UC Irvine

UC Irvine Previously Published Works

Title

The interface vibrational contribution to the thermodynamic functions of the system of liquid and solid separated by a planar interface

Permalink

<https://escholarship.org/uc/item/9hp2h25t>

Journal

Solid State Communications, 103(6)

ISSN

0038-1098

Authors

Hawkins, D
Maradudin, AA
Shchegrov, AV

Publication Date

1997

DOI

10.1016/S0038-1098(97)00141-5

Copyright Information

This work is made available under the terms of a Creative Commons Attribution License, available at <https://creativecommons.org/licenses/by/4.0/>

Peer reviewed

PII: S0038-1098(97)00141-5

THE INTERFACE VIBRATIONAL CONTRIBUTION TO THE THERMODYNAMIC FUNCTIONS OF THE SYSTEM OF LIQUID AND SOLID SEPARATED BY A PLANAR INTERFACE

D. Hawkins, A. A. Maradudin and A. V. Shchegrov

Department of Physics and Astronomy and Institute for Surface and Interface Science, University of California, Irvine, CA 92697, U.S.A.

(Received and; accepted 27 March 1997 by M. Cardona)

We study theoretically the interface vibrational contribution to thermodynamic properties of a system of solid and liquid in contact across a planar interface. We express thermodynamic quantities such as the free energy and specific heat in terms of the interface density of vibrational states. We derive a simple analytic expression for this function in terms of the static Green's tensor of the system. In order to calculate the free energy, we modify the Debye model to apply it to the interface problem. We also obtain the dynamical Green's tensor for the system of liquid and solid separated by a planar interface. ©1997 Elsevier Science Ltd

Keywords: A. surfaces and interfaces, D. elasticity, D. heat capacity.

1. INTRODUCTION

Since the 1970s a number of approaches have been developed to obtain the low-temperature interface vibrational contribution to the thermodynamic functions of compound systems such as a solid in contact with a different solid, or with a liquid, across a planar interface. Djafari-Rouhani and Dobrzynski [1] showed that the interface contribution $\Delta C_v^{(I)}(T)$ to the low-temperature specific heat of a system of two isotropic solids in contact can be expressed in terms of the static Green's tensor of the system and is proportional to ST^2 , where S is the interface area, and T is the temperature. Velasco and García-Moliner [2], studying the same problem, derived the density of vibrational states $G(\omega)$ of the system. This function is defined by $G(\omega) = \sum_n \delta(\omega - \omega_n)$, and is normalized to the total number of vibrational degrees of freedom of the system. The sum on n runs over all normal modes with frequencies ω_n . With the aid of $G(\omega)$, the authors of [2] obtained the same ST^2 law for $\Delta C_v^{(I)}(T)$. This dependence was also confirmed by Iosilevskii [3] and, very recently, by Shchegrov [4].

Interest in the thermodynamics of a solid-liquid interface has increased recently due to the experimental studies [5] of the wetting of the surfaces of alkali metals by liquid helium. In particular, the interface contribu-

tion to the zero-point energy $\Delta F_0^{(I)}$ could be of special importance in understanding this phenomenon. Under certain conditions, one can apply the same methods for the calculation of this quantity that were developed for the case of two solids in contact.

To calculate the density of states $G(\omega)$ for the system of solid and liquid separated by a planar interface, one can use any of the methods developed in [1–4], since all of them seem to give consistent results for $\Delta C_v^{(I)}(T)$ in the case of two solids in contact. However, as opposed to the calculation of $\Delta C_v^{(I)}(T)$, in obtaining $\Delta F_0^{(I)}$ one has to impose a cut-off on the integral over frequency ω . The well-known Debye model for an infinite solid has to be modified to be applicable to the interface problem. Such a modification was made by Velasco and García-Moliner [2, 6] for a semi-infinite solid bounded by a stress-free surface. However, they did not extend their analysis to the more complicated case of two media in contact. In the only work in which the density of states for a solid-liquid system was obtained, Iosilevskii [3] described the entire system by a single spectral function $G(\omega)$, which included bulk and interface contributions. The cut-off frequency ω_M was determined by the requirement that $G(\omega)$ be normalized to the total number of degrees of freedom in the system, $3N = 3(N_s + N_l)$, where N_s and N_l are the number of atoms in the solid and

in the liquid, respectively. However, one could expect that if the numbers of atoms $N_{s,l}$ and the volumes $V_{s,l}$ are sufficiently large, then only the ratios (N_s/V_s) and (N_l/V_l) would enter the expression for ω_M , which was not the case for the results obtained in [3].

A consistent approach to the determination of a cut-off frequency was suggested recently by Shchegrov [4] for the case of two solids in contact. He required that each medium involved in the problem be described by its own density of states $G(\omega)$ and cut-off frequency ω_M . In this case $G(\omega)$ for each medium is given by the sum of a "bulk" part, describing the vibrations of an unbounded medium, and an interface contribution. In the present work we will apply this method to a planar solid-liquid interface. We will show that, as in the case of two solids in contact, the interface contribution to the density of states is described by the static Green's tensor of the system. Then we will find the interface contribution to the free energy and specific heat.

2. THE INTERFACE DENSITY OF VIBRATIONAL STATES

We consider a system of a solid and a liquid separated by an interface $x_3 = 0$. The solid occupies the upper half-space $x_3 > 0$ and is described by its mass density ρ and the elastic modulus tensor $C_{\alpha\beta\mu\nu} = \rho(c_l^2 - 2c_t^2)\delta_{\alpha\beta}\delta_{\mu\nu} + \rho c_t^2(\delta_{\alpha\mu}\delta_{\beta\nu} + \delta_{\alpha\nu}\delta_{\beta\mu})$, where c_l and c_t are the speeds of longitudinal and transverse waves, respectively. The inviscid liquid whose density is ρ_0 supports longitudinal vibrations of speed c_0 , and occupies the lower half space $x_3 < 0$. Its elastic modulus tensor is given by $C_{\alpha\beta\mu\nu}^{(0)} = \rho c_0^2 \delta_{\alpha\beta} \delta_{\mu\nu}$. The density of vibrational states in this system can be obtained with the aid of the dynamical Green's tensor $D_{\alpha\beta}(x, x'|\omega)$ which satisfies the equations

$$\left[\rho \omega^2 \delta_{\alpha\mu} + C_{\alpha\beta\mu\nu} \frac{\partial^2}{\partial x_\beta \partial x_\nu} \right] D_{\mu\gamma}(x, x'|\omega) = \delta_{\alpha\gamma} \delta(x - x'), \quad x_3 > 0, \quad (1)$$

$$\left[\rho_0 \omega^2 \delta_{\alpha\mu} + C_{\alpha\beta\mu\nu}^{(0)} \frac{\partial^2}{\partial x_\beta \partial x_\nu} \right] D_{\mu\gamma}(x, x'|\omega) = \delta_{\alpha\gamma} \delta(x - x'), \quad x_3 < 0, \quad (2)$$

and the boundary conditions

$$D_{3\gamma}(x, x'|\omega) \Big|_{x_3=0+} = D_{3\gamma}(x, x'|\omega) \Big|_{x_3=0-}, \quad (3)$$

$$C_{\alpha 3 \mu \nu} \frac{\partial}{\partial x_\nu} D_{\mu \gamma}(x, x'|\omega) \Big|_{x_3=0+}$$

$$= C_{\alpha 3 \mu \nu}^{(0)} \frac{\partial}{\partial x_\nu} D_{\mu \gamma}(x, x'|\omega) \Big|_{x_3=0-}, \quad (4)$$

which express the continuity of the normal components of the velocities and the stresses across the interface. In addition, we require that $D_{\alpha\beta}(x, x'|\omega)$ obey outgoing or exponentially decaying wave conditions as $|x_3| \rightarrow \infty$. In Eqs. (1)–(4) and throughout the paper summation over repeated Greek indices is implied.

The partial differential operator appearing on the left hand side of Eqs. (1)–(2), supplemented by the boundary conditions (3)–(4), can be shown to be Hermitian in the entire coordinate space. The density of vibrational states $G(\omega)$ is then related to the trace of $D_{\alpha\beta}(x, x'|\omega)$ as follows [4]:

$$G(\omega) = \frac{2\omega}{\pi} \int_S d^2 x_{\parallel} \int_{-\infty}^{\infty} dx_3 \rho(x_3) \text{Im} D_{\alpha\alpha}(x, x|\omega - i\eta), \quad (5)$$

where $x_{\parallel} = (x_1, x_2, 0)$, $\rho(x_3) = \rho$ if $x_3 > 0$ and ρ_0 if $x_3 < 0$, S is the interface area, and η is a positive infinitesimal.

Due to the symmetry of the problem in the plane $x_3 = 0$, it is convenient to work with the 2D Fourier coefficients $g_{\alpha\beta}(k_{\parallel}, \omega|x_3, x'_3)$ of the Green's tensor [1, 4], where $k_{\parallel} = (k_1, k_2, 0)$. As in the case of two solids in contact [1, 4], the functions $g_{\alpha\beta}$ can be represented in the form

$$g_{\alpha\beta}(k_{\parallel}, \omega|x_3, x'_3) = g_{\alpha\beta}^{(B)}(k_{\parallel}, \omega|x_3, x'_3) + g_{\alpha\beta}^{(I)}(k_{\parallel}, \omega|x_3, x'_3). \quad (6)$$

Here $g_{\alpha\beta}^{(B)}$ corresponds to the Green's tensor for the infinite solid in the case when both x_3 and x'_3 lie in the upper half-space, and for the infinite liquid when x_3 and x'_3 are both negative. The interface Green's functions $g_{\alpha\beta}^{(I)}$ appear due to the presence of the interface and ensure the satisfaction of the boundary conditions. The explicit expressions for $g_{\alpha\beta}^{(B)}$ and $g_{\alpha\beta}^{(I)}$ are given in Appendices A and B, respectively. We next use Eq. (6) in Eq. (5) to obtain

$$G(\omega) = G_s^{(B)}(\omega) + G_s^{(I)}(\omega) + G_l^{(B)}(\omega) + G_l^{(I)}(\omega). \quad (7)$$

The functions $G_s^{(B)}$ and $G_s^{(I)}$ denote the bulk and interface densities of states in the solid, and arise from the integration of $g_{\alpha\beta}^{(B)}$ and $g_{\alpha\beta}^{(I)}$ over positive x_3 , i.e. the region occupied by the solid. The functions $G_l^{(B)}$ and $G_l^{(I)}$ are their analogs for the liquid, and come from the integration over $x_3 < 0$.

The densities of states $G_{s,l}^{(B)}$ of the unbounded solid and liquid are given by

$$G_s^{(B)}(\omega) = \frac{3\omega^2 V_s}{2\pi^2 c_B^3}, \quad G_l^{(B)}(\omega) = \frac{\omega^2 V_l}{2\pi^2 c_0^3}, \quad (8)$$

where V_s and V_l are the volumes occupied by the solid and liquid, respectively, and c_B is defined by $(3/c_B^3) = (2/c_l^3) + (1/c_0^3)$. The interface contribution to the density of states of the solid, which we consider first, has the form

$$G_s^{(I)}(\omega) = \frac{2\omega S\rho}{\pi} \int_0^\infty dx_3 \times \int \frac{d^2 k_\parallel}{(2\pi)^2} \text{Im} g_{\alpha\alpha}^{(I)}(k_\parallel, \omega - i\eta | x_3, x_3). \quad (9)$$

The integrals in Eq.(10) can be evaluated analytically [4]. We introduce the function

$$f(k_\parallel, \omega) = \frac{\rho S}{8\pi^3} \int_0^\infty dx_3 \int_0^{2\pi} d\theta g_{\alpha\alpha}^{(I)}(k_\parallel, \omega | x_3, x_3), \quad (10)$$

where θ is the polar angle in the (k_1, k_2) - plane. One can see from Appendix B that due to the simple exponential dependence of $g_{\alpha\alpha}^{(I)}(k_\parallel, \omega | x_3, x_3)$ on x_3 , we can immediately evaluate the integral over this variable in Eq.(10). The integral over θ gives 2π because the integrand does not depend on the direction of $k_\parallel$. Thus, we obtain an explicit expression for $f(k_\parallel, \omega)$, which we do not give here because of its cumbersome form. The interface density of states in the solid is given by

$$G_s^{(I)}(\omega) = 4\omega \int_0^\infty dk_\parallel k_\parallel \text{Im} f(k_\parallel, \omega - i\eta). \quad (11)$$

We notice that $g_{\alpha\alpha}^{(I)}(k_\parallel, \omega | x_3, x_3)$ depends on $k_\parallel$ and ω only through $k_\parallel^2$ and ω^2 , hence so does $f(k_\parallel, \omega)$. Therefore, it is convenient to introduce new variables $p = k_\parallel^2$, $\Omega = \omega^2$, which we will use later on as the arguments of f . Since $G_s^{(I)}(\omega) = 2\omega \tilde{G}_s^{(I)}(\omega^2)$, where $\tilde{G}_s^{(I)}(\omega^2)$ is the interface contribution to $\tilde{G}(\omega^2) = \sum_n \delta(\omega^2 - \omega_n^2)$, we obtain the result that

$$\tilde{G}_s^{(I)}(\Omega) = \int_0^\infty dp \text{Im} f(p, \Omega - i\eta). \quad (12)$$

We next transform this integral into a contour integral [4]:

$$\int_0^\infty dp \text{Im} f(p, \Omega - i\eta) = \frac{1}{2i} \int_{C_1} dp f(p, \Omega), \quad (13)$$

where the contour C_1 is depicted in Fig.1 by the solid line. We consider f as a function of complex p and real

Fig. 1. The complex p -plane and the integration contours C_1 and C_∞ used in the evaluation of the integral in Eq.(11).

Ω . The Riemann surface on which $f(p, \Omega)$ is single-valued consists of eight sheets, each of which is defined by a combination of the sheets of $\sqrt{p - (\Omega/c_l^2)}$, $\sqrt{p - (\Omega/c_t^2)}$, and $\sqrt{p - (\Omega/c_0^2)}$. The branch cuts associated with these square roots are shown in Fig.1. The singular points of $f(p, \Omega)$ come only from the singularities of the functions $g_{\alpha\alpha}^{(I)}(k_\parallel, \omega | x_3, x_3)$. As in the case of two solids in contact [4], if we close the contour C_1 by a circle C_∞ of an infinitely large radius, as shown in Fig.1, $f(p, \Omega)$ is regular and has no singularities inside the closed contour $C = C_1 + C_\infty$. Then, with the aid of Cauchy's theorem, we obtain,

$$\tilde{G}_s^{(I)}(\Omega) = -\frac{1}{2i} \int_{C_\infty} dp f(p, \Omega). \quad (14)$$

The analysis of the behavior of $f(p, \Omega)$ for large $|p|$ shows that this function decreases as $1/|p|$ as $|p| \rightarrow \infty$. Because $|p| \gg \Omega/c_{t,l}^2$, Ω can be put equal to zero, and $\tilde{G}^{(S)}(\Omega) = -\pi \{pf(p, 0)\}$, i.e. is independent of Ω . Using Eqs.(10)–(10), we obtain

$$G_s^{(I)}(\omega) = -\frac{\rho S \omega}{2\pi} \int_0^\infty dx_3 \left\{ k_\parallel^2 g_{\alpha\alpha}^{(I)}(k_\parallel, 0 | x_3, x_3) \right\}, \quad (15)$$

where the expression in curly brackets in Eq.(15) does not depend on $k_\parallel$. Thus, we have proved that the interface contribution to the density of vibrational states is determined by the static ($\omega = 0$) Green's tensor of the system. This tensor is much easier to calculate than its dynamical version, which advantage becomes especially important in the more complicated case of anisotropic solids [7]. We must point out here that a

contour integration technique, similar to the one applied here and in [4], was used earlier by Iosilevskii [3]. One can see that the results of Ref.3 also imply that the interface density of states is determined by the static limit of the corresponding Green's tensor, although this was not stated explicitly in that work.

Substituting the explicit expressions for $g_{\alpha\beta}^{(I)}$, given in Appendix B, in the limit $\omega = 0$, into Eq.(15), we obtain

$$G_s^{(I)}(\omega) = \frac{S\omega}{8\pi c_s^2}, \quad (16)$$

where $c_s^2 = c_l^2 c_t^2 (c_l^2 - c_t^2) / (3c_l^4 - 3c_t^2 c_l^2 + 2c_t^4)$. A similar analysis in the evaluation of the liquid interface contribution to the density of states $G_l^{(I)}(\omega)$ leads to

$$G_l^{(I)}(\omega) = \frac{S\omega}{8\pi c_0^2}. \quad (17)$$

Equations (8), (16), and (17) illustrate the well-known fact that the spectral density for acoustic phonons has a frequency dependence proportional to ω^{d-1} , where d is the dimensionality of the system. In our case $d = 3$ for the bulk and $d = 2$ for the interface.

The solid interface contribution $G_s^{(I)}(\omega)$ to the density of states, given by Eq.(16), coincides with the one corresponding to the stress-free surface of that solid [4]. On the other hand, the liquid interface contribution (17) is the one which would be obtained in the case of a liquid bounded by a hard wall. Thus, the contributions from the solid and the liquid enter additively, which phenomenon we ascribe to the fact that the solid supports both longitudinal and transverse vibrations, whereas the liquid supports only longitudinal vibrations. This decoupling does not occur in the case of two solids in contact [4].

3. THE INTERFACE DEBYE MODEL

Since all four terms that contribute to the density of states (10) are known now, we will try to develop an interface Debye model in the same spirit as this was done in Ref.5. First, by analogy with the case of a single medium, we could try to require that $G(\omega)$ be normalized to the total number of degrees of freedom in the system, i.e. to $3(N_s + N_l)$. However, the resulting cut-off frequency ω_M does not depend on N_s and N_l through the combinations (N_s/V_s) and (N_l/V_l) as one would expect. Therefore, we disregard this model.

A consistent approach is obtained with the aid of the following example, which was suggested in [4] for the case of two solids in contact. We consider the same solid and liquid, occupying the half-spaces $x_3 > 0$ and

$x_3 < 0$, respectively, but now separated by a very narrow gap, so that we have two stress-free surfaces instead of the interface. As in the interface problem, we can define the Green's tensor (6) for the entire system of solid and liquid with the interface contribution replaced by surface contributions. However, it would be quite unreasonable to describe the system by a single density of states (7), and to normalize it to $3(N_s + N_l)$. Instead, each medium has to be described by its own density of vibrational states.

Returning to the interface problem, we claim that both the solid and the liquid possess their own vibrational spectra, perturbed by the presence of the interface. Hence, we require that each medium (s or l) be described by its own density of states,

$$G_{s,l}(\omega) = G_{s,l}^{(B)}(\omega) + G_{s,l}^{(I)}(\omega). \quad (18)$$

Then we normalize the functions $G_s(\omega)$ and $G_l(\omega)$ to $3N_s$ and $3N_l$, respectively, and obtain the following cut-off frequencies,

$$\omega_{Ms} = \omega_s \left[1 - \frac{S}{V_s} \frac{(6\pi^2)^{2/3} c_B^2}{144\pi c_s^2} \left(\frac{V_s}{N_s} \right)^{1/3} \right], \quad (19)$$

$$\omega_{Ml} = \omega_l \left[1 - \frac{S}{V_l} \frac{(18\pi^2)^{2/3}}{144\pi} \left(\frac{V_l}{N_l} \right)^{1/3} \right], \quad (20)$$

where $\omega_s = (6\pi^2)^{1/3} (N_s/V_s)^{1/3} c_B$ and $\omega_l = (18\pi^2)^{1/3} (N_l/V_l)^{1/3} c_0$ are the Debye cut-off frequencies for the unperturbed solid and liquid, respectively. This completes the construction of the Debye interface model.

It is now straightforward to calculate the interface vibrational contribution to the zero-point energy, as was done in [4] for two solids in contact, and the result is

$$\Delta F_0^{(I)} = -S \frac{\hbar}{96\pi} \left[\frac{\omega_s^3}{c_s^2} + \frac{\omega_l^3}{c_0^2} \right]. \quad (21)$$

We notice that unlike the case of two solids in contact [4], the contributions from the liquid and from the solid are additive. When $k_B T$ is small compared to both $\hbar\omega_s$ and $\hbar\omega_l$, the interface contribution to the temperature-dependent part of the Helmholtz free energy is given by

$$\Delta F^{(I)}(T) = \left(\frac{k_B T}{\hbar} \right)^3 \frac{\hbar S}{8\pi} \left[\frac{1}{c_s^2} + \frac{1}{c_0^2} \right] \zeta(3), \quad (22)$$

where $\zeta(z)$ is the Riemann zeta function. Finally, the low-temperature interface vibrational contribution to the specific heat is

$$\Delta C_V^{(I)}(T) = 3\pi \zeta(3) \frac{k_B^3 T^2 S}{\hbar^2} \left[\frac{1}{c_s^2} + \frac{1}{c_0^2} \right]. \quad (23)$$

The law $\Delta C_V^{(I)}(T) \propto ST^2$ agrees with the results obtained for the surface contribution to the specific heat of a semi-infinite solid [2, 7] and for the interface contribution to the specific heat of two solids in contact [1, 4].

4. CONCLUSIONS

In this paper we have studied the interface vibrational contribution to the low-temperature thermodynamic properties of a system of solid and liquid in contact across a planar interface. We showed that the interface contribution to the density of vibrational states is determined by the static Green's tensor of the system.

The main result of this work is the construction of the interface Debye model. Following the analysis of [4], we described each medium by its own density of vibrational states, which we normalized to the total number of vibrational degrees of freedom in that medium. This requirement results in modified Debye cut-off frequencies for both media. We have obtained the interface zero-point energy, the temperature-dependent part of the free energy, and the interface contribution to the specific heat for the solid-liquid system. The results are consistent with those obtained for similar interface problems [1, 3, 4].

Another result of the paper, which can be useful in solving other problems, is the dynamical Green's tensor for the system of solid and liquid separated by a planar interface. The elements of this tensor are given in Appendices A and B.

Acknowledgements—This work was supported in part by NSF Grant No. DMR 93-19404.

REFERENCES

1. B. Djafari-Rouhani and L. Dobrzynski, *Rev. B.* **14**, 2296 (1976).
2. V. R. Velasco and F. García-Moliner, *Surf. Sci.*, **83**, 376 (1979).
3. Ya. A. Iosilevskii, *Phys. Rev. B*, **20**, 1489 (1979).
4. A. V. Shchegrov, *Phys. Rev. B*, **53**, 13753 (1996).
5. P. Tabor and J. E. Rutledge, *Phys. Rev. Lett.* **68**, 2184 (1992); **69**, 937 (1992).
6. F. García-Moliner, *Ann. Phys. (Paris)*, **2**, 179 (1977).
7. K. Portz and A. A. Maradudin, *Phys. Rev. B*, **16**, 3535 (1977).
8. A. A. Maradudin and D. L. Mills, *Ann. Phys. (N.Y.)* **100**, 292 (1976).

Appendix A. BULK GREEN'S FUNCTIONS

In this Appendix we present the Green's functions $g_{\alpha\beta}^{(B)}$ and $g_{\alpha\beta}^{(I)}$. The technique for calculating these functions is quite standard, and can be found in [8] for similar problems. In the expressions for the Green's functions we use the definitions $\alpha_{l,t,0}(k_{\parallel}, \omega) = [k_{\parallel}^2 - (\omega + i\eta)^2/c_{l,t,0}^2]^{1/2}$, where the positive infinitesimal η defines the appropriate choice of the branch of the square roots, when the branch cut for the square root is taken along the negative real axis. The bulk Green's functions are given by the following expressions.

A.1. For the Solid

$$g_{11}^{(B,s)} = -\frac{k_{\parallel}^2}{2\rho\alpha_l\omega^2} \left[e^{-\alpha_l|x_3-x'_3|} - \frac{\alpha_l\alpha_t}{k_{\parallel}^2} e^{-\alpha_t|x_3-x'_3|} \right] \quad (\text{A.1})$$

$$g_{31}^{(B,s)} = \frac{ik_{\parallel}}{2\rho\omega^2} \text{sgn}(x_3 - x'_3) \left[e^{-\alpha_l|x_3-x'_3|} - e^{-\alpha_t|x_3-x'_3|} \right] \quad (\text{A.2})$$

$$g_{13}^{(B,s)} = \frac{ik_{\parallel}}{2\rho\omega^2} \text{sgn}(x_3 - x'_3) \left[e^{-\alpha_l|x_3-x'_3|} - e^{-\alpha_t|x_3-x'_3|} \right] \quad (\text{A.3})$$

$$g_{33}^{(B,s)} = -\frac{k_{\parallel}^2}{2\rho\alpha_t\omega^2} \left[e^{-\alpha_t|x_3-x'_3|} - \frac{\alpha_t\alpha_l}{k_{\parallel}^2} e^{-\alpha_l|x_3-x'_3|} \right] \quad (\text{A.4})$$

$$g_{22}^{(B,s)} = -\frac{1}{2\rho\alpha_t c_t^2} e^{-\alpha_t|x_3-x'_3|} \quad (\text{A.5})$$

A.2. For the Liquid

$$g_{11}^{(B,l)} = -\frac{k_{\parallel}^2}{2\rho_0\alpha_0\omega^2} e^{-\alpha_0|x_3-x'_3|} + \frac{1}{\rho_0\omega^2} \delta(x_3 - x'_3) \quad (\text{A.6})$$

$$g_{31}^{(B,l)} = -\frac{ik_{\parallel}}{2\rho_0\omega^2} \text{sgn}(x_3 - x'_3) e^{-\alpha_0|x_3-x'_3|} \quad (\text{A.7})$$

$$g_{13}^{(B,l)} = -\frac{ik_{\parallel}}{2\rho_0\omega^2} \text{sgn}(x_3 - x'_3) e^{-\alpha_0|x_3-x'_3|} \quad (\text{A.8})$$

$$g_{33}^{(B,l)} = \frac{\alpha_0}{2\rho_0\omega^2} e^{-\alpha_0|x_3-x'_3|} \quad (\text{A.9})$$

$$g_{22}^{(B,l)} = \frac{1}{\rho_0\omega^2} \delta(x_3 - x'_3). \quad (\text{A.10})$$

Appendix B. INTERFACE GREEN'S FUNCTIONS

We present the expressions for the interface Green's functions. In them we use the definitions $r_{\pm} = \frac{4\alpha_l\alpha_t k_{\parallel}^2 \pm (\alpha_t^2 + k_{\parallel}^2)^2}{4\alpha_l\alpha_t(\alpha_t^2 + k_{\parallel}^2)}$, $R_{\pm} = r_{\pm} \pm \frac{\rho_0}{\rho} \frac{\omega^4}{4c_t^4\alpha_0\alpha_t(\alpha_t^2 + k_{\parallel}^2)}$, and $\tilde{R}_{-} = r_{-} + \frac{\rho_0}{\rho} \frac{\omega^4}{4c_t^4\alpha_0\alpha_t(\alpha_t^2 + k_{\parallel}^2)}$.

B.1. For $x_3 > 0, x'_3 > 0$

$$g_{11}^{(I)} = -\frac{k_{\parallel}^2}{2\rho\alpha_l\omega^2 R_+} \left[\tilde{R}_- e^{-\alpha_l(x_3+x'_3)} + e^{-\alpha_l x_3 - \alpha_l x'_3} + e^{-\alpha_l x_3 - \alpha_l x'_3} + \frac{\alpha_l \alpha_t}{k_{\parallel}^2} R_- e^{-\alpha_l(x_3+x'_3)} \right] \quad (\text{B.1})$$

$$g_{31}^{(I)} = -\frac{ik_{\parallel}}{2\rho\omega^2 R_+} \left[\tilde{R}_- e^{-\alpha_l(x_3+x'_3)} + e^{-\alpha_l x_3 - \alpha_l x'_3} + \frac{k_{\parallel}^2}{\alpha_l \alpha_t} e^{-\alpha_l x_3 - \alpha_l x'_3} + R_- e^{-\alpha_l(x_3+x'_3)} \right] \quad (\text{B.2})$$

$$g_{13}^{(I)} = \frac{ik_{\parallel}}{2\rho\omega^2 R_+} \left[\tilde{R}_- e^{-\alpha_l(x_3+x'_3)} + \frac{k_{\parallel}^2}{\alpha_l \alpha_t} e^{-\alpha_l x_3 - \alpha_l x'_3} + e^{-\alpha_l x_3 - \alpha_l x'_3} + R_- e^{-\alpha_l(x_3+x'_3)} \right] \quad (\text{B.3})$$

$$g_{33}^{(I)} = -\frac{k_{\parallel}^2}{2\rho\alpha_l\omega^2 R_+} \left[\frac{\alpha_l \alpha_t}{k_{\parallel}^2} \tilde{R}_- e^{-\alpha_l(x_3+x'_3)} + e^{-\alpha_l x_3 - \alpha_l x'_3} + e^{-\alpha_l x_3 - \alpha_l x'_3} + R_- e^{-\alpha_l(x_3+x'_3)} \right] \quad (\text{B.4})$$

$$g_{22}^{(I)} = -\frac{1}{2\rho\alpha_l c_t^2} e^{-\alpha_l(x_3+x'_3)}. \quad (\text{B.5})$$

B.2. For $x_3 < 0, x'_3 < 0$

$$g_{11}^{(I)} = -\frac{k_{\parallel}^2}{2\rho_0\alpha_0\omega^2 R_+} e^{\alpha_0(x_3+x'_3)} \quad (\text{B.6})$$

$$g_{31}^{(I)} = \frac{ik_{\parallel}}{2\rho_0\omega^2 R_+} e^{\alpha_0(x_3+x'_3)} \quad (\text{B.7})$$

$$g_{13}^{(I)} = -\frac{ik_{\parallel}}{2\rho_0\omega^2 R_+} e^{\alpha_0(x_3+x'_3)} \quad (\text{B.8})$$

$$g_{33}^{(I)} = \frac{\alpha_0}{2\rho_0\omega^2 R_+} e^{\alpha_0(x_3+x'_3)} \quad (\text{B.9})$$

$$g_{22}^{(I)} = 0. \quad (\text{B.10})$$

B.3. For $x_3 > 0, x'_3 < 0$

$$g_{11}^{(I)} = \frac{k_{\parallel}^2}{2\rho\alpha_0\alpha_l\alpha_t c_t^2 R_+} \left[\frac{1}{2} e^{(-\alpha_l x_3 + \alpha_0 x'_3)} - \frac{\alpha_l \alpha_t}{k_{\parallel}^2 + \alpha_t^2} e^{(-\alpha_l x_3 + \alpha_0 x'_3)} \right] \quad (\text{B.11})$$

$$g_{31}^{(I)} = \frac{ik_{\parallel}}{2\rho\alpha_0\alpha_l c_t^2 R_+} \left[\frac{1}{2} e^{(-\alpha_l x_3 + \alpha_0 x'_3)} - \frac{k_{\parallel}^2}{k_{\parallel}^2 + \alpha_t^2} e^{(-\alpha_l x_3 + \alpha_0 x'_3)} \right] \quad (\text{B.12})$$

$$g_{13}^{(I)} = \frac{ik_{\parallel}}{2\rho\alpha_l\alpha_t c_t^2 R_+} \left[\frac{1}{2} e^{(-\alpha_l x_3 + \alpha_0 x'_3)} - \frac{\alpha_l \alpha_t}{k_{\parallel}^2 + \alpha_t^2} e^{(-\alpha_l x_3 + \alpha_0 x'_3)} \right] \quad (\text{B.13})$$

$$g_{33}^{(I)} = -\frac{1}{2\rho\alpha_l c_t^2 R_+} \left[\frac{1}{2} e^{(-\alpha_l x_3 + \alpha_0 x'_3)} - \frac{k_{\parallel}^2}{k_{\parallel}^2 + \alpha_t^2} e^{(-\alpha_l x_3 + \alpha_0 x'_3)} \right] \quad (\text{B.14})$$

$$g_{22}^{(I)} = 0. \quad (\text{B.15})$$

B.4. For $x_3 < 0, x'_3 > 0$

$$g_{11}^{(I)} = \frac{k_{\parallel}^2}{2\rho\alpha_0\alpha_l\alpha_t c_t^2 R_+} \left[\frac{1}{2} e^{(\alpha_0 x_3 - \alpha_l x'_3)} - \frac{\alpha_l \alpha_t}{k_{\parallel}^2 + \alpha_t^2} e^{(\alpha_0 x_3 - \alpha_l x'_3)} \right] \quad (\text{B.16})$$

$$g_{31}^{(I)} = -\frac{ik_{\parallel}}{2\rho\alpha_l\alpha_t c_t^2 R_+} \left[\frac{1}{2} e^{(\alpha_0 x_3 - \alpha_l x'_3)} - \frac{\alpha_l \alpha_t}{k_{\parallel}^2 + \alpha_t^2} e^{(\alpha_0 x_3 - \alpha_l x'_3)} \right] \quad (\text{B.17})$$

$$g_{13}^{(I)} = -\frac{ik_{\parallel}}{2\rho\alpha_0\alpha_l c_t^2 R_+} \left[\frac{1}{2} e^{(\alpha_0 x_3 - \alpha_l x'_3)} - \frac{k_{\parallel}^2}{k_{\parallel}^2 + \alpha_t^2} e^{(\alpha_0 x_3 - \alpha_l x'_3)} \right] \quad (\text{B.18})$$

$$g_{33}^{(I)} = -\frac{1}{2\rho\alpha_l c_t^2 R_+} \left[\frac{1}{2} e^{(\alpha_0 x_3 - \alpha_l x'_3)} - \frac{k_{\parallel}^2}{k_{\parallel}^2 + \alpha_t^2} e^{(\alpha_0 x_3 - \alpha_l x'_3)} \right] \quad (\text{B.19})$$

$$g_{22}^{(I)} = 0. \quad (\text{B.20})$$