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A MATRIX FORMULATION OF THE LOWRY MODEL FOR INTRAMETROPOLITAN ACTIVITY ALLOCATION

Introduction

In the past fifteen years many models have been developed in the field of urban planning. The earlier attempts focused mostly on the transportation system—traffic distribution or assignment—mainly because they were proposed for application in metropolitan transportation studies. More recently, however, increasing attention has been directed toward models for the allocation of activities in urban areas, concerning the distribution of residential, commercial, industrial, and other land uses over space.

A most important one among these, the Lowry Model, uses an iterative procedure to generate a distribution of employment and population in a region from a given distribution of employment in the basic sector. (The basic sector here is defined as consisting of those activities which are locationally exogenous, those which are considered to locate independently of the location of other activities.)

At each iteration, an additional population and the non-basic employment necessary to serve it are generated and distributed over the area. The procedure is stopped as soon as total population in the region reaches some predetermined value. For instance, in the Bay Area Simulation Study test model, only five iterations were needed to satisfy this condition.

However, such a constraint added merely to prevent an infinite number of iterations of the system is arbitrary; it is not necessary for the conceptual framework of the model. The initial purpose of the approach proposed here is to find a more logical way to deal with the infinite number of iterations needed. It soon appeared that other benefits could be obtained from this simple matrix formulation.

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Formulation of the Model

Let us consider a region with n zones (numbered from 1 to n). Basic employment is assumed to be given for the moment, and can be represented as an n -vector.

$$E^b = (E_1^b, E_2^b, \dots, E_n^b)$$

Residential location of basic employees The residential location of basic employees is found by means of a *journey to home function* a'_{ij} which distributes the residences of people working in any zone i over all zones j in accordance with observed behavior or some model.

Further multiplication by the labor participation rate in zone j , a_j will give the basic population living in j .

$$P^b = E^b A = E^b A' [a] \quad (1)$$

where $P^b = (P_1^b, P_2^b, \dots, P_j^b, \dots, P_n^b)$ is the basic population vector,

$$A = [a_{ij}] = [a'_{ij} a_j],$$

$A' = [a'_{ij}]$ is the *journey to home* distribution matrix,

$$\text{and } [a] = \begin{bmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_n \end{bmatrix}$$

is the labor participation rate diagonal matrix.

Location of population-serving activities. Non-basic employment necessary to serve population P_i^b is obtained as a fraction of P^b and then distributed over all zones j in terms of *journey to shop function* b'_{ij} .

This operation can be written

$$E^{(1)} = P^b B = P^b [\beta] B' \quad (2)$$

where the index 1 in $E^{(1)}$ is the iteration number, and where $B = [b_{ij}] = [\beta_i b'_{ij}]$.

$B' = [b'_{ij}]$ is the *journey to shop* distribution matrix,

$$\text{and } [\beta] = \begin{bmatrix} \beta_1 & & & 0 \\ & \beta_2 & & \\ & & \ddots & \\ 0 & & & \beta_n \end{bmatrix}$$

is the population-serving employment ratio, diagonal matrix.

Combining (1) and (2) we obtain

$$P^b = E^b A$$

$$E^{(1)} = P^b B = E^b (AB).$$

We can now perform the same sequence of operations—residential location now of all employees, location of population-serving activities—with non-basic employment $E^{(1)}$. This will generate a new non-basic employment $E^{(2)}$.

$$P^{(1)} = E^{(1)} A = E^b (AB) A$$

$$E^{(2)} = P^{(1)} B = E^{(1)} (AB) = E^b (AB) (AB) = E^b (AB)^2$$

It becomes apparent that the successive iterations will provide additional employments and populations:

$$E^{(2)} = E^b (AB)^2 \rightarrow P^{(2)} = E^b (AB)^2 A$$

$$E^{(3)} = E^b (AB)^3 \rightarrow P^{(3)} = E^b (AB)^3 A$$

$$\vdots$$

$$E^{(n)} = E^b (AB)^n \rightarrow P^{(n)} = E^b (AB)^n A$$

Therefore total employment and population will be equal to the following sums, where I denotes the identity matrix:

$$E = E^b + E^{(1)} + E^{(2)} + \dots + E^{(n)} + \dots$$

$$= E^b(I + AB + (AB)^2 + \dots + (AB)^n + \dots) \quad (3)$$

$$P = P^b + P^{(1)} + \dots + P^{(n)} + \dots$$

$$= E^b(I + AB + (AB)^2 + \dots + (AB)^n + \dots)A \quad (3')$$

Under certain conditions on the product matrix AB, which are satisfied in our case (see Appendix for more complete discussion) the matrix series (between parentheses) in (3) and (3') converges to the inverse of the matrix (I-AB). We can now write:

$$E = E^b(I - AB)^{-1} \quad (4)$$

$$\text{or } P = E^b(I - AB)^{-1}A \quad (4')$$

A Numerical Example

As an illustration,¹ consider the following 10-zones example in which we take simple *journey to home* and *journey to shop* functions:

$$a'_{ij} = b'_{ij} = \frac{e^{-0.1 t_{ij}}}{\sum_j e^{-0.1 t_{ij}}}$$

(where t_{ij} is the travel time from zone i to zone j in minutes) and $a_i = 2.5$ ($i = 1, 2, \dots, 10$)
 $\beta_i = 0.2$ ($i = 1, 2, \dots, 10$)

Interzonal travel times:

	1	2	3	4	5	6	7	8	9	10
1	2.00	5.25	3.22	10.50	4.94	9.44	15.86	7.94	10.94	12.66
2		1.50	3.75	5.25	5.47	9.97	10.61	8.47	11.47	13.19
3			1.50	7.72	1.72	6.22	13.08	4.72	7.72	9.44
4				2.00	6.00	10.50	5.36	6.00	9.00	10.72
5					1.50	4.50	11.36	3.00	6.00	7.72
6						2.00	15.22	6.86	3.86	3.86
7							2.00	8.36	11.36	13.08
8								2.00	3.00	4.72
9									1.50	1.72
10										2.00

$$A' = \begin{bmatrix} .172 & .124 & .152 & .073 & .128 & .081 & .043 & .095 & .070 & .059 \\ .117 & .170 & .136 & .117 & .114 & .073 & .068 & .085 & .063 & .053 \\ .123 & .117 & .146 & .078 & .143 & .091 & .046 & .106 & .078 & .066 \\ .069 & .118 & .092 & .163 & .109 & .069 & .116 & .109 & .081 & .068 \\ .099 & .094 & .136 & .089 & .139 & .103 & .052 & .120 & .089 & .075 \\ .075 & .071 & .103 & .067 & .123 & .157 & .042 & .097 & .131 & .131 \\ .054 & .091 & .071 & .154 & .084 & .057 & .216 & .114 & .084 & .071 \\ .076 & .072 & .105 & .092 & .125 & .085 & .073 & .138 & .125 & .105 \\ .060 & .057 & .083 & .073 & .099 & .123 & .058 & .134 & .156 & .152 \\ .056 & .053 & .078 & .068 & .092 & .136 & .054 & .125 & .169 & .164 \end{bmatrix} = B'$$

$$(I - AB)^{-1} = \begin{bmatrix} 1.097 & .100 & .117 & .094 & .120 & .097 & .068 & .111 & .101 & .090 \\ .095 & 1.102 & .115 & .098 & .118 & .095 & .073 & .111 & .099 & .089 \\ .095 & .099 & 1.115 & .094 & .119 & .098 & .069 & .112 & .102 & .092 \\ .089 & .099 & .111 & 1.101 & .116 & .095 & .078 & .112 & .102 & .091 \\ .092 & .097 & .114 & .095 & 1.119 & .099 & .070 & .113 & .104 & .093 \\ .089 & .093 & .111 & .092 & .117 & 1.103 & .068 & .113 & .109 & .100 \\ .086 & .097 & .107 & .104 & .114 & .093 & 1.086 & .113 & .103 & .092 \\ .089 & .094 & .111 & .095 & .117 & .099 & .072 & 1.114 & .107 & .097 \\ .087 & .091 & .109 & .093 & .116 & .103 & .070 & .115 & 1.111 & .101 \\ .086 & .090 & .108 & .092 & .116 & .104 & .070 & .115 & .112 & 1.103 \end{bmatrix}$$

A basic employment vector

$$E^b = (4500, 0, 0, 0, 9000, 0, 1000, 0, 0)$$

would then lead to the following distribution of total employment and population:

Zone	Basic Employment	Total Employment	Population
1	4500	5835	7126
2	0	1387	6673
3	0	1640	8383
4	0	1351	6033
5	0	1718	8801
6	9000	10472	8303
7	0	1000	4217
8	1000	2639	7709
9	0	1549	7898
10	0	1405	7352

where $E = E^b(I - AB)^{-1}$,
and $P = EA$.

Journey from Work to Shop

For simplicity we have assumed so far that all shopping trips were originated at the home. That is, a population P_i residing in zone i calls for $P_i \times \beta_i$ population-serving employees, and these were allocated to all zones j , according to a *journey from home to shop* function b_{ij} .

There is no need to maintain this assumption, and we can now consider how shopping trips originated at the place of work can be incorporated into the model. For example, we could assume as Lowry did that all trips

from work to shop are walking trips and terminate in the same zone where they originate.

A number of employees $E_i^{(1)}$, working in zone i will demand $E_i^{(1)} \times \gamma_i$ non-basic employees, which in turn are allocated to their places of work j in terms of a *journey from work to shop* function $C'_{ij}[C]$.

In matrix notation:

$$E^2 = E^{(1)}[\gamma]C' = E^{(1)}C \quad \text{where } \gamma = \begin{bmatrix} \gamma_1 & & 0 \\ & \gamma_2 & \\ & & \ddots \\ 0 & & & \gamma_n \end{bmatrix}$$

is the "employment-serving" employment ratio diagonal matrix and,

$$C = [C_{ij}] = [\gamma_i C'_{ij}], \text{ and} \\ C' = [C'_{ij}]$$

is the *journey from work to shop* distribution matrix.

Going back to the formulation previously presented, we should now, at each iteration, add non-basic employment generated (1) from home-based and (2) from place of work-based shopping trips. This yields:

$$\begin{aligned} E^b & & P^b &= E^b A \\ E^{(1)} &= P^b B + E^b C = E^b AB + E^b C = E^b (AB + C) & P^{(1)} &= E^{(1)} A = E^b (AB + C) A \\ E^{(2)} &= P^{(1)} B + E^{(1)} C = E^{(1)} AB + E^{(1)} C \\ & & &= E^b (AB + C)^2 \\ & \dots & & \\ E^{(n)} &= P^{(n-1)} B + E^{(n-1)} C = \dots & P^{(n)} &= E^{(n)} A \\ & & &= E^b (AB + C)^n A \end{aligned}$$

And totalling:

$$\begin{aligned} E &= E^b(I + AB + C + (AB + C)^2 + \dots \\ & \quad + (AB + C)^n + \dots) \\ &= E^b[I - (AB + C)]^{-1} \\ P &= E^b A + E^b (AB + C) A + \dots + E^b (AB + C)^n A \\ &= E^b[I - (AB + C)]^{-1} A. \end{aligned}$$

Evaluation and Extensions

Starting with the same basic assumptions as the Lowry Model, this formulation leads to a very simple and concise expression of the overall relationship between the distributions of population, basic, and non-basic employment. (Equations (4) and (4').) It makes directly apparent the net effect of various factors, such as labor participation rates, accessibilities, or other factors, that influence the locational behavior of the firm and the individual.

Consequently, it is possible to test different forms for the *journey to home* and *journey to shop* function, by applying equation (4) and (4') to past or present data. (For example, "least squares," methods can be used to estimate some parameters from these $2 \times n$ equations.) It is worth emphasizing that the only information needed to "calibrate" the model are population and employment—basic and non-basic—per zone. Although very useful, origin-destination studies could be spared if time or budget constraints make it necessary.

As for future predictions and planning, whether the *journey to shop* and *journey to work* functions are assumed to change or not, population and employment in all zones of the region can be derived from an expected future distribution of basic industries. Or taking the opposite approach to the problem, what policies in terms of public investment or basic industrial location could achieve a proposed pattern of development? More precisely, given a "desirable" population-vector P , one can solve the linear system of equations (4) and (4') and find whether the corresponding basic employment distribution is feasible; or else determine what public investment in transportation, housing, etc.; that is, what changes in the matrices A and B , would lead to such a pattern.

Finally, extensions of the model can be envisioned. First, one could think of introducing more than one class of population (by income, age groups, etc.), with different "tastes" for their residential locations and shopping needs, and dividing population-serving activities into several sectors. While such an expanded model would be more cumbersome to operate—with n population classes and p activity sectors, each of the matrices A and B would be composed of $n \times p$ submatrices ($n \times n$)—its formulation is not changed, and it could provide more insight into the determinants of development patterns.

The other dimension to be added is time. So far the model is static; that is, it relates population and employ-

ment distribution at time t with basic industries locations, accessibilities and other factors at time t . Obviously, different activities would have to be included in the basic sector for long-term and short-term analyses. This leads to the belief that much could be gained by introducing the time element into this formulation to build a dynamic, recursive model.

Appendix Convergence of Matrix Series

Before discussing mathematical conditions for the matrix series (3) to converge, a preliminary remark imposes itself: it is a fact of common experience that urban populations are finite; therefore, if the description of the process by which population-serving employees are generated is to be right, it should converge. We shall give only sufficient conditions.

$$\text{Let } \overline{a\beta} = \max_{k=1,n} \alpha_k \beta_k$$

$$\text{Then } AB = A'[\alpha][\beta]B' \leq A' \times (\overline{a\beta} I) \times B' = (\overline{a\beta}) \cdot A'B'$$

But since A' and B' are distribution matrices, the sum of the elements in any row are equal to 1:

$$\sum_j a'_{ij} = 1$$

$$\sum_j b'_{ij} = 1$$

and the same property holds for the product matrix $A'B'$. Therefore, we have an upper bound for the row sums of matrix AB .

$$\sum_j (ab)_{ij} = \sum_j \left(\sum_k a'_{ik} \alpha_k \beta_k b'_{kj} \right) = \sum_k a'_{ik} \alpha_k \beta_k \leq \overline{a\beta} \quad (i=1, \dots, n)$$

When $\overline{a\beta} < 1$, the spectral radius of the matrix AB , is less than one (Ref. 3—Lemma 2.5). Consequently, $I - AB$ is non singular and $I + AB + (AB)^2 + \dots = (I - AB)^{-1}$ the series on the left converging (Ref. 3—Theorem 3.7).

Although an improved upper bound for the spectral radius of AB might be necessary in some cases, the above argument will hold normally in practical applications where $\overline{a\beta} < 1$.

NOTE

¹ Data on travel times and basic employment are those of the theoretical town of LARTSVILLE developed at the Los Angeles Regional Transportation Study.

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A Note on the Garin-Lowry Model

Andrei Rogers

Garin's matrix formulation of the Lowry model [2] succinctly exhibits that model's fundamental character: it is a cross-sectional, static model that is *iterated* to an *equilibrium* solution. As in input-output models, the matrix formulation conveniently allows one to find this solution in one step by inverting the appropriate matrix. Recent research efforts in matrix models of population growth suggest a way of introducing, via recursion, a dynamic dimension to the Garin-Lowry model. In the process, one may identify another kind of equilibrium: that of *stationarity*. This is the distribution that arises after repeated application of an unchanging *recursive* growth process.

1. The Dynamic Garin-Lowry Model

Let us denote basic employment at time t by the n -vector:

$$E_t^b = (E_{t1}^b, E_{t2}^b, \dots, E_{tn}^b),$$

and population at time t by the n -vector:

$$P_t = (P_{t1}, P_{t2}, \dots, P_{tn}).$$

Then, using Garin's Equation 4', we have:

$$P_t = E_t^b (I - AB)^{-1} A. \quad (1)$$

Now consider a growth matrix G which when applied to the basic employment vector E_t^b projects it forward one time period, i.e.,

$$E_{t+1}^b = E_t^b G. \quad (2)$$

Then we may combine Equations 1 and 2 to form the following dynamic relationship:

$$P_{t+1} = E_{t+1}^b (I - AB)^{-1} A = E_t^b G (I - AB)^{-1} A, \quad (3)$$

and, if we assume for the moment that A , B , and G remain constant over time,

$$P_{t+1} = E_t^b G^n (I - AB)^{-1} A. \quad (4)$$

It is, of course, clear to everyone that the matrices A , B , and G will not remain constant through time and that therefore in a forecasting effort they would need to be predicted at each recursion of the model. Nevertheless, it is of some interest to derive what may be referred to as a "speedometer" reading of where we are headed if things do not change. In this way we could compare today's "speedometer" reading with yesterday's and see if the "horizon" projection to which we are headed has shifted and, if so, in what direction. This may be done by recursing the model until it reaches a stationary state, that is, an unchanging growth rate and spatial distribution.

2. The Stationary Growth Rate and Spatial Distribution

Recall Equation 4 and consider what happens as n , the number of time periods, gets very large, that is, as n approaches infinity. It can be shown [3] that in such in-

stances the vector of basic employment, and hence the population vector as well, stabilizes around a distribution that grows at a constant rate of growth and remains proportionately constant, that is, each sub-region's share of the total population and employment remains the same over time, even though the total itself increases at a constant rate. (Moreover, it can be shown that this constant rate of growth is the dominant characteristic root of the matrix G , and that the stationary distribution is the associated characteristic vector.)

3. A Numerical Example of Stationarity

As an illustration, consider Garin's example. His hypothetical data describe the population and employment distributions at a particular point in time, 1960, for a theoretical town, Lartsville. From his data source [1], we may borrow the hypothetical 1970 distribution of basic employment and assume that it was predicted by the following growth process:¹

$$E_{1970}^b = E_{1960}^b G$$

$$= (4500, 0, 0, 0, 0, 9000, 0, 1000, 0, 0)$$

$$\begin{bmatrix} 2.00 & 0 & 0 & 0 & 0 & .08 & 0 & .02 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ .20 & 0 & 0 & 0 & 0 & 1.10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ .10 & 0 & 0 & 0 & 0 & .04 & 0 & 1.41 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= (10900, 0, 0, 0, 0, 10300, 0, 1500, 0, 0).$$

Hence we can easily solve for P_{1970} by applying Equation 1 with Garin's values for A and $(I - AB)^{-1}$:

$$P_{1970} = (12053, 10912, 13520, 9412, 13761, 12158,$$

$$6992, 11980, 11698, 10741).$$

Now consider the stationary rate of growth and spatial distribution. It can be demonstrated that raising G to higher and higher powers ultimately yields the relationship:

$$G^{s+1} = v G^s, \quad (5)$$

where v , a constant, is the stationary rate of growth [3]. This means that at and beyond $n=s$, the system is stable and

$$E_{s+1}^b = E_s^b G = E_1^b G^s = v (E_1^b G^{s-1}) = v E_s^b, \quad (6)$$

and, therefore,

$$P_{s+1} = v P_s. \quad (7)$$

That is, at stationarity, each sub-region's share of the total population (or employment) remains constant and grows at a rate of v every period. Denote this allocation among the sub-regions by the vector W where:

$$W_{s1} = \frac{P_{s1}}{\sum_i P_{s1}} \quad (8)$$

Then, for our hypothetical city, we have the following proportional allocations over time:

Zone	Basic Employment			Total Population		
	1960	1970	Stat.	1960	1970	Stat.
1	.310	.480	.892	.098	.107	.127
2	.0	.0	.0	.092	.096	.108
3	.0	.0	.0	.116	.119	.130
4	.0	.0	.0	.083	.083	.085
5	.0	.0	.0	.121	.122	.123
6	.621	.454	.079	.115	.107	.093
7	.0	.0	.0	.058	.062	.058
8	.069	.066	.029	.106	.106	.105
9	.0	.0	.0	.109	.103	.091
10	.0	.0	.0	.102	.095	.080
Total	1.000	1.000	1.000	1.000	1.000	1.000

The stationary ten-year growth rate $\nu=2.0$. (Notice that during the ten-year period 1960–1970 the growth rate was 1.6.) Thus we may conclude that, as of today (today being 1960 or 1970, depending on whether the G was observed or predicted), the City of Lartsville is heading toward an ultimate rate of growth of 2.0 per decade and an ultimate spatial allocation of people to sub-regions as described above. Tomorrow's "speedometer" reading may indicate a sudden shift.

4. A Relation Between Population Growth and Employment Growth

Consider a growth matrix S which when applied to the population vector P_t projects it forward one time period [4], that is,

$$P_{t+1} = P_t S. \quad (9)$$

Each diagonal element of such a matrix denotes the sum of *one*, the proportion of people who remain in a particular sub-region during a unit time interval, and *two*, the natural increase (births over deaths) rate in that sub-region. Off-diagonal elements denote the proportion of people who migrate from one sub-region to another during the unit time interval.

Now consider the relationship between the basic employment growth matrix G and the population growth matrix S . Combining Equations 3 and 9, we find:

$$P_{t+1} = E_t^b G (I - AB)^{-1} A = P_t S. \quad (10)$$

But by Equation 1,

$$P_t = E_t^b (I - AB)^{-1} A.$$

Therefore,

$$P_{t+1} = E_t^b G (I - AB)^{-1} A = E_t^b (I - AB)^{-1} A S$$

or

$$E_t^b G M = E_t^b M S$$

where

$$M = (I - AB)^{-1} A.$$

Hence,

$$G M = M S \quad (11)$$

and

$$G = M S M^{-1} \text{ or } S = M^{-1} G M. \quad (12)$$

Thus, given a population growth matrix S we may find the basic employment growth matrix G , or vice versa. This provides a consistency check on both growth processes. It is of interest to note that the stationary growth rates implied by the two growth matrices must be equal.

NOTE

¹ Each diagonal element of G denotes the sum of *one*, the employment that remained in a sub-region during the ten-year time interval and *two*, the excess there of the entry of new employment ("births" and migration into Lartsville) over the exit of old employment ("deaths" and migration out of Lartsville). The off-diagonal elements denote the intracity migration of employment. For an example of how these might be predicted in a recursive model see: David R. Seidman, "A Linear Interaction Model for Manufacturing Location," unpublished manuscript, Penn-Jersey Transportation Study, January 15, 1964.

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