assume that there is no surface charge on the crack surfaces. Let $q_D = 0$ The asymptotic fields become

$$\epsilon_{yy} = \frac{(\epsilon + \epsilon_0 r)\sigma_\infty}{\Delta} \frac{X}{\sqrt{X^2 - b^2}} + \left(\mathcal{E}_\infty - \frac{(\epsilon + \epsilon_0 r)\sigma_\infty}{\Delta} \right) + \text{higher order terms} \tag{111}$$

$$E_y = - \frac{(e\sigma_\infty - M q_\infty)}{\Delta} \frac{X}{\sqrt{X^2 - b^2}} + \left(E_\infty + \frac{e\sigma_\infty}{\Delta} \right) + \text{higher order terms} \tag{112}$$

$$\sigma_{yy} = \frac{\sigma_\infty X}{\sqrt{X^2 - b^2}} + \text{higher order terms} \tag{113}$$

$$D_y = \frac{e\epsilon_0 r \sigma_\infty}{\Delta} \frac{X}{\sqrt{X^2 - b^2}} + \left(q_\infty - \frac{e\epsilon_0 r \sigma_\infty}{\Delta} \right) + \text{higher order terms} \tag{114}$$

The field intensity factors are

$$K_I^S = (\epsilon + \epsilon_0 r)\sigma_\infty \frac{\sqrt{\pi b}}{\Delta} \tag{115}$$

$$K_I^E == -e\sigma_\infty \frac{\sqrt{\pi b}}{\Delta} \tag{116}$$

$$K_I^T = \sigma_\infty \sqrt{\pi b} \tag{117}$$

$$K_I^D = e\epsilon_0 r\sigma_\infty \frac{\sqrt{\pi b}}{\Delta} \tag{118}$$

Furthermore, let $\epsilon_0 = 0$, we recover all the results obtained by Zhang and Hack [48] for a mode III crack.

$$K_I^S = \frac{\epsilon}{\Delta_i}\sigma_\infty\sqrt{\pi b} \tag{119}$$

$$K_I^E == -\frac{e}{\Delta_i}\sigma_\infty\sqrt{\pi b} \tag{120}$$

$$K_I^T = \sigma_\infty\sqrt{\pi b} \tag{121}$$

$$K_I^D = 0\ . \tag{122}$$

4.1.1 Zero width crack solution ($h_0 = 0$)

Let $h_0 = 0$ and consequently $r \to \infty$. That is: the slit has zero initial width. The physical interpretation of this limit is that the upper and lower crack surfaces are constantly in close contact during fracture process, there is no dielectric medium inside the crack. The asymptotic fields become

$$\epsilon_{yy} = \frac{\sigma_\infty}{M}\frac{X}{\sqrt{X^2 - b^2}} - \frac{\sigma_\infty}{M} + \text{higher order terms} \tag{123}$$

$$E_y = E_\infty + \text{higher order terms} \tag{124}$$

$$\sigma_{yy} = \frac{\sigma_\infty X}{\sqrt{X^2 - b^2}} + \text{higher order terms} \tag{125}$$

$$D_y = \frac{e}{M}\frac{\sigma_\infty X}{\sqrt{X^2 - b^2}} - \frac{1}{M}(e\sigma_\infty - Mq_\infty) + \text{higher order terms} \tag{126}$$

This recovers the solution obtained by Yang and Kao [45] for a zero width crack in piezoelectric medium.

4.2 Surface charge distribution

4.2.1 Impermeable solution

Let $\epsilon_0 = 0$. $\Delta \to \Delta_i = M\epsilon + e^2$. Assume that the surface charge is uniformly distributed along the crack surface ($a = 0$) with the distribution intensity $q_D = -q_\infty$, i.e.

$$-q_D\left(\frac{2}{\pi}\right)\cos^{-1}\left(\frac{a}{b}\right) = q_\infty$$

We recover the impermeable solution, which has exactly the same structure as the mode III impermeable crack solution obtained by Pak [26].

$$\epsilon_{yy} = \frac{(\epsilon\sigma_\infty + eq_\infty)}{\Delta_i} \frac{X}{\sqrt{X^2 - b^2}} + \text{higher order terms} \tag{127}$$

$$E_y = -\frac{(e\sigma_\infty - Mq_\infty)}{\Delta_i} \frac{X}{\sqrt{X^2 - b^2}} + \text{higher order terms} \tag{128}$$

$$\sigma_{yy} = \frac{\sigma_\infty X}{\sqrt{X^2 - b^2}} + \text{higher order terms} \tag{129}$$

$$D_y = \frac{q_\infty X}{\sqrt{X^2 - b^2}} + \text{higher order terms} \tag{130}$$

This discovery reveals that the so-called impermeable solution is more than an approximation by just seting $\epsilon_0 = 0$. It is involved a double charge on the crack surface to shield off the interaction between the dielectric medium inside crack and the piezoelectric matrix. The discussion on its consequence to the energy release rate is deferred to the later section.

4.2.2 Surface charge negating the effect of applied electric field

Assume that surface charge distribution on the crack surface accumulates to a certain level, it negates the effects of applied electric field in a manner such that

$$\frac{q_\infty}{q_D} = \left(\frac{2}{\pi}\right) \cos^{-1}\left(\frac{a}{b}\right) \tag{131}$$

the asymptotic fields are still singular. They become

$$\epsilon_{yy} = \frac{\left((\epsilon + \epsilon_0 r)\sigma_\infty - eq_\infty\right)}{\Delta} \frac{X}{\sqrt{X^2 - b^2}} + \left(\mathcal{E}_\infty - \frac{1}{\Delta}\left((\epsilon + \epsilon_0 r)\sigma_\infty - eq_\infty\right)\right) + \text{higher order terms} \tag{132}$$

$$E_y = -\frac{\left(e\sigma_\infty + Mq_\infty\right)}{\Delta} \frac{X}{\sqrt{X^2 - b^2}} + \left(E_\infty + \frac{1}{\Delta}(e\sigma_\infty + Mq_\infty)\right) + \text{higher order terms} \tag{133}$$

$$\sigma_{yy} = \frac{\sigma_\infty X}{\sqrt{X^2 - b^2}} + \text{higher order terms} \tag{134}$$

$$D_y = \frac{\left(e\epsilon_0 r\sigma_\infty - (M\epsilon + e^2)q_\infty\right)}{\Delta} \frac{X}{\sqrt{X^2 - b^2}} - \frac{\epsilon_0 r}{\Delta}(e\sigma_\infty + Mq_\infty) + \text{higher order terms} \tag{135}$$

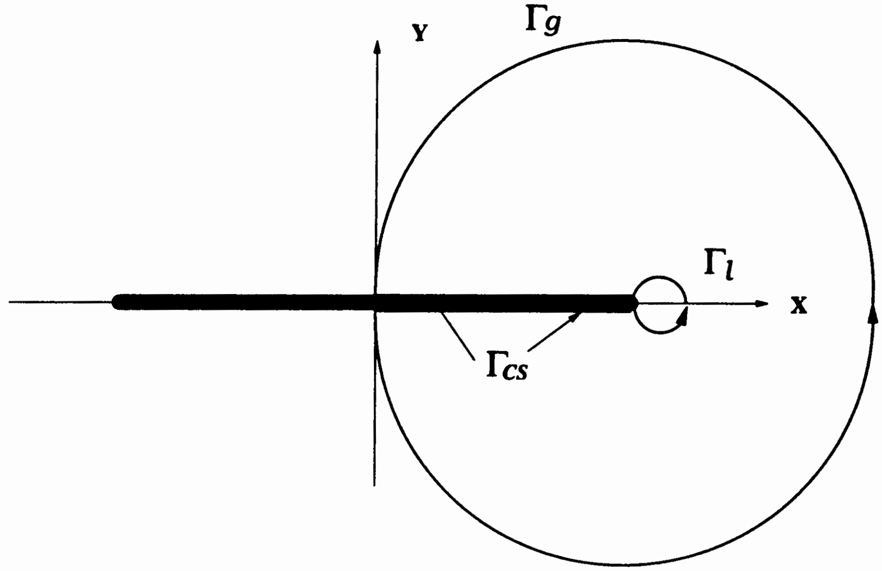

Fig. 4. J-integral contours for evaluating local and global energy release rates.

5 Energy release rate

Energy release rate during crack growth in a piezoelectric medium have been discussed by many authors, e.g. Cherepanov [2], Pak [26], Suo et al [41], Zhang and Hack [48], Dascalu and Maugin [3,4], Park and Sun [32,31], Gao and his co-workers [11,12], McMeeking [23] among many others.

It is generally believed that energy release rate, or J-integral, is a better criterion for crack growth than stress intensity factors. The J-integral in a piezoelectric medium is given by Cherepanov [2],

$$J = \int_{\Gamma} \Big(H n_x - \sigma_{ij} n_i u_{j,x} - n_i D_i \phi_{,x} \Big) dS \tag{136}$$

where H is the electric enthalpy density.

On the surface of a permeable crack, both the normal component of electric displacement as well as the electric potential are not zero, consequently, the contribution in the contour integral, J, along crack surfaces is not zero. Therefore, for permeable cracks, J-integral has two parts: *Local energy release rate*, which is defined as the local energy release rate and global energy release rate. The so-called local energy release rate is defined as the contour integral, J, along an infinitesimal circle around the crack tip, Γ_ℓ; *Global energy release rate*, which may be defined as any contour integral, J, starting at the center of the lower part of the crack surface and ending at the center of upper part of the crack surface (see Fig. 4). The total energy release rate, or the so-called global energy release rate, is the sum of the local energy release rate and the contour integral contribution along the crack surfaces, i.e.

$$J_g = J_\ell + J_{cs} \tag{137}$$

where J_{cs} denote the energy release rate contribution from crack surfaces.

Arguments have been made by Gao and his colleagues [11,12] about different roles that local and global energy release rates may play in the process of piezoelectric fracture.

5.1 Local energy release rate

We first consider the so-called *local energy release rate.* There are four electro-mechanical boundary condition combinations used in literature

(1) Case 1: $\sigma_{yy} = \sigma_\infty\ ,\quad D_y = q_\infty\ ,\qquad \forall\ y\ \to \infty;$
(2) Case 2: $\epsilon_{yy} = \mathcal{E}_\infty\ ,\quad E_y = E_\infty\ ,\qquad \forall\ y\ \to \infty;$
(3) Case 3: $\sigma_{yy} = \sigma_\infty\ ,\quad E_y = E_\infty\ ,\qquad \forall\ y\ \to \infty;$
(4) Case 4: $\epsilon_{yy} = \mathcal{E}_\infty\ ,\quad D_y = q_\infty\ ,\qquad \forall\ y\ \to \infty;$

The basic results regarding energy release rate in piezoelectric materials in the literature are

(1) Pak [26]:

$$\begin{aligned}
J_{P1} &= \frac{\pi b}{2}\Big[\frac{\epsilon\sigma_\infty^2 + 2e\sigma_\infty q_\infty - M q_\infty^2}{\Delta_i}\Big] \\
J_{P2} &= \frac{\pi b}{2}\Big[M\mathcal{E}_\infty^2 - 2eE_\infty\mathcal{E}_\infty - \epsilon E_\infty^2\Big] \\
J_{P3} &= \frac{\pi b}{2}\Big[\frac{\sigma_\infty^2 - (\epsilon M + e^2)E_\infty^2}{M}\Big] \\
J_{P4} &= \frac{\pi b}{2}\Big[\frac{(\epsilon M + e^2)\mathcal{E}_\infty^2 - q_\infty^2}{M}\Big]
\end{aligned}$$

(2) Zhang and Hack [48]:

$$\begin{aligned}
J_{ZH1} &= \frac{\pi b\epsilon}{2\Delta_i}\sigma_\infty^2 \\
J_{ZH2} &= \frac{\pi b\epsilon}{2\Delta_i}\Big(M\mathcal{E}_\infty - eE_\infty\Big)^2 \\
J_{ZH3} &= \frac{\pi b\epsilon}{2\Delta_i}\sigma_\infty^2 \\
J_{ZH4} &= \frac{\pi b}{2\epsilon\Delta_i}\Big(\Delta_i\mathcal{E}_\infty - eq_\infty\Big)^2
\end{aligned} \tag{138}$$

(3) Yang and Kao [45]:

$$J_{YK1} = \frac{\pi b}{2}\frac{\sigma_\infty^2}{M}$$
$$J_{YK2} = \frac{\pi b}{2}\frac{(M\mathcal{E}_\infty - eE_\infty)^2}{M}$$
$$J_{YK3} = \frac{\pi b}{2}\frac{\sigma_\infty^2}{M}$$
$$J_{YK4} = \frac{\pi b}{2M}\Big[\frac{(M\epsilon + e^2)\mathcal{E}_\infty^2 - eq_\infty^2}{\epsilon}\Big]^2$$

(4) Park and Sun [32]:

$$J_{PS3} = \frac{\pi b}{2M}\Big(\sigma_\infty^2 + e\sigma_\infty E_\infty\Big) \tag{139}$$

(5) Gao et al [12]:

$$J_{GZT3} = \frac{\pi b}{2M}\Big(1 + \frac{e^2}{M\epsilon}\Big)(\sigma_\infty + eE_\infty)^2 \tag{140}$$

Corresponding to the first kind boundary conditions, the local energy release rate of the present permeable crack model is

$$J_{NEW1} = \Big(\frac{\pi b}{2}\Big)\Big\{\Big(\frac{(\epsilon + \epsilon_0 r)}{\Delta} + \frac{e^2\epsilon_0 r}{\Delta^2}\Big)\sigma_\infty^2 - \frac{2e}{\Delta}\Big(1 - \frac{M\epsilon_0 r}{\Delta}\Big)\sigma_\infty q_D^{'} - \frac{M(M\epsilon + e^2)}{\Delta^2}q_D^{'\,2}\Big\} \tag{141}$$

where $q_D^{'} = q_D\Big(\frac{2}{\pi}\Big)\cos^{-1}\Big(\frac{a}{b}\Big)$. The energy release rates in other boundary conditions may be obtained by simple substitution of Eqs. (44)-(47).

Assume that there is no surface charge distribution. Let $q_D = 0$. It may be found that

$$J_{NEW1a} = \Big(\frac{\pi b}{2}\Big)\Big(\frac{(\epsilon + \epsilon_0 r)}{\Delta} + \frac{e^2\epsilon_0 r}{\Delta^2}\Big)\sigma_\infty^2 \tag{142}$$

Furthermore, letting $\epsilon_0 = 0$ in (142), one recovers the result obtained by Zhang and Hack [48], i.e.

$$J_{NEW1a} \Rightarrow \Big(\frac{\pi b}{2}\Big)\frac{\epsilon}{\Delta_i}\sigma_\infty^2 = J_{ZH1} \tag{143}$$

Let $h_0 = 0$ or $r \to \infty$. in Eq. (142). The result obtained by Yang and Kao [45] may be recovered,

$$J_{NEW1a} \Rightarrow \left(\frac{\pi b}{2}\right)\frac{\sigma_\infty^2}{M} = J_{YK1} \tag{144}$$

which is basically the purely elastic energy release rate, since there is no dielectric medium inside the crack.

This again confirms the fact that the intensity factors of a permeable crack are independent from the remote applied electric fields [50,51].

If there is a surface charge distribution on crack surfaces, the energy release rate may change significantly depending on the range and the intensity of the surface charge distribution. Assume that there is a double surface charge, i.e. $a = 0$ and $q_D = -q_\infty$,

$$q_D' = q_D \frac{2}{\pi} \cos^{-1}\left(\frac{a}{b}\right) = -q_\infty, \tag{145}$$

and let $\epsilon_0 = 0$. one recovers the energy release rate expression under impermeable approximation,

$$J_{NEW1} \Rightarrow J_{P1} = \left(\frac{\pi b}{2}\right)\frac{\epsilon\sigma_\infty^2 + 2e\sigma_\infty q_\infty - Mq_\infty^2}{M\epsilon + e^2} \tag{146}$$

5.2 Global energy release rate

When a permeable crack grows, energy release is not only consumed in creating surface energy for newly formed crack surfaces, but also consumed by supplying the electrostatic energy to the dielectric medium inside the crack. In fact, if surface charge is absent on the crack surfaces, the normal component of electric displacement in piezoelectric medium may equal to the normal component of electric displacement in the dielectric medium inside the crack. This suggests that the crack surface contribution to the J-integral is the part of energy release rate that may go directly into supplying the electrostatic energy increase in the dielectric medium inside the crack.

If discharge, or additional charge, occurs, surface charge will be present on crack surfaces, which may either enhance or reverse the direction of energy-moment flux. Therefore, additional energy release rate will be created that will influence to crack growth process.

By taking into account the effect of charge distribution on the crack surface, the so-called global energy release rate is,

$$J_g = J_\ell + J_{cs} \tag{147}$$

where J_{cs} denote the energy release contribution from crack surfaces, which can be calculated by

$$J_{cs} = -\int_{cs} n_i D_i \phi_{,x} dS \tag{148}$$

In order the evaluate J_{cs}, we first evaluate the normal component of the electric displacement on the crack surfaces.

$$\begin{aligned} D_y(X,h_0) \approx D_y(X,0) &= e\frac{\partial u}{\partial Y} + \epsilon\frac{\partial \phi}{\partial Y} \\ &= q_\infty - e\sqrt{\frac{2}{\pi}}\int_0^\infty A_1(\zeta)\cos(\zeta X)d\zeta + \epsilon\sqrt{\frac{2}{\pi}}\int_0^\infty B_1(\zeta)\cos(\zeta X)d\zeta \\ &= q_\infty - e\frac{d}{dX}\int_0^b \frac{f(t)tH(X-t)}{\sqrt{X^2-t^2}}dt + \epsilon\frac{d}{dX}\int_0^b \frac{g(t)tH(X-t)}{\sqrt{X^2-t^2}}dt \\ &= q_\infty - (eS - \epsilon T)\frac{d}{dX}\int_0^X \frac{tdt}{\sqrt{X^2-t^2}} \\ &\quad + \frac{(e^2+M\epsilon)}{\Delta}q_D\Big(\frac{2}{\pi}\Big)\frac{d}{dX}\int_0^X \frac{t\cos^{-1}\big(\frac{a}{t}\big)}{\sqrt{X^2-t^2}}dt \\ &= q_\infty - (eS-\epsilon T)\frac{d}{dX}X + \frac{(e^2+M\epsilon)q_D}{\Delta}\frac{d}{dX} < X - a > \qquad (149) \\ &= D_{Y1} + D_{Y2} \qquad (150) \end{aligned}$$

where $D_{Y1} := q_\infty - \frac{e\epsilon_0 r}{\Delta}\sigma_\infty$ and $D_{Y2} := \frac{e^2+M\epsilon}{\Delta}q_D H(X-a)$.

To evaluate the electric potential on permeable crack surface, it may be found that

$$\phi(X,h_0) \approx \phi(X,0) = \sqrt{\frac{2}{\pi}}\int_0^\infty B_1(\zeta)\cos(\zeta X)d\zeta = \int_x^b \frac{g(t)t}{\sqrt{t^2-x^2}}dt \tag{151}$$

where

$$g(t) = \frac{e\sigma_\infty}{\Delta} + \frac{Mq_D}{\Delta}\cos^{-1}\Big(\frac{a}{t}\Big)H(t-a), \quad 0 < t < b \tag{152}$$

Therefore,

$$\phi(X,0)=\frac{e}{\Delta}\sigma_\infty\int_x^b \frac{tdt}{\sqrt{t^2-X^2}}+\frac{Mq_D}{\Delta}\Big(\frac{2}{\pi}\Big)\int_x^b \frac{\cos^{-1}\big(\frac{a}{t}\big)H(t-a)}{\sqrt{t^2-X^2}}dt$$
$$=\frac{1}{\Delta}\left(e\sigma_\infty+Mq_D\Big(\frac{2}{\pi}\Big)\cos^{-1}\Big(\frac{a}{b}\Big)\right)\sqrt{b^2-X^2}$$
$$-\frac{Mq_D}{\pi\Delta}\left\{X\ln\Big|\frac{a\sqrt{b^2-X^2}-X\sqrt{b^2-a^2}}{a\sqrt{b^2-X^2}+X\sqrt{b^2-a^2}}\Big|\right.$$
$$\left.-a\ln\Big|\frac{a\sqrt{b^2-a^2}-X\sqrt{b^2-X^2}}{a\sqrt{b^2-a^2}+X\sqrt{b^2-X^2}}\Big|\right\} \tag{153}$$

Substituting Eq. (150) and Eq. (153) into Eq. (148) yield

$$J_{cs}=D_{Y1}(\phi(0^+,0)-\phi(0^-,0))+D_{Y2}(\phi(a^+,0)-\phi(a^-,0))$$
$$=\frac{2b}{\Delta}\Big(q_\infty-\frac{e\epsilon_0 r\sigma_\infty}{\Delta}\Big)\left\{\Big(e\sigma_\infty+Mq_D\Big(\frac{2}{\pi}\Big)\theta\Big)-\frac{2Mq_D}{\pi}\cos\theta\ln\Big|\frac{1+\sin\theta}{\cos\theta}\Big|\right\}$$
$$+\frac{2b}{\Delta}\Big[\frac{(e^2+M\epsilon)}{\Delta}q_D\Big]\left\{\Big[(e\sigma_\infty+Mq_D\Big(\frac{2}{\pi}\Big)\theta\Big]\sin\theta-\frac{2q_D}{\pi}\cos\theta\ln(\sec\theta)\right\} \tag{154}$$

where $\theta:=\cos^{-1}\big(\frac{a}{b}\big)$.

Assume that there is no surface charge distribution on the crack surface, i.e. $q_D=0$. In this situation, the energy release rate contribution from the crack surface becomes

$$J_{cs}=\frac{2b}{\Delta}\Big(q_\infty-\frac{e\epsilon_0 r\sigma_\infty}{\Delta}\Big)e\sigma_\infty \tag{155}$$

Let $\epsilon_0\to 0$ in Eq. (155). It yields

$$J_{cs}=\frac{2be}{M\epsilon+e^2}\sigma_\infty q_\infty=\frac{2be^2}{M(M\epsilon+e^2)}\sigma_\infty^2+\frac{2be}{M}\sigma_\infty E_\infty \tag{156}$$

On the other hand, letting $h_0\to 0$ in Eq. (155), one has

$$J_{cs}=\frac{2be}{M}\sigma_\infty E_\infty \tag{157}$$

The global energy release rate in the case $q_D = 0$ and $\epsilon_0 = 0$ has the form

$$\boxed{\text{Energy release rate I}: \ J^g_{cr1} = \Big(\frac{\pi b}{2M}\Big)\left(\frac{M\epsilon + \dfrac{4}{\pi}e^2}{M\epsilon + e^2}\sigma^2_\infty + \frac{4}{\pi}e\sigma_\infty E_\infty\right)} \tag{158}$$

In the case $q_D = 0$ and $h_0 \to 0$, the global energy release rate is

$$\boxed{\text{Energy release rate II}: \ J^g_{cr2} = \Big(\frac{\pi b}{2M}\Big)\Big(\sigma^2_\infty + \frac{4}{\pi}e\sigma_\infty E_\infty\Big)} \tag{159}$$

Since $\dfrac{4}{\pi} = 1.273238 \approx 1.0$, the newly derived results are fairly close to the empirical result proposed by Park and Sun [32,31].

6 Closure

The analysis presented in this paper shows that the interaction between crack and ite permeable environment can be crucial to crack growth and fracture process in a piezoelectric ceramic. The effect of the interaction may be captured by the J-integral along surfaces of a permeable crack.

Moreover, the surface charge on a permeable crack surfaces due to any possible charge-discharge mechanisms may either shield or assist energy-moment flux flow into or flow out the dielectric medium inside crack, which manifests the important role of the applied electric field in crack growth.

This paper is not intended to discredit the strip-saturation model and the related local energy release rate criterion. It is this writer's belief that the nature of crack growth, or fracture, in a piezoelectric ceramic is a complex physical phenomenon. It is probably true that several physical mechanisms are playing important roles in a fracture process simultaneously, such as the interaction between dielectric medium inside crack and the piezoelectric matrix, domain switching, discharge due to permeable environment or surface separation, etc. Which mechanism is the dominant factor that controls the crack growth may be still unknown, and it needs further study in both experimental as well as analytical researches, though the current status quo may be in favor of the so-called strip-saturation model and its associated local energy release rate criterion [11,12,8].

Some mathematical treatments of the strip-saturation model have been adopted in current approach, such as the Dugdale analogy. Nevertheless, the Dugdale analogy made in present context is not referred to as the so-called "*electrically yielding*", but the piecewise distribution of surface charge on the crack surfaces. A study to incorporate the strip-saturation model with this particular permeable crack geometry is undertaken, which shall be presented in the second part of this work.

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