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Black Holes, Branes, and Knots in String Theory

by

Kevin Alan Schaeffer

A dissertation submitted in partial satisfaction of the
requirements for the degree of

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in

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of the

University of California, Berkeley

Committee in charge:

Professor Mina Aganagic, Chair

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Spring 2013

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Kevin Alan Schaeffer

Abstract

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Professor Mina Aganagic, Chair

String theory has proven to be fertile ground for interactions between physical and mathematical ideas. This dissertation develops several new points of contact where emerging mathematical ideas can be applied to stringy physics, and conversely where stringy insights suggest new mathematical structures. In the first half, I explain how the formula of Kontsevich and Soibelman (KS) can be used to compute the spectrum of stable BPS particles in Calabi-Yau compactifications and I show that dimer models can be used to prove the KS formula for walls of the second kind in toric Calabi-Yau manifolds. I also explain how dimer models give a way of associating integrable systems to Calabi-Yau threefolds and show that this map agrees with an existing gauge-theoretic map. The second half of this dissertation focuses on a background in M-theory that defines the refined topological string. I explain how orientifolds can be introduced into this background, leading to new integrality conditions on amplitudes and to new invariants for torus knots. Finally, I introduce a new duality that relates the refined counting of supersymmetric black hole entropy and refined topological string theory.

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Chapter. 1

Introduction

String theory is an ambitious enterprise – not only does it give a consistent way of combining quantum mechanics and general relativity, it also gives a unified framework for explaining the origin of all fundamental forces and matter. Historically, it was first studied as a fundamental theory by perturbatively quantizing the classical string, leading to the requirement of ten microscopic spacetime dimensions. This perturbative approach was useful for understanding the most basic spectrum of stringy excitations and for obtaining a coarse picture of the theory, but non-perturbative effects were poorly understood. This changed dramatically during the second string revolution in the '90s, leading to remarkable non-perturbative dualities and breakthroughs in our understanding of deep questions in quantum gravity such as black hole entropy [261] and holography [146, 204, 263, 281]. Although these developments greatly enriched our perspective on string theory, they also raised questions about whether strings should really be viewed as the fundamental degrees of freedom in “string” theory.

Much of the subsequent research in string theory has focused on obtaining a better understanding of stringy dualities in order to find the deep structures that underpin the theory. Remarkably, ideas from mathematics have emerged as the structures controlling many of the dualities that appear in string theory – this has given convincing evidence that the mathematical languages of geometry, topology, and combinatorics are the correct ones for understanding many phenomena. At the same time, mathematics has also emerged as a useful testing ground for string theory – although experimental tests of string theory are still far away, string theorists can use dualities to make many nontrivial mathematical predictions. The verification of these predictions by mathematicians has given powerful evidence for the consistency and power of string theory.

In this dissertation, I will discuss several mathematical structures that have emerged from studying supersymmetric string backgrounds. This work is centered around string theory on geometries of the form $\mathbb{R}^{3,1} \times X$ and M-theory on $\mathbb{R}^{4,1} \times X$, where X is a Calabi-Yau threefold. Originally, such backgrounds were studied as a way of pre-

serving supersymmetry while reducing the macroscopic dimensionality of space-time from 10 to 4, in agreement with everyday observations. However, it has also been realized that such compactifications are extremely rich mathematically – for example, the topological string is critical when its target space is a Calabi-Yau threefold ($\hat{c} = 3$). It is an open question whether this particular connection between mathematical beauty and physical relevance is simply coincidental or if it gives some explanation for the macroscopic dimensionality of our universe.

In the Chapter 2, I focus on IIA string theory on $\mathbb{R}^{3,1} \times X$ in the case where X is a noncompact Calabi-Yau threefold. Such backgrounds preserve $\mathcal{N} = 2$ supersymmetry in four dimensions (8 supercharges). It is particularly interesting to study the spectrum of stable BPS branes wrapping the cycles of X . A remarkable fact is that as the moduli of X are varied, certain branes can become unstable and the spectrum can jump. These jumps occur along real codimension-one “walls” in the moduli space and a fundamental problem is to predict the form of these jumps under such “wall crossing.”

This wall crossing phenomenon was first pointed out by Seiberg and Witten in their solution of pure $\mathcal{N} = 2$ Yang-Mills – they realized that for their theory to be consistent, infinitely many dyons that are stable at large VEVs would have to decay in the interior of the moduli space [250]. Progress in understanding wall crossing from a physical perspective has been made in [12, 27, 28, 58, 73, 84, 95, 97, 124–126, 175, 236]. However, the greatest leap forward was made by Kontsevich and Soibelman (KS) [188], who proposed a mathematical formula that determines the jump in spectrum in complete generality. In this chapter, I explain how the KS formula can be tested by studying the intricate spectrum of BPS branes in toric Calabi-Yaus with compact four-cycles. For these geometries, I show that the KS formula for crossing walls of the second kind can actually be proven using the tool of dimer models which can be used to count BPS states. I also discuss the connections between crossing walls of the second kind and Seiberg duality in quiver quantum mechanics, and conclude by explaining how topological string theory emerges in the limit of large B-field. This work was done in collaboration with Mina Aganagic and appeared previously in [16]

In Chapter 3, I focus on Calabi-Yau compactifications in M-theory which engineer five-dimensional supersymmetric gauge theories. Starting with the work of [142, 222], it has been known that four- and five-dimensional gauge theories with 8 supercharges are intimately related to algebraic integrable systems. The connection is made through the low energy solution of these gauge theories, which naturally has the structure of a torus-fibration over the Coulomb branch. Since these gauge theories are geometrically engineered by compactifying M-theory or string theory on a Calabi-Yau, this gives a way of assigning integrable systems to threefolds.

Recent work by Goncharov and Kenyon (GK) gave a mathematically precise map from dimer models to integrable systems [137]. As explained in Chapter 2, there is also a very useful assignment of dimer models to Calabi-Yau manifolds. Combining these two observations gives a method for obtaining explicit integrable systems from

Calabi-Yau threefolds. This raises the natural question: do the two methods of finding integrable systems (the gauge-theoretic method and the mathematical GK method) give the same results? In this chapter, I explicitly show that this is true for $Y^{p,0}$ Calabi-Yaus which are known to engineer pure $SU(p)$ supersymmetric gauge theories. This work was carried out in collaboration with Richard Eager and Sebastian Franco, and previously appeared in [103].

Chapters 4 and 5 focus on a one-parameter generalization of the topological string, known as “refined topological string theory.” To put this generalization into context, it helps to recall the origin of topological string theory and its connection with string duality. In [276], it was discovered that two-dimensional $\mathcal{N} = (2, 2)$ sigma models could be topologically twisted, making them independent of the worldsheet metric. Depending on the kind of twist (A or B), the topological theory is also independent of the complex structure of the target space (A twist) or independent of the Kähler structure of the target (B twist). It was further explained in [276] that the twisted sigma model can be coupled to worldsheet gravity in a way analogous to the bosonic string, leading to a topological string theory.

Although originally defined on the worldsheet, it was quickly realized that topological string theory can also be defined as computing F-terms in low-energy supergravity after Type II string theory has been compactified on a Calabi-Yau threefold, X [45]. Equivalently, the topological string partition function can be computed in M-theory by counting $M2$ -brane bound states wrapping cycles in X [91, 138, 139]. It is this last perspective that is especially valuable for generalization – these $M2$ -branes can be counted in a more refined way, giving an implicit definition of refined topological string theory [162, 171, 225]. Ideally, there would also be an explicit worldsheet definition of the theory, but such a definition is currently unknown¹ and is not necessary to perform many computations.

Taking this definition as a starting point, much progress has been made in computing the partition function of the refined topological string in both the closed and open cases [2, 18, 20, 21, 60, 90, 91, 95, 172]. In Chapter 4, I explain how to introduce orientifolds into both refined and ordinary topological string computations – this leads to new integrality conditions for the partition functions in both cases. It also gives a way of defining open topological string theory, and therefore, a refinement of $SO(N)$ Chern-Simons theory. This can then be used to compute a refined generalization of the Kauffman polynomial for torus knots, and further leads to a new instance of large N duality. All of these computations are made possible by the intricate mathematical structure of D -type Macdonald Polynomials. This work was done in collaboration with Mina Aganagic and appeared previously in [15].

Finally, in Chapter 5 I discuss how refined topological string theory is related to black holes. This work generalizes the seminal discovery of Ooguri, Strominger, and Vafa (OSV) that the mixed partition function of BPS black holes is computed by the

¹For some recent progress in this direction see [29, 221]

square of the topological string partition function [235]. The OSV conjecture was explicitly tested in [4], where the authors studied noncompact Calabi-Yau threefolds given by two line bundles over a Riemann surface. In that paper they studied the analogue of large black holes by wrapping N D4-branes over the Riemann surface and one complex line bundle, and counting D2 and D0 branes bound to it. They found that this counting was in fact computed by a two-dimensional gauge theory that is closely related to Chern-Simons theory, which they called “q-deformed Yang-Mills.” In the large N limit, this gauge theory partition function is equal to the square of the topological string on that Calabi-Yau, in agreement with the OSV conjecture.

In fact, all aspects of this correspondence can be generalized to the refined case. I explain how a refined counting of $D4/D2/D0$ bound states can be defined and is equal to a two-dimensional gauge theory, (q, t) -deformed Yang Mills, which is closely related to refined Chern-Simons Theory. At the same time, it is straightforward to compute the partition function of refined topological string theory on these geometries. Taking the large N limit of (q, t) deformed Yang-Mills, I find that it is equal to the square of the refined string theory partition function, giving evidence for a refinement of the OSV conjecture. Mathematically, this duality suggests that in the limit of large rank, the Hodge numbers of certain instanton moduli spaces should encode refined Calabi-Yau invariants. This duality can also be viewed as a novel form of gauge/gravity duality, where both the gauge and gravity theories are defined by counting different BPS states in M-theory. This work was done in collaboration with Mina Aganagic and appeared previously in [17].

Chapter. 2

Wall Crossing, Quivers, and Crystals

2.1 Introduction

There has been remarkable recent progress in understanding the spectra of BPS states of $\mathcal{N} = 2$ theories in 4 dimensions, driven in part by the mathematical conjectures of Kontsevich and Soibelman [189]. KS conjectured how the degeneracies of BPS states change as we cross walls of marginal stability. In some cases, we have a physical understanding of why the conjectures of [189] are true. In particular, for BPS states in a gauge theory, the results of [189] have been explained in [124, 125], and from a different perspective in [58] (see also the very recent [126])¹.

In this chapter we do three things. First, we give the physical explanation for the “walls of the second kind” in [189]. Second, we provide further evidence that the conjectures of [189] are true – by proving that the spectrum of BPS bound states of a D6 brane with D4, D2 and D0 branes wrapping toric Calabi-Yau manifolds satisfies them, for certain walls. The spectrum here is in general very complicated (far more so than in the examples studied so far, corresponding either to four dimensional gauge theories [98], or Calabi-Yau manifolds without compact four cycles, where the generating function of the spectrum is computable in closed form [13, 73, 176, 239]). Finally, we use this to shed new light on a relation between familiar objects: topological strings, Calabi-Yau crystals, and BPS D-branes. These were studied previously in [173, 212, 213, 234, 240].

2.1.1 Walls of the Second Kind and Seiberg Dualities

There are two distinct phenomena that affect the BPS spectrum. One, which was most studied in the literature, e.g. in [84], corresponds to crossing a wall of marginal

¹See [236] for a matrix model perspective, and [71, 260, 265] for more mathematical progress.

stability where central charges of a pair of states align. There, the degeneracies of bound states of the pair change. When the BPS states are described in a quiver gauge theory, we cross the wall by varying the Fayet-Iliopoulos terms.

The other phenomenon is the “wall of the second kind” of [189], which we interpret as a kind of Seiberg duality that changes the basis of BPS branes and the split of the spectrum into the branes and the anti-branes. In the quiver gauge theory, this can be made precise. The basis of the BPS branes is provided by the nodes of the quiver. The branes are described as linear combinations of these with positive coefficients which are the ranks of the quiver gauge groups. The walls of the second kind correspond to varying the gauge couplings g^{-2} of the nodes. When one of them passes through zero, we need to change the description to a new quiver, related to the original one by Seiberg duality [41, 44, 54, 109]. This replaces the brane of the corresponding node with its anti-brane. The action of Seiberg duality on the nodes of the quiver is exactly the change of basis that appears in [189]. Once the basis vectors change, the possible brane bound states they can form change as well. It is worth emphasizing that the combined spectrum of BPS branes and anti-branes does not change at a wall of the second kind. No central charges have aligned, so no states can decay across this wall. However, the spectrum of only the BPS branes (which we define as states with a central charge in the upper half-plane) will be affected by these walls.

We will work out examples of this, starting from a simple one, with an acyclic quiver, in section 2.2. In section 2.3, we set this up in more detail, in the context of D3 branes wrapping three-cycles of a local Calabi-Yau Y . Y is related by mirror symmetry to a toric Calabi-Yau X , which maps the D3 branes to D6, D4, D2 and D0 branes, as we review in section 2.5.

2.1.2 Dimer models, Seiberg Dualities, and BPS Degeneracies

Recently, [218] gave a remarkably simple way of computing degeneracies of BPS bound states of one D6 brane on a toric Calabi-Yau X , with D4 branes, D2 branes and D0 branes, generalizing the earlier work of [264] to essentially arbitrary toric Calabi-Yau singularities. Adding a D6 brane corresponds to extending the D4-D2-D0 quivers² by a node of rank 1 [239], in a manner which we make precise. While in principle BPS degeneracies of a quiver are computable for any given choice of ranks as an Euler characteristic of the moduli space, the direct computations become cumbersome as ranks increase. Instead, as we review in section 2.4, [218] give a combinatorial way to compute the degeneracies for any ranks, by counting perfect matchings of certain dimer models on a plane. (This can be rephrased in terms of counting melting crystals, as we will explain shortly.) The relevant dimers are the lift to $\mathbf{R}^2$ of dimers on $\mathbf{T}^2$ that correspond to D4-D2-D0 quivers in [110, 117].

²These were studied extensively in [33, 35, 41, 44, 54, 86, 100, 109, 117, 152, 156, 158–160, 274, 275].

We show that crossing the walls of the second kind that take a quiver Q to its Seiberg dual Q' corresponds to a simple geometric transition in the dimer. In the case of the dimer on $\mathbf{T}^2$, this was known from [117], and what we have here is a simple lift of this from the dimers on $\mathbf{T}^2$ to the dimers on the plane. Since perfect matchings of the planar dimer count BPS states, this gives a geometric description for how the spectrum jumps. We show (in section 2.4) that this can be used to prove the KS wall crossing formula in the context of the quivers of [218].

In section 2.6 we give another example of wall crossing of the second kind corresponding to varying the B-field on X . Since shifting the B-field brings us back to the same point in the moduli space of X , this should leave the spectrum of BPS states invariant. Indeed, we show that if one turns on compact B-field, $B \in H_{\text{compact}}^2(X, \mathbf{Z})$, this corresponds to a sequence of Seiberg dualities of the quiver, but in the end the quiver comes back to itself. Shifting by a B-field in $B \in H^2(X, \mathbf{Z})/H_{\text{compact}}^2(X, \mathbf{Z})$ also corresponds to a change of basis of branes and Seiberg duality, but this time the quiver does not come back to itself. Instead, this change of basis permutes the quivers describing D6 branes with different amounts of noncompact D4 brane charge.

2.1.3 Calabi-Yau Crystals, Quivers and Topological Strings

There is an intriguing connection between the dimers that appear in [218] and an earlier appearance of dimers in the context of the closed topological string, in [234]. As explained in [181, 218, 234, 239] there is a close relation between dimer models in the plane, with suitable boundary conditions, and three dimensional melting crystals. We show that the melting crystals of [218] have a beautifully simple geometric description: the crystal sites are a discretization of the Calabi-Yau geometry. The shape of the crystal is determined by the geometry of the Calabi-Yau base, which is a singular cone at the point in the moduli space where the quiver is defined. The crystal sites are the integral points of the Calabi-Yau. The precise microscopic structure of the crystal depends on the quiver (and changes under Seiberg dualities). The refinement comes from the fact that we are not counting bound states of the D6 brane with D0 branes, but the more general bound states with D4, D2 and D0 branes, corresponding to splitting of the D0 branes into fractional branes. We show that, increasing the B-field by $D \in H^2(X, \mathbf{Z})$, D6 brane bound states are counted by the crystal that takes the shape of the Calabi-Yau X with Kahler class D , while the microscopic structure of the crystal does not change.

The relation of topological string amplitudes on X with certain melting crystals was observed in [234]. The Calabi-Yau crystals that arise in this chapter are the same as those in [173, 234], but only in the limit of infinite D , where the microscopic structure of the crystal is lost. The crystals in [234] were interpreted in [173] as counting bound states of a D6 brane on X , formulated as the Witten index of a non-commutative $\mathcal{N} = 2$ SYM on X . The observations of [173] are known as the Gromov-Witten/Donaldson-Thomas correspondence [212, 213]. We thus provide a

new derivation of the Gromov-Witten/Donaldson-Thomas correspondence from the perspective of the $0+1$ dimensional quiver quantum mechanics, but only in the limit of large D . This is in agreement with [84], which pointed out that the correspondence of [173] can hold only in the limit of infinite B-field (this was also verified in [13] when X has no compact 4-cycles). This is described in section 2.7.

2.2 Walls of the Second Kind and Seiberg Duality

Consider BPS states from D-branes wrapping cycles in a Calabi-Yau Y . For definiteness, take IIB string theory, so that the BPS particles are labeled by charge

$$\Delta \in H_3(Y, \mathbf{Z}).$$

The mass of the BPS state and the supersymmetry it preserves are determined by the central charge,

$$Z(\Delta) = \int_{\Delta} \Omega,$$

where Ω is the $(3,0)$ form on Y . For an arbitrary Calabi-Yau Y , the BPS spectrum will be quite complicated and may not have a simple quiver description. However, in the neighborhood of a shrinking two- or four-cycle, we can take Y to be a local Calabi-Yau and the D-brane spectrum can be described by quiver gauge theories in $0+1$ dimensions. The nodes of the quiver correspond to a basis of $H_3(Y, \mathbf{Z})$. Any bound state of the branes with charge Δ can be written as

$$\Delta = \sum_{\alpha} N_{\alpha} \Delta_{\alpha}, \quad N_{\alpha} \geq 0. \quad (2.2.1)$$

The corresponding quiver quantum mechanics is a

$$G_{\Delta} = \prod_{\alpha} U(N_{\alpha})$$

gauge theory. The quiver is a good description when all the central charges of the nodes are nearly aligned. By choosing an overall phase of Ω , we can write

$$Z(\Delta_{\alpha}) = \frac{i}{l_s^3 g_{\alpha}^2} + l_s \theta_{\alpha},$$

corresponding to all Z 's being close to imaginary. Above, θ_{α} is the Fayet-Iliopoulos parameter, and g_{α} is the gauge coupling in the quantum mechanics. The BPS degeneracy of a state Δ is determined by the Witten index

$$\Omega_Q(\Delta) = \text{Tr}_{\Delta, Q}(-1)^F, \quad (2.2.2)$$

of the quiver with gauge group G_Δ .³ Thus, the quiver gauge theory provides both a basis of D-branes, and a means to compute the degeneracies.

At walls in the moduli space, the spectrum of BPS states can change. One kind of wall is a wall of marginal stability, where central charges of two states, for example of two nodes, align. At the wall, even though (2.2.2) is an index, it can jump, since the one particle states it is counting can split, or pair up. In the quiver, since we are near the intersection of walls anyhow, crossing this wall corresponds to some combination of FI parameters passing through zero.

Before we go on, note that the quiver only describes the states with non-negative N_α . This is just as well, since the central charges of anti-branes $-\Delta_\alpha$ would be anti-parallel to the rest of the states in the quiver. The states with some of the N_α 's negative and some positive do not form bound states, and the quiver does not miss anything⁴. Moreover, this remains true even when the central charges of Δ_α are *not* nearly aligned, as long as they all remain in the upper half of the complex plane – simply because there are no walls where Δ_β and $-\Delta_\alpha$ can align. The fact that only the states in the upper half of the complex Z -plane bind, implies that the spectrum of BPS particles can change as a node Δ_* leaves the upper half of the complex plane, and correspondingly $-\Delta_*$ enters it.⁵ The node Δ_* leaves the upper half of the complex plane when g_*^{-2} passes through zero. This wall, in real codimension one of the moduli space, was called the “wall of the second kind” in [189].

On the wall, $g_*^{-2} = 0$, the gauge coupling of the node is infinitely strong. Even though in $0 + 1$ dimensions the gauge fields have no local dynamics, the coupling enters the action as the coefficient of the kinetic terms for the fields on the brane, e.g., $\frac{1}{g_*^2} \int |\partial\phi|^2$. We can continue past infinite coupling since clearly nothing special happens to the D-branes, as long as we stay away from $Z(\Delta_*) = 0$, but we need to change description. To make g_*^2 positive, we need to flip

$$\Delta_* \rightarrow \Delta'_* = -\Delta_*. \quad (2.2.3)$$

The existence of the two dual descriptions of the same state, related by a continuation past infinite coupling (2.2.3), is a Seiberg duality [41, 44, 54, 109, 249]. In fact, Δ_* is not the only node that changes. Were we to complete the circle around $Z(\Delta_*) = 0$, all of the nodes would have changed due to monodromy, which maps Δ to $\Delta \pm (\Delta \circ \Delta_*) \Delta_*$, for any Δ , with the sign depending on the path. Going halfway around the circle, there are partial monodromies, which are Seiberg dualities [44]. On the other side of

³From the space time perspective, we are computing $-2\text{Tr}F^2(-1)^F$. The F^2 factor serves to absorb the contribution of zero modes on $\mathbf{R}^{3,1}$. This reduces then to $\text{Tr}(-1)^F$ where one traces internal degrees of freedom only [84].

⁴There are of course the bound states with all N_α negative, but these are just CPT conjugates of the states at hand.

⁵We reiterate that the full spectrum of BPS particles and anti-particles does not change, but the spectrum of states in the upper half-plane does.

the wall, the theory is described by a dual gauge theory, based on $\{\Delta'_\alpha\}$ and a new quiver Q' .

A state Δ in (2.2.1), can have descriptions both in terms of Q and Q' . In addition to the one based on Q and G_Δ , it may have another one, in terms of Q' with gauge group

$$G'_\Delta = \prod_{\alpha} U(N'_\alpha)$$

where

$$\Delta = \sum_{\alpha} N'_\alpha \Delta'_\alpha, \quad (2.2.4)$$

if N'_α are non-negative. Crossing the “wall of the second kind,” the spectrum can change because the spaces of BPS particles one can obtain from Q and Q' (by varying FI parameters, while staying in the upper half of the complex plane) are different. In the terminology of KS, this is a change of t -structure.

2.2.1 KS conjecture and Seiberg duality

Kontsevich and Soibelman [189] predict how the degeneracies $\Omega_Q(\Delta)$ change as the central charges Z are varied in some way. For each charge $\Delta \in H_3(Y, \mathbf{Z})$, associate an operator e_Δ , satisfying

$$[e_\Delta, e_{\Delta'}] = \hbar(-1)^{\Delta \circ \Delta'} \Delta \circ \Delta' e_{\Delta + \Delta'} \quad (2.2.5)$$

and

$$e_\Delta e_{\Delta'} = (-1)^{\Delta \circ \Delta'} e_{\Delta + \Delta'},$$

where $\Delta \circ \Delta'$ is the intersection product on $H_3(Y)$, and an operator

$$A_\Delta = \exp\left(\frac{1}{\hbar} \sum_{n=1}^{\infty} \frac{e_n \Delta}{n^2}\right).$$

Consider the product ⁶

$$A_Q = \prod_{\Delta}^{\rightarrow} A_\Delta^{\Omega_Q(\Delta)} \quad (2.2.6)$$

taken over all states Δ with central charges in the upper half of the complex plane, and in the order of increasing phase of the central charges $Z(\Delta)$. The KS conjecture states that if we cross a wall of the first kind, the BPS degeneracies adjust so that the product, taken over the states with central charges in the upper half plane, is

⁶In general, the degeneracies $\Omega_Q(\Delta)$ in the KS formula may differ from the physical BPS degeneracies by a Δ -dependent sign. This additional sign factor will be important in section 2.4, when we use the KS formula to compute wall crossing for quiver gauge theories.

invariant⁷. However, it is important to note that this KS product is only invariant if no states enter or leave the upper half plane.

Now consider what happens as we cross a wall of the second kind where the coupling g_*^2 of node Δ_* flips sign. Near the wall, the chamber corresponding to the quiver is as in the Figure 2.1. As we cross the wall, the state Δ_* leaves the upper half plane from the left, and the state $-\Delta_*$ enters it from the right, so that the new degeneracies should satisfy

$$A'_Q = A_{\Delta_*}^{-1} A_Q A_{-\Delta_*}. \quad (2.2.7)$$

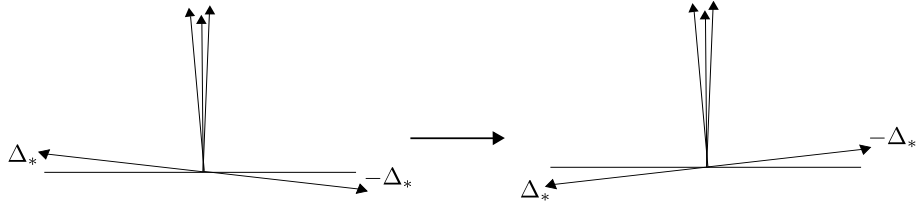


Figure 2.1: Wall of the Second Kind. As we cross this wall, the central charge of Δ_* leaves the upper half plane.

Crossing the wall as above, we generally do not come back to the same point in the moduli space, so A_Q and A'_Q are not themselves equal, not even after a change of basis. However, the spectrum should be determined uniquely by the point in the moduli space we are at. Going around a closed loop in the moduli space, the spectrum of BPS states must come back to itself, up to a monodromy that relabels the branes. In particular, it should not matter which way we go around the singularity.

2.2.2 A simple example

As a simple example, consider the quiver Q with two nodes, Δ_1 and Δ_* , and one arrow, corresponding to the intersection number $\Delta_1 \circ \Delta_* = 1$. With the central charges as in Figure 2.2, there are only three BPS states, and the operator A_Q is simply [189]

$$A_Q = A_{\Delta_1} A_{\Delta_1 + \Delta_*} A_{\Delta_*}. \quad (2.2.8)$$

Rotating the central charge of Δ_* counterclockwise, eventually $Z(\Delta)$ and $Z(\Delta_*)$ align, and we cross a wall of the first kind. The bound state $\Delta_1 + \Delta_*$ decays, and the

⁷More precisely, [189] restrict to a “strict” wedge, meaning one that subtends an angle less than 180° . This simply tells us how to define A_Q exactly on the wall, when states Δ_α and $-\Delta_\alpha$ both have central charges on the real line.

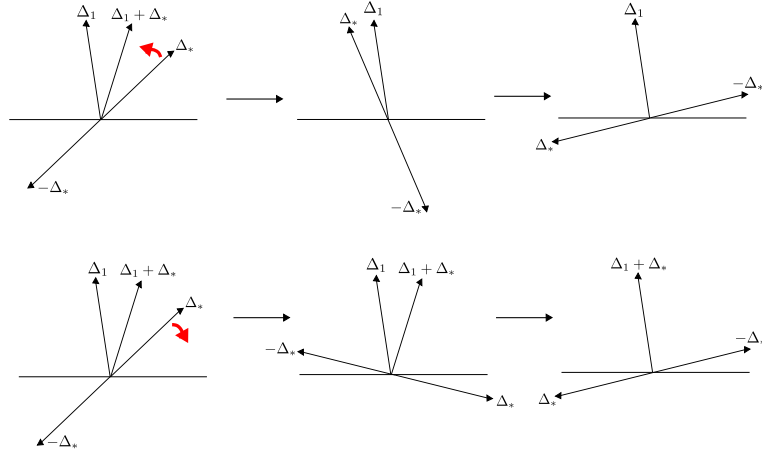


Figure 2.2: Rotating the central charge of Δ_* counterclockwise (above) and clockwise (below). Note that in both cases we cross a wall of the first kind and a wall of the second kind.

product becomes⁸

$$A_Q = A_{\Delta_*} A_{\Delta_1}.$$

Continuing past this, eventually the gauge coupling of Δ_* becomes negative, so Δ_* leaves the upper half of the complex plane, and $-\Delta_*$ enters it. To get a good description, we need to change the quiver from Q , to Q' with nodes Δ_1 , $-\Delta_*$. This corresponds to Seiberg duality on the node Δ_* (This quiver and its Seiberg dualities were studied in detail in [44].) with

$$A_{Q'} = A_{\Delta_*}^{-1} A_Q A_{-\Delta_*} = A_{\Delta_1} A_{-\Delta_*}.$$

Now, we could have reached the same point in the moduli space by starting with (2.2.8) and rotating $Z(\Delta_*)$ clockwise instead. Then Δ_* leaves the upper half of the complex plane, $-\Delta_*$ enters it. We again need to dualize node Δ_* , but now we get the basis $\Delta_1 + \Delta_*$, $-\Delta_*$ corresponding to quiver Q'' . Moreover,

$$A_{Q''} = A_{-\Delta_*} A_Q A_{\Delta_*}^{-1} = A_{-\Delta_*} A_{\Delta_1} A_{\Delta_1 + \Delta_*}.$$

To get this to correspond to the same point in the moduli space as Q' above, we need to keep rotating $Z(\Delta_*)$, until $A_{Q''}$ becomes

$$A_{Q''} = A_{\Delta_1 + \Delta_*} A_{-\Delta_*}.$$

Since Q' and Q'' now correspond to two quivers describing physics at exactly the same point in the moduli space, they are of course equivalent. The non-trivial relation

⁸We used the pentagon identity $A_{\Delta_1} A_{\Delta_1 + \Delta_*} A_{\Delta_*} = A_{\Delta_*} A_{\Delta_1}$ which holds for any two states Δ_1 , Δ_* with intersection number +1 [189].

between $A_{Q''}$ and $A_{Q'}$ is a consequence of the fact that to relate them, we need to go once around the $Z(\Delta_*) = 0$. In doing so, there is a monodromy acting on the cycles that maps

$$\Delta \rightarrow \Delta + (\Delta \circ \Delta_*)\Delta_*,$$

looping counter-clockwise, which is precisely how these are related. As an aside, note that, taking for example the state $\Delta_1 + \Delta_*$ of quiver Q , depending on which way we go around the singularity there are two different interpretations for its fate. Along the path corresponding to Q' , the state decays into Δ_1 and Δ_* on the wall of marginal stability where Δ_* and Δ_1 align. From the perspective of the split attractor flows [81], the flow corresponding to $\Delta_1 + \Delta_*$ splits on this wall into a flow corresponding to Δ_* , which crashes at $Z(\Delta_*) = 0$, and an honest black hole attractor corresponding to Δ_1 . Along the path corresponding to Q'' , the state crosses no walls, but the monodromy changes $\Delta_1 + \Delta_*$ to Δ_1 . Following either flow, the attractor point is the same, as expected from [81].⁹

2.3 Quivers from Calabi-Yau Threefolds

In this section we will review (following [54]) the quiver gauge theories Q that arise for certain choices of a Calabi-Yau Y , and its moduli. One reason we choose these theories is that one has a very direct, geometric understanding of what happens to Q as the central charges are varied, and the theory undergoes Seiberg dualities. The second reason is that the choice we make implies that the quiver theory has extra symmetries – Q are the so called toric quiver gauge theories of [110, 117] (see [180] for an excellent review). The torus symmetries will allow us to extract very precise information about the quantum BPS spectra of Q , in the next section. The presence of the extra symmetries is related to the fact, aspects of which we will review in the next section, that Y is mirror to a toric Calabi-Yau X .

Consider a Calabi-Yau Y given by

$$W(e^x, e^y) = w \quad uv = w. \quad (2.3.9)$$

We can view Y as a fibration over the w plane. At a generic point in the w -plane, the fiber is a product of a cylinder $uv = w$, and a Riemann surface $W(e^x, e^y) = w$. Over special points, the fiber degenerates. Over a point q with $w(q) = 0$, the S^1 of the cylinder pinches. The 1-cycles of the Riemann surface degenerate over critical points of $W(e^x, e^y)$,

$$p_\alpha : \partial_x W = 0 = \partial_y W, \quad \alpha = 1, \dots, r \quad (2.3.10)$$

⁹The change of basis of BPS states was also recently discussed in [27], from the attractor viewpoint.

where $w(p_\alpha) = w_\alpha$. For each p_α , a path in the w -plane connecting it with q , together with an $S^1 \times S^1$ fiber over it, gives an S^3 which we will denote

$$\Delta_\alpha, \quad \alpha = 1, \dots, r$$

Above, one of the S^1 's corresponds to the cylinder, while the other corresponds to the 1-cycle of the Riemann surface pinching at p_α). The three-cycles Δ_α provide a basis for the compact homology of Y . We can associate a quiver to the above singularity by considering D3 branes wrapping the cycles Δ_α . Since the Δ_α 's are spheres, the D-branes wrapping them have no massless adjoint matter. For a collection of

$$\Delta = \sum_{\alpha} N_{\alpha} \Delta_{\alpha}, \quad N_{\alpha} \geq 0. \quad (2.3.11)$$

branes, we get a

$$G = \prod_{\alpha} U(N_{\alpha}) \quad (2.3.12)$$

quiver gauge theory, with Δ_α corresponding to node α of the quiver. Moreover, for every point of intersection of Δ_α and Δ_β we get a massless chiral bifundamental in either $(N_\alpha, \bar{N}_\beta)$, or $(N_\beta, \bar{N}_\alpha)$ representation of the gauge group. Pairs of these can get mass, so the net number of chiral multiplets going *from* node α to node β is

$$n_{\alpha\beta} = \Delta_\alpha \circ \Delta_\beta.$$

As we vary the moduli of Y , the locations of critical points p_α in the w plane change. If the critical point p_α crosses the cycle Δ_β , due to monodromy, the homology of the cycle Δ_β changes to Δ'_β

$$\Delta'_\beta = \Delta_\beta \pm (\Delta_\beta \circ \Delta_\alpha) \Delta_\alpha, \quad (2.3.13)$$

where in (2.3.13), Δ_β stands for the original homology class of the cycle (see Figure 2.3).

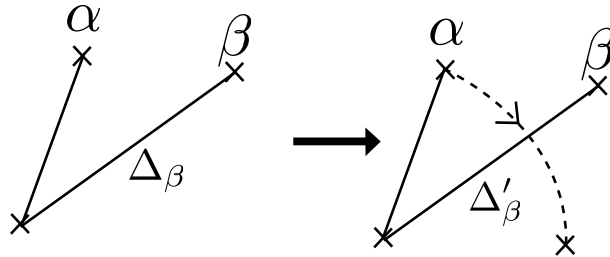


Figure 2.3: Picard lefschetz monodromy as α passes through the Δ_β cycle.

The fact that the homology classes of the cycles change implies that the quiver changes. For example, the intersection numbers $n_{\alpha\beta}$ change, and with them the number of arrows connecting the two nodes of the quiver. Consider varying the moduli so that the gauge coupling of the nodes Δ_* becomes negative. In the process, p_* crosses the cycles Δ_{β_i} , so that the basis of branes changes to

$$\begin{aligned}\Delta'_* &= -\Delta_* \\ \Delta'_{\beta_j} &= \Delta_{\beta_j} \pm n_{\beta_j*}\Delta_* \\ \Delta'_{\gamma_k} &= \Delta_{\gamma_k}\end{aligned}\tag{2.3.14}$$

This implies that the intersection numbers change as

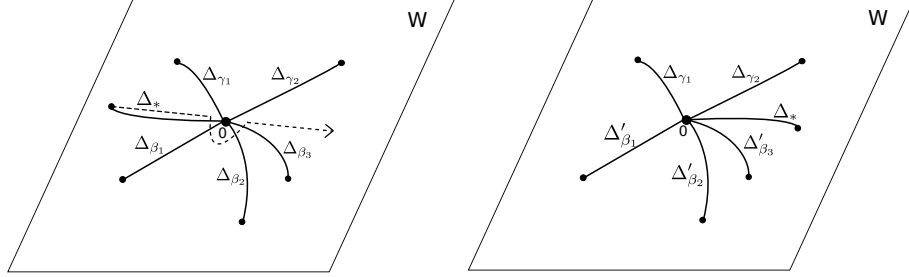


Figure 2.4: The Δ_* cycle is shown shrinking and then growing in the opposite direction. In the process, Δ_* crosses the $\{\Delta_{\beta_i}\}$.

$$\begin{aligned}n'_{*\beta_k} &= -n_{*\beta_k} \\ n'_{\beta_k\gamma_j} &= n_{\beta_k\gamma_j} \pm n_{\beta_j*}n_{*\gamma_k} \\ n'_{\gamma_j*} &= -n_{\gamma_j*}\end{aligned}\tag{2.3.15}$$

In addition, for the D3 brane charge to be conserved, the ranks of the quiver have to change, for the D-brane charge to be conserved. The numbers N_α of the branes on the node α have to change to $N_{\alpha'}$, so that

$$\Delta = \sum_{\alpha} N_{\alpha} \Delta_{\alpha} = \sum_{\alpha} N'_{\alpha} \Delta'_{\alpha},$$

in order to be consistent with (2.3.13).

Note that (2.3.14) and (2.3.15) are *exactly* the same changes of basis as on p. 134 of [189]. The transformation is a Seiberg duality of the quiver gauge theory [35, 157] when either

$$n_{*\beta_k} > 0, \quad n_{*\gamma_j} \leq 0,$$

for all β_k, γ_j , or with the direction of inequalities reversed – depending on the sign in (2.3.14).¹⁰ The choice of sign determines which way around the singularity we go. The new quiver Q' is obtained from Q by (i) reversing the arrows beginning or ending on the node $*$ we dualized. The reversed arrows correspond to new chiral fields associated with the dual node, all of whose intersection numbers have flipped signs. (ii) The original bifundamentals transforming under the node $*$ are confined in the bifundamental mesons that no longer transform under gauge transformations on node $*$. (iii) There is an additional gauge invariant coupling of the mesons to the new bifundamentals charged under the dual node. These can however pair up with the existing bifundamentals of opposite orientations and disappear, since only the net intersection number, counted with signs is invariant. The net effect of (ii) and (iii) is to give a mass to all but $n_{\beta_k \gamma_j} \pm n_{\beta_j * n_{* \gamma_k}}$ bifundamentals between the node β_k and γ_j .

2.3.1 The mirror of $\mathbf{P}^1 \times \mathbf{P}^1$ example

Consider for example,

$$W(e^x, e^y) = e^x + z_t e^{-x} + e^y + z_s e^{-y} + 1, \quad (2.3.16)$$

where $z_t = e^{-t}$, $z_s = e^{-s}$. There are four critical points in the W plane, $w_\alpha = \pm 2\sqrt{z_t} \pm 2\sqrt{z_s} + 1$.

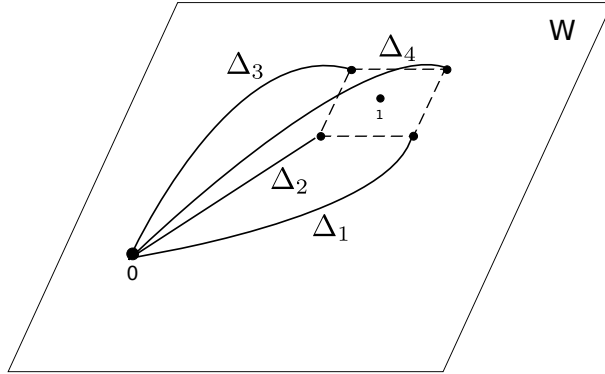


Figure 2.5: The W plane for the mirror of $\mathbf{P}^1 \times \mathbf{P}^1$.

The four corresponding S^3 's are drawn in Figure 2.5, in the limit

$$z_s, z_t \rightarrow 0.$$

¹⁰When this is not the case, one should still get a dual description, though it may not correspond to the quiver gauge theory one would naively obtain. For example due to the presence of tachyons, the intersection numbers from the geometry need not count chiral multiplets. See, e.g. [35, 274].

The intersection numbers of the cycles were determined in [153] . The quiver Q that results has four nodes, connected by arrows

$$n_{32} = n_{34} = n_{21} = n_{41} = 2, \quad n_{13} = 4.$$

The theory also has a superpotential, computed by the topological A-model on a disk,

$$\mathcal{W} = \sum_{i,j,a,b=1,2} \epsilon^{ij} \epsilon^{ab} (\text{Tr} B_a A_i D_{jb} + \text{Tr} \tilde{A}_i \tilde{B}_a D_{jb}).$$

where the fields are labeled in Figure 2.7. As we vary the complex structure moduli of Y , the gauge coupling of one of the nodes, say node 2 can become negative. This can be achieved by sending

$$z_s, z_t \rightarrow \infty,$$

keeping the ratio $z_s/z_t = e^{-T}$ fixed. As we vary z 's the cycles deform as in the Figure 2.6 [54] , corresponding to

$$\begin{aligned} \Delta'_1 &= \Delta_1 + 2\Delta_2, \\ \Delta'_2 &= -\Delta_2, \\ \Delta'_3 &= \Delta_3, \\ \Delta'_4 &= \Delta_4. \end{aligned} \tag{2.3.17}$$

We chose the labeling of the new nodes in a manner that will be useful later.

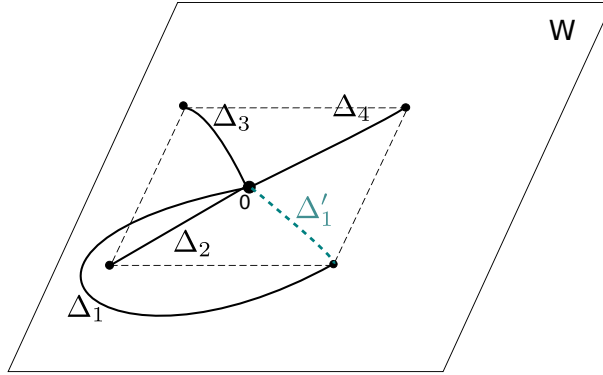


Figure 2.6: Deformation of the cycles as $z_s, z_t \rightarrow \infty$. Since Δ_1 is deformed, the good cycle in this limit is now Δ'_1 .

This implies that the non-vanishing intersection numbers are now

$$n'_{12} = n'_{23} = n'_{34} = n'_{41} = 2.$$

This results in a new quiver, Q' , given in the Figure 2.7. The superpotential of the theory also changes, and becomes¹¹

$$W = \sum_{i,j,\alpha,\beta} \epsilon^{ij} \epsilon^{\alpha\beta} \text{Tr} B_\alpha A_i B'_\beta A'_j.$$

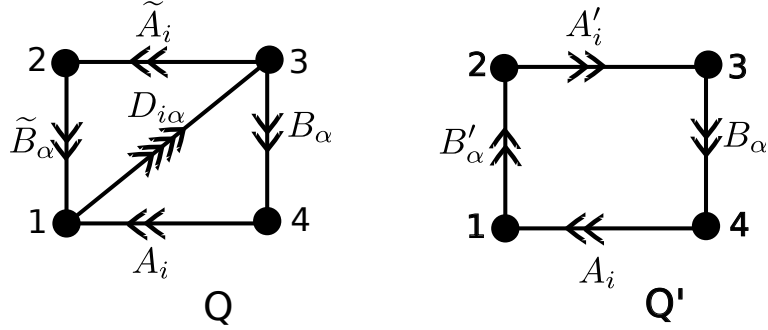


Figure 2.7: The Quivers for $\mathbf{P}^1 \times \mathbf{P}^1$. The Q phase corresponds to $z_s, z_t \rightarrow 0$ while the Q' phase corresponds to $z_s, s_t \rightarrow \infty$.

2.3.2 Toric Quivers and Dimers on a Torus

The two quiver gauge theories above are examples of the “toric” gauge theories [117] ([180] contains an excellent summary). The quiver Q of a toric gauge theory can be represented as a periodic quiver on a T^2 torus. The periodic quiver gives a tiling of the torus which turns out to encode not just the quiver, but also the superpotential W . The terms in the superpotential correspond to the plaquettes on the torus defined by the quiver, whose boundaries are the bifundamental matter fields and where the orientation of the boundary of a plaquette determines the sign of the term. This implies, for example, that a given matter field enters precisely two superpotential terms, with opposite signs¹². The dual graph, with faces and nodes exchanged, is per definition a bipartite graph on the torus. The bi-coloring of the nodes is determined

¹¹The dual theory is obtained by Seiberg duality – instead of $\tilde{A}_i, \tilde{B}_\alpha$ we introduce two new pairs of fields A'_i, B'_α , with opposite orientation. The original fields are confined in mesons $M_{\alpha i} = \tilde{B}_\alpha \tilde{A}_i$, and the superpotential becomes

$$W = \sum_{i,j,\alpha,\beta} \epsilon^{ij} \epsilon^{\alpha\beta} (\text{Tr} B_\alpha A_i D_{j\beta} + \text{Tr} M_{i\alpha} D_{j\beta} + \text{Tr} B'_\alpha A'_i M_{j\beta}).$$

The second term above makes both M and D massive, and they can be integrated out. This results in the effective superpotential we wrote.

¹²Any additional coefficients can be set to 1 by a field redefinition.

by the sign of the corresponding superpotential term. Moreover, the edges of the dual graph connect pairs of nodes of different colors.

The structure is in part a consequence of mirror symmetry [110], and the fact that Y is fibered by three-tori T^3 . If we view the T^3 as an S^1 fibration over T^2 , where the S^1 fiber corresponds to the $uv = w$ cylinder, and the T^2 is mapped out by the phases of x and y coordinates. Consider a D3 brane, wrapping a generic T^3 fiber (this is mirror to a D0 brane on X), and let it approach $w = 0$ in the base. There, the S^1 fibration degenerates over a graph on the T^2 cut out by the Riemann surface $W(e^x, e^y) = 0$. In the nice cases, corresponding to the quiver being toric, the fibration is such that this sections the D3 brane out into plaquettes, which are the plaquettes of the bipartite graph [5, 110]. In particular,

$$T^3 = \sum_{\alpha} \Delta_{\alpha}$$

this implies that the moduli space of the $U(1)^r$ quiver gauge theory is the mirror manifold X , since mirror symmetry maps D3 brane on T^3 to D0 brane on X .

The toric quiver gauge theories have a global

$$T = U(1)^2 \times U(1)_R$$

symmetry. The $U(1)^2$ symmetry is inherited from the $U(1)^2$ symmetry of the Calabi-Yau, and leaves the superpotential invariant. The $U(1)_R$ is an R -symmetry, under which the superpotential is homogenous, of degree 2.

There is a simple geometric relation between bipartite graphs of a pair of dual quivers [117]. The transformation is local, acting only on the face of the bipartite graph we dualize. We replace the face corresponding to the dualized node of the quiver (in the present case, this is node 2), with a dual face. The dual face is the copy of the original, but with the colors of all the vertices reversed. For this to fit into the original bipartite graph consistently, we add a link connecting each original vertex bonding the face to its dual of opposite color. The new links that appear in this way correspond to mesons. Finally, we can erase links corresponding to massive fields. The result is the bipartite graph corresponding to the dual gauge theory. An example of this, relating the dimers of quivers Q and Q' is in the figure 2.8.

2.4 BPS Degeneracies and Wall Crossing from Crystals and Dimers

There is a combinatorial way to compute the BPS degeneracies of toric quiver theories [218, 239, 262] in terms of enumerating certain melting crystal configurations, or equivalently [218, 234], by counting dimer configurations. In the language of dimers,

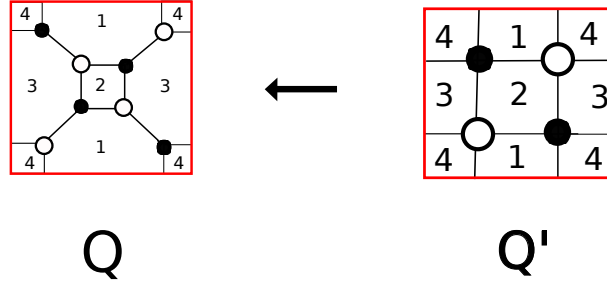


Figure 2.8: The dimer models on T^2 for Q and Q' are related by dualizing face 2.

Seiberg duality becomes geometric. We will use this to prove that the BPS degeneracies of two Seiberg dual quivers satisfy the relations (2.2.7) for a certain infinite class of states. (As we will see in the next section, these will turn out to be bound states of a single D6 brane bound to D4, D2 and D0 brane wrapping cycles in a mirror toric Calabi-Yau X .)

2.4.1 BPS states of Quivers and Melting Crystals

The BPS degeneracy of a state Δ is the Witten index

$$\Omega_Q(\Delta) = \text{Tr}_{Q,\Delta}(-1)^F$$

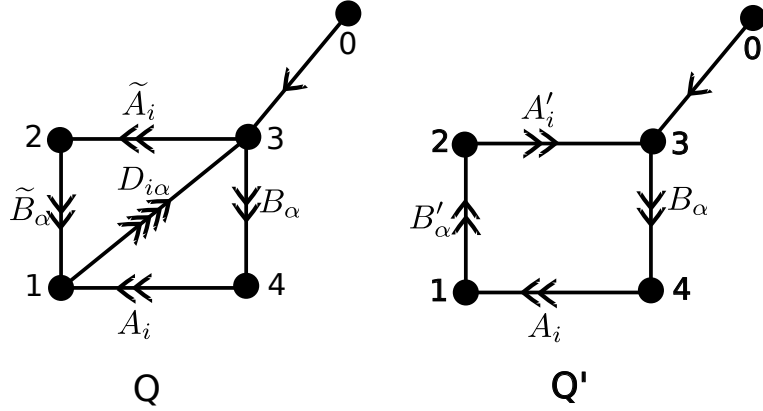
of the quiver Q with gauge group G_Δ . It can be computed as the Euler character of the moduli space of the quiver, defined by setting F- and D-terms to zero, and dividing by the gauge group. In practice, this is doable for any fixed Δ , but it quickly gets cumbersome.

There is a combinatorial way to compute the degeneracies $\Omega_Q(\Delta)$, for any Δ , for a toric gauge theory, using the torus $\mathbf{T}$ symmetry and localization. The price to pay is that the results correspond to degeneracies not of the quiver Q alone, but of its extension by adding one extra node of rank 1. In Figure 2.9 we show the extension of the two dual quivers we considered above by an node $\alpha = 0$ and an arrow.

The extended quiver has the same superpotential as before, since there are no gauge invariant operators we can add. We will return to the physical meaning of this extension in a moment¹³, but for now we simply explain the statement [218, 239].

Recall that a quiver defines a path algebra A , whose elements are all paths on the quiver obtained by joining arrows in the obvious way, where we consider equivalent two paths related by F-term constraints. Since we allow paths on the quiver Q that wind around the torus arbitrary numbers of times, consider the lift to a periodic quiver on the plane, Q_{R^2} . Let A_0 be the subspace of A , corresponding to paths

¹³This was discussed in [73] in the case of the conifold, and in [239] for toric Calabi-Yau without compact 4 cycles.

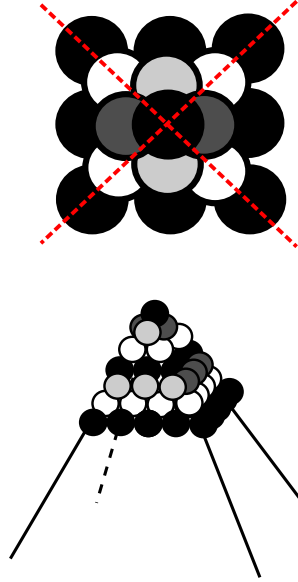
Figure 2.9: The extended quivers for $\mathbf{P}^1 \times \mathbf{P}^1$.

starting on node 0. The $\mathbf{T}$ -charge of the arrows in the quiver assigns $\mathbf{T}$ charge to paths in A_0 . A theorem of [42, 218] states that any two elements of A_0 with the same $\mathbf{T}$ charge are equivalent modulo F-terms. The set of $\mathbf{T}$ charges of endpoints of A_0 is a three-dimensional crystal $\mathcal{C}$, which is a cone (see Figure 2.10). Keeping track of only the $U(1)^2$ charge, $\mathcal{C}$ projects down to the two-dimensional planar quiver we started with. The $U(1)_R$ charge defines height of the nodes, making the crystal three dimensional.¹⁴

Starting from $\mathcal{C}$ one can explicitly construct $\mathbf{T}$ -invariant solutions to F-term equations. A melting of the crystal is an ideal $\mathcal{C}_\Delta$ such that if a path p is in $\mathcal{C}_\Delta$, then pa is also in $\mathcal{C}_\Delta$ for any path a in A . $\mathcal{C}_\Delta$ is obtained from $\mathcal{C}$ by removing N_α sites corresponding to node α where $\Delta = \sum_\alpha N_\alpha \Delta_\alpha$. The melting crystal configurations $\mathcal{C}_\Delta$ are in one-to-one correspondence with $\mathbf{T}$ -fixed solutions to F-term equations corresponding to quiver Q , with gauge group G_Δ .¹⁵ We will choose the FI parameters θ_α in such a way that every solution we constructed is stable – they can solve the D-term constraints. This corresponds to $\theta_0 > 0$ and $\theta_{\alpha \neq 0} < 0$. Physically, this means

¹⁴It is crucial for the structure of the crystal that there exist a $U(1)_R$ symmetry so that every path has positive R-charge. One choice is the $U(1)_R$ symmetry that can be geometrically realized in the dimer model as an “isoradial embedding” of the bipartite graph in the plane [180].

¹⁵To sketch this, consider the finite set of sites we removed to get $\mathcal{C}_\Delta$ from $\mathcal{C}$. The set of sites corresponding to node α give vector spaces V_α of Chan-Paton factors of rank $\dim(V_\alpha) = N_\alpha$. The algebra A is represented on this by matrices, with non-zero entries corresponding to paths in the crystal. By construction, these satisfy F-term constraints. The vector spaces V_α come with the grading by the torus $\mathbf{T}$ charge assigned to them by paths in A_0 . The solutions to the F-term equations we obtained above are fixed under the torus action that transforms the V_α and elements of A by their corresponding weights. We still need to impose D-term constraints, and divide by the subgroup of the gauge group G_Δ that is preserved by the solution. Since the nodes of V_α all carry different $\mathbf{T}$ charge, G_Δ is broken to the maximal abelian subgroup. We thank D. Jafferis for discussions and explanations of this point.

Figure 2.10: The crystal for local $\mathbf{P}^1 \times \mathbf{P}^1$.

that the central charges of the nodes Δ_α are all roughly aligned, and at an angle with the central charge of the node Δ_0 . Then, setting D-terms to zero and dividing by the gauge group is equivalent to dividing by the complexified gauge group. Doing so, the fixed points in the quiver moduli space are isolated, and counting them reduces to enumerating crystal configurations $\mathcal{C}_\Delta$.

Counting T -fixed points in the classical moduli space gives $Tr_\Delta(-1)^F$ up to a sign $(-1)^{d(\Delta)}$, corresponding to the fermion number of the fixed point (which determines whether the BPS multiplet is bosonic or a fermionic). Thus, the BPS degeneracies of the quiver are obtained by counting melting crystal configurations, up to sign

$$\Omega_Q(\Delta) = \sum_{\mathcal{C}_\Delta} (-1)^{d(\Delta)} = (-1)^{d(\Delta)} \chi(\mathcal{M}_Q^\Delta),$$

where the sum is over crystals $\mathcal{C}_\Delta$ with charge Δ nodes removed, and where we have also written the degeneracies in terms of the euler characteristic of the quiver moduli space $\mathcal{M}_Q^\Delta$ with fixed ranks given by Δ . The sign is computed by

$$d(\Delta) = \sum_{\alpha} N_{\alpha}^2 + \sum_{\alpha \rightarrow \beta} N_{\alpha} N_{\beta} \quad (2.4.18)$$

where the last sum is over all the arrows in the quiver Q connecting nodes α , and β , including node 0. This counts the dimension of the tangent space to the fixed point set.

Finally, let us define a generating function for the degeneracies

$$Z_Q(q) = \sum_{\Delta, c_\Delta} (-1)^{d(\Delta)} q^\Delta \quad (2.4.19)$$

where q^Δ is the chemical potential, induced by giving weight q_α to node α , i.e.

$$q^\Delta = \prod_{\alpha > 0} q_\alpha^{N_\alpha}.$$

2.4.2 Wall Crossing and Crystals

We have seen that the degeneracies of $N_0 = 1$ BPS states of a toric quiver Q extended by a node, can be computed by counting crystal configurations. Consider now two toric quivers Q, Q' related by dualizing a node Δ_* . We will prove that the degeneracies computed by the corresponding crystals satisfy the wall crossing formulas of [189] .

The wall crossing formula (2.2.7) , predicts that degeneracies corresponding to two quivers Q and Q' are related by

$$A_{Q'} = A_{\Delta_*}^{-1} A_Q A_{-\Delta_*}, \quad (2.4.20)$$

together with a change of basis following from (2.3.14)

$$\begin{aligned} e'_{\Delta_*} &= e_{-\Delta_*} \\ e'_{\Delta_{\beta_j}} &= e_{(\Delta_{\beta_j} + n_{\beta_j*} \Delta_*)} \\ e'_{\Delta_{\gamma_k}} &= e_{\Delta_{\gamma_k}} \end{aligned} \quad (2.4.21)$$

where $n_{\beta_j*} > 0$, and $n_{\gamma_k*} < 0$. (In (2.4.20) we made a particular choice of the route around the singularity. The other choice is related to this by monodromy, which does not affect the degeneracies, but relabels the charges.) In the above, A_Q contains the information about the BPS states with any N_0 , and not just $N_0 = 1$ states that are counted by the crystal. To apply this to the present context, consider a truncation of the algebra (2.2.5) to operators e_Δ with $N_0 = 0$. Denote by $A_Q^{(0)}$ the restriction of A_Q operator product to states with vanishing N_0 . This can be implemented by setting $e_{\Delta_0} = 0$. Then, (2.4.20) reads the same, just restricted to $A_Q^{(0)}$.

Next, restrict to operators e_Δ , with $N_0 = 0, 1$. (Note that this implies that any two operators with $N_0 = 1$ commute.) By our choice of the FI parameters, the central charges of all the states with $N_0 = 0$ are approximately aligned, and the central charges of all the states with $N_0 = 1$ are also aligned, but at an angle to the former.¹⁶ To this order, A_Q reduces to a product $A_Q^{(0)} A_Q^{(1)}$. Then, (2.4.20) implies that

¹⁶Note that this implies that $|Z(\Delta_\alpha)| \ll |Z(\Delta_0)|$, an assumption that we will justify in the next section, when we identify node 0 with a D6 brane wrapping X .

$A_Q^{(1)}$ transforms by conjugation with $A_{-\Delta_*}$:

$$A_{Q'}^{(1)} = A_{-\Delta_*}^{-1} A_Q^{(1)} A_{-\Delta_*}. \quad (2.4.22)$$

To compare $A_{Q'}^{(1)}$ and $A_Q^{(1)}$, we in addition need to redefine the variables using (2.4.21). Note that, in addition to crossing the wall of the second kind, corresponding to a Seiberg duality, we have adjusted the FI terms so that the computation of [218] applies. In other words, all the central charges of all the $\Delta'_{\alpha \neq 0}$ are aligned, and at an angle to Δ'_0 .

Since we are only interested in states with $N_0 = 1$, and using that conjugation by A_{Δ} acts as

$$A_{\Delta} e_{\Delta'} A_{\Delta}^{-1} = (1 - e_{\Delta})^{\Delta \circ \Delta'} e_{\Delta'}$$

for any two Δ, Δ' , we can equivalently rewrite (2.4.22) as

$$\sum_{\Delta} \chi(\mathcal{M}_{Q'}^{\Delta}) e_{\Delta} = \sum_{\Delta} \chi(\mathcal{M}_Q^{\Delta}) (1 - e_{-\Delta_*})^{\Delta_* \circ \Delta} e_{\Delta}.$$

Note that we have written this KS formula in terms of the (unsigned) moduli space euler characteristics rather than the true (signed) BPS invariants. As noted in section 2, this is because KS naturally counts the euler characteristic moduli spaces without the additional sign. However, this sign is naturally restored in the change of variables below. The sum is over all Δ with $N_0 = 1$. Finally, let us write how the partition function transforms. Writing

$$e_{\Delta} \rightarrow (-1)^{d(\Delta)} q^{\Delta},$$

and noting that

$$e_{\Delta} e_{\Delta'} = (-1)^{\Delta \circ \Delta'} e_{\Delta + \Delta'}$$

we can write

$$Z_{Q'}(q') = \sum_{\Delta} \Omega_{Q'}(\Delta) q'^{\Delta} = \sum_{\Delta} \Omega_Q(\Delta) (1 - (-1)^{\Delta_* \circ \Delta} q_*^{-1})^{\Delta_* \circ \Delta} q^{\Delta}, \quad (2.4.23)$$

where we change the variables consistent with (2.4.21), in other words

$$\begin{aligned} q'_* &= (q_*)^{-1} \\ q'_{\beta_j} &= q_{\beta_j} (q_*)^{n_{\beta_j^*}} \\ q'_{\gamma_k} &= q_{\gamma_k}. \end{aligned} \quad (2.4.24)$$

Note that this is the semi-primitive wall crossing formula of [84], because always one of the products has $N_0 = 1$. When we consider passing through several walls of the second kind this will end up involving general decays, because the degeneracies of decay products at one wall can jump on the next. This gives us an explicit prediction for the degeneracies computed from one crystal in terms of the other, which one can check term by term. We can however do better, and *prove* that the BPS degeneracies of the two quivers are indeed related by (2.4.22). The proof is elementary, using the dimer point of view on the crystals.

2.4.3 Dimers and Wall Crossing

Recall that Seiberg duality relating quivers Q and Q' has a simple geometric realization in terms of bipartite graphs on T^2 . This will allow us to give a geometric proof of the wall crossing formula (2.4.23) in terms of dimers. But, for this we need to translate the counting of BPS states in the quiver from the language of crystals, which we used so far, to dimers.

Consider the bipartite graph dual to the periodic quiver Q_{T^2} on the torus. The lift of this to the covering space is a bipartite graph in the plane. The path set A_0 gives rise to a canonical perfect matching of the bipartite graph. Consider the paths in A_0 that lie on the surface of the crystal. These correspond to short paths in the planar quiver Q_{R^2} , paths containing no loops. These short paths define a set of paths on the bipartite graph, where they pick out a subset of edges crossed by them. The edges in the complement of this define a perfect matching m_0 of the bipartite graph [218, 239].

The finite melting crystal configurations are in one to one correspondence with perfect matchings m which agree with m_0 outside of a finite domain. Removing an atom in the crystal is rotating the dimers around the corresponding face in the bipartite graph. The difference $m - m_0$ of the two perfect matchings defines closed level sets on the bipartite graph. We can use them to define a height function whose value is 0 at infinity, and increases by ± 1 each time we cross a level set, depending on the orientation. Defined this way, the height function lets us keep track of the height of the sites we melt from the crystal. The figure 2.11 shows the planar dimers and their backgrounds m_0 corresponding to the two quivers Q and Q' . The

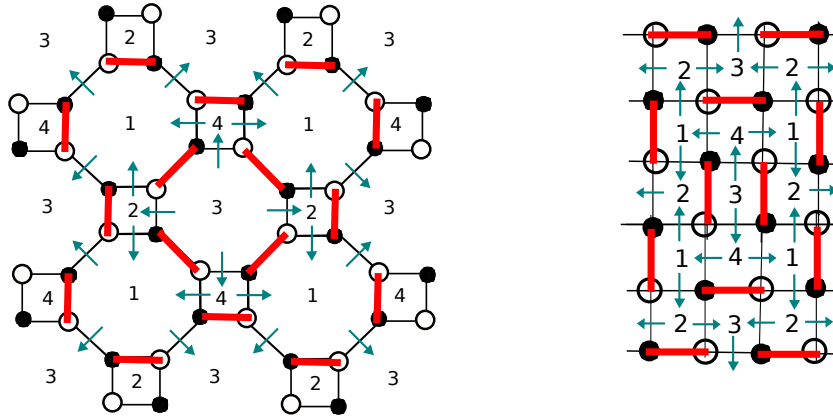


Figure 2.11: The vacuum perfect matchings, m_0 , for the dimer model associated to the quivers Q and Q' of local $\mathbf{P}^1 \times \mathbf{P}^1$.

partition function (2.4.19) can thus be written in terms of the dimer model as

$$Z_Q(q) = \sum_{m \in \mathcal{D}_Q} (-1)^{d(\Delta(m))} q^{\Delta(m)} \quad (2.4.25)$$

where $q^{\Delta(m)}$ is the weight of the dimer configuration, chosen to agree with the weights of the corresponding crystal [181, 234]. We assign a fixed weight $w(e)$ to every dimer e on the plane, in such a way that the product of weights of edges around a face, corresponding to node α of Q_{R^2} equals q_α . The edges have a natural orientation, from a white vertex say to black, and contribute to the product by $w(e)$ or $w(e)^{-1}$ depending on whether orientation of the edge agrees or disagrees with the orientation of the cycle. The weight of the dimer configuration m is a product over the weights of the edges in the dimer, $w(m) = \prod_{e \in m} w(e)$, normalized by $w(m_0) = \prod_{e \in m_0} w(e)$, and

$$q^{\Delta(m)} = \frac{w(m)}{w(m_0)}$$

The sign in (2.4.25) does not come from the weights (i.e. for a general Q it cannot be absorbed into the weights), but is added in by hand.

The duality transformation relating the two quivers Q and Q' has a simple geometric interpretation in terms of the bipartite graph, corresponding to replacing a face of the node we dualize, with the dual face. This operation lifts to perfect matchings of the corresponding bipartite graphs as well – from the perfect matching of one graph, we can obtain a perfect matching of the dual graph. Were the operation one to one, the degeneracies computed from one dimer and its dual would have been the same. The operation is in fact almost one to one, except for an ambiguity in the mapping of certain dimers on the faces we dualized.

Let $\mathcal{D}_Q$ and $\mathcal{D}_{Q'}$ be the sets of the perfect matchings of the two dual bipartite graphs in the plane (with the suitable asymptotics). Any two perfect matchings in $\mathcal{D}_{Q'}$ differing by “configurations of type 1” in Figure 2.12 come from the same configuration in $\mathcal{D}_Q$. In addition, any two perfect matchings in $\mathcal{D}_Q$ containing a “configuration of type 2” correspond to the same dimer in $\mathcal{D}_{Q'}$.

This means we can use perfect matchings of $\mathcal{D}_Q$ to enumerate perfect matchings of $\mathcal{D}_{Q'}$, provided we know the numbers $\#1(m)$, $\#2(m)$ of configurations of type 1 and type 2 in each perfect matching m of $\mathcal{D}_Q$. To count the perfect matchings in $\mathcal{D}_{Q'}$, we need to sum over all the perfect matchings of $\mathcal{D}_Q$, and compensate for configurations over- or under-counted. Schematically, in terms of counting dimer configurations without signs (later, we will be precise about the weights), this gives

$$\sum_{m \in \mathcal{D}_{Q'}} (q')^{\Delta(m')} = \sum_{m \in \mathcal{D}_Q} (1 + q_*)^{I(m)} q^{\Delta(m)} \quad (2.4.26)$$

where

$$I(m) = \#1(m) - \#2(m).$$

The factor

$$(1 + q_*)^{I(m)} = \frac{(1 + q_*)^{\#1(m)}}{(1 + q_*)^{\#2(m)}}$$

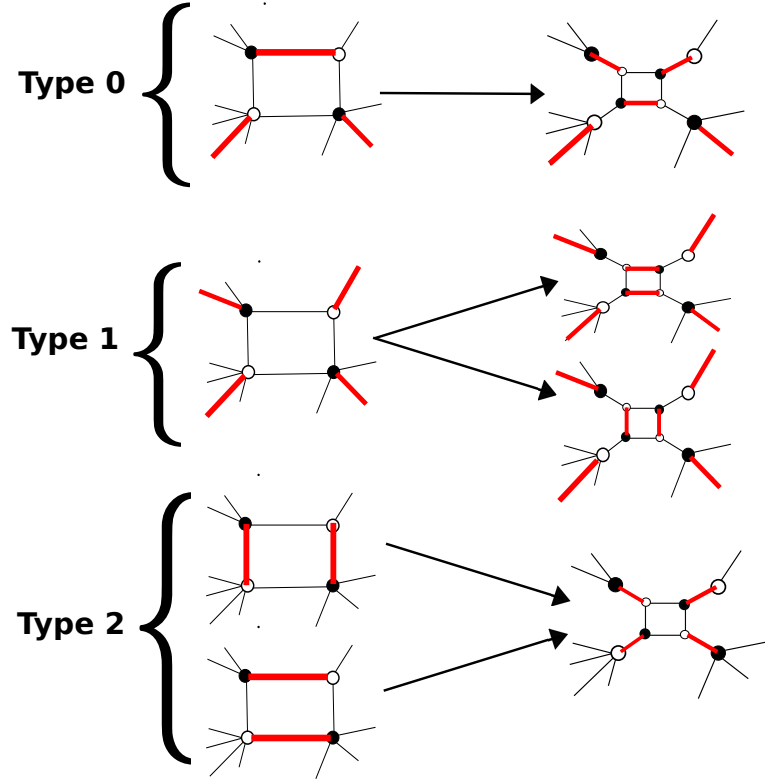


Figure 2.12: How dimer configurations transform under Seiberg Duality.

appears since in $\mathcal{D}_Q$ there are too many configurations of type 2, and too few of type configurations of type 1.

Notice that (2.4.23) and (2.4.26) are of the same form, provided $I(m)$, which we will call the “dimer intersection number”, depends only on the charge Δ of the dimer configuration m , and not on m itself, and equals the intersection number of Δ_* and Δ ,

$$I(m) = \Delta_* \circ \Delta,$$

where Δ_* is the node being dualized. In the next subsection, we will show that this is indeed the case. Using this, we will prove that the degeneracies of two Seiberg dual toric quivers indeed satisfy (2.4.23).

2.4.4 The proof

To be able to count the BPS states using dimers, both the original and the dual quiver need to be toric. The conditions for this were studied in [117]. The Seiberg duality needs to preserve the fact that a D3 brane wrapping the T^3 fiber corresponds to all ranks of the quiver being one. (This is required by the stringy derivation of

the relation of quivers and dimer models [110] , as we reviewed earlier.) For this to be the case, as is easy to see from (2.3.14) and charge conservation of $T^3 = \sum_{\alpha} \Delta_{\alpha}$, the node we dualize has to have two incoming and two outgoing arrows. Under this restriction, the most general face of the dimer model that can be dualized is shown in Figure 2.13.

Now consider some arbitrary matching m corresponding to charge Δ , with

$$\Delta = \sum_{\alpha} N_{\alpha} \Delta_{\alpha}.$$

If we denote the face to be dualized by, Δ_* , then it follows that its intersection with Δ is

$$\Delta_* \circ \Delta = \sum_{\alpha} N_{\alpha} n_{*\alpha} = N_2 + N_4 - N_1 - N_3 - n_{0*} \quad (2.4.27)$$

where we used the fact that $n_{*\alpha}$ is non-zero only for faces that share an edge with $*$, and its value, including the sign can be read off from the bipartite graph. We are using here the conventions of figure 2.13. The last term is not geometric, n_{0*} is the number of framing nodes from 0 to node $*$. In our setup so far, this is either 0 or 1.

We will use induction to find the dimer intersection number $I(m)$, and show it equals the physical intersection number (2.4.27). Starting with some arbitrary perfect matching, we consider the effect of melting an additional node. We will show that the $I(m)$ and $\Delta_* \circ \Delta$ change in the same way. In the end, we will show that the canonical perfect matching m_0 also satisfies the relation, so in fact any perfect matching does so as well.

In the dimer model, melting a node simply corresponds to exchanging occupied and unoccupied bonds along the perimeter of that face. Further, from the ingoing and outgoing arrows at face $*$, a bond on edge a corresponds to face 1 being melted and bond c corresponds to 3 melted, while b corresponds to 2 unmelted, and d corresponds to 4 unmelted. Now we observe that removing a bond from the perimeter of $*$ always changes the dimer intersection number by $+1$, since it either removes a type 2 configuration or adds a type 1 configuration. Thus we find that melting faces 1 or 3 both change the dimer intersection number by -1 and melting faces 2 or 4 change the dimer intersection number by $+1$.

To complete the inductive argument, we need to compute the dimer intersection number for the vacuum configuration. If $n_{0*} = 0$ then there are no “removable” $*$ faces in the initial dimer configuration so no type 2 configurations appear. Since the vacuum configuration can be seen as the complement of those bonds that intersect arrows in the planar quiver, there are also no type 1 configurations. Such a type 1 configuration would have both incoming and outgoing arrows present, but such a configuration cannot appear on the surface of the crystal. One way to understand this is to note that the vacuum dimer configuration breaks the rotational and translational symmetry of the bipartite graph by specifying the tip of the crystal. For any face in the bipartite graph (which must correspond to an atom on the surface of the crystal),

the direction toward the tip of the crystal is a preferred direction, and the dimer model reflects this preference. However, a type 1 configuration has symmetric arrows with no preferred direction. Thus, it cannot exist in the vacuum configuration.¹⁷ If $n_{0*} = 1$ then there is exactly one “removable” $*$ face in m_0 , which corresponds to a type 2 configuration, giving a dimer intersection number of -1. Combining these results, we find,

$$I(m) = -n_{0*} + N_2 + N_4 - N_1 - N_3$$

We will now tie this all together, and show that, explicitly mapping the dimer configurations of $\mathcal{D}_Q$ to configurations in $\mathcal{D}'_Q$ *together with their weights*, we reproduce the change of degeneracies (2.4.23), together with the change of basis (2.4.24). If we denote the dimer weight for configuration m by $w(m)$ and the partition function variables by $q^{\Delta(m)}$, then by definition,

$$q^{\Delta(m)} = \frac{w(m)}{w(m_0)}$$

Now we must decide how our weights transform under the duality. There is a large redundancy in the weight assignments, since the partition function (normalized by the weight of the canonical perfect matching) does not depend on the weights of the individual edges, but only on gauge invariant information, the products of weights around closed loops. A convenient choice (Figure 2.13) is one which does not change the weighting of Type 0 configurations, so that the weighting of the vacuum, $w(m_0)$ remains the same. We accomplish this by flipping the edge weights across the dualized face and assigning weight 1 to the meson edges. If we denote the old variables by $\{q_\alpha\}$ and the new variables by $\{q'_\alpha\}$ and take into account the change in orientation of arrows, we find,

$$\begin{aligned} q'_1 &= q_1(w_a w_c)^{-1} \\ q'_2 &= q_2(w_b w_d) \\ q'_3 &= q_3(w_a w_c)^{-1} \\ q'_4 &= q_4(w_b w_d) \\ q'_* &= q_*^{-1} = (w_a w_c)(w_b w_d)^{-1} \\ q'_\gamma &= q_\gamma \end{aligned} \tag{2.4.28}$$

where q_γ are correspond to nodes with no intersection with node $*$. Now we can

¹⁷We can also see this by breaking the planar dimer into a tiling of T^2 dimers, so that the vacuum configuration corresponds to a set of T^2 dimer configurations. These T^2 dimer matchings are in one-to-one correspondence with points on the toric diagram [117]. It has been conjectured [239] that only matchings corresponding to external points on the toric diagram appear along the surfaces of m_0 . A general T^2 matching containing a type 1 or type 2 configuration will always be an internal point in the toric diagram, and thus can only appear at the apex of the crystal.

explicitly transform the dimer configuration, together with its weight,

$$w(m) \rightarrow w'(m) = w(m) \frac{(w_b w_d + w_a w_c)^{\#_1}}{(w_b w_d + w_a w_c)^{\#_2}}.$$

We still have leftover arbitrariness in the weights, since nothing depends on w_a, w_b, w_c, w_d separately. We can set for example

$$w_b w_d = 1, \quad w_a w_c = q_*^{-1},$$

and then we get the correct change of variables for the Seiberg duality corresponding to $+$ sign in (2.3.14), and going around the singularity clockwise. This is because n_{3*}, n_{1*} are positive, n_{2*}, n_{4*} negative, and $n_{\gamma*}$ vanishes. This is also the choice we have been making in this section. Setting $w_a w_c = 1$, instead, the change of variables is correct for a Seiberg duality with a $-$ sign in (2.3.14). The two choices are related by full monodromy around $Z(\Delta_*) = 0$, which is just the change of variables at hand. Proceeding with the $+$ sign choice, (2.4.28) is the change of basis in (2.4.24). In addition, the contribution of dimer configuration $m \in \mathcal{D}_Q$ to $\mathcal{D}_{Q'}$ is obtained by replacing and

$$w(m) \rightarrow w'(m) = w(m)(1 + q_*^{-1})^{\#_1 - \#_2}.$$

To find the contribution to the partition function, we still need to divide by the weight of the vacuum configuration, so

$$w(m)/w(m_0) = q^{\Delta(m)} \rightarrow w'(m)/w'(m_0) = q^{\Delta(m)}(1 + (q_*)^{-1})^{\Delta_* \circ \Delta}.$$

Finally, to compute BPS degeneracies from the dimer configurations, we need to reintroduce the sign twists, which means replacing q^Δ by $(-1)^{d(\Delta)} q^\Delta$, and sum over all matchings m . This precisely reproduces (2.4.23) and (2.4.24), since $(-1)^{d(\Delta - \Delta_*)} = -(-1)^{d(\Delta)}(-1)^{(\Delta_* \circ \Delta)}$. Thus we have derived the KS wall crossing formula purely from how perfect matchings in the dimer model transform.

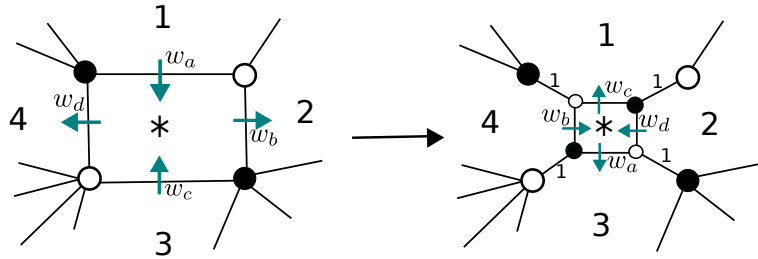


Figure 2.13: The effect of Seiberg Duality on face $*$ for a general brane tiling. Note that the dimer weights, w_i are flipped on the inner square and are 1 on the new legs.

2.5 Mirror Symmetry and Quivers

Mirror symmetry provides a powerful perspective on the quivers in section 2.3 and 2.4. The mirror of the manifold Y of section 2.3 is a toric Calabi-Yau manifold X . Mirror symmetry also maps D3 branes wrapping three-cycles in IIB on Y to D0-D2-D4-D6-branes wrapping holomorphic submanifolds in IIA string on X . In the mirror, many aspects of the quiver construction become more transparent. In particular, computing the quiver gauge theory on the branes becomes a question in the topological B-model on X , with the answers provided by a vast machinery of the derived category of coherent sheaves on X (see [32, 257] for excellent reviews). The goal of this section is to explain what are the BPS D-branes counted by the crystals in section 2.4. For the “small quivers” of section 2.2, this is well understood. The D3 branes on compact three cycles of Y , are mirror to D4, D2, and D0 branes wrapping compact submanifolds of X [54, 164]. The specific combinations of branes involved correspond to collections of spherical sheaves on the surface S as we will review. The extended quivers of section 2.4 correspond to adding a D6 brane wrapping all of X [239]. We will show that, which node of the D4-D2-D0 quiver ends up extended, is determined by a choice of a suitable bundle on the D6 brane. These are essentially the tilting line bundles of [33, 35, 215].¹⁸ Having understood this, the mirror perspective gives a simple interpretation to some of the Seiberg dualities, as turning on B-field on X . The effect on the quiver ends up depending on the class of B in $H^2(X, \mathbf{Z})/H^2_{\text{compact}}(X, \mathbf{Z})$, as we will explain in the next section.

2.5.1 Mirror Symmetry

The Calabi-Yau manifold Y given by (2.3.9)

$$W(e^x, e^y) = w \quad uv = w$$

is mirror to IIA on X , where X is a toric Calabi-Yau threefold. The monomials $w_i = e^{m_i x + n_i y}$ in W satisfy relations

$$\prod_i w_i^{Q_i^a} = e^{-t_a}$$

for some complex constants t_a , and integers Q_i^a , satisfying $\sum_i Q_i^a = 0$, since Y is Calabi-Yau. The mirror X is given in terms of coordinates z_i , one for each monomial w_i , satisfying

$$\sum_i Q_i^a |z_i|^2 = r_a, \quad (2.5.29)$$

¹⁸The fact that the D-brane in question is a D6 brane was proposed in [239], however the specific choices of bundles are very important.

and modulo gauge transformations

$$z_i \sim z_i e^{i\theta_a Q_i^a}. \quad (2.5.30)$$

Above, $r_a = \text{Re}(t_a)$, and the imaginary part of t_a , gets related to the NS-NS B-field on X . For each a , we get a curve class $[C_a] \in H_2(X, \mathbf{Z})$, whose volume is r_a . Dual to this are divisor classes D_a , corresponding to 4-cycles with $D_a \circ C^b = \delta_a^b$. The toric divisors D_i , obtained by setting $z_i = 0$, are given in terms of these by

$$D_i = \sum_a Q_i^a D_a$$

By Poincare duality, we can think of D_a as spanning $H^2(X, \mathbf{Z})$. Among the toric divisors are compact ones, D_S , which restrict to compact surfaces S in $H_4(X, \mathbf{Z}) \approx H_{\text{compact}}^2(X, \mathbf{Z})$. For simplicity, we will restrict to the case when X is a local del Pezzo surface, i.e. when there is only one compact S .

2.5.2 Mirror Symmetry, D-branes and Quivers

Mirror symmetry maps D3 branes on three-spheres Δ_α in Y to a collection of sheaves¹⁹ E_α , supported on S , for $\alpha = 1, \dots, r$. Physically, E_α are D4 branes wrapping S , with some specific bundles turned on, giving the branes specific D2 and D0 brane charges. (More precisely, we need to include both the D4 branes and the anti-D4 branes.) The specific bundles turned on correspond to E_α being an exceptional collection of spherical sheaves. Spherical means that the sheaf cohomology group $\text{Ext}_X^k(E_\alpha, E_\alpha)$ is the same as $H^k(S^3)$, so both sets of D-branes have no massless adjoint-valued excitations beyond the gauge fields. The collection of sheaves is in addition such that, for α and β distinct, $\text{Ext}_X^k(E_\alpha, E_\beta)$ is non-zero only for $k = 1, 2$, corresponding to chiral bifundamental matter.²⁰

The net number of chiral minus anti-chiral multiplets is an index

$$n_{\alpha\beta} = \sum_{k=0}^3 (-1)^k \dim \text{Ext}_X^k(E_\alpha, E_\beta)$$

On Y , this corresponded to the intersection numbers of cycles $n_{\alpha\beta} = \Delta_\alpha \circ \Delta_\beta$. In fact, $n_{\alpha\beta}$ also has geometric interpretation on X . A D-brane corresponding to E_α has charge

$$\Delta_\alpha = \text{ch}(E_\alpha) \sqrt{td(X)},$$

¹⁹By E_α we will really mean a sheaf $i_* E_\alpha$ induced on X from a sheaf E_α on S by the embedding $i : S \rightarrow X$ of S in X .

²⁰Non-zero $\text{Ext}^{0,3}(E_\alpha, E_\beta)$ would have corresponded to the presence of ghosts.

where we made use of Poincare duality to express the Chern class in terms of the dual homology class. Using an index theorem, we can relate the $n_{\alpha\beta}$ to computing intersections on X ,

$$n_{\alpha\beta} = \int_X \text{ch}(E_\alpha)^v \text{ch}(E_\beta) td(X),$$

where ω^v denotes $(-1)^n \omega$ for any $2n$ -form ω .

In this case, we have an additional simplification²¹ that the sheaves E_α on X correspond to line bundles V_α on S . This implies [54]

$$Ext_X^k(E_\alpha, E_\beta) = Ext_S^k(V_\alpha, V_\beta) \oplus Ext_S^{3-k}(V_\alpha, V_\beta). \quad (2.5.31)$$

Relative to the naive expectations, the relevant vector bundles on S are twisted [179] by $K_S^{-1/2}$. This accounts for the fact that the theory on the brane is naturally twisted if S is curved. The relation to bundles on S simplifies things, since for a holomorphic vector bundle $Ext_S^k(V_\alpha, V_\beta) = H^k(S, V_\alpha^{-1} \otimes V_\beta)$. The superpotential can also be computed by fairly elementary means, at least if it is cubic [54, 275]. For the more general case, see [34]. We can then write (see for example [32])

$$\Delta_\alpha = ch(V_\alpha) \sqrt{\frac{td(S)}{td(N)}} = ch(V_\alpha) \left(1 + \frac{e(S)}{24}\right). \quad (2.5.32)$$

where N is the normal bundle to S in X . We are implicitly using mirror symmetry and Poincare Duality to identify a Chern class on X with a cycle on Y .

When we add a D6 brane wrapping all of X , the quiver gets extended by one node corresponding to E_0 , a sheaf supported on all of X . The charges of the brane are

$$\Delta_0 = ch(E_0) \sqrt{td(X)} = ch(E_0) \left(1 + \frac{c_2(X)}{24}\right)$$

We want to choose E_0 so that $Ext^k(E_0, E_0)$ is non-vanishing only for $k = 0, 3$, and extending the quiver by this node does not introduce exotic matter in the quiver, which means $Ext^k(E_0, E_\alpha)$ is non-vanishing only for $k = 1, 2$. For the quivers in section 2.4 where $n_{0\alpha} = 1$ for only one node α and zero for all the others, there is a natural construction of E_0 . There is a “duality” that pairs up sheaves E_α supported on S , with the dual line bundles L_β on X [33, 35, 215], such that $\chi(E_\alpha, L^\beta) = \delta_\alpha^\beta$. Thus, the requisite extension corresponds simply to choosing $E_0 = L_\alpha$.

²¹As explained in [32, 179] even though the sheaf E_α on X comes from a vector bundle E_α on S (see footnote 17), it does not correspond to a D-brane on S with a vector bundle E_α . It corresponds to a D-brane with a vector bundle V_α , where $V_\alpha = E_\alpha \otimes K_S^{-\frac{1}{2}}$, where K_S is the canonical line bundle of S .

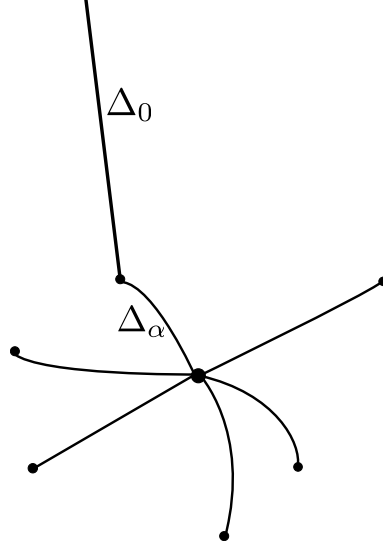


Figure 2.14: The D6 brane corresponds to a semi-infinite line, Δ_0 , in the W -plane. In the mirror this corresponds to a dual sheaf, $E_0 = L_\alpha$.

2.5.3 $\mathbf{P}^1 \times \mathbf{P}^1$ example

Consider local $\mathbf{P}^1 \times \mathbf{P}^1$. In this case, $Q_t = (1, 1, 0, 0, -2)$, $Q_s = (0, 0, 1, 1, -2)$,

$$|z_1|^2 + |z_2|^2 = 2|z_0|^2 + r_t, \quad |z_3|^2 + |z_4|^2 = 2|z_0|^2 + r_s,$$

and the corresponding curve classes C_t and C_s generate $H_2(X)$. The two dual divisor classes, D_t and D_s , with the property that $C_t \circ D_t = 1 = C_s \circ D_s$, and $C_t \circ D_s = 0 = C_s \circ D_t$, can be taken to correspond to divisors $D_t = D_1$ and $D_s = D_3$ on Figure 2.15. There is one compact divisor $D_S = D_0$ with

$$D_S = -2D_s - 2D_t$$

The divisor D_S restricts to the surface $S = \mathbf{P}^1 \times \mathbf{P}^1$.

The four three-cycles Δ_α on Y map to four exceptional sheaves supported on X , whose charges span the compact homology of X . The D4, D2 and the D0 branes correspond to a collection of exceptional sheaves

$$E_3 = \mathcal{O}_S(-2, -2), \quad E_2 = \mathcal{O}_S(-1, -2)[-1], \quad E_4 = \mathcal{O}_S(-2, -1)[-1], \quad E_1 = \mathcal{O}_S(-1, -1)[-2],$$

supported on S . To compute the charges, we use the fact that a D4-brane with a $\mathcal{O}_S(m, n)$ line bundle has Chern class

$$D_S e^{nD_t + mD_s - \frac{1}{2}K_S} \left(1 + \frac{c_2(S)}{24}\right),$$

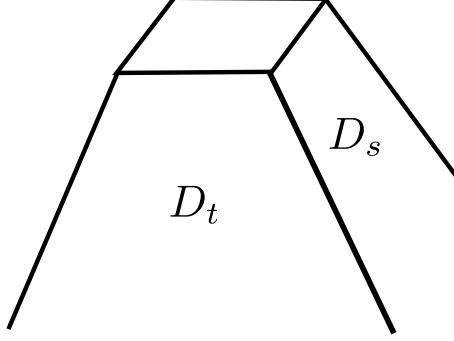


Figure 2.15: The toric base of local $\mathbf{P}^1 \times \mathbf{P}^1$, with the non-compact divisors labeled by D_s and D_t .

which equals²²

$$(D_S + (m+1)C_t + (n+1)C_s + (m+1)(n+1)\text{pt})\left(1 + \frac{c_2(S)}{24}\right).$$

This gives the charges of fractional D-branes are

$$\begin{aligned} \Delta_3 &= (D_S - C_t - C_s + \text{pt})\left(1 + \frac{c_2(S)}{24}\right) \\ \Delta_2 &= (-D_S + C_t)\left(1 + \frac{c_2(S)}{24}\right), \end{aligned} \tag{2.5.33}$$

$$\begin{aligned} \Delta_4 &= (-D_S + C_s)\left(1 + \frac{c_2(S)}{24}\right), \\ \Delta_1 &= D_S\left(1 + \frac{c_2(S)}{24}\right). \end{aligned} \tag{2.5.34}$$

The non-vanishing intersection numbers can be derived from [54] and found to be

$$n_{32} = n_{34} = n_{21} = n_{41} = 2, \quad n_{13} = 4,$$

in agreement with mirror symmetry and section 2.3.

Now we add a D6 brane corresponding to the simplest choice,

$$E_0 = \mathcal{O}_X[-1],$$

²²We are using that the canonical class K_S of the surface S equals the class of the divisor D_S restricting to it in a Calabi-Yau, $K_S = D_S$ and $D_S \cdot D_s = C_t$, $D_S \cdot D_t = C_s$, $D_t \cdot D_t \cdot D_t = D_s \cdot D_s \cdot D_s = \frac{1}{4}$, $D_t \cdot D_t \cdot D_s = D_t \cdot D_s \cdot D_s = -\frac{1}{4}$. Moreover $D_S c_2(S)$ is just the Euler characteristic of S , which is 4 in this case.

a trivial line bundle on the Calabi-Yau, with charge

$$\Delta_0 = -X(1 + \frac{1}{24}c_2(X)).$$

Clearly, $Ext_X^k(E_0, E_0) = 0$, except for $k = 0, 3$. We can compute the spectrum of bifundamentals using ²³

$$Ext_X^k(\mathcal{O}_X, E_\alpha) = H^k(S, E_\alpha),$$

$$Ext_X^k(E_\alpha, E_\beta) = Ext_X^{k-p+q}(E_\alpha[p], E_\beta[q]),$$

and $\dim H^0(\mathbf{P}^1, \mathcal{O}(m)) = m + 1$, $\dim H^1(\mathbf{P}^1, \mathcal{O}(-m)) = -m - 1$, we find that

$$Ext_X^1(E_0, E_3) = \mathbf{C},$$

while the rest $Ext^{0,1}$'s vanish in this sector, so there is precisely one chiral bifundamental multiplet between E_0 and E_3 .²⁴

$$n_{03} = 1.$$

2.6 Monodromy and B -field

One particularly simple example of monodromy in the moduli space corresponds to changing B by integer values. Shifts of the NS-NS B-field by a two form $D \in H^2(X, \mathbf{Z})$

$$B \rightarrow B + D, \quad D \in H^2(X, \mathbf{Z}) \quad (2.6.35)$$

are a symmetry of string theory. The closed string theory is invariant under this, essentially per definition since we come back to the same point in the moduli space. This does not mean that the states are invariant - in particular, the D-branes are not invariant. Turning on a B-field is the same as turning on lower dimensional brane charges. The induced charges are given by

$$\Delta \rightarrow \Delta e^D, \quad (2.6.36)$$

for any sheaf with chern class $\Delta \in H^*(X, \mathbf{Z})$, as shifting the B -field is the same as shifting the field strength F on the brane by D . The state thus does not come back

²³Recall, see footnote 17, that the sheaf E_α is really i_*E_α , inherited from S by the embedding $i : S \rightarrow X$. Therefore, what we call $Ext_X^k(\mathcal{O}_X, E_\alpha)$ is $Ext_X^k(\mathcal{O}_X, i_*E_\alpha) = Ext_S^k(i^*\mathcal{O}_X, E_\alpha) = Ext_S^k(\mathcal{O}_S, E_\alpha) = H^k(S, E_\alpha)$. We thank E. Sharpe for an explanation of this point.

²⁴The intersection numbers in the D4-D2-D0 quiver follow similarly. Using that $Ext_S^k(V_\alpha, V_\beta) = H^k(S, V_\alpha^* \otimes V_\beta)$, since V_α 's are bundles on $S = \mathbf{P}^1 \times \mathbf{P}^1$. This implies that, for example $Ext_X^1(E_3, E_2) = H^0(S, \mathcal{O}(1, 0)) = \mathbf{C}^2$, and $Ext_X^2(E_3, E_1) = H^2(S, \mathcal{O}(1, 1)) = \mathbf{C}^4 = Ext_X^1(E_1, E_3)$, as claimed.

to itself. Since we come back to the same point in the moduli space, we expect the partition function to change as

$$Z(q) = \sum_{\Delta} \Omega(\Delta) q^{\Delta} \rightarrow Z'(q) = \sum_{\Delta} \Omega(\Delta e^D) q^{\Delta} \quad (2.6.37)$$

simply corresponding to the fact that Δ is mapped to a different state Δe^D at the same point in the moduli space. Z and Z' are of course equivalent, up to a change of variables. This looping around the moduli space can be represented in terms of crossing a sequence of walls of the “second kind” and of Seiberg dualities.

In the present context, there is a subtlety in the realization of (2.6.37), due to the fact that the Calabi-Yau is non-compact, and that the quiver does not describe all the possible D-branes on X . Start with an extended quiver describing bound states of a D6 brane, say corresponding to $\mathcal{O}_X$, with compact D4-D2-D0 branes. Shifting B-field as in (2.6.35) corresponds to adding non-compact D4 brane charge to the D6 brane, if the divisor corresponding to D is non-compact. In this case, we get (2.6.37) only after summing the contributions of different quivers in which the Δ_0 node corresponds to $\mathcal{O}_X(D)$, for different $D \in H^2(X, \mathbf{Z})$. If however, we consider shifts by two forms with compact support,

$$D \in H_{\text{compact}}^2(X, \mathbf{Z}),$$

then this is a symmetry of the fixed quiver gauge theory as well. This is because the D6-D4-D2-D0 quiver describes all the BPS states corresponding to a D6 brane bound to compact D-branes on X . Thus, the different D6-D4-D2-D0 quiver gauge theories we can get on X , at the same point in the moduli space, are classified by

$$D \in H^2(X, \mathbf{Z}) / H_{\text{compact}}^2(X, \mathbf{Z}).$$

As an aside, note that if we consider only the D4-D2-D0 quiver, as in section 2.3, having no non-compact charge to begin with, any shift of B-field (2.6.35) is a symmetry.

2.6.1 The $\mathbf{P}^1 \times \mathbf{P}^1$ example

Consider the effect of changing the B-field on the quiver Q in Figure 2.7.

$$B \rightarrow B + n_t D_t + n_s D_s, \quad n_s, n_t \in \mathbf{Z}.$$

Since $H_{\text{compact}}^2(X, \mathbf{Z})$ is generated by the divisor D_S

$$D_S = -2D_t - 2D_s.$$

shifts by $(n_s, n_t) = (2n, 2n)$ for n an integer, should be a symmetry both of the quiver and of the spectrum. Thus, the different quiver gauge theories one can get by shifting B-field are classified by (n_t, n_s) modulo shifts by $(2, 2)$.

Consider increasing the B -field by $(n_t, n_s) = (1, 0)$. In the language of IIB on Y , shifting B by D_t corresponds to shifting

$$z_t, z_s \rightarrow z_t e^{2\pi i}, z_s.$$

This deforms the 3-cycles Δ_α as in the Figure 2.16 to Δ'_α , related to the good basis we started out with by

$$\begin{aligned} \Delta_3 &\rightarrow \Delta'_3 = -\Delta_4 \\ \Delta_4 &\rightarrow \Delta'_4 = \Delta_3 + 2\Delta_4 \\ \Delta_2 &\rightarrow \Delta'_2 = -\Delta_1 \\ \Delta_1 &\rightarrow \Delta'_1 = \Delta_2 + 2\Delta_1 \end{aligned} \tag{2.6.38}$$

The flip of orientation of Δ_1 and Δ_4 is necessary for their orientations to agree with Δ'_2 and Δ'_3 .

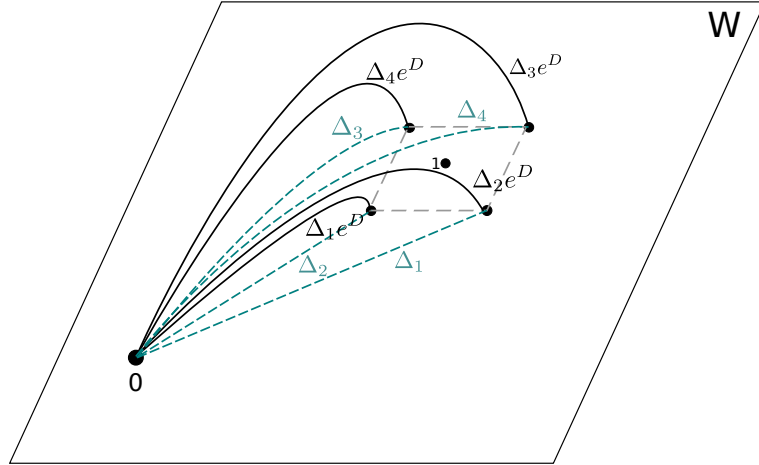


Figure 2.16: The W-Plane after adding one unit of B-field, $(1,0)$, for local $\mathbf{P}^1 \times \mathbf{P}^1$. The deformed cycles $\{\Delta_i e^D\}$ are shown in black. while the original cycles are shown in blue.

Viewed from X , Δ are sheaves on X . The shift of B-field corresponds to tensoring with a line bundle of first chern class D_t . The chern classes change by

$$\Delta \rightarrow \Delta' = \Delta e^{D_t},$$

where above describes how components of Δ change in a fixed basis for $H_*(X)$. To begin with, for example, Δ_3 , and Δ_2 correspond to $\mathcal{O}_S(-2, -2)$ and $\mathcal{O}_S(-1, -2)[-1]$ respectively. After we change the B-field, they pick up charge, and get mapped to $\mathcal{O}_S(-2, -1)$ and $\mathcal{O}_{D_0}(-1, -1)[-1]$, which correspond to $-\Delta_4$ and $-\Delta_1$, respectively,

in agreement with (2.6.38). Similarly, Δ_4 , corresponding to $\mathcal{O}_S(-2, -1)[-1]$ gets mapped to $\mathcal{O}_S(-2, 0)[-1]$. The chern class of this, Δ'_4 equals

$$\Delta'_4 = -(D_S + C_t - C_s - \text{pt})(1 - \frac{c_2(S)}{24}) = \Delta_3 + 2\Delta_4,$$

in agreement with (2.6.38).

In the quiver, the shift of the B -field by D_t corresponds to a sequence of two Seiberg dualities, where we dualize first the node Δ_2 , (as we did in the last section) and then the node Δ_3 (Figure 2.17). It is easy to see from this that the D4-D2-D0 quiver is invariant under the shift of the B-field, although the D-branes on the nodes get replaced by a linear combinations of the ones that we had started out with, given by (2.6.38).

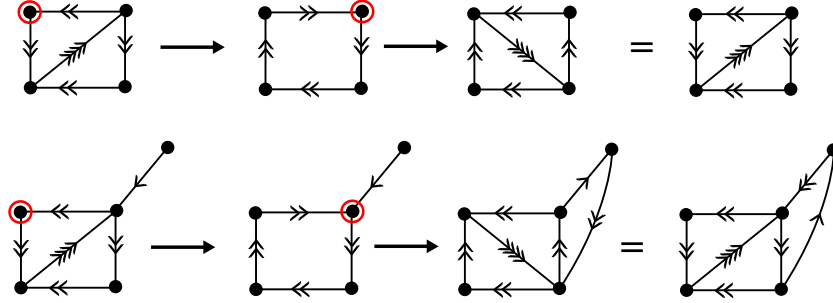


Figure 2.17: Seiberg Duality corresponding to a shift in the B-field (1,0) for local $\mathbf{P}^1 \times \mathbf{P}^1$. The unframed quiver on top is invariant up to a permutation of the nodes, while the framed quiver develops new framing arrows.

Now consider the theory with the D6 brane, by including the node Δ_0 , which has the bundle $\mathcal{O}_X[-1]$. The shift of the B-field acts on all the D-branes, so it acts on the D6 brane as well, mapping it to $E'_0 = \mathcal{O}_X(D_t)[-1]$. This implies that the D6 brane picks up D4 brane charge corresponding to the non-compact divisor D_t . Since no other nodes carry this charge, E'_0 becomes the new framing node, with charge Δ'_0 induced by the B-field

$$\Delta_0 = -X \rightarrow \Delta'_0 = -X e^{D_t}$$

The extended quiver, with the Δ_0 node included also transforms by two Seiberg dualities, Figure 2.17. The quiver in this case is *not* the same as before, since Δ_α have different intersection numbers with Δ'_0 than with Δ_0 . In particular

$$n'_{03} = \Delta'_0 \circ \Delta_3 = 2, \quad n'_{40} = \Delta_4 \circ \Delta'_0 = 1.$$

This can also be verified directly from sheaf cohomology, showing that for example $\text{Ext}^1(E'_0, E_3) = H^0_S(\mathcal{O}(-2, -3)) = 2$. The quiver gauge theory also has a new

superpotential term

$$\mathcal{W} = \sum_{i,j=1,2} \epsilon^{ij} \text{Tr } p A_i q_j + \sum_{i,j,a,b=1,2} \epsilon^{ij} \epsilon^{ab} (\text{Tr } A_i B_a D_{jb} + \text{Tr } \tilde{A}_i \tilde{B}_a D_{jb}).$$

In the previous section, we showed that dualizing a node Δ_* in the quiver is realized as a geometric transition in the dimer model counting the BPS degeneracies, where the face corresponding to Δ_* is dualized. We showed that this implies the [189] wall crossing formula

$$A_{Q'}^{(1)} = A_{-\Delta_*}^{-1} A_Q^{(1)} A_{-\Delta_*}, \quad (2.6.39)$$

together with a change of variables. Here, we apply this twice, corresponding to a sequence of two Seiberg dualities:

$$A_{Q'}^{(1)} = A_{-\Delta_3}^{-1} A_{-\Delta_2}^{-1} A_Q^{(1)} A_{-\Delta_2} A_{-\Delta_3}.$$

In the dimer model, this is a change of the boundary conditions at infinity, replacing the top of the cone by an edge two sites long (see Figure 2.18 and 2.19). It is easy to see that this is consistent with the crystal one would have derived from the quiver Q' by localization.²⁵

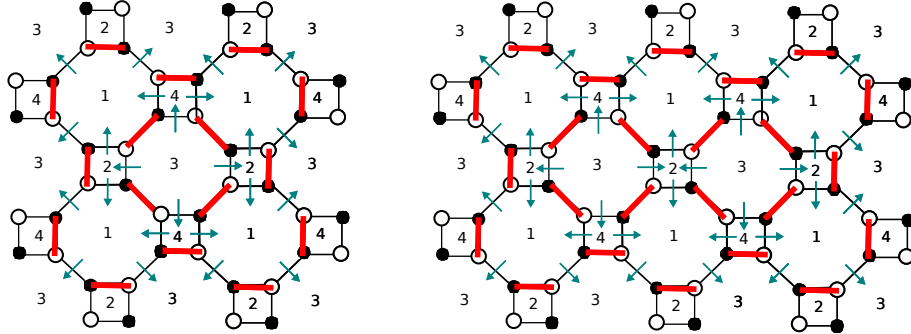


Figure 2.18: The vacuum dimer configuration for $\mathbf{P}^1 \times \mathbf{P}^1$ grows an edge corresponding to a shift in the B-field by $(1,0)$.

Repeating this m times, shifting B to $B + mD_t$ we end up with a quiver Q_m with $n_{03} = m + 1$, and $n_{40} = m$. This corresponds to growing a ridge in the crystal, $m + 1$ atoms long as in [73, 264]. In the next section, we will give a physical interpretation to this observation.

More generally, all the possible inequivalent quivers one can obtain in this way correspond to shifts of B-field by the inequivalent choices in $H^2(X, \mathbf{Z})/H_{\text{compact}}^2(X, \mathbf{Z})$. For example, it is easy to show (either by Seiberg duality, or direct computation) that

²⁵The relation of Seiberg duality, wall crossing and crystals was first noted in [73]. That work was one inspiration for this research.

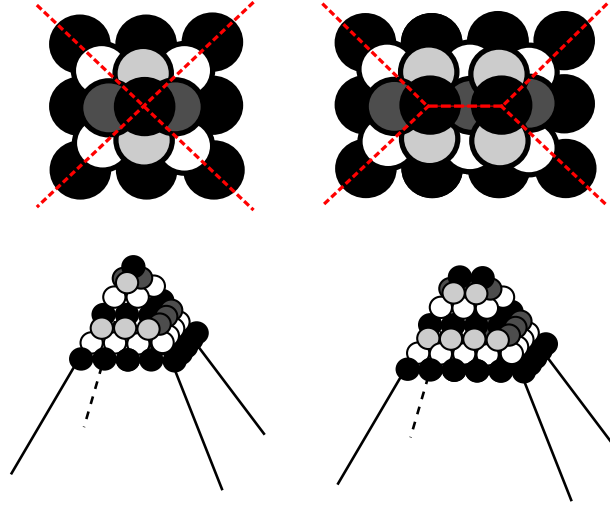


Figure 2.19: The apex of the $\mathbf{P}^1 \times \mathbf{P}^1$ crystal grows an edge corresponding to a shift in the B-field by $(1,0)$.

shifting the B-field by $-D_t$, replaces the node E_0 by $\mathcal{O}_X(-D_t)[-1]$, keeping everything else the same. This means that only $n_{04} = 1$ is nonzero for arrows beginning or ending on Δ_0 . Similarly, replacing E_0 by $\mathcal{O}_X(-D_s)[-1]$, we get only $n_{02} = 1$. Finally, we get only $n_{01} = 1$, by taking $E_0 = \mathcal{O}_X(-D_t - D_s)[-1]$, see figure 2.20.

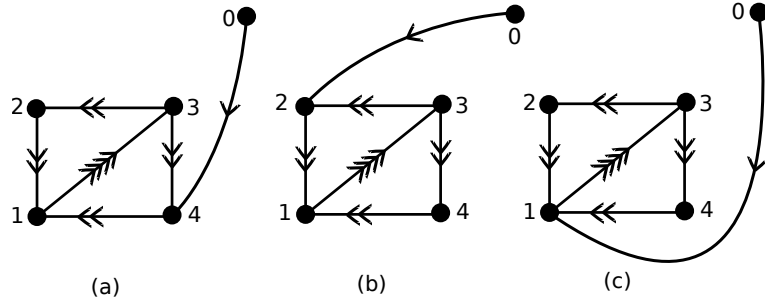


Figure 2.20: Framings for local $\mathbf{P}^1 \times \mathbf{P}^1$, corresponding to negative shifts of the B-field that replace $\mathcal{O}_X[-1]$ by (a) $\mathcal{O}_X(-D_t)[1]$, (b) $\mathcal{O}_X(-D_s)[1]$, and (c) $\mathcal{O}_X(-D_t - D_s)[1]$.

Now consider shifts of the B-field by $D = -D_S \in H_{\text{cmpt}}^2(X, \mathbf{Z})$,

$$B \rightarrow B - D_S.$$

This is a symmetry of the quiver. It can be implemented by changing the B-field by D_t , D_s , D_t and D_s , corresponding to a sequence of four Seiberg dualities on the quiver, which in the end leaves the quiver invariant, except that the charges

of the nodes of the quiver change.²⁶ This can also be seen from the perspective of the Kontsevich-Soibelman algebra. This corresponds to conjugation of $A_Q^{(1)}$ by $A_{-\Delta_2}A_{-\Delta_3}$, $A_{-\Delta'_4}A_{-\Delta'_3}$, $A_{-\Delta''_2}A_{-\Delta''_3}$, and $A_{-\Delta'''_4}A_{-\Delta'''_3}$, respectively, where the four Δ_3 's, for example, correspond to the four Δ_3 nodes in the four quivers we get along the way. This can be rewritten as conjugation by

$$M = M_{\Delta_3}M_{\Delta'_3}M_{\Delta''_3}M_{\Delta'''_3}, \quad (2.6.40)$$

in terms of the operator

$$M_\Delta = A_{-\Delta}A_\Delta.$$

M_Δ is a monodromy operator, implementing the shift of the charge²⁷

$$M_\Delta : e_{\Delta'} \rightarrow (-1)^{\Delta \cdot \Delta'} e_{(\Delta' - (\Delta' \cdot \Delta)\Delta)}.$$

If we denote by

$$M : e_\Delta \rightarrow e_{f(\Delta)},$$

the net effect on the partition function is that

$$M : \sum_{\Delta} \Omega_Q(\Delta) q^\Delta \rightarrow \sum_{\Delta} \Omega_Q(\Delta) q^{f(\Delta)} = \sum_{\Delta} \Omega_Q(f^{-1}(\Delta)) q^\Delta,$$

where we changed variables in the last line. It can be shown that $f^{-1}(\Delta)$ is exactly what is needed for this to correspond to turning on the B-field, $-D_S$, namely, $f(\Delta) = \Delta e^{D_S}$.

2.7 Geometry of the D6 brane bound states

In this section, we begin by considering a fixed state, a D6 brane bound to D4 branes, D2 branes and D0 branes of charge Δ , and ask how its degeneracies change as we shift the B-field by $D \in H^2(X, \mathbf{Z})$. Turning on a B-field is the same as turning on lower dimensional brane charges on the D6 brane. The induced charges are given by

$$\Delta \rightarrow \Delta e^D, \quad (2.7.41)$$

as shifting the B-field is the same as shifting the field strength F on the brane by D . When we come back to the same point in the moduli space, the states have been re-shuffled by (2.7.41). So the degeneracies of any one state will change (at the same time, the spectrum as a whole is invariant, as we emphasized in the previous section).

²⁶The order in which we cross the walls matters, only in so much that the description is simplest for one particular order, and for that ordering we can describe this by Seiberg duality. Otherwise, this does not have a simple interpretation in the gauge theory. The end result, however, is independent of the order.

²⁷Since $M_\Delta^{-1} e_{\Delta'} M_\Delta = (1 - e_{-\Delta})^{-\Delta \cdot \Delta'} (1 - e_\Delta)^{\Delta \cdot \Delta'} e_{\Delta'}$.

The degeneracies, and how they change, can be found from the quivers and crystals describing the brane. A state of arbitrary charge Δ , in general, corresponds to a complicated configuration of the melting crystal $\mathcal{C}$.

It turns out that the vacuum of the crystal $\mathcal{C}$, and certain subset of its states have a beautifully simple geometric description, closely related to the geometry of the Calabi-Yau. We will spend some time explaining this, and then use this result to show that, in the limit of the large B-field,

$$D \rightarrow \infty$$

counting of quiver degeneracies reduces to the Donaldson-Thomas theory as formulated in [173, 212, 213], in terms of a counting ideal sheaves on X . The latter was discovered by trying to give a physical interpretation to the combinatorics of the topological vertex [7] and Gromov-Witten theory. In this way, we will be able to derive the famous Donaldson-Thomas/Gromov-Witten correspondence directly from considering quiver representations in this limit. The proposal that the DT/GW correspondence holds in the large B-field limit was put forward in [84] using split attractors and verified in [13] for Calabi-Yau manifolds without compact 4-cycles, using M-theory.

2.7.1 Geometry of D6 branes

To begin with, consider the D6 brane itself, the sheaf $\mathcal{O}_X[-1]$, which corresponds to the entire crystal, and

$$\Delta = -X = \Delta_0.$$

The crystal $\mathcal{C}$ is a cone in the lattice $\Lambda = \mathbf{Z}_{\geq 0}^3$, whose geometry is closely related to the geometry of the Calabi-Yau. Namely, consider the intersection of $\mathcal{C}$ with a sublattice $\Lambda_0 \subset \Lambda$ corresponding to points of color α , where α is the framing node. The subset of lattice points $\mathcal{C}_0 = \mathcal{C} \cap \Lambda_0$ correspond to holomorphic functions on X .

This can be seen as follows [42, 152, 180]. Recall that the cone $\mathcal{C}$ is generated by the $\mathbf{T}^3$ weights of the paths in A_0 , starting at the framing node. The subcone $\mathcal{C}_0$ corresponds to paths ending on node α_0 . Because $n_{0\alpha} = 1$, such paths can be viewed as starting and ending at the node of color α . These correspond to single trace operators $\text{Tr } O$ in the quiver, where we take the ranks to infinity, so that there are no relations between the traces. The chiral operators O are generated by a set of monomials M_i corresponding to the shortest loops in the quiver, beginning and ending on node α . The monomials M_i all commute, since two paths in the quiver of the same R -charge and the same endpoints are equivalent in the path algebra. Thus, we can simply write O as $O = M_1^{n_1} M_2^{n_2} \dots M_k^{n_k}$. Since the order is irrelevant, the space of operators $\text{Tr } O$ is the same as considering all gauge invariant chiral operators in the abelian quiver theory corresponding to a single D0 brane on X , described in

the quiver by taking all ranks to be 1. Since the moduli space of a D0 brane is X , the later is the space of holomorphic functions on X .

The space of holomorphic functions on X is also a lattice, closely related to the geometry of X . As we reviewed in section 2.5, X comes with a set of coordinates z_i , satisfying (2.5.29)

$$\sum_i Q_i^a |z_i|^2 = r_a, \quad (2.7.42)$$

modulo gauge transformations (2.5.30). We can view X as a fibration over the toric base obtained by forgetting the phases of z_i and using $|z_i|^2$ as coordinates. Consider the integral points in the base, where

$$N_i = |z_i|^2, \quad N_i \in \mathbf{Z}_{\geq 0}.$$

For each such point, we get a monomial in z_i

$$\prod_i z_i^{N_i}$$

which is a function on X , since

$$\sum_i Q_i^a N_i = 0. \quad (2.7.43)$$

We have set $r_a = 0$ in (2.5.29), as we are starting out with singular X . This is because the quiver we started out with has just one framing node, and so $\mathcal{C}$ was a cone with an apex at the origin. Since the Calabi-Yau is three dimensional, the space of all such monomials is a three dimensional lattice, the singular cone $\mathcal{C}_0$.

Now consider what happens to the D6 brane as we increase the B -field

$$B \rightarrow B + D.$$

Increasing the B -field is the same as turning on flux on the D6 brane, tensoring $\mathcal{O}_X[-1]$ with a line bundle of first Chern class D . The D6 brane becomes $\mathcal{O}_X(D)[-1]$, and the charge of the state becomes

$$\Delta = -X e^D.$$

Here, we assume

$$D = \sum_a n_a D_a, \quad n_a \geq 0 \quad (2.7.44)$$

where D_a generate the Kahler cone. We claim that the state Δ corresponds to the deformed crystal $\mathcal{C}(D)$ whose lattice sites $\mathcal{C}_0(D) = \mathcal{C}(D) \cap \Lambda_0$ are holomorphic sections of an $\mathcal{O}(D)$ bundle over X instead. This corresponding to solving

$$\sum_i Q_i^a N_i = n_a. \quad (2.7.45)$$

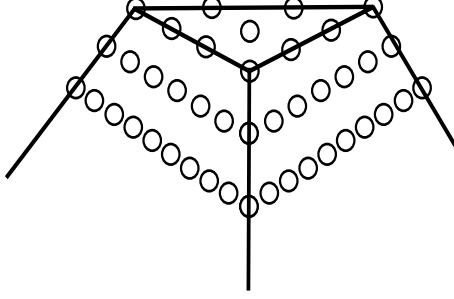


Figure 2.21: Integral points in the toric base of local $\mathbf{P}^2$ correspond to holomorphic sections of $\mathcal{O}(D)$.

As $\mathcal{C}_0$ is a discretized version of the base of Calabi-Yau, modifying $\mathcal{C}_0$ as in (2.7.45) modifies the moduli.

The cases $D \in H_{\text{compact}}^2(X, \mathbf{Z})$ and $D \in H^2(X, \mathbf{Z})/H_{\text{compact}}^2(X, \mathbf{Z})$, should be discussed separately, since they differ in character. When

$$D = D_0 \in H^2(X, \mathbf{Z})/H_{\text{compact}}^2(X, \mathbf{Z}), \quad (2.7.46)$$

the quiver changes since the D6 brane node becomes $\mathcal{O}_X(D)[-1]$, so

$$\Delta = (-X)e^{D_0} = \Delta_0$$

corresponds to the “vacuum” configuration of the crystal $\mathcal{C}(D_0) \subset \Lambda$ associated with this new quiver. We are growing the edges of the crystal, corresponding to increasing the lengths of curves, but no faces open up. In other words, shifts of the B-field by non-compact divisors (2.7.46) correspond to purely non-normalizable deformations of the crystal and the Calabi-Yau. Next, consider shifting the B-field by D , such that

$$D - D_0 \in H_{\text{compact}}^2(X, \mathbf{Z}), \quad (2.7.47)$$

This $D - D_0 \neq 0$ corresponds to shifting the normalizable moduli of the Calabi-Yau, opening up faces in the toric base, and deforming $\mathcal{C}_0$ without changing its asymptotics. This adds compact D4 brane charge to the D6 brane, and this can always be described in the quiver we had started with. To verify that the configuration of the crystal corresponds to $\mathcal{O}_X(D)[-1]$, we need to express its Chern character

$$\Delta = -Xe^D$$

in terms of the Chern characters of the nodes of the quiver

$$\Delta = \Delta_0 + \sum_{\alpha} N_{\alpha} \Delta_{\alpha},$$

where Δ_0 corresponds to the D6 brane node,

$$\Delta_0 = -Xe^{D_0}.$$

Note that ranks N_α depend both on D and D_0 . The set of N_α 's we obtain in this way corresponds to how many atoms of color α we need to remove to describe $\mathcal{O}_X(D)[-1]$. In specific examples, this is completely straight forward, the only subtlety being that, to get integers, one has to keep careful track of the curvature contributions to the charges, which we have been mostly suppressing so far. It would be nice to find a general proof of this.

2.7.2 An example

Consider the local $\mathbf{P}^1 \times \mathbf{P}^1$. In this case, $Q^1 = (1, 1, 0, 0, -2)$, $Q^2 = (0, 0, 1, 1, -2)$ so the “pure” D6 brane $\mathcal{O}_X[-1]$ on X corresponds to the set of points

$$N_1 + N_2 = 2N_0, \quad N_3 + N_4 = 2N_0,$$

in Λ_0 . The cone direction here is parameterized by N_0 . At fixed N_0 , we get a square with $(2N_0 + 1) \times (2N_0 + 1)$ integral points. It is easy to see that this agrees with we get by considering either of the two quivers in section 2.3, and a subset of points corresponding to node $\alpha_0 = 3$, in this case. Of course, the finer structure of the crystal $\mathcal{C}$ is different in the two cases, but the geometry of $\mathcal{C}_0$ is the same.

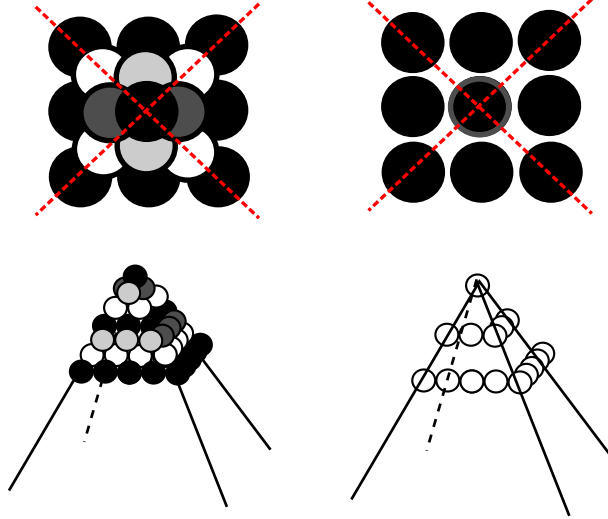


Figure 2.22: The crystal for local $\mathbf{P}^1 \times \mathbf{P}^1$.

Take now $\mathcal{O}_X(mD_t)[-1]$. This is the D6 brane node of the quiver Q_m in the previous section. The corresponding crystal $\mathcal{C}$ gets deformed to $\mathcal{C}(mD_t)$ by replacing

the apex of the cone with an edge $m + 1$ sites long. The crystal sites of $\mathcal{C}_0(mD_t)$ are given by

$$N_1 + N_2 = 2N_0 + m, \quad N_3 + N_4 = 2N_0, \quad (2.7.48)$$

which corresponds to giving the curve C_t length $m + 1$. Add to this $-n$ units of the compact D4 brane charge, by taking $D = (m + 2n)D_t + 2nD_s$ instead. From what we had said above, we expect a face of $(m + 2n + 1) \times (2n + 1)$ nodes of color Δ_3 to open up at the appex

$$N_1 + N_2 = 2N_0 + m + 2n, \quad N_3 + N_4 = 2N_0 + 2n. \quad (2.7.49)$$

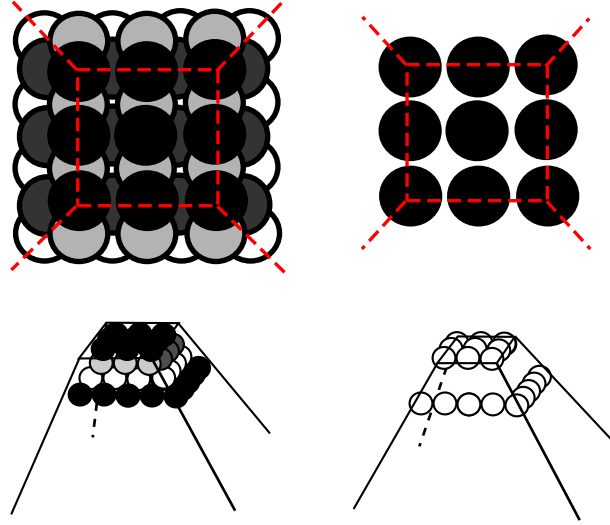


Figure 2.23: The crystal for local $\mathbf{P}^1 \times \mathbf{P}^1$ with compact B-field.

Now we will show that this precisely describes the state $\mathcal{O}_X(D)[-1]$, whose charge is

$$\Delta = -X \left(1 + \frac{c_2(X)}{24}\right) e^{(m+2n)D_t + 2nD_s}.$$

in terms of the crystal associated with the quiver Q_m . The quiver has the D6 brane node $\mathcal{O}_X(mD_t)[-1]$ of charge

$$\Delta_0 = -X \left(1 + \frac{c_2(X)}{24}\right) e^{mD_t}.$$

The difference of these charges

$$\Delta - \Delta_0 = nD_0 + (n + m)nC_s + n^2C_t - \left(\frac{4}{3}n^3 + \frac{1}{2}mn^2 - \frac{1}{6}n\right)\text{pt}, \quad (2.7.50)$$

has to be carried by the D4-D2-D0 nodes of the quiver, and moreover, should correspond to the nodes we needed to remove to go from (2.7.48) to (2.7.49). The number of sites we remove from the crystal $\mathcal{C}$ corresponding to the quiver Q_m is

$$\sum_{i=0}^{n-1} (2i+1+m)(2i+1)\Delta_3 + (2i+1)(2i+2+m)\Delta_2 \quad (2.7.51)$$

$$+ (2i+2)(m+2i+1)\Delta_4 + (2i+2)(2i+2+m)\Delta_1. \quad (2.7.52)$$

Recall that (2.5.33)

$$\begin{aligned} \Delta_3 &= (D_S - C_t - C_s + \text{pt})(1 + \frac{1}{24}c_2(S)) \\ \Delta_4 &= (-D_S + C_s)(1 + \frac{1}{24}c_2(S)) \\ \Delta_2 &= (-D_S + C_t)(1 + \frac{1}{24}c_2(S)) \\ \Delta_1 &= D_S(1 + \frac{1}{24}c_2(S)). \end{aligned} \quad (2.7.53)$$

Adding up (2.7.51) and using that, for a divisor D_S which restricts to a surface S in the Calabi-Yau

$$c_2(X)D_S = (c_2(S) - c_1(S)^2)D_S \quad (2.7.54)$$

and

$$c_2(S)D_S = \chi(S)\text{pt}, \quad c_1(S)^2D_0 = (12 - \chi(S))\text{pt},$$

we recover (2.7.50). Above $\chi(S)$ is the euler characteristic of S . Here, $S = \mathbf{P}^1 \times \mathbf{P}^1$, and $\chi(S) = 4$.

While we did the explicit computation for the quiver Q_m , we could have just as well used the dual quiver Q'_m , obtained by dualizing node 2. This changes the microscopics of the crystal, so $\mathcal{C}(D)$ changes, however the shape of the crystal and $\mathcal{C}_0(D)$ stays the same. This had better be the case, as $\mathcal{C}_0$ does not know about the full quiver, but only about a subset of its nodes that are untouched by the quiver mutation. Explicitly, the charges of the nodes of Q_m and Q'_m are related by

$$\begin{aligned} \Delta'_2 &= \Delta_1 + 2\Delta_2 \\ \Delta'_1 &= -\Delta_2 \end{aligned} \quad (2.7.55)$$

$$\begin{aligned} \Delta'_3 &= \Delta_3 \\ \Delta'_4 &= \Delta_4 \end{aligned} \quad (2.7.56)$$

while $\Delta_0 = \Delta'_0$. Using this to rewrite (2.7.51), we get

$$\begin{aligned} \sum_{i=0}^{n-1} (2i+1+m)(2i+1)\Delta'_3 + (2i+2)(2i+1+m)\Delta'_4 \\ + (2i+2)(2i+2+m)\Delta'_2 + (2i+3)(2i+2+m)\Delta'_1. \end{aligned} \quad (2.7.57)$$

It is easy to see that this counts the nodes in the cut-off top of the crystal based on the quiver Q'_m .

In the next subsection, we will consider meltings of $\mathcal{C}(D)$. This counts a subset of states of the original crystal $\mathcal{C}$, which naturally should correspond to bound states of D6 brane with an $\mathcal{O}_X(D)$ bundle on it, with lower dimensional branes. As an aside, note that, while this problem can be phrased in terms of counting a subset of states of the original quiver, we can try to define it more directly as well, in terms of a D4-D2-D0 quiver, with a framing node corresponding to $\mathcal{O}_X(D)[-1]$. However, such quivers appear to contain bifundamentals in $Ext^{0,3}(E_\alpha, E_\beta)$, which correspond to ghosts, not chiral multiplets [155, 274].

2.7.3 Geometric interpretation of the D6 brane bound states

Take a D6 brane $-X$, with any number of *i.* D4 branes wrapping surfaces S in $H_4(X, \mathbf{Z})$, *ii.* D2 branes wrapping curves C in $H_2(X, \mathbf{Z})$ and *iii.* D0 branes where we require D_S , C to be positive²⁸, and the number of D0 branes to be non-negative.

Changing the B-field by D (2.7.41), gives this state a simple geometric interpretation in terms of removing *i.* faces corresponding to toric divisors D_S , restricting to S , *ii.* edges corresponding to C and *iii.* vertices of $\mathcal{C}_0(D)$. This holds for any D large enough that the “edges”, “faces” and “vertices” of $\mathcal{C}_0(D)$ have an unambiguous meaning, and the positivity constraint comes from the fact that we can only remove, but not add sites along the edges, faces and vertices.²⁹

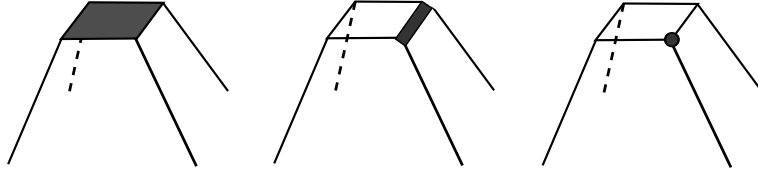


Figure 2.24: Melting a face (left) corresponds to adding compact D4 brane charge, melting an edge (middle) adds D2 brane charge, and melting a node (right) in $\mathcal{C}_0(D)$ adds D0 brane charge.

As we discussed above, the D6 brane $\mathcal{O}_X(D)[-1]$ in the background of B-field D is described by the crystal $\mathcal{C}(D)$ such that $\mathcal{C}_0(D)$ are integral points in the Calabi-Yau with Kahler class D . Adding to this a D4 brane on a divisor D_S corresponds to changing the bundle on the D6 brane to $\mathcal{O}_X(D - D_S)[-1]$, and hence changing $\mathcal{C}(D)$

²⁸More precisely, we require C to be in the Mori Cone of X , and we require S to be a very ample divisor so that $S \cdot D > 0$ for any C, S satisfying these conditions [205].

²⁹To be completely clear, arbitrary meltings of the crystal correspond to a larger set of charges that do not obey this, but the other states do not have such an intuitive description.

to $\mathcal{C}(D - D_S)$. This simply changes the Kahler class of the Calabi-Yau base by D_S . From the perspective of the crystal $\mathcal{C}(D)$, $\mathcal{C}(D - D_S)$ is obtained by removing sites along the face corresponding to D_S . If D_S is a compact divisor, we remove a finite number of sites. Consistency requires that the charge carried by these be the charge of the D4 brane on D_S in this background. In other words, if S is the surface the divisor D_S restricts to, the charge should be that of $\mathcal{O}_S(D)$, obtained from the pure D4 brane $\mathcal{O}_S$ on D_S by shifting the B-field by D .

Explicitly, the crystals $\mathcal{C}(D)$ and $\mathcal{C}(D - D_S)$ carry the charges of the D6 branes corresponding to $\mathcal{O}_X(D)[-1]$ and $\mathcal{O}_X(D - D_S)[-1]$,

$$-Xe^D(1 + \frac{1}{24}c_2(X))$$

and

$$-Xe^{D-D_S}(1 + \frac{1}{24}c_2(X)).$$

Thus, the sites that we remove in going from one crystal to the other must carry the difference of the charges. This is

$$D_S(1 - \frac{1}{2!}D_S + \frac{1}{3!}D_S^2 + \frac{1}{24}c_2(X))e^D = D_S(1 + \frac{1}{24}c_2(S))e^{D-\frac{1}{2}K_S}$$

which equals the charge of a D4 brane on S in the background B-field $\mathcal{O}_S(D)$. We used here the fact that one can think of the sheaf supported on a surface S as a bundle on S twisted by $-1/2$ of the canonical line bundle K_S of the surface, and that $K_S = D_S$, on a Calabi-Yau.

Similarly, a D2 brane wrapping a curve C in X corresponding to $\mathcal{O}_C(D)$ is obtained by removing an edge in the crystal along C . To see this, suppose that C lies on the intersection of two divisors $C = D_S D_T$. Then, analogously to what we had done for the D6 branes and the D4 branes, we can express the D2 brane as a difference of the D4 branes on D_S when we change the background from D to $D - D_T$. The corresponding charges are

$$D_S e^{D-\frac{1}{2}K_S}(1 + \frac{1}{24}c_2(S)) \tag{2.7.58}$$

and

$$D_S e^{D-D_T-\frac{1}{2}K_S}(1 + \frac{1}{24}c_2(S)). \tag{2.7.59}$$

The difference of (2.7.58) and (2.7.59) is

$$C\left(1 - \frac{1}{2}(D_T + D_S)\right)e^D = C\left(1 + \frac{1}{2}c_1(C)\right)e^D,$$

which is the Chern class of the sheaf $\mathcal{O}_C(D)$ supported on C in the background B-field D .

We could also remove n edges along C in the crystal, but then we have the choice of melting additional edges along the face or down the side of the crystal. The difference in charges corresponds exactly to $n\mathcal{O}_C(D) + k[pt]$ for some k determined by the crystal structure, where the additional D0 charge arises from the fact that in a general crystal, the lengths of multiple melted edges will not all be the same.

2.7.4 An Example

In our local $\mathbf{P}^1 \times \mathbf{P}^1$ example, consider X with the B-field shifted by $D = (m + 2n)D_t + 2nD_s$. Removing a face worth of sites from $\mathcal{C}_0$ (and the relevant sites from its refinement $\mathcal{C}$),

$$(2n + 1 + m)(2n + 1)\Delta_3 + (2n + 1)(2n + 2 + m)\Delta_2 + (2n + 2)(m + 2n + 1)\Delta_4 + (2n + 2)(2n + 2 + m)\Delta_1 \quad (2.7.60)$$

and adding up the charges, we get

$$D_S + (m + 2n + 1)C_s + (2n + 1)C_t + \left((2n + 1 + m)(2n + 1) + \frac{1}{6}\right)\text{pt}$$

which is the same as

$$D_S(1 + \frac{1}{24}c_2(S))e^{D - \frac{1}{2}K_S},$$

the chern character of $\mathcal{O}_S(D)$, where $S = \mathbf{P}^1 \times \mathbf{P}^1$, and $K_S = -2D_s - 2D_t = D_S$.

Similarly, removing an edge, corresponding to C_t say, we remove sites of net charge

$$(2n + 1 + m)\Delta_3 + (2n + 2 + m)\Delta_2 + (m + 2n + 1)\Delta_4 + (2n + 2 + m)\Delta_1$$

Adding this up, the net charge is

$$C_t + (2n + 1 + m)\text{pt} = C_t(1 + \frac{1}{2}c_1(C_t))e^D$$

corresponding to $\mathcal{O}_{C_t}(D)$.

To summarize, the states that have a simple geometric description are those we obtain from “pure” D6, D4, D2 (and of course D0 branes) $\mathcal{O}_X$, $\mathcal{O}_S$ and $\mathcal{O}_C$ by turning on a B-field $B \rightarrow B + D$, in other words $\mathcal{O}_X(D)$, $\mathcal{O}_S(D)$ and $\mathcal{O}_C(D)$.

2.7.5 The large B -field limit and DT/GW correspondence

Now consider $\mathcal{C}(D)$ in the limit of large $D \rightarrow \infty$, or more precisely

$$n_a \rightarrow \infty \quad (2.7.61)$$

where

$$D = \sum_a n_a D_a, \quad n_a \geq 0. \quad (2.7.62)$$

This is the large radius limit of X , where the lengths of all edges in $\mathcal{C}(D)$ go to infinity. When we consider meltings of this crystal, we need to specify what we keep fixed in this limit. Without doing anything, working in terms of the fixed basis of $H_*(X, \mathbf{Z})$, corresponding to X , D_a , C_a and pt the states corresponding to melting of $\mathcal{C}(D)$ would all have infinite charges. Moreover, because the edges of the crystal are infinitely long, the counting problem itself is not well defined.

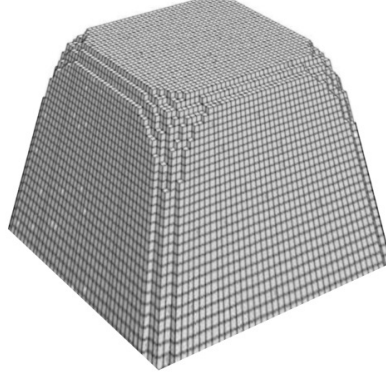


Figure 2.25: Melting the local $\mathbf{P}^1 \times \mathbf{P}^1$ crystal with a large compact B-field turned on.

To remedy this, consider changing the basis to

$$X' = Xe^D, \quad D'_a = D_a e^D, \quad C'_a = C_a e^D,$$

and counting states whose charges in terms X' , D'_a , C'_a and pt are finite, i.e. states of the form

$$X' + k_a D'_a + \ell_a C'_a + m \text{ pt} \quad (2.7.63)$$

where k, ℓ and m are all finite. In terms of the quiver we started with, this is implemented by taking a limit in the weight space, where the weights of all but one node go to zero,

$$q_\alpha \rightarrow 0, \quad \alpha = 1, \dots, k-1 \quad (2.7.64)$$

keeping the product of all the weights fixed,

$$\prod_{\alpha=1}^k q_\alpha = q.$$

Since the sum of the charges of the nodes is one unit of D0 brane charge, states weighted by q^m carry m units of D0 brane charge.

Taking the limits (2.7.61) and (2.7.64) together, makes the state counting well defined. In the limit, the only local excitations of the crystal $\mathcal{C}(D)$ that survive

are those where equal numbers of nodes of all colors are excited – corresponding to removing pure D0 branes, weighted by q^m . One can think of this as a kind of a phase transition in the crystal $\mathcal{C}(D)$ where excitations of individual nodes freeze out, and one can only remove atoms in groups weighted by q . In this limit, $\mathcal{C}(D)$ becomes $\mathcal{C}_0(D)$. Moreover, the crystal can melt only from the vertices, and near each vertex [173], the crystal $\mathcal{C}_0(D)$ looks like a copy of the $\mathbf{C}^3$ crystal in [234]. In $\mathbf{C}^3$ we only have D0 branes to begin with, so the crystals $\mathcal{C}$ and $\mathcal{C}_0$ for this coincide. Note that, while we were originally counting signed partitions, with signs $(-1)^{d(\Delta)}$ from (2.4.18), in the large D limit the signs become simply $(-1)^{m^2} = (-1)^m$, up to an overall sign, since we are restricting to configurations where all ranks are equal.

But, in addition, some large excitations survive as well. There are edge excitations which carry finite number of D2 brane charge ℓ_a in (2.7.63), and also excitations along faces carrying D4 brane charges k_a . We have shown in the previous section that, even at finite D , excitations along an edge $\sum_a \ell_a C_a$ in $H_2(X, \mathbf{Z})$ carry charge

$$\sum_a \ell_a C'_a = \left(\sum_a \ell_a C_a \right) e^D.$$

Similarly, removing the face in $\mathcal{C}_0(D)$ in the class $\sum_k D_a$ corresponds to adding charge

$$\sum_a k_a D'_a = \left(\sum_a k_a D_a \right) e^D$$

to the D6 brane $-X'$. We also showed that adding the D4 brane is the same as shifting D by $\sum_a k_a D_a$. This clearly does not affect the degeneracies of D0-D2-D6 branes in the limit where we take D to infinity – it only shifts what we mean by D , in agreement with [84, 92].

The relation of topological string amplitudes on X with certain melting crystals was observed in [234]. The combinatorics of the topological vertex [7] and the A-model topological string on X is the same [234] as the combinatorics of $\mathbf{C}^3$ crystals glued together over the edges between $\mathbf{C}^3$ patches in X . In [173] a physical explanation of this was proposed, by relating the crystals to D6 brane bound states. Starting with the crystal $\mathcal{C}_0(D)$ corresponding to integral points in the base of a toric Calabi-Yau X , [173] showed that in the limit where one takes the Kahler class D of X to infinity, the crystal degenerates to $\mathbf{C}^3$ crystals glued together over long legs, exactly as in [234]. On the other hand, it was shown that in the same limit, the crystal counts bound states of a D6 brane on X , with D2 and D0 branes (in the language of sheaves, these are ideal sheaves on X). The count in [173] was formulated in terms of the maximally supersymmetric SYM on X , topologically twisted, and non-commutative. The conjecture of [173] relating the topological string on X to counting bound states of a single D6 brane on X with D0 and D2 branes, for any Calabi-Yau X , is known as the Gromov-Witten/Donaldson-Thomas correspondence [212, 213]. Recently, it was proven for toric threefolds by [214].

We have shown that the bound states of a D6 brane on X , with D4, D2 and D0 branes described by a quiver Q and in the background of B-field D , are counted by crystals $\mathcal{C}(D)$, at any D . The crystal roughly corresponds to integral points in the base of the Calabi- Yau with Kahler class D , though the precise microscopic details depend on Q . In the limit of infinite D , the microscopic structure is lost, and $\mathcal{C}(D)$ becomes the same as the crystal $\mathcal{C}_0(D)$ – and hence the same as the crystal in [173,234]. Thus, the count of the D6 brane bound states from the six dimensional perspective of [173] and the $0 + 1$ dimensional quiver quantum mechanics of D6 branes bound to D4-D2 and D0 branes, agree – but only in this limit.³⁰ We have thus re-derived the Gromov-Witten/Donaldson-Thomas correspondence of [173,212,213] from the quiver perspective.

More than that, we provided an answer to the question raised in [232]: what is the crystal $\mathcal{C}_0(D)$ in [173] is counting at finite D ? The crystal $\mathcal{C}_0(D)$, or more precisely its refinement $\mathcal{C}(D)$, is counting Donaldson-Thomas invariants defined as the Witten indices of the quiver quantum mechanics describing one D6 brane on X , bound to D4, D2 and D0 branes in the background B field D .

³⁰This is in accord with [84], which pointed out that the correspondence of [173] can hold only in the limit of infinite B-field. One should be able to understand this a consequence of essentially infinite non-commutativity turned on in [173].

Chapter. 3

Dimer Models and Integrable Systems

3.1 Introduction

Integrable systems have a dense web of connections with gauge theories and string theory. Integrable systems appear in a variety of contexts including Seiberg-Witten theory [142], recent connections to the vacua of supersymmetric theories [226], and the calculation of the spectrum of anomalous dimensions [143] and scattering amplitudes [26] in super-Yang-Mills, to name a few.

Recently, Goncharov and Kenyon discovered an exciting correspondence between integrable systems and dimer models [137]. According to their correspondence, every dimer model defines an integrable system, whose conserved charges can be systematically calculated from perfect matchings.

The correspondence sheds new light on integrable systems, with applications and implications yet to be investigated. It provides new perspectives on integrable systems that are naturally related to dimer models, such as 4d quiver gauge theories, D3-branes probing toric Calabi-Yau 3-folds [118], mirror symmetry [111] and quantum Teichmüller space [115, 137].

In this chapter, we investigate various aspects of the correspondence from a physical perspective. This work is organized as follows. Section 2 briefly reviews the relation between dimer models and quiver theories, toric Calabi-Yaus and integrable systems. Section 3 discusses the connection between integrable systems and 5d $\mathcal{N} = 1$ and 4d $\mathcal{N} = 2$ gauge theories. In Section 4, we apply the correspondence, constructing a relativistic generalization of the periodic Toda chain from the dimer models associated to $Y^{p,0}$ manifolds. We study the relation between the Kasteleyn and Lax operators, the non-relativistic limit of the integrable system and how dimer models for general $Y^{p,q}$ geometries produce alternative relativistic generalizations of the periodic Toda chain. We also discuss the connection to twisted $\mathfrak{sl}(2)$ XXZ spin chains

with impurities. In Section 5, we introduce a practical method for generating new integrable systems based on higgsing and illustrate the method with explicit examples. We conclude and mention future directions in Section 6.

3.2 Some Background

In this section we provide a lightning review of various concepts used throughout this chapter. When necessary, we indicate references for more thorough explanations.

Dimers and Quivers

Brane tilings, to which we will also refer to as dimer models, are bipartite graphs embedded in a two-torus. The dual of a brane tiling is a planar, periodic quiver. There is a one-to-one correspondence between brane tilings and periodic quivers [118] that is summarized in the following dictionary:

Gauge Theory		Brane Tiling
gauge group	$\leftrightarrow$	face
chiral superfield	$\leftrightarrow$	edge
superpotential term	$\leftrightarrow$	node

Every term in the superpotential of the gauge theory is encoded in an oriented plaquette of the periodic quiver. Figure 3.1 exemplifies the correspondence for phase I of F_0 [108].

Dimers and Calabi-Yau 3-folds

Quiver gauge theories that are described by brane tilings arise on the worldvolume of stacks of D3-branes probing singular, toric Calabi-Yau (CY) 3-folds. The CY geometry emerges as the moduli space of vacua of the quiver gauge theory. The connection between dimer models and quivers has trivialized the determination of the corresponding CY geometry. GLSM fields in the toric description of the CY are in one-to-one correspondence with perfect matchings of the dimer model. As a result, points in the toric diagram correspond to (sets of) perfect matchings. This correspondence reduces the task of finding the CY geometry to computing the determinant of the Kasteleyn matrix [118].

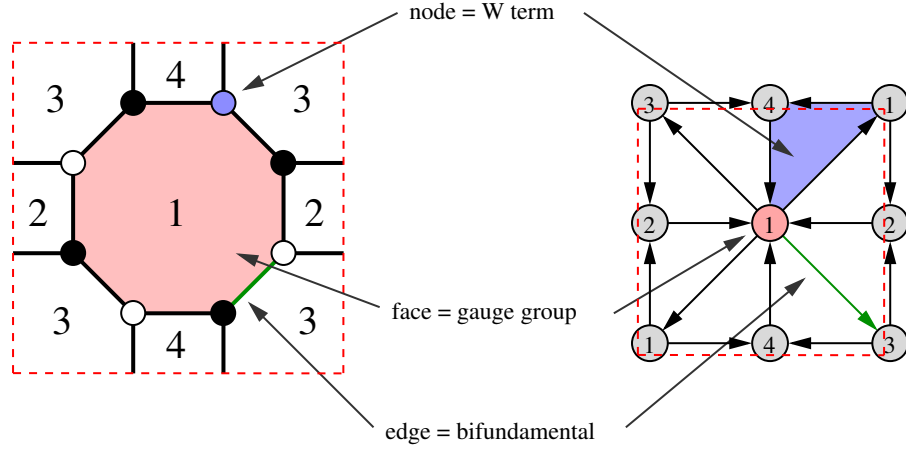


Figure 3.1: Correspondence between the brane tiling and the periodic quiver for phase I of F_0 .

Dimers and Integrable Systems

The dynamical variables of the integrable system correspond to oriented loops in the brane tiling. One basis for such loops is given by the cycles going clockwise around each face w_i ($i = 1, \dots, N_g$, with N_g the number of gauge groups in the quiver) and the cycles z_1 and z_2 wrapping the two directions of the 2-torus.^{1,2}

The Poisson brackets between basis cycles are

$$\begin{aligned} \{w_i, w_j\} &= \epsilon_{w_i, w_j} w_i w_j \\ \{z_1, z_2\} &= (\langle z_1, z_2 \rangle + \epsilon_{z_1, z_2}) z_1 z_2 \\ \{z_a, w_i\} &= \epsilon_{z_a, w_i} z_a w_i \end{aligned} \quad (3.2.1)$$

where $\epsilon_{x,y}$ is the number of edges on which the loops x and y overlap, counted with orientation. Then, ϵ_{w_i, w_j} is simply the antisymmetric oriented incidence matrix of the quiver. In addition, $\langle z_1, z_2 \rangle$ is the intersection number in homology of the cycles z_1 and z_2 .

The classical integrable system can be quantized replacing the Poisson brackets by a q -deformed algebra of the form

$$X_i X_j = q^{\{x_i, x_j\}} X_j X_i, \quad (3.2.2)$$

where $X_i = e^{x_i}$ and $q = e^{-i2\pi\hbar}$.

¹Since $\prod_{i=1}^{N_g} w_i = 1$, one of the w_i 's is redundant. This identity can also be exploited for simplifying expressions.

²The analysis of some models, such as the ones in Section 3.4, can be considerably simplified by choosing a different basis.

Every perfect matching corresponds to a point in the toric diagram and defines a closed loop by subtraction of a reference perfect matching. This loop can then be expressed in terms of the basic cycles. When multiple perfect matchings correspond to a given toric diagram point, their contributions must be added. Goncharov and Kenyon showed that the commutators defined by (3.2.2) and (3.2.1) result in a $(0+1)$ -dimensional quantum integrable system in which the conserved charges are given by:

- **Casimirs:** they commute with everything. They are defined as the ratio between contributions of consecutive points on the boundary of the toric diagram.
- **Hamiltonians:** they commute with each other and correspond to internal points in the toric diagram.

In this chapter we will not discuss the choice of reference perfect matching in detail, we instead refer the interested reader to [137]. Different choices of the reference perfect matching correspond to shifts in the toric diagram. These overall shifts do not affect Casimirs (since they are defined as ratios of points in the toric diagram) but they modify the Hamiltonian(s). Models with zero or one internal point are insensitive to this choice, since they have zero or one Hamiltonians. The choice of reference perfect matching becomes important for models with more than one internal point and can be straightforwardly determined by demanding the Hamiltonians to commute.

Following [182] (see also [121] for applications), we define the magnetic flux through a loop, γ , in terms of edges in the tiling as

$$v(\gamma) = \prod_{i=1}^{k-1} \frac{X(\mathbf{w}_i, \mathbf{b}_i)}{X(\mathbf{w}_{i+1}, \mathbf{b}_i)}, \quad (3.2.3)$$

where the product runs over the contour γ and $\mathbf{b}_i$ and $\mathbf{w}_j$ denote black and white nodes. With this definition, the elements in our basis are $w_j \equiv v(\gamma_{w_j})$, $z_1 \equiv v(\gamma_{z_1})$, and $z_2 \equiv v(\gamma_{z_2})$. Another brief but more complete summary of the work in [137] can be found in [115].

3.3 Integrable Systems from Dimers, 5d and 4d

5d Gauge Theories and Dimers

The integrable systems we are discussing can be derived from either dimer models or 5d gauge theories. The main object underlying all constructions is the spectral curve Σ .

Dimer models encode the quiver gauge theory (and also the geometry) on D3-branes probing a singular, toric Calabi-Yau 3-fold X . The toric singularity has a

characteristic polynomial $P(z_1, z_2) = \sum a_{n_1, n_2} z_1^{n_1} z_2^{n_2}$, where (n_1, n_2) runs over points in the toric diagram. The mirror Calabi-Yau is given by $P(z_1, z_2) = W$, $W = uv$. In this case, Σ corresponds to the Riemann surface sitting at $W = 0$ in the mirror.

Nekrasov [222] proposed a non-perturbative solution for 5d gauge theories compactified on a circle using relativistic integrable systems. From the Seiberg-Witten solution [250, 251] a connection to integrable systems was proposed in [142]. In particular, the spectral curve of the integrable system matches the Seiberg-Witten curve. Nekrasov's insight was to generalize the integrable system to a relativistic integrable system in order to determine the corresponding Seiberg-Witten curve for the 5d gauge theory. There are two possible ways to engineer such compactified 5d theories and Σ plays a prominent role in both of them. First, we can construct them by wrapping an M5-brane on Σ . This M5-brane is a de-singularization of a web of (p, q) 5-branes in Type IIB, obtained after compactifying in a circle of radius β to pass to Type IIA and then lifting to M-theory [25]. Alternatively, these theories can be obtained in the low energy limit of M-theory on a Calabi-Yau 3-fold with vanishing cycles to decouple gravity [101, 132, 169, 217]. The Calabi-Yau is the same X of the dimer construction, so we see that Σ is also relevant from this perspective. Particle-like states arise from M2-branes wrapped around SUSY 2-cycles.

4d Gauge Theories and the Non-Relativistic Limit

The integrable systems constructed using dimer models (equivalently 5d gauge theories compactified on a circle) are naturally *relativistic*. This is reflected, for example, in the exponential dependence on momenta of the conserved charges. This fact is manifest in the form of Poisson brackets (3.2.1).

The radius, β , of the circle on which the 5d gauge theory is compactified plays the role of the inverse speed of light. The non-relativistic limit corresponds to taking $\beta \rightarrow 0$.³ In this limit, we obtain a 4d, $\mathcal{N} = 2$ gauge theory whose Seiberg-Witten curve is the spectral curve of the non-relativistic integrable system, which we denote σ .⁴ Figure 3.2 summarizes the connection between 5d gauge theories, dimer models, integrable systems and their non-relativistic (4d) limits.

³Also interesting is the decompactification limit $\beta \rightarrow \infty$, in which the 5d perturbative solution [31] is recovered

⁴As explained in Section 3.2, dimer models are in one-to-one correspondence with 4d (generically $\mathcal{N} = 1$) quiver theories. These quivers thus provide other 4d gauge theories naturally associated to the integrable systems. In what follows, whether we refer to the 4d $\mathcal{N} = 2$ or quiver gauge theories should be clear from the context.

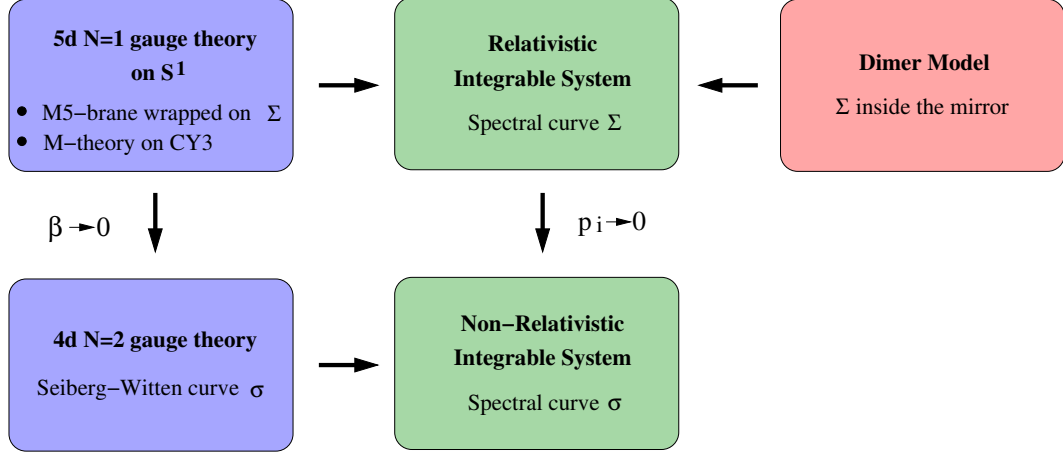


Figure 3.2: Dimer models and 5d, $\mathcal{N} = 1$ gauge theories compactified on an S^1 correspond to different perspectives on the spectral curve Σ . The 4d, $\mathcal{N} = 2$ limit of the gauge theory corresponds to a non-relativistic limit of the integrable system.

3.4 The Periodic Toda Chain

In this section we show the correspondence from [137] at work in explicit examples, using dimer models to construct (relativistic generalizations of) the periodic Toda chain. The relativistic, periodic Toda chain was first introduced in [245] and studied in connection with 5d gauge theories in [222]. These integrable systems arise from dimer models, equivalently quiver gauge theories, associated to $Y^{p,q}$ manifolds [43]. The associated spectral curves, Σ , correspond to 5d, $\mathcal{N} = 1$, $SU(p)$ gauge theory with no flavors and different values of a quantized parameter controlling the cubic couplings in the prepotential.

3.4.1 $Y^{p,0}$ Integrable Systems

Integrable system

Let us now consider general $Y^{p,0}$ geometries. Our goal is to make contact with the relativistic Toda chain for arbitrary p . For concreteness, let us focus on the case in which p is even. The shape of the unit cell depends on whether p is even (rectangle) or odd (rhombus). The dimer model for the conifold corresponds to a square lattice [154]. The cone over $Y^{p,0}$ is a $\mathbb{Z}_p$ orbifold of the conifold. As a result, its dimer model is given by a square lattice with an enlarged unit cell, given by p copies of the one for the conifold, as shown in Figure 3.3.⁵

⁵A similar configuration for a discrete time integrable system related to dimer models was considered in [190].

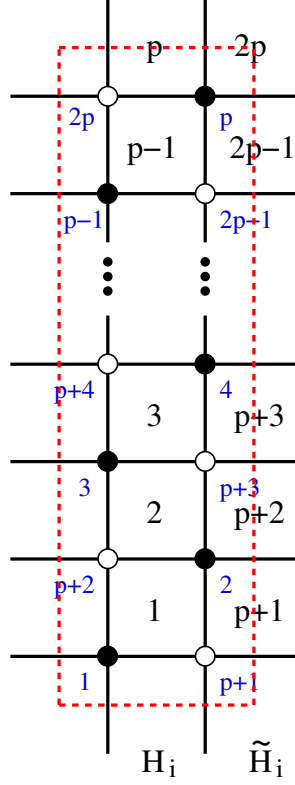


Figure 3.3: Brane tiling for $Y^{p,0}$ with even p . We indicate the numbering of nodes in blue.

The reference perfect matching and z_1 and z_2 paths for $Y^{p,0}$ are shown in Figure 3.4. Figure 3.5 shows the toric diagram for $Y^{p,0}$, with even p . The system has a $\mathbb{Z}_2$ symmetry corresponding to choosing the opposite corner of the toric diagram as the reference perfect matching. The $\mathbb{Z}_2$ symmetry interchanges the Hamiltonian with its dual [183].

The construction of the associated integrable system is considerably simplified by an appropriate choice of basis of $2p + 2$ cycles, instead of w_i and z_j . Figure 3.6 shows $2p$ of the loops. There are two additional cycles, with winding numbers $(-p/2 - 1, 1)$ and $(-p/2 - 1, -1)$ around the z_1 and z_2 directions, which correspond to the green points in Figure 3.5. They are mapped to two Casimirs and are hence fixed.

The non-vanishing Poisson brackets are

$$\{c_k, d_k\} = c_k d_k \quad \{c_k, d_{k+1}\} = -c_k d_{k+1} \quad \{c_k, c_{k+1}\} = -c_k c_{k+1} \quad (3.4.4)$$

The Hamiltonian corresponds to the $(-1, 0)$ point in the toric diagram and is given

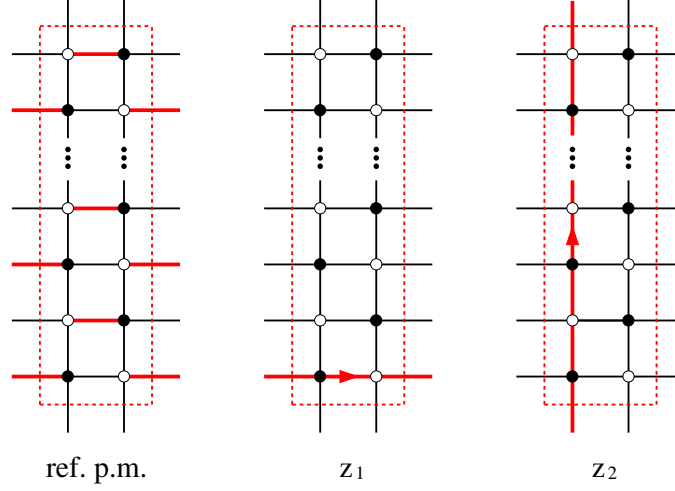


Figure 3.4: Reference perfect matching and z_1 and z_2 paths for $Y^{p,0}$.

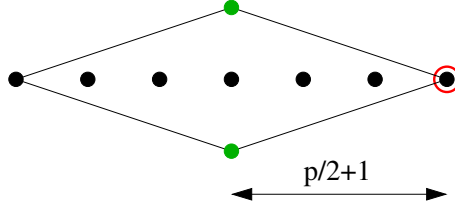


Figure 3.5: Toric diagram for $Y^{p,0}$ (shown in the figure for $p = 6$). The reference perfect matching is circled in red. By construction, its position in the (z_1, z_2) plane is $(0, 0)$. The green dots correspond to cycles with windings $(-p/2 - 1, 1)$ and $(-p/2 - 1, -1)$.

by

$$H_1 = \sum_{i=1}^p (c_i + d_i). \quad (3.4.5)$$

Equations (3.4.4) and (3.4.5) precisely agree with the Poisson brackets and Hamiltonian for the general periodic relativistic Toda chain [52, 183]. The c_i and d_i variables can be expressed in terms of position and momentum variables with canonical commutation relations as follows

$$\begin{aligned} c_i &= \exp(p_i - q_i + q_{i+1}) \\ d_i &= \exp p_i. \end{aligned} \quad (3.4.6)$$

Using the c_i and d_i cycles, determining the additional $(p - 2)$ higher Hamiltonians

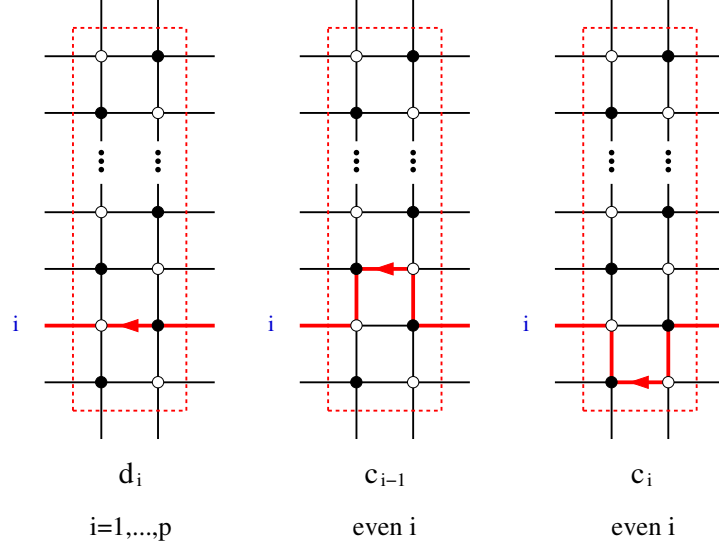


Figure 3.6: $2p$ of the cycles in a convenient basis for $Y^{p,0}$. Notice that the c cycles only exist for even i .

reduces to a straightforward combinatorial problem. The H_n Hamiltonian, associated with the $(-n, 0)$ point in the toric diagram, corresponds to the sum of all possible combinations of n of these cycles with the condition that they do not overlap or touch at any vertex of the tiling. For example, for $Y^{4,0}$ we have:

$$H_2 = (c_1 c_3 + c_2 c_4) + (c_1(d_3 + d_4) + c_2(d_4 + d_1) + c_3(d_1 + d_2) + c_4(d_2 + d_3)) + (d_1 d_2 + d_1 d_3 + d_1 d_4 + d_2 d_3 + d_3 d_4), \quad (3.4.7)$$

$$H_3 = (d_2 d_3 d_4 + d_1 d_3 d_4 + d_1 d_2 d_4 + d_1 d_2 d_3) + (c_1 d_3 d_4 + c_2 d_4 d_1 + c_3 d_1 d_2 + c_4 d_2 d_3).$$

3.4.2 Kasteleyn matrix

Let us construct the Kasteleyn matrix for the dimer model in Figure 3.3. It is convenient to label edges according to whether they are horizontal (H and $\tilde{H}$) or vertical (V and $\tilde{V}$). H edges are those at the center of the tiling and $\tilde{H}$ are those crossing the edge of the unit cell. In addition:

- V : vertical with black node at top endpoint
- $\tilde{V}$: vertical with white node at top endpoint

Finally, let us call H_i and $\tilde{H}_i$ the edges connecting nodes i and $n+i$, $V_i \equiv V_{i+1,p+i}$ and $\tilde{V}_i \equiv \tilde{V}_{i,p+i+1}$. Subindices indicate nodes on the tiling and are identified mod $(2p)$. In these variables, the Kasteleyn matrix is

$$K_p = \begin{pmatrix} & p+1 & p+2 & p+3 & \cdots & 2p-1 & 2p \\ \begin{matrix} 1 \\ 2 \\ 3 \\ \vdots \\ p-1 \\ p \end{matrix} & \begin{matrix} -H_1 - \tilde{H}_1 z_1 \\ V_1 \\ \\ \\ \tilde{V}_p z_2^{-1} \end{matrix} & \begin{matrix} \tilde{V}_1 \\ H_2 + \tilde{H}_2 z_1^{-1} \\ V_2 \\ \\ \\ \end{matrix} & \begin{matrix} \tilde{V}_2 \\ \ddots \\ \\ \end{matrix} & \begin{matrix} \cdots \\ \ddots \\ \\ \end{matrix} & \begin{matrix} \\ \\ \ddots \\ V_{p-1} \end{matrix} & \begin{matrix} V_p z_2 \\ \tilde{V}_{p-1} \\ H_p + \tilde{H}_p z_1^{-1} \end{matrix} \end{pmatrix} \quad (3.4.8)$$

Notice the alternating overall sign of the terms on the diagonal.

Non-Relativistic Limit

The Kasteleyn matrix (3.4.8) and the Lax operator of the non-relativistic periodic Toda chain [113, 114] are strikingly similar. We now make this connection explicit by first expressing the edge variables in terms of coordinates and momenta and then taking the non-relativistic (i.e. small momentum) limit.⁶ We define

$$\begin{aligned} V_i &= \tilde{V}_i \equiv e^{q_i - q_{i+1}} \\ H_i &= -\tilde{H}_i^{-1} \equiv e^{(-1)^i p_i / 2} \end{aligned} \quad (3.4.9)$$

and

$$z_1 \equiv e^{-z} \quad z_2 \equiv w \quad (3.4.10)$$

In the next section, we explain how these definitions and canonical Poisson brackets $\{p_i, q_j\} = \delta_{ij}$ are in agreement with the Goncharov-Kenyon Poisson brackets.⁷

Taking the small momenta limit (i.e. linear order in p_i and z) of the Kasteleyn matrix (3.4.8), we conclude that

$$\det K_p = \det(L_p(w) - z) \quad (3.4.11)$$

⁶The definitions in (3.4.9) are reasonable, rather symmetric and simple. We later check that they are indeed consistent with the w_i commutation relations.

⁷Changing from the basis of coordinates and momenta in (3.4.9) to the one in (3.4.6) is straightforward.

with

$$L_p(w) = \begin{pmatrix} p_1 & e^{q_1-q_2} & & & & e^{q_p-q_1} w \\ e^{q_1-q_2} & p_2 & e^{q_2-q_3} & & & \\ & e^{q_2-q_3} & \ddots & & & \\ & & & \ddots & & \\ & & & & \ddots & \\ & & & & & e^{q_{p-1}-q_p} \\ e^{q_p-q_1} w^{-1} & & & & e^{q_{p-1}-q_p} & p_p \end{pmatrix} \quad (3.4.12)$$

which is precisely the Lax operator of the non-relativistic periodic Toda chain. Notice that the previous analysis nicely associates coordinates and momenta with the vertical and horizontal directions of the square lattice, respectively.

Loop Poisson Brackets from Edges

In the previous section we have expressed edges of the brane tiling in term of coordinates and momenta in such a way that the Lax operator of the periodic Toda chain is obtained from the Kasteleyn matrix of the dimer model by taking the non-relativistic limit. We now show how the commutation relations among loop variables given by the Goncharov-Kenyon rules are recovered from our edge definitions and the $\{p_i, q_j\}$ Poisson brackets.

Consider $Y^{p,0}$ with even p . Using equation (3.2.3), we have

$$\begin{aligned} \text{odd } i: \quad w_i &= H_i \quad V_i^{-1} \quad H_{i+1} \quad \tilde{V}_i^{-1} & w_{p+i} &= \tilde{H}_i^{-1} \quad \tilde{V}_i \quad \tilde{H}_{i+1}^{-1} \quad V_i \\ \text{even } i: \quad w_i &= H_i^{-1} \quad V_i \quad H_{i+1}^{-1} \quad \tilde{V}_i & w_{p+i} &= \tilde{H}_i \quad \tilde{V}_i^{-1} \quad \tilde{H}_{i+1} \quad V_i^{-1} \end{aligned} \quad (3.4.13)$$

$i = 1, \dots, p$. From (3.4.13) and the canonical Poisson brackets $\{p_i, q_j\} = \delta_{ij}$ (which become $[p_i, q_j] = -i\hbar \delta_{ij}$ in the quantum theory) we can calculate

$$\frac{\{A, B\}}{AB} = \begin{pmatrix} & w_1 & w_2 & \cdots & \cdots & \cdots & w_p & w_{p+1} & w_{p+2} & \cdots & \cdots & \cdots & w_{2p} \\ w_1 & & 2 & & & & 2 & -4 & & & & & \\ w_2 & -2 & & -2 & & & & & 4 & & & & \\ w_3 & & 2 & & 2 & & & & & -4 & & & \\ \vdots & & & \ddots & & \ddots & & & & & \ddots & & \\ w_{p-1} & & & & 2 & & 2 & & & & & -4 & \\ w_p & -2 & & & & -2 & & & & & & & 4 \\ w_{p+1} & 4 & & & & & & 2 & -2 & & & & -2 \\ w_{p+2} & & -4 & & & & & & & 2 & & & \\ w_{p+3} & & & 4 & & & & 2 & & -2 & & & \\ \vdots & & & & \ddots & & & & & & \ddots & & \\ w_{2p-1} & & & & & 4 & & & & -2 & & -2 & \\ w_{2p} & & & & & & -4 & 2 & & & 2 & & \end{pmatrix} \quad (3.4.14)$$

which, modulo an unimportant overall scaling, are exactly the Poisson brackets that follow from Goncharov-Kenyon prescription! This scaling can be absorbed in the value of $\hbar$ in the quantum theory. It is important to notice that the final result depends crucially on the details of (3.4.9), such as the sign of the exponents, which are also vital for obtaining the correct non-relativistic limit. An interesting example that depends on these details is $\{w_i, w_{p+i+1}\} = 0$.⁸

3.4.3 More Relativistic Generalizations of Toda from $Y^{p,q}$

The integrable models based on $Y^{p,0}$ are not the only possible relativistic generalizations of the periodic Toda chain. In fact, as we now discuss, all $Y^{p,q}$ geometries give rise to valid generalizations. The corresponding toric diagram is shown Figure 3.7.⁹ These integrable systems were conjectured to exist in [51].

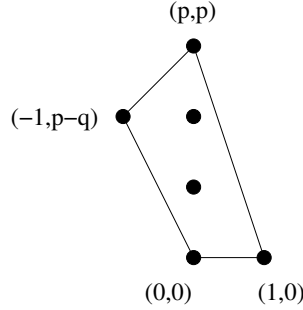


Figure 3.7: Toric diagram for the cone over $Y^{p,q}$.

M-theory on the CY cones over $Y^{p,q}$ geometries gives rise to 5d, $\mathcal{N} = 1$, pure $SU(p)$ gauge theories. The theories differ in the value of c_{cl} , which is a quantized parameter of the theory that controls the cubic couplings in the exact quantum prepotential, related to a five dimensional Chern-Simons term [169]. For $Y^{p,q}$, we have $c_{cl} = q$.

The distinction between theories with different values of c_{cl} disappears when taking the non-relativistic limit. This fact is clearly manifest at the level of the spectral curve. The spectral curve associated with $Y^{p,q}$ and the manipulations for taking its

⁸Of course, the variables in (3.4.9) also reproduce the Poisson brackets (3.4.4) for the basis of cycles we considered in Section 3.4.1. Those cycles are typically written in terms of coordinates and momenta such that horizontal lines are equal to e^{p_i} and squares are equal to $e^{q_i - q_{i+1}}$. The p_i and q_i considered in this section are the ones in which the Lax operator takes the form (3.4.12), but can be mapped to p'_i and q'_i by an appropriate change of variables.

⁹The toric diagram in Figure 3.5 can be put into this form by an $SL(2, \mathbb{Z})$ transformation.

non-relativistic limit have repeatedly appeared in the literature (see for example [162], which we now follow).¹⁰ The spectral curve, Σ , can be written as

$$z_1 + \alpha \frac{z_2^{p-q}}{z_1} + P_p(z_2) = 0 \quad (3.4.15)$$

with $P_p(z_2)$ a degree p polynomial. We can rewrite Σ as

$$y^2 = \prod_{i=1}^p (z_2 - e^{\phi_i})^2 - 4e^{-t_B} z_2^{p-q} \quad (3.4.16)$$

In order to take the non-relativistic (4d) limit, we define

$$z_2 = e^{\beta x}, \quad e^{\phi_i} = e^{\beta a_{i,i+1}}, \quad e^{-t_B} = \left(\frac{\beta \Lambda}{2}\right)^{2p}, \quad (3.4.17)$$

with $a_{i,i+1} = a_i - a_{i+1}$. From our perspective, it is clear that not only z_2 but also the ϕ_i variables and t_B must be rescaled when taking the non-relativistic limit since they are controlled by the w_i variables, which are momentum dependent. In the $\beta \rightarrow 0$ limit, Σ becomes

$$y^2 = \prod_{i=1}^p (x - a_{i,i+1})^2 - 4 \left(\frac{\Lambda}{2}\right)^{2p}, \quad (3.4.18)$$

which is the Seiberg-Witten curve for the pure $\mathcal{N} = 2$ $SU(N)$ gauge theory. As we have anticipated, the dependence on q has disappeared. Reversing the reasoning, we conclude that all $Y^{p,q}$ manifolds (i.e. for arbitrary values of q) give rise to integrable systems that can be considered relativistic generalizations of the periodic Toda chain. In the next section we discuss the $Y^{p,p}$ geometry and its corresponding integrable system.

3.4.4 $Y^{p,p}$ Integrable Systems

We now construct a new infinite family of relativistic integrable systems associated with $Y^{p,p}$. From the discussion in section 3.4.3, we know that these systems also reduce to the periodic Toda chain in the non-relativistic limit.

The cone over $Y^{p,p}$ is the $\mathbb{C}^3/\mathbb{Z}_{2p}$ orbifold. Its brane tiling consist of two columns of p hexagons [118] as shown in Figure 3.8.

Figure 3.9 shows the toric diagram for $Y^{p,p}$, where the reference perfect matching is indicated with a red circle. As for $Y^{p,0}$, the analysis of these models is simplified by a convenient choice of basis for closed cycles. Figure 3.8 shows $2p$ of them. There are two additional cycles with windings $(-p, 1)$ and $(-p, -1)$ along the (z_1, z_2) directions. They correspond to the green points in Figure 3.9 and are fixed by the Casimirs.

¹⁰In fact, this correspondence of the spectral curve is the original reason why we identify $Y^{p,q}$ dimer models as giving rise to relativistic generalizations of the periodic Toda chain.

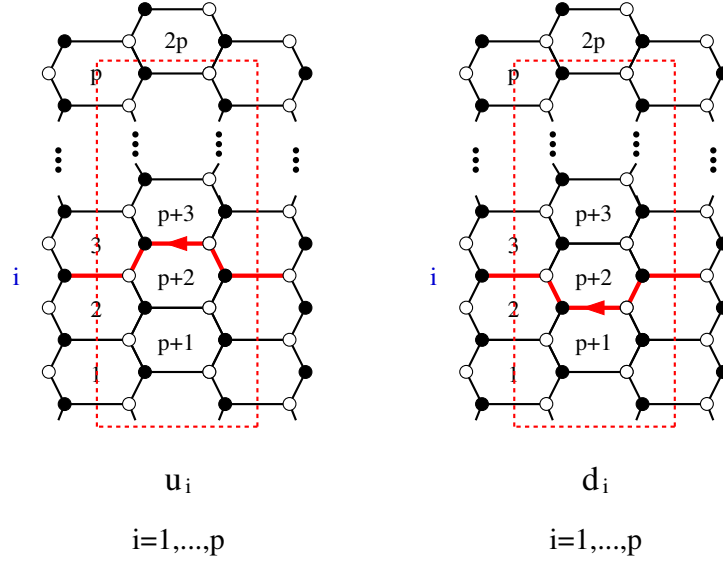


Figure 3.8: Brane tiling for $Y^{p,p}$. A convenient basis for $Y^{p,p}$ is given by the u_i (up) and d_i (down) cycles, $i = 1, \dots, p$.

The Hamiltonian associated to the $(-1, 0)$ point in the toric diagram takes the simple form

$$H_1 = \sum_{i=1}^p (u_i + d_i). \quad (3.4.19)$$

Similar to our analysis of $Y^{p,0}$, determining the Hamiltonians for the $(-n, 0)$ points is very simple in the u_i and d_i basis. Finding the n^{th} Hamiltonian reduces to determining all possible combinations of n cycles that do not overlap or intersect at nodes of the tiling. For example, the higher Hamiltonians for $Y^{4,4}$ are

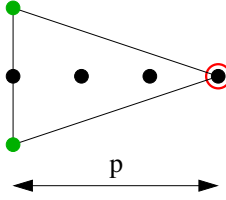


Figure 3.9: Toric diagram for $Y^{p,p}$ (shown in the figure, $p = 3$). The reference perfect matching is circled in red. By construction, its position in the (z_1, z_2) plane is $(0, 0)$. The green dots correspond to cycles with windings $(-p, 1)$ and $(-p, -1)$.

$$\begin{aligned}
H_2 &= u_1 u_2 + u_1 u_3 + u_1 u_4 + u_2 u_3 + u_2 u_4 + u_3 u_4 \\
&+ d_1 d_2 + d_1 d_3 + d_1 d_4 + d_2 d_3 + d_2 d_4 + d_3 d_4 \\
&+ u_1 d_3 + u_1 d_4 + u_2 d_4 + u_2 d_1 + u_3 d_1 + u_3 d_2 + u_4 d_2 + u_4 d_3, \\
\\
H_3 &= u_1 u_2 u_3 + u_1 u_2 u_4 + u_1 u_3 u_4 + u_2 u_3 u_4 \\
&+ u_1 d_3 d_4 + u_2 d_4 d_1 + u_3 d_1 d_2 + u_4 d_2 d_3 \\
&+ d_1 d_2 d_3 + d_1 d_2 d_4 + d_1 d_3 d_4 + d_2 d_3 d_4 \\
&+ d_1 u_2 u_3 + d_2 u_3 u_4 + d_3 u_4 u_1 + d_4 u_1 u_2, \\
\\
H_4 &= u_1 u_2 u_3 u_4 + d_1 d_2 d_3 d_4.
\end{aligned} \tag{3.4.20}$$

3.4.5 $Y^{p,q}$ Integrable Systems as Spin Chains

We have identified the family of integrable systems associated to the $Y^{p,q}$ dimer models in the previous section. Previously the integrable systems identified with the $Y^{p,q}$ spectral curves were described as twisted $\mathfrak{sl}(2)$ XXZ spin chains with impurities [141]. We will now explain the equivalence of these apparently different descriptions.

The $\mathfrak{sl}(2)$ spin chains are described by N “spins” Ψ_i where $i = 1, \dots, N$. The spin operators satisfy the commutation relations

$$\{S_{\pm}, S_0\} = \pm S_{\pm}, \quad \{S_+, S_-\} = \sinh 2S_0$$

where the raising and lowering operators $S_{\pm} = S_1 \pm iS_2$ are defined as usual. Integrability of the spin chain can be shown starting from the auxiliary linear problem for the Lax matrix

$$L_i(\mu)\Psi_i(\mu) = \Psi_{i+1}(\mu)$$

with twisted boundary conditions implemented by the identification

$$\Psi_{i+N}(\mu) = -w\Psi_i(\mu).$$

The spectral curve of the spin chain is given by the determinant of the transfer matrix,

$$\det(T(\lambda) + w 1) = 0,$$

where the transfer matrix is defined by the product of the two-by-two Lax matrices

$$T(\lambda) \equiv L_N(\lambda) \dots L_1(\lambda).$$

Impurities are added to the spin chain by performing a site-dependent shift of variables for the Lax matrices $L_j(\mu)$. $Y^{p,q}$ quiver gauge theories can be constructed by starting from $Y^{p,p}$ and adding $(p - q)$ impurities [43]. Amusingly, the spin chain and quiver impurities are precisely the same. This reflects the well-known phenomena that the same integrable system can have different Lax representations. Since the Lax matrices for all of the $Y^{p,q}$ quiver gauge theories are tridiagonal [118], we can re-write the determinant of the Lax matrix in spin-chain form using the following identity

$$\det \begin{pmatrix} a_1 & b_1 & & c_0 \\ c_1 & \ddots & \ddots & \\ & \ddots & \ddots & b_{N-1} \\ b_0 & & c_{N-1} & a_N \end{pmatrix} = (-1)^{N-1} \left(\prod_j b_j + \prod_j c_j \right) + \text{Tr} L_N L_{N-1} \dots L_1$$

where

$$L_j = \begin{pmatrix} a_j & -b_{j-1}c_{j-1} \\ 1 & 0 \end{pmatrix}.$$

Thus we can re-write the the spectral curve as

$$\det(T(\lambda) + w1) = w^2 + w \text{Tr} T(\lambda) + \det T(\lambda).$$

Under a suitable change of variables, it should be straightforward to show that this representation matches the XXZ form proposed in [141]. For the relativistic, periodic Toda chain this change of variables appears in [196].

3.5 Generating New Integrable Systems via Partial Resolution

In this section we explain how to determine the integrable system associated with a partial resolution, given the one for the parent theory. This is a useful way of obtaining new integrable systems from known ones. Starting from a relatively complicated example, partial resolution provides a practical way of deriving new integrable systems, much faster in practice than going through the process of expressing the new system in terms of loop variables. This method is so efficient that it is natural to expect that it has a counterpart in the integrable system literature.

We will focus on minimal partial resolutions, which correspond to removing extremal perfect matchings (i.e those located at corners of the toric diagram) one at a time.¹¹ In addition, it might be necessary to simultaneously remove non-extremal perfect matchings. The result of this process is a new toric diagram in which the multiplicity of each new extremal perfect matching is one and some of the internal multiplicities might also change. From a quiver point of view, partial resolution corresponds to turning on non-zero vacuum expectation values (vevs) for bifundamental fields X_{ij} and then higgsing nodes i and j .¹² This operation might result in mass terms for some fields, which can be integrated out using their equations of motion. From a dimer model perspective, partial resolution corresponds to removing from the tiling the edges associated to the fields with non-zero vevs. As a result, some of the adjacent faces in the tiling are merged into a single new face. The integration of massive fields maps to the removal of 2-valent vertices that might be generated in the process by condensing the nodes at the endpoints of the two edges terminating in them.

The standard understanding of partial resolutions using dimer models is that all perfect matchings containing the edge associated with X_{ij} are removed [118]. Following the connection between bifundamental fields and perfect matchings given in [118], we see that all these perfect matchings (which are interpreted as GLSM fields) need to acquire a non-zero vev in order for the bifundamental to get one.

In the integrable systems context, the fundamental objects are closed loops rather than individual perfect matchings. In fact, even the reference perfect matching might disappear when applying the discussion in the previous paragraph. It is straightforward to adapt the previous reasoning to loops: partial resolution removes all loops that contain an edge that gets a non-zero vev.

From a brane tiling perspective, the edge associated to X_{ij} is deleted and faces i and j are combined into a single face. Consequently, we start from two cycles w_i and w_j and end with a combined cycle $w_{i/j} = w_i w_j$ as shown in Figure 3.10.

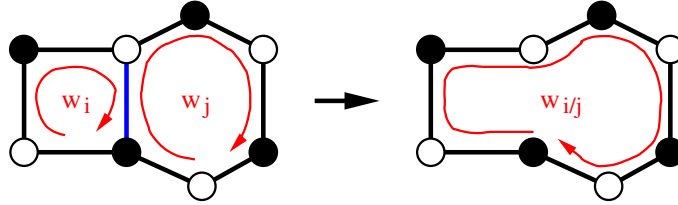


Figure 3.10: Combination of gauge group cycles when higgsing by a vev associated to the blue edge.

¹¹The implementation of more general partial resolutions using dimer models has been discussed in great detail in [133]. We will restrict ourselves to minimal ones in this chapter.

¹²Notice that, given a quiver, not all possible higgsings correspond to consistent partial resolutions.

The following rules produce the integrable system for the partially resolved geometry:

- 1) Remove loops that contain an edge with a non-zero vev.
- 2) Re-express the surviving loops with the replacement $(w_i w_j) \rightarrow w_{i/j}$.

More practically, the loops that are removed in rule (1) are those that cannot be re-written using rule (2).

In some cases, a z_i path can involve an edge that is removed when higgsing, as shown in Figure 3.11. If so, the path can be redefined by using a w_j to make the path wiggle appropriately, avoiding this edge. Equivalently, we could have chosen a different set of z_i paths in the parent theory such that they do not involve higgsed edges.

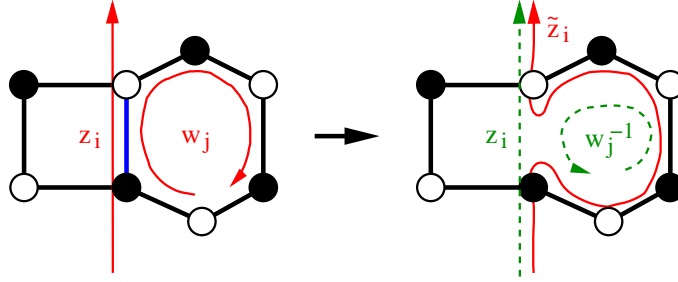


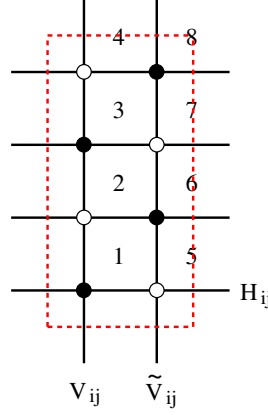
Figure 3.11: Redefinition of one of the paths that winds around the $\mathbb{T}^2$, $z_i = \tilde{z}_i w_j$, after partial resolution by turning on a vev for the blue edge.

3.5.1 Examples: Partial Resolutions of $Y^{4,0}$

We now illustrate our ideas in the explicit case of partial resolutions of $Y^{4,0}$. Partial resolution can either preserve or reduce the genus of the spectral curve (i.e. the number of Hamiltonians). The examples that follow exhibit the latter behavior.

In this section we depart from the notation for edges we used for general $Y^{p,0}$ in Section 3.4.2, which was specially devised for giving the Kasteleyn matrix a nice form. Here our emphasis is on higgsing, so we explicitly indicate the gauge groups under which bifundamentals are charged using subindices. V_{ij} and $\tilde{V}_{ij}$ indicate vertical edges in the first and second columns of the brane tiling, respectively. Horizontal edges are denoted H_{ij} .

Figure 3.13 shows the resolutions we will consider. The number of gauge groups in the associated quivers is given by twice the area of the toric diagrams. This implies

Figure 3.12: Brane tiling for $Y^{4,0}$.

that the number of bifundamental expectation values that need to be turned on is equal to twice the decrease in area of the toric diagram.

The starting point is the integrable system for $Y^{4,0}$. In the table below, (n_1, n_2) gives the $(z_1^{n_1} z_2^{n_2})$ contribution. Every term in a given contribution arises from a loop in the tiling or equivalently from a perfect matching. Hamiltonians correspond to internal points and Casimirs are given by the ratio of consecutive points on the boundary. Instead of the basis of cycles used in Section 3.4.1, here we use the w_i (which correspond to gauge groups), z_1 and z_2 basis, since it makes higgsing more transparent.

(n_1, n_2)	Loops
$(0, 0)$	1
$(-1, 0)$	$w_4 + w_4 w_8 + w_4 w_7 w_8 + w_3 w_4 w_7 w_8 + w_2 w_3 w_4 w_7 w_8$ $+ w_2 w_3 w_4 w_6 w_7 w_8 + w_2 w_3 w_4 w_5 w_6 w_7 w_8 + 1$
$(-2, 0)$	$w_1^{-1} w_5^{-1} w_4 + w_4 w_8 + w_1^{-1} w_4 w_8 + w_1^{-1} w_5^{-1} w_4 w_8$ $+ w_1^{-1} w_5^{-1} w_6^{-1} w_4 w_8 + w_4 w_7 w_8 + w_1^{-1} w_4 w_7 w_8 + w_1^{-1} w_5^{-1} w_4 w_7 w_8$ $+ w_3 w_4 w_7 w_8 + w_1^{-1} w_3 w_4 w_7 w_8 + w_1^{-1} w_5^{-1} w_3 w_4 w_7 w_8 + w_2 w_3 w_4 w_7 w_8$ $+ w_1^{-1} w_5^{-1} + w_3 w_4^2 w_7 w_8 + w_2 w_3 w_4^2 w_7 w_8 + w_3 w_4^2 w_7 w_8^2$
$(-3, 0)$	$w_1^{-1} w_5^{-1} w_4 w_8 + w_1^{-1} w_5^{-1} w_6^{-1} w_4 w_8 + w_1^{-1} w_5^{-1} w_4 w_7 w_8 + w_1^{-1} w_5^{-1} w_3 w_4 w_7 w_8$ $+ w_1^{-1} w_5^{-1} w_3 w_4^2 w_7 w_8 + w_3 w_4^2 w_7 w_8^2 + w_1^{-1} w_3 w_4^2 w_7 w_8^2 + w_1^{-1} w_5^{-1} w_3 w_4^2 w_7 w_8^2$
$(-4, 0)$	$w_1^{-1} w_5^{-1} w_3 w_4^2 w_7 w_8^2$
$(-2, 1)$	$w_1^{-1} w_4 w_7 w_8$
$(-2, -1)$	$w_2 w_3 w_4^2 w_7 w_8$

(3.5.21)

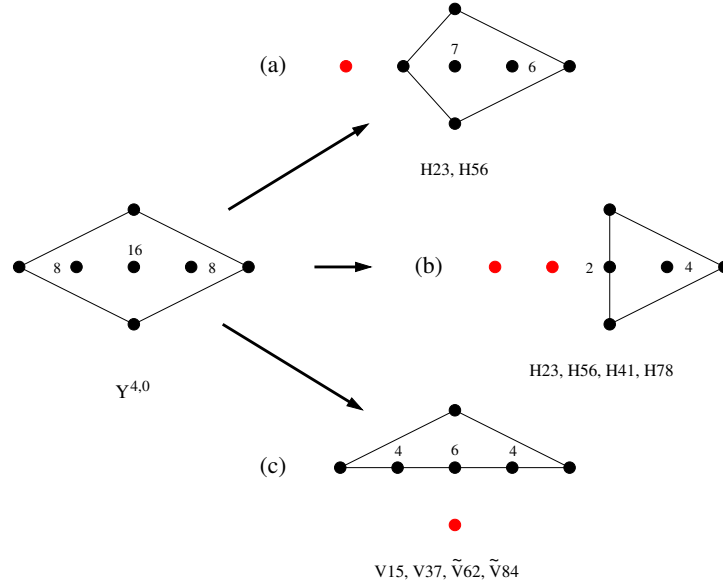


Figure 3.13: Toric diagrams for $Y^{4,0}$ and various of its partial resolutions. We indicate the multiplicity associated with each point and the non-zero vevs that need to be turned on. Higgsing takes the number of Hamiltonians from 3 in the original theory to a) 2, b) 1 and c) 0.

Higgsing a

Model (a) in Figure 3.13 is obtained by giving non-zero vevs to both H_{23} and H_{56} . There are other choices of expectation values that lead to the same result. Figure 3.14 shows the corresponding brane tiling.

The resulting integrable model is given by:

(n_1, n_2)	Loops	(3.5.22)
$(0, 0)$	1	
$(-1, 0)$	$w_4 + w_4 w_8 + w_4 w_7 w_8 + w_{2/3} w_4 w_7 w_8 + w_{2/3} w_4 w_{5/6} w_7 w_8 + 1$	
$(-2, 0)$	$w_4 w_8 + w_1^{-1} w_4 w_8 + w_1^{-1} w_{5/6}^{-1} w_4 w_8 + w_4 w_7 w_8 + w_1^{-1} w_4 w_7 w_8 + w_{2/3} w_4 w_7 w_8 + w_{2/3} w_4^2 w_7 w_8$	
$(-3, 0)$	$w_1^{-1} w_{5/6}^{-1} w_4 w_8$	
$(-2, 1)$	$w_1^{-1} w_4 w_7 w_8$	
$(-2, -1)$	$w_{2/3} w_4^2 w_7 w_8$	

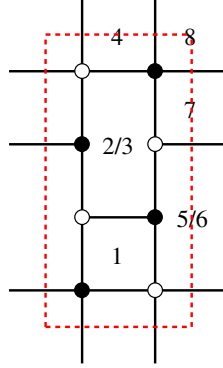


Figure 3.14: Brane tiling for model (a) obtained by higgsing $Y^{4,0}$.

Higgsing b

Model (b) corresponds to turning on vevs for H_{23} , H_{56} , H_{78} and H_{41} . The resulting brane tiling is shown in Figure 3.15.

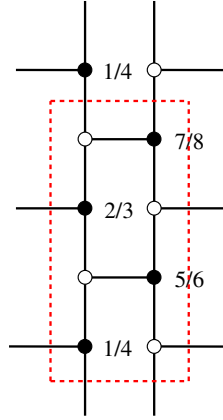


Figure 3.15: Brane tiling for model (b) obtained by higgsing $Y^{4,0}$.

The edge associated to H_{41} is contained in the original z_1 path, which thus needs to be redefined. We can consider a new path $\tilde{z}_1$ given by $z_1 = \tilde{z}_1 w_1^{-1}$. The integrable system is summarized in (3.5.23), where now (n_1, n_2) corresponds to the $(\tilde{z}_1^{n_1} z_2^{n_2})$ contributions.

(n_1, n_2)	Loops
$(0, 0)$	1
$(-1, 0)$	$w_{4/1} + w_{4/1}w_{7/8} + w_{2/3}w_{4/1}w_{7/8} + w_{2/3}w_{4/1}w_{5/6}w_{7/8}$
$(-2, 0)$	$w_{4/1}w_{7/8} + w_{2/3}w_{4/1}^2w_{7/8}$
$(-2, 1)$	$w_{4/1}w_{7/8}$
$(-2, -1)$	$w_{2/3}w_{4/1}^2w_{7/8}$

(3.5.23)

Higgsing c

Finally, model (c) follows from turning on vevs for V_{15} , V_{37} , $\tilde{V}_{62}$ and $\tilde{V}_{84}$. The corresponding brane tiling is shown in Figure 3.16.

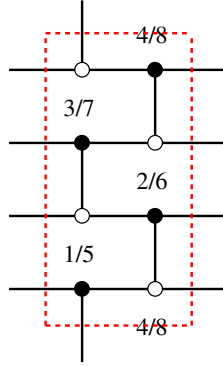


Figure 3.16: Brane tiling for model (c) obtained by higgsing $Y^{4,0}$.

This time both V_{15} and V_{37} overlap with the original z_2 path, which can be redefined according to $z_2 = \tilde{z}_2 w_1 w_3$. Denoting (n_1, n_2) the $(z_1^{n_1} \tilde{z}_2^{n_2})$ contribution, (3.5.24) summarizes the resulting integrable system.

(n_1, n_2)	Loops
$(0, 0)$	1
$(-1, 0)$	$w_{8/4} + w_{3/7}w_{8/4} + w_{6/2}w_{3/7}w_{8/4} + 1$
$(-2, 0)$	$w_{8/4} + w_{1/5}^{-1}w_{8/4} + w_{3/7}w_{8/4} + w_{1/5}^{-1}w_{3/7}w_{8/4} + w_{1/5}^{-1} + w_{3/7}w_{8/4}^2$
$(-3, 0)$	$w_{1/5}^{-1}w_{8/4} + w_{1/5}^{-1}w_{3/7}w_{8/4} + w_{3/7}w_{8/4}^2 + w_{1/5}^{-1}w_{3/7}w_{8/4}^2$
$(-4, 0)$	$w_{1/5}^{-1}w_{3/7}w_{8/4}^2$
$(-2, 1)$	$w_{3/7}w_{8/4}$

(3.5.24)

From an integrability point of view, this model is trivial, i.e. it only consists of Casimirs. From a quiver perspective, the reason for this is that, as one can deduce from Figure 3.16, the associated quiver is fully non-chiral. This implies that all commutators vanish.

3.6 Conclusions and Outlook

We have investigated various applications of the correspondence between dimer models and integrable system introduced by Goncharov and Kenyon in [137]. We used it to explicitly construct relativistic generalizations of the periodic Toda chain associated to $Y^{p,0}$ and $Y^{p,p}$ geometries. In these models, the calculation of commuting Hamiltonians reduces to the combinatorics of non-intersecting paths on the brane tiling. We investigated the connection between the Kasteleyn matrix and the Lax operator, the non-relativistic limit of the integrable systems, additional relativistic versions of the periodic Toda chain based dimer models for general $Y^{p,q}$ geometries, and the identification of quiver impurities and spin chain impurities. Finally, we introduced a method for generating new integrable systems based on higgsing. We can envision, and are currently pursuing, multiple directions in which the correspondence between dimer models and integrable systems can be exploited. We discuss some of them below.

We have explained how higgsing is an efficient tool for generating new integrable systems. The characteristic polynomials for the dimer models associated to $\mathbb{Z}_n \times \mathbb{Z}_m$ orbifolds of arbitrary geometries, which correspond to $n \times m$ arrays of copies of the original unit cell, can be determined using simple formulas [182]. These expressions have been used for calculating the multiplicity of perfect matchings associated to points in the toric diagrams of orbifolds [154]. It would be interesting to investigate whether there are analogous expressions for the integrable systems associated to orbifolds starting from the integrable system for the unorbifolded geometry. New integrable systems could then be generated by higgsing, using the orbifold theories as starting points.

It would be interesting to study the continuous limit of integrable systems arising from dimer models. By this, we not only mean the infinite length limit of integrable chains but also the (1+1)-dimensional field theory limit of fixed length systems. It is reasonable to conjecture that field theories such as A_n Toda field theories can be constructed in this way. Furthermore, it is natural to expect that dimer models are useful for classifying (0+1)-dimensional, integrability-preserving defects and interfaces that can be added to such field theories.

The work of [93] investigated the quantization of Riemann surfaces defined by the vanishing of the A-polynomials of three-manifolds M that are the complement of (thickened) knots or links. In the case of knots, the boundary of M is a 2-torus. The decomposition of M into glued tetrahedra with truncated vertices gives rise to

a triangulation of the 2-torus, the developing map, whose dual is reminiscent of a dimer model. On the other hand, [137] and [115] discussed the relation between dimer models and quantum Teichmüller space. It is then natural to ask whether the existing similarities indicate the existence of a true connection that associates dimer models to three-manifolds that are knot complements.

One clear direction for further research is to explore the connection between quantum integrable systems and gauge theories proposed by Nekrasov and Shatashvili [226]. By considering the 5d gauge theory on an Ω -background with $\epsilon_1 = \epsilon$ and $\epsilon_2 = 0$, the classical integrable systems we have investigated become quantized. We now review the interpretation of the quantization in terms of a B-brane on the spectral curve using the refined topological string [2]. The mirror Calabi-Yau geometry takes the form

$$uv + H(x, p) = 0.$$

When the Ω -background is turned on, the classical spectral curve, Σ defined by $H(x, p) = 0$, is promoted to a quantum spectral curve

$$\hat{H}(x, p)\Psi(x) = 0$$

where $\Psi(x)$ is the wave-function for a B-brane. The quantum spectral curve is the Baxter equation for the relativistic Toda chain [196]. The coefficients of $H(x, p)$ are the energy eigenvalues of the Hamiltonians (and Casimirs) of the quantum integrable system. In the relativistic case, the differential operators $e^{\hat{p}}$ become shift operators, acting on $\Psi(x)$ by

$$e^{\hat{p}}\Psi(x) \rightarrow \Psi(x + \hbar).$$

Solutions to this differential equation were obtained from the refined topological string partition function in [2]. Thus the refined topological string should provide a way to solve the Baxter equation for quantum integrable systems. We plan to elucidate this connection in future work.

A graphical representation for the finite non-periodic Toda lattice, similar to dimer models, was developed [134] using the Poisson geometry of planar directed networks in an annulus [135]. The geometry of planar directed networks was developed in order to study the totally nonnegative Grassmannian [243]. Our work suggests an intriguing connection between the geometry of brane tilings and the totally nonnegative part of the double loop Grassmannian.

In general, gauge theories arising from brane tilings have sequences of periodic Seiberg dualities known as duality cascades [116, 119, 120, 161, 187]. According to [137], Seiberg dualities correspond to canonical transformations of the integrable system. In terms of the new variables the Hamiltonians will typically take a different functional form. However, since cascades are periodic, we are interested in special canonical transformations that preserve the functional form of the commuting Hamiltonians.

Such canonical transformations are known as auto-Bäcklund-Darboux transformations. For example, the transformation [107]

$$\tilde{c}_i = c_i \frac{d_i + c_{i-1}}{d_{i+1} + c_i}, \quad \tilde{d}_i = d_{i+1} \frac{d_i + c_{i-1}}{d_{i+1} + c_i}$$

is an auto-Bäcklund-Darboux transformation of the relativistic Toda chain because it is canonical and the new Hamiltonian

$$\tilde{H} = \sum_i \left(\tilde{c}_i + \tilde{d}_i \right)$$

takes the same functional form as the original Hamiltonian. The theory of auto-Bäcklund-Darboux transformations is closely related to the theory of separation of variables and discrete-time integrable systems [195, 272]. Thus we expect a fruitful interplay between duality cascades and integrable systems.

Chapter. 4

Orientifolds and the Refined Topological String

4.1 Introduction

Recently in [19], a refinement of $SU(N)$ Chern-Simons theory on Seifert manifolds was constructed by studying certain M-theory backgrounds. There it was also shown that in the large N limit, the partition function of refined Chern-Simons theory on S^3 is equal to the partition function of the refined topological string on the resolved conifold, thus providing a refinement of the celebrated Gopakumar-Vafa duality [140].

It was also shown that the refined Chern-Simons theory can be used to explicitly compute new two-variable polynomials associated to torus knots. These invariants generalize the one-variable quantum $SU(N)$ knot invariants. Further, the authors of [19] discovered that under appropriate changes of variables, these new polynomials could be used to determine the superpolynomial [102] of certain torus knots. The superpolynomial is the Poincare polynomial of a knot homology theory categorifying the HOMFLY polynomial. It encodes the large N behavior of Khovanov-Rozansky knot invariants.

In analogy with the unrefined case, refined Chern-Simons on a three-manifold M is equivalent to open refined string theory on the Calabi-Yau, T^*M . Thus refined Chern-Simons theory is part of the broader program of understanding the refinement of topological strings in both the closed and open settings. In the absence of a worldsheet definition, this refinement is best understood in terms of counting BPS states in M-theory.

In this chapter, we analyze refined topological string theory when there is an orientifold acting on the spacetime. Again, we find that refinement is most naturally understood by studying an index that counts second-quantized BPS contributions in M-theory. This analysis leads us to propose new integrality structures for orientifolds of both the refined string partition function, and the unrefined string studied

previously in [1, 49, 50, 191, 192, 209, 259, 273].

Our analysis also sheds light on the conjecture [90, 194] that the refined topological string at $\beta = \frac{1}{2}, 2$ is equal to the ordinary topological string in the presence of an $SO(N) / Sp(N)$ orientifold. From the second-quantized M-theory perspective, it is clear that in the noncompact part of spacetime, the trace that computes the refined topological A-model at $\beta = \frac{1}{2}, 2$ is the same as the unrefined trace when an orientifold acts on two of the spacetime directions. However, on the internal Calabi-Yau, X , the two computations are different – in the refined case, there is an internal $U(1)_R$ rotation and in the unrefined case, there is an anti-holomorphic involution acting on X .

For simple geometries, such as the Dijkgraaf-Vafa models that engineer $SO(N)$ and $Sp(N)$ gauge theory with matter in the symmetric or antisymmetric representation, we expect there will be no significant difference between the presence or absence of the involution and the conjectured correspondence will hold [3, 55, 89, 168, 199]. However, from this line of reasoning, we expect that the correspondence will not hold more generally unless we drop the involution acting on the internal Calabi-Yau.

Having understood the general structure of refined topological strings in the presence of orientifolds, we turn to the $SO(2N)$ refined Chern-Simons theory that arises from wrapping branes on an orientifold plane. We explain how to solve the theory, and study the invariants that come from the expectation value of torus knots. We also study the large N limit of the theory and show that it is consistent with the expected form of refined closed strings propagating on an orientifold of the resolved conifold. This gives a new refinement of the $SO(N)$ Chern-Simons geometric transition studied in [259].

The organization of this chapter is as follows. In section 4.2, we review the construction of open and closed refined topological string theory from M-theory. We also review the connection to instanton counting, vortex counting, and knot theory. In section 4.3, we introduce orientifolds and explain how refinement can be extended to unoriented strings. From this analysis we propose a new integrality condition in Section 4.4 for both unrefined and refined topological strings on orientifolds.

In section 4.5, we review the definition of refined Chern-Simons theory and its connection with refined open string theory. We also explain the straightforward generalization of refined Chern-Simons theory to all ADE gauge groups. In section 4.6 we clarify the connection between refined Chern-Simons theory and knot homology, and use the $SO(2N)$ refined Chern-Simons theory to compute invariants of torus knots that generalize the Kauffman polynomials. We obtain new knot invariants associated to the $SO(2N)$ gauge group in the fundamental and spinor representations. Unlike the $SU(N)$ case, we find that these polynomials cannot generally be related to the Kauffman superpolynomial [150] of $SO(N)$ knot homology by a change of variables. This is not unexpected as, by construction, refined Chern-Simons theory computes an index, and not a Poincare polynomial.

Finally, in section 4.7 we study the large N limit of the $SO(N)$ refined Chern-Simons theories on S^3 . The result is naturally interpreted as the partition function

of refined closed strings on an orientifold of $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbb{P}^1$.

In Appendix C we give a detailed description of the topologically twisted $(2,0)$ theory on Seifert manifolds. Some useful facts about Macdonald polynomials are reviewed in Appendix D, and specific results about $SO(2N)$ and its Macdonald polynomials are given in Appendix E. Finally, in Appendix F we review the refined indices in five and three dimensions that compute the refined topological string.

4.2 M-Theory and Refined Topological Strings

In this chapter we will mainly focus on a one-parameter deformation of topological string theory that is known as the refined topological string. Before discussing refinement, we review some useful facts about unrefined topological string theory. Recall that the (A-model) closed topological string localizes on holomorphic maps from Riemann surfaces into a Calabi-Yau, X , and is only sensitive to the Kahler structure of X . We can also introduce branes wrapping Lagrangian three-cycles, $L \subset X$, so that the open topological string localizes on holomorphic two-chains with boundary on L . Although topological string theory was originally defined from this worldsheet point of view, it was later realized that both open and closed topological strings naturally compute certain physical indices in M-theory [24, 57, 60, 91, 138, 139, 162, 237]. In many ways, this modern viewpoint is advantageous since it reveals an integrality structure that is hidden on the worldsheet.

Although there is not yet a worldsheet definition of the refined topological string, it has a very natural definition in M-theory. Refining the topological string simply corresponds to computing a more general trace, which is a five-dimensional analogue of the refined spin character of [127].

In this section we review the definition of the refined topological string in both the closed and open cases. Further, we explain how this is connected to K-theoretic instanton counting and knot homology, and emphasize the integrality properties that arise from M-theory. This will lay the foundation for introducing Calabi-Yau orientifolds in the next section.

In the pioneering work of [138, 139], the topological A-model was reinterpreted in terms of integer Gopakumar-Vafa invariants that count BPS M2 branes. The connection was made by starting with IIA physical string theory on the geometry, $X \times \mathbb{R}^{3,1}$ and considering the low-energy theory in the four spacetime dimensions. It is known that the topological A-model computes terms in the low energy effective action of the form,

$$\int d^4x \int d^4\theta F_g(t_i) (\mathcal{W}^2)^g \quad (4.2.1)$$

where $F_g(t_i)$ is the genus g free energy of the topological A-model, the t_i are the vector multiplets whose lowest components parametrize the Kahler moduli space of

X , and $\mathcal{W}$ is the $\mathcal{N} = 2$ Weyl multiplet. Expanding this out in components gives,

$$\int d^4x F_g(t_i) (\lambda^2)^{g-1} R_+^2 + \dots \quad (4.2.2)$$

where λ is the self-dual graviphoton field strength and R_+ is the self-dual part of the Riemann tensor. The crucial observation of Gopakumar and Vafa was that by turning on a background graviphoton field-strength, $\lambda = g_s dx_1 \wedge dx_2 + g_s dx_3 \wedge dx_4$, these terms could be reproduced by integrating out massive BPS matter. This charged matter comes precisely from D2-D0 bound states wrapping two-cycles in X . It is natural to go to large coupling so that IIA becomes M-theory, and the D2-D0 states lift to M2 branes with momentum around the M-theory circle. Integrating out these states by a Schwinger-type calculation gives a new way of writing the topological string free energy,

$$F_{top} = \sum_{\beta, s_L, s_R} \sum_{d=1}^{\infty} \sum_{j_L=-s_L}^{s_L} \sum_{j_R=-s_R}^{s_R} \frac{1}{d} \frac{(-1)^{2s_L+2s_R} N_{\beta}^{s_L, s_R} q^{2dj_L} Q^{\beta d}}{(q^{d/2} - q^{-d/2})^2} \quad (4.2.3)$$

where $q = e^{ig_s}$ and $N_{\beta}^{s_L, s_R}$ counts M2 branes wrapping the cycle $\beta \in H_2(X, \mathbb{Z})$ with intrinsic spin, (s_L, s_R) . It is important to note that the Gopakumar-Vafa invariants $N_{\beta}^{j_L, j_R}$ are in fact *integers*, since they count BPS states. Thus by integrating out M2-branes, this gives a new integrality structure underlying topological string theory.

Also note that F_{top} only depends on the combinations,

$$N_{\beta}^{s_L} \equiv \sum_{s_R} (-1)^{2s_R} (2s_R + 1) N_{\beta}^{s_L, s_R} \quad (4.2.4)$$

so these are usually defined as the Gopakumar-Vafa invariants. The starting point for refining the topological string is the desire to define a more general theory that keeps track of the full spin information, and not just the s_L spin content of M2 branes. This can be done by turning on a non-self-dual graviphoton background, $\lambda = \epsilon_1 dx_1 \wedge dx_2 - \epsilon_2 dx_3 \wedge dx_4$. Then we define the refined topological string to be the result of performing the same Schwinger calculation in this modified background,

$$F_{ref\ top} = \sum_{\beta, s_L, s_R} \sum_{d=1}^{\infty} \sum_{j_L=-s_L}^{s_L} \sum_{j_R=-s_R}^{s_R} \frac{1}{d} \frac{(-1)^{2s_L+2s_R} N_{\beta}^{s_L, s_R} (q/t)^{dj_R} (qt)^{dj_L} Q^{\beta d}}{(q^{d/2} - q^{-d/2})(t^{d/2} - t^{-d/2})} \quad (4.2.5)$$

where $q = e^{i\epsilon_1}$ and $t = e^{-i\epsilon_2}$. This gives a working definition of the refined topological string, and makes it clear that refinement captures much more information about the spin structure of the BPS spectrum. In the rest of this chapter, we will often refer to this picture as the “first-quantized” perspective, since the free-energy is related to counting of single BPS states, rather than the “second-quantized” perspective which we now introduce.

As explained in [91, 162], the partition function, $Z_{top} = \exp(F_{top})$ of the unrefined topological A-model on a Calabi-Yau threefold, X , can be computed as a trace over

the second-quantized Hilbert space of BPS states. We take M-theory on $\mathbb{C}^2 \times S^1 \times X$ and compute the five-dimensional trace,

$$Z_{\text{M-theory}}(X, q) = \text{Tr} (-1)^F q^{S_1 - S_2} e^{-\beta H} \quad (4.2.6)$$

where S_1 and S_2 are rotations in the z_1 and z_2 planes respectively, and β is the radius of the thermal circle. This trace only receives contributions from BPS states (see Appendix F for details), and is equivalent to the geometry,

$$(X \times TN \times S^1)_q \quad (4.2.7)$$

where TN denotes the Taub-NUT space. The Taub-NUT is twisted along the “thermal” S^1 so that upon going around the S^1 we have the rotation,

$$\begin{aligned} z_1 &\rightarrow q z_1 \\ z_2 &\rightarrow q^{-1} z_2 \end{aligned} \quad (4.2.8)$$

where $q = e^{ig_s}$ as before. Note that the integrality structure of the Gopakumar-Vafa invariants in the graviphoton background computation is precisely what ensures that the result can be interpreted as a second-quantized trace.

This relation between M-Theory and the topological A-model extends to the *open* topological string sector as follows [24, 57, 60, 237]. Consider the open topological A-model with N A-branes wrapping a special lagrangian 3-cycle, L , inside a Calabi-Yau threefold, X . The corresponding M-theory partition function comes from wrapping N M5-branes on,

$$(L \times \mathbb{C} \times S^1)_q \quad (4.2.9)$$

where the $\mathbb{C}$ is the cigar-shaped submanifold, $\{z_2 = 0\}$, sitting inside the Taub-NUT space. Note that the M5-brane partition function is given explicitly by the same index,

$$Z_{M5}(L, X, q) = \text{Tr} (-1)^F q^{S_1 - S_2} e^{-\beta H} \quad (4.2.10)$$

Note however, that now S_2 has the interpretation of R-charge from the perspective of the M5 brane while S_1 corresponds to a rotation along the brane. For more details on the indices that are relevant for the open and closed, refined and unrefined topological strings, see Appendix F.

Now we would like to introduce a refined index for both the closed and open A-model. Instead of 4.2.7, consider the refined M-Theory geometry,

$$(X \times TN \times S^1)_{q,t} \quad (4.2.11)$$

where upon going around the S^1 , the Taub-NUT space is twisted by,

$$\begin{aligned} z_1 &\rightarrow q z_1 \\ z_2 &\rightarrow t^{-1} z_2 \end{aligned} \quad (4.2.12)$$

As explained in [2, 19], when $t \neq q$ this configuration will break supersymmetry. However, when X is a *non-compact* Calabi-Yau threefold, M-Theory on X geometrically engineers a five-dimensional theory which has a conserved $U(1)_R \subset SU(2)_R$ symmetry. Then we can modify the construction to preserve supersymmetry by including an R-symmetry twist as we go around the S^1 . When X is non-compact, this $U(1)_R$ is actually realized geometrically by a Killing vector in the Calabi-Yau (see [221] for a recent discussion of this symmetry). This geometry then computes the index,

$$Z_{\text{refined top}}(X, q, t) \equiv Z_{\text{M-theory}}(X, q, t) = \text{Tr} (-1)^F q^{S_1 - S_R} t^{S_R - S_2} e^{-\beta H} \quad (4.2.13)$$

and gives a definition of the refined closed topological string on X .¹

It is instructive to consider the two possible dimensional reductions along either the thermal S^1 or the S^1 of the Taub-NUT space, as studied in the unrefined case in [91, 129]. If we consider X to be noncompact, then this geometry engineers a five dimensional gauge theory and by reducing on the thermal S^1 we precisely obtain the Nekrasov partition function of the gauge theory at $(i\epsilon_1, i\epsilon_2) = (\log q, -\log t)$ [225].

If we instead reduce along the Taub-NUT S^1 we obtain IIA string theory on the geometry,

$$X \times \mathbb{R}^3 \times S^1 \quad (4.2.14)$$

with a D6 brane wrapping $X \times S^1$ and sitting at the origin of $\mathbb{R}^3$. Here it is helpful to recall some useful facts about the Taub-NUT geometry. The geometry has a $U(1)_L \times SU(2)_R$ isometry, which we have used above in the definition of the index. Asymptotically, Taub-NUT looks like $S^1 \times \mathbb{R}^3$ and the $U(1)_L$ isometry rotates the S^1 , while the $SU(2)_R$ rotates the base geometry. So upon dimensional reduction, the charge under $U(1)_L$ becomes the $D0$ charge, while the charge under $U(1)_R$ becomes the spin in the base $\mathbb{R}^3$. Explicitly, $\sqrt{qt}$ becomes the $D0$ chemical potential, while $\sqrt{q/t}$ is the chemical potential for a combination of spin and R-charge, and the index on the D6 brane can be rewritten as,

$$Z_{\text{refined top}}(X, q, t) = \text{Tr}_{D6} (-1)^F q_1^{Q_0} q_2^{2J_3 - 2S_R} \quad (4.2.15)$$

where $q_1 = \sqrt{qt}$, $q_2 = \sqrt{q/t}$, Q_0 is the $D0$ brane charge, and J_3 is the generator of the rotation group in $\mathbb{R}^3$. Upon setting $q_2 = 1$, this gives the unrefined topological string, which is known to be equivalent to the topologically twisted theory living on a D6 brane wrapping X . In the refined case, this computation of refined BPS states bound to one $D6$ brane should be equivalent to the refined topological string.

¹It is important to emphasize that the refined topological string computes this protected spin character rather than simply counting the refined BPS multiplicities (as it would without the additional $U(1)_R$ twist). The refined BPS multiplicities themselves, $N_\beta^{(j_L, j_R)}$, are not invariant under changes of the complex structure [95], but the protected spin character *is* invariant. When the Calabi-Yau has no complex structure deformations, the naive and protected indices agree if no “exotic” BPS states (states with nonzero R-charge) are present in the spectrum. The absence of such exotic BPS states in the four-dimensional field theory limit was conjectured in [127].

It is natural to extend this construction to the open string case by inserting a stack of N M5-branes wrapping a special lagrangian L inside X . We will also require that L is fixed by the geometric $U(1)_R$ killing vector in X . Then the full geometry wrapped by the M5 branes is given by,

$$(L \times \mathbb{C} \times S^1)_{q,t} \quad (4.2.16)$$

where again $\mathbb{C}$ denotes the locus $\{z_2 = 0\}$ inside the Taub-NUT space. Now we can compute the same index as in the closed case,

$$Z_{M5}(L, X, q, t) = \text{Tr} \left((-1)^F q^{S_1 - S_R} t^{S_R - S_2} e^{-\beta H} \right) \quad (4.2.17)$$

This gives a definition of the refined open topological string theory on X in the presence of N refined A-branes wrapping L . Now recall that in the unrefined case, this M-theory partition function was related to ordinary Chern-Simons theory for the choice, $X = T^*L$. In the refined case, we do not yet have a path integral definition of the refined Chern-Simons theory, so it is natural to take this M-theoretic construction as the *definition* of $SU(N)$ refined Chern-Simons theory,

$$Z_{ref\ CS}(L, q, t; SU(N)) := Z_{N\ M5}(L, T^*L, q, t) \quad (4.2.18)$$

Again we can gain further insight by alternatively reducing the M-theory geometry on either the thermal S^1 or the Taub-NUT S^1 . Reducing on the thermal S^1 we obtain the omega background in the presence of a surface operator. Of course, it is worth noting that in some examples ($X = T^*S^3$), the “geometrically engineered” gauge theory in the bulk will be trivial and the only nontrivial dynamics will live on the surface operator. However, more generally we will obtain a surface operator coupled to a gauge theory, with the Omega-deformed theory computing a coupled instanton-vortex partition function [96]. The wall crossing behavior of such coupled 2d-4d systems has recently been studied in [128].

Alternatively, reducing on the Taub-NUT S^1 we are left with a $D4$ brane wrapping $S^1 \times \mathbb{R}_+ \times L$ and ending on the $D6$ brane. This is precisely the geometry considered by Witten in connection with Khovanov homology [285]. The only difference is that here we compute an index by utilizing the additional $U(1)_R$ symmetry, whereas Witten studies Khovanov homology directly, which *cannot* be computed as an index as it only involves the gradings, S_1 and S_2 .

4.3 Orientifolds and M-Theory

Having reformulated refined topological string theory as a trace in M-theory, we now introduce orientifolds. We will restrict to orientifolds that act as an anti-holomorphic involution, $I : X \rightarrow X$, on the Calabi-Yau, including both possibilities of I having fixed points or acting freely.

In physical string theory, orientifolds are defined by specifying a simultaneous involution on the spacetime, I , and orientation reversal, Ω , on the worldsheet. This definition can be extended to the ordinary topological string [1], by starting with the covering space, Σ , of an unorientable worldsheet. Then the A-model will localize on holomorphic maps, $\phi : \Sigma \rightarrow X$ satisfying the additional constraint that $I \circ \phi = \phi \circ \Omega$. Mathematically, this means that in the presence of an orientifold, the topological string is counting holomorphic maps which are $\mathbb{Z}_2$ -equivariant. Note that since orientation reversal on the worldsheet is anti-holomorphic, this constraint only makes sense if I acts anti-holomorphically on X as we have required. To fully specify an orientifold both in the physical and the topological string, we must make a choice of the sign of the cross-cap amplitude. In this chapter we will restrict to the negative sign case (for open strings, this leads to an $SO(N)$ gauge group), as this choice simplifies the lift to M-theory.

This gives a working worldsheet definition of unoriented topological strings, but it was further conjectured in [49, 50, 259, 273] that the unoriented topological string can be rewritten by counting single-particle BPS states. There it was argued that the orientifolded topological string computes terms in the low energy effective action of IIA string theory on $X \times \mathbb{C}^2$, with the involution acting simultaneously as I on X and as a reflection on two of the spacetime coordinates, $(z_1, z_2) \rightarrow (z_1, -z_2)$. For future reference, we refer to this combined action as $\tilde{I}$. In the case when I has a fixed locus, L , this corresponds to wrapping an $O4$ plane on $L \times \mathbb{C}$. As in the ordinary case, these terms in the effective action should also be computable by introducing a self-dual graviphoton background and studying the contribution of wrapped branes². This results in the free energy of the closed topological A-model taking the general form,

$$\mathcal{F}(X/I, g) = \frac{1}{2} \mathcal{F}(X, g) + \mathcal{F}(X/I, g)_{unor} \quad (4.3.19)$$

$$\begin{aligned} &= \frac{1}{2} \sum_{d=1}^{\infty} \sum_{g=0} \sum_{\beta} \frac{1}{d} \frac{N_{\beta}^g}{(q^{d/2} - q^{-d/2})^{2-2g}} Q^{\beta d} \\ &+ \sum_{d \text{ odd/even}} \sum_{g=0} \sum_{\beta} \frac{1}{d} \frac{N_{\beta}^{g,c=1}}{(q^{d/2} - q^{-d/2})^{1-2g}} Q^{\beta d} \\ &+ \sum_{d \text{ odd/even}} \sum_{g=0} \sum_{\beta} \frac{1}{d} N_{\beta}^{g,c=2} (q^{d/2} - q^{-d/2})^{2g} Q^{\beta d} \end{aligned} \quad (4.3.20)$$

²In order to show this more rigorously, we would need to determine exactly which terms in the orientifolded $\mathcal{N} = 1$ low energy effective action are computed by the topological string. Then by integrating out wrapped brane contributions to these terms, we could derive this integrality structure. This was done at genus zero in [1], but it would be interesting to study the higher genus amplitudes in more detail.

where the $N_\beta^{g,c}$ are integers that count branes wrapping the cycle, $\beta \in H_2(X/I, \mathbb{Z})$.³ In the unoriented sector, the sums are over either odd d or even d , depending on the details of the orientifold action I . Note that the orientifold action explicitly breaks the spacetime rotational symmetry, $SU(2)_L \times SU(2)_R$ down to $U(1)_1 \times U(1)_2$. Because of this breaking, it is no longer guaranteed that the BPS states will come in full spin multiplets. However, the $(q^{d/2} - q^{-d/2})^{2g}$ factors come from the moduli space of flat connections on a wrapped $D2$ brane, even if the brane is wrapping an unorientable cycle. As explained below, this moduli space generically takes the form of T^{2g} so we expect that the BPS contributions will continue to sit in full $SU(2)_L$ multiplets.

To understand this conjecture physically, note that except for the factor of $\frac{1}{2}$, the first term is the same as the free energy for the topological string on X in the absence of an orientifold. This term counts BPS states that wrap oriented cycles in the X and are free to move in the four spacetime directions. Since these states can move in $\mathbb{C}^2$, we obtain a factor of $(q^{d/2} - q^{-d/2})^2$ in the denominator. Although we do not know of a convincing target space interpretation for the factor of $\frac{1}{2}$, [259] argued that it arises on the worldsheet from dividing by the orientation reversal symmetry, Ω .

Now we turn to the unoriented contributions to the free energy. These come from branes wrapping BPS cycles in X/I . These cycles will lift to surfaces with boundaries in the covering space X (if they did lift to closed BPS two-cycles, then this would be an oriented rather than an unoriented contribution). Now recall that the orientifold acts on the full geometry as I on X and as $z_2 \rightarrow -z_2$ in two of the spacetime dimensions. When we take into account this full orientifold action, in order to get a genuine closed BPS state with no boundaries, it is necessary that the brane sits at $z_2 = 0$. This means that the unoriented branes effectively propagate in only two noncompact dimensions, so the denominator of the unoriented contributions will give only one power of $(q^{d/2} - q^{-d/2})$.

The first unoriented contribution to the free energy comes from D-branes wrapping unorientable surfaces with one crosscap. Recall that a crosscap is inserted into a surface by cutting open a hole and identifying antipodal points on the boundary. The lowest order contribution comes from curves with genus 0 and one crosscap, which have the topology of $\mathbb{RP}^2$.

To understand the structure in more detail, it is helpful to recall that the first

³In [191, 192, 273], it was found experimentally that for some orientifold actions with a non-toric fixed locus, the integrality of the BPS states only holds when a D-brane is introduced wrapping the same locus as the orientifold fixed plane. For example, such a D-brane must be introduced in the local $\mathbb{P}^2$ geometry when the orientifold fixed plane wraps a real line bundle over the real locus, $\mathbb{RP}^2$. In such cases, it was argued that only the combined contribution from open strings and unoriented strings gives the expected BPS integrality structure, and the resulting integers are known as *real* Gopakumar-Vafa invariants. Although we will not discuss these cases explicitly, we will find that our M-theory integrality conjecture, explained in the following section, applies to these geometries as well. In fact, the equivalent of our strong integrality property was originally stated for real GV invariants by Walcher in [273].

homology of a genus g surface with one crosscap, Σ_g^1 , is given by,

$$H_1(\Sigma_g^1, \mathbb{Z}) = \mathbb{Z}^{2g} \oplus \mathbb{Z}/2\mathbb{Z} \quad (4.3.21)$$

This tells us that the continuous part of the moduli space of flat connections on Σ_g^1 is generically given by T^{2g} as in the original analysis of Gopakumar and Vafa. This accounts for the $(q^{d/2} - q^{-d/2})^{2g}$ factor in the numerator of the $n_Q^{g,c=1}$ term. Because of the $\mathbb{Z}/2\mathbb{Z}$ factor, we also have the option of turning on one unit of discrete flux, which corresponds to dissolving half a unit of $D0$ brane charge in the wrapped $D2$ brane. As argued in [259], this half unit of $D0$ brane flux will shift $t \rightarrow t + \pi i$, or equivalently, $Q^d \rightarrow (-1)^d Q^d$. Depending on how I acts, this may change the fermion number assignment and introduce an additional overall minus sign. Adding together both choices of discrete flux, we obtain the cancellation that accounts for summation over only even/odd d .

Finally, we can do the same analysis on the terms with two crosscaps. It is helpful to recall that the first homology of a genus g , $c = 2$ surface is given by,

$$H_1(\Sigma_g^2, \mathbb{Z}) = \mathbb{Z}^{2g+1} \oplus \mathbb{Z}/2\mathbb{Z} \quad (4.3.22)$$

As in the single crosscap case, this means that a genus g surface will have a moduli space of flat connections given by, T^{2g+1} , which leads to a factor of $(q^{d/2} - q^{-d/2})^{2g+1}$. Note that for the genus 0 surface with two crosscaps, the Klein bottle, this factor in the numerator cancels the factor in the denominator from moving in two noncompact dimensions. Again, the additional torsion factor $\mathbb{Z}/2\mathbb{Z}$ accounts for the restriction to even/odd d . Note that we have exhausted all possible unorientable surfaces, since in the presence of an additional crosscap, two crosscaps can be traded for a handle. Although we will not discuss it in detail here, by similar arguments we expect the open topological string in the presence of orientifolds to possess nice integrality properties from counting BPS states ending on $D4$ branes.

Now that we have given the first-quantized BPS interpretation, it is natural to conjecture that, as in the oriented case, the partition function of the topological string, Z_{top} is computed by a second quantized trace in M-theory. As the first step, we must explain how the IIA orientifold action lifts to M-theory.

Of course, since there is no worldsheet in M-theory, the *IIA* orientifolds must lift to orbifolds in eleven dimensions. The *IIA* orientifold, with $\tilde{I}$ acting on the spacetime and Ω acting on the worldsheet, lifts to the M-theory orbifold by $\tilde{I}$. To fully specify the orbifold, we need to explain how the three-form, $C_{(3)}$ is affected by the orbifold action. As explained in [78, 80, 279], $\tilde{I}$ acts as $C_{(3)} \rightarrow -C_{(3)}$. One simple way to see this is to recall that the low-energy action of M-theory contains a Chern-Simons-like term,

$$\int C \wedge dC \wedge dC \quad (4.3.23)$$

All terms in the action should be invariant under the orbifold action, and since $\tilde{I}$ is orientation-reversing, the Chern-Simons term will only be invariant if $C \rightarrow -C$.

In the case where I has a fixed locus, the string theoretic $O4^-$ plane will lift to the six-dimensional fixed locus of the M-theory orbifold, which we refer to as an $MO5$ -plane. However, when I acts freely, the orientifold action lifts to M-theory on the unorientable, smooth geometry, $(X \times \mathbb{C})/\tilde{I} \times \mathbb{C} \times S^1$. It is important to remember that the action of $\tilde{I}$ multiplies C by -1 , so that in this unorientable M-theory geometry, C is actually a *twisted* three-form rather than an ordinary three-form [282].

Having explained the relevant objects in M-theory, we are now ready to study the topological string. We would like to compute the M-theory trace by taking the geometry,

$$\left(\mathbb{C}_1 \times (\mathbb{C}_2 \times X) / \tilde{I} \times S^1 \right)_q \quad (4.3.24)$$

where as before, $(\cdots)_q$ indicates that as we go around the S^1 , we rotate the $\mathbb{C}_1 \times \mathbb{C}_2$ by $(z_1, z_2) \rightarrow (qz_1, q^{-1}z_2)$. Note that the orbifold takes $\theta_2 \rightarrow \theta_2 + \pi$, while the S^1 twist takes $\theta_2 \rightarrow \theta_2 - g_s$, so these operations commute with each other, as they must in order to define a sensible geometry. This defines the closed topological string in the presence of orientifolds by representing it as an M-theory trace. Similarly, the open topological string is computed exactly as before by introducing $M5$ branes wrapping special lagrangain submanifolds in X , but now in the presence of orientifolds.

Now we are finally ready to give a definition of the refined topological string in the presence of orientifolds. Although it may be possible to give a definition from the first-quantized graviphoton picture, we have found that refinement is simpler and clearer in the second-quantized picture.

As in the oriented case, to refine we simply need to compute a more general M-theory partition function. This leads us to consider the geometry,

$$\left(\mathbb{C}_1 \times (\mathbb{C}_2 \times X) / \tilde{I} \times S^1 \right)_{q,t} \quad (4.3.25)$$

where as in the oriented case, the $\mathbb{C}^2$ geometry is rotated as we go around the S^1 ,

$$\begin{aligned} z_1 &\rightarrow qz_1 \\ z_2 &\rightarrow t^{-1}z_2 \end{aligned} \quad (4.3.26)$$

where we also must include a rotation by the $U(1)_R$ to preserve supersymmetry⁴. Then we define the refined topological string in the presence of orientifolds to be equal to this M-theory partition function,

$$Z_{ref\ closed}(X/I; q, t) = \text{Tr}_{\text{M-Theory}} (-1)^F q^{S_1 - S_R} t^{S_R - S_2} e^{-\beta H} \quad (4.3.27)$$

⁴For this to make sense, we must also require that the orientifold action I is compatible with the isometry generating the $U(1)_R$ symmetry.

Similarly, the refined open topological string is defined by introducing $M5$ branes wrapping special lagrangians and computing the same trace.

Before moving on to explicit computations, it is interesting to look at the general form of this M-theory partition function. We will focus on the closed case, although the open case is completely analogous. There are two types of BPS states contributing to the partition function. The first contribution comes from BPS M2-branes wrapping closed, orientable two-cycles. Each BPS state gives a field in four dimensions, $\Phi(z_1, \bar{z}_1, z_2, \bar{z}_2)$, and this field has additional excitations which are BPS, provided that Φ depends holomorphically on z_1 and z_2 . It is natural to decompose Φ into modes as,

$$\Phi = \sum_{l_1, l_2} \alpha_{l_1, l_2} z_1^{l_1} z_2^{l_2} \quad (4.3.28)$$

In the full M-theory partition function, we must include the contributions of these modes, which carry angular momentum in the $\mathbb{C}_1$ and $\mathbb{C}_2$ planes. However, here it is important to remember that the M-theory orbifold acts nontrivially on the spacetime as $z_2 \rightarrow -z_2$. The field, Φ , should have a well-defined transformation property under this orbifold, so that $\Phi(z_1, -z_2) = \pm \Phi(z_1, z_2)$, where the choice of $\pm$ is related to how the orbifold acts in X . This means that we should only keep either the even or odd modes of l_2 .

Thus, the contribution to $Z_{ref\ top}$ from an $M2$ brane with intrinsic spin, (m_1, m_2) , wrapping a two-cycle in the class $\beta \in H_2(X, \mathbb{Z})$ takes the form,

$$\prod_{l_1, l_2=1}^{\infty} \left(1 - q^{m_1+l_1} t^{m_2+2l_2} Q^{\beta} \right) \quad (4.3.29)$$

The second type of contribution comes from M2 branes ending on the orbifold (when I has fixed points) or M2 branes wrapping unorientable cycles (when I does not have fixed points). These $M2$ branes are frozen at the fixed locus, $z_2 = 0$ (since otherwise they would have a boundary). This means that they make a contribution in the form of a quantum dilogarithm,

$$\prod_{l_1=1}^{\infty} \left(1 - q^{m_1+l_1} t^{m_2} Q \right) \quad (4.3.30)$$

Putting these contributions together, we find that Z takes the form,

$$Z_{ref}(X/I; q, t) = \prod_{l_1, l_2=1}^{\infty} \left(1 - q^{m_1+l_1} t^{m_2+2l_2} Q^{\beta} \right)^{M_{\beta}^{(m_1, m_2)}} \prod_{l_1=1}^{\infty} \left(1 - q^{m_1+l_1} t^{m_2} Q \right)^{\widetilde{M}_{\beta}^{(m_1, m_2)}} \quad (4.3.31)$$

where the partition function is completely determined by the integer invariants, $M_{\beta}^{(m_1, m_2)}$ which counts oriented $M2$ branes and $\widetilde{M}_{\beta}^{(m_1, m_2)}$, which counts the unoriented branes in X/I .

4.4 A New Integrality Conjecture from M-Theory

It is interesting to take equation 4.3.31 and return to the unrefined case, $t = q$. We then have integrality properties following from both the second-quantized M-theory picture and from the first-quantized Gopakumar-Vafa picture. When no orientifolds are present, these integrality properties are equivalent. More explicitly, the first-quantized integer invariants appearing in the free energy, F , guarantee the integrality in the second-quantized partition function, $Z = e^F$.

However, once we introduce orientifolds, the first-quantized structure (equation 4.3.20) includes overall factors of $1/2$ in the free energy. This poses a serious problem since such half-integers will result in terms of the form $\sqrt{1 - q^n Q^\beta}$ in the partition function. Such terms have no sensible interpretation as counting second-quantized states in a free Fock space.

Thus, if the topological string partition function can truly be defined as an index in M-theory, we require a stronger integrality constraint on the Gopakumar-Vafa invariants $N_\beta^{g,c}$ appearing in the free energy. To see precisely which constraints are needed, it is helpful to rewrite the unoriented contributions to the free energy as ⁵

$$\begin{aligned} & \sum_{d \text{ odd}} \sum_{g=0}^{\infty} \sum_{\beta} \frac{1}{d} \frac{N_\beta^{g,c}}{(q^{d/2} - q^{-d/2})^{1-2g}} Q^{\beta d} \\ &= \sum_{d=1}^{\infty} \sum_{g=0}^{\infty} \sum_{\beta} \frac{1}{d} \frac{N_\beta^{g,c}}{(q^{d/2} - q^{-d/2})^{1-2g}} Q^{\beta d} - \sum_{d=1}^{\infty} \sum_{g=0}^{\infty} \sum_{\beta} \frac{1}{2d} \frac{N_\beta^{g,c}}{(q^d - q^{-d})^{1-2g}} Q^{2\beta d} \end{aligned} \quad (4.4.32)$$

This form makes it clear that half-integers can appear in both the oriented and unoriented sector. Of course, the simplest way to guarantee integrality is if all the Gopakumar-Vafa invariants, N_β^g and $N_\beta^{g,c}$, are even. However, generically, we do not expect this to be the case. Instead, integrality can be achieved more generally if the half-integer contributions in the oriented and unoriented sector partially cancel each other to give integers.

To see explicitly how this occurs, let us focus on the oriented genus g contribution of an M2 brane wrapping β , and take g to be even. Now we can combine this with the one-crosscap ($c = 1$), genus g amplitude for curves wrapping $\beta/2$. Grouping the half-integer pieces of each term gives,

$$\frac{1}{2d} \left(\frac{N_\beta^g (q^{d/2} - q^{-d/2})^{2g}}{(q^{d/2} - q^{-d/2})^2} - \frac{N_{\beta/2}^{g/2, c=1} (q^d - q^{-d})^g}{(q^d - q^{-d})} \right) Q^{\beta d} \quad (4.4.33)$$

⁵If we instead looked at the case of an even sum over d , only the second term would appear, with opposite sign. Since the second term is the crucial one in our analysis, precisely the same conclusions hold regardless of whether the sum is over even or odd d .

This can be rewritten as,

$$\frac{Q^{\beta d}(q^{d/2} - q^{-d/2})^g}{2d(q^{d/2} - q^{-d/2})(q^d - q^{-d})} \left(N_{\beta}^g (q^{d/2} - q^{-d/2})^g (q^{d/2} + q^{-d/2}) - N_{\beta/2}^{g/2, c=1} (q^{d/2} + q^{-d/2})^g (q^{d/2} - q^{-d/2}) \right) \quad (4.4.34)$$

Now observe that the q -factors in front of N_{β}^g and $N_{\beta/2}^{g/2, c=1}$ are identical except for signs. When we expand these terms out, we will obtain a polynomial in q whose coefficients are multiples of either $N_{\beta}^g + N_{\beta/2}^{g/2, c=1}$ or $N_{\beta}^g - N_{\beta/2}^{g/2, c=1}$. So to cancel the factor of $1/2$ in front of the expression, we require that the Gopakumar-Vafa invariants are both even or both odd.

We can repeat the same analysis by combining the odd genus contributions with the $c = 2$ terms. Putting everything together, we find that the second quantized M-theory interpretation of the topological string implies the strong integrality property,

$$N_{\beta}^g \equiv \begin{cases} N_{\beta/2}^{g/2, c=1} & \text{for } g \text{ even} \\ N_{\beta/2}^{(g-1)/2, c=2} & \text{for } g \text{ odd} \end{cases} \pmod{2} \quad (4.4.35)$$

Note that in some cases, there cannot be any BPS state wrapping the class $\beta/2$. In this case, the strong integrality property tells us that N_{β}^g itself must be even. We have verified that this integrality property is satisfied for all of the geometries studied in [49, 192, 273], including orientifolds of the conifold, the quintic, local $\mathbb{P}^2$, and more general toric geometries.

An equivalent observation was made by Walcher for the case of certain real orientifolds, with a D-brane wrapping the fixed locus [273]. Walcher argued that if we pretend the moduli space of oriented wrapped branes is a collection of points, then I acts on this set. The points that are fixed by I are precisely the BPS states wrapping the fixed real locus, while the other points must come in pairs that are exchanged by I . This implies that the “real” (unoriented) and oriented Gopakumar-Vafa invariants must differ by a multiple of two. Here we have generalized this statement to include arbitrary orientifolds, and have given it a natural M-theory interpretation.

At this point we could go one step further and obtain expressions for the natural M-theoretic integer invariants, M_{β}^s and $\widetilde{M}_{\beta}^s$, in terms of the $N_{\beta}^{g, c}$ by expanding out the polynomials in q . It is interesting to note that these variables are complementary, in the sense that the $M, \widetilde{M}$ variables make integrality manifest, but obscure the organization of invariants in full spin multiplets. In contrast, the natural free energy variables, $N_{\beta}^{g, c}$ make manifest the spin multiplet structure, while hiding the full integrality properties.

We can also consider the integrality structure when we have branes wrapping a special Lagrangian 3-cycle, L , inside X . When the orientifold action, I , has fixed

points, we will assume that the new branes do not sit at the fixed locus. As explained in [50], the open topological string will receive contributions from an oriented and an unoriented sector and the free energy will include half-integer contributions. As discussed above, the partition function, $Z = e^F$, of the open topological string can be written as a second-quantized trace in M-theory counting $M2$ branes ending on $M5$ branes wrapping L . For this M-theory interpretation to hold, the half-integers in the free energy must cancel, leading us to a new integrality condition in the open case.

To explain this in more detail, we begin with the first-quantized form of the open string free energy conjectured in [50],

$$F(X/I, V) = \frac{1}{2} \sum_{R_1, R_2} \sum_{d=1}^{\infty} \frac{1}{d} f_{R_1, R_2}^{cov}(q^d, Q^d) \text{Tr}_{R_1 \otimes R_2} V^d - \sum_R \sum_{d \text{ odd/even}} \frac{1}{d} f_R^{unor}(q^d, Q^d) \text{Tr}_R V^d \quad (4.4.36)$$

The first term comes from oriented open strings and can be computed by going to the covering space. Since the branes are not fixed by I , in the covering space there will be branes wrapping both L and its image under the involution, $I(L)$. However, since these two stacks of branes are related by I , their open string moduli must be equal to each other, giving the trace $\text{Tr}_{R_1} V \cdot \text{Tr}_{R_2} V = \text{Tr}_{R_1 \otimes R_2} V$. The second term comes from unoriented strings, with the sum over d restricted to odd or even positive integers depending on the orientifold action, as in the closed case.

It is convenient to rewrite the oriented piece as,

$$F^{or}(X/I, V) = \frac{1}{2} \sum_R \sum_{d=1}^{\infty} \frac{1}{d} \left(\sum_{R_1, R_2} f_{R_1, R_2}^{cov}(q^d, Q^d) N_{R_1, R_2}^R \right) \text{Tr}_R V^d \quad (4.4.37)$$

where N_{R_1, R_2}^R is the Littlewood-Richardson coefficient for decomposing the tensor product $R_1 \otimes R_2$. We can also rewrite the unoriented piece (assuming the sum is over odd d) as,

$$F^{unor}(X/I, V) = - \sum_R \sum_{d=1}^{\infty} \frac{1}{d} f_R^{unor}(q^d, Q^d) \text{Tr}_R V^d + \sum_R \sum_{d=1}^{\infty} \frac{1}{2d} f_R^{unor}(q^{2d}, Q^{2d}) \sum_{R'} c_{2;R}^{R'} \text{Tr}_{R'} V^d \quad (4.4.38)$$

where $c_{2;R}^{R'}$ is the coefficient of the second Adams Operation defined by,

$$\text{Tr}_R(V^2) = \sum_{R'} c_{2;R}^{R'} \text{Tr}_{R'}(V) \quad (4.4.39)$$

It is natural to combine the half-integer pieces in the oriented and unoriented amplitudes to give,

$$\frac{1}{2} \sum_R \sum_{d=1}^{\infty} \frac{1}{d} \left(\sum_{R_1, R_2} N_{R_1, R_2}^R f_{R_1, R_2}^{or}(q^d, Q^d) + \sum_{R'} c_{2;R'}^R f_{R'}^{unor}(q^{2d}, Q^{2d}) \right) \text{Tr}_R V^d \quad (4.4.40)$$

We can expand the f functions as,

$$f_{R_1, R_2}^{cov}(q, Q) = \sum_{\beta, s} N_{(R_1, R_2), \beta, s} Q^\beta q^s \quad (4.4.41)$$

$$f_R^{unor}(q, Q) = \sum_{\beta, s} \tilde{N}_{R, \beta, s} Q^\beta q^s \quad (4.4.42)$$

where $\tilde{N}_{R, \beta, s}$ and $N_{(R_1, R_2), \beta, s}$ are integers counting BPS states with spin s , wrapping the relative homology class, $\beta \in H_2(X, L)$, and in representation R . Then the absence of half-integers in the free energy imposes the condition,

$$\sum_{R_1, R_2} N_{R_1, R_2}^R N_{(R_1, R_2), \beta, s} \equiv \sum_{R'} c_{2; R'}^R \tilde{N}_{R', \beta/2, s/2} \pmod{2} \quad (4.4.43)$$

It is important to note that the integers $\tilde{N}_{R, \beta, s}$ and $N_{(R_1, R_2), \beta, s}$ that we have used above are not the most fundamental BPS invariants. To exhibit the full BPS structure it is necessary to include the structure of the moduli space of flat connections and geometric deformations of an open $D2$ brane [197]. In general this structure is more complicated and involves the Clebsch-Gordon coefficients of the symmetric group. For simplicity, we focus on the case of $R = \square$, which leads to a simple integrality constraint on the fundamental BPS invariants, $\hat{N}_{R, \beta, g}$.

We start by expanding out Equation 4.4.40 in traces over V ,

$$\begin{aligned} &= \frac{1}{2} (f_{\square, \cdot}^{cov} + f_{\cdot, \square}^{cov}) \text{Tr}_{\square} V + \left(\frac{1}{2} (f_{\square, \cdot}^{cov}(q, Q) + f_{\cdot, \square}^{cov}(q, Q)) + \frac{1}{2} (f_{\square, \cdot}^{cov}(q^2, Q^2) + f_{\cdot, \square}^{cov}(q^2, Q^2)) \right. \\ &\quad \left. + f_{\square, \square}^{cov}(q, Q) + f_{\square}^{unor}(q^2, Q^2) \right) \text{Tr}_{\square} V + \dots \end{aligned} \quad (4.4.44)$$

Now since I exchanges the two stacks of branes in the covering space, it follows that $f_{R, \cdot}^{cov} = f_{\cdot, R}^{cov}$. Therefore, the $\text{Tr}_{\square} V$ term and the first two terms of $\text{Tr}_{\square} V$ do not contribute any half integers, but the last two terms could. Imposing integrality leads to the condition,

$$f_{\square, \square}^{or}(q, Q) \equiv f_{\square}^{unor}(q^2, Q^2) \pmod{2} \quad (4.4.45)$$

Since we are working with the relatively simple $\square$ representation, these f functions can be expanded in terms of the fundamental BPS invariants,

$$f_{\square, \square}^{or}(q, Q) = \sum_{\beta, g} \hat{N}_{(\square, \square), g, \beta} (q^{1/2} - q^{-1/2})^{2g} Q^\beta \quad (4.4.46)$$

$$f_{\square}^{unor}(q, Q) = \sum_{\beta, g} \hat{N}_{\square, g, \beta}^{c=1} (q^{1/2} - q^{-1/2})^{2g} Q^\beta + \sum_{\beta, g} \hat{N}_{\square, g, \beta}^{c=2} (q^{1/2} - q^{-1/2})^{2g+1} Q^\beta \quad (4.4.47)$$

The invariant $\widehat{N}_{R,g,\beta}^c$ counts BPS states of genus g with c crosscaps wrapping the class β and in representation R . We can group terms exactly as we did in the closed case, and we find the integrality condition,

$$\widehat{N}_{(\square,\square),g,\beta} \equiv \begin{cases} \widehat{N}_{\square,g/2,\beta/2}^{c=1} & \text{for } g \text{ even} \\ \widehat{N}_{\square,(g-1)/2,\beta/2}^{c=2} & \text{for } g \text{ odd} \end{cases} \pmod{2} \quad (4.4.48)$$

It was argued in [209], that when X is the resolved conifold and $L = \mathcal{L}_\mathcal{K}$ is the special lagrangian corresponding to a knot $\mathcal{K}$, then the composite BPS invariants, $\widehat{N}_{(R_1,R_2),g,\beta}(\mathcal{K})$ are related to the HOMFLY polynomial in the composite representation (R_1, R_2) . By making use of this connection, the BPS invariants have been computed for many knots in [209, 241, 242]. Using this data, we have explicitly verified our strong integrality conjecture for the unknot, trefoil, and $T(2, 5)$ knots. It would be interesting to perform these checks for more complicated geometries.

In this section we have focused on unrefined amplitudes. In the case of refined amplitudes, we do not yet have a first-quantized definition, but we still demand integrality from the second quantized M-theory index that defines the theory. In Section 4.7, we will test this integrality by studying the large N limit of $SO(2N)$ refined Chern-Simons theory, and interpreting the result as refined closed strings on the resolved conifold.

4.5 Open Strings from Refined Chern-Simons Theory

Now that we have given a definition of the refined topological string for oriented geometries and in the presence of orientifolds, we would like to solve the theory. In the case of open, refined topological strings, this was done in [19] by carefully analyzing the contributions of BPS states to the refined index. As explained above, if we choose our Calabi-Yau to be the cotangent bundle over a three-manifold, T^*M , with branes wrapping M , then we expect that the open refined string theory should reduce to a three-dimensional field theory living on M . This theory is a refined version of Chern-Simons theory, and was defined for oriented strings in [19]. The crucial idea [19] used to compute the M-theory indices was to cut up M (along with T^*M) into pieces on which one can solve the theory explicitly, and glue the pieces back together. In this way, the S and T matrices of refined $SU(N)$ Chern-Simons theory could be deduced from M-theory.

In the unoriented case, the natural anti-holomorphic involution of T^*M is given by reversing the fibers of the cotangent bundle, so that $p_i \rightarrow -p_i$. This is equivalent to inserting an orientifold plane that wraps M . As is standard in both physical and topological string theory [259], the inclusion of an orientifold plane (with the appropriate choice of crosscap sign) simply changes the gauge group from $SU(N)$ to

$SO(2N)$. We can now follow the rest of the steps from [19] in the present context. Taking M to be a solid torus, the refined M-theory index can be computed explicitly in the presence of orientifold action. Instead of going through this here, we refer the reader to Appendix C for a detailed discussion of the derivation in the $SO(2N)$ case. We will simply state the answer. We find that the $SO(2N)$ theory is solved, analogously to the $SU(N)$ case, by replacing characters of $SO(2N)$ with D-type Macdonald polynomials associated to the root system of $SO(2N)$.

Moreover, in Appendix C, we also extend this to refined Chern-Simons theory with any simply-laced gauge group. Although there is no known brane realization of the E_a gauge groups, the same analysis still works in the corresponding six-dimensional $(2,0)$ theory. In the appendix, we also give an explicit example of the connection with three dimensional field theory when $M = \mathbb{R}^2 \times S^1$.

In this section we describe the refined Chern-Simons theory in detail from the perspective of topological field theory. This perspective will be especially helpful when we begin studying knot invariants in Section 4.6. Although a Lagrangian for refined Chern-Simons theory is not yet known, we can still define the theory by describing its amplitudes on simple geometries.

4.5.1 Refined Chern-Simons as a Topological Field Theory

Recall that any three-dimensional topological field theory should assign a number to any closed three manifold, $Z(M)$, and an element of the appropriate Hilbert space to any three manifold with boundary, $\Psi(M) \in \mathcal{H}_\Sigma$, where $\Sigma = \partial M$. Further, diffeomorphisms that act geometrically on the boundary should be represented by unitary operators acting on the Hilbert space, $\mathcal{O} : \mathcal{H}_\Sigma \rightarrow \mathcal{H}_\Sigma$.

The refined Chern-Simons theory can be thought of as a restricted topological field theory in three-dimensions. It is only well-defined on Seifert three-manifolds, M , to which it assigns a number, $Z(M)$. This restriction can be understood by recalling the definition of open refined string theory, which requires the existence of a $U(1)$ isometry on M . The refined Chern-Simons theory also assigns an element of the Hilbert space to any “Seifert manifold with boundary,” possessing a nondegenerate S^1 fibration. Necessarily, the boundary of such a manifold will be a disjoint union of two-tori. Finally, the modular group acts geometrically on T^2 , so the refined Chern-Simons theory gives a unitary representation of $SL(2, \mathbb{Z})$ acting on $\mathcal{H}_{T^2}$.

To fully specify the refined Chern-Simons theory, we must choose a simply-laced compact gauge group, G , an integer level, $k \in \mathbb{Z}$, and a continuous deformation parameter, $\beta \in \mathbb{R}^{\geq 0}$. It will be useful in the following formulas to define equivalent

variables,

$$q = \exp\left(\frac{2\pi i}{k + \beta y}\right) \quad (4.5.49)$$

$$t = \exp\left(\frac{2\pi i \beta}{k + \beta y}\right) \quad (4.5.50)$$

where y is the dual Coxeter number of G .⁶ Note that the unrefined limit corresponds to $\beta \rightarrow 1$.

We start by describing the Hilbert space, $\mathcal{H}_{T^2}$, associated to a T^2 boundary. Recall that in ordinary Chern-Simons theory, with gauge group, G , and coupling, k , the Hilbert space is given by the space of conformal blocks on T^2 of the associated Wess-Zumino-Witten model. This Hilbert space has a natural orthonormal basis, given by integrable representations of G at level k . In the Chern-Simons theory, this basis can be understood physically by taking the solid torus, $D \times S^1$ and inserting a Wilson line in the representation λ running along $\{0\} \times S^1$. Then performing the path integral on this geometry gives the state $|\lambda\rangle \in \mathcal{H}_{T^2}$.

Note that to define this basis, we must decide which cycle of the T^2 boundary will be filled in to make a solid torus. This gives two natural bases associated with filling in the A and B cycles respectively. The modular transformation, $S \in SL(2, \mathbb{Z})$ then will be represented as a unitary operator that transforms the A basis into the B basis.

The vector space structure of $\mathcal{H}_{T^2}$ does not change under refinement. This is expected heuristically, since β is a continuous parameter that can be adiabatically changed from the unrefined theory $\beta = 1$ to the refined theory $\beta \neq 1$. This can be seen more precisely by noting that the metric, g_i , defined below, vanishes for any representations that are not integrable at level k .

However, under refinement the inner product on these spaces does change. In ordinary Chern-Simons theory, the basis $|\lambda\rangle$ is an orthonormal one so that,

$$\langle \lambda_i | \lambda_j \rangle = \delta_{ij} \quad (4.5.51)$$

In the refined theory, this natural basis remains orthogonal but the normalization is nontrivial,

$$\langle \lambda_i | \lambda_j \rangle = g_i \delta_{ij} \quad (4.5.52)$$

where g_i is the metric defined by taking the Macdonald inner product of the Macdonald polynomial, M_{λ_i} with itself (see Appendix D for background on Macdonald polynomials). In the special case when $\beta \in \mathbb{Z}^{>0}$, the metric factor is given explicitly by,

$$g_i \equiv \prod_{\alpha \in R_+} \prod_{m=0}^{\beta-1} \frac{1 - t^{\langle \rho, \alpha \rangle} q^{\langle \lambda_i, \alpha \rangle + m}}{1 - t^{\langle \rho, \alpha \rangle} q^{\langle \lambda_i, \alpha \rangle - m}} \quad (4.5.53)$$

⁶The dual Coxeter number is given by $y = N$ for $SU(N)$, $2N - 2$ for $SO(2N)$, 12 for E_6 , 18 for E_7 , and 30 for E_8 .

where the product is over the positive roots, α , and ρ is the Weyl vector, $\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha$. For more general choices of β , there exists a combinatorial formula for g_i , which is given for the $SO(2N)$ case in Appendix E. Note that we could rescale the $|\lambda_i\rangle$ to obtain a normalized basis, but for subsequent formulas it is actually more convenient to leave the inner product nontrivial.

Now that we have described the Hilbert space associated to each T^2 boundary, we should consider arbitrary Seifert three-manifolds, M with boundaries. The refined Chern-Simons theory should assign a specific element of the Hilbert space, $\Psi(M) \in \mathcal{H}_{T^2}^{\otimes n}$, to each three-manifold.

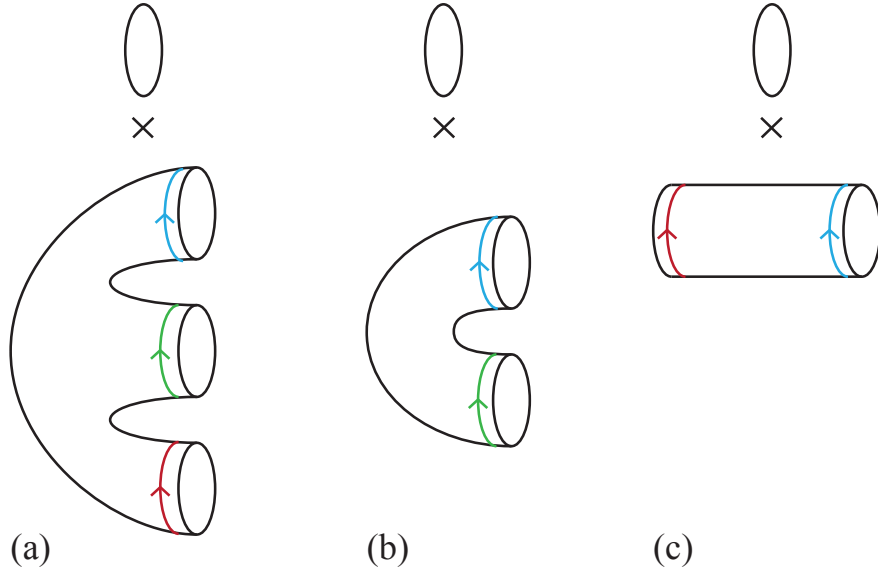


Figure 4.1: The geometric building blocks for the refined Chern-Simons theory are shown. The 3-punctured sphere times a circle, $P \times S^1$, is shown in part (a), with the orientations of the three boundary T^2 's indicated by arrows. The two propagators, which have the topology of $\mathbb{C}^* \times S^1$ are shown in parts (b) and (c). Note that they differ by the orientation of one of the boundary components.

We begin by defining the theory on the “pair of pants” geometry given by the three-punctured sphere times a circle, $P \times S^1$, shown in Figure 4.1(a). In Figure 4.1, we have included Wilson lines wrapping the A-cycles of the T^2 boundaries. This depiction is useful for two reasons: first, we will use these Wilson lines to keep track of the orientation of the boundaries, since in a TQFT changing the orientation of a boundary takes the Hilbert Space $\mathcal{H}$ to its dual, $\mathcal{H}^*$.

Second, recall from the discussion above that to specify a basis $|\lambda_i\rangle$ of $\mathcal{H}$, we must choose a cycle on T^2 . In all of the following discussion, the element $|\lambda_i\rangle$ will

correspond to taking the T^2 boundary and gluing in a solid torus, where the fiber S^1 becomes contractible. Then we insert a Wilson loop, in the representation λ_i running along one of the boundaries of the base pair of pants, P or annulus, $\mathbb{C}^*$, and sitting at a point in the now-contractible S^1 fiber. Computing the path integral then gives the state $|\lambda_i\rangle$. We have depicted this choice of basis graphically by showing the Wilson loops on P .

Then the wavefunction of the refined Chern-Simons theory on $P \times S^1$ is given by,

$$\Psi(P) = \sum_i \frac{1}{g_i S_{0i}} \langle \lambda_i | \langle \lambda_i | \langle \lambda_i | \quad (4.5.54)$$

where g_i is the metric defined above and S_{0i} can be thought of as the (q, t) -dimension of the representation λ_i . This (q, t) -dimension is a refinement of the quantum-dimension that appears in ordinary Chern-Simons theory, and is given by,

$$S_{0i} = S_{00} \dim_{q,t}(\lambda_i) \equiv S_{00} M_{\lambda_i}(t^\rho) = S_{00} \prod_{m=0}^{\beta-1} \prod_{\alpha>0} \frac{q^{\frac{\langle \lambda_i, \alpha \rangle + m}{2}} t^{\frac{\langle \rho, \alpha \rangle}{2}} - q^{-\frac{\langle \lambda_i, \alpha \rangle + m}{2}} t^{-\frac{\langle \rho, \alpha \rangle}{2}}}{q^{\frac{m}{2}} t^{\frac{\langle \rho, \alpha \rangle}{2}} - q^{-\frac{m}{2}} t^{-\frac{\langle \rho, \alpha \rangle}{2}}} \quad (4.5.55)$$

where M_{λ_i} is the Macdonald polynomial for the gauge group G and representation λ_i , and S_{00} is a normalization factor defined below.

We must also define the theory on propagator geometries of the form $\mathbb{C}^* \times S^1$ as shown in Figure 4.1(b) and 4.1(c). The propagators differ by the orientation of one of the boundary components. The first propagator, shown in Figure 4.1(b), is given by,

$$\Psi(\eta) = \sum_i \frac{1}{g_i} \langle \lambda_i | \langle \lambda_i | \quad (4.5.56)$$

while the second propagator, shown in Figure 4.1(c), is the same except for a reversal of orientation,

$$\Psi(\delta) = \sum_i \frac{1}{g_i} |\lambda_i\rangle \langle \lambda_i| \quad (4.5.57)$$

With this information, we can now compute the partition function of refined Chern-Simons theory on manifolds of the form $\Sigma \times S^1$. Remembering that $\langle \lambda_i | \lambda_i \rangle = g_i$, we find for a genus g Riemann surface Σ ,

$$Z(\Sigma \times S^1) = \sum_i \frac{(g_i)^{g-1}}{(S_{0i})^{2g-2}} \quad (4.5.58)$$

To obtain more complicated geometries, we must now understand how the modular group, $SL(2, \mathbb{Z})$ acts on the Hilbert Space $\mathcal{H}$. Recall that the modular group can be described by the generators, S and T , subject to the relations,

$$S^4 = 1 \quad (ST)^3 = S^2 \quad (4.5.59)$$

The refined Chern-Simons theory gives a representation of this group acting on $\mathcal{H}$, which we can describe by computing the matrix elements of S and T , provided they satisfy the above relations.⁷ As before, these matrices are deformations of the Wess-Zumino-Witten S and T matrices for ordinary Chern-Simons. The S -matrix is given by,

$$\langle \lambda_i | S | \lambda_j \rangle = S_{ij} \equiv S_{00} M_{\lambda_i}(t^{-\rho}) M_{\lambda_j}(t^{-\rho} q^{-\lambda_i}) \quad (4.5.60)$$

where S_{00} is given by,

$$S_{00} = i^{|\Delta_+|} |P/Q|^{-1/2} (k + \beta y)^{-\frac{r}{2}} \prod_{m=0}^{\beta-1} \prod_{\alpha > 0} (q^{-m/2} t^{-(\alpha, \rho)/2} - q^{m/2} t^{(\alpha, \rho)/2}) \quad (4.5.61)$$

where $|\Delta_+|$ is the number of positive roots, y is the dual Coxeter number, and r is the rank of G . Here P is the weight lattice and Q is the root lattice, so that $|P/Q|$ is the number of points in the fundamental cell of this quotient lattice. The T matrix is given by,

$$\langle \lambda_i | T | \lambda_j \rangle = T_{ij} \equiv g_i q^{\frac{1}{2}(\lambda_i, \lambda_i)} t^{(\lambda_i, \rho)} t^{\frac{\beta-1}{2}(\rho, \rho)} q^{-\frac{k}{2y}(\rho, \rho)} \delta_{ij} \quad (4.5.62)$$

For the $SU(N)$ case, Kirillov has proven in [185] that the S and T matrices satisfy the defining relations of $SL(2, \mathbb{Z})$. In [62], Cherednik generalized this result to arbitrary root systems by studying the $SL(2, \mathbb{Z})$ action on the corresponding Double Affine Hecke Algebras.⁸

As a simple application of this $SL(2, \mathbb{Z})$ action, we can compute the amplitude on S^3 . Using the Heegaard splitting of S^3 , its geometry is given by taking two solid tori (with no Wilson loops inserted), and gluing them after acting with the S operator. This gives,

$$Z(S^3) = \langle 0 | S | 0 \rangle = S_{00} \quad (4.5.63)$$

⁷Because the inner product is nontrivial in these conventions, inserting a complete basis of states is given by, $1 = \sum_i \frac{1}{g_i} |\lambda_i\rangle \langle \lambda_i|$. To keep track of these additional g_i^{-1} factors when computing matrix elements, it is sometimes convenient to think of g_i as a lowering metric and g_i^{-1} as a raising metric, so that $K_i^j = g_j^{-1} K_{ij}$. We will use this notation below when we discuss knot computations.

⁸In this chapter we will only explicitly discuss the simply laced gauge groups, since these are simplest both mathematically and physically. To obtain non-simply laced gauge groups, it should be possible to start with the simply laced ADE (2,0) theory and introduce outer automorphism twists along the S^1 , as in [266, 285]. Although we will not study that construction here, the work of Cherednik provides further evidence that the refined Chern-Simons theory exists for general gauge groups. It is interesting to note that in order for this $SL(2, \mathbb{Z})$ representation to exist, and to ensure that the Hopf Link invariants are symmetric in the two representations, we must use the *symmetric* rather than the ordinary Macdonald polynomials associated to the root system, R . These are the Macdonald polynomials used in Cherednik's work and were first defined by Macdonald under the name of $(R, R^\vee)$ polynomials. Of course, these polynomials agree with ordinary Macdonald polynomials for simply laced groups, but for non-simply laced groups, they are defined by using a modified inner product (see [61] for details).

We have now given the full structure of the refined Chern-Simons theory as a restricted topological quantum field theory. However, for computations it is helpful to explain another set of operators. We define the operators $\mathcal{O}_i$ by the property that,

$$\mathcal{O}_i|0\rangle = |\lambda_i\rangle \quad (4.5.64)$$

Then we can ask what happens when we collide two of these operators. The result should have the form,

$$\mathcal{O}_i\mathcal{O}_j|0\rangle = \sum_k N_{ij}^k \mathcal{O}_k|0\rangle \quad (4.5.65)$$

In the unrefined case, this corresponds to placing two Wilson lines in representations λ_i and λ_j on top of each other. Then the N_{ij}^k are simply the Littlewood-Richardson coefficients that arise from decomposing the tensor product, $\lambda_i \otimes \lambda_j$. In the refined case, these coefficients are deformed to the (q,t) Littlewood-Richardson coefficients, associated with decomposing the product of Macdonald polynomials (see Appendix D).

Note that we could also understand these coefficients by taking the pant amplitude, $P \times S^1$, but with Wilson loops wrapping the fiber S^1 instead of the base. This can be achieved by acting with the modular S matrix that exchanges the cycles of the T^2 boundaries. As explained in [19], this line of reasoning leads directly to the Verlinde formula for N_{ij}^k .

4.6 Refined Kauffman Invariants

Now that we have explained the structure of refined Chern-Simons theory as a TQFT, we can use it to compute new knot invariants. In ordinary Chern-Simons theory, it is well known that if we introduce some Wilson loop in representation R along a knot $\mathcal{K}$, then the expectation value of the Wilson loop gives a topological invariant of the knot.

As explained above, we can also introduce Wilson loops in the refined Chern-Simons theory, but we are restricted to only choosing loops that are preserved by the $U(1)$ action on the Seifert Manifold. In this section we will focus on knots inside S^3 , so the $U(1)$ condition restricts us to considering torus links. To see this more explicitly, let us describe S^3 as the locus in $\mathbb{C}^2$, with coordinates z_1, z_2 , where we require,

$$|z_1|^2 + |z_2|^2 = 1 \quad (4.6.66)$$

Here the S^3 is naturally realized as a T^2 fibration over the interval, where the coordinate on the interval is given by $|z_1|^2$. The $(1,0)$ cycle of the T^2 comes from phase rotations of z_1 , while the $(0,1)$ cycle comes from phase rotations of z_2 . Now we can consider the intersection of the S^3 with the locus,

$$z_1^n = z_2^m \quad (4.6.67)$$

The intersection is simply the (n, m) torus link in S^3 . We have not yet specified the $U(1)$ action, but the most natural choice is $(z_1, z_2) \rightarrow (e^{i\theta m} z_1, e^{i\theta n} z_2)$ ⁹. Recall that in the definition of Seifert manifolds, the $U(1)$ action was only required to be semi-free, which is important here since every point on the circle $z_2 = 0$ is fixed by the $\mathbb{Z}_m$ subgroup, generated by $e^{2\pi i/m}$.

As explained above, the expectation value of the refined Chern-Simons theory in the presence of a torus knot can be computed using only the S and T matrices, and the metric, g_i . Above, we defined a knot operator that inserts a Wilson loop in the interior of the solid torus geometry, M_L , so that,

$$\mathcal{O}_i^{(0,1)}|0\rangle = |i\rangle \quad (4.6.68)$$

Then the expectation value of the unknot is simply given by,

$$\langle 0|\mathcal{O}_i^{(0,1)}S|0\rangle = S_{i0} \quad (4.6.69)$$

The operator that inserts the (n, m) torus knot of interest is obtained by acting with an element of $SL(2, \mathbb{Z})$ that maps $(0, 1)$ to (n, m) ,

$$K = \begin{pmatrix} a & n \\ b & m \end{pmatrix} \in SL(2, \mathbb{Z}) \quad (4.6.70)$$

Then the representation of K acting on the torus Hilbert Space can be written explicitly as a string of S and T matrices. Then the (n, m) operator is given by,

$$\mathcal{O}^{(n,m)} = K\mathcal{O}^{(0,1)}K^{-1} \quad (4.6.71)$$

In order to expand this out, we must use the (q, t) Littlewood-Richardson coefficients, N_{ij}^k ,

$$\mathcal{O}_i^{(0,1)}|j\rangle = \sum_k N_{ij}^k |k\rangle \quad (4.6.72)$$

Putting these ingredients together we find an explicit formula for the (n, m) torus knot invariant,

$$Z(T(n, m)) = \langle 0|\mathcal{O}_i^{(n,m)}S|0\rangle = \langle 0|K\mathcal{O}_i^{(0,1)}K^{-1}S|0\rangle = \sum_{j,k,l} K_{0k} N_{ij}^k (K^{-1})^j_l S^l_0 \quad (4.6.73)$$

We will use this formula below to compute refined Chern-Simons $SO(2N)$ knot invariants, but before doing so we clarify the relationship between refined Chern-Simons and other knot invariants.

⁹Note that we also could have chosen the action $(z_1, z_2) \rightarrow (e^{i\theta m f} z_1, e^{i\theta n f} z_2)$ for any $f \in \mathbb{Z}$, $f \neq 0$. For any choice of f , the $U(1)_f$ action cannot be continuously deformed to the canonical $f = 1$ choice without passing through configurations that break supersymmetry, where the refined Chern-Simons theory is not defined. Thus a priori, it is difficult to prove that the resulting Wilson loop expectation values are independent of the choice of f . However, in all the examples that we have checked the resulting expectation values do not depend on f , up to trivial framing factors.

4.6.1 Relation to Knot Homology

It is worth explaining here the relation of this approach to previous studies of knot homology in string theory [102, 147–150, 285]. The starting point for the refined Chern-Simons theory comes from M-theory, where a stack of M5-branes wrapping the S^3 intersects another stack of M5 branes along the knot $\mathcal{K}$. If we kept the full space of BPS states at this intersection, then in accordance with the proposal of [148], this should describe the full knot homology associated to $\mathcal{K}$. Here the choice of the knot homology group ($G = SU(N), SO(N), \dots$) corresponds to the insertion of N M5-branes wrapping S^3 , with the possible addition of orientifold planes for the orthogonal groups and the action of outer automorphisms for non-simply laced groups. We denote this space by $\mathcal{H}_{BPS}^G$.

This space of BPS states comes equipped with two natural gradings, S_1 and S_2 , coming from the spins of the BPS states in $\mathbb{C} \times \mathbb{C}$. The Poincare polynomial in a given knot homology theory comes from computing the trace,

$$\mathcal{P}_G(\mathcal{K}) = \text{Tr}_{\mathcal{H}_{BPS}^G} \mathbf{q}^{2(S_1-S_2)} \mathbf{t}^{2S_1} \quad (4.6.74)$$

where we have made the change of variables, $\mathbf{q} = \sqrt{t}$ and $\mathbf{t} = -\sqrt{q/t}$. However, as emphasized in [285], this trace cannot be computed as an index. If we were to extend the trace to the entire M-theory Hilbert space, $\mathcal{H}$, then non-BPS states would contribute. Using only these gradings, the only genuine index that can be created is the euler characteristic,

$$P_G(\mathcal{K}) = \text{Tr}_{\mathcal{H}_{BPS}^G} (-1)^{2S_2} \mathbf{q}^{2(S_1-S_2)} \quad (4.6.75)$$

which simply computes the ordinary quantum knot invariants coming from Chern-Simons theory.

The key to the refined Chern-Simons construction is the additional grading for torus knots that comes from the $U(1)_R$ symmetry. With this new grading it now becomes possible to compute a new index that contains refined information about the knot homology,

$$Z_G(\mathcal{K}) = \text{Tr}_{\mathcal{H}_{BPS}^G} (-1)^{2S_R} \mathbf{q}^{2(S_1-S_2)} \mathbf{t}^{2(S_1-S_R)} \quad (4.6.76)$$

Thus, the existence of the refined Chern-Simons theory makes the mathematical prediction that there should exist a new grading on the knot homology of torus knots. A promising avenue for identifying this grading is the recent work of [231, 244], connecting the representation theory of Rational Double Affine Hecke Algebras (DAHA) to the HOMFLY and Khovanov-Rozansky invariants of torus knots. The DAHA plays a central role in the theory of Macdonald polynomials, so it is natural to suspect that their work may connect directly with refined Chern-Simons theory.

Another method for computing the HOMFLY polynomial and Superpolynomial has been recently proposed by Cherednik [67]. This method has the computational

advantage that it directly uses the $SL(2, \mathbb{Z})$ action on the DAHA, and does not require multiplying large matrices, as in the current refined Chern-Simons approach. It would be interesting to understand Cherednik's method from a physics perspective.

In general, the index computed by the refined Chern-Simons theory will include negative signs, and will simply be different from the Poincare polynomial computed by Khovanov-Rozansky theory. Surprisingly, however, it was found in [19] that the large N behavior of Khovanov-Rozansky theory, encoded in the superpolynomial, could be reconstructed from the $SU(N)$ refined Chern-Simons theory by using a special change of variables.

For the $SO(2N)$ case, a simple change of variables exists for the Hopf link (as in the $SU(N)$ case), connecting with the Kauffman Homology of [150, 184]. However, we find that no such change of variables exists for general torus knots, suggesting that the $SO(2N)$ refined Chern-Simons theory computes genuinely new invariants for torus knots.

4.6.2 Example: The Hopf Link

Below we compute the $SO(2N)$ refined Chern-Simons invariant for the Hopf link with both components colored by the fundamental representation, following a computational procedure similar to that used in [19]. Recall that the Hopf link knot invariant can be computed simply by evaluating an element of S ,

$$\bar{Z}(\text{Hopf}, SO(2N)) = S_{V_V} \quad (4.6.77)$$

where V denotes the fundamental representation and the bar indicates that this is an unnormalized amplitude. To simplify our results, it is helpful to normalize by the unknot amplitude,

$$\bar{Z}(\bigcirc, SO(2N)) = S_{\cdot V} = \frac{t^{(2N-1)/2} - t^{-(2N-1)/2}}{t^{1/2} - t^{-1/2}} + 1 \quad (4.6.78)$$

Normalizing by the unknot gives the general answer,

$$\begin{aligned} Z(\text{Hopf}, SO(2N)) &= \frac{\bar{Z}(\text{Hopf}, SO(2N))}{\bar{Z}(\bigcirc, SO(2N))} \\ &= qt^{N-1} + 1 + q^{-1}t^{-(N-1)} + \frac{t^{(2N-3)/2} - t^{-(2N-3)/2}}{t^{1/2} - t^{-1/2}} \end{aligned} \quad (4.6.79)$$

Let us make the following change of variables,

$$\begin{aligned} \mathbf{q} &= \sqrt{t} \\ \mathbf{t} &= -\sqrt{q/t} \\ \mathbf{a} &= t^{(2N-1)/2} \end{aligned} \quad (4.6.80)$$

Note that from the perspective of the large N dual, this is very natural since $\mathbf{a}$ should correspond to the Kahler class of the resolved conifold. This is the natural generalization of the $SU(N)$ change of variables for the Hopf link used in [19]. Making this substitution, we obtain the superpolynomial,

$$Z(\text{Hopf}, SO(2N)) = \mathbf{t}^2 \mathbf{q} \mathbf{a} + 1 + \mathbf{t}^{-2} \mathbf{q}^{-1} \mathbf{a}^{-1} + \frac{\mathbf{a} \mathbf{q}^{-2} - \mathbf{a}^{-1} \mathbf{q}^2}{\mathbf{q} - \mathbf{q}^{-1}} \quad (4.6.81)$$

As a check of our methods, following Gukov and Walcher [150] we use the fact that $SO(4) \cong SU(2) \times SU(2)$ and the fundamental representation of $SO(4)$ corresponds to the $(\mathbf{2}, \mathbf{2})$ representation. This implies that the $SO(4)$ knot homology invariant should be equal to the square of the $SU(2)$ Khovanov poicare polynomial, and we find perfect agreement. Thus, for the Hopf Link, the refined Chern-Simons $SO(2N)$ invariant agrees with the expected Kauffman Homology after a simple change of variables.

4.6.3 Example: The Trefoil Knot

We can follow a similar procedure to compute the refined $SO(2N)$ invariant in the fundamental representation associated to the Trefoil, or $T(2, 3)$ knot. As explained above, we must evaluate

$$\langle 0 | \mathcal{O}_V^{(2,3)} S | 0 \rangle \quad (4.6.82)$$

where the subscript V indicates that the knot is in the fundamental representation of $SO(2N)$.

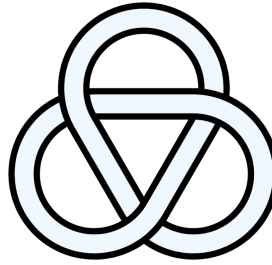


Figure 4.2: The Trefoil Knot, $T(2, 3)$

The result for the general normalized polynomial is,

$$\begin{aligned} Z_V(T(2, 3), SO(2N)) = & \mathbf{t}^6 \mathbf{q}^2 \mathbf{a}^4 + \mathbf{t}^4 \mathbf{q}^{-2} \mathbf{a}^4 + \mathbf{t}^4 \mathbf{q} \mathbf{a}^3 - \mathbf{t}^4 \mathbf{q}^{-1} \mathbf{a}^3 - \mathbf{t}^4 \mathbf{q}^2 \mathbf{a}^2 + \mathbf{t}^2 \mathbf{a}^2 \\ & - \mathbf{t}^2 \mathbf{q}^{-2} \mathbf{a}^2 - \mathbf{t}^2 \mathbf{q} \mathbf{a} + \mathbf{t}^2 \mathbf{q}^{-1} \mathbf{a} \end{aligned} \quad (4.6.83)$$

This refined Chern-Simons answer is structurally similar to the conjectured Kauffman homology result [150] (as it must be, since they both reduce to the quantum $SO(2N)$ invariant in the limit $\mathbf{t} \rightarrow -1$), but it can be seen straightforwardly that there does not exist a change of variables relating the two.

By the same method, we can also compute the normalized invariants associated to the spinor representation, S . For small values of N we find,

$$\begin{aligned} Z_S(T(2, 3), SO(4)) &= (-\mathbf{qt})^{-3}(\mathbf{t}^6\mathbf{q}^{10} + \mathbf{t}^4\mathbf{q}^6 - \mathbf{t}^4\mathbf{q}^4) \\ Z_S(T(2, 3), SO(6)) &= (-\mathbf{qt})^{-3/2}(\mathbf{t}^6\mathbf{q}^{18} + \mathbf{t}^4\mathbf{q}^{14} - \mathbf{t}^4\mathbf{q}^8) \\ Z_S(T(2, 3), SO(8)) &= \mathbf{q}^6\mathbf{t}^2 - \mathbf{q}^8\mathbf{t}^2 - \mathbf{q}^{12}\mathbf{t}^2 + \mathbf{q}^{14}\mathbf{t}^2 - \mathbf{q}^{16}\mathbf{t}^4 - \mathbf{q}^{20}\mathbf{t}^4 + \mathbf{q}^{22}\mathbf{t}^4 \\ &\quad + \mathbf{q}^{26}\mathbf{t}^4 + \mathbf{q}^{30}\mathbf{t}^6 \end{aligned} \tag{4.6.84}$$

4.6.4 Example: The General $T(2, 2m + 1)$ Torus Knot

By studying the invariants of the fundamental representation, V , for the $T(2, 2m + 1)$ torus knots, we find the following general formula for the refined Chern-Simons invariants,

$$\begin{aligned} Z_V(T(2, 2m + 1), SO(2N)) &= \mathbf{a}^2\mathbf{t}^2 + \mathbf{a}^{2m+2}\mathbf{q}^{2m}\mathbf{t}^{4m+2} \left(1 - \mathbf{a}^{-2}\mathbf{t}^{-2}\right) \left(\sum_{i=0}^m \mathbf{q}^{-4i}\mathbf{t}^{-2i}\right. \\ &\quad \left. + \sum_{j=1}^{2m-1} \mathbf{a}^{-j}\mathbf{q}^{-j}\mathbf{t}^{-2j} \left(\sum_{i=1}^{\lceil(2m+1-j)/2\rceil} (\mathbf{q}^4\mathbf{t}^2)^{-i+1} - \mathbf{q}^{-2} \sum_{i=1}^{\lfloor(2m+1-j)/2\rfloor} (\mathbf{q}^4\mathbf{t}^2)^{-i+1}\right)\right) \end{aligned}$$

4.6.5 Example: The $T(3, 4)$ Knot

Finally, we have studied the $T(3, 4)$ knot for invariants in the fundamental representation, V . For small gauge groups, we find,

$$Z_V(T(3, 4), SO(4)) = \mathbf{q}^{-24}\mathbf{t}^{-12} \left(1 + \mathbf{q}^4\mathbf{t}^2 - \mathbf{q}^6\mathbf{t}^2 + \mathbf{q}^6\mathbf{t}^4 - \mathbf{q}^{10}\mathbf{t}^4\right)^2 \tag{4.6.85}$$

$$\begin{aligned} Z_V(T(3, 4), SO(6)) &= \mathbf{q}^{-36}\mathbf{t}^{-12} \left(1 + \mathbf{q}^4\mathbf{t}^2 + \mathbf{q}^6\mathbf{t}^2 - \mathbf{q}^8\mathbf{t}^2 - \mathbf{q}^{10}\mathbf{t}^2 + \mathbf{q}^6\mathbf{t}^4 \right. \\ &\quad \left. + 2\mathbf{q}^8\mathbf{t}^4 - \mathbf{q}^{12}\mathbf{t}^4 - 2\mathbf{q}^{14}\mathbf{t}^4 - \mathbf{q}^{16}\mathbf{t}^4 + \mathbf{q}^{18}\mathbf{t}^4 + 2\mathbf{q}^{12}\mathbf{t}^6 \right. \\ &\quad \left. + \mathbf{q}^{14}\mathbf{t}^6 - 3\mathbf{q}^{16}\mathbf{t}^6 - 2\mathbf{q}^{18}\mathbf{t}^6 + \mathbf{q}^{22}\mathbf{t}^6 + \mathbf{q}^{24}\mathbf{t}^6 + \mathbf{q}^{16}\mathbf{t}^8 \right. \\ &\quad \left. - 2\mathbf{q}^{22}\mathbf{t}^8 - \mathbf{q}^{24}\mathbf{t}^8 + 2\mathbf{q}^{26}\mathbf{t}^8\right) \end{aligned} \tag{4.6.86}$$

4.7 The Large N Limit

In this section we study the large N limit of $SO(2N)$ refined Chern-Simons theory on S^3 . Recall that the large N limit of ordinary $SU(N)$ Chern-Simons theory on

S^3 is given by closed topological string theory on the resolved conifold [140]. The parameters on each side are related by,

$$\begin{aligned} g_{s \text{ closed}} &= \frac{2\pi i}{k+N} \\ t = N g_{s \text{ closed}} &= \frac{2\pi i N}{k+N} \end{aligned} \quad (4.7.87)$$

where t is the Kahler parameter of the base $\mathbb{P}^1$ in the resolved conifold. Since $SU(N)$ Chern-Simons theory is equivalent to open topological string theory on T^*S^3 with N A-branes wrapping the S^3 , this is an example of a topological open-closed string duality analogous to the celebrated AdS/CFT correspondence. This interpretation is reinforced by the observation that $t = N g_s$ takes the form of the usual 't Hooft parameter, since from the Chern Simons action, $g_s = \frac{2\pi i}{k+N} = g_{open}^2$. The corresponding unrefined duality for $SO(N)$ Chern Simons theory has also been studied in [259], where it was interpreted as an open-closed topological string duality in the presence of orientifolds.

In studying the refined $SU(N)$ Chern-Simons theory, it was shown that a similar duality exists between $SU(N)$ refined Chern-Simons and the refined topological string on the resolved conifold [19]. Here, we study this refined geometric transition in the presence of orientifolds. We find that the large N limit of $SO(2N)$ refined Chern-Simons is dual to refined, closed topological string theory on the resolved conifold, $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbb{P}^1$ in the presence of an orientifold, I' , that acts freely.

To describe the action of I' , recall that in the linear sigma model description of the resolved conifold, we take four coordinates, X_i with charges $(1, 1, -1, -1)$ under a $U(1)$ action. Then the resolved conifold is given by,

$$\{|X_1|^2 + |X_2|^2 - |X_3|^2 - |X_4|^2 = r\}/U(1) \quad (4.7.88)$$

The involution, I' acts by,

$$I' : (X_1, X_2, X_3, X_4) \rightarrow (\bar{X}_2, -\bar{X}_1, \bar{X}_4, -\bar{X}_3) \quad (4.7.89)$$

Note that I' acts freely on X , so that in X/I' the base $\mathbb{CP}^1$ becomes $\mathbb{RP}^1$.

To see the duality explicitly, we take the partition function of $SO(2N)$ refined Chern-Simons theory on S^3 ,

$$Z = S_{00} = \frac{1}{2(k + \beta(2N - 2))^{\frac{N}{2}}} \prod_{m=0}^{\beta-1} \prod_{\alpha>0} (q^{-m/2} t^{-(\alpha,\rho)/2} - q^{m/2} t^{(\alpha,\rho)/2}) \quad (4.7.90)$$

We are interested in the free energy, $F = -\log Z$ and will only keep the factors that have non-trivial q and t dependence,

$$F = \dots - \sum_{m=0}^{\beta-1} \sum_{\alpha>0} \log (1 - q^m t^{(\alpha,\rho)}) \quad (4.7.91)$$

Now using the properties of the D root system, this can be rewritten as,

$$F = - \sum_{m=0}^{\beta-1} \sum_{k=1}^{2N-1} f(k) \log(1 - q^m t^k) \quad (4.7.92)$$

where $f(k)$ is given by,

$$f(k) = \begin{cases} \frac{2N+1-k}{2} & \text{if } k < N, k \text{ odd} \\ \frac{2N-1-k}{2} & \text{if } k \geq N, k \text{ odd} \\ \frac{2N-k}{2} & \text{if } k < N, k \text{ even} \\ \frac{2N-2-k}{2} & \text{if } k \geq N, k \text{ even} \end{cases} \quad (4.7.93)$$

After some algebraic manipulation, the free energy can then be written as,

$$\begin{aligned} &= \sum_{d=1}^{\infty} \frac{1}{2d} \frac{t^{(2N-1)d} t^{d/2} q^{-d/2}}{(q^{d/2} - q^{-d/2})(t^{d/2} - t^{-d/2})} + \sum_{d=1}^{\infty} \frac{1}{d} \frac{t^{(N-\frac{1}{2})d} t^{d/2} q^{-d/2}}{(q^{d/2} - q^{-d/2})} \\ &\quad - \sum_{d=1}^{\infty} \frac{1}{2d} \frac{t^{(2N-1)d} t^{d/2} q^{-d/2}}{(q^{d/2} - q^{-d/2})(t^{d/2} + t^{-d/2})} \end{aligned} \quad (4.7.94)$$

Note that this looks like a refined version of the unoriented first-quantized structure seen in Equation 4.3.20. It would be interesting to understand the presence of plus signs in the denominator from the first-quantized perspective, in terms of a graviphoton background.

However, recall that in Section 4.3, we gave a definition of refinement in terms of computing a second-quantized M-theory trace. For this interpretation, it is more natural to write the free energy as,

$$F = \sum_{d=1}^{\infty} \frac{1}{d} \frac{t^{(2N-1)d} q^{-d/2}}{(q^{d/2} - q^{-d/2})(t^d - t^{-d})} + \sum_{d=1}^{\infty} \frac{1}{d} \frac{t^{(N-\frac{1}{2})d} t^{d/2} q^{-d/2}}{(q^{d/2} - q^{-d/2})} \quad (4.7.95)$$

Note that this expression precisely satisfies the integrality properties of Section 4.4 for refined topological string theory in the presence of orientifolds. We can identify the Kahler class of the resolved conifold as,

$$Q = t^{2N-1} \quad (4.7.96)$$

and rewrite the free energy as,

$$F = \sum_{d=1}^{\infty} \frac{1}{d} \frac{Q^d q^{-d/2}}{(q^{d/2} - q^{-d/2})(t^d - t^{-d})} + \sum_{d=1}^{\infty} \frac{1}{d} \frac{Q^{d/2} t^{d/2} q^{-d/2}}{(q^{d/2} - q^{-d/2})} \quad (4.7.97)$$

Then the first term comes from oriented $M2$ branes that move freely in the four noncompact dimensions, and the second term comes from unoriented $M2$ branes

wrapping $\mathbb{RP}^2$. The q and t dependent shifts in the numerator arise because the presence of the orientifold breaks $SU(2)_L \times SU(2)_R$ down to $U(1)_1 \times U(1)_2$. For this reason, as discussed in Section 4.3, the BPS contributions do not have to appear in full spin multiplets.

Altogether, we have found a refined geometric transition for unoriented strings, given by,

$$Z_{open\ ref}(T^*S^3/I; q, t, N) = Z_{closed\ ref}(X/I'; q, t) \quad (4.7.98)$$

This transition has also given us a nontrivial test of the conjectured integrality properties of the refined topological string in the presence of orientifolds. It would be interesting to use these refined geometric transitions to compute refined amplitudes for more complicated geometries, such as the orientifold of local $\mathbb{P}^2$ considered in [49].

Chapter. 5

Refined Black Hole Ensembles and Topological Strings

5.1 Introduction

The Ooguri-Strominger-Vafa (OSV) conjecture gives a beautiful relation between the partition function of four-dimensional BPS black holes in a type IIA string theory compactified on a Calabi-Yau and the topological A-model string partition function [235]. Consider the BPS black hole partition function, Z_{BH} , in a mixed ensemble given by fixing the magnetic charge, p_Λ , and summing over the electric charge with electric potential, ϕ_Λ . The OSV conjecture relates the exact microscopic entropy of the black hole, captured by Z_{BH} to the macroscopic entropy, computed in terms of the topological string

$$Z_{BH}(p^\Lambda, \phi^\Lambda) \sim |Z_{\text{top}}(X^\Lambda)|^2 \quad (5.1.1)$$

where $X^\Lambda = p^\Lambda + \frac{i}{\pi}\phi^\Lambda$. This reproduces the Bekenstein-Hawking entropy/area law to the leading order, but otherwise it computes quantum gravitational corrections to it. The entropy on the left is a supersymmetric index. The topological string on the right computes F-terms in the four-dimensional low energy effective action of the IIA string theory on the Calabi-Yau. But from Wald's formula these F-terms are precisely the corrections to the Bekenstein-Hawking black hole entropy. The OSV relation is an example of gauge/gravity duality, where the gauge theory is the theory on the D-branes comprising the black hole. This aspect of the correspondence was emphasized in [11, 39, 76, 77, 88, 235, 238, 254–256, 267].¹ Note that both sides of (5.1.1) depend only on the Kähler moduli and not the complex structure moduli - on the right, this is a well-known property of A-model topological strings, and on the left this is a consequence of computing a well-behaved BPS index.

¹The large N dual of a black hole with fixed both magnetic and electric charges is naturally the real version of the topological string, recently studied in [268]. Its partition function has both the holomorphic Z_{top} and the anti-holomorphic piece $\bar{Z}_{\text{top}}$.

It is natural to ask if there are meaningful ways to generalize the conjecture (5.1.1). The most general possible black hole partition function would include a chemical potential for angular momentum. This partition function, also known as a spin character, has been extensively studied in the context of motivic wall crossing in four-dimensional $\mathcal{N} = 2$ field theory [59, 95, 97, 127, 188] and supergravity [27, 28].² For compact Calabi-Yau manifolds, this spin character will depend sensitively on both the Kähler and complex structure of the manifold. On a *noncompact* Calabi-Yau, however, the situation improves. We can form a *protected spin character* by utilizing the $SU(2)_R$ -symmetry of four-dimensional $\mathcal{N} = 2$ theory [127]. The protected spin character is a genuine index that only depends on the Kähler moduli and is constant except at real codimension-one walls of marginal stability. Therefore, this protected spin character gives a well-behaved and computable definition for the refined BPS black hole partition function $Z_{\text{ref BH}}(p^\Lambda, \phi^\Lambda, y)$, depending on one extra parameter y to keep track of the spin.

There is also a natural candidate for what may replace the topological string partition function. The refined topological string is a one-parameter deformation of topological string theory. Just like the refined black-hole partition function, the refined topological string partition function $Z_{\text{ref top}}(X^I, y)$ also makes sense only in the non-compact limit, and also utilizes the $SU(2)_R$ symmetry to be defined. The refined topological string is defined as an index in M-theory [162, 171]. There are several equivalent ways to compute the index, either by counting spinning M2-branes [171], or alternatively, as the refined BPS index for D2 and D0-branes bound to a single D6 brane [91, 95, 172], to name two. Yet another way to compute the index is as the Nekrasov partition function of the five dimensional gauge theory that arises from M-theory at low energies.³

Having found a generalization of both Z_{BH} and Z_{top} , defined in the same circumstances, and depending on the same set of parameters, it is natural to conjecture that there is in fact a refinement of the OSV conjecture that relates the two:

$$Z_{\text{ref BH}}(p^\Lambda, \phi^\Lambda, y) = |Z_{\text{ref top}}(k^\Lambda, \epsilon_1, \epsilon_2)|^2. \quad (5.1.2)$$

Despite the fact that the ingredients for the conjecture fit naturally, one would still like to have a rationale for why the conjecture should hold. Because the Calabi-Yau is non-compact, the BPS black holes we are discussing are really BPS particles with large entropy. While one can imagine the system arising by taking a limit where we take the mass of the black-hole to infinity, at the same rate as we take the Planck mass to infinity so that entropy stays finite, the $SU(2)_R$ symmetry we are using makes

²Rotating single-centered black holes in four dimensions cannot be supersymmetric. The black holes that one is studying here are multicentered configurations that carry intrinsic angular momentum in their electromagnetic fields. Note that despite the fact that these are multi-centered, they still correspond to bound states [82].

³Alternatively, it has a B-model formulation, at least in some cases, in terms of β -deformed matrix models [2, 60, 90].

sense only in the non-compact limit. Correspondingly some of the justifications for the OSV conjecture, for example those based on Wald’s formula, may no longer be sound in the refined setting, since the supergravity solution is singular.

However, as we mentioned, the original OSV relation is fundamentally a large N duality. For the BPS states that came from a theory on N D-branes, we always get an $SU(N)$ gauge theory describing the particles. In the ’t Hooft large N limit of this theory one expects to get a string theory, whether or not there are dynamical gravitons in the theory. Indeed, many famous examples of large N duality are of this kind, see for example [140]. In particular, the duality should still hold even for the non-compact Calabi-Yau; it is merely difficult to check beyond the protected quantities. Finally, from the perspective of large N duality, once we modify one side of the correspondence, the duality, at least in principle, fixes what the other side has to be. Furthermore, there is a way to understand directly from the large N duality *why* refining the black hole ensemble has to correspond to a refinement of the topological string on the other side.

To precisely test the refined OSV conjecture, one would like to compute both sides of the relation and compare explicitly. For the unrefined OSV conjecture, this was done for local, non-compact Calabi-Yau manifolds in [4, 11, 267], and agreement was found. These local Calabi-Yaus can be thought of as the limit of compact Calabi-Yaus in the neighborhood of a shrinking two- (or four-) cycle. In this limit, gravity decouples so we need to be precise about what we mean by the black hole partition function. We must require that the D-branes forming the BPS black hole wrap a cycle that becomes noncompact in this limit so that their entropy remains nonzero. In this limit, the black hole partition function is simply computed by a Witten index on the noncompact brane worldvolume. In the chapter, we will run a similar test in the refined case. To this end, we will develop techniques to solve for the refined partition functions on both sides of the conjecture. Remarkably, the refined OSV conjecture passes the tests just as well as the original OSV.⁴ This provides strong evidence in support of our conjecture.

This chapter is organized as follows. In section 2, we review the ingredients of the original OSV conjecture for both compact and non-compact Calabi-Yaus. In

⁴In subsequent work, [10, 74, 75, 83, 88, 258, 267] several aspects of the conjecture were clarified. For one, while the relation holds to all orders in perturbation theory, the exact relation requires summing over nonperturbative corrections to the macroscopic entropy, taking the form of “baby universes” [10, 88]. Second, at the perturbative level the right side of equation 5.1.1 should include a summation over $\phi \rightarrow \phi + 2\pi i n$ so that the periodicities of ϕ match on both sides. Additionally, the right side generically contains an additional measure factor of $g_{top}^{-2} e^{-K}$, which is natural from the viewpoint of Kähler quantization since Z_{top} transforms as a wavefunction [83, 238, 271]. Such a measure factor did not appear in the unrefined local curve examples of [11, 267]. One way to understand this is to observe that these geometries do not have a holomorphic anomaly, which implies that Z_{top} actually transforms as a function rather than a wavefunction – therefore the additional measure factors from Kähler quantization are absent. In this chapter when we study the refined OSV relation on these local geometries, we will again find that no measure factors appear.

section 3, we motivate the refined OSV conjecture first from the perspective of the AdS_2/CFT_1 correspondence, and second by studying the wall crossing of $D4$ branes splitting into $D6 - \overline{D6}$ bound states. In the noncompact setting these arguments are necessarily heuristic, but we believe they capture the correct physics. In section 4, we explain how the refined topological string on Calabi-Yau manifolds of the form $X = \mathcal{L}_1 \oplus \mathcal{L}_2 \rightarrow \Sigma_g$ can be completely solved by a two-dimensional topological quantum field theory (TQFT). We then use this TQFT to compute refined topological string amplitudes for the above geometries. We also show that our results agree precisely with the five-dimensional Nekrasov partition function of $U(1)$ gauge theories with g adjoints that are engineered by these Calabi-Yaus. In section 5, we study the black hole side of the correspondence. Mathematically, the refined partition function for $D4/D2/D0$ -branes computes the χ_y genus of the relevant instanton moduli spaces. As originally suggested in [285], this can be thought of as a categorification of the euler characteristic invariants computed by the $\mathcal{N} = 4$ Vafa-Witten theory. We then specialize to $D4$ branes wrapping the geometry $C_4 = (\mathcal{L}_1 \rightarrow \Sigma_g)$ and study the refined BPS partition function of bound states with $D2$ and $D0$ -branes. We propose that this partition function is computed by a (q, t) -deformation of two-dimensional Yang-Mills, which is closely related to the refined Chern-Simons theory studied in [15, 18, 123]. Wrapping branes on the geometry $C = (\mathcal{O}(-1) \rightarrow \mathbb{P}^1)$, we show that (q, t) -deformed Yang Mills precisely reproduces a mathematical result of Yoshioka and Nakajima for the χ_y genus of instanton moduli space [220, 286]. In section 6 we connect the black hole and topological string perspectives by studying the large N limit of the (q, t) -deformed Yang Mills theory. We find that the theory factorizes to all orders in $1/N$ into two copies of the refined topological string partition function. This gives a nontrivial check of our refined OSV conjecture. Finally, in section 7 we explain an alternative way to compute the refined black hole partition function on $\mathcal{O}(-1) \rightarrow \mathbb{P}^1$. The refined bound states are counted by using the semi-primitive refined wall-crossing formula [95], thus giving a refined extension of the techniques used in [229, 230].

5.2 The OSV Conjecture: Unrefined and Refined

We start by reviewing the remarkable conjecture of Ooguri, Strominger, and Vafa (OSV) connecting four-dimensional BPS black holes with topological strings [235]. Consider IIA string theory compactified on a Calabi-Yau, X , with four-dimensional black holes arising from D-branes wrapping holomorphic cycles in X . In terms of four-dimensional gauge fields, the $D0$ and $D2$ branes are electrically charged while the $D4$ and $D6$ branes are magnetically charged.

The object of interest for the OSV conjecture is the mixed black hole partition function given by fixing the magnetic charges and summing over electric charges with

chemical potentials,

$$Z_{BH}(P_6, P_4; \phi_2, \phi_0) = \sum_{Q_2, Q_0} \Omega(P_6, P_4, Q_2, Q_0) e^{-\phi_2 Q_2 - \phi_0 Q_0} \quad (5.2.3)$$

where we have denoted $D6$ charges by P_6 , $D4$ charges by P_4 , $D2$ charges by Q_2 , and $D0$ charges by Q_0 . Here ϕ_2 and ϕ_0 are chemical potentials associated to the electrically charged D-branes. $\Omega(P, Q)$ is computed by the Witten index in the corresponding charge sector,

$$\Omega(P_6, P_4, Q_2, Q_0) = \text{Tr}_{\mathcal{H}_{P,Q}} (-1)^F \quad (5.2.4)$$

and only receives contributions from BPS black holes.

The OSV conjecture states that this mixed black hole partition function is equal to the square of the A-model topological string partition function on X ,

$$Z_{BH}(P_6, P_4; \phi_2, \phi_0) = |Z_{top}(g_{top}, k)|^2 \quad (5.2.5)$$

where k is the complexified Kähler form on X . The projective coordinates on moduli space are given by,

$$X_I = P_I + i \frac{\phi_I}{\pi} \quad (5.2.6)$$

This implies that the string coupling constant and Kähler moduli are determined by the magnetic charges and electric potentials,

$$g_{top} = \frac{4\pi i}{X_0} = \frac{4\pi i}{P_6 + i \frac{\phi_0}{\pi}} \quad (5.2.7)$$

$$k_I = 2\pi i \frac{X_I}{X_0} = \frac{1}{2} g_{top} \left(P_{4,I} + \frac{i \phi_{2,I}}{\pi} \right) \quad (5.2.8)$$

Here the real part of X_I is fixed by the attractor mechanism which determines the near-horizon Calabi-Yau moduli in terms of the black hole charge. Both sides of the relation should be considered as expansions in $1/Q$ where Q is the total graviphoton charge of the black hole. From the change of variables, we have a perturbative expansion in g_{top} if either P_6 or ϕ_0 is large. We will usually set $P_6 = 0$ so it is natural to take both ϕ_0 and P_4 to infinity in such a way that g_{top} becomes small and the Kähler form remains constant.

One way to further understand the OSV relation is by inverting it,

$$\Omega(P_6, P_4, Q_2, Q_0) = \int d\phi_0 d\phi_2 e^{Q_2 \phi_2 + Q_0 \phi_0} |Z_{top}|^2 \quad (5.2.9)$$

so that black hole degeneracies are formally computed by the topological string. It is known that because the topological string partition function obeys the holomorphic anomaly equations [45], it transforms as a wavefunction under changes of polarization on the Calabi-Yau moduli space [278]. Thus, from quantum mechanics the appearance

of $|Z_{top}|^2$ is very natural. In fact, this interpretation implies that $\Omega(P, Q)$ is the Wigner quasi-probability function on (P, Q) phase space [235].

The original OSV conjecture focused on the case of compact Calabi-Yau manifolds, so that the wrapped D-branes correspond to black holes in four-dimensional $\mathcal{N} = 2$ supergravity. However, as explored in [4, 9–11, 88, 267], it is interesting to study the OSV conjecture for local Calabi-Yau manifolds which can be thought of as the decompactification limit of the compact case. In this limit gravity decouples which means that the four dimensional planck mass goes to infinity. Since the Bekenstein-Hawking entropy of a black hole is proportional to M_{BH}^2/M_{Pl}^2 , to obtain a finite entropy in this limit we should also take $M_{BH} \rightarrow \infty$. This can be accomplished simply by wrapping $D4$ or $D6$ -branes on cycles that become non-compact in this limit. The precise OSV relation remains the same in this limit, except that now the black hole partition function is naturally computed by a partition function on the worldvolume of the noncompact $D4$ or $D6$ -branes.

The advantage of taking this limit is that both sides of the OSV relation are exactly solvable, leading to a highly nontrivial check of the conjecture. The conjecture has been tested perturbatively to all orders in [4, 11, 267], and non-perturbative corrections in the form of baby universes have been computed in [10, 88].

5.2.1 Refining the Conjecture

Now that we have reviewed the OSV conjecture, a natural question to ask is whether the black hole degeneracy computed by $\Omega(P, Q)$ is the most general index that counts four-dimensional BPS black holes. In fact, we could include information about spin by replacing the Witten index,

$$\text{Tr}_{\mathcal{H}_{BPS}}(-1)^F \quad (5.2.10)$$

by the spin character,

$$\text{Tr}_{\mathcal{H}_{BPS}}(-1)^F \exp(-2\gamma J_3) \quad (5.2.11)$$

where J_3 is the three-dimensional generator of rotations and γ is the conjugate chemical potential.⁵ These spin-dependent BPS traces and their wall-crossing behavior have been studied extensively in the context of $\mathcal{N} = 2$ field theory [59, 95, 97, 127, 188] and supergravity [27, 28]. However, this trace has the drawback of not being an index, which means that it will be sensitive to both the complex and Kähler moduli.

If we consider a local Calabi-Yau manifold by taking the gravity decoupling limit, there is a preserved $SU(2)$ R-symmetry that appears. As explained in [127], this can

⁵Rotating single-centered black holes in four dimensions cannot be supersymmetric. The black holes that we are studying here are multicentered configurations that carry intrinsic angular momentum in their electromagnetic fields. Note that despite the fact that these are multi-centered, they still typically correspond to bound states [82].

be used to form a protected spin character that is a genuine index, and only depends on the Kähler moduli through wall-crossing,

$$\mathrm{Tr}_{\mathcal{H}_{P,Q}}(-1)^{2J_3} e^{-2\gamma(J_3-R)} = \sum_{J_3,R} \Omega(P, Q; J_3, R) e^{-2\gamma(J_3-R)} \quad (5.2.12)$$

Now we can form the mixed ensemble of black holes counted with spin where, as in the ordinary case, we fix the magnetic charge and sum over the electric charge,

$$Z_{\mathrm{ref\ BH}}(P_6, P_4; \phi_2, \phi_0; \gamma) = \sum_{Q_2, Q_0, J_3, R} \Omega(P_6, P_4, Q_2, Q_0; J_3, R) e^{-2\gamma(J_3-R) - \phi_2 Q_2 - \phi_0 Q_0} \quad (5.2.13)$$

We will refer to this as the refined black hole partition function.

We would like to know whether there exists a generalization of the topological string whose square is equal to $Z_{\mathrm{ref\ BH}}$. A natural candidate for this one-parameter deformation is well-known, and is given by the refined topological string.

Recall the definition of the refined topological string as the index of M-theory, depending on Kahler moduli k and two additional parameters ϵ_1 and ϵ_2 . The refined topological string partition function [2, 162, 171] on a Calabi-Yau X is given by computing the index of M-theory on the geometry,

$$(X \times TN \times S^1)_{\epsilon_1, \epsilon_2} \quad (5.2.14)$$

where TN denotes the Taub-NUT spacetime, and upon going around the S^1 the Taub-NUT is twisted by,

$$(z_1, z_2) \rightarrow (q z_1, t^{-1} z_2) \quad (5.2.15)$$

where

$$q = e^{-\epsilon_1} \quad t = e^{-\epsilon_2}.$$

In addition, we must include an R-symmetry twist to preserve supersymmetry. This twist is implemented by a geometric Killing vector on the non-compact Calabi-Yau, X . Note that, in our notation, the unrefined limit is $\epsilon_1 = \epsilon_2$, unlike in much of the literature, where one typically defines ϵ_2 with a different sign. The partition function of M-theory in this geometry is computing the index of the resulting theory on $TN \times S^1$,

$$Z_{\mathrm{ref\ top}}(\epsilon_1, \epsilon_2; k) = \mathrm{Tr}(-1)^{2S_1+2S_2} q^{S_1-R} t^{R-S_2} e^{-k_I Q_2^I} \quad (5.2.16)$$

where S_1 and S_2 are the spins in the z_1 and z_2 directions, respectively, and R is the R-charge of the state. We have schematically indicated that the partition function depends on the Kahler moduli k , via the M2 brane contributions to the index. Note that although the trace is over all states, only BPS states will make a contribution.

The index can be computed in several different ways: by counting spinning M2-branes [162, 171] or as the Omega-deformed instanton partition function [225]. In analogy with the unrefined case [172], the refined topological string can also be written in terms of the refined Donaldson-Thomas invariants that compute the BPS protected spin character for D2 and D0-branes bound to a single D6 brane [95]. Given that the refined topological string also counts BPS particles with spin and only depends on the Kähler moduli of X , it should be related to the refined black hole partition function.

The index is related to the ordinary topological string partition function if we set $\epsilon_1 = \epsilon_2 = g_s$, and reduce on the thermal S^1 to IIA. Then, we get IIA string theory on $X \times TN$, whose partition function is the same as the ordinary topological string partition function, where g_s is the topological string coupling constant [91]. In particular, M2 branes wrapping holomorphic curves in X and the thermal S^1 become the worldsheet instantons of the topological string. For $\epsilon_1 \neq \epsilon_2$, the theory has no known worldsheet formulation, at the moment.

There is yet another way to view the partition function (5.2.16), which will be useful for us. This corresponds to the TST dual formulation, where instead, we go down to IIA string theory on the S^1 in the Taub-Nut space [91]. This turns the Taub-Nut space into a single D6 brane wrapping $X \times S^1$. In this case, the refined topological string partition function has the interpretation as the refined spin character, counting the bound states of the D6 brane on X with D0 and D2 branes. In terms of the $SO(4) = SU(2)_\ell \times SU(2)_r$ rotation symmetry of the Taub-Nut space, the D0 brane charge Q_0 is the $2J_3^\ell$ component of the $SU(2)_\ell$ spin in M-theory, the $SO(3) = SU(2)$ rotation symmetry in IIA is identified, under the dimensional reduction, with the $SU(2)_r$ symmetry in M-theory, while the $SU(2)_R$ R-symmetry is manifestly the same in both IIA and M-theory. In particular, the refinement is associated with the diagonal $SU(2)_d \subset SU(2)_r \times SU(2)_R$ spin [223]. This allows us to rewrite 5.2.16 as

$$Z_{ref\ top}(\epsilon_1, \epsilon_2; k) = \text{Tr}(-1)^{2J_3} e^{\frac{\epsilon_1 + \epsilon_2}{2} Q_0} e^{\frac{\epsilon_1 - \epsilon_2}{2} (2J_3 - 2R)} e^{k_I Q_2^I}. \quad (5.2.17)$$

In writing this, we used the fact that $2J_3^\ell = S_1 - S_2$, $2J_3^r = S_1 + S_2$, which is obvious from the way the $SU(2)_\ell \times SU(2)_r$ acts on the coordinates z_1, z_2 of the Taub-Nut space, and furthermore, as we just reviewed, that $Q_0 = 2J_3^\ell$ and $2J_3 = 2J_3^r$.

This leads us to propose the refined OSV conjecture relating the protected spin character of black holes to the refined topological string⁶,

$$Z_{ref\ BH}(P_6, P_4; \phi_2, \phi_0; \gamma) = |Z_{ref\ top}(\epsilon_1, \epsilon_2, k)|^2 \quad (5.2.18)$$

⁶For an alternate proposal relating the Nekrasov partition function to non-supersymmetric extremal black holes, see [246].

To complete the conjecture, we propose that the variables are related by,

$$\begin{aligned} k_I &= \frac{2\pi i(\frac{i\phi_{2,I}}{\pi} + \beta P_{4,I})}{\frac{i\phi_0}{\pi} + \beta P_6} \\ \epsilon_1 &= \frac{4\pi i C}{i\frac{\phi_0}{\pi} + \beta P_6} \\ \epsilon_2 &= \frac{4\pi i C \beta}{i\frac{\phi_0}{\pi} + \beta P_6} \end{aligned} \tag{5.2.19}$$

where we have defined the variable $\beta \equiv 1 - \frac{\gamma}{2\pi i}$, and we have included an additional constant C .⁷ The most natural choice for C , as we explain in section 5.3 is

$$C = \frac{2\epsilon_1}{\epsilon_1 + \epsilon_2} \tag{5.2.20}$$

so that when $P_6 = 0$ we have,

$$\phi_0 = \frac{8\pi^2}{\epsilon_1 + \epsilon_2} \tag{5.2.21}$$

for the D0 brane chemical potential, and moreover

$$\gamma = 2\pi i \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2}$$

for the spin chemical potential.

In the next section we will motivate the conjecture, and explain the origin of the change of variables. Note that, in the specific example that we study in sections 5.4–5.6, we will find that one gets a slightly different effective value of C , for the reason we will explain (having to do with a shift in the zero of the spin for the D0 branes).

5.3 Motivating the Refined Conjecture

As explained in [39, 76, 77, 88, 235, 238, 254–256, 267], the OSV conjecture is an instance of large N duality. In this case, the gauge theory is the $SU(N)$ gauge theory on the N D-branes wrapping cycles of the Calabi-Yau manifold X and comprising the black holes. The large N dual of this theory is a string theory in the back-reacted geometry. In the full physical string theory, the near-horizon geometry of a BPS black hole in four dimensions takes the form

$$AdS_2 \times S^2 \times X. \tag{5.3.22}$$

⁷As explained in [235], an arbitrary constant C is needed in the compact case due to the fact that X_I are not functions on the moduli space, but sections of a line bundle. In the non-compact case, which we study, this degree of freedom is fixed. In the unrefined case, it is typically set to 1. Our choice of the refined value is such that it reduces to 1 when we set $\epsilon_{1,2}$ to be equal.

The OSV conjecture deals with supersymmetric sub-sectors of the theory. On the black-hole side, we consider the Witten index of the theory on N D-branes; this is typically a partition function of a topological $SU(N)$ gauge theory in one dimension less. On the large N dual side, the partition function ends up depending only on F-terms in the low energy effective action, which are captured by the topological string partition function. The OSV conjecture can also be thought of as a consequence of large N duality in the topological setting alone. The 't Hooft large N duality, relating a $SU(N)$ gauge theory to a string theory is a very general phenomenon. It should encompass any $SU(N)$ gauge theory, including topological ones, and requires a string theory on the dual side, though not one containing dynamical gravity. In particular, in [11, 267] it was shown that OSV conjecture holds even for non-compact Calabi-Yau manifolds. In the physical version of the theories studied there, the Planck mass is infinite, and the black holes horizon will have zero area, making it difficult to study the large N duality in the full physical theory.

In the refined context, we have to take the Calabi-Yau to be non-compact, since the R-symmetry which is necessary to compute the protected index exists only in that case. However, the theory on N D-branes is still a $SU(N)$ gauge theory, with large entropy at large N . $Z_{\text{ref BH}}$ is simply a one parameter deformation of the ordinary black-hole partition function Z_{BH} . The dual description of the theory at large N has to be a string theory, on general grounds, and moreover, a suitable one-parameter deformation of the topological string theory. Now, we will explain why this one-parameter deformation is the refined topological string.

The statement of the refined OSV conjecture is that we can refine both sides of the OSV duality, by keeping track of the $J_3 - R$ charge. Why this should be true is most transparent in yet another way to understand the OSV, namely, using wall crossing [83]. In this case, we do not take the near-horizon limit but instead we study general D4/D2/D0-brane bound states. We can then perform TST -duality on this partition function so that it is dominated by “polar” states. Generically, these polar states can be made to decay by varying the background Kähler parameters. Along a real co-dimension one wall, the state will decay into a D6/D4/D2/D0 state and a $\overline{D6}/D4/D2/D0$ -state. By a chain of dualities (lifting to M-theory, then reducing on a different circle), the $AdS_2 \times S^2 \times X$ geometry (see [39, 83] for details) can be related to IIA string theory on X , with a $D6 - \overline{D6}$ pair.

Further, from the primitive wall-crossing formula we know that the degeneracies will factorize,

$$\Omega(D4 + \dots) \sim \Omega(D6 + \dots)\Omega(\overline{D6} + \dots) \quad (5.3.23)$$

Now the key observation is that the degeneracies of D6/D4/D2/D0 brane bound states are precisely computed by Donaldson-Thomas invariants, which are further identified with the topological string. On the topological string side, the S-duality used is precisely what relates the D6 brane partition function to the topological string [224], as we reviewed in the previous section. Therefore, we can identify the D6-brane

bound states with Z_{top} and the $\overline{D6}$ -brane bound states with $\overline{Z}_{top}$. Therefore, the semiprimitive wall-crossing formula gives precisely the factorization expected from OSV,

$$Z_{BH} \sim |Z_{top}|^2 \quad (5.3.24)$$

In the refined setting, the argument goes through in precisely the same way as in the unrefined case; we simply replace, on both sides of the duality, the Witten index of the D4 brane and the D6 branes by the protected spin character. Then we simply use the refined primitive wall-crossing formula which also factorizes. On the black hole side, the protected spin character of the D4 branes is the refined black hole partition function $Z_{\text{ref BH}}$, and on the topological string side, the protected spin character of the D6 branes is the topological string partition function – moreover, we get both $Z_{\text{top ref}}$ and $\overline{Z}_{\text{top ref}}$ from the D6 branes and the $\overline{D6}$ branes, and thus

$$Z_{\text{ref BH}} \sim |Z_{\text{ref top}}|^2. \quad (5.3.25)$$

This argument makes it obvious that the OSV conjecture should hold in the refined setting, as we conjectured.

The one subtlety in this argument is that on a noncompact Calabi-Yau geometry, there is actually no place in moduli space where the $D4$ -branes can be made to split into $D6 - \overline{D6}$ constituents, since D6 and $\overline{D6}$ branes will always have opposite central charges because of the noncompactness of the Calabi-Yau. However, this issue was already present in the unrefined case, where it did not affect the validity of the conjecture, as was shown in [11, 267]. Thus, there is no reason to think it would affect our refined conjecture either. Thus, we believe that this $D6 - \overline{D6}$ decomposition captures the correct physics of D4/D2/D0-brane bound states in both the unrefined and the refined case, and this moreover leads to a refined OSV formula.

In the rest of this section, we will provide further support for the conjecture, and explain the identification of the parameters we gave previously.

5.3.1 Refined OSV and The Wave Function on the Moduli Space

The refined topological string partition function is a wave function on the moduli space, just like in the ordinary topological string case [2, 166, 193, 194]. The quantum mechanics on the moduli space played a central role in understanding the original conjecture, and the same is true in the refined case. In this respect, there only two differences between the refined and the unrefined topological string thing: for one, the effective value of the Plank's constant of the theory g_s^2 , becomes $\epsilon_1 \epsilon_2$ (recall that, in the unrefined case, ϵ_1 and ϵ_2 coincide),

$$g_s^2 \quad \rightarrow \quad g_s^2 = \epsilon_1 \epsilon_2.$$

Secondly, the wave function that the topological string partition function computes changes: in the refined case, this wave function depends on the additional parameter $\beta = \epsilon_2/\epsilon_1$.

For this discussion of the quantum mechanics, it is useful to switch to the mirror perspective and study the refined B-model [2]. The refined B-model only depends on the complex structure moduli space, which can be parametrized by the holomorphic three form $\Omega \in H_3(X)$. We can choose a symplectic basis for $H_3(X)$ such that $A_I \cap B^J = \delta_I^J$, and define coordinates,

$$X_I = \int_{A_I} \Omega, \quad F^J = \int_{B^J} \Omega \quad (5.3.26)$$

From special geometry, we know that classically these variables are not independent and that there exists a prepotential, $F^{(0)}$, such that,

$$F^J = \frac{\partial F^{(0)}}{\partial X_J} \quad (5.3.27)$$

But now it is important to recognize that this prepotential is the genus zero contribution of the refined topological string,

$$Z_{\text{ref top}} = \exp\left(\frac{1}{\epsilon_1 \epsilon_2} F\right) = \exp\left(\frac{1}{\epsilon_1 \epsilon_2} F^{(0)} + \dots\right) \quad (5.3.28)$$

Therefore in the full quantum theory, we can represent F^J as the operator $F^J = \epsilon_1 \epsilon_2 \frac{\partial}{\partial X_J}$ and this leads to the commutation relations,

$$[F^J, X_I] = \epsilon_1 \epsilon_2 \delta_I^J \quad (5.3.29)$$

We could have applied this same reasoning to the conjugated theory, $\overline{Z}_{\text{ref top}}$ which gives,

$$[\overline{F}^J, \overline{X}_I] = \epsilon_1 \epsilon_2 \delta_I^J \quad (5.3.30)$$

and finally all of the barred variables commute with all of the unbarred variables. Note that, in this case, because the Calabi-Yau is non-compact, the moduli space is always governed by the rigid special geometry of the $\mathcal{N} = 2$ field theory, rather than the local special geometry of $\mathcal{N} = 2$ supergravity.

Now consider formally inverting the refined OSV relation,

$$\Omega(P_6, P_4, Q_2, Q_0; \gamma) = \int d\phi_0 d\phi_2 e^{Q_2 \phi_2 + Q_0 \phi_0} |Z_{\text{ref top}}|^2 \quad (5.3.31)$$

where

$$\Omega(P_6, P_4, Q_2, Q_0; \gamma) = \sum_{J_3, R} \Omega(P_6, P_4, Q_2, Q_0; J_3, R) e^{-2\gamma(J_3 - R)}. \quad (5.3.32)$$

We say that this is a formal inversion, since in the non-compact case ϕ_0 is always just a parameter, so in particular, it does not really make sense integrating over it. This aside, note that for the relation such as (5.3.31) to make sense, it *has to be the case* that $Z_{\text{ref top}}$ is indeed a wave function on the moduli space. This is because while the left hand side is independent of the choice of polarization, i.e. the choice of basis of A - and B -cycles, for an arbitrary function on the moduli space, the right hand side *would* depend on such a choice, and the conjecture would not have a chance to hold. Because $Z_{\text{ref top}}$ is a wave function, while all the terms on the right hand side depend on the choice of polarization, the integral does not depend on such a choice.

More precisely, for this to hold, one has to have the following commutation relations. We follow the reasoning in [235], and formally introduce magnetic potentials, χ^I in addition to the electric potential that we have already used. In the refined black hole partition function, we cannot specify both the electric charge and the electric potential at the same time, so they must have nontrivial commutation relations. Similarly, we require that the new magnetic potentials are conjugate to the magnetic charges. Therefore, we find

$$\begin{aligned} [\phi_I, Q^J] &= [P_I, \chi^J] = \frac{i\pi}{2} \delta_I^J \\ [\phi_I, P_J] &= [\chi^I, Q^J] = 0 \\ [Q^I, P_J] &= [\chi^I, \phi_J] = 0 \end{aligned} \tag{5.3.33}$$

where for convenience we have included an extra normalization factor above. For the right hand side of 5.3.31 to be invariant under symplectic transformations one needs the black hole commutation relations and the topological string commutation relations, to be consistent. This requires,

$$\begin{aligned} X_I &= C' \epsilon_2 P_I + i \frac{\epsilon_1}{C'} \frac{\phi_I}{\pi} \\ F^I &= C' \epsilon_2 Q^I + i \frac{\epsilon_1}{C'} \frac{\chi^I}{\pi} \end{aligned} \tag{5.3.34}$$

for some arbitrary constant, C' . Note that is in perfect agreement with the refined OSV change of variables in equation 5.2.19 upon fixing the constant to $C' = 1$. In fact, these relations were our main motivation for the change of variables we proposed in section 3, as a part of our conjecture. Notice that $\Omega(P, Q; \gamma)$ still has the interpretation as a Wigner quasi-probability distribution on phase space, just as it did in [235], but now it depends on the additional auxiliary parameter, γ .

To understand the identification of ϵ_1 and ϵ_2 with the D0 brane chemical potential ϕ_0 and the spin fugacity γ , we can use the wall crossing derivation, which forces the identification of parameters. The only subtlety is that, to relate the refined topological string to the black-hole ensemble, we need to perform the TST duality. The TST duality relates this to the chemical potentials before and after as follows:

$$\phi_0 \rightarrow \phi'_0 = \frac{4\pi^2}{\phi_0}, \quad \phi_2 \rightarrow \phi'_2 = 2\pi i \frac{\phi_2}{\phi_0}, \quad \gamma \rightarrow \gamma' = 2\pi i \frac{\gamma}{\phi_0} \quad (5.3.35)$$

The derivation of this is presented in appendix J.

As we reviewed in the previous section, the chemical potential for the D0 branes bound to the D6 brane is

$$\phi'_0 = \frac{\epsilon_1 + \epsilon_2}{2}, \quad (5.3.36)$$

and the spin is captured by

$$\gamma' = \frac{\epsilon_1 - \epsilon_2}{2}, \quad (5.3.37)$$

For this to be consistent with TST duality, the chemical potentials in the black hole ensemble need to be

$$\phi_0 = \frac{4\pi^2}{\epsilon_1 + \epsilon_2}, \quad (5.3.38)$$

for the D0 brane charge, and moreover the spin needs to be captured by

$$\gamma = 2\pi i \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2} \quad (5.3.39)$$

just as we gave in the previous section. In particular, $\gamma/2\pi i = 1 - \beta$.

The rest of this chapter is devoted to testing our conjecture. There is a class of geometries where both sides of duality are computable explicitly. These correspond to Calabi-Yau manifolds that are complex line bundles over a Riemann surface. After developing the necessary tools to precisely compute both sides of the refined OSV formula, we show that the refined OSV conjecture holds true perturbatively to all orders for these geometries.

5.4 Refined Topological String on $\mathcal{L}_1 \oplus \mathcal{L}_1 \rightarrow \Sigma$

For some simple Calabi-Yau manifolds, the refined topological string partition function is exactly computable by cutting the Calabi-Yau into simple pieces, and sewing them back together. The open-string version of this index was computed explicitly on simple geometries in [18] and used to solve the refined Chern-Simons theory completely. We will follow a similar approach here in the closed string case, for Calabi-Yau manifolds of the form

$$\mathcal{L}_1 \oplus \mathcal{L}_1 \rightarrow \Sigma$$

To obtain a Calabi-Yau manifold, the degrees of the line bundles must satisfy the property,

$$\deg(\mathcal{L}_1) + \deg(\mathcal{L}_2) = -\chi(\Sigma) = 2g - 2 \quad (5.4.40)$$

The key idea is to chop up our geometries by introducing stacks of infinitely many M5 brane/anti-brane pairs wrapping Lagrangian three-cycles as in the original topological vertex [8].⁸ Then the computation of the refined index on these chopped geometries reduces to counting M2 branes ending on these M5 branes. In this chapter, we simply explain the structure of the TQFT, and refer the reader to [22] for the details of computing these amplitudes by counting M2-brane contributions.

5.4.1 A TQFT for the Refined Topological String

The basic building blocks of the TQFT are given by the annulus (A), cap (C), and pant (P) geometries. Since degrees of bundles and euler characteristics add upon gluing, this gives a way of building up more complicated bundles over Riemann surfaces.

We start by considering the simplest geometry, which is the annulus (shown in Figure 5.1) with two trivial complex line bundles over it given by $A^{(0,0)} = \mathbb{C}^* \times \mathbb{C}^2$.

$$Z_{\text{ref top}}(A^{(0,0)}) = \sum_R \frac{1}{g_R(q, t)} M_R(U; q, t) M_R(V; q, t) \quad (5.4.41)$$

Here we are summing over all $U(\infty)$ representations R , and $M_R(U; q, t)$ is the associated Macdonald polynomial which reduces on setting $q = t$ to the simpler $\text{Tr}_R(U)$. The Macdonald metric, g_R computes the inner product of a Macdonald polynomial with itself and is given by,⁹

$$g_R(q, t) = (t/q)^{|R|/2} \prod_{(i,j) \in R} \frac{1 - t^{R_j^T - i} q^{R_i - j + 1}}{1 - t^{R_j^T - i + 1} q^{R_i - j}} \quad (5.4.42)$$

$$= \prod_{(i,j) \in R} \frac{t^{\frac{R_j^T - i}{2}} q^{\frac{R_i - j + 1}{2}} - t^{-\frac{R_j^T - i}{2}} q^{-\frac{R_i - j + 1}{2}}}{t^{\frac{R_j^T - i + 1}{2}} q^{\frac{R_i - j}{2}} - t^{-\frac{R_j^T - i + 1}{2}} q^{-\frac{R_i - j}{2}}} \quad (5.4.43)$$

Since we are working with $U(\infty)$ representations, this metric is the $N \rightarrow \infty$ limit of the ordinary $SU(N)$ Macdonald metric. We refer the reader to Appendix G for our Macdonald polynomial conventions.

⁸In the refined setting, we must choose whether to wrap these M5 branes on the z_1 or z_2 plane of the Taub-NUT space. This gives two types of refined A-branes, which can be denoted as q -branes and t -branes. At each boundary of our geometry we can place either type of brane, leading to different choices of basis for each Hilbert space. In this chapter we will not need this rich structure, and we will implicitly place q -branes at each puncture. We refer the reader to [22] for details on general q/t -brane amplitudes.

⁹Note we are including additional (q/t) factors in our definition of g_R and $M_R(U)$ compared to the standard definitions in [202]. The advantage of these factors is that they restore the symmetry, $g_R(q, t) = g_R(q^{-1}, t^{-1})$ (see also [38] for a similar shift). We refer the reader to Appendix G for more details on our Macdonald polynomial conventions.

Having discussed the simplest geometry, we should now explain how building blocks are glued together. Recall that ordinarily when we want to glue two boundaries together, we should set their holonomies to be equal except that the boundaries should have opposite orientation. This orientation reversal simply flips one of the holonomies from U to U^{-1} . Finally, to glue together the boundaries we must integrate over the Hilbert space at the boundaries.

This is also true in the refined setting, except that the integration measure is deformed to the one natural for Macdonald polynomials. If we denote the eigenvalues of U by e^{u_i} , then the Macdonald measure is given by,

$$\Delta(U; q, t) = \prod_{m=0}^{\infty} \prod_{i \neq j} \frac{(1 - q^m e^{u_i - u_j})}{(1 - tq^m e^{u_i - u_j})} \quad (5.4.44)$$

Then gluing two boundaries gives,

$$\int du_i \Delta(U; q, t) M_{R_1}(U) M_{R_2}(U^{-1}) = g_{R_1}(q, t) \delta_{R_1 R_2} \quad (5.4.45)$$

where g_R is the Macdonald metric for infinitely many variables introduced above, which is to be contrasted with the finite N Macdonald metric which we will define below in equation 5.5.134. Thus, the M_R give an orthogonal but not orthonormal basis for the boundary Hilbert space. Although we could remove the explicit metric factors g_R by choosing a different normalization for M_R , it will actually be more convenient in this chapter to keep them.

As a simple consistency check, gluing two annuli with trivial bundles should give back the original annulus amplitude. But this is clearly true, since the two annuli contribute a total factor of g_R^{-2} while the gluing process contributes a factor of g_R so that the resulting amplitude is equal to the original annulus amplitude.

Now that we have explained gluing, we also want to know how to introduce non-trivial bundles. Note that since $\chi(A) = 0$, any choice of line bundles over the annulus must satisfy $\deg(\mathcal{L}_1) = -\deg(\mathcal{L}_2)$. The simplest nontrivial choice is the geometry $A^{(1,-1)}$. This geometry can be alternatively understood as implementing a change in framing, which has been studied for the refined topological vertex in [171]. The resulting amplitude is given by,

$$Z_{\text{ref top}}(A^{(1,-1)}) = \sum_R \frac{1}{g_R(q, t)} q^{\frac{1}{2}\|R\|^2} t^{-\frac{1}{2}\|R^T\|^2} M_R(U; q, t) M_R(V; q, t) \quad (5.4.46)$$

where $\|R\|^2 = \sum_i R_i^2$ and $\|R_i^T\|^2 = \sum_i (R_i^T)^2 = \sum (2i - 1) R_i$.

Next we study the cap geometries (shown in Figure 5.1), which are given by two complex line bundles over the disc. Since the euler characteristic of the disc is equal to 1, the degrees of the line bundles must satisfy $\deg(\mathcal{L}_1) + \deg(\mathcal{L}_2) = -1$. In practice, it suffices to determine the $(0, -1)$ amplitudes, since the rest can be obtained by gluing.

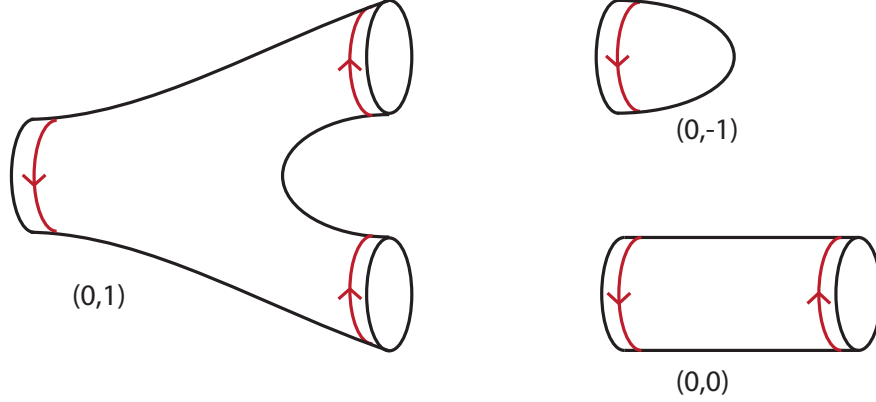


Figure 5.1: The building blocks of the refined TQFT are the pant, cap, and annulus geometries, along with complex bundles of degree (d_1, d_2) over each Riemann surface.

It is helpful to notice that this geometry is equivalent to $\mathbb{C}^3$ with a stack of branes inserted on one leg of the vertex. Thus the cap amplitude can be computed by the refined topological vertex amplitude, $C_{R..}$, with branes on the q -leg or from refined Chern-Simons [22]. The result is,

$$Z(C^{(0,-1)}) = \sum_R \frac{1}{g_R} \dim_{q,t}(R) M_R(U; q, t) \quad (5.4.47)$$

where we have defined the (q, t) -dimension of a representation R by,

$$\begin{aligned} \dim_{q,t}(R) &= (q/t)^{\frac{1}{2}|R|} M_R(t^\rho; q, t) \\ &= q^{\frac{1}{4}\|R\|^2} t^{-\frac{1}{4}\|R^T\|^2} \prod_{\square \in R} \left(q^{\frac{a(\square)}{2}} t^{\frac{l(\square)+1}{2}} - q^{-\frac{a(\square)}{2}} t^{-\frac{l(\square)+1}{2}} \right)^{-1} \end{aligned} \quad (5.4.48)$$

where $(\rho)_i = -i + 1/2$ is the $U(\infty)$ Weyl vector, while

$$\begin{aligned} a(\square) &= R_i - j \\ l(\square) &= R_j^T - i \end{aligned} \quad (5.4.49)$$

are the arm- and leg-lengths, respectively, of a box in the Young Tableau of R . The (q, t) -dimension can be understood as a (q, t) -deformation of the dimension of the symmetric group representation specified by R .¹⁰ To obtain the cap with a different choice of line bundles we can simply glue on the $A^{(1,-1)}$ or $A^{(-1,1)}$ annuli.

¹⁰Our notation differs slightly from the notation used for the unrefined case in [11]. In the limit $t = q$, our (q, t) -dimension is related to their $d_q(R)$ by, $\dim_{q,q}(R) = s_R(q^\rho) = q^{\frac{1}{4}\kappa_R} d_q(R)$.

Finally, we must specify the three-punctured sphere amplitude (see Figure 5.1), which we refer to as the “pant.” Since the three-punctured sphere has the euler characteristic $\chi = -1$, the degree of the line bundles must add to one in this case. To compute this amplitude it is helpful to recall some general properties that our TQFT must satisfy. Since it computes the refined topological A-model, the TQFT must be independent of complex structure moduli. Specifically this means that the amplitude for a Riemann surface should not depend on how it is formed by gluing simpler geometries. For this to be true, the pant amplitude should be symmetric in the three punctures, and thus should be diagonal in the Macdonald basis,

$$Z_{\text{ref top}}(P^{(0,1)}) = \sum_R P_R M_R(U_1) M_R(U_2) M_R(U_3) \quad (5.4.50)$$

Now it is helpful to recognize that the pant, cap, and annulus are not all independent. We can form the annulus by capping off one of the punctures of the pant. By consistency and using the fact that $Z(P^{(0,1)})$ is diagonal, we can solve for the pant amplitude,

$$Z_{\text{ref top}}(P^{(0,1)}) = \sum_R \frac{1}{g_R \dim_{q,t}(R)} M_R(U_1) M_R(U_2) M_R(U_3) \quad (5.4.51)$$

So far we have described the structure of the A-model on these geometries as a TQFT, but it is important to remember that the theory is not purely topological since it depends on the Kähler moduli. For Calabi-Yaus of the form $\mathcal{L}_1 \oplus \mathcal{L}_2 \rightarrow \Sigma_g$, there is only one Kähler modulus, k , that measures the area of the Riemann surface. In fact, as is familiar from the topological vertex [8], the partition function depends on this modulus only by introducing a term, $e^{-k|R|}$ in the sum over representations.

Altogether, we have given the necessary data to solve the theory completely. As an application of these results, we can study geometries of the form $\mathcal{L}_1 \oplus \mathcal{L}_2 \rightarrow \Sigma_g$ where Σ_g is a genus g Riemann surface. For this to be a Calabi-Yau manifold we must have $\deg(\mathcal{L}_1) = 2g - 2 + p$ and $\deg(\mathcal{L}_2) = -p$. Then the refined amplitude on this geometry is given by,

$$Z_{\text{ref top}}^{(g,p)}(q, t) = \sum_R \left(\frac{\dim_{q,t}(R)^2}{g_R} \right)^{1-g} q^{\frac{(2g-2+p)}{2} \|R\|^2} t^{-\frac{(2g-2+p)}{2} \|R^T\|^2} Q^{|R|} \quad (5.4.52)$$

where we have defined the exponentiated Kähler modulus as $Q = e^{-k}$. It can be checked that this has the expected symmetry,

$$Z_{\text{ref top}}^{(g,p)}(q, t) = Z_{\text{ref top}}^{(g,p)}(t^{-1}, q^{-1}) \quad (5.4.53)$$

which implies that the Gopakumar-Vafa invariants come in complete multiplets of $SU(2)_\ell$ (as was the case in the unrefined limit). However, the amplitude is not symmetric under the exchange $q \leftrightarrow t$. This implies that the Gopakumar-Vafa invariants

for these Calabi-Yaus do not come in full representations of $SU(2)_r$, but only carry $U(1)_r \subset SU(2)_r$ charge. The BPS states come from quantizing the moduli space of curves in X , together with the $U(1)$ bundle on them. The $SU(2)_r$ spin content comes from cohomologies of the moduli of the curve itself, while the $SU(2)_\ell$ comes from the bundle. In the present case, the curve is Σ itself. Its moduli space is in general non-compact, as typically one of the two line bundles over Σ has positive degree. Correspondingly, the Lefschetz $SU(2)_r$ action on the cohomologies of the moduli space does not have to result in complete multiplets – there can be contributions that escape to infinity. (See Appendix H for some sample computations of Gopakumar-Vafa invariants for these geometries.) In fact, the only case when the moduli space is compact is when $\Sigma = \mathbb{P}^1$, and both line bundles are $\mathcal{O}(-1)$. It is easy to see that in this case the amplitude does in fact have the $q \leftrightarrow t$ symmetry as well.

In addition, the amplitude is not symmetric under exchange of the two line bundles, which is equivalent to taking $p \rightarrow 2 - 2g - p$. This tells us that one of the line bundles is distinguished from the other in the refined setting. In fact, this arises because the index in equation 5.2.16 includes an R -symmetry twist that rotates a specific bundle in the noncompact Calabi-Yau (for more details see [22]). Equivalently, as will be explained in section 5.6, these refined topological string amplitudes can be obtained by taking the large N limit of $D4$ branes wrapping one of the bundles. In the unrefined case, the large N limit does not retain information about which bundle the $D4$ branes wrapped, but in the refined case this choice has an effect on the closed string amplitude. This is related to the above observation since in both the $D4$ construction and in the closed string construction we must choose an R -symmetry rotation to preserve supersymmetry. However, these symmetries are not completely lost since the amplitude is symmetric under simultaneously exchanging $q \leftrightarrow t$ and exchanging the bundles,

$$Z_{\text{ref top}}^{(g,p)}(q, t) = Z_{\text{ref top}}^{(g, 2-2g-p)}(t, q) \quad (5.4.54)$$

As we will explain in section 5.4.3, p specifies the five-dimensional Chern-Simons coupling of the geometrically engineered gauge theory. In [38], it was similarly observed that for geometries that engineer five-dimensional $SU(N)$ gauge theories, the refined topological string is only symmetric under the simultaneous exchange of $q \leftrightarrow t$ and inverting the Chern-Simons level, $k \rightarrow -k$.

So far, we have computed all the non-trivial contributions to the refined topological string. However, we should also include by hand the additional pieces that appear at genus zero and one. In the unrefined case, these arise from constant maps. In the refined case, for geometries that engineer five-dimensional gauge theories, these contributions arise from the classical prepotential and the one-loop determinant of the instanton partition function. These degree zero pieces take the form,

$$Z_0(q, t) = \left(M(q, t) M(t, q) \right)^{x/4} \exp \left(\frac{1}{\epsilon_1 \epsilon_2} \frac{ak^3}{6} + \frac{\epsilon_2}{\epsilon_1} b \frac{k}{24} \right) \quad (5.4.55)$$

where $M(q, t)$ is the refined MacMahon function,

$$M(q, t) = \prod_{i,j=1}^{\infty} (1 - t^i q^{j-1}) \quad (5.4.56)$$

and χ is the euler characteristic of the Calabi-Yau, while a is related to the triple intersection of the Kähler class and b is the second Chern class of the Calabi-Yau. These numbers are ambiguous because of the non-compactness of our geometries, but it was argued in [11] that for the connection with black holes the natural values are,

$$\chi = 2 - 2g, \quad a = -\frac{1}{p(p+2g-2)}, \quad b = \frac{p+2g-2}{p} \quad (5.4.57)$$

Note, that we have split the MacMahon function into two pieces, related by interchanging q and t . This split naturally appears in section 5.6, when making the connection with the refined black hole partition function. A similar splitting was recently observed for the motivic Donaldson-Thomas invariants of the conifold in [216].

In section 5.4.3, we will give further evidence that this refined amplitude is the correct one by comparing it with the equivariant instanton partition function of the geometrically engineered five-dimensional field theory. Before doing so, however, it will be helpful to discuss one final aspect of the TQFT that arises when D-branes are included in the fiber of the complex line bundles.

5.4.2 Branes in the Fiber

So far we have solved for the refined string on bundles over closed Riemann surfaces and Riemann surfaces with boundaries. These boundaries naturally end on branes wrapping an S^1 in the base and two dimensions in the fiber. However, for understanding the refined OSV conjecture, it will be helpful to also consider introducing branes in the fiber. For unrefined topological strings, this was studied in [11], and our analysis will follow a similar approach.

We consider a lagrangian brane at a point, z , in the base Riemann surface, Σ_g . Since the brane is local in the base, we only need to study a neighborhood of z . Thus it is natural to introduce branes in the base that chop up the geometry into a disc, D , containing z , and its complement, $\Sigma \setminus D$. The full amplitude is given by,

$$Z = \sum_{R,Q} Z_R(\Sigma \setminus D) Z_{RQ}(D) M_Q(V; q, t) \quad (5.4.58)$$

where V is the holonomy around the branes in the fiber. The amplitude on the complement, $\Sigma \setminus D$, can be solved by gluing using the amplitudes in the previous section, but we still need to solve for the disc amplitude with two sets of branes.

This can be accomplished by noticing that D has the topology of $\mathbb{C}^3$ with the base and fiber branes on two legs of the vertex. Thus, the full cap amplitude,

$$Z(D) = \sum_{R,Q} Z_{RQ}(D) M_R(U) M_Q(V) \quad (5.4.59)$$

is simply computed by the refined topological vertex amplitude with two stacks of branes on different legs, as shown in Figure 5.2.

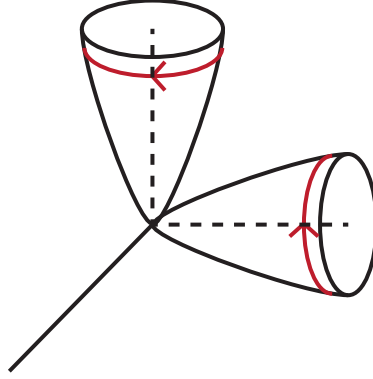


Figure 5.2: The full cap amplitude with branes in both the fiber and the base.

Alternatively, as will be explained in [22], this amplitude can be solved by following the refined Chern-Simons theory through a geometric transition. From this perspective, Z_{RQ} is computed by the large N limit of the refined Chern-Simons S-matrix,

$$W_{RQ} = \lim_{N \rightarrow \infty} t^{-\frac{N(|R|+|Q|)}{2}} (q/t)^{\frac{|R|+|Q|}{4}} \frac{S_{RQ}(q, t; N)}{S_{00}(q, t; N)} \quad (5.4.60)$$

$$= (q/t)^{\frac{|R|+|Q|}{2}} M_R(t^\rho) M_Q(t^\rho q^R) \quad (5.4.61)$$

where we have used the symmetrized definition of the infinite-variable Macdonald polynomials in Appendix G. By including the appropriate metric factors, we obtain,

$$Z_{RQ} = \frac{1}{g_R g_Q} W_{RQ} \quad (5.4.62)$$

Note that if we set the representation of the fiber brane to be trivial, $Q = 0$, then this geometry is the same as the cap (C) that we studied above. This is consistent with the fact that $W_{R0} = \dim_{q,t}(R)$.

As an example, take the geometry, $\mathcal{O}(2g - 2 + p) \oplus \mathcal{O}(-p) \rightarrow \Sigma_g$ with branes in the fiber over h points. Then the refined amplitude is given by,

$$Z_{\text{ref top}}^{(g,p,h)}(q, t) = \sum_{R, R_1, \dots, R_h} \frac{g_R^{g-1}}{W_{R0}^{2g-2+h}} \frac{W_{RR_1} \cdots W_{RR_h}}{g_{R_1} \cdots g_{R_h}} q^{\frac{(2g-2+p)}{2} \|R\|^2} t^{-\frac{(2g-2+p)}{2} \|R^T\|^2} Q^{|R|} \cdot M_{R_1}(V_1) \cdots M_{R_h}(V_h) \quad (5.4.63)$$

It is also useful to understand how an anti-brane can be introduced that wraps the fiber. Recall that in the unrefined topological string, converting a brane into an anti-brane corresponds to taking,

$$s_R(U) \rightarrow (-1)^{|R|} s_{R^T}(U) \quad (5.4.64)$$

where $s_R(U)$ is the Schur function. The analogue of this reversal in the refined setting corresponds to taking,

$$M_R(U; q, t) \rightarrow \iota M_R(U; q, t) \quad (5.4.65)$$

where ι is defined by how it acts on power sums, $p_n(x)$,

$$\iota(p_n) = -p_n \quad (5.4.66)$$

We refer the reader to [22] for more details on this construction. This implies that the disc amplitude with an anti-brane in the fiber is given by,

$$\tilde{Z}(D) = \sum_{R, Q} \frac{1}{g_R g_Q} W_{RQ} M_R(U) \iota M_Q(V) \quad (5.4.67)$$

If we want to rewrite this in the $M_Q(V)$ basis, this can be done by using a generalized Cauchy identity (see Appendix G),

$$\tilde{Z}(D) = \sum_{R, Q} \frac{1}{g_R g_Q} (q/t)^{\frac{|R|+|Q|}{2}} M_R(t^\rho) M_Q(t^\rho q^R) M_R(U) \iota M_Q(V) \quad (5.4.68)$$

$$= \sum_R \frac{1}{g_R} M_R(t^\rho) M_R(U) \sum_Q \frac{1}{g_Q} (q/t)^{\frac{|R|+|Q|}{2}} M_Q(t^\rho q^R) \iota M_Q(V) \quad (5.4.69)$$

$$= \sum_R \frac{1}{g_R} M_R(t^\rho) M_R(U) \sum_Q \frac{1}{g_Q} (q/t)^{\frac{|R|+|Q|}{2}} \iota M_Q(t^\rho q^R) M_Q(V) \quad (5.4.70)$$

$$= \sum_{R, Q} \frac{1}{g_R g_Q} \widetilde{W}_{RQ} M_R(U) M_Q(V) \quad (5.4.71)$$

where we have defined $\widetilde{W}_{RQ}$ by,

$$\widetilde{W}_{RQ} = (q/t)^{\frac{|R|+|Q|}{2}} M_R(t^\rho) \iota M_Q(t^\rho q^R) \quad (5.4.72)$$

This amplitude will be particularly important for studying the genus $g = 0$ OSV conjecture in section 5.6.2.

It is important to note that the fiber brane has a modulus, k_f . If we take u to be a coordinate for one of the fibers, then the fiber brane sits at $|u|^2 = \text{const.}$ As is standard, this real modulus combines with the holonomy to form the complexified Kähler parameter, k_f . Including this modulus simply modifies the partition function as,

$$M_R(U) \rightarrow e^{-k_f |R|} M_R(U) \quad (5.4.73)$$

This modulus will appear in section 5.6 when we discuss the “ghost branes” that appear in tests of the refined OSV conjecture.

5.4.3 Refined Topological Strings on $\mathcal{L}_1 \oplus \mathcal{L}_2 \rightarrow \Sigma_g$ and $5d$ $U(1)$ Gauge Theories

Now that we have defined a TQFT that computes refined topological string amplitudes, we would like to verify our proposal. A simple check is that the Gopakumar-Vafa invariants are integers. We have verified this in general, and we present a few examples in Appendix H.

We can perform a much stronger check of our proposal by using geometric engineering. We consider M-theory on the Calabi-Yau, $X = \mathcal{O}(p) \oplus \mathcal{O}(2g - 2 - p) \rightarrow \Sigma_g$. This is known to engineer five dimensional $U(1)$ gauge theory with g hypermultiplets in the adjoint representation, and with a level $k_{CS} = 1 - g - p$ five-dimensional Chern-Simons term turned on [70, 72, 178].¹¹

Now we consider the K-theoretic equivariant instanton partition function for these theories.¹² The original index in [225] that computes the K-theoretic instanton partition function is exactly the same index that we have used to compute the refined A-model in equation 5.2.16, so the two partition functions must agree.

¹¹The motivic Donaldson-Thomas invariants of these geometries were also studied recently in [70, 72]. In general, the motivic invariants of a Calabi-Yau, X , will differ from the refined invariants that we compute in this chapter. Motivic invariants depend on the motive of X and thus are sensitive to its complex structure. In contrast, our refined invariants are computed by a supersymmetric index which makes them invariant under complex structure deformations. These differences are reflected in the connection with geometric engineering. In [70, 72], the motivic invariants for these geometries were related to the instanton partition function with the adjoint mass equal to $m = (\epsilon_1 - \epsilon_2)/2 \leftrightarrow \tilde{y} = \sqrt{q/t}$. Our refined invariants are identified with the different parameter choice, $m = 0 \leftrightarrow \tilde{y} = 1$. We thank Emanuel Diaconescu for helpful discussions on this point.

¹²Ordinarily, such counting would not be sensible because $U(1)$ instantons are singular and because the adjoint representation of $U(1)$ is trivial. However, this instanton counting is performed by turning on background noncommutativity which both resolves $U(1)$ instantons and causes fields in the $U(1)$ adjoint representation to transform nontrivially.

As explained in [70], the instanton partition function for this five-dimensional field theory is given by,

$$Z_{U(1)}^{g,k_{CS}}(q, t, \tilde{Q}) = \sum_{\mu} \prod_{\square \in \mu} \left(q^{-l(\square)-1/2} t^{a(\square)+1/2} \right)^{k_{CS}} \left(1 - q^{-l(\square)} t^{-a(\square)-1} \right)^{g-1} \cdot \left(1 - q^{l(\square)+1} t^{a(\square)} \right)^{g-1} (t/q)^{\frac{(g-1)|\mu|}{2}} \tilde{Q}^{|\mu|} \quad (5.4.74)$$

where q and t^{-1} are the equivariant parameters rotating the z_1 and z_2 planes respectively. The sum is over all Young Tableaux, μ , and the arm and leg length of a box in such a tableau (defined in equation 5.4.49) are denoted by $a(\square)$ and $l(\square)$, respectively.

By using the definitions of the metric and (q, t) -dimension in equations 5.4.42 and 5.4.48, we can rewrite the equivariant instanton partition function as,

$$Z_{U(1)}^{g,k_{CS}}(q, t, \tilde{Q}) = \sum_{\mu} \left(\frac{\dim_{q,t}(\mu)^2}{g(\mu)} \right)^{1-g} \left(q^{-\frac{1}{2}\|\mu\|^2} t^{\frac{1}{2}\|\mu^T\|^2} \right)^{k_{CS}+1-g} (-1)^{|R|} \tilde{Q}^{|\mu|} \quad (5.4.75)$$

But now it is clear that this agrees with the refined topological string partition function of equation 5.4.52,

$$Z_{U(1)}^{g,k_{CS}}(q, t, \tilde{Q}) = Z_{\text{ref top}}^{g,p}(q, t, Q) \quad (5.4.76)$$

upon making the change of variables,

$$\begin{aligned} \tilde{Q} &= (-1)^{g-1} Q \\ k_{CS} &= 1 - g - p \end{aligned}$$

This verifies in general our proposed refinement of the Bryan-Pandharipande TQFT for arbitrary line bundles over a Riemann surface.

5.5 Refined Black Hole Entropy

In this section we study BPS bound states of N D4 branes wrapping a four-cycle inside a Calabi-Yau, and carrying D2 and D0 brane charge. We start by explaining that the refined counting of D4/D2/D0-brane BPS bound states computes the χ_y -genus of the moduli space of instantons on the four-cycle wrapped by the D4 branes. We then specialize to the case of interest in this chapter – IIA string theory compactified to four-dimensions on the class of Calabi-Yau manifolds, X , that consist of two complex line bundles over a Riemann surface, and show how to compute the χ_y genus in the examples that arise there. Finally, as a check of our results in this section, we compare our answers in the case when the four-cycle is $\mathcal{O}(-1) \rightarrow \mathbb{P}^1$ against a direct

computation of the cohomologies of the moduli space of instantons, by Yoshioka and Nakajima in [220, 286], and find a perfect agreement.

In the unrefined case, the black hole partition function is the index

$$Z_{BH} = \text{Tr}_{\mathcal{H}_{BPS}} (-1)^F e^{-\phi_2 Q_2} e^{-\phi_0 Q_0} \quad (5.5.77)$$

where Q_0 and Q_2 are the D0 and D2 charges, while ϕ_0 and ϕ_2 are the respective chemical potentials. Since we are working in the large volume limit, we can identify D0/D2/D4 bound states with nontrivial $U(N)$ bundles, V , over C_4 . The D-brane charges and Chern classes of this bundle are related by,

$$Q_2 = c_1(V), \quad Q_0 = ch_2(V) \quad (5.5.78)$$

Therefore, calculating degeneracies will reduce to field theoretic computations on the D4-brane worldvolume. Since the D4 brane wraps $\mathbb{R}_t \times C_4$, we can associate to C_4 a Hilbert space $\mathcal{H}$, which is graded by D2/D0-brane charge, angular momentum, J_3 , and R-charge, R . Now we would like to compute the BPS degeneracies as a trace over this entire Hilbert space. This can be done easily by using the Witten index, since non-BPS contributions will cancel out. Therefore we must simply compute the D4-brane path integral on $S^1 \times C_4$,

$$Z_{BH} = \text{Tr}_{\mathcal{H}} (-1)^F e^{-\phi_2 c_1} e^{-\phi_0 ch_2} \quad (5.5.79)$$

Since the D-branes are wrapping a curved geometry, the gauge theory is topologically twisted along C_4 [46]. In our case, $\mathcal{H}_{BPS}$ is equal to the cohomology of instanton moduli space for the corresponding topological sector. Therefore, computing the Witten index reduces to computing the euler characteristic, $\chi(\mathcal{M})$, for the moduli space of instantons on C_4 .

We have presented this computation entirely from a five-dimensional perspective because this approach will easily generalize to the refined setting. However, in the unrefined case we could also reduce on the S^1 and study the four-dimensional gauge theory. This leads to four-dimensional topologically twisted $\mathcal{N} = 4$ Yang-Mills [269] on C_4 with the observables,

$$S = \frac{\phi_0}{8\pi^2} \int \text{Tr } F \wedge F + \frac{\phi_2}{2\pi} \int \text{Tr } F \wedge \omega_{\Sigma} \quad (5.5.80)$$

inserted into the action. Here, ω_{Σ} is the Kähler class of the Riemann surface, Σ_g . Since this is the Vafa-Witten [269] twist of $\mathcal{N} = 4$, the four-dimensional perspective explains why we are computing the euler characteristic of instanton moduli space.

Now that we have discussed the unrefined case, we would like to count BPS states while keeping information about angular momentum and R-charge. As explained in section 5.2, our goal is to compute the protected spin character of D2 and D0-branes bound to the D4-branes,

$$Z_{BH} = \text{Tr}_{\mathcal{H}_{BPS}} (-1)^{2J_3} y^{J_3 - R} e^{-\phi_2 Q_2} e^{-\phi_0 Q_0} \quad (5.5.81)$$

where we have used the variable, $y \equiv e^{-2\gamma}$. Since this is an index, it only receives contributions from BPS states. This means we can extend the trace to be over the full D4-brane Hilbert space $\mathcal{H}$,

$$Z_{BH} = \text{Tr}_{\mathcal{H}}(-1)^{2J_3} y^{J_3-R} e^{-\phi_2 Q_2} e^{-\phi_0 Q_0}. \quad (5.5.82)$$

This also means that Z_{BH} can be computed by the five-dimensional path integral and will be invariant under small deformations.¹³

To understand precisely what the protected spin character computes, it helps to remember that the Hilbert space $\mathcal{H}_{BPS}$ can be identified with the cohomology of the moduli space $\mathcal{M}$ of instantons on C_4 . Once we fix the topological charges c_1 and ch_2 , the most general geometric quantity that can be computed from the cohomology of $\mathcal{M}$ is the Hodge polynomial,

$$e(\mathcal{M}; x, y) = \sum_{p,q} (-1)^{p+q} x^p y^q \dim H^{p,q}(\mathcal{M}) \quad (5.5.83)$$

As explained in [87], the degrees, (p, q) are related to the R-charge and spin by,

$$J_3 = \frac{p+q}{2}, \quad R = \frac{p-q}{2} \quad (5.5.84)$$

Therefore, the refined black hole partition function in equation 5.5.82 computes the generating function for the χ_y genus of instanton moduli space,

$$Z_{BH} = \sum_{c_1, ch_2} e^{-\phi_0 ch_2 - \phi_2 c_1} \sum_{p,q} (-1)^{p+q} y^q h^{p,q}(\mathcal{M}_{c_1, ch_2}) \quad (5.5.85)$$

$$= \sum_{c_1, ch_2} e^{-\phi_0 ch_2 - \phi_2 c_1} \chi_y(\mathcal{M}_{c_1, ch_2}) \quad (5.5.86)$$

One more aspect of the protected spin character that we will need is its transformation properties under S -duality. In the unrefined case, the transformation properties are well known [269]. The partition function 5.5.77 transforms like a theta function, with modular parameter ϕ_0 . We show in appendix J that in the refined case S -duality corresponds to replacing

$$\phi_0 \rightarrow \frac{4\pi^2}{\phi_0}, \quad \phi_2 \rightarrow 2\pi i \frac{\phi_2}{\phi_0}, \quad \gamma \rightarrow 2\pi i \frac{\gamma}{\phi_0} \quad (5.5.87)$$

¹³One should contrast this with the most general trace,

$$Z_{BH} = \text{Tr}_{\mathcal{H}_{BPS}}(-x)^{J_3+R}(-y)^{J_3-R} e^{\phi_2 Q_2 - \phi_0 Q_0}$$

where J_3 is the generator of the $Spin(3)$ rotation group in the (3+1)-dimensional spacetime, and R is the $U(1)$ R-charge of the four-dimensional BPS states. Unfortunately, this trace cannot be extended to the full Hilbert space, since non-BPS states will contribute nontrivially. This means that the doubly-refined trace in equation 13 cannot be computed by a five-dimensional path-integral, and is therefore analogous to the five-dimensional Khovanov-Rozansky construction of [285].

where the variable y in the χ_y genus is related to γ by $y = e^{-2\gamma}$. In the rest of this section, we will show that, in the simple example of the family of Calabi-Yaus we have been studying, the χ_y genus of the instanton moduli space is computable explicitly in terms of a topological theory on the base Riemann surface Σ .

5.5.1 D4 branes on $\mathcal{L}_1 \oplus \mathcal{L}_2 \rightarrow \Sigma$

In our local Calabi-Yau manifold,

$$\mathcal{L}_1 \oplus \mathcal{L}_2 \rightarrow \Sigma_g$$

consider N D4 branes wrapping the zero section of $\mathcal{L}_2$. The world-volume of the brane is

$$\mathcal{D} = (\mathcal{L}_1 \rightarrow \Sigma_g)$$

As before, we take $\mathcal{L}_1$ to have the first Chern class $-p$, so $\mathcal{L}_1$ is an $\mathcal{O}(-p)$ bundle over Σ . In the unrefined case studied in [11], the partition function of the Vafa-Witten twisted $\mathcal{N} = 4$ $U(N)$ Yang-Mills on $\mathcal{D}$ was shown to be computed by q -deformed two-dimensional bosonic Yang-Mills on Σ_g . Roughly speaking, one can use localization on the fiber over the Riemann surface to reduce the four-dimensional theory down to a theory on Σ . The basic observation is that one can use localization along the fiber of $\mathcal{O}(-p) \rightarrow \Sigma_g$ to reduce the four dimensional theory to a two dimensional theory on the fixed point set. This reduces the observables,

$$S = \frac{\phi_0}{8\pi^2} \int \text{Tr } F \wedge F + \frac{\phi_2}{2\pi} \int \text{Tr } F \wedge \omega_\Sigma \quad (5.5.88)$$

to

$$S = \frac{\phi_0}{4\pi^2} \int_{\Sigma_g} \text{Tr } \Phi F + \frac{\phi_2}{2\pi} \int_{\Sigma_g} \text{Tr } \Phi \omega_\Sigma - p \frac{\phi_0}{8\pi^2} \int_{\Sigma_g} \text{Tr } \Phi^2, \quad (5.5.89)$$

which is the action of the bosonic two dimensional Yang-Mills. Here, Φ is the holonomy of the circle at infinity of the $\mathbb{C}$ fiber – the action becomes a boundary term. The last term reflects the topology of the fibration over Σ_g . The way it arises from four dimensions was explained in [11]. This is not quite the end of the story, as one has to be careful about the measure of the path integral. The fact that Φ comes from the holonomy around the S^1 turns it into a periodic variable – this is why the theory is q -deformed 2d Yang-Mills, instead of ordinary 2d Yang Mills. In the limit where the $D2$ brane chemical potential ϕ_2 is turned off, the q -deformed Yang-Mills on the above geometry reduces to an analytic continuation of ordinary Chern-Simons theory on a degree p S^1 bundle over Σ_g . It is important to note that the $\mathcal{N} = 4$ YM and Chern-Simons couplings are the same.

In this chapter, we would like to solve for the corresponding refined amplitudes. Since all of the arguments in the derivation so far were topological, the only thing that can change in a non-trivial way is the measure of the two-dimensional path

integral. While deriving the measure is straightforward in the unrefined case, it is more challenging in the refined theory. Instead, we will give pursue a different path. We will find another derivation of the fact that the Vafa-Witten partition function in this background is computed by q -deformed 2d Yang-Mills, in which understanding the deformation we need will be easy. It will turn out that the theory we get is related to the refined Chern-Simons theory of [18, 22].

5.5.2 From 4d $\mathcal{N} = 4$ Yang-Mills to 2d (q, t) -deformed Yang-Mills

The idea of the derivation is to look at the same D4 brane background in a slightly different way – by pairing the coordinates differently to get a Calabi-Yau four-fold instead. Doing so will make it manifest that, in the unrefined case, the theory we get is the same as Chern-Simons theory at $\phi_2 = 0$, or more generally, 2d q -deformed Yang Mills.

To begin with, we will consider the case $p = 0$ case, so the Calabi-Yau manifold is simply

$$(\mathcal{O}(0) \oplus \mathcal{O}(2g - 2) \rightarrow \Sigma_g) = \mathbb{C} \times T^*\Sigma_g$$

and the D4 branes wrap the divisor

$$\mathcal{D} = (\mathcal{O}(0) \rightarrow \Sigma_g) = \mathbb{C} \times \Sigma_g.$$

Thus, all together, the Vafa-Witten theory we are interested in, as we explained above, arises from studying the partition function of the N D4-branes wrapping,

$$\mathbb{C} \times \Sigma_g \times S_t^1 \tag{5.5.90}$$

in IIA string theory on,

$$\mathbb{C} \times T^*\Sigma_g \times \mathbb{R}^3 \times S_t^1. \tag{5.5.91}$$

As we go around the temporal circle, S_t^1 , we compute the index,

$$Z = \text{Tr}(-1)^F e^{-\phi_0 Q_0} \tag{5.5.92}$$

where Q_0 is the $D0$ -brane charge bound to the $D4$ -branes. We have temporarily set ϕ_2 to zero. Our goal is to explain why this construction leads to the partition function of analytically continued Chern-Simons on $S^1 \times \Sigma_g$. Recall moreover that after a modular transformation, the partition function becomes manifestly equal to the partition function of Chern-Simons theory, with $q = e^{g_s} = e^{\frac{4\pi^2}{\phi_0}}$.

In [285], analytically continued $SU(N)$ Chern-Simons theory on $S^1 \times \Sigma_g$ was obtained from a string theory construction involving a stack of N D4 branes wrapping

$$\mathbb{C} \times \Sigma_g \times S_t^1 \tag{5.5.93}$$

in IIA string theory on the Calabi-Yau fourfold $T^*(\mathbb{C} \times \Sigma_g)$, or more precisely, on

$$T^*(\mathbb{C} \times \Sigma_g) \times \mathbb{R} \times S_t^1 = \mathbb{C} \times T^*\Sigma_g \times \mathbb{R}^3 \times S_t^1. \quad (5.5.94)$$

At infinity of the D4 brane, we impose the D6 brane boundary conditions along

$$T^*(S^1 \times \Sigma_g) \times \{0\} \times S_t^1 \quad (5.5.95)$$

We view the $\mathbb{C} \times \Sigma_g$ as the base of the cotangent space, and the D6 brane wraps the S^1 at the boundary of $\mathbb{C}$. The theory on the D4 branes (forgetting the temporal circle) on a Lagrangian cycle in a Calabi-Yau fourfold, is the Langlands twist of $\mathcal{N} = 4$ $U(N)$ Yang-Mills on $\mathbb{C} \times \Sigma_g$. Upon going around the first S^1 , we compute the index on the $D4$ brane worldvolume,

$$Z = \text{Tr}(-1)^F e^{-\phi'_0 Q_0} \quad (5.5.96)$$

where $\phi'_0 = \frac{4\pi^2}{\phi_0} = g_s$. Using string dualities, [285] argued that the theory on the D4 branes is $U(N)$ Chern-Simons theory, with $q = e^{g_s}$.

It may be surprising at first, but these two constructions are effectively the same. In the case studied in [285] one has $\mathcal{N} = 4$ theory with the Langlands twist, while we are a priori interested in the Vafa-Witten twist. While the two are not the same on a generic four manifold V , if we take $V = \mathbb{C} \times \Sigma_g$, the difference disappears. We can argue that this is the case by recalling that topologically twisting merely implements the twisted version of supersymmetry imposed by the string background. As is manifest from 5.5.90 and 5.5.94, the string backgrounds end up being the same in our case. The other apparent difference is that the Witten construction involves $D4$ -branes ending on a $D6$ -brane at infinity. In contrast, the first construction naively only involves $D4$ -branes. This discrepancy can be resolved by remembering that in the our setup, we still must impose boundary conditions at infinity on the noncompact $\mathbb{C}$. If we were to choose the boundary conditions S -dual to those of the $D6$ -brane boundary conditions for the S^1 at infinity, then the two setups agree. The S -duality is here simply to account for the fact that with D6 brane boundary conditions it is $q = e^{g_s}$ that keeps track of the instanton charge, where q is parameter in terms of which the one naturally writes the Chern-Simons amplitudes – while with the S -dual boundary conditions instead, it is e^{-1/g_s} that keeps track of the instanton charge in the gauge theory on the four-manifold.

As preparation for understanding the refined theory (and to ultimately make contact with the definition of refined Chern-Simons in [18]), it is helpful to also consider the unrefined setup in a slightly different geometry. On very general grounds [285], we expect the partition function of Langlands-twisted $SU(N)$ $\mathcal{N} = 4$ theory on a four-manifold V , to be equal to the partition function of the $SU(N)$ Chern-Simons theory on the boundary ∂V of the four manifold. The choice of the bulk geometry, V , only potentially affects the integration contour of the analytically continued Chern-Simons partition function. In the previous setup we studied $V = \mathbb{C} \times \Sigma$, but instead

we could choose $V = \mathbb{R}_+ \times S^1 \times \Sigma$. More explicitly we could take IIA string theory on the geometry,

$$T^*(\mathbb{C}^* \times \Sigma) \times \mathbb{R} \times S_t^1 = \mathbb{C}^* \times T^*\Sigma \times \mathbb{R}^3 \times S_t^1 \quad (5.5.97)$$

with $D4$ branes wrapping

$$\mathbb{R}_+ \times S^1 \times \Sigma \times S_t^1 \quad (5.5.98)$$

and with D6 brane boundary condition along,

$$\{0\} \times T^*(S^1 \times \Sigma) \times \{0\} \times S_t^1. \quad (5.5.99)$$

This is the more familiar realization of Chern-Simons that appears in the study of topological strings and in [285].

Now we would like to understand the effect of refinement on these setups. Firstly, as we argued, the setups are effectively the same for the purposes of the index, so if we understand refinement in any one of these, we will have understood it in all the others as well. Moreover, note that already in the unrefined case, the first and the second (or third) setup, are related by S -duality. Thus, if, in the second and third constructions, we are computing the index

$$Z_{ref} = \text{Tr}(-1)^F \exp \left(-\phi'_0 Q_0 - 2\gamma'(J_3 - R) \right), \quad (5.5.100)$$

where J_3 rotates the $\mathbb{R}^3$ spacetime and the R -symmetry acts geometrically by rotating the fiber of $T^*\Sigma_g$, the index in the first setup is related to this by S -duality, (5.5.87)

$$\phi'_0 \rightarrow \phi_0 = \frac{4\pi^2}{\phi'_0}, \quad \gamma' \rightarrow \gamma = 2\pi i \frac{\gamma'}{\phi'_0} \quad (5.5.101)$$

and equals

$$Z_{ref} = \text{Tr}(-1)^F \exp \left(-\phi_0 Q_0 - 2\gamma(J_3 - R) \right). \quad (5.5.102)$$

This would in principle simply provide alternate setups to compute the index, but it would not save us the work of actually evaluating it. Fortunately, however, in the third setup, the index was already computed. The problem of evaluating the refined index in this context was solved in [15,18], in terms of the refined Chern-Simons theory on $S^1 \times \Sigma_g$. Since all three different setups, with the identification of parameters as in 5.5.101 give rise to the same partition function, we conclude that the partition function in the first setup is simply the refined Chern-Simons partition function! For $\gamma' = 0$, refined Chern-Simons becomes the same as ordinary Chern-Simons theory, analytically continued away from the integer level. As shown in [11], this in turn is the same as the 2d q -deformed Yang-Mills theory on Σ , upon reduction on the S^1

factor. Thus, we have derived the result of [11], by different means. Moreover, we have explained how to generalize it to the refined case. We will explain below that there is a two dimensional theory related to refined Chern-Simons theory the same way the q -deformed Yang-Mills is related to the ordinary Chern-Simons theory; we will call this theory the q, t -deformed 2d Yang-Mills.

The only thing that remains to do is identify ϕ_0 and γ , with the parameters q, t that appear in the refined Chern-Simons partition function. To do this, we need to back up slightly, and recall how the refined Chern-Simons theory was defined originally. The refined Chern-Simons partition function is defined as the index of M-theory in the background that arises by simply uplifting the third setup to M-theory. In this case, D6 branes lift to Taub-Nut space, and D4 branes lift to M5 branes. All together, we get M-theory on

$$T^*(S^1 \times \Sigma) \times TN \times S^1 = \mathbb{C}^* \times T^*\Sigma \times TN \times S^1 \quad (5.5.103)$$

with M5 branes on

$$S^1 \times \Sigma \times \mathbb{C} \times S^1 \quad (5.5.104)$$

In addition, as we go around the S^1 , the Taub-NUT is twisted by,

$$(z_1, z_2) \rightarrow (q z_1, t^{-1} z_2). \quad (5.5.105)$$

So, from this perspective, the index we are computing takes the form similar to that in equation (5.2.16):

$$Z_{ref} = \text{Tr}(-1)^F q^{S_1 - R} t^{R - S_2}, \quad (5.5.106)$$

where S_1 rotates the complex plane z_1 wrapped by the M5 brane and S_2 generates the rotation of the z_2 plane, transverse to M5 brane and R is another R-symmetry, coming from rotations of the fiber of $T^*\Sigma$. Moreover, let

$$q = e^{-\epsilon_1}, \quad t = e^{-\epsilon_2}.$$

We will see that these will turn out to be exactly the ϵ_1 and the ϵ_2 parameters that arise in the refined topological string. If we reduce this to IIA, we get the third setup back with the definition of refined Chern-Simons we started the discussion with. Naively, following arguments similar to those in section 2, one would have expected that we simply have $e^{-\phi_0} = \sqrt{qt} = e^{-\frac{\epsilon_1 + \epsilon_2}{2}}$ while $e^{\gamma'} = \sqrt{q/t} = e^{\frac{\epsilon_1 - \epsilon_2}{2}}$. However, this would have treated q and t symmetrically, while in refined Chern-Simons theory the symmetry is badly broken. The fact that it is broken is very natural from the M-theory perspective, as on the M5 brane wrapping the z_1 plane S_1 corresponds to angular momentum, while S_2 is an R-symmetry, rotating the space transverse to the brane. Correspondingly, had we considered the M5 brane wrapping the z_2 plane instead, S_2 would have been the momentum on the brane.

To reconcile these two perspectives, one from M-theory with M5 branes and the other from IIA with D4 branes, we must understand how the choice of q/t -branes appears in the IIA geometry. From the geometry of Taub-NUT space, it can be seen that this corresponds to whether the $D4$ -brane runs along the positive or negative half-line in $\mathbb{R}^3$. More precisely, the choice of q/t -branes is translated into whether it fills,

$$\{0\} \times \{z > 0\} \in \mathbb{R}^3 \quad \text{or} \quad \{0\} \times \{z < 0\} \in \mathbb{R}^3 \quad (5.5.107)$$

Now to see how this affects the identification of $D0$ -brane charge, recall that $D0$ -branes are magnetically dual to $D6$ -branes. In the presence of both kind of branes, the electromagnetic fields carry one unit of angular momentum along the vector connecting their positions. But the $D6$ -brane in our geometry is frozen at $\{0\} \in \mathbb{R}^3$ and the $D0$ -branes that bind to the $D4$ -brane must sit at $\{z > 0\}$ or $\{z < 0\}$ depending on the type of refined brane. Therefore, we find that the $D0$ -brane always carries either $+\frac{1}{2}$ unit of angular momentum, J_3 , or $-\frac{1}{2}$ unit of angular momentum, depending on the type of $D4$ -branes used. When we compute the above trace, it is natural to only count the angular momentum that comes from other physics, and absorb this intrinsic angular momentum into the weighting of $D0$ -brane charge. From equation this implies that each $D0$ -brane is weighted by

$$(\sqrt{qt})^{Q_0} (\sqrt{q/t})^{Q_0} = q^{Q_0} \quad \text{or} \quad (\sqrt{qt})^{Q_0} (\sqrt{q/t})^{-Q_0} = t^{Q_0} \quad (5.5.108)$$

in perfect agreement with the M-theory perspective. Putting

$$q = e^{-\epsilon_1} \quad t = e^{-\epsilon_2}.$$

this implies we should identify (for the $D4$ brane ending from $\{z > 0\}$ on the $D6$ brane)

$$\phi'_0 = \epsilon'_1, \quad \gamma' = \frac{\epsilon_1 - \epsilon_2}{2}.$$

We can use this, and S -duality, to identify the parameters ϕ_0 and γ in terms of q and t , as:

$$\phi_0 = \frac{4\pi^2}{\epsilon_1}, \quad \gamma = \pi i \frac{\epsilon_1 - \epsilon_2}{\epsilon_1}$$

So far our discussion has focused on the $p = 0$ case. Once we consider nontrivial circle or line bundles over Σ , the setups will differ since the Vafa-Witten and Langlands twists are not equivalent when $p \neq 0$. As discussed in [15], the framing factors in refined Chern-Simons can be understood as arising from a topological term. We expect that such topological terms should be present, regardless of which setup we use. Finally, we should remember that our goal is to count both $D0$ -brane charge and $D2$ -brane charge. Therefore, we must reintroduce a term in the index, $e^{-\phi_2 Q_2}$. This term is unaffected by refinement and takes precisely the same form as before.

5.5.3 Refined Chern-Simons Theory and a (q, t) -deformed Yang-Mills

To summarize, the refined black hole partition function,

$$Z_{ref} = \text{Tr}(-1)^F \exp \left(-\phi_0 Q_0 - 2\gamma(J_3 - R) \right). \quad (5.5.109)$$

corresponding to N D4 branes on the divisor

$$\mathcal{D} = (\mathcal{O}(-p) \rightarrow \Sigma_g)$$

is computed by refined $U(N)$ Chern-Simons theory on an S^1 bundle over Σ_g of first Chern-Class $-p$, where the $q = e^{-\epsilon_1}$, $t = e^{-\epsilon_2}$ parameters of refined Chen-Simons are related to ϕ_0 and γ as

$$\epsilon_1 = \frac{4\pi^2}{\phi_0}, \quad \epsilon_2 = \frac{4\pi^2}{\phi_0} \left(1 - \frac{\gamma}{2\pi i}\right), \quad \theta = \frac{2\pi\phi_2}{\phi_0} \quad (5.5.110)$$

It will be useful for us to formulate the refined Chern-Simons theory as a two dimensional one, refining q -deformed 2d Yang-Mills, in particular since we still need to turn on ϕ_2 , the D2 brane chemical potential (we could have done this in the refined Chern-Simons theory as well using the natural contact structure, but we will not do that here). As we explained earlier, the topological terms in four dimensional action are unchanged by refinement; only the values of ϕ_0 , ϕ_2 change.

$$S = \frac{\phi_0}{8\pi^2} \int \text{Tr} F \wedge F + \frac{\phi_2}{2\pi} \int \text{Tr} F \wedge \omega_\Sigma \quad (5.5.111)$$

Namely, localization on the $\mathcal{L}_1 = \mathcal{O}(-p)$ fiber, relates the 4d observables, to 2d ones

$$S = \frac{\phi_0}{4\pi^2} \int_{\Sigma_g} \text{Tr} \Phi F + \frac{\phi_2}{2\pi} \int_{\Sigma_g} \text{Tr} \Phi \omega_\Sigma - p \frac{\phi_0}{8\pi^2} \int_{\Sigma_g} \text{Tr} \Phi^2 \omega_\Sigma. \quad (5.5.112)$$

Here, Φ is the holonomy of the four-dimensional gauge field around the circle at infinity of the $\mathcal{L}_1$ fiber. We can also think of it as the holonomy of the Chern-Simons gauge field around the S^1 . The origin of the last term, from the Chern-Simons perspective was reviewed in [15, 20]. Here ω_Σ is a volume form on Σ_g , normalized to unit volume. The presence of this form in the action means that the 2d YM is invariant under area preserving diffeomorphisms only.

Thus, we still get the action of the ordinary 2d Yang-Mills, but the measure has to be deformed – both because of the periodicity of Φ , and now because also of the q, t dependence of the index. The measure factor was in fact the only difference between the refined and ordinary Chern-Simons theory, as well. Let $\mathcal{D}_{q,t}A$ be the refined Chern-Simons measure. This induces a measure on the gauge fields in two dimensions, but also on the holonomy. We will explain what the measure is in some

detail later on, for now, let us leave it schematic. All together, the path integral of the theory is

$$Z_{\text{ref BH}} = \int \mathcal{D}_{q,t} A \mathcal{D}_{q,t} \Phi \exp \left(\frac{\phi_0}{4\pi^2} \int_{\Sigma_g} \text{Tr} \Phi F + \frac{\phi_2}{2\pi} \int_{\Sigma_g} \text{Tr} \Phi \omega_\Sigma - p \frac{\phi_0}{8\pi^2} \int_{\Sigma_g} \text{Tr} \Phi^2 \omega_\Sigma \right)$$

We will refer to this theory as (q, t) -deformed Yang-Mills theory.¹⁴

We will use this path integral to derive the answer for the partition function, in the next subsection. For now, let us simply state the answer: For N D4 branes wrapping a degree $-p$ complex line bundle fibered over a genus g Riemann surface Σ_g , the resulting refined partition function for bound states with D2-D0 branes is given by,

$$Z_{\text{ref BH}} = \sum_{\mathcal{R}} \left(\frac{(S_{\mathcal{R}0})^2}{G_{\mathcal{R}}} \right)^{1-g} q^{\frac{p(\mathcal{R}, \mathcal{R})}{2}} t^{p(\rho, \mathcal{R})} Q^{\sum_i \mathcal{R}_i} \quad (5.5.113)$$

where the sum is over representations, $\mathcal{R}$, of $U(N)$, ρ is the Weyl vector $(\rho)_i = \frac{N+1}{2} - i$, and $Q = e^{-\phi_2}$ is the D2-brane chemical potential. Here we have also used the definitions,

$$\begin{aligned} S_{\mathcal{R}0} = S_{00} \dim_{q,t}(\mathcal{R}) &= \prod_{m=0}^{\beta-1} \prod_{1 \leq i < j \leq N} [\mathcal{R}_i - \mathcal{R}_j + \beta(j-i) + m]_q \\ G_{\mathcal{R}} &= \prod_{m=0}^{\beta-1} \prod_{1 \leq i < j \leq N} \frac{[\mathcal{R}_i - \mathcal{R}_j + \beta(j-i) + m]_q}{[\mathcal{R}_i - \mathcal{R}_j + \beta(j-i) - m]_q} \end{aligned} \quad (5.5.114)$$

Note that $G_{\mathcal{R}}$ is naturally the finite N version of the metric that appeared in equation 5.4.43.

5.5.4 A Path Integral for (q, t) -deformed Yang-Mills

Let us now explain in more detail what the q, t -deformed 2d Yang-Mills is, and how to compute its partition function. The most straight-forward way to proceed is to note that, because the theory is essentially topological, we can simply formulate the theory on pieces of the Riemann surface, explain how to glue them together, and show that the answer is independent of the decomposition. This has essentially been done in [18, 20], only from the 3d perspective of the refined Chern-Simons theory on S^1 fibration over the Riemann surface. To avoid simply repeating the derivation of [18, 20], we will instead compute the path integral directly. We will begin by

¹⁴This (q, t) -deformed Yang-Mills theory has also appeared as a limit of the TQFT that computes the four-dimensional $\mathcal{N} = 2$ superconformal index in [123]. It is also worth noting that in the limit $q \rightarrow 1, t \rightarrow 1$ this theory reduces to ordinary 2d Yang-Mills at zero coupling.

recalling some of the results of [11], where the $U(N)$ q -deformed 2d Yang-Mills theory was studied. This section will not be entirely self contained, but will build on [11].

In [11], it was shown that the path integral of the unrefined theory can be abelianized, so that we are left with $U(1)^N$ gauge fields, A_k , and N compact scalar fields, ϕ_k , which are the eigenvalues of Φ . Starting from the original integral:

$$Z_{\text{BH}} = \int \mathcal{D}A \mathcal{D}\Phi \exp\left(\frac{1}{g_s} \int \text{Tr} \Phi F + \frac{\theta}{g_s} \int \text{Tr} \Phi \omega_\Sigma - \frac{p}{g_s} \int \text{Tr} \Phi^2 \omega_\Sigma\right)$$

the abelianized version becomes

$$Z_{\text{qYM}} = \frac{1}{N!} \int' \prod_i \mathcal{D}\phi_i \mathcal{D}A_i \left(\Delta(\phi)\right)^{1-g} \exp\left(\sum_i \frac{1}{g_s} \int_\Sigma d^2\sigma \left(\frac{p}{2} \phi_i^2 - \theta \phi_i\right) - \frac{1}{g_s} \int_\Sigma F_i \phi_i\right) \quad (5.5.115)$$

where we have used the measure factor,

$$\Delta(\phi) = \prod_{i \neq j} \left(e^{\frac{\phi_i - \phi_j}{2}} - e^{\frac{\phi_j - \phi_i}{2}}\right) \quad (5.5.116)$$

and where $\int'$ indicates that the path integral omits those values of ϕ for which $\Delta(\phi) = 0$. As was argued in [11], this partition function is precisely equal to the black hole partition function of equation 5.5.77 with the identification,

$$\phi_0 = \frac{4\pi^2}{g_s}, \quad \phi_2 = \frac{2\pi\theta}{g_s} \quad (5.5.117)$$

Since this action is quadratic and very simple, the path integral of q -deformed Yang-Mills can be solved exactly.

As we discussed, in the refined case, the only thing that changes are the chemical potentials and the measure factors. The chemical potentials become

$$\phi_0 = \frac{4\pi^2}{\epsilon_1}, \quad \phi_2 = \frac{2\pi\theta}{\epsilon_1} \quad (5.5.118)$$

and γ enters through the measure of the path integral that depends on both ϵ_1 and ϵ_2 :

$$\epsilon_2 = \frac{4\pi^2}{\phi_0} \left(1 - \frac{\gamma}{2\pi}\right). \quad (5.5.119)$$

In the refined theory, the measure becomes

$$\Delta(\phi) \rightarrow \Delta_{q,t}(\phi) = \prod_{m=0}^{\beta-1} \prod_{j \neq k} \left(q^{-\frac{m}{2}} e^{\frac{\phi_i - \phi_j}{2}} - q^{\frac{m}{2}} e^{\frac{\phi_j - \phi_i}{2}}\right) \quad (5.5.120)$$

where $\beta = \epsilon_2/\epsilon_1$, and we have taken it to be a positive integer for computational convenience. The path integral is thus given by:

$$Z_{qtYM}(\Sigma) = \frac{1}{N!} \int' \prod_i \mathcal{D}\phi_i \mathcal{D}A_i \left(\Delta_{q,t}(\phi) \right)^{1-g} \exp \left(\sum_i \int_{\Sigma} d^2\sigma \left(\frac{p}{2\epsilon_1} \phi_i^2 - \frac{\theta}{\epsilon_1} \phi_i \right) - \frac{1}{\epsilon_1} \int_{\Sigma} F_i \phi_i \right) \quad (5.5.121)$$

Since this path integral is abelian, we can evaluate it explicitly following the approach of [11, 47]. We begin by evaluating the path integral over abelian gauge fields. It is helpful to first change integration variables from $\mathcal{D}A_i$ to $\mathcal{D}F_i$. However, we should only integrate over those two-forms F_k that are genuine bundles over Σ_g , which means that we must impose,

$$\int_{\Sigma_g} F_k \in 2\pi\mathbb{Z} \quad (5.5.122)$$

This can be accomplished by inserting a periodic delta function,

$$\sum_{\{n_k\}} \delta^{(N)} \left(\int_{\Sigma_g} F_k - 2\pi n_k \right) \quad (5.5.123)$$

which can be rewritten using Poisson resummation,

$$\sum_{\{n_k\}} \delta^{(N)} \left(\int_{\Sigma_g} F_k - 2\pi n_k \right) = \sum_{\{n_k\}} \exp \left(i n_k \int_{\Sigma_g} F_k \right) \quad (5.5.124)$$

Therefore the path integral takes the form,

$$Z_{qtYM}(\Sigma) = \frac{1}{N!} \int' \prod_i \mathcal{D}\phi_k \mathcal{D}F_k \left(\Delta_{q,t}(\phi) \right)^{1-g} \cdot \exp \left(\sum_i \int_{\Sigma} d^2\sigma \left(\frac{p}{2\epsilon_1} \phi_k^2 - \frac{\theta}{\epsilon_1} \phi_k \right) - \frac{1}{\epsilon_1} \int_{\Sigma} F_k (\phi_k - i\epsilon_1 n_k) \right) \quad (5.5.125)$$

When performing the path integral over F_k we obtain a delta function from the last term, and the path integral over ϕ only receives contributions from constant fields,

$$\phi_k = i\epsilon_1 n_k \quad (5.5.126)$$

Therefore, the path integral evaluates to,

$$Z_{qtYM}(\Sigma) = \frac{1}{N!} \sum'_{\{n_k\}} \left(\Delta_{q,t}(i\epsilon_1 n_k) \right)^{1-g} \exp \left(-\frac{p\epsilon_1}{2} \sum_k n_k^2 - i\theta \sum_k n_k \right) \quad (5.5.127)$$

where $\sum'$ indicates that we should only sum over those field configurations where $\Delta_{\epsilon_1, \beta}(\phi)$ is nonzero. From equation 5.5.120, this means that we must impose $n_i \neq n_j$, but we also require,

$$\begin{aligned} n_i &\neq n_j \pm 1 \\ n_i &\neq n_j \pm 2 \\ &\vdots \\ n_i &\neq n_j \pm (\beta - 1) \end{aligned} \quad (5.5.128)$$

It is also helpful to notice that both the refined measure and the action are invariant under Weyl reflections, so we can restrict the sum to the fundamental Weyl chamber so that $n_1 > n_2 > \dots > n_N$.

We would like to use these observations to rewrite equation 5.5.127 as a sum over $U(N)$ representations. Recall that a $U(N)$ representation is specified by highest weights satisfying $\mathcal{R}_1 \geq \mathcal{R}_2 \geq \dots \geq \mathcal{R}_N$. Therefore, in order to satisfy the constraints of equation 5.5.128, we should shift each n_k by βk . For convenience, we can also shift all n_k by a constant amount. This leads us to the identification,

$$n_k = \mathcal{R}_k + \beta \rho_k \quad (5.5.129)$$

where $\rho_k = \frac{N+1}{2} - k$. Then the partition function takes the form,

$$Z_{q,t\text{YM}}(\Sigma) = \sum_{\mathcal{R}} \left(\Delta_{q,t}(\mathcal{R}) \right)^{1-g} q^{\frac{p}{2}(\mathcal{R}, \mathcal{R})} t^{p(\rho, \mathcal{R})} Q^{|\mathcal{R}|} \quad (5.5.130)$$

where we have defined $q = e^{-\epsilon_1}$, $t = e^{-\beta \epsilon_1}$, and $Q = e^{-i\theta}$, and where,

$$\Delta_{q,t}(\mathcal{R}) = \prod_{m=0}^{\beta-1} \prod_{i < j} [\mathcal{R}_i - \mathcal{R}_j + \beta(j-i) + m]_q [\mathcal{R}_i - \mathcal{R}_j + \beta(j-i) - m]_q \quad (5.5.131)$$

where we have used the notation, $[n]_q = q^{n/2} - q^{-n/2}$.

We can now rewrite $\Delta_{q,t}(\mathcal{R})$ in a form that clarifies the relationship to refined Chern-Simons theory,

$$\Delta_{q,t}(\mathcal{R}) = \frac{S_{0\mathcal{R}} S_{0\mathcal{R}}}{G_{\mathcal{R}}} \quad (5.5.132)$$

where S_{PQ} is the S-matrix for refined Chern-Simons theory, but analytically continued away from integer level, and $G_{\mathcal{R}}$ is the finite N Macdonald metric. These elements take the form,

$$S_{\mathcal{R}0} = S_{00} \dim_{q,t}(\mathcal{R}) = \prod_{m=0}^{\beta-1} \prod_{1 \leq i < j \leq N} [\mathcal{R}_i - \mathcal{R}_j + \beta(j-i) + m]_q \quad (5.5.133)$$

$$G_{\mathcal{R}} = \prod_{m=0}^{\beta-1} \prod_{1 \leq i < j \leq N} \frac{[\mathcal{R}_i - \mathcal{R}_j + \beta(j-i) + m]_q}{[\mathcal{R}_i - \mathcal{R}_j + \beta(j-i) - m]_q} \quad (5.5.134)$$

Putting this together, we can rewrite the entire partition function as,

$$Z_{\text{ref BH}} = \sum_{\mathcal{R}} \frac{(S_{\mathcal{R}0})^{2-2g}}{(G_{\mathcal{R}})^{1-g}} q^{\frac{p(\mathcal{R}, \mathcal{R})}{2}} t^{p(\rho, \mathcal{R})} Q^{\sum_i \mathcal{R}_i} \quad (5.5.135)$$

in agreement with equation 5.5.113.

Finally, for comparison with refined topological string theory, it is helpful to include an overall Q -independent normalization factor,

$$\alpha_{BH} = \exp \left(-\frac{\epsilon_2^2 \rho^2 (p+2g-2)^2}{\epsilon_1 2p} + \frac{N\theta^2}{2p\epsilon_1} + \frac{\epsilon_2^2}{\epsilon_1} (2g-2)\rho^2 \right) \left((t; q)_{\infty} (q; q)_{\infty} \right)^{N(g-1)} \quad (5.5.136)$$

Therefore, our final result takes the form,

$$Z_{qt\text{YM}}(\Sigma) = \alpha_{BH} \sum_{\mathcal{R}} \frac{(S_{\mathcal{R}0})^{2-2g}}{(G_{\mathcal{R}})^{1-g}} q^{\frac{p(\mathcal{R}, \mathcal{R})}{2}} t^{p(\rho, \mathcal{R})} Q^{\sum_i \mathcal{R}_i} \quad (5.5.137)$$

In conclusion, (q, t) -deformed Yang-Mills gives a precise prediction for the refined black hole partition function for D4 branes wrapping an arbitrary complex line bundle over Σ . From the discussion above, this can also be rephrased as a mathematical prediction for the χ_y genus of instanton moduli space.

It is important to notice that, as written, $Z_{qt\text{YM}}$ is an expansion in $q = e^{-\epsilon_1}$ and $t = e^{-\epsilon_2}$, while the original black hole index of equation 5.5.82 should be expanded in e^{-1/ϵ_1} and $e^{-\gamma}$. Therefore, to extract D4/D2/D0 refined degeneracies, we must use TST duality to resum the partition function Z_{BH} so that it is written in the appropriate expansion.

5.5.5 Example: $\mathcal{O}(-1) \rightarrow \mathbb{P}^1$

In this section we focus on D4 branes wrapping the bundle $\mathcal{O}(-1)$ over a genus $g = 0$ Riemann surface.

$$\mathcal{D}_0 = \mathcal{O}(-1) \rightarrow \mathbb{P}^1$$

This geometry is simply given by blowing up $\mathbb{C}^2$ at a point, which means that the corresponding instanton moduli space is especially simple. In the mathematical work of Yoshioka and Nakajima [220, 286], the authors proved an explicit formula for the Hodge polynomial of $U(N)$ instantons on the blow-up geometry,

$$P_{\text{blow-up}}(\phi_0, \phi_2; x, y) = \sum_{ch_2, c_1, i} e^{-\phi_0 ch_2 - \phi_2 c_1} (-1)^{i+j} x^i y^j h_{i,j}(\mathcal{M}_{ch_2, c_1}) \quad (5.5.138)$$

$$= \frac{1}{\eta(e^{-\phi_0})^N} \sum_{\{n_i\}=-\infty}^{\infty} e^{-\phi_0 \frac{(n, n)}{2}} (xy)^{(\rho, n)} e^{-\phi_2 \sum_i n_i} \quad (5.5.139)$$

Note that since this geometry is toric, $h_{i,j}$ is only nonzero if $i = j$.¹⁵ Setting $x = 1$ we obtain the χ_y genus of interest,

$$P_{\chi_y, \text{blow-up}}(\phi_0, \phi_2, y) = \frac{1}{\eta(e^{-\phi_0})^N} \sum_{\{n_i\}=-\infty}^{\infty} e^{-\phi_0 \frac{(n,n)}{2}} y^{(\rho,n)} e^{-\phi_2 \sum_i n_i} \quad (5.5.140)$$

We would like to compare this answer against the prediction of (q, t) -deformed Yang-Mills,

$$Z_{\text{ref}}(\mathcal{D}_0) = \alpha_{BH} \sum_{n_1 \geq n_2 \geq \dots \geq n_N} q^{\frac{(n,n)}{2}} t^{(\rho,n)} Q^{\sum_i n_i} \frac{\dim_{q,t}(\{n_i\})^2}{g(\{n_i\})} \quad (5.5.141)$$

This partition function can be rewritten using the remarkable identity,¹⁶

$$\sum_{n_1 \geq n_2 \geq \dots \geq n_N} q^{\frac{(n,n)}{2}} t^{(\rho,n)} Q^{\sum_i n_i} \frac{\dim_{q,t}(\{n_i\})^2}{g(\{n_i\})} = \left(\prod_{k=1}^{N-1} \prod_{j=1}^{\infty} \frac{1 - q^j t^k}{1 - q^j} \right) \sum_{n_i=-\infty}^{\infty} q^{\frac{(n,n)}{2}} t^{(\rho,n)} Q^{\sum_i n_i} \quad (5.5.142)$$

Since the summation on the right is over all integers $\{n_i\}$, we can shift the definition of n by any integer amount. Using this freedom we can finally write the partition function as,

$$Z_{\text{ref}}(\mathcal{D}_0) = \alpha_{BH}(q, t) \left(\prod_{k=1}^{N-1} \prod_{j=1}^{\infty} \frac{1 - q^j t^k}{1 - q^j} \right) \sum_{n_i=-\infty}^{\infty} q^{\frac{(n,n)}{2}} (t/q)^{(\rho,n)} Q^{\sum_i n_i} \quad (5.5.143)$$

The Q -independent prefactor in this expression is related to $D4/D0$ degeneracies and is ambiguous because our geometry is non-compact. For this reason, in subsequent formulas we will drop it.

To compare our result with the χ_y -genus of instanton moduli space, we must remember that the partition function of (q, t) -deformed Yang-Mills is an expansion in $e^{-\epsilon_k}$, while D-brane degeneracies arise as coefficients of an expansion in $e^{-\frac{1}{\epsilon_1}}$. To relate these two expansions, we can rewrite Z_{ref} as a product of Jacobi theta functions,

$$Z_{\text{ref}}(\mathcal{D}_0) \sim \prod_{j=1}^N \vartheta \left(\frac{(\epsilon_1 - \epsilon_2)(N + 1 - 2j)}{4\pi i} - \frac{\theta}{2\pi}, -\frac{\epsilon_1}{2\pi i} \right) \quad (5.5.144)$$

where the Jacobi theta function is defined by $\vartheta(z, \tau) = \sum_n e^{\pi i \tau n^2 + 2\pi i n z}$. It is helpful to recall that the Jacobi theta function has the modular property,

$$\vartheta\left(\frac{z}{\tau}, -\frac{1}{\tau}\right) = (-i\tau)^{1/2} \exp\left(\pi i z^2 / \tau\right) \vartheta(z, \tau) \quad (5.5.145)$$

¹⁵From equation 5.5.84, it follows that all D4/D2/D0 BPS states in this geometry have zero R-charge ($R = 0$). Therefore, the ordinary and protected spin characters agree.

¹⁶An analogue of this identity for the $SU(N)$ case has been proven by Cherednik in [63, 65, 68].

Applying this transformation to the black hole partition function we obtain,

$$Z_{\text{ref}}(\mathcal{D}_0) \sim \sum_{n_i=-\infty}^{\infty} e^{-\phi_0 \frac{(n,n)}{2}} e^{-\phi_2 \sum_i n_i} e^{-2\gamma(\rho,n)} \quad (5.5.146)$$

where ϕ_0 , ϕ_2 , and γ are given by the definitions in equation 5.5.110. As discussed above, physically this modular transformation arises from performing TST duality on the D-brane configuration.

This result precisely agrees with the expected χ_y -genus in equation 5.5.140. In section 5.7, we will give an independent physical derivation of equation 5.5.140 by using the refined semi-primitive wall crossing formula.

5.6 Large N Factorization and the Refined OSV Conjecture

Now that we have explained how to compute both the refined black hole partition function and the refined topological string, we would like to see how they are connected. As explained in section 5.2, the refined OSV conjecture predicts that the refined partition function of N D4-branes wrapping,

$$\mathcal{O}(-p) \rightarrow \Sigma_g \quad (5.6.147)$$

should be equal to the square of the partition function of refined topological string on

$$\mathcal{O}(-p) \oplus \mathcal{O}(2g - 2 + p) \rightarrow \Sigma_g \quad (5.6.148)$$

to all orders in the $1/N$ expansion. In equations, this implies

$$Z_{qtYM}(\phi_0, \phi_2, \gamma) \sim |Z_{\text{ref top}}(\epsilon_1, \epsilon_2, k)|^2 \quad (5.6.149)$$

with the change of variables,

$$\begin{aligned} k &= \frac{2\pi^2}{\phi_0} \left(\beta P + i \frac{\phi_2}{\pi} \right) \\ \epsilon_1 &= \frac{4\pi^2 C}{\phi_0} \\ \frac{\epsilon_2}{\epsilon_1} &= \beta = 1 - \frac{\gamma}{2\pi i} \end{aligned} \quad (5.6.150)$$

Note that we have included the additional constant factor, C . It will become clear (see equation 5.5.110) that in our example the value for C is 1 so that,

$$\phi_0 = \frac{4\pi^2}{\epsilon_1} \quad (5.6.151)$$

This is in contrast with the more symmetric choice of

$$\phi_0 \sim \frac{4\pi^2}{\epsilon_1 + \epsilon_2} \quad (5.6.152)$$

that was used in our discussion of the refined OSV conjecture in section 5.2. This apparent discrepancy can be resolved by postulating that $D0$ -branes in our setup carry intrinsic charge under $(J_3 - R)$, which will change their effective weighting in the refined partition function. As explained in section 5.5.2, this is expected since in the dual refined Chern-Simons construction, $D0$ -branes carry angular momentum which causes them to be weighted by either q or t , but not $\sqrt{qt}$.

We can make the change of variables in equation 5.6.150 even more explicit for the geometries we are studying. It might seem that the $D4$ -brane charge, P , is simply the number of $D4$ branes, N , that wrap the bundle, $C_4 = \mathcal{O}(-p) \rightarrow \Sigma_g$. However, we should really measure this charge in electric $D2$ brane units. As explained in [11] these charges differ because of the nontrivial intersection number of the Riemann surface Σ with the four-cycle wrapped by the $D4$ branes, C_4 ,

$$\#(\Sigma \cap C_4) = 2g - 2 + p \quad (5.6.153)$$

leading to the identification,

$$P = N(2g - 2 + p) \quad (5.6.154)$$

Now we want to test the above predictions by using the results that we have built up in sections 5.4 and 5.5, where we solved for both the refined black hole partition function and the refined topological string on these geometries. We found in section 5.5, that the refined brane partition function is computed by the two-dimensional (q, t) -deformed Yang-Mills, whose partition function depends on two coupling constants (ϵ_1, ϵ_2) and a theta-term, θ . As explained in equation 5.5.110, these gauge theory variables are related to the refined black hole chemical potentials by,

$$\phi_0 = \frac{4\pi^2}{\epsilon_1}, \quad \phi_2 = \frac{2\pi\theta}{\epsilon_1}, \quad \gamma = \frac{2\pi i(\epsilon_2 - \epsilon_1)}{\epsilon_1} \quad (5.6.155)$$

Putting this together with the predictions of the refined OSV conjecture, we find that the large N limit of (q, t) -deformed Yang-Mills should be equal to the square of the refined topological string with the Kähler modulus equal to,

$$k = 2\pi i \frac{X_1}{X_0} = \frac{1}{2}(2g - 2 + p)N\epsilon_2 + i\theta \quad (5.6.156)$$

and the topological string couplings (ϵ_1, ϵ_2) identified with the same variables in the Yang-Mills theory.

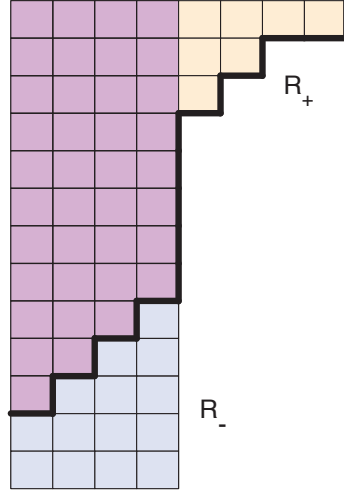


Figure 5.3: The composite Young Tableau, $R_+ \overline{R_-}$, is shown for two $SU(N)$ representations R_+ and R_- .

To test this precise prediction, we must carefully study the large N limit of (q, t) -deformed Yang-Mills. The key observation, first made for ordinary two-dimensional Yang-Mills in [145], is that representations of $SU(N)$ for large N can be viewed as composites of two Young Tableaux as shown in Figure 5.3. This splitting can be thought of as a splitting of the Hilbert space as $N \rightarrow \infty$ into two chiral pieces,

$$\mathcal{H} \sim \mathcal{H}_+ \otimes \overline{\mathcal{H}}_- \quad (5.6.157)$$

If the dynamics of the theory respect this splitting, then the partition function will also factorize into two pieces as predicted by the refined OSV conjecture. In this section we will show that the partition function does indeed satisfy this factorization, after properly accounting for the sum over RR fluxes and asymptotic boundary conditions. It is important to note that this splitting is valid to all orders in $1/N$, but it is not valid nonperturbatively since the two Young Tableaux will interfere with each other at finite N . It would be interesting to compute the nonperturbative refined corrections to our result.

As in the unrefined case, it is helpful to split our discussion into the genus $g \geq 1$ and the genus $g = 0$ cases. We begin with the higher genus geometries.

5.6.1 Genus $g \geq 1$ Case

As explained in equation 5.5.137, the refined black hole partition function for N $D4$ branes wrapping $\mathcal{O}(-p) \rightarrow \Sigma_g$ is given by,

$$Z_{qtYM}(\Sigma_g, p) = \alpha_{BH} \sum_{\mathcal{R}} \left(\frac{(S_{\mathcal{R}0})^2}{G_{\mathcal{R}}} \right)^{1-g} q^{\frac{p(\mathcal{R}, \mathcal{R})}{2}} t^{p(\rho, \mathcal{R})} Q_{\sum_i \mathcal{R}_i} \quad (5.6.158)$$

The sum is over all $U(N)$ representations which we denote by $\mathcal{R}$. The normalization constant α_{BH} defined in equation 5.5.136 is included for convenience when making the connection with refined topological strings.

When we take the large N limit, it will be helpful to decompose each $U(N)$ representation into an $SU(N)$ and $U(1)$ representation. Recall that representations of $U(N)$ are labeled by integers, $\mathcal{R}_i$, such that $\mathcal{R}_1 \geq \mathcal{R}_2 \geq \dots \geq \mathcal{R}_N$ where the $\mathcal{R}_i$ can take positive or negative values. We can decompose this into a representation of $SU(N)$ and a representation of $U(1)$ by rewriting it as,

$$\begin{aligned} \mathcal{R}_i &= R_i + r & i = 1, \dots, N-1 \\ \mathcal{R}_N &= r \end{aligned} \quad (5.6.159)$$

where the R_i label an $SU(N)$ representation. Then the $U(1)$ charge is given by

$$m = |R| + Nr \quad (5.6.160)$$

where $r \in \mathbb{Z}$. We can rewrite the terms appearing in the partition function, as,

$$\begin{aligned} |\mathcal{R}| &= \sum_k \mathcal{R}_k = m \\ (\mathcal{R}, \mathcal{R}) &= \sum_k \mathcal{R}_k^2 = \sum_k R_k^2 + \frac{m^2}{N} - \frac{|R|^2}{N} \\ 2(\rho, \mathcal{R}) &= \sum_k \mathcal{R}_k(N+1-2k) = \sum_k R_k(1-2k) + N|R| \end{aligned} \quad (5.6.161)$$

For genus $g \neq 1$, the partition function also involves the metric and (q, t) -dimension. These quantities are the same for the $U(N)$ representation $\mathcal{R}$ and its $SU(N)$ part R ,

$$\begin{aligned} \dim_{q,t}(\mathcal{R}) &= \dim_{q,t}(R) \\ g_{\mathcal{R}} &= g_R \end{aligned} \quad (5.6.162)$$

From equations 5.5.134 and 5.5.133, this is true because both quantities can be written as functions only of the differences $\mathcal{R}_i - \mathcal{R}_j$ which are independent of the $U(1)$ charge.

Now we would like to study the large N limit of this theory. As mentioned above, in this limit each $SU(N)$ representation can be decomposed into a composite of two

representations as depicted in Figure 5.3. If we form the composite representation of R_+ and R_- , which we denote by $R_+\overline{R}_-$, then its row lengths are given by,

$$(R_+\overline{R}_-)_i = (R_-)_1 + (R_+)_i - (R_-)_{N+1-i} \quad (5.6.163)$$

Note that for this to be a good composite representation the two representations should not interact with each other, so that either $(R_+)_i$ or $(R_-)_{N+1-i}$ is equal to zero for each i . As explained above, we can neglect these interactions provided that we only study the perturbative expansion in $1/N$.

Now we want to study how the quantities in equation 5.6.161 decompose for a $U(N)$ representation $\mathcal{R}$ whose $SU(N)$ part consists of the composite representation $R_+\overline{R}_-$. In this case the $U(1)$ charge is given by, $m = Nl + |R_+| - |R_-|$, where we have defined $l = r + (R_-)_1$. Then we find the decomposition,

$$\begin{aligned} |\mathcal{R}| &= \sum_i \mathcal{R}_i = |R_+| - |R_-| + Nl \\ (\mathcal{R}, \mathcal{R}) &= \sum_i \mathcal{R}_i^2 = ||R_+||^2 + ||R_-||^2 + Nl^2 + 2l(|R_+| - |R_-|) \\ 2(\rho, \mathcal{R}) &= \sum_k \mathcal{R}_k(N+1-2k) = -||R_+^T||^2 - ||R_-^T||^2 + N|R_+| + N|R_-| \end{aligned} \quad (5.6.164)$$

where $||R||^2 = \sum_k R_k^2$ and $||R^T||^2 = \sum_k (2k-1)R_k$.

Putting together these results, we can rewrite the refined black hole partition function as,

$$\begin{aligned} Z_{\text{ref BH}} &= \alpha_{\text{BH}} \sum_{l \in \mathbb{Z}} \sum_{R_+, R_-} \left(\frac{G_{R_+\overline{R}_-}}{(S_{0R_+\overline{R}_-})^2} \right)^{g-1} q^{\frac{p}{2} (||R_+||^2 + ||R_-||^2)} \\ &\quad t^{-\frac{p}{2} (||R_+^T||^2 + ||R_-^T||^2)} e^{-i\theta(|R_+| - |R_-|)} q^{lp(|R_+| - |R_-|)} t^{\frac{Np}{2} (|R_-| + |R_+|)} q^{\frac{Npl^2}{2}} e^{-i\theta Nl} \end{aligned} \quad (5.6.165)$$

With the refined OSV relation in mind, we can use the formula for the Kähler modulus in equation 5.6.156 to rewrite α_{BH} as,

$$\begin{aligned} \alpha_{\text{BH}} &= \left((t; q)_\infty (q; q)_\infty \right)^{N(g-1)} \\ &\quad \cdot \exp \left(-\frac{1}{\epsilon_1 \epsilon_2} \frac{k^3 + \bar{k}^3}{6p(p+2g-2)} + \beta \frac{(k + \bar{k})(p+2g-2)}{24p} + \epsilon_1 \beta^2 \rho^2 (2g-2) \right) \end{aligned} \quad (5.6.166)$$

So far we have explained how the framing factor and θ -dependent terms factorize, but we still need to understand the metric and (q, t) -dimension. Their factorization properties are derived in Appendix I, with the result that,

$$\begin{aligned} \frac{G_{R_+\overline{R}_-}}{q^{2\beta^2 \rho^2} (S_{0R_+\overline{R}_-})^2} &= \left(M(q, t) M(t, q) \right)^{-1} \left((t; q)_\infty (q; q)_\infty \right)^{-N} T_{R_+}^2 T_{R_-}^2 Q^{|R_+| + |R_-|} \\ &\quad \cdot \frac{g_{R_+} g_{R_-} K_{R_+ R_-} (Q_t^q) K_{R_+ R_-} (Q)}{W_{R_+}^4 W_{R_-}^4} \end{aligned} \quad (5.6.167)$$

where we have defined $Q = t^N$ and used the framing factor, $T_R = q^{\|R\|^2/2} t^{-\|R^T\|^2/2}$, and where

$$K_{R_+R_-}(Q; q, t) = \sum_P \frac{1}{g_P} Q^{|P|} (t/q)^{|P|} W_{PR_+}(q, t) W_{PR_-}(q, t) \quad (5.6.168)$$

From these factorization formulas, we can write the entire refined black hole partition function as a sum over chiral blocks,

$$Z_{\text{ref BH}} = \sum_{l \in \mathbb{Z}} \sum_{R_1, \dots, R_{2g-2}} Z_{R_1, \dots, R_{2g-2}}^+(k + pl\epsilon_1) Z_{R_1, \dots, R_{2g-2}}^+(\bar{k} - pl\epsilon_1) \quad (5.6.169)$$

where the chiral block is defined by,

$$\begin{aligned} Z_{R_1, \dots, R_{2g-2}}^+(k) &= Z_0(q, t) \cdot \left(t^{\frac{N}{2}}\right)^{|R_1| + \dots + |R_{g-1}|} \left(t^{\frac{N+1}{2}} q^{-\frac{1}{2}}\right)^{|R_g| + \dots + |R_{2g-2}|} \\ &\quad \sum_R (T_R)^{p+2g-2} e^{-k|R|} \frac{W_{R_1 R}(q, t) \cdots W_{R_{2g-2} R}(q, t)}{W_{0R}^{4g-4}} \frac{(g_R)^{g-1}}{g_{R_1} \cdots g_{R_{2g-2}}} \end{aligned} \quad (5.6.170)$$

where $k = \frac{1}{2}(p + 2g - 2)N\epsilon_2 + i\theta$ and where we used the definition of the degree zero piece, Z_0 , defined in equation 5.4.55.

Now we come to the physical interpretation of these chiral blocks. First notice that the chiral block is precisely the refined topological string amplitude for the geometry

$$\mathcal{O}(2g - 2 + p) \oplus \mathcal{O}(-p) \rightarrow \Sigma_g \quad (5.6.171)$$

with branes in the fibers over $2g - 2$ points, as explained in section 5.4.2,

$$Z_{R_1, \dots, R_{2g-2}}^+(k) = Z_{\text{ref top}, R_1, \dots, R_{2g-2}}(k) \quad (5.6.172)$$

The factors, $t^{\frac{1}{2}N|R_i|}$ appear because these branes have been moved in the fiber away from the origin. These “ghost branes” were explained in [9] as parametrizing noncompact Kähler moduli in the fiber directions. Naively, one might expect there to be noncompact moduli over every point on the Riemann surface. However, these moduli can be localized by using the symmetries corresponding to meromorphic vector fields on Σ . Since these vector fields have $2g - 2$ poles generically, we find ghost branes over precisely $2g - 2$ points.

This picture must be modified slightly in the refined case, since not all of these branes are the same. They split into two groups of $g - 1$ branes, which differ only by their fiber Kähler parameters which are either

$$k_f = \frac{1}{2}N\epsilon_2, \quad \text{or} \quad k_f = \frac{1}{2}N\epsilon_2 + \frac{1}{2}(\epsilon_2 - \epsilon_1) \quad (5.6.173)$$

It would be interesting to derive this splitting from first principles. Another important aspect of our formula for large N factorization is the sum over the $U(1)$ charge, l . As in the unrefined case [267], we interpret this as arising from the sum over RR -flux through the Riemann surface. Alternatively, this sum arises because the black hole partition function is trivially invariant under shifts $\phi_2 \rightarrow \phi_2 + 2\pi i p n$.¹⁷ The sum over $U(1)$ charge enforces this same periodicity on the topological string side of the correspondence.

5.6.2 Genus $g = 0$ Case

The genus zero case works much the same way as the higher genus case. By decomposing the $U(N)$ representations into $U(1)$ and composite $SU(N)$ representations, we can write the corresponding brane partition function as,

$$Z_{\text{ref BH}} = \alpha \sum_{l \in \mathbb{Z}} \sum_{R_+, R_-} \frac{(S_{0R_+ \bar{R}_-})^2}{G_{R_+ \bar{R}_-}} q^{\frac{p}{2}(\|R_+\|^2 + \|R_-\|^2)} t^{-\frac{p}{2}(\|R_+^T\|^2 + \|R_-^T\|^2)} e^{-i\theta(\|R_+| - \|R_-|)} q^{lp(\|R_+| - \|R_-|)} t^{\frac{Np}{2}(\|R_-| + \|R_+|)} q^{\frac{Npl^2}{2}} e^{-i\theta Nl} \quad (5.6.174)$$

However, since $\dim_{q,t}(R_+ \bar{R}_-)$ appears in the numerator, we will use the other set of identities from appendix I to give,

$$\frac{q^{2\beta^2 \rho^2} (S_{0R_+ \bar{R}_-})^2}{G_{R_+ \bar{R}_-}} = M(q, t) M(t, q) \left((t; q)_\infty (q; q)_\infty \right)^N T_{R_+}^{-2} T_{R_-}^{-2} Q^{-\|R_+| - \|R_-|} \cdot \frac{N_{R_+ R_-}(Q \frac{q}{t}) N_{R_+ R_-}(Q)}{g_{R_+} g_{R_-}} \quad (5.6.175)$$

where,

$$N_{R_+ R_-}(Q, q, t) := \sum_P \frac{1}{g_P} Q^{|P|} (t/q)^{|P|} \widetilde{W}_{RP}(q, t) W_{PS}(q, t) \quad (5.6.176)$$

and where $\widetilde{W}_{RP}$ is the cap amplitude for placing a brane in the base and an anti-brane in the fiber as explained in section 5.4.2.

Therefore, we can rewrite the black hole partition function as a sum over chiral blocks,

$$Z_{\text{ref BH}} = \sum_{l \in \mathbb{Z}} \sum_{R_1, R_2} Z_{R_1, R_2}^+(k + pl\epsilon_1) Z_{R_1, R_2}^-(\bar{k} - pl\epsilon_1) \quad (5.6.177)$$

where the Kähler parameter is given by,

$$k = \frac{1}{2}(p-2)N\epsilon_2 + i\theta \quad (5.6.178)$$

¹⁷The extra factor of p arises because the black holes have charges, $Q_2 \in \frac{1}{p}\mathbb{Z}$, as explained in [11].

It is important to notice that in the genus zero case, the chiral and anti-chiral blocks are not precisely the same. The chiral block is equal to,

$$Z_{R_1, R_2}^+(k) = Z_0(q, t) t^{\frac{1}{2}N|R_1|} \left(t^{\frac{N+1}{2}} q^{-\frac{1}{2}} \right)^{|R_2|} \sum_R (T_R)^{p-2} e^{-k|R|} \frac{W_{R_+R_1}(q, t) W_{R_+R_2}(q, t)}{g_R g_{R_1} g_{R_2}} \quad (5.6.179)$$

while the anti-chiral block takes the slightly different form,

$$Z_{R_1, R_2}^-(k) = Z_0(q, t) t^{\frac{1}{2}N|R_1|} \left(t^{\frac{N+1}{2}} q^{-\frac{1}{2}} \right)^{|R_2|} \sum_R (T_R)^{p-2} e^{-k|R|} \frac{\widetilde{W}_{R_+R_1}(q, t) \widetilde{W}_{R_+R_2}(q, t)}{g_R g_{R_1} g_{R_2}} \quad (5.6.180)$$

As in the higher genus case, the refined ghost branes split into two types depending on their fiber Kähler moduli and we obtain a sum over RR-flux. The chiral block computes precisely the refined topological string amplitude on $\mathcal{O}(p-2) \oplus \mathcal{O}(-p) \rightarrow \mathbb{P}^1$ with two “ghost” branes in the fiber.

The main difference between the higher genus and genus zero cases is that here the anti-chiral amplitude is the refined topological string on the same geometry but with two anti-branes rather than branes in the fiber. We can use the same argument as before for why there are precisely two ghost branes. However, now localization by using a generic vector field on the $\mathbb{P}^1$ will have two zeros rather than poles.

5.7 Black Hole Entropy and Refined Wall Crossing

In section 5.5.5, we used (q, t) -deformed Yang-Mills to compute the entropy of D-branes wrapping the blow-up geometry $\mathcal{O}(-1) \rightarrow \mathbb{P}^1$ and found agreement with the mathematical result of [286]. In this section, we explain an alternative way to compute this black hole partition function by using refined wall-crossing. We follow the approach of [229, 230], giving a refined generalization of their unrefined computations.

We start by studying the resolved conifold, $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbb{P}^1$, with N D4-branes wrapping the four cycle $C_4 = \mathcal{O}(-1) \rightarrow \mathbb{P}^1$. We would like to compute the refined degeneracies of these $D4$ branes bound to lower dimensional branes. Note that since C_4 contains the compact two-cycle, $\mathbb{P}^1$, D2 branes can form bound states with the stack of $D4$ s.

The key insight of [230] is that by varying the Kähler modulus of the $\mathbb{P}^1$, which we denote by z , the refined partition function will simplify in certain chambers. Specifically, by sending $z \rightarrow -\infty$, the conifold will undergo a flop transition, and the four-cycle, C_4 will become a cycle wrapping only the fiber directions, as in shown in Figure 5.4. Most importantly, this new four-cycle is topologically $\mathbb{C}^2$ and does not

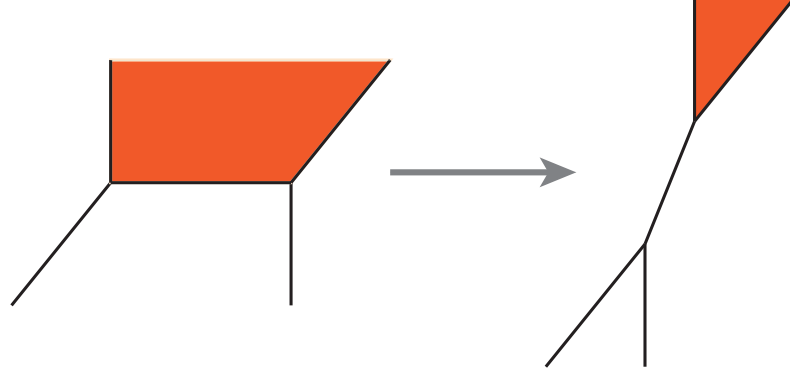


Figure 5.4: The flop transition for the conifold. N D4-branes wrap the shaded four-cycle, which changes topology under the flop.

have any compact two-cycles. Thus, in this chamber, only D4 and D0 branes can bind, and the refined partition function simplifies dramatically.

Now by starting in this simple chamber, we can vary the Kähler parameter z and keep track of how the refined partition function jumps. The partition function is locally constant, but along real codimension-one walls of marginal stability it will jump. Since the conifold geometry is fairly simple, we can identify all walls of marginal stability and jumps as we take z from $-\infty$ to ∞ . This will allow us to explicitly compute the refined partition function at $z = \infty$ and find agreement with our (q, t) -deformed Yang-Mills computation in section 5.5.5.

To find these walls, we must first compute the central charge of BPS bound states. Note that since our geometry is noncompact, the central charge of the $D4$ branes is infinite and its phase is not well-defined. As explained originally in [175], this can be remedied by starting with a compact geometry and including a component of the complexified Kähler form along the direction that is becoming noncompact. The result is that the central charge of the $D4$ brane is given by,

$$Z(D4) = -\frac{1}{2}\Lambda^2 e^{2i\phi} \quad (5.7.181)$$

where $\Lambda \gg 1$, and ultimately we want to take $\Lambda \rightarrow \infty$ to obtain the resolved conifold. Note that ϕ is still a free real parameter, but for our purposes we will keep it fixed and only vary the complexified Kähler parameter, z .

Now we can consider a more general D4/D2/D0 bound state with charges,

$$\Gamma = (P_6, P_4, Q_2, Q_0) = (0, N, m, n) \quad (5.7.182)$$

Its central charge is given by,

$$Z(\Gamma) = -\frac{1}{2}N\Lambda^2 e^{2i\phi} + mz + n \quad (5.7.183)$$

This state can decay as $\Gamma \rightarrow \Gamma_1 + \Gamma_2$ precisely when $Z(\Gamma_1)$ and $Z(\Gamma_2)$ are aligned. Depending on the charges of Γ_i , there are two types of walls that we must consider.

First, Γ could decay into two states that each have nonzero $D4$ -brane charge. Such fragmentation of the stack of $D4$ branes was discussed in this context in [229]. Although the alignment of central charges seems to suggest that these fragments can form, we argue that fragmentation walls are not physical in our setup.

The crucial point is that the $D4$ branes remain noncompact throughout moduli space. From a field theory perspective fragmentation corresponds to changing a scalar field's vev. But for a field theory on noncompact spacetime, this vev is a background parameter of the theory and changing it would cost infinite energy. Therefore, we conclude that because of the noncompactness of the $D4$ -branes, there can be no binding or decay across these walls, and they can be consistently ignored.

The second type of wall is much more interesting for us and involves the decay $\Gamma \rightarrow \Gamma_1 + m\Gamma_2$ where Γ_2 only has $D2/D0$ charge. From the Gopakumar-Vafa invariants of the conifold, it follows that the only BPS $D2/D0$ bound states are,

$$\begin{aligned}\Omega_0(0, 0, 0, n; y) &= -2 \\ \Omega_0(0, 0, \pm 1, n; y) &= 1\end{aligned}\tag{5.7.184}$$

where $\Omega_j(\Gamma)$ is the refined degeneracy for states with spin j .

Now we would like to find the walls of marginal stability where $\Gamma = (0, N, m, n)$ can decay into these bound states. Since $\Lambda \gg 1$, the phase of the central charge is equal to $\arg(Z(\Gamma)) = 2\phi + \pi$, which means that the walls of marginal stability for Γ will be independent of N , m , and n , provided $N > 0$.

First we can consider the pure $D0$ -brane decay channel where $\Gamma_2 = (0, 0, 0, n)$. However, the central charge of Γ_2 is always real, which means that for a generic fixed choice of ϕ , the central charges of Γ and Γ_2 will never align. This means that there are no walls of marginal stability associated with $D0$ decay.

Next, we consider the second possibility of a decay involving $\Gamma_2 = (0, 0, \pm 1, n)$. In this case, the phase of the central charge is given by, $\arg(Z(\Gamma_2)) = \arg(\pm z + n)$, which implies that a wall of marginal stability can occur as we vary z . We will denote these walls of marginal stability by $W_n^{\pm 1}$, and they are given explicitly by,

$$W_n^1 : \quad \phi = \frac{1}{2} \arg(-z - n) \tag{5.7.185}$$

$$W_n^{-1} : \quad \phi = \frac{1}{2} \arg(z - n) \tag{5.7.186}$$

These walls of marginal stability are shown in Figure 5.5.

Now that we have identified all walls of marginal stability, we want to study how the partition function jumps across these walls. For general decays, this jump can be quite complicated and is determined by the formula of Kontsevich and Soibelman [188]. However, for our purposes we only must deal with semi-primitive decays of the

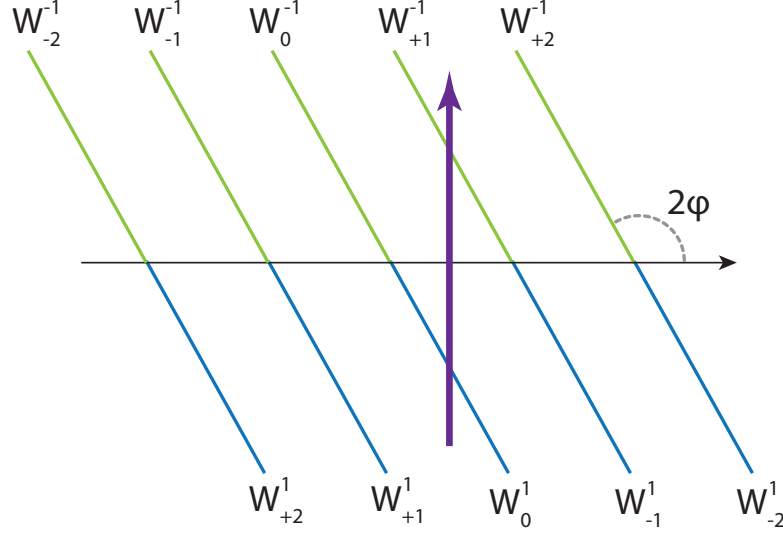


Figure 5.5: Walls of marginal stability for the conifold moduli space. The path we follow is shown by the arrow.

form, $\Gamma \rightarrow \Gamma_1 + m\Gamma_2$. The refined semi-primitive wall-crossing formula was computed in [95], and is given by,

$$\sum_{n=0}^{\infty} \Omega(\Gamma_1 + n\Gamma_2; y) x^n = \Omega(\Gamma_1; y) \prod_{k=1}^{\infty} \prod_{j=1}^{k|\langle \Gamma_1, \Gamma_2 \rangle|} \prod_n \left(1 + (-1)^n x^k y^{\frac{k|\langle \Gamma_1, \Gamma_2 \rangle| + 1 - 2j}{2}} \right)^{(-1)^n \Omega_n(k\Gamma_2)} \quad (5.7.187)$$

where we have used the intersection form $\langle \Gamma_1, \Gamma_2 \rangle$. In the mirror IIB geometry, this intersection form is simply the geometric intersection number of the corresponding lagrangian three-cycles. In IIA, it is equal to,

$$\langle \Gamma, \Gamma' \rangle = P_6 \cdot Q'_0 - P'_6 \cdot Q_0 + P_4 \cdot Q'_2 - P'_4 \cdot Q_2 \quad (5.7.188)$$

Finally, in the exponent of the semi-primitive wall-crossing formula we have used the refined degeneracies, $\Omega_n(\Gamma)$, that compute the index of states with spin n .

Now we that we have explained the refined semi-primitive wall-crossing formula, we want to apply this to our setup. We start with some BPS state $\Gamma = (0, N, m, n)$. At the W_n^1 wall, our state will decay into $\Gamma_1 + \Gamma_2$ where $\Gamma_2 = (0, 0, 1, n)$. This means that the intersection is given by $|\langle \Gamma_1, \Gamma_2 \rangle| = N$. From our discussion above, the semi-primitive wall-crossing formula simplifies since $\Omega(k\Gamma_2) = 0$ for $k > 1$, and $\Omega_n(\Gamma_2) = \delta_{0,n}$.

Putting this together, we can write the refined black hole partition function using chemical potentials $\tilde{q}$ and $\tilde{Q}$ for the D0 and D2-brane charge respectively. Then we

find that the partition function jumps across the wall W_n^1 as,

$$Z_{\text{ref BH}}(\tilde{Q}, \tilde{q}, y; z) \rightarrow Z_{\text{ref BH}}(\tilde{Q}, \tilde{q}, y; z) \prod_{j=1}^N \left(1 + y^{\frac{N+1}{2}-j} \tilde{Q} \tilde{q}^n\right) \quad (5.7.189)$$

Note that the wall-crossing factor takes the form of the Fock space character for a spin $\frac{N-1}{2}$ multiplet.

Similarly, crossing the wall W_n^{-1} results in the jump,

$$Z_{\text{ref BH}}(\tilde{Q}, \tilde{q}, y; z) \rightarrow Z_{\text{ref BH}}(\tilde{Q}, \tilde{q}, y; z) \prod_{j=1}^N \left(1 + y^{\frac{N+1}{2}-j} \tilde{Q}^{-1} \tilde{q}^n\right) \quad (5.7.190)$$

Now having understood how to cross individual walls, we want to follow a path in moduli space that connects the flopped geometry to the the large volume limit of interest. As shown in Figure 5.5, we can take $z = \frac{1}{2} + ir$ and follow the path from $r = -\infty$ to $r = +\infty$. This path crosses all W_n^1 and W_n^{-1} walls for $n > 0$, along with the wall W_0^1 . This implies the relationship,

$$Z_{+\infty}(\tilde{q}, \tilde{Q}, y) = Z_{-\infty}(\tilde{q}, y) \prod_{j=1}^N \left\{ (1 + y^{\frac{N+1}{2}-j} \tilde{Q}) \prod_{n=1}^{\infty} (1 + y^{\frac{N+1}{2}-j} \tilde{Q} \tilde{q}^n) (1 + y^{j-\frac{N+1}{2}} \tilde{Q}^{-1} \tilde{q}^n) \right\} \quad (5.7.191)$$

Using the Jacobi Triple product, this can be rewritten as,

$$\begin{aligned} Z_{+\infty}(\tilde{q}, \tilde{Q}, y) &= Z_{-\infty}(\tilde{q}, y) \left(\prod_{n=1}^{\infty} (1 - \tilde{q}^n)^{-N} \right) \sum_{\{n_i\}} \tilde{q}^{\frac{1}{2}(n,n)} (\tilde{Q} \tilde{q}^{-1/2})^{\sum_i n_i} y^{\sum_i (\frac{N+1}{2}-i)n_i} \\ &\sim \sum_{\{n_i\}} e^{-\phi_0 \frac{1}{2}(n,n)} e^{-\phi_2 \sum_i n_i} y^{\sum_i (\frac{N+1}{2}-i)n_i} \end{aligned} \quad (5.7.192)$$

where we have identified $e^{-\phi_0} = \tilde{q}$ and $e^{-\phi_2} = \tilde{Q} \tilde{q}^{-1/2}$.¹⁸ Up to the D0/D4-brane bound states which are determined by the $z = -\infty$ chamber, this formula agrees precisely with the partition function computed in equation 5.5.146. Thus, we have given two independent derivations of this mathematical formula, from gauge theory and from wall-crossing.

¹⁸This shift in charges arises because of the Freed-Witten anomaly [122], which implies that the spacetime D-brane charge, Γ , is related to the chern character of a vector bundle, E , over a four-cycle S , by $\Gamma = \text{ch}(E) e^{\frac{1}{2}c_1(S)}$. This leads to the above nontrivial relationship between the D-brane charges seen by wall-crossing, and the instanton charges seen by (q, t) -deformed Yang-Mills.

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Appendix A

The Conifold

We will show how we can use Seiberg duality and dimer mapping explained in section 2.4, to reproduce the results of [264, 287] on the partition function of the conifold. Recall that the conifold has only two nodes, which makes this model especially simple. The conifold quiver is shown for chamber n (with n framing nodes) in Figure A.1 while the conifold dimer is shown in Figure A.2.

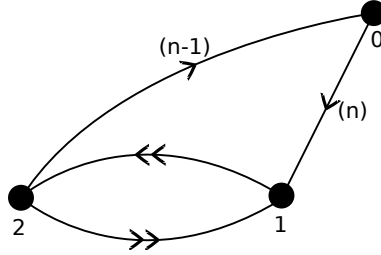


Figure A.1: The conifold quiver for chamber n .

Starting from the configuration shown in Figure A.2(a), we can perform Seiberg Duality on Node 1. Seiberg Duality results in a dual face for 1, as explained in section 2.4. However, the resulting brane tiling has two-valent vertices, which correspond to mass terms in the superpotential. Integrating out results in the quiver shown in Figure A.2(c). This transformation takes the quiver from chamber n to chamber $n+1$. As an explicit example of the techniques in section 2.4, we will derive the exact wall crossing formula, along with the change of variables from the dimer model.

The exact partition function for the conifold is known [264, 287]. In chamber n , it is given by,

$$Z(n, q_1, q_2) = M(1, -q_1 q_2)^2 \prod_{k \geq 1} (1 + q_1^k (-q_2)^{k-1})^{k+n-1} \prod_{k \geq n} (1 + q_1^k (-q_2)^{k+1})^{k-n+1}$$

where $M(x, q)$ is the MacMahon function,

$$M(x, q) \equiv \prod_{m=1}^{\infty} \left(\frac{1}{1 - xq^m} \right)^m$$

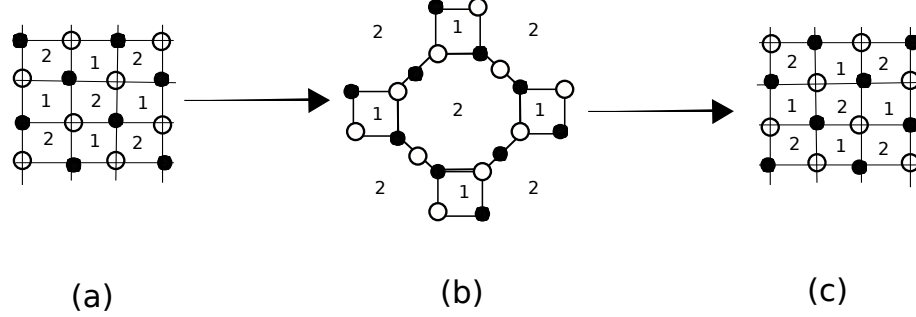


Figure A.2: The effect of Seiberg Duality on the conifold dimer. In the second step, we have integrated out the fields with a mass term in the superpotential coming from the 2-valent vertices.

Now consider crossing the wall from chamber n to chamber $n+1$. This gives,

$$Z(n+1) = Z(n, q_1, q_2)(1 + q_1)^{\langle q_1, q_\Delta \rangle} = Z(n, q_1, q_2)(1 + q_1)^{-n}$$

since for the conifold, every melting configuration must have the same intersection number, $\Delta_1 \circ \Delta = -n$.¹ By a simple change of variables²

$$\begin{aligned} q_1 &= -Q_2^{-1} \\ q_2 &= -Q_1 Q_2^2 \end{aligned}$$

we find,

$$Z(n+1) = M(1, -Q_1 Q_2)^2 \prod_{k \geq 1} (1 + Q_1^k (-Q_2)^{k-1})^{k+n+1} \prod_{k \geq n+1} (1 + Q_1^k (-Q_2)^{k+1})^{k-n}$$

which agrees with the general formula for the conifold partition function in chamber $(n+1)$.

¹Note that in Section 2.4 we only considered dualizing on nodes with $n_{0*} = 1$ or 0 framing arrows. However, the conifold is simple enough to explicitly check that the dimer intersection number equals the quiver intersection number for arbitrary n_{0*} .

²This change of variables includes additional minus signs relative to (2.4.24). This is because in this infinite product form, we have implicitly absorbed signs in the $\{q_i\}$. These signs flip (from the $(-1)^{d(\Delta)}$ factors) when we cross the wall from chamber n to chamber $n+1$.

Appendix B

The local $\mathbf{P}^2$ example

As another example, consider the toric quiver corresponding to $X = \mathcal{O}(-3) \rightarrow \mathbf{P}^2$. The D4-D2-D0 quiver has three nodes, corresponding to $\tilde{E}_3 = \mathcal{O}_S(-3)$, $\tilde{E}_1 = \mathcal{O}_S(-2)[-1]$, $\tilde{E}_2 = \mathcal{O}_S(-1)[-2]$. The charge of $\mathcal{O}_S(n)$ is

$$D_S e^{nD_t - \frac{1}{2}K_S} (1 + \frac{1}{24}c_2(S)) \quad (\text{B.0.1})$$

where D_S is the divisor corresponding to the surface, and D_t generates the Kahler class, so $D_S = -3D_t$. Also, $D_t D_S = C_t$ and per definition $D_t C_t = 1$. Thus, (B.0.1) can be rewritten as

$$D_S + (n + \frac{3}{2})C_t + \frac{1}{2}(n+1)(n+2) \text{ pt} - \frac{1}{8} \text{ pt} \quad (\text{B.0.2})$$

We can write

$$\begin{aligned} \tilde{\Delta}_3 &= D_S + \text{pt} + (-\frac{3}{2}C_t + \frac{1}{4}\text{pt}) \\ \tilde{\Delta}_2 &= D_S + 2C_t + (-\frac{3}{2}C_t + \frac{1}{4}\text{pt}) \\ \tilde{\Delta}_1 &= -D_S - C_t + (\frac{3}{2}C_t - \frac{1}{4}\text{pt}). \end{aligned}$$

This has $n_{31} = n_{12} = 3$, $n_{23} = 6$, where n_{ij} is the number of arrows from node i to node j . Add to this the D6 brane, corresponding to $\mathcal{O}_X[-1]$, whose charge is

$$\tilde{\Delta}_0 = -X(1 + \frac{c_2(X)}{24}),$$

Since $c_2(X) \cdot D_S = (c_2(S) - c_1^2(S))D_S = -6$, it is easy to see that $n_{03} = 1$ and $n_{02} = n_{01} = 0$. The intersection numbers can be computed by setting up the usual *Ext* machinery, and then reducing this to cohomology calculations on $S = \mathbf{P}^2$. For

example, $Ext_X^1(\mathcal{O}_X[-1], \mathcal{O}_S(-3)) = Ext_X^2(\mathcal{O}_X, \mathcal{O}_S(-3)) = H^2(\mathbf{P}^2, \mathcal{O}_S(-3)) = \mathbf{C}$, while all the other Ext groups vanish in this sector.¹

This is not a toric quiver yet, we need to dualize node $\tilde{\Delta}_1$. We get,

$$\begin{aligned}\Delta_3 &= \tilde{\Delta}_3 = D_S + \text{pt} + (-\frac{3}{2}C_t + \frac{1}{4}\text{pt}) \\ \Delta_2 &= \tilde{\Delta}_2 + 3\tilde{\Delta}_3 = -2D_S - C_t + 2(\frac{3}{2}C_t - \frac{1}{4}\text{pt}) \\ \Delta_1 &= -\tilde{\Delta}_1 = D_S + C_t - (\frac{3}{2}C_t - \frac{1}{4}\text{pt}).\end{aligned}$$

and $\Delta_0 = \tilde{\Delta}_0$, unchanged. The new bundles are given by, $E_3 = \mathcal{O}_S(-3)$, $E_1 = \mathcal{O}_S(-2)[-2]$, $E_2 = \tilde{\mathcal{O}}_S$. This has $n_{13} = n_{21} = n_{32} = 3$, $n_{03} = 1$, and the superpotential

$$W = \sum_{i,j,k=1}^4 \epsilon_{ijk} \text{Tr} A_i B_j C_k$$

where n_{ij} is the number of arrows from node i to node j . The resulting quiver is in Figure B.1.

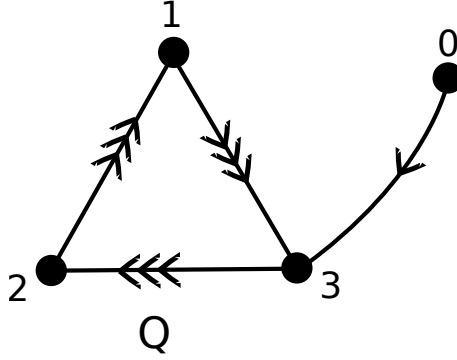


Figure B.1: The quiver for the orbifold phase of local $\mathbf{P}^2$.

The crystal $\mathcal{C}$ corresponding to this quiver is on the left in the Figure B.2. The corresponding crystal $\mathcal{C}_0$ is on the right. This corresponds to a set of points

$$N_1 + N_2 + N_3 = 3N_0,$$

in agreement with the fact that for local $\mathbf{P}^2$, $Q = (1, 1, 1, -3)$.

¹The last step follows from $\dim H^0(\mathbf{P}^k, \mathcal{O}(m)) = \binom{m+k}{k}$, $\dim H^k(\mathbf{P}^k, \mathcal{O}(m)) = \binom{-m-1}{-k-m-1}$, and $\dim H^n(\mathbf{P}^k, \mathcal{O}(m)) = 0$, for $n \neq 0, k$, see e.g. [32].

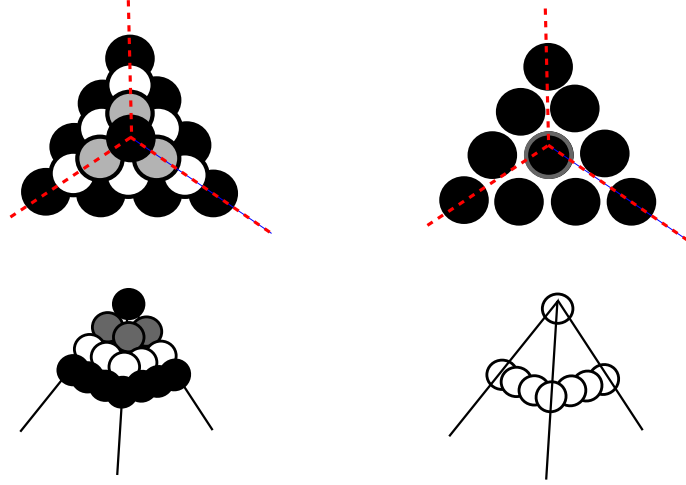


Figure B.2: The full crystal, $\mathcal{C}$ for local $\mathbf{P}^2$ is shown on the left, while the subcrystal, $\mathcal{C}_0$ corresponding to holomorphic functions is shown on the right.

Changing the B-field to $D = -nD_S = 3nD_t$, the lattice $\mathcal{C}_0(D)$ becomes (see Figure B.3)

$$N_1 + N_2 + N_3 = 3N_0 + 3n.$$

Getting $\mathcal{C}(D)$ from $\mathcal{C}$ corresponds to removing sites

$$\sum_{i=0}^{n-1} \frac{(3i+1)(3i+2)}{2} \Delta_3 + \frac{(3i+2)(3i+3)}{2} \Delta_2 + \frac{(3i+3)(3i+4)}{2} \Delta_1$$

Summing up the charges, we find

$$nD_S + \frac{3}{2}n^2C_t + (\frac{3}{2}n^3 - \frac{1}{4}n)\text{pt}$$

This accounts for the charge the D6 brane picks up by putting it in the background B-field, which takes it to $\mathcal{O}_X(D)[-1]$, with charge

$$\Delta = -Xe^D(1 + \frac{c_2(X)}{24}),$$

Namely, it is easy to see that this agrees with the difference

$$\Delta - \Delta_0 = nD_S - \frac{n^2}{2}D_S D_S + \frac{n^3}{3!}D_S D_S D_S + \frac{n}{24}C_2(X)D_S$$

using

$$D_S D_S = -3C_t, \quad D_S D_S D_S = 9,$$

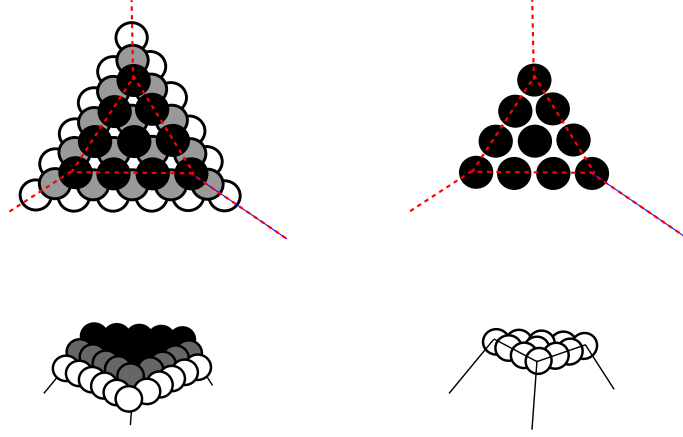


Figure B.3: The full crystal, $\mathcal{C}(D)$, and the subcrystal, $\mathcal{C}_0(D)$ for local $\mathbf{P}^2$ after turning on a large B-field.

Similarly, the face of the crystal carries charge

$$\frac{(3n+1)(3n+2)}{2}\Delta_3 + \frac{(3n+2)(3n+3)}{2}\Delta_2 + \frac{(3n+3)(3n+4)}{2}\Delta_1$$

which equals

$$D_S + (3n + \frac{3}{2})C_t + (\frac{(3n+1)(3n+2)}{2} + \frac{1}{4})\text{pt.}$$

From above, we see that this is the charge of

$$\mathcal{O}_S(D),$$

as we claimed in the text. Similarly, the charge of an edge is

$$(3n+1)\Delta_3 + (3n+2)\Delta_2 + (3n+3)\Delta_3.$$

This equals

$$C_t + (3n+1)\text{pt},$$

the charge of

$$\mathcal{O}_{C_t}(D).$$

Appendix C

The $(2, 0)$ Theory on Seifert Three-Manifolds

In this appendix we explain in more detail how the refined Chern-Simons theory arises from wrapping $M5$ branes on Seifert three-manifolds.

We begin by considering the $(2, 0)$ theory on three-manifolds of the special form, $M = \Sigma \times S^1$, where Σ is a Riemann surface, and we take the rest of the worldvolume directions to be $\mathbb{R}^2 \times S^1_\beta$. Since Σ is curved the theory will be partially topologically twisted along Σ .

We can either work in M-theory, where the partial topological twist is implemented by the geometry, $M \subset T^*M$, or in the $(2, 0)$ theory by performing the topological twist directly. For maximal generality, we will consider the A , D , and E -type $(2, 0)$ theories together. Recall that the D -type $(2, 0)$ theory arises from placing $M5$ branes on an orbifold, as we have considered in the body of this dissertation. An $M5$ -brane construction of the E -type theory is not known, but we can still describe it in terms of the $(2, 0)$ theory. In the following, we denote the choice of gauge group by G .

The construction is simplified by working instead with five-dimensional $\mathcal{N} = 2$ Super-Yang-Mills theory, obtained by reducing on the trivial S^1 . The $\mathcal{N} = 2$ supersymmetry algebra in five dimensions contains the maximal bosonic subalgebra, $SO(4, 1)_E \times SO(5)_R$. Under the $SO(4, 1)_E$ rotation group and the $SO(5)_R$ -symmetry, the supercharges transform as $(\mathbf{4}, \mathbf{4})$.

$M5$ branes wrapping $S^1_\beta \times \mathbb{C} \times \Sigma \times S^1 \subset S^1_\beta \times \mathbb{C}^2 \times T^*(\Sigma \times S^1)$ correspond to Yang-Mills on $S^1_\beta \times \mathbb{R}^2 \times \Sigma$, so it is natural to study the subgroups $SO(2)_{S_1} \times SO(2)_\Sigma \subset SO(4, 1)_E$ and $SO(2)_R \times SO(2)_{S_2} \subset SO(5)_R$. Geometrically, $SO(2)_{S_1}$ corresponds to rotations along the brane in $\mathbb{R}^2$, $SO(2)_\Sigma$ rotates the Riemann surface, $SO(2)_R$ rotates the directions in the Calabi-Yau transverse to the brane, and $SO(2)_{S_2}$ corresponds to rotations in the two noncompact transverse directions.¹ The sixteen supercharges

¹The convention of using S_2 to denote part of the R-symmetry may seem strange from a field theory perspective, but from the brane picture it is quite natural since this R-symmetry rotates the the transverse noncompact directions. This notation is especially natural when the $M5$ branes are

have the quantum numbers, $(\pm\frac{1}{2}, \pm\frac{1}{2}; \pm\frac{1}{2}, \pm\frac{1}{2})$ under this subgroup.

The partial topological twist comes from taking the diagonal combination $SO(2)'_{\Sigma} = (SO(2)_{\Sigma} \times SO(2)_R)_{diag}$. It is important to note that the groups involved in the twist are abelian; for this reason the supercharges will simultaneously have well-defined quantum numbers under both $SO(2)'_{\Sigma}$ and $SO(2)_R$. Upon performing the topological twist, we keep only those supercharges that are neutral under $SO(2)'_{\Sigma}$ so we are left with eight supercharges whose quantum numbers under $SO(2)_{S_1} \times SO(2)_R \times SO(2)_{S_2}$ are, $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$ (see Table C.1). These supercharges form an $\mathcal{N} = 4$ superalgebra in three dimensions. $\mathcal{N} = 4$ supersymmetry in three dimensions has an $SU(2)_R \times SU(2)_{S_2}$ R-symmetry, and the above $SO(2)_R \times SO(2)_{S_2}$ symmetries are simply the respective abelian subgroups of the full R-symmetry.

	$2S_1$	$2S_2$	$2S_R$	$S_R - S_2$
Q_+^{11}	+1	+1	+1	0
Q_-^{11}	-1	+1	+1	0
Q_+^{22}	+1	-1	-1	0
Q_-^{22}	-1	-1	-1	0
Q_+^{21}	+1	-1	+1	+1
Q_-^{21}	-1	-1	+1	+1
Q_+^{12}	+1	+1	-1	-1
Q_-^{12}	-1	+1	-1	-1

Table C.1: Supercharge Quantum Numbers in Three-Dimensional $\mathcal{N} = 4$ Supersymmetry. The addition of the supersymmetric Chern-Simons term breaks this to $\mathcal{N} = 2$ supersymmetry, preserving only the supercharges that are neutral under $S_R - S_2$.

Now recall that the refined Chern-Simons theory is defined by computing the index in the three-dimensional theory given by,

$$Z_{ref\ open}(X; q, t) = \text{Tr} \left((-1)^{2S_1} q^{S_1 - S_R} t^{S_R - S_2} e^{-\beta H} \right) \quad (\text{C.0.1})$$

Note that this index counts states annihilated by both Q_+^{11} and Q_-^{22} , as can be seen by reading off the quantum numbers in Table C.1. Here we have given a three-dimensional field-theoretic interpretation to the index, but we would also like to match up these symmetries with those in the original M-theory definition of the index in [19].

From the geometric picture given above, it should be clear that the S_1 and S_2 symmetries agree with those defined in [19]. Originally the S_R symmetry was identified with a rotation in the fibers of T^*M that is transverse to the $U(1)$ isometry of the Seifert manifold, M . In our case, this Seifert manifold isometry is given by simply rotating the S^1 in $M = \Sigma \times S^1$. This means that the S_R symmetry must rotate the

surface operators in a non-trivial geometrically engineered five-dimensional theory.

cotangent fiber over Σ , but this is precisely the $U(1)_R$ rotation that appeared above in our topological twist. Thus, we have given a purely field-theoretic identification of the symmetries involved in the refined Chern-Simons index.

Now we consider the more general case of M5 branes on a Seifert three manifold obtained by fibering the S^1 non-trivially over Σ . In this case, half of the supersymmetry of the theory is broken: after the partial topological twist, the theory on $\mathbb{R}^{2,1}$ has only $\mathcal{N} = 2$ supersymmetry in three dimensions. This also corresponds to the fact that in this case T^*M is an honest Calabi-Yau three-fold (its holonomy group is precisely $SU(3)$ and not a subgroup). Recall that $\mathcal{N} = 2$ supersymmetry has a $U(1)$ R-symmetry, but if M is an arbitrary three-manifold no additional symmetries will be present in the problem. Of the symmetries described above, only the rotation symmetry S_1 and the transverse R-symmetry, S_2 , will survive. However, the key is that for the special case of a Seifert manifold, the S_R symmetry will also be preserved by the breaking from $\mathcal{N} = 4$ to $\mathcal{N} = 2$. In this case all the supercharges are uncharged under $S_2 - S_R$, which becomes a new global symmetry.

One way to argue for this is to use the geometric description of the $U(1)_R$ symmetry when M is a Seifert three-manifold. The argument presented in [19] is based on the fact that one can use a nowhere vanishing vector field V rotating the S^1 fiber of the Seifert three-manifold to define, at each point on Σ a two-plane in the fiber T^*M – this two plane is co-normal to V in the natural symplectic structure on T^*M .

However, we would also like to see directly that this symmetry survives in the partial twisting of the $(2, 0)$ theory. We start by asking about the effect of fibering the S^1 over Σ in the $(2, 0)$ theory and in the dimensionally reduced five-dimensional Yang-Mills theory – where we reduce on the S^1 fiber of the Seifert three-manifold, as we did above. The answer is that for an S^1 bundle of degree p fibered over Σ , we obtain an $\mathcal{N} = 2$ Chern-Simons coupling on $\mathbb{R}^{2,1}$,

$$p \int_{\mathbb{R}^{2,1}} d^2\theta \operatorname{Tr}(\Sigma \mathcal{V}) \quad (\text{C.0.2})$$

where $\mathcal{V}$ is the $\mathcal{N} = 2$ vector multiplet, and $\Sigma = \epsilon^{\alpha\beta} \bar{D}_\alpha D_\beta \mathcal{V}$ is the linear superfield.² The crucial point is that this coupling is neutral under both $U(1)_r$ and $U(1)_R$, so turning it on does not break either symmetry. This can be seen by expanding out the Chern-Simons term in components,

$$p \int_{\mathbb{R}^{2,1}} d^3x d^2\theta \operatorname{Tr}(\Sigma \mathcal{V}) = p \int_{\mathbb{R}^{2,1}} d^3x \operatorname{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A + i \bar{\chi} \chi - 2D\sigma \right) \quad (\text{C.0.3})$$

For example, since A_μ is uncharged under both $U(1)_r$ and $U(1)_R$, the Chern-Simons term does not break these symmetries.

²Recently, in [56] it was argued that when wrapping M5 branes on a three-manifold, Chern-Simons terms should arise from torsion in the first homology of M . Specifically, they argued that the factor $\mathbb{Z}_p \in H_1(M, \mathbb{Z})$ should correspond to a Chern-Simons term at level p . This is perfectly consistent with our result since the first homology of a degree p circle bundle over a genus g Riemann surface, Σ , is given by $H_1(M, \mathbb{Z}) = \mathbb{Z}^{2g} \oplus \mathbb{Z}_p$.

The origin of the Chern-Simons coupling is purely topological. The fastest way to see this is to take the circle fiber of M as the M-theory circle, and reduce the six dimensional $M5$ brane theory to the five dimensional Yang-Mills theory living on a $D4$ brane. Now recall that the $D4$ brane action includes a coupling,

$$\int F_{RR} \wedge \text{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \quad (\text{C.0.4})$$

Using the relationship between IIA and M-theory, F_{RR} is simply the curvature of the M-theory circle bundle. In our present case, we have

$$\int_{\Sigma} F_{RR} = p$$

so we obtain the expected Chern-Simons term. The rest of Equation C.0.3 is fixed by $\mathcal{N} = 2$ supersymmetry.

It may seem that we are using some very particular facts about the couplings on the $D4$ branes, but this is not the case. The term in Equation C.0.4 really appears because the $(2, 0)$ theory has a propagating self-dual two-form tensor. In any attempt to write down the action for such a theory, there is a peculiar Wess-Zumino type term that arises – albeit involving the metric on the six-dimensional worldvolume [14, 248]. Using dimensional reduction to get the five dimensional Yang-Mills theory, the term C.0.4 arises with F_{RR} as the curvature of the circle bundle that we reduced on. This more general argument also makes it clear that the Chern-Simons term appears for the E -type $(2, 0)$ theory even though it does not have a known M-theoretic brane construction.

Thus, in the specific case when M is a Seifert three manifold, the theory has, an R-symmetry given by S_2 and an additional $U(1)$ flavor symmetry generated by $S_2 - S_R$. Therefore, when M is a Seifert three manifold we can define the refined index that computes refined Chern-Simons theory.

It may seem that by using the dimensional reduction on the S^1 fiber, the above topological twist is different from the usual geometric one obtained by wrapping a brane on $M \subset T^*M$. To see that they are the same, we would need to perform the three-dimensional partial topological twist directly in the $(2, 0)$ theory. This is rather difficult, but we can instead reduce the entire theory on a trivial circle and then study the twisting of the $D4$ -brane theory on M .

This gives two different ways of reducing the theory. First, we can use the above construction to give an $\mathcal{N} = 2$ theory in three dimensions, and then further reduce to an $\mathcal{N} = (2, 2)$ two-dimensional field theory. Alternatively, we can reduce on the trivial circle first, and then perform the standard topological twist of the $D4$ brane theory on the three-manifold, M , leaving us with a two-dimensional field theory. These procedures should agree, giving us a consistency check that the above twist is really the same as the standard twist.

Starting with the first approach, the dimensional reduction of the supersymmetric Chern-Simons term in Equation C.0.2 is given by an $\mathcal{N} = (2, 2)$ twisted superpotential,

$$\widetilde{W} = p \int d\theta^+ d\bar{\theta}^- \Sigma^2 = p \left(4\sqrt{2}H\text{Re}(\sigma) + 4\sqrt{2}F_{01}\text{Im}(\sigma) + 4\text{tr}(\bar{\lambda}_+\lambda_- + \lambda_+\bar{\lambda}_-) \right) \quad (\text{C.0.5})$$

where Σ is the two-dimensional super-field strength given by,

$$\Sigma = \sigma + i\sqrt{2}(\theta^+\bar{\lambda}_+ - \bar{\theta}^-\lambda_-) + \sqrt{2}\theta^+\bar{\theta}^-(H - iF_{01}) \quad (\text{C.0.6})$$

where σ is a complex scalar whose real part comes from the three-dimensional real scalar and whose imaginary part comes from the Wilson line of the three-dimensional gauge field, $\oint_{S^1_\beta} A$. Upon integrating out the auxillary field H and properly taking F_{01} into account, we obtain the potential terms,

$$U = g^2 p^2 (\text{Re}(\sigma))^2 + g^2 p^2 (\text{Im}(\sigma))^2 \quad (\text{C.0.7})$$

where g^2 is the coupling constant coming from the kinetic term.

Now we would like to understand how these quadratic terms are reproduced by the partial topological twist of D4 branes wrapping M . More generally, we must understand what is special about Seifert manifolds from this perspective. For example, a naive group theory analysis of the topological twist would not detect the additional symmetry that appears when M is a Seifert manifold.

Both of these problems can be resolved by making full use of the structure of M as a Seifert manifold. By definition this means that M has a nowhere-vanishing vector field, v , that acts as an isometry on M . By using the metric, this also implies the existence of a nowhere-vanishing one-form on M , which we denote by κ .

Upon choosing v , the structure group of the tangent bundle, TM , is reduced from $SO(3)$ to $SO(2)$, since the transition functions must respect the globally defined vector field, v . Because the structure group is reduced, the spin bundle over M will also split into two line bundles corresponding to the one-dimensional irreducible representations of $SO(2)$ (see [219, 228] for a discussion of this splitting in the context of three-dimensional Seiberg-Witten Floer theory). Finally, the same arguments imply that the cotangent bundle, T^*M will also split. This splitting is crucial since it explains why the fields living on the brane have well-defined quantum numbers under the geometric symmetry, S_R .

The topological twist coming from a brane wrapping $M \subset T^*M$ means that in addition to the gauge fields, A_μ , the three scalars fields corresponding to fluctuations of the brane in the fiber directions of T^*M will also now transform as a one form. Now by using the splitting of the cotangent bundle, any one-form on M can be decomposed as,

$$A = \phi\kappa + A_\Sigma \quad (\text{C.0.8})$$

where A_Σ is a one-form orthogonal to κ .

If we focus on the topologically twisted scalar fields, this means that we obtain one scalar and two vector components instead of the three vector components that we would get by topologically twisting on an arbitrary three-manifold. Note that this looks like the field content that we would expect from performing a topological twist only on the base two-manifold, Σ . The difference is that here we have a globally nontrivial circle fibration, which is responsible for the additional mass terms that break half of the supersymmetry.

Now we want to see how these twisted superpotential terms of equation C.0.7 arise. By using the above decomposition of A , the gauge field kinetic term includes,

$$\int_{\mathbb{R}^{1,1} \times M} d(\phi\kappa) \wedge \star d(\phi\kappa) \quad (\text{C.0.9})$$

But as explained in [40, 48, 177], $d\kappa = p(\star\kappa)$ and $\int \kappa \wedge d\kappa = p$, where p again is the Euler class of the circle bundle of M .³ Thus we obtain the term,

$$\int_{\mathbb{R}^{1,1}} \phi^2 \int_M d\kappa \wedge \kappa = p^2 \int_{\mathbb{R}^{1,1}} \phi^2 \quad (\text{C.0.10})$$

Likewise, under the topological twisting, three scalar fields become one-forms on M . Again, we can split the one-form bundle, and write the scalar part as σ . Then the kinetic term includes,

$$\int_{\mathbb{R}^{1,1}} \varphi^2 \int_M d\kappa \wedge \kappa = p^2 \int_{\mathbb{R}^{1,1}} \varphi^2 \quad (\text{C.0.11})$$

Thus we end up with quadratic mass terms for both ϕ and φ . But since these scalars are by definition the real and imaginary parts of σ , this is precisely the same as the potential generated by the twisted superpotential in equation C.0.7. Thus we have given an alternate way to understand the appearance of the quadratic terms in the twisted superpotential that break $\mathcal{N} = (4, 4)$ to $\mathcal{N} = (2, 2)$.

C.1 Computing the refined index

The basic building block for the derivation of refined Chern-Simons theory in [19], is the value of the index on $M = R^2 \times S^1$. We will now explain how to compute it, in an arbitrary ADE case.

³One way to see this is to recall that, as argued in [40, 48, 177], $\int \kappa \wedge d\kappa = p$. Further, if κ is normalized so that $\langle \kappa, \kappa \rangle = 1$ then we have, $\int \kappa \wedge \star \kappa = \int \langle \kappa, \kappa \rangle d\mu = V$ where V is the volume of the three-manifold. Finally, we know that $d\kappa = f(\star\kappa)$ where f is some arbitrary function. By choosing κ appropriately, f becomes a constant, and from the above facts we can deduce that $d\kappa = (p/V)(\star\kappa)$. We have dropped the volume dependence above, since it can be absorbed in the overall coupling constant which we have also omitted for simplicity.

Note that with a flat metric on $\mathbb{R}^2$ this geometry actually preserves even more supersymmetry, but we will allow an arbitrary metric on $\mathbb{R}^2$ so that topological twisting is necessary and only $\mathcal{N} = 4$ supersymmetry is preserved. Recall that the refined index is given by,

$$Z_{ref\ open}(X; q, t) = \text{Tr} (-1)^{2S_1} q^{S_1 - S_R} t^{S_R - S_2} e^{-\beta H} \quad (\text{C.1.12})$$

and the contribution of an $\mathcal{N} = 2$ BPS state with charges (S_1, S_2, S_R) is given by a quantum dilogarithm,

$$\exp \left(\sum_{n=1}^{\infty} \frac{1}{n} \frac{(-1)^{2S_1} q^{n(S_1 - S_R)} t^{n(S_R - S_2)}}{1 - q^n} e^{nx} \right) \quad (\text{C.1.13})$$

where x is the complexified mass of the BPS state. Now we can consider moving onto the Coulomb branch of the five-dimensional Yang-Mills theory; in the M-theory picture, this corresponds to separating the branes in the fiber direction over the S^1 . This geometric deformation will become complexified by the Wilson line around the S^1 , so that altogether we have a complex scalar deformation.

On the Coulomb Branch, after performing the partial topological twist, there will be a massive three-dimensional $\mathcal{N} = 4$ BPS state for every W -boson in the Yang-Mills theory. This $\mathcal{N} = 4$ state will decompose into two $\mathcal{N} = 2$ BPS states. These two states are related by acting with supercharges in the $\mathcal{N} = 4$ algebra, but not in the $\mathcal{N} = 2$ algebra. But from the discussion above, these supercharges will raise or lower the charge $S_R - S_2$. Therefore, altogether we have two BPS states of charges $(0, 0, 0)$ and $(+\frac{1}{2}, -\frac{1}{2}, +\frac{1}{2})$ under the $SO(2)_{S_1} \times SO(2)_{S_2} \times SO(2)_R$ symmetry, up to an overall ambiguous shift in the ground state charge. Therefore, the total partition function for the type- G $(2, 0)$ theory is given by,

$$Z(\mathbb{R}^2 \times S^1; G, x, q, t) = \exp \left(\sum_{\alpha > 0} \sum_{n=1}^{\infty} \frac{1}{n} \frac{1 - t^n}{1 - q^n} e^{n\langle \alpha, x \rangle} \right) \quad (\text{C.1.14})$$

where the x variables are the complex Coulomb-branch parameters, and α are the roots of the algebra G .

We can also use this simple geometry to make another connection with the field theory interpretation of the refined Chern-Simons theory. It is interesting to consider the $\epsilon_1 \rightarrow 0$, $(q \rightarrow 1)$ limit. In this limit the Ω -deformation along the M5-brane is turned off, but it remains nonzero along the noncompact directions transverse to the brane. For general three-manifolds, the free energy of the partition function then simply computes the twisted effective superpotential, as in the unrefined case [237]. The only difference is that here, the mass of the BPS states contributing to $\mathcal{W}_{eff}$ also depends on the R-charge because of the remaining Ω -deformation in the transverse directions. In fact, this corresponds to turning on a twisted mass equal to $\epsilon_2(S_R - S_2)$

for each field in the $\mathcal{N} = 2$ theory. In our simple geometry, $\mathbb{R}^2 \times S^1$, taking this limit gives,

$$\lim_{\epsilon_1 \rightarrow 0} Z_{CS}(\mathbb{R}^2 \times S^1, G, x, q, t) = \exp \left(- \frac{1}{\epsilon_1} \sum_{\alpha > 0} \sum_{n=1}^{\infty} \frac{1}{n^2} \left(e^{n\langle \alpha, x \rangle} - e^{n(\langle \alpha, x \rangle + \epsilon_2)} \right) \right). \quad (9.15)$$

But these are precisely the contributions to the twisted effective superpotential from a vector and chiral multiplet in three dimensions [227, 237], if we identify the x_I as the twisted chiral superfield strengths.

Altogether, we have derived the contribution to the index for the solid torus geometry, which can be identified with $\mathbb{R}^2 \times S^1$, and we have also understood the effect of fibering the S^1 nontrivially. Putting this information together, we can glue together two solid tori to give the S^3 geometry. The gluing process leads to the refined Chern-Simons matrix model that was the basis for deriving more general amplitudes in [19].

Appendix D

Background on Macdonald polynomials

Here we review some useful properties of the Macdonald polynomials associated to any semisimple lie algebra, $\mathfrak{g}$ [201]. For more details see [186, 203].

We will denote the rank of $\mathfrak{g}$ by r , the root system of $\mathfrak{g}$ by R and its positive part by R_+ . Similarly, we denote the root lattice by Q and its positive part by Q_+ , and the weight lattice by P and its positive part by P_+ . The root system lives in an r -dimensional vector space, $V = \mathbb{R}^r$ and we denote the standard basis of V by $\{\epsilon_i\}$, with the normalization that the $\{\epsilon_i\}$ are the smallest vectors, such that they all sit inside P .

We introduce r variables, $\{x_1, \dots, x_r\}$. Formally, we can relate the x_i to the basis of V by the relationship, $x_i \sim e^{\epsilon_i}$. Since the Weyl group, W , has a well defined action on the weight lattice of $\mathfrak{g}$, this allows us to define an action of the Weyl group on $\mathbb{Z}[x_i, x_i^{-1}]$.

Then we say that a polynomial, $f(x_1, \dots, x_r)$, is symmetric if f is invariant under the action of the Weyl group on the $\{x_i\}$. Note that in the case of $\mathfrak{g} = \mathfrak{su}(N)$, this reduces to the usual definition of a symmetric polynomial since the Weyl group of $\mathfrak{su}(N)$ is simply the symmetric group.

The simplest example of such a symmetric polynomial is given by the monomial symmetric polynomials, m_λ , which are associated to a representation of $\mathfrak{g}$ with highest weight $\lambda = \sum_i \lambda_i \epsilon_i$. We use the notation that $x^\lambda := x_1^{\lambda_1} \dots x_r^{\lambda_r}$. Then m_λ is defined by,

$$m_\lambda = \sum_{w \in W} x^{w(\lambda)} \quad (\text{D.0.1})$$

Note that this polynomial is symmetric by construction because of the sum over the Weyl group.

A slightly more sophisticated example is given by the character of a representation, χ_λ . Recall that from the Weyl character formula, the character of λ can be written

as,

$$\chi_\lambda = \frac{\sum_{w \in W} \epsilon(w) x^{w(\lambda+\rho)}}{\sum_{w \in W} \epsilon(w) x^{w(\rho)}} \quad (\text{D.0.2})$$

Although χ_λ is usually described as a character, it can also be uniquely defined in another way that is closer to the definition of Macdonald polynomials. We start by defining an inner product on the space of symmetric polynomials given by,

$$\langle f, g \rangle = \frac{1}{|W|} \int_0^{2\pi} d\phi_1 \cdots \int_0^{2\pi} d\phi_r \Delta(e^{i\phi_1}, \dots, e^{i\phi_r}) f(e^{i\phi}) g(e^{-i\phi}) \quad (\text{D.0.3})$$

where

$$\Delta(x_1, \dots, x_r) = \prod_{\alpha \in R} (1 - x^\alpha) \quad (\text{D.0.4})$$

and where we use the shorthand $f(x)$ for $f(x_1, \dots, x_r)$.

Then we can uniquely define the χ_λ to be the symmetric functions obeying a condition on the leading behavior and an orthogonality condition,

$$\chi_\lambda = m_\lambda + \sum_{\mu < \lambda} K_{\lambda\mu} m_\mu \quad (\text{D.0.5})$$

$$\langle \chi_\lambda, \chi_\nu \rangle = 0 \quad \text{for } \lambda \neq \nu \quad (\text{D.0.6})$$

where $K_{\lambda\mu}$ are arbitrary coefficients in the decomposition of the character. By $\mu < \lambda$, we mean that $\lambda - \mu \in Q_+$. Thus the first condition only restricts the leading behavior of χ_λ .

Now we are ready to finally define the Macdonald Polynomials in a similar way. The Macdonald polynomials are Laurent polynomials in the $\{x_i\}$, but also depend rationally on two additional variables, q and t . First we define a new measure,

$$\Delta_{q,t} = \prod_{\alpha \in R} \frac{(x^\alpha; q)_\infty}{(x^\alpha t; q)_\infty} \quad (\text{D.0.7})$$

where we have used the q-Pochhammer symbol,

$$(a; q)_\infty = \prod_{r=0}^{\infty} (1 - aq^r) \quad (\text{D.0.8})$$

Although Macdonald polynomials are defined for arbitrary values of q and t , the formulas simplify in the case when $t = q^\beta$, with β a positive integer,

$$\Delta_{q,q^\beta} = \prod_{\alpha \in R} \prod_{m=0}^{\beta-1} (1 - q^m x^\alpha) \quad (\text{D.0.9})$$

Using this measure we can define a new inner product on the space of symmetric polynomials,

$$\langle f, g \rangle_{q,t} = \frac{1}{|W|} \int_0^{2\pi} d\phi_1 \cdots \int_0^{2\pi} d\phi_r \Delta_{q,t}(e^{i\phi_1}, \dots, e^{i\phi_r}) f(e^{i\phi}) g(e^{-i\phi}) \quad (\text{D.0.10})$$

Then the Macdonald polynomials, M_λ are uniquely defined by the same leading order condition as in D.0.5, and the orthogonality condition with respect to the new inner product,

$$M_\lambda = m_\lambda + \sum_{\mu < \lambda} u_{\lambda\mu} m_\mu \quad (\text{D.0.11})$$

$$\langle M_\lambda, M_\nu \rangle_{q,t} = 0 \quad \text{for } \lambda \neq \nu \quad (\text{D.0.12})$$

Note that this definition does not explicitly specify the inner product of M_λ with itself. For the case of $t = q^\beta$ with β a positive integer, this inner product, which we sometimes refer to as the metric, g_λ , is equal to,

$$\langle M_\lambda, M_\lambda \rangle = g_\lambda \equiv \prod_{\alpha \in R_+} \prod_{m=0}^{\beta-1} \frac{1 - t^{\langle \rho, \alpha^\vee \rangle} q^{\langle \lambda, \alpha^\vee \rangle + m}}{1 - t^{\langle \rho, \alpha^\vee \rangle} q^{\langle \lambda, \alpha^\vee \rangle - m}} \quad (\text{D.0.13})$$

where ρ is the Weyl vector, $\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha$. It will also be useful for knot computations to have a combinatorial expression for the metric that holds for general q and t . For the $SO(2N)$ case, such a formula is given in Appendix B. There exist relatively efficient algorithms for computing Macdonald polynomials for general root systems, such as the determinantal formula of [270].

As in the case of characters, the product of two Macdonald polynomials can be decomposed into a sum of Macdonald polynomials, where the coefficients are known as (q, t) Littlewood-Richardson coefficients,

$$M_\lambda M_\nu = \sum_{\gamma} N_{\lambda\nu}^\gamma M_\gamma \quad (\text{D.0.14})$$

In general the computation of the $N_{\lambda\nu}^\gamma$ is difficult, but for specific choices of representations, the Pieri formula gives an explicit expression for N [200].

A weight, λ , is known as minuscule if $\langle \lambda, \alpha \rangle = 0$ or 1 , for all $\alpha \in R_+$. Note that the fundamental weights, ω_i , are not necessarily minuscule since they obey, $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$ only for the dual of the simple roots, α_i , and not necessarily for all positive roots. In fact, for the cases of E_8, F_4 , and G_2 , it is known that no minuscule weights exist.

However, when a minuscule weight, ω , does exist, then its product with any other weight, λ is given by,

$$M_\omega M_\lambda = \sum_{\substack{\tau \in W(\omega) \\ \lambda + \tau \in P^+}} N_{\omega, \lambda}^{\lambda + \tau} M_{\lambda + \tau} \quad (\text{D.0.15})$$

where,

$$N_{\omega,\lambda}^{\lambda+\tau} = \prod_{\substack{\alpha \in R_+ \\ \langle \tau, \alpha^\vee \rangle = -1}} \frac{1 - t^{-1}q^{\langle \lambda+\rho, \alpha^\vee \rangle}}{1 - q^{\langle \lambda+\rho, \alpha^\vee \rangle}} \frac{1 - tq^{\langle \lambda+\rho, \alpha^\vee \rangle - 1}}{1 - q^{\langle \lambda+\rho, \alpha^\vee \rangle - 1}} \quad (\text{D.0.16})$$

Appendix E

Facts about $SO(2N)$

Here we collect some useful facts about the lie algebra $\mathfrak{so}(2N)$ and its Macdonald polynomials. For $\mathfrak{so}(2N) = D_N$ recall that the positive roots are given by,

$$\{e_i - e_j, e_i + e_j\}, \quad i < j \quad (\text{E.0.1})$$

and the Weyl vector is given by,

$$\rho = \sum_{i=1}^N (N - i)e_i \quad (\text{E.0.2})$$

The highest root is equal to,

$$\theta = e_1 + e_2 \quad (\text{E.0.3})$$

Finally, the fundamental weights, ω_i , are,

$$\begin{aligned} \{\omega_i &\equiv e_1 + \cdots + e_i\}, \quad 1 \leq i \leq N-2 \\ \omega_{N-1} &= \frac{1}{2}(e_1 + \cdots + e_N) \\ \omega_N &= \frac{1}{2}(e_1 + \cdots + e_{N-1} - e_N) \end{aligned} \quad (\text{E.0.4})$$

In this section, we will write the weight of a generic representation as,

$$\lambda = \sum_{i=1}^N \lambda_i e_i = \sum_{i=1}^N \gamma_i \omega_i \quad (\text{E.0.5})$$

This means that the representations of D_N are indexed by $\{\lambda_i\}$ which are either all integers or all half-integers, and satisfy,

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{N-1} \geq |\lambda_N| \quad (\text{E.0.6})$$

In ordinary Chern-Simons theory, the Hilbert space of the theory on T^2 is spanned by the integrable representations at level k . This result also holds true for the refined Chern-Simons theory, so for computations it is useful to know precisely which representations are integrable. Recall that for a representation to be integrable, we require that,

$$(\lambda, \theta) \leq k \quad (\text{E.0.7})$$

where λ is the highest weight for the integrable representation, and θ is the highest root. Thus for D_N , the integrability condition becomes,

$$\gamma_1 + 2(\gamma_2 + \cdots + \gamma_{N-2}) + \gamma_{N-1} + \gamma_N \leq k \quad (\text{E.0.8})$$

For computations in the refined Chern-Simons theory, the Macdonald polynomials of D_N also play an important role. The framework outlined in Appendix A is generally sufficient, but it is useful to have an explicit combinatorial formula for the metric even when β is not an integer in $t = q^\beta$.

Note from the structure of the fundamental weights, either all of the $\{\lambda_i\}$ are integers or none of them are integers. Also note that λ_N may be negative. If we define, $\tilde{\lambda}_i = |\lambda_i|$, then the $\tilde{\lambda}_i$ can naturally be described as a Young tableau when all the λ_i are integers. The metric can then be expressed as a sum over the boxes of this tableau,

$$\begin{aligned} \langle M_\lambda, M_\lambda \rangle = g_\lambda &= \prod_{(i,j) \in \tilde{\lambda}} \frac{(1 - t^{2N-2i} q^{2j-1})(1 - t^{N-i+1} q^{j-1})(1 + t^{N-i} q^{j-1})(1 - t^{N-i-1} q^j)}{(1 - t^{2N-2i-1} q^{2j})(1 - t^{2N-2i-1} q^{2j-1})(1 - t^{N-i} q^j)} \\ &\quad \cdot \frac{1 - t^{\tilde{\lambda}_j^T - i} q^{\tilde{\lambda}_i - j + 1}}{1 - t^{\tilde{\lambda}_j^T - i + 1} q^{\tilde{\lambda}_i - j}} \frac{1 - t^{2N - \tilde{\lambda}_j^T - i - 1} q^{\tilde{\lambda}_i + j}}{1 - t^{2N - \tilde{\lambda}_j^T - i} q^{\tilde{\lambda}_i + j - 1}} \end{aligned} \quad (\text{E.0.9})$$

Here, i is the vertical coordinate of the tableau and runs from 1 to N , while j is the horizontal coordinate. $\tilde{\lambda}_i$ is the length of the row i , while $\tilde{\lambda}_j^T$ denotes the number of boxes in column j .

For the case when all of the λ_i are half-integers, we define, $\tilde{\lambda}_i = |\lambda_i| - 1/2$, and again $\tilde{\lambda}$ can be interpreted as a Young tableau. Then the corresponding formula for the metric is given by,

$$\begin{aligned} \langle M_\lambda, M_\lambda \rangle = g_\lambda &= C \prod_{(i,j) \in \tilde{\lambda}} \frac{1 - t^{2N-2i} q^{2j}}{1 - t^{2N-2i-1} q^{2j+1}} \frac{1 - t^{2N-2i} q^{2j-1}}{1 - t^{2N-2i-1} q^{2j}} \frac{1 - t^{N-i+1} q^{j-1}}{1 - t^{N-i} q^j} \frac{1 - t^{N-i-1} q^{j+1}}{1 - t^{N-i} q^j} \\ &\quad \cdot \frac{1 - t^{\tilde{\lambda}_j^T - i} q^{\tilde{\lambda}_i - j + 1}}{1 - t^{\tilde{\lambda}_j^T - i + 1} q^{\tilde{\lambda}_i - j}} \frac{1 - t^{2N - \tilde{\lambda}_j^T - i - 1} q^{\tilde{\lambda}_i + j + 1}}{1 - t^{2N - \tilde{\lambda}_j^T - i} q^{\tilde{\lambda}_i + j}} \end{aligned} \quad (\text{E.0.10})$$

where C is an additional factor given by,

$$C = \prod_{k=1}^{\lfloor N/2 \rfloor} \frac{1 - t^{2k-2} q}{1 - t^{2k-1}} \frac{1 - t^{2k-2+2\lceil N/2 \rceil}}{1 - t^{2k-3+2\lceil N/2 \rceil} q} \quad (\text{E.0.11})$$

Appendix F

Refined Indices

Here we give details on the precise three- and five-dimensional indices that compute refined topological string partition functions. In five dimensions this is simply the index studied by Nekrasov in [225]. Both indices are analogues of the four-dimensional protected spin character studied in [127].

F.1 Five Dimensional Indices

Recall that the five-dimensional $\mathcal{N} = 1$ supersymmetry algebra consists of eight supercharges and includes an $Sp(1)_r = SU(2)_r$ R-symmetry. Upon dimensional reduction, the algebra is equivalent to four-dimensional $\mathcal{N} = 2$ supersymmetry. For our purposes, it will be useful to rewrite the algebra in terms of four-dimensional notation, so that the supercharges can be organized as,

$$Q_\alpha^I, Q_{\dot{\alpha}}^I \tag{F.1.1}$$

where $I = 1, 2$ is the $SU(2)_R$ index, while α and $\dot{\alpha}$ are the $SO(4) = SU(2)_l \times SU(2)_r$ indices under rotations in four dimensions. Then the supersymmetry algebra is given by,

$$\{Q_\alpha^I, \bar{Q}_{\dot{\beta}}^J\} = 2\delta^{IJ}\sigma_{\alpha\dot{\beta}}^\mu P_\mu \tag{F.1.2}$$

$$\{Q_\alpha^I, Q_\beta^J\} = 2\epsilon_{\alpha\beta}\epsilon^{IJ}(Z - iP_5) \tag{F.1.3}$$

where Z is the *real* five dimensional central charge. Now we can study massive representations of this algebra, whose little group is $SO(4) = SU(2)_l \times SU(2)_r$. Then a generic long multiplet (with $M \geq |Z|$) transforms under $SU(2)_l \times SU(2)_r \times SU(2)_R$ as,

$$(J_l, J_r; I_R) \otimes \left((0, 0; 0) \oplus (0, 0; 1) \oplus (0, \frac{1}{2}; \frac{1}{2}) \oplus (\frac{1}{2}, 0; \frac{1}{2}) \oplus (\frac{1}{2}, \frac{1}{2}; 0) \right) \tag{F.1.4}$$

where $(J_l, J_r; I_R)$ is an arbitrary representation. In contrast, short left-handed BPS multiplets (with $M = Z$) take the form,

$$(J_l, J_r; I_R) \otimes \left((0, 0; \frac{1}{2}) \oplus (\frac{1}{2}, 0; 0) \right) \quad (\text{F.1.5})$$

while those with $M = -Z$ will have the same structure but with the chirality flipped.

The unrefined index, which is related to the ordinary topological string, is given by

$$\text{Tr}(-1)^{2(j_l+j_r)} q^{2j_l} e^{-\beta H} \quad (\text{F.1.6})$$

The contribution of a long multiplet to this index is 0. In addition, a short right-handed multiplet contributes 0, while the fundamental left-handed multiplet contributes

$$-(q^{1/2} - q^{-1/2})^2 e^{-\beta M} \quad (\text{F.1.7})$$

Thus, this gives a good index, since it only receives contributions from left-handed BPS states, while long multiplets cancel out of the trace. This is precisely the five-dimensional index in M-theory compactified on a Calabi-Yau that computes the ordinary topological string.

In order to extend this to a more refined index, we must use the R-symmetry, as in [225], which gives the index,

$$\text{Tr}(-1)^{2(j_l+j_r)} q_1^{2j_l} q_2^{2(j_r-S_R)} e^{-\beta H} \quad (\text{F.1.8})$$

Again, we find that the long multiplets and the right-handed multiplets do not contribute to the index, while the fundamental left-handed multiplet now contributes,

$$(q_2 + q_2^{-1} - q_1 - q_1^{-1}) e^{-\beta M} \quad (\text{F.1.9})$$

By using the definitions $j_l = \frac{1}{2}(S_1 - S_2)$, $j_r = \frac{1}{2}(S_1 + S_2)$, and taking $q_1 = \sqrt{qt}$, $q_2 = \sqrt{q/t}$, we can rewrite this in the form more natural for refined topological string theory,

$$\text{Tr}(-1)^F q^{S_1-S_R} t^{S_R-S_2} e^{-\beta H} \quad (\text{F.1.10})$$

and the contribution from a BPS multiplet becomes,

$$-\left(\sqrt{q} - \frac{1}{\sqrt{q}}\right)\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) e^{-\beta H} \quad (\text{F.1.11})$$

F.2 Three Dimensional Indices

Here we collect the details on indices for three dimensional $\mathcal{N} = 2$ supersymmetry, since this is the case of primary interest for refined Chern-Simons theory. The

supersymmetry algebra is given by dimensionally reducing $\mathcal{N} = 1$ supersymmetry in four dimensions,

$$\{Q_\alpha, \bar{Q}_\beta\} = 2\sigma_{\alpha\beta}^\mu P_\mu + 2i\epsilon_{\alpha\beta} Z \quad (\text{F.2.12})$$

$$\{Q_\alpha, Q_\beta\} = 0 \quad (\text{F.2.13})$$

$$\{\bar{Q}_\alpha, \bar{Q}_\beta\} = 0 \quad (\text{F.2.14})$$

The Q are complex spinors with charge $\pm\frac{1}{2}$ under the two-dimensional rotation group, $U(1)_S$, and as in four dimensions there is a $U(1)_R$ symmetry that rotates Q_α and $\bar{Q}_\alpha$ ¹. Then the simplest massive long ($M > |Z|$) representation transforms under $U(1)_S \times U(1)_R$ as $(0; \pm\frac{1}{2}) \oplus (\pm\frac{1}{2}; 0)$. As usual, a generic long representation is given by tensoring this with an arbitrary representation of $U(1)_S \times U(1)_R$,

$$(S; R) \otimes \left((0; \pm\frac{1}{2}) \oplus (\pm\frac{1}{2}; 0) \right) \quad (\text{F.2.15})$$

There are also short BPS multiplets, each containing one bosonic and one fermionic degree of freedom. The right short representations are given by $(S; R) \otimes ((0; -1/2) \oplus (+1/2; 0))$ and the left short representations are given by $(S; R) \otimes ((0; -1/2) \oplus (-1/2; 0))$.

Now consider the unrefined index,

$$\text{Tr}(-1)^{2S} q^{S-R} e^{-\beta H} \quad (\text{F.2.16})$$

It can be seen that the long representation makes a contribution of 0 to this index, as do the right short multiplets, but the simplest left short representation contributes $(q^{1/2} - q^{-1/2})e^{-\beta M}$. This index is related to the open topological string, and thus, to unrefined Chern-Simons theory.

Generically, this unrefined index is the best that we can do. However, as explained in Appendix C, an additional symmetry, $U(1)_r$, appears upon compactifying the $(2, 0)$ theory on a Seifert Manifold. In this case, the supercharges have the quantum numbers shown in Table F.1.

We can now form a refined index

$$\text{Tr}(-1)^{2S} q^{S-r} t^{r-R} \quad (\text{F.2.17})$$

Note that $(r - R)$ is a flavor symmetry, since none of the supercharges are charged under it. Then it is straightforward to show that long multiplets and right short multiplets make no contribution to this improved index. The only contribution comes from the fundamental left short multiplet, which contributes $(q^{1/2} - q^{-1/2})$ as before.

However, there will generically still be t -dependence, since the BPS states may be charged under the new flavor symmetry. It is precisely this index that computes the refined Chern-Simons theory.

¹In this appendix, we use notations that are more appropriate for the three-dimensional perspective, instead of the natural M-theory notations used in the body of this dissertation. The relationship is $(S, R, r) \leftrightarrow (S_1, S_2, S_R)$.

	$2S$	$2R$	$2r$
Q_+^1	+1	+1	+1
Q_-^1	-1	+1	+1
Q_+^2	+1	-1	-1
Q_-^2	-1	-1	-1

Table F.1: Quantum Numbers for Three-Dimensional $\mathcal{N} = 2$ Supersymmetry

Appendix G

Identities for Macdonald Polynomials

In this appendix we fix our conventions and collect some useful results on Macdonald polynomials. We refer the reader to [202] for more details.

Macdonald polynomials form a special class of symmetric functions. They are rational functions of q and t , and are symmetric functions of N variables, x_i . The simplest way to understand Macdonald polynomials comes from defining an inner product on the space of symmetric functions,

$$\langle f, g \rangle = \frac{1}{N!} \oint dz_1 \cdots \oint dz_N \Delta_{q,t}(z_1, \cdots, z_N) f(z_1, \cdots, z_N) g(z_1^{-1}, \cdots, z_N^{-1}) \quad (\text{G.0.1})$$

where the measure is given by,

$$\Delta_{q,t}(z_1, \cdots, z_N) = \prod_{1 \leq i < j \leq N} \frac{(z_i/z_j; q)_\infty (z_j/z_i; q)_\infty}{(z_i t/z_j; q)_\infty (z_j t/z_i; q)_\infty} \quad (\text{G.0.2})$$

where $(x; q)_\infty = \prod_{m=0}^{\infty} (1 - xq^m)$.

Then we can uniquely associate a Macdonald polynomial, $M_R(z; q, t)$, to every $SU(N)$ representation, R , by requiring the following two properties,

$$\langle M_R, M_S \rangle = 0 \quad \text{if } R \neq S \quad (\text{G.0.3})$$

$$M_R = m_R + (\text{lower order}) \quad (\text{G.0.4})$$

where m_R is the monomial symmetric polynomial given by,

$$m_R(z_1, \cdots, z_N) = \sum_{\sigma} z_1^{R_{\sigma(1)}} \cdots z_N^{R_{\sigma(N)}} \quad (\text{G.0.5})$$

where the sum is over all elements σ of the symmetric group.

Therefore, orthogonality and a condition on the leading behavior completely determine the Macdonald polynomials. It is important to note that in the limit $t = q$, Macdonald polynomials reduce to the more familiar Schur functions (which are independent of q),

$$M_R(x_i; q, q) = s_R(x_i) \quad (\text{G.0.6})$$

In this dissertation, we use Macdonald polynomials with either finitely many variables or infinitely many variables - the finite polynomials appear in (q, t) -deformed Yang-Mills and the infinite polynomials appear in the closed refined topological string. We begin by reviewing the finite case.

G.1 $SU(N)$ Macdonald Polynomials

For finitely many variables, the inner product of a Macdonald polynomial with itself can be written either in a combinatorial way

$$g_R := \frac{G_R}{G_0} = \frac{\langle M_R, M_R \rangle}{\langle M_0, M_0 \rangle} = \prod_{(i,j) \in \lambda} \frac{1 - q^{R_i-j+1} t^{R_j^T-i}}{1 - q^{R_i-j} t^{R_j^T-i+1}} \frac{1 - q^{j-1} t^{N+1-i}}{1 - q^j t^{N-i}} \quad (\text{G.1.7})$$

or for the case when $\beta \in \mathbb{Z}_{\geq 0}$, in a Lie-theoretic way,

$$g_R = \prod_{m=0}^{\beta-1} \prod_{1 \leq i < j \leq N} \frac{[R_i - R_j + \beta(j-i) + m]_q [\beta(j-i) - m]_q}{[R_i - R_j + \beta(j-i) - m]_q [\beta(j-i) + m]_q} \quad (\text{G.1.8})$$

We will often refer to g_R as the Macdonald metric for R .

We can also give an explicit formula for Macdonald polynomials evaluated at $z_k = t^{\rho_k}$. This gives a generalization of the quantum dimension of a representation, which we refer to as the (q, t) -dimension,

$$\dim_{q,t}(R) := M(t^\rho) = t^{\frac{(N+1)|R|}{2}} \prod_{(i,j) \in R} \frac{t^{i-N-1} - q^{j-1}}{1 - q^{R_i-j} t^{R_j^T-i+1}} \quad (\text{G.1.9})$$

This formula can also be rewritten when $\beta \in \mathbb{Z}_{\geq 0}$ in a Lie-theoretic way,

$$M(t^\rho) = \prod_{m=0}^{\beta-1} \prod_{1 \leq i < j \leq N} \frac{[R_i - R_j + \beta(j-i) + m]_q}{[\beta(j-i) + m]_q} \quad (\text{G.1.10})$$

Now that we have explained the finite N formulas, we would like to take the $N \rightarrow \infty$ limit.

G.2 $SU(\infty)$ Macdonald Polynomials

For studying the closed refined topological string, it is also useful to collect some formulas for Macdonald polynomials with infinitely many variables. Starting with the metric in equation G.1.7, and taking the $N \rightarrow \infty$ limit naively gives,

$$g_R^{(mac)} = \prod_{(i,j) \in R} \frac{1 - t^{R_j^T - i} q^{R_i - j + 1}}{1 - t^{R_j^T - i + 1} q^{R_i - j}} \quad (\text{G.2.11})$$

which is the standard formula for the metric given in [202]. However, unlike its finite N counterpart, this expression is not symmetric under $(q, t) \rightarrow (q^{-1}, t^{-1})$. We can fix this by using a slightly different definition,

$$g_R(q, t) := \prod_{(i,j) \in R} \frac{t^{\frac{R_j^T - i}{2}} q^{\frac{R_i - j + 1}{2}} - t^{-\frac{R_j^T - i}{2}} q^{-\frac{R_i - j + 1}{2}}}{t^{\frac{R_j^T - i + 1}{2}} q^{\frac{R_i - j}{2}} - t^{-\frac{R_j^T - i + 1}{2}} q^{-\frac{R_i - j}{2}}} = (t/q)^{\frac{|R|}{2}} \cdot g_R^{(mac)} \quad (\text{G.2.12})$$

In order to preserve the relationship between the metric and Macdonald polynomials, $g_R = \langle M_R, M_R \rangle$, we must also rescale M_R in the infinite-variable limit,

$$M_R(x_1, x_2, \dots; q, t) := (t/q)^{\frac{|R|}{4}} \cdot M_R^{(mac)}(x_1, x_2, \dots; q, t) \quad (\text{G.2.13})$$

We can also give a formula for evaluating the infinite-variable Macdonald polynomial at a certain value,

$$M_R(t^\rho; q, t) = q^{\frac{1}{4}\|R\|^2} t^{-\frac{1}{4}\|R^T\|^2} (t/q)^{\frac{|R|}{2}} \prod_{(i,j) \in R} \left(q^{\frac{R_i - j}{2}} t^{\frac{R_j^T - i + 1}{2}} - q^{-\frac{R_i - j}{2}} t^{-\frac{R_j^T - i + 1}{2}} \right)^{-1} \quad (\text{G.2.14})$$

where $(\rho)_k = -k + 1/2$

There are also generalized Cauchy identities for Macdonald polynomials. The most useful ones are,

$$\sum_R \frac{1}{g_R} M_R(x; q, t) M_R(y; q, t) = \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} \frac{1 - t^n}{1 - q^n} p_n(x) p_n(y) \right) \quad (\text{G.2.15})$$

$$\sum_R M_R(x; q, t) M_{R^T}(y; t, q) = \prod_{i,j} (1 + x_i y_j) \quad (\text{G.2.16})$$

where $p_n(x) = \sum_i x_i^n$ is the n 'th power sum.

Finally, when understanding anti-branes in this dissertation, it is useful to use the operation that flips the sign of the power sums,

$$\iota(p_n(x)) = -p_n(x) \quad (\text{G.2.17})$$

This operation acts on Schur functions as,

$$\iota s_R(x) = (-1)^{|R|} s_{R^T}(x) \quad (\text{G.2.18})$$

Its action on Macdonald polynomials is more complicated. However, from equation G.2.15 and the definition of ι , it is straightforward to write down a generalized Cauchy identity for ιM ,

$$\sum_R \frac{1}{g_R} \iota M_R(x; q, t) M_R(y; q, t) = \exp \left(- \sum_{n=1}^{\infty} \frac{1}{n} \frac{1-t^n}{1-q^n} p_n(x) p_n(y) \right) \quad (\text{G.2.19})$$

Appendix H

Gopakumar-Vafa Invariants

In section 5.4 we presented a TQFT that computes the refined topological string amplitudes for geometries of the form $\mathcal{L}_1 \oplus \mathcal{L}_2 \rightarrow \Sigma_g$. As explained originally by Gopakumar and Vafa in [138, 139], the topological string partition function can be rewritten as counting spinning M2-branes which wrap a two-cycle in the Calabi-Yau and are free to move in the noncompact $\mathbb{R}^{4,1}$. Since these M2-branes are massive, they should transform in a definite representation of the (4+1)-dimensional little group, $SO(4) = SU(2)_l \times SU(2)_r$. BPS states will also fall in representations of the $SU(2)_R$ R-symmetry. As explained in section 5.4, when tracing over BPS states we should study the diagonal subgroup $SU(2)'_r = \text{diag}(SU(2)_r \times SU(2)_R)$. It is necessary to use this diagonal subgroup $SU(2)'_r$ rather than $SU(2)_r$ in order to guarantee that the Gopakumar-Vafa invariants will be invariant under complex structure deformations.

If we denote these spins as J_l and J'_r , then the refined free energy (defined by $F = -\log Z$) takes the general form,

$$F_{\text{ref top}}(X; q, t) = \sum_{d=1}^{\infty} \sum_{\beta, J_l, J'_r} \frac{N_{J_l, J'_r}^{\beta} \left((qt)^{dJ_l} + \dots + (qt)^{-dJ_l} \right) \left((q/t)^{dJ'_r} + \dots + (q/t)^{-dJ'_r} \right) Q^{d\beta}}{d(q^{d/2} - q^{-d/2})(t^{d/2} - t^{-d/2})} \quad (\text{H.0.1})$$

where the $N_{J_l, J'_r}^{\beta}(X)$ are the Gopakumar-Vafa (GV) invariants associated to the Calabi-Yau. Since these invariants count M2-branes, they are expected to be integers on general grounds.

One nontrivial check of our proposed TQFT is the integrality of the corresponding GV invariants. We have found that the invariants arising from the TQFT are always integral as expected, but that they do not sit in full representations of $SU(2)_l \times SU(2)'_r$. Instead, they generally sit in representations of $SU(2)_l \times U(1)'_r$. This is generally true for any Calabi-Yau manifold that engineers a field theory with a nonzero five-dimensional Chern-Simons level, as has been observed previously in [38, 70]. It would be interesting to understand this breaking more precisely from a spacetime perspective.

Since the symmetry group has been reduced to $SU(2)_l \times U(1)'_r$, it is more convenient to write the free energy as,

$$F_{\text{ref top}}(X; q, t) = \sum_{d=1}^{\infty} \sum_{\beta, J_l, m'_r} \frac{N_{J_l, m'_r}^{\beta} \left((qt)^{dJ_l} + \dots + (qt)^{-dJ_l} \right) (q/t)^{dm'_r} Q^{d\beta}}{d(q^{d/2} - q^{-d/2})(t^{d/2} - t^{-d/2})} \quad (\text{H.0.2})$$

where (J_l, m'_r) labels an irreducible representation of $SU(2)_l \times U(1)'_r$. Some examples of these invariants are listed in Tables H.1 and H.2,

β	$\sum_{J_l, m'_r} N_{(J_l, m'_r)}^{\beta} (j_l, m'_r)$
1	$-(\frac{1}{2}, 1) + (0, 1) + (0, 0)$
2	$-(1, \frac{3}{2}) + (\frac{1}{2}, 2) + 2(\frac{1}{2}, 1) - 2(0, \frac{3}{2}) - 2(0, \frac{1}{2})$
3	$-(2, 3) + (\frac{3}{2}, \frac{7}{2}) + (\frac{3}{2}, \frac{5}{2}) + (1, 2) - (\frac{1}{2}, \frac{7}{2}) - 3(\frac{1}{2}, \frac{5}{2}) - (\frac{1}{2}, \frac{3}{2}) + 2(0, 3) + 2(0, 2)$

Table H.1: Genus 1 Gopakumar-Vafa Invariants for the geometry $\mathcal{O}(1) \oplus \mathcal{O}(-1) \rightarrow T^2$.

β	$\sum_{J_l, m'_r} N_{(j_l, m'_r)}^\beta(j_l, m'_r)$
1	$-(1, 1) + 2(\frac{1}{2}, \frac{3}{2}) + 2(\frac{1}{2}, \frac{1}{2}) - (0, 2) - 3(0, 1) - (0, 0)$
2	$ \begin{aligned} & -(3, \frac{7}{2}) + 2(\frac{5}{2}, 4) + 2(\frac{5}{2}, 3) - (2, \frac{9}{2}) - 2(2, \frac{7}{2}) - 2(\frac{3}{2}, 4) - 4(\frac{3}{2}, 3) - (\frac{3}{2}, 2) \\ & + 2(1, \frac{9}{2}) + 7(1, \frac{7}{2}) + (1, \frac{5}{2}) - (1, \frac{3}{2}) - 2(\frac{1}{2}, 4) + 3(\frac{1}{2}, 3) + 8(\frac{1}{2}, 2) + (\frac{1}{2}, 1) \\ & - (0, \frac{9}{2}) - 4(0, \frac{7}{2}) - 11(0, \frac{5}{2}) - 5(0, \frac{3}{2}) \end{aligned} $
3	$ \begin{aligned} & -(\frac{13}{2}, \frac{15}{2}) + 2(6, 8) + 2(6, 7) - (\frac{11}{2}, \frac{17}{2}) - 2(\frac{11}{2}, \frac{15}{2}) - 2(5, 8) - 3(5, 7) - (5, 6) \\ & + 2(\frac{9}{2}, \frac{17}{2}) + 5(\frac{9}{2}, \frac{15}{2}) - (\frac{9}{2}, \frac{11}{2}) - (4, 8) + 3(4, 7) + 4(4, 6) \\ & - (\frac{7}{2}, \frac{17}{2}) - 2(\frac{7}{2}, \frac{15}{2}) - (\frac{7}{2}, \frac{13}{2}) + 3(\frac{7}{2}, \frac{11}{2}) + 3(\frac{7}{2}, \frac{9}{2}) - 5(3, 7) - 7(3, 6) - 7(3, 5) - 4(3, 4) \\ & + 2(\frac{5}{2}, \frac{15}{2}) + 3(\frac{5}{2}, \frac{13}{2}) + 3(\frac{5}{2}, \frac{11}{2}) + 8(\frac{5}{2}, \frac{9}{2}) + 2(\frac{5}{2}, \frac{7}{2}) \\ & + (2, 8) + 3(2, 7) + 3(2, 6) - 6(2, 5) - 3(2, 4) + 2(2, 3) \\ & - 2(\frac{3}{2}, \frac{15}{2}) - 2(\frac{3}{2}, \frac{13}{2}) + 5(\frac{3}{2}, \frac{11}{2}) - 2(\frac{3}{2}, \frac{9}{2}) - 6(\frac{3}{2}, \frac{7}{2}) - 2(\frac{3}{2}, \frac{5}{2}) \\ & - 5(1, 6) + 4(1, 5) + 8(1, 4) - 3(1, 3) - (1, 2) \\ & + 3(\frac{1}{2}, \frac{13}{2}) + 2(\frac{1}{2}, \frac{11}{2}) - 3(\frac{1}{2}, \frac{9}{2}) + 19(\frac{1}{2}, \frac{7}{2}) + 13(\frac{1}{2}, \frac{5}{2}) + (\frac{1}{2}, \frac{3}{2}) \\ & - (0, 7) - 4(0, 6) - 3(0, 5) - 18(0, 4) - 24(0, 3) - 6(0, 2) \end{aligned} $

Table H.2: Genus 2 Gopakumar-Vafa Invariants for the geometry $O(3) \oplus O(-1) \rightarrow \Sigma_{g=2}$.

Appendix I

Factorization of the (q, t) -Dimension and Metric

In this appendix we derive formulas for the (q, t) -dimension and metric of a composite representation. As a warm-up exercise, it is helpful to recall how this works for the quantum dimension of an $SU(N)$ representation, R ,

$$\dim_q(R) = \prod_{1 \leq i < j \leq N} \frac{[R_i - R_j + j - i]}{[j - i]} \quad (\text{I.0.1})$$

where we are using the definition, $[n] = q^{n/2} - q^{-n/2}$. For the composite representation coming from R and S , we obtain,

$$\dim_q(R\bar{S}) = \dim_q(R)\dim_q(S) \prod_{i=1}^{c_R} \prod_{j=1}^{c_S} \frac{[S_j + R_i + N + 1 - j - i][N + 1 - i - j]}{[S_j + N + 1 - i - j][R_i + N + 1 - j - i]} \quad (\text{I.0.2})$$

where c_R and c_S are the number of rows in R and S .

Now we can perform the same calculation for the (q, t) -dimension. For computational convenience, we will specialize to $t = q^\beta$ where $\beta \in \mathbb{Z}_{>0}$, but our final results will be valid for any q and t .

Recall that the (q, t) -dimension is given by the expression,

$$\dim_{(q,t)}(R) = \prod_{m=0}^{\beta-1} \prod_{1 \leq i < j \leq N} \frac{[R_i - R_j + \beta(j - i) + m]}{[\beta(j - i) + m]} \quad (\text{I.0.3})$$

A short calculation shows that we obtain the same type of splitting,

$$\begin{aligned} \dim_{(q,t)}(R\bar{S}) &= \dim_{(q,t)}(R) \cdot \dim_{(q,t)}(S) \cdot \prod_{m=0}^{\beta-1} \prod_{i=1}^{c_R} \prod_{j=1}^{c_S} \frac{[R_i + S_j + \beta(N + 1 - i - j) + m]}{[R_i + \beta(N + 1 - i - j) + m]} \\ &\quad \cdot \frac{[\beta(N + 1 - i - j) + m]}{[S_j + \beta(N + 1 - i - j) + m]} \end{aligned} \quad (\text{I.0.4})$$

Now we would like to understand the additional factors that appear in this formula, which are interpreted in section 5.6 as arising from ghost branes. To do so, it is helpful first to convert the $SU(N)$ (q, t) -dimension into an expression for the $SU(\infty)$ (q, t) -dimension, since it is the $N \rightarrow \infty$ formulas that appear in the refined topological string. We find,

$$\dim_{q,t}(R) = \prod_{m=0}^{\beta-1} \prod_{1 \leq i < j \leq N} \frac{[R_i - R_j + \beta(j-i) + m]}{[\beta(j-i) + m]} \quad (\text{I.0.5})$$

$$= W_R(q, t) T_R^{-1} (-1)^{|R|} (q/t)^{|R|/4} Q^{-\frac{|R|}{2}} \prod_{i=1}^{c_R} \prod_{j=1}^{R_i} (1 - t^{-i+1} q^{j-1} Q) \quad (\text{I.0.6})$$

where $Q = t^N$ and we have used the definitions,

$$W_R(q, t) = (q/t)^{\frac{|R|}{2}} M_R(t^\rho; q, t) \quad (\text{I.0.7})$$

$$T_R = q^{\frac{1}{2}\|R\|^2} t^{-\frac{1}{2}\|R^T\|^2} \quad (\text{I.0.8})$$

where $(\rho)_i = -i + 1/2$.

Next, it is helpful to rewrite the factors in I.0.4 and I.0.6 in exponential form,

$$\prod_{m=0}^{\beta-1} \prod_{i=1}^{c_R} \prod_{j=1}^{c_S} \left(1 - q^{R_i + S_j + \beta(N+1-i-j)+m}\right) = \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} Q^n g_1(q^n, t^n)\right) \quad (\text{I.0.9})$$

where $g_1(q, t) = \sum_{m=0}^{\beta-1} \sum_{i=1}^{c_R} \sum_{j=1}^{c_S} q^{R_i + S_j + \beta(1-i-j)+m}$. Similarly,

$$\prod_{m=0}^{\beta-1} \prod_{i=1}^{c_R} \prod_{j=1}^{c_S} \left(1 - q^{R_i + \beta(N+1-i-j)+m}\right) = \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} Q^n g_2(q^n, t^n)\right) \quad (\text{I.0.10})$$

where $g_2(q, t) = \sum_{m=0}^{\beta-1} \sum_{i=1}^{c_R} \sum_{j=1}^{c_S} q^{R_i + \beta(1-i-j)+m}$.

$$\prod_{m=0}^{\beta-1} \prod_{i=1}^{c_R} \prod_{j=1}^{c_S} \left(1 - q^{S_j + \beta(N+1-i-j)+m}\right) = \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} Q^n g_3(q^n, t^n)\right) \quad (\text{I.0.11})$$

where $g_3(q, t) = \sum_{m=0}^{\beta-1} \sum_{i=1}^{c_R} \sum_{j=1}^{c_S} q^{S_j + \beta(1-i-j)+m}$.

$$\prod_{m=0}^{\beta-1} \prod_{i=1}^{c_R} \prod_{j=1}^{c_S} \left(1 - q^{\beta(N+1-i-j)+m}\right) = \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} Q^n g_4(q^n, t^n)\right) \quad (\text{I.0.12})$$

where $g_4(q, t) = \sum_{m=0}^{\beta-1} \sum_{i=1}^{c_R} \sum_{j=1}^{c_S} q^{\beta(1-i-j)+m}$.

$$\prod_{i=1}^{c_R} \prod_{j=1}^{R_i} \left(1 - t^{-i+1} q^{j-1} Q\right) = \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} Q^n f_R(q^n, t^n)\right) \quad (\text{I.0.13})$$

where $f_R(q, t) = \sum_{i=1}^{c_R} \sum_{j=1}^{R_i} q^{\beta(-i+1)+j-1}$. Putting all of these results together, we can rewrite the composite (q, t) -dimension as,

$$\begin{aligned} \dim_{q,t}(\overline{RS}) &= W_R(q, t)W_S(q, t)T_R^{-1}T_S^{-1}(-1)^{|R|+|S|}(q/t)^{\frac{1}{4}(|R|+|S|)} \\ &\quad \cdot Q^{-\frac{|R|+|S|}{2}} \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} Q^n M_{RS}(q^n, t^n)\right) \end{aligned} \quad (\text{I.0.14})$$

where we have defined

$$M_{RS}(q, t) = g_1(q, t) + g_4(q, t) - g_2(q, t) - g_3(q, t) + f_R(q, t) + f_S(q, t) \quad (\text{I.0.15})$$

Next, we would like to simplify M_{RS} . To do so, it is helpful to notice that f_R can be rewritten as,

$$f_R(q, t) = \frac{t}{q-1} \sum_{i=1}^{c_R} \left(q^{R_i - \beta i} - q^{-\beta i} \right) \quad (\text{I.0.16})$$

Then a short calculation reveals that M_{RS} can be rewritten as,

$$M_{RS}(q, t) = \frac{(1-q)(1-t)}{t} f_R(q, t) f_S(q, t) + f_R(q, t) + f_S(q, t) \quad (\text{I.0.17})$$

We would like to use this rewriting to make contact with the refined topological vertex amplitudes, W_{RQ} and $\widetilde{W}_{RQ}$. As discussed in section 5.4, these amplitudes can be computed from the large N limit of the refined Chern-Simons S-matrix and are equal to,

$$W_{RQ} = (q/t)^{\frac{|R|+|Q|}{2}} M_R(t^\rho; q, t) M_Q(t^\rho q^R; q, t) \quad (\text{I.0.18})$$

$$\widetilde{W}_{RQ} = (q/t)^{\frac{|R|+|Q|}{2}} M_R(t^\rho; q, t) \iota M_Q(t^\rho q^R; q, t) \quad (\text{I.0.19})$$

We define the following refined quantity,

$$K_{RS}(Q, q, t) := \sum_P \frac{1}{g_P} Q^{|P|} (t/q)^{|P|} W_{PR}(q, t) W_{PS}(q, t) \quad (\text{I.0.20})$$

$$\begin{aligned} &= W_R(q, t) W_S(q, t) \sum_P \frac{1}{g_P} Q^{|P|} M_P(t^\rho q^R; q, t) M_P(t^\rho q^S; q, t) \\ &= W_R(q, t) W_S(q, t) \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \frac{1-t^n}{1-q^n} Q^n p_n(x) p_n(y)\right) \end{aligned} \quad (\text{I.0.21})$$

where in the last line we have used the generalized Cauchy identity of G.2.15 and where $x_i = q^{R_i} t^{-i+1/2}$ and $y_j = q^{S_j} t^{-j+1/2}$. A straightforward evaluation shows that,

$$K_{RS}(q, t) = K_{..}(q, t) W_R(q, t) W_S(q, t) \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} Q^n M_{RS}(q^n, t^n)\right) \quad (\text{I.0.22})$$

This means that we can rewrite the composite (q, t) -dimension as,

$$\dim_{q,t}(R\bar{S}) = T_R^{-1}T_S^{-1}(-1)^{|R|+|S|}Q^{-\frac{|R|+|S|}{2}}(q/t)^{\frac{1}{4}(|R|+|S|)}\frac{K_{..}(Q, q, t)W_R(q, t)^2W_S(q, t)^2}{K_{RS}(Q, q, t)} \quad (\text{I.0.23})$$

In section 5.6, we use this result to understand the genus $g > 1$ geometries.

It will also be useful to rewrite the (q, t) -dimension in another way. Define the refined quantity,

$$\begin{aligned} N_{RS}(Q, q, t) &:= \sum_P \frac{1}{g_P} Q^{|P|} (t/q)^{|P|} \widetilde{W}_{RP}(q, t) W_{PS}(q, t) \\ &= W_R(q, t) W_S(q, t) \sum_P \frac{1}{g_P} Q^{|P|} {}_\iota M_P(t^\rho q^R; q, t) M_P(t^\rho q^S; q, t) \\ &= W_R(q, t) W_S(q, t) \exp \left(- \sum_{n=1}^{\infty} \frac{1}{n} \frac{1-t^n}{1-q^n} Q^n p_n(x) p_n(y) \right) \end{aligned} \quad (\text{I.0.24})$$

where as before, $x_i = q^{R_i} t^{-i+1/2}$ and $y_j = q^{S_j} t^{-j+1/2}$. Then using the same analysis as above, we can rewrite this expression as,

$$N_{RS}(Q, q, t) = N_{..}(q, t) W_R(q, t) W_S(q, t) \exp \left(- \sum_{n=1}^{\infty} \frac{1}{n} Q^n M_{RS}(q^n, t^n) \right) \quad (\text{I.0.25})$$

Therefore, we can alternatively rewrite the composite (q, t) -dimension as,

$$\dim_{q,t}(R\bar{S}) = T_R^{-1}T_S^{-1}(-1)^{|R|+|S|}(q/t)^{\frac{1}{4}(|R|+|S|)}Q^{-\frac{|R|+|S|}{2}}\frac{N_{RS}(Q, q, t)}{N_{..}(Q, q, t)} \quad (\text{I.0.26})$$

This formula is useful in section 5.6 for studying the genus $g = 0$ geometries. It is also helpful to notice that $N_{..}(Q, q, t) = (K_{..}(Q, q, t))^{-1}$.

It is important to note here that by using the above definition of $\dim_{q,t}(R)$, we are rewriting the expression in equation 5.5.137 as,

$$\frac{S_{0R}^2}{G_R} = \frac{S_{00}\tilde{S}_{00}\dim_{q,t}(R)^2}{g_R} \quad (\text{I.0.27})$$

where we used the definition from above that $g_R = G_R/G_0$ and have also introduced the normalization factors, $S_{00}\tilde{S}_{00}$. In the large N limit, these additional factors can

be rewritten as,

$$q^{2\beta^2\rho^2} S_{00}(q, t) \tilde{S}_{00}(q, t) := q^{2\beta^2\rho^2} \prod_{m=0}^{\beta-1} \prod_{1 \leq i < j \leq N} [\beta(j-i) + m][\beta(j-i) - m] \quad (\text{I.0.28})$$

$$\begin{aligned} &= \left(K_{..}(Q) K_{..}\left(Q \frac{q}{t}\right) \right)^{-1} \prod_{k=0}^{\infty} \left(1 - tq^k\right)^N \left(1 - q^{k+1}\right)^N \quad (\text{I.0.29}) \\ &\quad \cdot \prod_{j,k=1}^{\infty} \left(1 - t^k q^{j-1}\right)^{-1} \left(1 - q^k t^{j-1}\right)^{-1} \\ &= \left(K_{..}(Q) K_{..}\left(Q \frac{q}{t}\right) \right)^{-1} \left((t; q)_{\infty} (q; q)_{\infty} \right)^N M(q, t) M(\mathbf{1}, \mathbf{q}) \end{aligned}$$

where we have written the expression in terms of the refined MacMahon function,

$$M(q, t) = \prod_{j,k=1}^{\infty} \left(1 - t^k q^{j-1}\right)^{-1} \quad (\text{I.0.31})$$

that appears in the refined topological vertex [171].

Now we move on to studying the large N factorization of the metric, g_R . Recall that the metric is given by the formula,

$$g_R^{(N)} = \prod_{m=0}^{\beta-1} \prod_{1 \leq i < j \leq N} \frac{[R_i - R_j + \beta(j-i) + m][\beta(j-i) - m]}{[R_i - R_j + \beta(j-i) - m][\beta(j-i) + m]} \quad (\text{I.0.32})$$

As a first step, we must express this in terms of the $N \rightarrow \infty$ metric,

$$g_R^{(N)} = (q/t)^{\frac{|R|}{2}} g_R^{(\infty)} \prod_{m=0}^{\beta-1} \prod_{i=1}^N \prod_{j=1}^{\infty} \frac{[R_i + \beta(N+j-i) - m][\beta(N+j-i) + m]}{[R_i + \beta(N+j-i) + m][\beta(N+j-i) - m]} \quad (\text{I.0.33})$$

where $g_R^{(\infty)}$ is the symmetrized metric defined by G.2.12. We also find that the metric for the composite representation $S\bar{R}$ is given by,

$$\begin{aligned} g_{S\bar{R}}^{(N)} &= g_R^{(N)} \cdot g_S^{(N)} \cdot \prod_{m=0}^{\beta-1} \prod_{i=1}^{c_S} \prod_{j=1}^{c_R} \frac{[S_i + R_j + \beta(N+1-j-i) + m][S_i + \beta(N+1-i-j) - m]}{[S_i + R_j + \beta(N+1-j-i) - m][S_i + \beta(N+1-i-j) + m]} \\ &\quad \cdot \frac{[R_j + \beta(N+1-i-j) - m][\beta(N+1-j-i) + m]}{[R_j + \beta(N+1-i-j) + m][\beta(N+1-j-i) - m]} \end{aligned} \quad (\text{I.0.34})$$

Now as with the analysis of the (q, t) -dimension, we want to write all of the factors in I.0.33 and I.0.34 in terms of exponentials,

$$\prod_{m=0}^{\beta-1} \prod_{i=1}^{c_R} \prod_{j=1}^{c_S} \frac{1 - q^{R_i + S_j + \beta(N+1-i-j) + m}}{1 - q^{R_i + S_j + \beta(N+1-i-j) - m}} = \exp \left(- \sum_{n=1}^{\infty} \frac{1}{n} Q^n h_1(q^n, t^n) \right) \quad (\text{I.0.35})$$

where $h_1(q, t) = \sum_{m=0}^{\beta-1} (q^m - q^{-m}) \sum_{i=1}^{c_R} \sum_{j=1}^{c_S} q^{R_i + S_j + \beta(1-i-j)}$.

$$\prod_{m=0}^{\beta-1} \prod_{i=1}^{c_R} \prod_{j=1}^{c_S} \frac{1 - q^{S_j + \beta(N+1-i-j)+m}}{1 - q^{S_j + \beta(N+1-i-j)-m}} = \exp \left(- \sum_{n=1}^{\infty} \frac{1}{n} Q^n h_2(q^n, t^n) \right) \quad (\text{I.0.36})$$

where $h_2(q, t) = \sum_{m=0}^{\beta-1} (q^m - q^{-m}) \sum_{i=1}^{c_R} \sum_{j=1}^{c_S} q^{S_j + \beta(1-i-j)}$.

$$\prod_{m=0}^{\beta-1} \prod_{i=1}^{c_R} \prod_{j=1}^{c_S} \frac{1 - q^{R_i + \beta(N+1-i-j)+m}}{1 - q^{R_i + \beta(N+1-i-j)-m}} = \exp \left(- \sum_{n=1}^{\infty} \frac{1}{n} Q^n h_3(q^n, t^n) \right) \quad (\text{I.0.37})$$

where $h_3(q, t) = \sum_{m=0}^{\beta-1} (q^m - q^{-m}) \sum_{i=1}^{c_R} \sum_{j=1}^{c_S} q^{R_i + \beta(1-i-j)}$.

$$\prod_{m=0}^{\beta-1} \prod_{i=1}^{c_R} \prod_{j=1}^{c_S} \frac{1 - q^{\beta(N+1-i-j)+m}}{1 - q^{\beta(N+1-i-j)-m}} = \exp \left(- \sum_{n=1}^{\infty} \frac{1}{n} Q^n h_4(q^n, t^n) \right) \quad (\text{I.0.38})$$

where $h_4(q, t) = \sum_{m=0}^{\beta-1} (q^m - q^{-m}) \sum_{i=1}^{c_R} \sum_{j=1}^{c_S} q^{\beta(1-i-j)}$.

$$\prod_{m=0}^{\beta-1} \prod_{i=1}^N \prod_{j=1}^{\infty} \frac{1 - q^{R_i + \beta(N+j-i)-m}}{1 - q^{R_i + \beta(N+j-i)+m}} \frac{1 - q^{\beta(N+j-i)+m}}{1 - q^{\beta(N+j-i)-m}} = \exp \left(- \sum_{n=1}^{\infty} \frac{1}{n} Q^n h_R(q^n, t^n) \right) \quad (\text{I.0.39})$$

where

$$h_R(q, t) = \sum_{m=0}^{\beta-1} (q^{-m} - q^m) \sum_{i=1}^{c_R} \sum_{j=1}^{\infty} \left(q^{R_i + \beta j - \beta i} - q^{\beta j - \beta i} \right) \quad (\text{I.0.40})$$

$$= \frac{q - t}{1 - q} \sum_{i=1}^N \left(q^{R_i - \beta i} - q^{-\beta i} \right) \quad (\text{I.0.41})$$

$$= \left(1 - \frac{q}{t} \right) f_R(q, t) \quad (\text{I.0.42})$$

Putting this all together we can write the composite metric as,

$$g_{R\bar{S}}^{(N)} = (q/t)^{\frac{|R|+|S|}{2}} g_R^{(\infty)} g_S^{(\infty)} \exp \left(- \sum_{n=1}^{\infty} \frac{1}{n} Q^n H_{RS}(q^n, t^n) \right) \quad (\text{I.0.43})$$

where H_{RS} is given by,

$$H_{RS} = h_1(q, t) + h_4(q, t) - h_2(q, t) - h_3(q, t) + \left(1 - \frac{q}{t} \right) \left(f_R(q, t) + f_S(q, t) \right) \quad (\text{I.0.44})$$

A little algebra shows,

$$H_{RS} = \left(1 - \frac{q}{t} \right) \frac{(1-q)(1-t)}{t} f_R f_S + \left(1 - \frac{q}{t} \right) (f_R + f_S) \quad (\text{I.0.45})$$

$$= \left(1 - \frac{q}{t} \right) M_{RS}(q, t) \quad (\text{I.0.46})$$

where M_{RS} is the quantity that we defined in equation I.0.15 while studying the (q, t) -dimension. Therefore,

$$g_{R\bar{S}}^{(N)} = (q/t)^{\frac{|R|+|S|}{2}} g_R^{(\infty)} g_S^{(\infty)} \exp \left(- \sum_{n=1}^{\infty} \frac{1}{n} Q^n M_{RS}(q^n, t^n) + \sum_{n=1}^{\infty} \frac{1}{n} Q^n \frac{q^n}{t^n} M_{RS}(q^n, t^n) \right) \quad (\text{I.0.47})$$

Using the results from above, this means that we can write the composite metric in two ways,

$$g_{R\bar{S}}^{(N)} = (q/t)^{\frac{|R|+|S|}{2}} g_R^{(\infty)} g_S^{(\infty)} \frac{K_{RS}(Q_t^q, q, t)}{K_{..}(Q_t^q, q, t)} \frac{K_{..}(Q, q, t)}{K_{RS}(Q, q, t)} \quad (\text{I.0.48})$$

$$g_{R\bar{S}}^{(N)} = (q/t)^{\frac{|R|+|S|}{2}} g_R^{(\infty)} g_S^{(\infty)} \frac{N_{..}(Q_t^q, q, t)}{N_{RS}(Q_t^q, q, t)} \frac{N_{RS}(Q, q, t)}{N_{..}(Q, q, t)} \quad (\text{I.0.49})$$

Appendix J

Refined S-Duality

From string theory we can study the action of TST duality on the D4/D2/D0 system. Our goal is to understand how the chemical potentials in the index transform under this duality. If we simply wrapped our $D4$ -branes on a euclidean time circle of circumference λ , we would be computing the trace,

$$\mathrm{Tr}(-1)^F e^{-\lambda H} \quad (\text{J.0.1})$$

The contribution of a D4/D2/D0-brane bound state to this index is given by, $e^{-\lambda M}$, where M is the mass of the bound state and is equal to the magnitude of the central charge, $M = |Z|$.

We would like to identify the $D0$ and $D2$ chemical potentials with the corresponding $D0$ and $D2$ -brane masses. However, for a generic choice of the Kähler parameter, $k = B + iJ$, we are studying genuine D4/D2/D0 bound states whose mass is not equal to the sum of the constituent masses. This can be fixed by studying the special limit when $J = 0$, so that the D2-branes get all of their mass from a background B-field.¹ In this limit the central charges of the D0 and D2-branes align, which means that they form marginal bound states. In this case, the total mass of the bound state is simply equal to the sum of the D2/D0 masses.² Therefore, we can identify the D-brane masses with the chemical potentials,³

$$\phi_0 \leftrightarrow \frac{2\pi\lambda}{g_s}, \quad \phi_2 \leftrightarrow \frac{2\pi\lambda k}{g_s} \quad (\text{J.0.2})$$

To determine their transformation properties under TST duality, it is helpful first to recall how the coupling, g_s , transforms,

$$g_s \xrightarrow{T} \frac{g_s}{\lambda} \xrightarrow{S} \frac{\lambda}{g_s} \xrightarrow{T} \frac{\lambda^2}{g_s} \quad (\text{J.0.3})$$

¹A similar limit was used to derive wall-crossing formulas from M-theory in [12, 24].

²More precisely, the central charge for a bound state of N D4-branes, m D2-branes, and n D0-branes is given by $Z = N\Lambda^2 e^{2i\phi} + mk + n$ (see section 5.7 for details). Then using the fact that $\Lambda \gg 1$, the mass is given by, $M = |Z| \approx \frac{1}{2}N\Lambda^2 - (mk + n)\cos(2\phi)$.

³Throughout this section we set $2\pi\sqrt{\alpha'} = 1$.

Therefore,

$$\frac{2\pi\lambda}{g_s} \xrightarrow{TST} 4\pi^2 \left(\frac{g_s}{2\pi\lambda} \right) \quad (\text{J.0.4})$$

which implies that the D0-brane chemical potential transforms as,

$$\phi_0 \rightarrow \frac{4\pi^2}{\phi_0} \quad (\text{J.0.5})$$

To determine the transformation of the $D2$ -brane charge, we must simply follow the background B-field, k , which transforms as,

$$(B_{ij} = k) \xrightarrow{T} B_{ij} \xrightarrow{S} C_{ij} \xrightarrow{T} \left(C_{ijt} = \frac{k}{\lambda} \right) \quad (\text{J.0.6})$$

Therefore, the TST duality converts the background B-field into a background RR three-form which couples to the D2-brane as,

$$2\pi i \int C_{(3)} = 2\pi i \frac{k}{\lambda} \int dt = 2\pi i k \quad (\text{J.0.7})$$

Therefore, we have the transformation property,

$$\frac{2\pi\lambda k}{g_s} \xrightarrow{TST} 2\pi i k \quad (\text{J.0.8})$$

which implies that the D2-brane chemical potential transforms as,

$$\phi_2 \rightarrow 2\pi i \frac{\phi_2}{\phi_0} \quad (\text{J.0.9})$$

Now that we have explained how the unrefined potentials transform, we want to study the transformation properties of the spin character chemical potential, γ . First, recall that the inclusion of $e^{-\lambda\gamma J_3}$ in the trace is equivalent to considering the geometry,

$$(\mathbb{R}^2 \times S^1)_\gamma \times \dots \quad (\text{J.0.10})$$

where we rotate the $\mathbb{R}^2$ plane by γ as we go around the thermal circle. The metric for this geometry can be written explicitly as,

$$ds^2 = \frac{\lambda^2}{4\pi^2} dt^2 + (dx^i + \Omega^i dt)(dx^i + \Omega^i dt) \quad (\text{J.0.11})$$

where Ω is given by $\Omega = \gamma r^2 d\theta$ and (r, θ) are the coordinates on the $\mathbb{R}^2$ plane.

Applying T-duality, this metric is converted to first order into a background B-field,

$$B_{\theta t} = \alpha' \frac{g_{\theta t}}{g_{tt}} = \frac{\gamma r^2}{\lambda^2} \quad (\text{J.0.12})$$

Applying S-duality converts this into a RR two-form, $C_{\theta t}$, and after the final T-duality we are left with a RR one-form,

$$C_\theta = \frac{\gamma r^2}{\lambda} \quad (\text{J.0.13})$$

Now it is important to remember that the RR one-form couples to D0-branes as,

$$2\pi i \int C_{(1)} = 2\pi i \int \frac{\gamma r^2}{\lambda} d\theta = \frac{2\pi i}{\lambda} \int \gamma r^2 \frac{d\theta}{dt} dt \quad (\text{J.0.14})$$

Therefore, we can rewrite this coupling as,

$$\frac{2\pi i \gamma}{\lambda m_{\text{D0}}} \int J_3 dt = \frac{g_s i}{\lambda} \gamma \int J_3 dt = \frac{g_s i}{\lambda} \gamma \lambda J_3 \quad (\text{J.0.15})$$

This implies that under TST duality, γ transforms as $\gamma \rightarrow \gamma \frac{g_s}{\lambda}$. In terms of chemical potentials, this implies,

$$\gamma \rightarrow 2\pi i \frac{\gamma}{\phi_0} \quad (\text{J.0.16})$$

It might seem that after the TST duality, γ only couples to the angular momentum of D0-branes. However, the RR one-form also couples to D2-branes in the presence of a background B-field, and the same arguments go through,

$$2\pi i \int B_{(2)} \wedge C_{(1)} = 2\pi i k \int \frac{\gamma r^2}{\lambda} d\theta = \frac{2\pi i k \gamma}{\lambda m_{\text{D2}}} \int J_3 dt = \frac{g_s i}{\lambda} \gamma \int J_3 dt \quad (\text{J.0.17})$$

So far, we have only studied the transformation properties of the J_3 chemical potential, and have not studied the full combination $(J_3 - R)$. However, the R -symmetry can be realized as a geometric rotation in the Calabi-Yau, so the same arguments apply. Further, by supersymmetry we know that the chemical potential for R must transform in precisely the same way as the chemical potential for J_3 .

Combining all of these results together we find the potentials transform under TST duality as,

$$\phi_0 \rightarrow \frac{4\pi^2}{\phi_0}, \quad \phi_2 \rightarrow 2\pi i \frac{\phi_2}{\phi_0}, \quad \gamma \rightarrow 2\pi i \frac{\gamma}{\phi_0} \quad (\text{J.0.18})$$

This general result agrees nicely with the explicit transformation properties found in section 5.5.5.