Some Case Studies in Algebra Motivated by Abstract Problems of Language

by

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Abstract

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This thesis concerns three different topics. The first has to do with axiomatizing the universal theory of certain classes of multisorted algebras arising from intersection, union, and other first order operations on relations. The second has to do with axiomatizing certain classes of actions arising from intersection and union, and axiomatizing certain classes of posets arising from actions arising from intersection. The third has to do with understanding under what conditions morphisms between two structures can be finitely determined.

Contents

Chapter 1

Introduction

1.1 Overview of Results

There are three different parts to this thesis. The first two parts both have to do with axiomatizing certain naturally arising classes of algebras, while the third part investigates conditions under which knowing finitely many values of a morphism determines all its values. The motivation for looking at each of these three situations lies for me ultimately in understanding language. Each situation deals implicitly with a particular model of some issue in language, and the results of the thesis help us to understand this model and its mathematical properties. Chapter 2 has to do with logical connectives like "and", "or", and "there exists". We model these as operations on relations. For example,"and" corresponds to intersection. Chapter 3 has to do with how the state of a conversation changes as sentences are asserted. The states are modeled as sets, and the assertion of a sentence amounts to intersecting and/or unioning this set with others corresponding to the sentence. Finally Chapter 4 has to do with acquiring only finitely much information yet managing to still learn a language and comprehend its infinitely many sentences. In this chapter we take as a starting point the view the that if a structure A represents the syntax of the language and a structure B in the same signature represents the semantics, then a morphism $f\colon A \to B$ is a way of compositionally assigning meaning (elements of B) to the sentences (elements of A). In each part of the thesis we are thus approaching an abstract problem of language by looking at mathematical questions about relevant models.

Having understood something of the general motivation for the investigations of this thesis, we now proceed to summarize the novel mathematical results. Chapter 2 gives an axiomatization of the universal theory of certain classes of multisorted algebras arising from intersection, union, and other first order operations on relations. A reasonable axiomatization of the Horn clause theory of these classes was already known [3]. In Section 2.3 we introduce an axiom (axiom (0)) which spells the difference between the Horn clause theory and the universal theory. Theorems 18, 22, 25, and 26 establish an axiomatization of the universal theory for various situations depending on which first order operations are included in the signature. The material for this chapter is based on my paper [17].

Chapter 3 has to do with axiomatizing certain classes of actions arising from intersection and union, and axiomatizing certain classes of posets arising from actions arising from intersection. The material having to do with actions comes from the paper [12] which is joint work with Alex Kruckman, while the material having to do with posets does not. An action is a pair of sets, C and S, and a function $f\colon C \times S \to C$. Rothschild and Yalcin gave a simple axiomatic characterization of those actions arising from set intersection, i.e. for which the elements of C and S can be identified with sets in such a way that elements of S act on elements of C by intersection [14]. These actions give rise to posets in a natural way (essentially put $cs \leq c$), and Section 3.5 presents a reasonable second order characterization of these posets, and shows that there is no first order axiomatization. Meanwhile, Theorems 47 and 54 axiomatically characterize two natural classes of actions which arise from both intersection and union. In the first class, the $\updownarrows$-actions, each element of S is identified with a pair of sets (s^-, s^+), which act on a set c by intersection with s^- and union with s^+. In the second class, the $\updownarrows$-biactions, each element of S is labeled as an intersection or a union, and acts accordingly on C. The class of $\updownarrows$-actions is closely related to a class of single-sorted algebras, which was previously treated by Margolis, Saliola, and Steinberg, albeit in another guise (hyperplane arrangements), and we note this connection [13].

Chapter 4 has to do with understanding under what conditions morphisms between two structures can be *finitely determined*. A pair of structures (A, B) is finitely determined when there is a finite subset A_0 of A so that if f and g are two morphisms from A to B that agree on A_0, then they agree on all of A. We examine a few related conditions and note their interrelationships in Proposition 92. We also introduce some questions concerned with generalizing finitely determined to a couple situations involving free monoids.

1.2 Background Information

We now present some general and background information that is relevant to the chapters that follow. The material here is generally speaking well known.

Chapters 2 and 3 involve finding axiomatizations for certain natural classes of structures. Suppose we are interested in a class of structures K. Then we have an *axiomatization problem*: Find a set of axioms T (generally of a specified form) which characterizes the structures in K up to isomorphism. Having selected a candidate set of axioms T, we are faced with a *representation problem*: Show that every "abstract" model of T is isomorphic to one of the "concrete" structures in K. Familiar examples include Cayley's theorem, which says that every abstract group is isomorphic to a subgroup of the full permutation group on some set, and Stone's theorem, which says that every abstract Boolean algebra is isomorphic to an algebra of sets. In this thesis we will be presenting more examples. However, let's first familiarize ourselves with the different kinds of axioms we'll be concerned with. We will review universal, Horn clause, and equational theories. We will examine fields as an example of how these theories differ in strength. In this introductory section it is also appropriate to review distributive lattices and Boolean algebras and give a version of Stone's result. Finally, we will examine one particular way natural classes of structures arise that's relevant to the $\upharpoonleft\!\downharpoonright$-actions and $\upharpoonleft\!\downharpoonright$-biactions of Chapter 3.

Universal, Horn Clause, and Equational Theories

In this subsection we review some well-known definitions and facts concerning universal, Horn clause, and equational theories.

For us signatures may contain relation and function symbols (including constants) of various finite arities. Sometimes we restrict attention to just relational signatures, and sometimes just functional signatures, which we call algebraic signatures.

A **universal sentence** is one of the form $\forall \bar{x}\varphi(\bar{x})$ where φ is quantifier-free. A universal theory is a collection of universal sentences. It's obvious that universal theories are closed under substructures, in the sense that any substructure of a model of a universal theory is again a model. In fact, there is a close relationship between the universal theory of a class of structures K and $S(K)$, the class of substructures of structures in K. We now proceed to examine this relationship in a bit more detail.

If you have a class K of structures (in some signature), and you are able to universally axiomatize $S(K)$, then this also axiomatizes the universal theory of K. The converse to this is true if you assume K is a pseudo-elementary class. It isn't true generally.

Definition 1. Let L be a signature. Let K be a class of L-structures. K is a *pseudo-elementary class* means that there is some signature $L^+ \supseteq L$ and some (first order) L^+-theory T^+ such that $\{M \upharpoonright L \mid M \models T^+\} = K$, i.e. K is the set of reducts of the models of T^+.

Proposition 2. *Let K be a pseudo-elementary class of L-structures. Let $T_\forall$ consist of the universal L-sentences true of every member of K. Then $T_\forall$ axiomatizes $S(K)$.*

Proof. If $A \in S(K)$, then of course $A \models T_\forall$.

Let $A \models T_\forall$. Because K is pseudo-elementary, there are L^+ and T^+ such that L is a subsignature of L^+ and T^+ is an L^+-theory whose class of L-reducts is K. Consider the L^+-theory $U := T^+ \cup \mathrm{diag}_L(A)$, where $\mathrm{diag}_L(A)$ denotes the quantifier-free theory of A where there is a new constant for each element of A. This theory is satisfiable lest $T^+ \models \forall \bar{x} \neg\varphi(\bar{x})$ with $\varphi(\bar{a}) \in \mathrm{diag}_L(A)$. So A embeds into a structure of K. □

Corollary 3. *A pseudo-elementary class of structures is closed under substructures iff it is universally axiomatizable.*

Proposition 4. *It is possible to have a class K of structures such that $T_\forall$ does not axiomatize $S(K)$.*

Proof. Let K consist of all the finite structures in the language with just equality. Then $S(K) = K$ and by the compactness theorem no first order theory, let alone universal theory, can axiomatize the class of finite structures. □

The following proposition will be used in Chapter 3.

Proposition 5. *Let K be a pseudo-elementary class which is closed under substructures, and let T be a universal theory. If every finitely generated model of T is in K, then every model of T is in K.*

Proof. By Corollary 3, K is elementary, axiomatized by a universal theory T_K. Given a model $A \models T$, we need to show that $A \models T_K$.

Let $\psi \in T_K$, written as $\forall \overline{x}\, \varphi(\overline{x})$, with φ quantifier-free, and let $\overline{a}$ be from A. Let $B_{\overline{a}}$ be the substructure of A generated by $\overline{a}$. Then $B_{\overline{a}} \models T$, since T is universal, and hence $B_{\overline{a}} \in K$, since it is finitely generated. So $B \models \psi$, $B_{\overline{a}} \models \varphi(\overline{a})$, and since φ is quantifier-free, $A \models \varphi(\overline{a})$. □

Now we turn to Horn clauses. A **Horn clause** is a formula of the form

$$(A_1 \wedge \cdots \wedge A_n) \to B$$

where the A_i and B are atomic formulas. We allow the case where $n = 0$, which just gives the atomic formula B. We often identify Horn clauses with the corresponding universal sentences. Of course Horn clause theories are closed under substructures (being universal), but Horn clause theories are also closed under products. To understand this, imagine you have a bunch of structures that satisfy an if-then Horn clause and they all satisfy the premises at some location, then they all must satisfy the conclusion too. For comparison, consider a formula of the form $A \vee B$. You could have a bunch of structures that each satisfy this, but different structures may make different choices as to which of A or B they satisfy.

Whereas universal theories are associated with $S(K)$, Horn clause theories are associated with $SP(K)$, the substructures of the products of members of K. (Of course in general, $P(K)$ denotes the class consisting of the products of members of K.) If K is a class of structures and you have a Horn clause axiomatization of $SP(K)$, then this also axiomatizes the Horn clause theory of K. Once again, for the converse we want some kind of additional information about K.

Proposition 6. *Assume that K is a class of structures such that $P(K)$ is pseudo-elementary. Let T_H be the set of Horn clauses true in all members of K. Then T_H axiomatizes $SP(K)$.*

Proof. If $M \in SP(K)$, then of course $M \models T_H$.

Let $M \models T_H$. Let $T_\forall$ be the universal theory of $P(K)$, which axiomatizes $SP(K)$ by Proposition 2. Assume to get a contradiction that there is some formula $\forall \bar{x} \theta(\bar{x}) \in T_\forall$ with $M \models \neg\theta(\bar{m})$ for some tuple $\bar{m}$ from M. We may assume that $\theta(\bar{x})$ has the form

$$\neg A_1(\bar{x}) \vee \cdots \vee \neg A_k(\bar{x}) \vee B_1(\bar{x}) \vee \cdots \vee B_l(\bar{x})$$

where the A_i and the B_j are atomic, and so

$$M \models A_1(\bar{m}) \wedge \cdots \wedge A_k(\bar{m}) \wedge \neg B_1(\bar{m}) \wedge \cdots \wedge \neg B_l(\bar{m})$$

Since $\forall\bar{x}\bigwedge_{i=1}^{k} A_i(\bar{x}) \leftrightarrow B_j(\bar{x})$ is a Horn clause for each $j = 1, \ldots, l$, and M doesn't satisfy them, there are structures $M_1, \ldots, M_l$ in K which don't satisfy these clauses. The product $\prod_{j=1}^{l} M_j$ of these structures doesn't satisfy $\forall\bar{x}\theta(\bar{x})$, a contradiction. □

Corollary 7. *A pseudo-elementary class of structures is closed under products and substructures iff it is Horn clause axiomatizable.*

The class $SP(K)$ is the smallest class containing K and closed under S and P. (The only thing to really check here is that $PSP(K) = SP(K)$.) This may be contrasted with $PS(K)$, which is not necessarily closed under S (e.g. let K be all the finite structures in the signature of just equality – then $S(K) = K$, and $PS(K)$ contains infinite sets but no countably infinite sets).

We may (vaguely) summarize the above propositions by saying that universal theories correspond to closure under substructures, and Horn clause theories correspond to closure under substructures and products. However, Horn clauses are further useful in defining the notion "recursively enumerable". There are other ways to define recursively enumerable, but this way seems particularly natural to the author.

Definition 8. Let τ be a finite algebraic signature (i.e. a signature with just function (and constant) symbols). Let F_τ be the τ-term algebra, i.e. the free τ-algebra generated by just the constants of the signature. Then a relation $r \subseteq F_\tau^n$ is recursively enumerable iff there is a finite Horn clause theory T in some signature (expanding τ and containing relation symbols) with a designated n-ary relation symbol R such that for all $\bar{t} \in F_\tau^n$ we have

$$T \vdash R(\bar{t}) \iff \bar{t} \in r$$

In fact, as is shown in [2], the signature of the finite Horn clause theory can even be assumed to be a relational expansion of τ and we still arrive at the same notion of recursively enumerable (which agrees with one given through say Turing machines).

Intuitively, recursively enumerable means that we may algorithmically list the elements of the relation, being sure we will eventually list everything that is actually in the relation and not list anything that is not. Given the finite Horn clause theory T we may start listing all the proofs of atomic sentences using T as axioms (these proofs are particularly simple, just being trees where atomic sentence nodes are justified by their successors using a substitution instance of a Horn clause in T), and in this way obtain a list of the relation r. Other notions of algorithms to list relations tend to be easy to simulate by the finite Horn clause theories.

Finally, we turn to equations. An **equation** is an atomic formula whose relation symbol is the equality symbol in a signature with equality. That is, something of the form $t_1 = t_2$ where t_1 and t_2 are terms. We often identify equations with the corresponding universal sentences. Equational theories are closed under substructures and products, being Horn clauses after all, and further they are closed under homomorphic images. Let K be a class of structures. If you have an equational axiomatization of $HSP(K)$ — H stands for homomorphic images — then you also have an axiomatization of the equational theory of K. In fact, the converse is also true without any additional assumptions. This is essentially Birkhoff's HSP theorem. For a reference, one may look at the textbook [4] Theorem 11.9, p. 83.

A motto to remember about equational theories is that they are the ones closed under substructures, products, and homomorphic images.

An Example: Fields

For some classes of structures K the universal, Horn clause, and equational theories are all equivalent. For example, this happens with Boolean algebras. For some classes, only some are equivalent. For example, the first order algebras we'll consider later have equivalent Horn clause and equational theories, but these are different from the universal theory. To better understand the difference between the universal, Horn clause, and equational theories, let's look at a (hopefully familiar) example where they all are inequivalent. Let K be the class of fields, in the signature $(0, -, +, 1, \cdot)$. Fields are commutative rings with identity (which I call rings) with the property that nonzero elements have multiplicative inverses.

The universal theory of fields (in this signature) is the theory of integral domains. Integral domains are rings that also satisfy:

$$\forall x, y(xy = 0 \implies (x = 0 \vee y = 0))$$

Note that this is not a Horn clause. Since this is universal and true of fields, this is also true of all subalgebras of fields (and the ring axioms are all universal too). Conversely, given an integral domain, we may form its field of fractions, which the integral domain embeds into. Thus, we have universally axiomatized the class $S(\text{Fields})$, and so we have found (as pointed out in the previous section) the universal theory of fields.

The Horn clause theory of fields (in this signature) is the theory of reduced rings, which are rings with no nilpotents. In other words, rings plus the following axioms:

$$\forall x(x^n = 0 \implies x = 0) \quad (n \geq 1)$$

Since this is a Horn clause true of fields, it is true of $SP(\text{Fields})$. Conversely, suppose that we have a ring R with no nilpotents (other than 0). We wish to find an embedding of R into a product of fields. It suffices to find for each distinct pair $r \neq s \in R$ a morphism φ from R to a field with $\varphi(r) \neq \varphi(s)$. Since integral domains embed into fields, it is actually sufficient to find a morphism φ from R to an integral domain with $\varphi(r) \neq \varphi(s)$.

Any quotient of a ring by a prime ideal is an integral domain. So we're done if we can find a prime ideal P of R such that $r - s \notin P$. Let t denote the nonzero element $r - s$. Let $\mathbb{I}$ consist of the ideals I of R such that $t^n \notin I$ for $n \geq 0$. Then $\mathbb{I}$ is nonempty because $I = \{0\}$ is in it. As $\mathbb{I}$ is closed under unions of chains, by Zorn's lemma there is a maximal element P of $\mathbb{I}$. We claim P is in fact prime. Assume to get a contradiction that we have $xy \in P$ with $x, y \notin P$. Let $A := \{z \mid xz \in P\}$. Then A is an ideal containing P, but also $y \notin P$, so there is an n such that $t^n \in A$. Next let $B := \{z \mid t^n z \in P\}$. Then B is an ideal containing P, but also $x \notin P$, so there is an m such that $t^m \in B$. Thus, $t^n t^m = t^{n+m} \in P$, a contradiction.

Finally, the equational theory of fields is rings. To see this it suffices to show that HSP(Fields) is the class of rings. Of course everything in HSP(Fields) is a ring because the ring axioms are equational. To show the other direction, we need only show that every ring is a homomorphic image of a reduced ring, in view of our above result. But this is easy because the free rings have no nilpotents.

Distributive Lattices

Now we turn to a class of structures more closely related to the algebras of Chapter 2. Consider the propositional formula φ with two proposition letters P and Q given by $P \wedge \neg Q$. We can think of such a formula φ as giving rise to an operation on subsets of a set. If p and q are subsets of some set W, then $\varphi(p, q) := p \cap (W - q)$ is also a subset of W. This function $\varphi\colon \mathcal{P}(W) \times \mathcal{P}(W) \to \mathcal{P}(W)$ accepts as input two subsets of W and outputs a subset. Indeed, every propositional formula gives rise to a finitary operation $\mathcal{P}(W)^n \to \mathcal{P}(W)$. In this way we arrive at a functional signature τ where there is a function symbol of arity n for every propositional formula involving n proposition letters, and we have for every set W a τ-algebra we call a *powerset algebra*.[1] Of course, we do not work in practice with the full signature τ, but rather we isolate just some of the operations which compositionally generate all the others in the class of algebras of interest. One convenient choice is $0, 1, \vee, \wedge, \neg$ where 0 and 1 are the constants interpreted by $\emptyset$ and W in the powerset algebra determined by W. If instead of all propositional formulas we want to focus on the positive propositional formulas, then one convenient choice is $0, 1, \vee, \wedge$.

In this section we start by considering the structures $(\mathcal{P}(W); \emptyset, W, \cup, \cap)$ where W is any set, the powerset algebras (in the positive signature). We will find that the universal theory is axiomatized by distributive lattices (some straightforward equations). If we add complementation, then the universal theory is axiomatized by Boolean algebras. This motivates interest in trying to do the same thing in more expressive logics involving relations of higher arity. Furthermore, it turns out this result (Proposition 10) is not only used in the more general case, but it also serves as a model for other arguments. So it is crucial to understanding the general argument.

Definition 9. A **distributive lattice** is an algebra in the signature $(0, 1, \vee, \wedge)$ that satisfies the following axioms.

1. $\vee$ and $\wedge$ are both idempotent, commutative, and associative (so that they each define a partial order: $r \leq_\wedge s \iff r \wedge s = r$ and $r \leq_\vee s \iff r \vee s = s$)

2. $\vee$ and $\wedge$ satisfy "absorption", i.e. $r \wedge (r \vee s) = r$ and $r \vee (r \wedge s) = r$ (so that the partial orders they define are the same; $\wedge$ gives the meets and $\vee$ gives the joins)

3. $0 \leq r \leq 1$

[1] By "algebra" I mean a structure in a signature with only function symbols (including constants)

4. Distributivity: $r \vee (s \wedge t) = (r \vee s) \wedge (r \vee t)$ (in fact $\wedge$ distributing over $\vee$ follows from this, as we'll see)

If we're able to show that every distributive lattice embeds into a powerset algebra, then we will have found an equational axiomatization of the subalgebras of powersets algebras, because of course the distributive lattice axioms are true in powersets.

Proposition 10. *Every distributive lattice embeds into a powerset algebra.*

The class of powerset algebras is closed under products, because $\mathcal{P}(\biguplus W_i) \cong \prod \mathcal{P}(W_i)$. So it suffices to show that for any two distinct points $r \neq s$ in a distributive lattice L, there is a morphism φ from L to some powerset algebra with $\varphi(r) \neq \varphi(s)$. The following lemma gives us a way to achieve this.

Definition 11. A **filter** is a subset of a distributive lattice that is closed under $\wedge$, upward-closed, contains 1, and doesn't contain 0. A **prime filter** is a filter F which satisfies $r \vee s \in F$ implies $r \in F$ or $s \in F$.

Here is an example of a filter which is not a prime filter. Let W be a set containing at least two distinct elements x and y. Let F be the subsets of W which contain both x and y. Then F is a filter of $\mathcal{P}(W)$. However, it is not prime: $\{x\} \cup \{y\} \in F$ but neither $\{x\}$ nor $\{y\}$ is in F.

The idea behind prime filters is that they are the possible types of an element. (Let me explain more formally what I mean: If we have a distributive lattice L which is a subalgebra of a powerset algebra $\mathcal{P}(W)$ then the collection of things in L containing some fixed element $x \in W$ (the "type" of x) forms a prime filter and conversely given a prime filter F of some distributive lattice $L \subseteq \mathcal{P}(W)$ we may add a new element x to W forming W' and get a distributive lattice $L' \subseteq \mathcal{P}(W')$ isomorphic to L for which F is determined in this way by x.) Filters are an approximation to prime filters, useful in building prime filters with certain properties.

Lemma 12. *Let F be a prime filter for some distributive lattice L. Let $W = \{F\}$. Then the function $\varphi \colon L \to \mathcal{P}(W)$ defined by*

$$F \in \varphi(r) \iff r \in F$$

is a morphism.

Proof.

1. $F \in \varphi(0)$ iff $0 \in F$, so $\varphi(0) = \emptyset$.
2. $F \in \varphi(1)$ iff $1 \in F$, so $\varphi(1) = W$.
3. $F \in \varphi(r \wedge s)$ iff $r \wedge s \in F$ iff $r, s \in F$ iff $F \in \varphi(r) \cap \varphi(s)$
4. $F \in \varphi(r \vee s)$ iff $r \vee s \in F$ iff $r \in F$ or $s \in F$ (because F is prime) iff $F \in \varphi(r) \cup \varphi(s)$

□

Proof of Proposition 10

From the above lemma, we see that it is enough to find, given two distinct elements $r \neq s \in L$, a prime filter F containing one but not the other. Without loss of generality, let $s \not\geq r$. We will find a prime filter F with $r \in F$ and $s \notin F$. Let $\mathbb{F}$ consist of the filters on L that contain r and don't contain s. This is nonempty because $\{t \in L \mid t \geq r\}$ is such a filter. As $\mathbb{F}$ is closed under unions of chains, there is a maximal element F. We claim that F is prime. Let $t_1 \vee t_2 \in F$.

I claim that if $s \not\geq t_i \wedge f$ for every $f \in F$, then in fact $t_i \in F$. To see this, let $F^+ := \{u \in L \mid \exists f \in F \text{ with } u \geq t_i \wedge f\}$. Then F^+ is a filter containing F, t_i, and not s, so $F^+ \in \mathbb{F}$ and in fact $t_i \in F^+ = F$.

So suppose that we have $f_1, f_2 \in F$ such that $s \geq t_1 \wedge f_1$ and $s \geq t_2 \wedge f_2$ in order to get a contradiction. We get

$$\begin{aligned} s &\geq (t_1 \wedge f_1) \vee (t_2 \wedge f_2) \\ &= (t_1 \vee t_2) \wedge (t_1 \vee f_2) \wedge (f_1 \vee t_2) \wedge (f_1 \vee f_2) \\ &\in F \end{aligned}$$

And so $s \in F$, a contradiction.
End of Proof of Proposition 10

We may easily modify the above argument so as to include negation. The only part that needs extra work is in the lemma. We need to make sure the function defined by $F \in \varphi(r) \iff r \in F$ satisfies $\varphi(\neg r) = \neg\varphi(r)$. I.e., we want it to be the case that $\neg r \in F$ iff $r \notin F$. We can accomplish this by adding to the distributive lattice axioms two additional axioms: $r \vee \neg r = 1$ and $r \wedge \neg r = 0$. The resulting theory is called the theory of Boolean algebras.

We used Zorn's lemma to prove Proposition 10. However, it is known that the reverse implication (modulo ZF) is not true [7]. Proposition 10 is also known to be independent of ZF [6].This proposition has a few equivalent reformulations that we'll make use of in what follows. These are: (i) the compactness theorem for first order logic, (ii) the fact that there are prime filters extending filters in any distributive lattice, and similarly (iii) when we have a filter disjoint from an ideal in a distributive lattice, there is a prime filter extending the filter and not containing anything from the ideal. Proposition 10 and each of (i), (ii), and (iii) are known to be equivalent [8].

Turning Disjoint Unions Into Products

In the examples of Cayley's theorem and Stone's theorem, as well as in the first order algebras of Chapter 2 and the $\Updownarrow$-actions and $\Updownarrow$-biactions of Chapter 3, the class K to be axiomatized is the class of substructures of some "full" structures. Then the representation problem becomes the problem of embedding each model of T, the candidate axioms, into one of these full structures.

When the full structures are obtained from sets by a construction which turns disjoint unions of sets into products of structures (e.g. in the case of Boolean algebras, but not in the case of groups), the class K is controlled by the full structure on the one element set, in a way we will now make precise.

Let L be a signature. Fix an operation F associating to each set X an L-structure $F(X)$, such that

1. If there is a bijection between X and Y, then there is an isomorphism between $F(X)$ and $F(Y)$, and

2. F turns disjoint unions of sets into products of structures, i.e.

$$F\left(\coprod_{i\in I} X_i\right) \cong \prod_{i\in I} F(X_i).$$

Call the structures in the image of F full, and let K be the class of (structures isomorphic to) substructures of full structures.

Proposition 13. *Let F and K be as defined above, and let 1 be the one element set $\{*\}$.*

1. *The class K is closed under substructure and product.*

2. *If K is pseudo-elementary, then it is elementary, axiomatized by the Horn clause theory of the structure $F(1)$.*

Proof. (1): K is closed under substructure by definition. If $\{A_i\}_{i\in I}$ is a collection of structures in K, then each A_i embeds in some full structure $F(X_i)$. Then $\prod_{i\in I} A_i$ embeds in $\prod_I F(X_i) \cong F(\coprod_I X_i)$, so $\prod_I A_i$ is in K.

(2): Any pseudo-elementary class closed under substructure and product is axiomatizable by a Horn clause theory, by Corollary 7. Observe that for all X, X can be expressed as an X-indexed disjoint union of copies of 1: $X = \coprod_{x\in X} 1$. So $F(X) \cong F(\coprod_{x\in X} 1) \cong \prod_{x\in X} F(1)$. Hence every structure A in K embeds into a product of copies of $F(1)$. Let φ be a Horn clause. If φ is true in every structure in K, then clearly it is true of $F(1)$. Conversely, if φ is true of $F(1)$, then since every A in K is isomorphic to a substructure of a product of copies of $F(1)$, and Horn clauses are preserved under substructures and products, φ is true of A. $\square$

This proposition does not apply to the first order algebras of Chapter 2 but it will apply to the $\upharpoonleft\!\downharpoonright$-actions and the $\upharpoonleft\!\downharpoonright$-biactions considered in Chapter 3.

Chapter 2

First Order Algebras and Various Reducts

In this chapter we investigate logical connectives like "and" ($\wedge$), "or" ($\vee$), "not" ($\neg$), and "there exists" ($\exists$) in an algebraic way. We regard logical connectives as operations on relations. For example "and" corresponds to intersection of relations. Our goal is to understand better how these operations relate to each other. An example of one well-known fact about the connectives is that $\neg(R \wedge S) = \neg R \vee \neg S$. But many other facts are true too, and so we are in fact interested in finding some axioms from which all such facts follow. This was done in the case of the propositional connectives (and, or, not) by Stone in 1936 [16]. Our main topic of first order connectives has been studied too. Our formalism of choice for discussing this topic is a multisorted one. We will be dealing with first order algebras, to be defined in the next section, which are multisorted algebras of relations with the possible arities of the relations being the sorts. Schwartz [15] and Börner [3] have studied these structures, but they focused on the Horn clause theory of them: we shall be axiomatizing the universal theory.

Cylindric algebras provide another formalism for investigating first order operations on relations. The locally finite regular cylindric set algebras, in symbols $\mathrm{Cs}_\omega^{\mathrm{reg}} \cap \mathrm{Lf}_\omega$, roughly correspond to our multisorted algebras. There is a characterization of their universal theory, due to Andréka and Németi, findable in the second volume of Cylindric Algebras (Thm 4.1.48, p. 127 and 129) [9]. However, it's not clear how to translate between the two formalisms, and the axiomatizations and proofs seem different. Also, these results have all the first order operations present, while the argument here explicitly addresses various reducts as well.

2.1 Definition of First Order Algebras

Instead of having an operation for every propositional formula, as was discussed in the Distributive Lattices portion of Section 1.2, let us have an operation for every first order formula in a finite relational signature. For example, if θ is the formula $\exists y[R_1(x, y) \wedge R_2(x, y)]$, then as an operation on relations θ accepts as input two binary relations $r_1, r_2 \subseteq W^2$ and outputs the unary relation $\{x \in W \mid \exists y[r_1(x, y) \wedge r_2(x, y)]\}$ which is a projection of their intersection. For the next example it is important to note that we consider a formula to come specifically equipped with a *variable context* that contains the free variables of the formula but could contain variables otherwise not explicitly occurring in the formula. Let θ now be the formula $R(x, y)$ in the variable context (x, y, z). Then as an operation θ accepts as input a binary relation $r \subseteq W^2$ and outputs the 3-ary relation $\{(x, y, z) \in W^3 \mid (x, y) \in r\}$. We will be calling such an operation a *cylindrification*. More generally, let σ be the finite relational signature consisting of the relation symbols $R_1, \ldots, R_n$ of arities $m_1, \ldots, m_n$ respectively. Let $\theta(\bar{x})$ be a first order σ-formula in the variable context $x_1, \ldots, x_k$. Let W be any set. Then θ induces a function on relations $\theta\colon \mathcal{P}(W^{m_1}) \times \cdots \times \mathcal{P}(W^{m_n}) \to \mathcal{P}(W^k)$ defined by

$$\theta(r_1, \ldots, r_n) := \{\bar{x} \in W^k \mid (W, r_1, \ldots, r_n) \models \theta(\bar{x})\}$$

where $(W, r_1, \ldots, r_n)$ denotes the σ-structure where each relation symbol R_i is interpreted as r_i and $\models$ denotes the usual notion of satisfaction.

We see that the operations arising from first order formulas are a little bit different from those arising from propositional formulas in that now there are different sorts $\mathcal{P}(W^0)$, $\mathcal{P}(W^1)$, $\mathcal{P}(W^2)$, etc. Instead of a single-sorted algebraic signature, the first order formulas naturally give rise to a multisorted algebraic signature where there is a sort for each natural number. Every set W gives rise to a multisorted algebra in this signature, with the operations as defined above. We call the algebras that arise in this way *first order algebras*.

Instead of dealing with a signature where there is a function symbol for every first order formula, it suffices to deal with a subsignature which will compositionally generate all the operations of interest. There is of course some degree of choice here, and we have generally chosen so as to make our axioms and arguments to follow more conveniently stated. Below is the (largest) signature we will use. Note the convention that "$x\colon A$" indicates x is a constant of sort A, and "$x\colon A \to B$" indicates that x is a function symbol with domain A and codomain B – in this case A may be a sort or a product of sorts.

Definition 14. The multisorted **signature of first order algebras** (with equality) is given as follows.

- We have a sort n for each natural number $n \in \{0, 1, 2, \ldots\}$. The sort n is intended to consist of n-ary relations on some set.
- For each function $\alpha\colon \{1, \ldots, n\} \to \{1, \ldots, k\}$ we have a function symbol $\alpha\colon n \to k$. These are called **substitutions** and will correspond to the operations arising from atomic formulas.
- For each n we have a constant symbol 0^n belonging to sort n, which we may write as $0^n\colon n$. Likewise we have constant symbols $1^n\colon n$ and function symbols $\vee^n\colon n \times n \to n$, $\wedge^n\colon n \times n \to n$, and $\neg^n\colon n \to n$. We usually omit the superscript and write simply $0, 1, \vee, \wedge, \neg$, leaving the arity implicit to the context.

- For each n we have a function symbol $\exists^n \colon n+1 \to n$, which we will generally write $\exists$. This will correspond to projection or existential quantification of the last coordinate.
- For each n with $1 \leq i, j \leq n$ we have a constant $\Delta^n_{i,j} \colon n$. These will correspond to equality of various coordinates.

Before going further, let us introduce some notation involving the substitutions. A function $\alpha \colon \{1, \ldots, n\} \to \{1, \ldots, k\}$ can also be described as a sequence of length n with repetition allowed taken from a k-element set. Thus, α gives a way of transforming any k-tuple into an n-tuple. Let W be a set. Define $\alpha^{\text{tuple}} \colon W^k \to W^n$ to be the obvious function induced on tuples. In detail, $\alpha^{\text{tuple}}(x_1, \ldots, x_k) := (x_{\alpha(1)}, \ldots, x_{\alpha(n)})$. This function on tuples in turn induces a function on relations of particular interest, the inverse image. I.e., define $\alpha^{\text{relation}} \colon \mathcal{P}(W^n) \to \mathcal{P}(W^k)$ by $\alpha^{\text{relation}}(r) := \{\bar{x} \mid \alpha^{\text{tuple}}(\bar{x}) \in r\}$. An atomic formula like $R(x, x, y, x)$ in the variable context (x, y, z) corresponds to the substitution $\alpha^{\text{tuple}}(x, y, z) = (x, x, y, x)$. Both give rise to the same operation on relations which accepts as input a 4-ary relation r and outputs a 3-ary relation $\{(x, y, z) \mid (x, x, y, x) \in r\}$. We generally use lower case Greek letters near the beginning of the alphabet to denote substitutions, e.g. α, β, γ.

Definition 15. We say a **cylindrification** is a substitution where the function $\alpha \colon \{1, \ldots, n\} \to \{1, \ldots, k\}$ is increasing. In other words, as a tuple of symbols $\alpha^{\text{tuple}}(x_1, \ldots, x_k)$ is a subtuple of $(x_1, \ldots, x_k)$. We often use lowercase c to denote cylindrifications. A collection $c_1, \ldots, c_m$ of cylindrifications are called **partitioning cylindrifications** when they take the form $c_i^{\text{tuple}}(\bar{x}_1, \cdots, \bar{x}_m) = \bar{x}_i$. I.e., we have $c_i \colon k_i \to n$ where $n = k_1 + \cdots + k_m$, and the function $c_i \colon \{1, \ldots, k_i\} \to \{1, \ldots, n\}$ is given by $c_i(l) = l + \sum_{j=1}^{i-1} k_j$.

Here is an example of some partitioning cylindrifications: $c_1^{\text{tuple}}(x, y) = x$ and $c_2^{\text{tuple}}(x, y) = y$, so that $c_1^{\text{relation}}(r) = \{(x, y) \mid x \in r\}$ and $c_2^{\text{relation}}(s) = \{(x, y) \mid y \in s\}$.

We use id to denote the identity substitution $\text{id} \colon n \to n$ for each n. Let $\alpha \colon n \to k$ and $\beta \colon k \to m$ be two composable substitutions. Note that $(\beta \circ \alpha)^{\text{relation}} = \beta^{\text{relation}} \circ \alpha^{\text{relation}}$. To be able to say this axiomatically, we need different notation for the two compositions. We shall use $(\beta \circ \alpha)(r)$ for the former and $\beta(\alpha(r))$ for the latter.

Now we define the classes of algebras of interest to us (in our leaner signature).

Definition 16. A **first order algebra** (with equality) is an algebra in the multisorted signature specified in Defintion 14 that arises from some set W in the following way:

- The interpretation of the sort n is $\mathcal{P}(W^n)$, the collection of all n-ary relations on W.
- The interpretation of a substitution $\alpha \colon n \to k$ is

$$\alpha^{\text{relation}} \colon \mathcal{P}(W^n) \to \mathcal{P}(W^k)$$

  as defined above.

- The Boolean operations $0, 1, \vee, \wedge, \neg$ are interpreted as usual on each sort.
- Projection $\exists^n : n + 1 \to n$ is interpreted as expected. In detail,

$$\exists^n(r) := \{(x_1, \ldots, x_n) \mid \exists y((x_1, \ldots, x_n, y) \in r)\}$$

- The constant $\Delta^n_{i,j} : n$ is interpreted as

$$\Delta^n_{i,j} := \{(x_1, \ldots, x_n) \mid x_i = x_j\}$$

We will have occasion to look at various reducts of our signature, and the corresponding reducts of the first order algebras are given appropriate names. E.g., the **positive existential algebras** (without equality) are the reducts of the first order algebras to the signature not containing negation (for any sort), and not containing the constants for equality, but otherwise containing all the symbols. The **positive quantifier-free algebras** (without equality) are when we restrict attention to just the substitutions and the lattice operations $0, 1, \vee, \wedge$ for each sort.

Just as all first order formulas can be constructed from the atomic formulas using the Boolean connectives and existential quantification, so too is every operation on relations arising from a first order formula equivalent to a term in our signature when looking at the first order algebras. Similarly, there are terms for every positive existential formula in the positive existential algebras, etc.

2.2 Completeness Theorem and Horn Clause Theory

In this section we point out a well known connection between the completeness theorem for first order logic and the first order algebras introduced in the previous section. Another way to understand the completeness theorem for first order logic is as a (reasonable) axiomatization of the Horn clause theory of first order algebras.

It's important for this section to realize that terms in the first order algebra signature are essentially the same as the usual (relational) first order formulas.

Consider a completeness theorem which says that $\Sigma \models \varphi$ iff $\Sigma \vdash \varphi$ where Σ is a finite collection of first order formulas and φ is a first order formula, $\models$ is inductively given the usual meaning, and $\vdash$ is defined via some proof calculus. We assume that there is some (recursively enumerable) collection of axioms and some (recursively enumerable) collection of proof rules (e.g. modus ponens) yielding the definition of $\vdash$. $\Sigma \vdash \varphi$ iff there is a (finite) tree with the property that each leaf is an axiom or among Σ, and each non-leaf is justified by its predecessors using a proof rule.

Given such a completeness theorem, we may obtain an axiomatization of the Horn clause theory of first order algebras as follows. First of all, we write down two Horn clause axioms

$$r = s \iff (r \leftrightarrow s) = 1$$

Here $r \leftrightarrow s$ is an abbreviation for $(r \wedge s) \vee (\neg r \wedge \neg s)$ because recall technically we don't have "$\leftrightarrow$" in our signature. Next we write down

$$(\sigma_1 = 1) \wedge \cdots \wedge (\sigma_n = 1) \implies \varphi = 1$$

whenever

$$\{\sigma_1, \ldots, \sigma_n\} \vdash \varphi$$

This is a Horn clause axiomatization of the Horn clause theory of first order algebras. First observe that these are both true of first order algebras: If $\{\sigma_1, \ldots, \sigma_n\} \vdash \varphi$, then by assumption $\{\sigma_1, \ldots, \sigma_n\} \models \varphi$, which is the same as the corresponding Horn clause being true in first order algebras. Let

$$\bigwedge(\sigma_i = \tau_i) \implies \varphi = \chi$$

be in the Horn clause theory. Then

$$\bigwedge((\sigma_i \leftrightarrow \tau_i) = 1) \implies (\varphi \leftrightarrow \chi) = 1$$

i.e.

$$\{\sigma_i \leftrightarrow \tau_i\} \models \varphi \leftrightarrow \chi$$

and so by our assumption of a completeness theorem we get a $\vdash$ proof of this. So

$$\bigwedge((\sigma_i \leftrightarrow \tau_i) = 1) \implies (\varphi \leftrightarrow \chi) = 1$$

is among our Horn clause axioms, and we thus get a proof of

$$\bigwedge(\sigma_i = \tau_i) \implies \varphi = \chi$$

Conversely, given a (recursively enumerable) Horn clause axiomatization T of the Horn clause theory of first order algebras, we may get a completeness theorem for first order logic as follows. Given an atomic Horn clause axiom $\varphi = \chi$, we write an axiom $\varphi \leftrightarrow \chi$. Given a proper non-atomic Horn clause axiom $\bigwedge_i(\sigma_i = \tau_i) \implies \varphi = \chi$, we write a proof rule "From $\sigma_i \leftrightarrow \tau_i$ obtain $\varphi \leftrightarrow \chi$". For both of these we allow substitution instances. We also add in proof rules to go between $\theta \leftrightarrow 1$ and θ and also rules essentially saying $\leftrightarrow$ is a congruence relation. The resulting proof system is certainly sound. If $\sigma_1, \ldots, \sigma_n \models \varphi$, then $\bigwedge(\sigma_i = 1) \implies \varphi = 1$ is in the Horn clause theory, and so the Horn clause axioms T imply it by our assumption that T axiomatizes the Horn clause theory of first order algebras. An easy to verify "Horn clause completeness theorem" (for a reference see, e.g., [10]) then yields a nice proof thereof consisting of a tree whose root is $\varphi = 1$, whose leaves are $\sigma_i = 1$ or instances of atomic Horn clauses in T, and whose interior nodes are justified by their predecessors using an instance of a Horn clause in T or justified by the fact that $=$ is a congruence relation. This proof is also a "proof calculus" proof of $\varphi \leftrightarrow 1$ from $\sigma_i \leftrightarrow 1$, and so we get a proof of φ from the σ_i as well.

2.3 Positive Quantifier-free Algebras

The core of our argument about the first order algebras and various reducts can already be illustrated with the positive quantifier-free algebras, where our signature is restricted to the substitutions and the lattice operations $0, 1, \vee, \wedge$ for each sort. The kind of operations an relations you can get in this situation is limited; for example, you can't express composition of binary relations. We begin by presenting the universal axioms which we will see axiomatize the subalgebras of the positive quantifier-free algebras — this is our goal in this section. Note that I have placed a list of all the axioms considered in this chapter (for the various reducts) at the end of the chapter for ease of reference. Also note that each of these "axioms" is actually an axiom schema.

The Positive Quantifier-free Axioms

(0) When $c_1, \ldots, c_m$ are partitioning cylindrifications of arities $c_i \colon k_i \to (k_1 + \cdots + k_m)$ we have the axiom: For all $r_1, s_1 \colon k_1$, and all $r_2, s_2 \colon k_2$, ..., and all $r_m, s_m \colon k_m$ we have

$$\text{If } \bigvee_{i=1}^{m} c_i(s_i) \geq \bigwedge_{i=1}^{m} c_i(r_i), \text{ then } s_i \geq r_i \text{ for some } i = 1, \ldots, m.$$

Note that by $y \geq x$ we actually mean $x = x \wedge y$ or equivalently (in the presence of the next axiom) $y = x \vee y$.

(1) $0, 1, \vee, \wedge$ form a (bounded) distributive lattice in each sort. In particular, each sort comes with a partial order $\leq$ defined by $r \leq s$ just in case $r = r \wedge s$ or equivalently $s = r \vee s$.

(2) Substitutions preserve $0, 1, \vee, \wedge$. E.g., when $\alpha \colon k \to n$ we write: For all $r, s \colon k$ we have $\alpha(r \wedge^k s) = \alpha(r) \wedge^n \alpha(s)$.

(3) When $\alpha \colon k \to n$ and $\beta \colon n \to m$ we have the axiom: For all $r \colon k$ we have

$$(\beta \circ \alpha)(r) = \beta(\alpha(r)).$$

Recall that by "$\beta \circ \alpha$" we mean the function symbol which is the composition of these two substitution function symbols, while $\beta(\alpha(\bullet))$ is the usual composition within the algebra.

(4) For each identity substitution $\mathrm{id} \colon n \to n$ we have the axiom: For all $r \colon n$ we have

$$\mathrm{id}(r) = r.$$

Remark 17. Here are some notes on the axioms, and intuitive explanations.

- Intuitively, axiom (0) says that if you have a union of "orthogonal cylinders" covering a "rectangle", then one of the cylinders has width larger than the width of the corresponding side of the rectangle. Note that the instances of axiom (0) are not Horn clauses (they are universal implications where the conclusion is a disjunction of atomic formulas).

- Axioms (1)-(4), which are equations, axiomatize the Horn clause theory of positive quantifier-free algebras. When axiomatizing the algebras of larger signatures, we will see that the difference between the universal and the Horn clause theories is still just axiom (0).

- Axiom (4) is redundant in the context of axioms (0), (1), and (3). However, I include it because we will have occasion to omit axiom (0) when considering the Horn clause theory. To see this redundancy, note that $\mathrm{id}\colon n \to n$ just by itself is trivially a partitioning cylindrification. Thus by axiom (0) we have $\mathrm{id}(s) \geq \mathrm{id}(r)$ implies $s \geq r$. But axiom (3) gives $\mathrm{id}(\mathrm{id}(t)) = \mathrm{id}(t)$, and so we can get $\mathrm{id}(t) = t$.

- It is straightforward to check that the axioms are all true in positive quantifier-free algebras. For illustration, let us verify axiom (0). Suppose that $s_i \not\geq r_i$ for each $i = 1, \ldots, m$. Then there are $\bar{x}_i \in r_i - s_i$, and so $(\bar{x}_1 \cdots \bar{x}_m) \in \bigwedge_i c_i(r_i) - \bigvee_i c_i(s_i)$.

- It follows from axioms (0), (1), and (3) that everything in sort zero is either 0 or 1. To see this, note that $c_1, c_2\colon 0 \to 0$ are partitioning cylindrifications, where $c_1 = c_2 = \mathrm{id}$. Then given any element r of sort zero, we have

  $$c_1(r) \vee c_2(0) = r \vee 0 \geq 1 \wedge r = c_1(1) \wedge c_2(r)$$

  So either $r \geq 1$ (and hence $r = 1$) or $0 \geq r$ (and hence $r = 0$).

- We may consider $0 \not\geq 1$ in sort zero to be a special case of axiom (0), because the empty collection of cylindrifications trivially forms a partitioning cylindrification (of sort zero). The right hand side of the axiom in this case becomes an empty disjunction and therefore is considered as FALSE. If this offends the reader's sensibilities, then they may specifically add an axiom asserting that $0 \not\geq 1$ in sort zero. Taken together with the previous remark, we see that an algebra satisfying axioms (0), (1), and (3) will have exactly two elements in sort zero.

The main result of this chapter will be the following theorem.

Theorem 18. *Axioms (0)-(4) axiomatize the subalgebras of the positive quantifier-free algebras.*

A basic step in our proof of Theorem 18 will be the observation that if L is an abstract algebra that satisfies the axioms above then a prime filter on any one of the sorts of L gives rise to a morphism from L to a concrete positive quantifier-free algebra.[1] If you think of

[1] Axiom (1) ensures that each sort is a distributive lattice, and so it makes sense to speak of a prime filter on a sort.

the abstract algebra as a theory, and the morphism to a concrete algebra as a model of this theory, then intuitively Lemma 19 says that any prime filter F on sort n is the type of an n-tuple $\bar{F} = (F_1, \ldots, F_n)$ in some model. In fact, we can take the model to just consist of this n-tuple. Lemma 21 will allow us to realize finitely many types at once, and so then by the compactness theorem we will be able to realize all types at once, yielding an embedding.

Lemma 19. *Let F be a prime filter on sort n of some algebra L satisfying the axioms (1), (2), and (3). Let $F_1, \ldots, F_n$ be distinct symbols. Let $W = \{F_1, \ldots, F_n\}$. Let $A(W)$ denote the positive quantifier-free algebra of relations on the set W. Define a function (on each sort) $\varphi\colon L \to A(W)$ by putting, for each r in sort k of L and each substitution $\alpha\colon k \to n$,*

$$\varphi(r) = \{\alpha^{\text{tuple}}(F_1, \ldots, F_n) \mid \alpha(r) \in F\}$$

Then φ is a morphism.

Proof. Note that

$$\alpha^{\text{tuple}}(F_1, \ldots, F_n) \in \varphi(r) \iff \alpha(r) \in F$$

because every tuple (of any length) from W can be expressed as $\alpha^{\text{tuple}}(\bar{F})$ for a unique substitution α. We now proceed to check that φ is a morphism. First observe that φ preserves 0, i.e. $\varphi(0) = \emptyset$ for each sort, because $\alpha(0) = 0 \notin F$ by axiom (2). Similarly, $\varphi(1) = W^k$ because $\alpha(1) = 1 \in F$.

The preservation of $\vee$ and $\wedge$ also follow from axiom (2) via the following calculations:

$$\begin{aligned} \alpha^{\text{tuple}}(\bar{F}) \in \varphi(r) \cup \varphi(s) &\iff \alpha(r) \in F \text{ or } \alpha(s) \in F \\ &\iff \alpha(r \vee s) \in F \\ &\iff \alpha^{\text{tuple}}(\bar{F}) \in \varphi(r \vee s) \end{aligned}$$

and

$$\begin{aligned} \alpha^{\text{tuple}}(\bar{F}) \in \varphi(r) \cap \varphi(s) &\iff \alpha(r) \in F \text{ and } \alpha(s) \in F \\ &\iff \alpha(r \wedge s) \in F \\ &\iff \alpha^{\text{tuple}}(\bar{F}) \in \varphi(r \wedge s) \end{aligned}$$

Finally, we check the preservation of substitutions using axiom (3):

$$\begin{aligned} \alpha^{\text{tuple}}(\bar{F}) \in \beta^{\text{relation}}(\varphi(r)) &\iff \beta^{\text{tuple}}(\alpha^{\text{tuple}}(\bar{F})) \in \varphi(r) \\ &\iff (\alpha \circ \beta)^{\text{tuple}}(\bar{F}) \in \varphi(r) \\ &\iff (\alpha \circ \beta)(r) \in F \\ &\iff \alpha(\beta(r)) \in F \\ &\iff \alpha^{\text{tuple}}(\bar{F}) \in \varphi(\beta(r)) \end{aligned}$$

□

Proposition 20. *Axioms (1)-(4) axiomatize the class of subalgebras of products of positive quantifier-free algebras. Thus, we have found an equational axiomatization of the Horn clause theory of positive quantifier-free algebras.*

Proof. Let L be an algebra satisfying axioms (1)-(4). We want to find an embedding of L into a product of positive quantifier-free algebras. Using axiom (4), we can make sure to separate any two distinct points using a morphism from Lemma 19: Let $r \neq s$ in the same sort. Then there is a prime filter F containing, say, r and not s.[2] Let φ be the morphism obtained from Lemma 19. Then $\bar{F} = \mathrm{id}(\bar{F}) \in \varphi(r) - \varphi(s)$ because $\mathrm{id}(r) = r \in F$ and $\mathrm{id}(s) = s \notin F$, by axiom (4). Taking a product of a bunch of such morphisms, we actually get an embedding of an algebra satsifying axioms (1)-(4) into a product of positive quantifier-free algebras. □

Observe that this does not automatically give us the universal theory, because the positive quantifier-free algebras are not closed under products (even just look at the zero sort and observe that there must be exactly two elements in it). Note that while $\mathcal{P}(\biguplus W_i) = \prod \mathcal{P}(W_i)$ it is not the case that $\mathcal{P}((\biguplus W_i)^n) = \prod \mathcal{P}(W_i^n)$.

In order to have an embedding into an actual positive quantifier-free algebra instead of a product of them, we show that we can deal with all the prime filters at once by showing a certain first order theory is satisfiable.

First Part of Proof of Theorem 18. Given an algebra L that satisfies axioms (0)-(4), let us introduce a relation symbol r of arity n for each element r of L of each sort n. We also introduce constants $F_1, \ldots, F_n$ for each prime filter F of L on sort n. Let T be the first order theory in this language with the following axiom schemata:

(A) $r(\bar{F})$ when $r \in F$ and $\neg r(\bar{F})$ when $r \notin F$

(B) The **morphic conditions**, i.e.

- (i) $\forall \bar{x}\ \neg 0(\bar{x})$. We have such a sentence for the 0 of each sort.
- (ii) $\forall \bar{x}\ 1(\bar{x})$
- (iii) $\forall \bar{x}\ (r \vee s)(\bar{x}) \iff (r(\bar{x}) \text{ or } s(\bar{x}))$. Note that in $(r \vee s)(\bar{x})$ the "$\vee$" is an operation of the algebra, while in $(r(\bar{x}) \text{ or } s(\bar{x}))$ the "or" is a logical symbol of the ambient first order logic. We have such a sentence for every pair (r, s) of elements from the same sort.
- (iv) $\forall \bar{x}\ (r \wedge s)(\bar{x}) \iff (r(\bar{x}) \text{ and } s(\bar{x}))$
- (v) $\forall \bar{x}\ (\alpha(r))(\bar{x}) \iff r(\alpha^{\text{tuple}}(\bar{x}))$. We have such a sentence for every substitution $\alpha\colon k \to n$ and element r in L of sort k.

It is straightforward to verify that a model of the morphic conditions of T is (essentially) the same thing as a morphism from the algebra L to a positive quantifier-free algebra. Item

[2] Recall we have taken the Boolean prime ideal theorem as an assumption.

(A) of T ensures that this morphism is 1-1 (on each sort). To show that this theory is satisfiable, we show that the theory is finitely satisfiable and then use the compactness theorem. Thus, it suffices to find a model satisfying all of item (B) but the instances of item (A) involving only finitely many prime filters $F^1, \ldots, F^m$. (We use superscript here to avoid confusion of the prime filters with the constants associated to each of them.)
End of First Part of Proof of Theorem 18

The key idea for how to proceed is to assemble these finitely many prime filters into one master prime filter on a larger sort. We formalize this in the following lemma.

Lemma 21. *Let L be an algebra that satisfies axioms (0), (1), and (2). Let $k_1+\cdots+k_m = n$. Let $c_i\colon k_i \to n$ be partitioning cylindrifications. Let F^i be prime filters on sort k_i respectively. Then there is a prime filter G on sort n such that for each $i = 1, \ldots, m$, for all r in L of sort k_i we have $c_i(r) \in G$ if and only if $r \in F^i$.*

Proof. Let A be the distributive lattice which is sort n of L. Define

$$G_F := \{z \in A \mid z \geq \bigwedge_{i=1}^{m} c_i(r_i) \text{ for some } r_i \in F^i\}$$

and

$$G_I := \{z \in A \mid \bigvee_{i=1}^{m} c_i(s_i) \geq z \text{ for some } s_i \notin F^i\}$$

We claim that G_F is a filter, G_I is an ideal, and they are disjoint. First we show they are disjoint. Suppose, to get a contradiction, that $\bigvee_i c_i(s_i) \geq z \geq \bigwedge_i c_i(r_i)$ where $s_i \notin F^i$ and $r_i \in F^i$. Then $s_i \geq r_i$ for some i by axiom (0), implying that $s_i \in F^i$ because F^i is upward-closed, but as noted $s_i \notin F^i$, and we have a contradiction.

Next, observe that $1 \in G_F$ since $1 \in F^i$ for each i and $1 \geq \bigwedge_i c_i(1)$. Similarly $0 \in G_I$. It follows at once that $0 \notin G_F$ and $1 \notin G_I$ because G_F and G_I are disjoint.

It is obvious from the definitions that G_F is upward-closed and G_I is downward closed.

Finally, suppose $z, z' \in G_F$. Let $z \geq \bigwedge_i c_i(r_i)$ and $z' \geq \bigwedge_i c_i(r'_i)$ where $r_i, r'_i \in F^i$. Then $z \wedge z' \geq \bigwedge_i c_i(r_i \wedge r'_i)$ by axiom (2), and $r_i \wedge r'_i \in F^i$ for each i. So $z \wedge z' \in G_F$. The argument that $z, z' \in G_I$ implies $z \vee z' \in G_I$ is similar.

Because G_F and G_I are disjoint, there is a prime filter G such that $G_F \subseteq G$ and $G \cap G_I = \emptyset$. This G is a prime filter satisfying the desired property. If $r \in F^i$, then $c_i(r) = c_i(r) \wedge \bigwedge_{j \neq i} c_j(1) \in G_F$, and so $c_i(r) \in G$. If $r \notin F^i$, then $c_i(r) = c_i(r) \vee \bigvee_{j \neq i} c_j(0) \in G_I$, and so $c_i(r) \notin G$. □

Continuation of Proof of Theorem 18. Armed with this lemma, we may return to showing that the theory T is finitely satisfiable. Given our finitely many prime filters $F^1, \ldots, F^m$, there is by Lemma 21 a prime filter G such that

$$c_i(r) \in G \iff r \in F^i$$

for each i. Introduce distinct symbols $G_1, \ldots, G_n$, where $n = k_1 + \cdots + k_m$, the sum of the arities of the F^i. Let $W = \{G_1, \ldots, G_n\}$. We will interpret the constants corresponding to each prime filter F^i by $c_i^{\text{tuple}}(\bar{G})$ respectively. By Lemma 19 we know that $\varphi\colon L \to A(W)$ defined by

$$\alpha^{\text{tuple}}(\bar{G}) \in \varphi(r) \iff \alpha(r) \in G$$

is a morphism. Further,

$$c_i^{\text{tuple}}(\bar{G}) \in \varphi(r) \iff c_i(r) \in G \iff r \in F^i$$

as desired. So φ yields the desired model of the small portion of T we gave ourselves.
End of Proof of Theorem 18

Unlike powerset algebras which have equivalent equational, Horn clause, and universal theories, the positive quantifier-free algebras (and other first order algebra reducts considered below) have only equivalent equational and Horn clause theories.

2.4 Adding Negation, Projection, Equality

Negation

It is relatively easy to extend the results of Section 2.3 to algebras with negation, yielding an axiomatization of the subalgebras of the quantifier-free algebras.

Theorem 22. *Axioms (0)-(6) axiomatize the subalgebras of the quantifier-free algebras. (Axioms (5) and (6) are given below.)*

Examining the argument of Section 2.3, the only place where there needs to be significant change is for Lemma 19, where we need to now also verify that the function defined is morphic for negation. In other words, we want $\varphi(\neg r) = \neg\varphi(r)$. I.e., we want $\alpha(\neg r) \in F \iff \alpha(r) \notin F$. One way to accomplish this is to add the following two equational axiom schemata to our list:

Axioms for Negation

(5) When $\alpha\colon k \to n$ is a substitution we have the axiom: For all $r\colon k$ we have

$$\alpha(\neg r) = \neg\alpha(r).$$

(6) For each sort n, we have the axiom: For all $r\colon n$ we have

$$r \vee \neg r = 1 \text{ and } r \wedge \neg r = 0.$$

It is easy to check that these axioms are true in quantifier-free algebras.

Lemma 23. *Let F be a prime filter in sort n of some algebra L satisfying the axioms (1)-(3), and (5)-(6). Let $F_1, \ldots, F_n$ be distinct symbols. Let $W = \{F_1, \ldots, F_n\}$. Let $A(W)$ denote the quantifier-free algebra of relations on the set W. Define a function (on each sort) $\varphi\colon L \to A(W)$ by putting*

$$\alpha^{\text{tuple}}(\bar{F}) \in \varphi(r) \iff \alpha(r) \in F$$

Then φ is a morphism.

Proof. The proof is the same as that for Lemma 19, except now we must also check that $\varphi(\neg r) = \neg\varphi(r)$. Axiom (6) ensures that for any prime filter F, $\neg r \in F$ if and only if $r \notin F$. Then using axiom (5) we have $\alpha(\neg r) \in F$ if and only if $\neg\alpha(r) \in F$ if and only if $\alpha(r) \notin F$. □

As before we get the following proposition:

Proposition 24. *Axioms (1)-(6) axiomatize the class of subalgebras of products of quantifier-free algebras. Thus, we have found an equational axiomatization of the Horn clause theory of quantifier-free algebras.*

Proof. This follows from Lemma 23 in the same way that Proposition 20 follows from Lemma 19. □

Proof of Theorem 22. The proof is the same as that for Theorem 18, except that the theory T used in that proof changes in a very minor way: we must add preservation of negation to the morphic conditions. In detail, we add the following sentences to T:

(B) (vi) $\forall \bar{x}\ (\neg r)(\bar{x}) \iff \neg(r(\bar{x}))$

The addition of this to the theory T does not change the rest of the argument because Lemma 23 handles negation. Note that Lemma 21 remains unchanged by an expansion of the signature.
End of Proof of Theorem 22

Projection

Adding projection takes more work than adding negation. As in the quantifier-free case, our argument below works whether negation is present or not, and so we will obtain the following two theorems. (The new axioms (7)-(10) are presented later in this section.)

Theorem 25. *Axioms (0)-(4) and (7)-(10) axiomatize the subalgebras of the positive existential algebras.*

Theorem 26. *Axioms (0)-(10) axiomatize the subalgebras of the first order algebras.*

The proofs of these two theorems are essentially the same, and so for convenience we will focus attention on the positive existential case, i.e. Theorem 25. The signature under consideration thus includes the substitutions, the lattice operations, and the projections, but not negation. Our general approach is to find a 1-1 function from the abstract algebra satisfying the axioms to a concrete positive existential algebra which is not quite a morphism because there are not enough witnesses, but then we modify the function to obtain an actual embedding by adding witnesses.

Often when dealing with a projection $\exists\colon n+1 \to n$ we wish to also speak of the **associated cylindrification** $c\colon n \to n+1$ given by $c(i) = i$, i.e. $c(\bar{x}y) = \bar{x}$. The operations $\exists$ and c form a Galois connection, which is a special case of how direct image and inverse image form a Galois connection. However, we present the situation equationally with the following axioms, which imply more than just this Galois connection.

Axioms for Projection

(7) $\exists$ preserves 0 and $\vee$

(8) For each projection $\exists\colon (n+1) \to n$ and associated cylindrification $c\colon n \to (n+1)$ we have the axiom: For all $r\colon (n+1)$ we have

$$r \leq c(\exists(r))$$

(9) For each projection $\exists\colon (n+1) \to n$ and associated cylindrification $c\colon n \to (n+1)$ we have the axiom: For all $r\colon (n+1)$ and all $s\colon n$ we have

$$\exists(r \wedge c(s)) = \exists(r) \wedge s.$$

(10) Let $\alpha_i\colon k_i \to m$ be substitutions for $i = 1, \ldots, n$. Let $\beta_i\colon (k_i+1) \to (m+n)$ be the substitutions defined by $\beta_i^{\text{tuple}}(\bar{x}y_1 \cdots y_n) = \alpha_i(\bar{x})y_i$. Then we have the axiom: For all $r_1\colon k_1+1$, ..., and all $r_n\colon k_n+1$ we have

$$\exists^{(n)}(\bigwedge_{i=1}^{n} \beta_i(r_i)) = \bigwedge_{i=1}^{n} \alpha_i(\exists(r_i))$$

where $\exists^{(n)}$ means we apply projection n times.

Remark 27. Here are some notes on the axioms for projection, and intuitive explanations.

- It is straightforward to check that these axioms are all true in the positive existential algebras. For illustration, consider axiom (10). Intuitively, this axiom says that casting an ensemble for a theatrical production involving n roles is equivalent to finding a good actor for each role, as long as you do not care about how the team works together. A tuple $\bar{x}$ is in $\exists^{(n)}(\bigwedge_{i=1}^{n} \beta_i(r_i))$ if and only if there are $y_1, \ldots, y_n$ such that for each

$i = 1, \dots, n$ we have $\alpha_i^{\text{tuple}}(\bar{x})y_i = \beta_i^{\text{tuple}}(\bar{x}\bar{y}) \in r_i$. However, since each y_i occurs on its own, this is equivalent to saying that for each $i = 1, \dots, n$ there is some y_i with $\alpha_i^{\text{tuple}}(\bar{x})y_i \in r_i$, which is to say $\bar{x}$ is in $\bigwedge_{i=1}^{n} \alpha_i(\exists(r_i))$.

- Note that $r \leq s$ implies $\exists(r) \leq \exists(s)$ follows from axiom (7). Of course the substitutions are also increasing in this way because of axiom (2).

- It might appear that, by taking $r = 1$ in axiom (9), we could conclude that $\exists(c(s)) = s$. This is usually correct, but not always. If we consider the algebra of relations on the empty set, and $s = 1$ in sort zero, then $c(s) = 1 = 0$ in sort one, and $\exists(c(s)) = 0 \neq 1 = s$. On the other hand, we do always have $\exists(c(s)) \leq s$.

Definition 28. Let L be an algebra in the positive existential signature, and let $A(W)$ be the positive existential algebra of relations on some set W. An **almost morphism** is a function (on each sort) $\varphi\colon L \to A(W)$ such that

1. φ is morphic for the substitutions and the lattice operations

2. $\exists(\varphi(r)) \subseteq \varphi(\exists(r))$

I.e., an almost morphism is a morphism except for the possibility it might not satisfy $\exists(\varphi(r)) \supseteq \varphi(\exists(r))$.

The modified version of Lemma 19 is as follows.

Lemma 29. *Let F be a prime filter in sort n of some algebra L satisfying the axioms (1), (2), (3), and (8). Let $F_1, \dots, F_n$ be distinct symbols. Let $W = \{F_1, \dots, F_n\}$. Let $A(W)$ denote the positive existential algebra of relations on the set W. Define a function (on each sort) $\varphi\colon L \to A(W)$ by putting*

$$\alpha^{\text{tuple}}(\bar{F}) \in \varphi(r) \iff \alpha(r) \in F$$

Then φ is an almost morphism.

Proof. The new thing we need to verify is that $\exists(\varphi(r)) \subseteq \varphi(\exists(r))$. Let $\alpha^{\text{tuple}}(\bar{F}) \in \exists(\varphi(r))$. There is a substitution β such that $c^{\text{tuple}}(\beta^{\text{tuple}}(\bar{F})) = \alpha^{\text{tuple}}(\bar{F})$ and $\beta^{\text{tuple}}(\bar{F}) \in \varphi(r)$. Thus, $\beta(r) \in F$. We want to check that $c^{\text{tuple}}(\beta^{\text{tuple}}(\bar{F})) \in \varphi(\exists(r))$, i.e. $(\beta \circ c)(\exists(r)) \in F$. Well,

$$\begin{aligned}(\beta \circ c)(\exists(r)) &= \beta(c(\exists(r))) \\ &\geq \beta(r) \\ &\in F\end{aligned}$$

□

Lemma 29 does not immediately yield an axiomatization of the Horn clause theory, but rather the following lemma.

Lemma 30. *Let L be an algebra satisfying axioms (1)-(4) and (8). Let $r \neq s$ be two distinct elements in the same sort. Then there is an almost morphism φ from L to a positive existential algebra such that $\varphi(r) \neq \varphi(s)$.*

Proof. Follows from Lemma 29 in the same way that (a portion of) Proposition 20 follows from Lemma 19. □

Similarly, the argument in the proof of Theorem 18 applied to this situation does not immediately yield Theorem 25, but rather the following lemma.

Lemma 31. *Let L be an algebra satisfying axioms (0)-(4), and (8). Then there is a 1-1 almost morphism from L to some positive existential algebra.*

Proof. The proof is the same as that for Theorem 18, except that the morphic conditions of the theory T become the **almost morphic conditions**. That is, instead of adding $\forall \bar{x}\ (\neg r)(\bar{x}) \iff \neg(r(\bar{x}))$ as we did in the proof of Theorem 22, we add

(B) (vi) $\forall \bar{x}, y\ \ (r(\bar{x}y) \implies (\exists(r))(\bar{x}))$

□

To go further, we need a way of turning an almost morphism into an actual morphism. The following lemmas help us accomplish this.

Lemma 32. *Let L be an algebra that satisfies the axioms (1), (2), (7), and (9). Let F be some prime filter on sort n and let r be an element in sort $n+1$ such that $\exists(r) \in F$. Then there is some prime filter G on sort $n+1$ such that $r \in G$ and for all u in sort n we have $c(u) \in G$ if and only if $u \in F$.*

Proof. We use the same approach as in the proof of Lemma 21. Let A be the distributive lattice which is sort $n+1$ of L. Define

$$G_F := \{z \in A \mid z \geq r \wedge c(u) \text{ for some } u \in F\}$$

and

$$G_I := \{z \in A \mid c(u) \geq z \text{ for some } u \notin F\}$$

Then as in Lemma 21 we have that G_F is a filter, G_I is an ideal, and they are disjoint. These things are easy to check, and we here only deal with disjointness for illustration. Suppose, to get a contradiction, that $c(u) \geq z \geq r \wedge c(t)$ where $t \in F$ and $u \notin F$. Then by axioms (7) and (9) we get

$$\begin{aligned} u &\geq \exists(c(u)) \\ &\geq \exists(r \wedge c(t)) \\ &= \exists(r) \wedge t \\ &\in F \end{aligned}$$

putting $u \in F$, a contradiction.

So there is a prime filter G extending G_F and disjoint from G_I. This works. □

Given a prime filter G on sort $n+1$, note that $c^{-1}(G) := \{u \mid c(u) \in G\}$ is always a prime filter on sort n such that $u \in c^{-1}(G)$ if and only if $c(u) \in G$. The above lemma asserts that given any prime filter F on sort n and element r with $\exists(r) \in F$, there is some prime filter G such that $r \in G$ and $c^{-1}(G) = F$.

Recall that when L is an algebra in the positive existential signature, an almost morphism from L to a positive existential algebra is (essentially) the same thing as a model of the almost morphic conditions. Note that every tuple $(a_1, \ldots, a_n)$ from a model M of the almost morphic conditions gives rise to a prime filter $\mathfrak{p}(\bar{a}) := \{r \mid M \models r(\bar{a})\}$ of L on sort n.

Definition 33. If M_1 and M_2 are models of the almost morphic conditions (associated to some algebra L in the positive existential signature), then we say that M_2 **has witnesses over** M_1 when M_1 is a substructure of M_2, written $M_1 \subseteq M_2$, and whenever $(a_1, \ldots, a_n)$ is a tuple from M_1 and G is a prime filter of L on sort $n+1$ with $c^{-1}(G) = \mathfrak{p}(\bar{a})$, then there is some element b in M_2 such that $\bar{a}b$ weakly realizes G, i.e. $M_2 \models r(\bar{a}b)$ for every $r \in G$. Note that I say "weakly realizes" instead of "realizes" because we do not require that $M_2 \models \neg r(\bar{a}b)$ when $r \notin G$.

Lemma 34. *Let L be an algebra that satisfies axioms (1)-(3), (7)-(10). Let M_1 be a model of the almost morphic conditions. Then there is some model $M_2 \supseteq M_1$ of the almost morphic conditions which has witnesses over M_1.*

Proof. Consider the following first order theory U, the signature for which contains a relation symbol for each element of L with arity corresponding to its sort, and also some constants as indicated below:

(A) The almost morphic conditions.

(B) The literal diagram of M_1. I.e. for every tuple $\bar{a}$ from M_1 and every r in L we write $r(\bar{a})$ when $M_1 \models r(\bar{a})$ and $\neg r(\bar{a})$ when $M_1 \models \neg r(\bar{a})$

(C) For each prime filter G in sort $n+1$ of L, and each n-tuple $\bar{a}$ from M_1 with $\mathfrak{p}(\bar{a}) = c^{-1}(G)$, we introduce a new constant $y_{G,\bar{a}}$, and then for each $r \in G$, we write

$$r(\bar{a}y_{G,\bar{a}})$$

Items (A) and (B) of the theory ensure that a model satisfies the almost morphic conditions and is a superstructure of M_1. Item (C) ensures that a model will have witnesses over M_1. By the compactness theorem, we'll thus be done if we can find a model of any given finite amount of items (A), (B), and (C).

Let U^- be consist of finitely many sentences from items (B) and (C). Only finitely many elements of M_1 appear in U^-. Collect them together in one big tuple $\bar{a}$, say of length m, without duplicates. We will be satisfying all of item (A). Let $(G_1, \bar{a}_1), \ldots, (G_n, \bar{a}_n)$ be the tuple/prime filter pairs that occur in U^- and item (C). We may assume $n \geq 1$ (otherwise we can let $M_2 = M_1$). Finitely many of the elements $r \in G_i$ will occur, but we will actually be ensuring things work for all $r \in G_i$, for each i. We have $c^{-1}(G_i) = \mathfrak{p}(\bar{a}_i)$ where $\alpha_i^{\text{tuple}}(\bar{a}) = \bar{a}_i$ for some substitution α_i. Let k_i denote the length of $\bar{a}_i$. Define substitutions $\beta_0^{\text{tuple}}(\bar{a}y_1 \cdots y_n) = \bar{a}$ and $\beta_i^{\text{tuple}}(\bar{a}\bar{y}) = \bar{a}_i y_i$ for $1 \leq i \leq n$.

Now I claim that there is a prime filter H on sort $m+n$ such that

(I) For all r in sort m we have $\beta_0(r) \in H$ if and only if $r \in \mathfrak{p}(\bar{a})$, and

(II) For each $i = 1, \ldots, n$, for each r in sort $k_i + 1$ we have $r \in G_i$ implies $\beta_i(r) \in H$.

Suppose for now that there is such a prime filter. Then we define a model M_2^- with underlying set $\{H_1, \ldots, H_{m+n}\}$ as follows:

$$M_2^- \models r(\gamma(\bar{H})) \iff \gamma(r) \in H$$

We interpret the constants $a_1, \ldots, a_m$ by $H_1, \ldots, H_m$, and the constants $y_{G_1,\bar{a}_1}, \ldots, y_{G_n,\bar{a}_n}$ by $H_{m+1}, \ldots, H_{m+n}$. Of course item (A) is satisfied by Lemma 29. Now consider the sentences in U^- and item (C). Let $r \in G_i$. We want $M_2^- \models r(\bar{a}_i y_{G_i,\bar{a}_i})$, i.e. $\beta_i(r) \in H$ (because $\beta_i^{\text{tuple}}(\bar{a}\bar{y}) = \bar{a}_i y_i$). But this is implied by $r \in G_i$, according to (II).

Finally consider the sentences in U^- and item (B). We show that for any tuple $\gamma^{\text{tuple}}(\bar{a})$ assembled from $\bar{a}$, and for any r of the appropriate sort, we have $M_2^- \models r(\gamma^{\text{tuple}}(\bar{a}))$ if and only if $M_1 \models r(\gamma^{\text{tuple}}(\bar{a}))$. Note that

$$\gamma^{\text{tuple}}(\bar{a}) = \gamma^{\text{tuple}}(\beta_0(\bar{a}\bar{y})) = (\beta_0 \circ \gamma)^{\text{tuple}}(\bar{a}\bar{y})$$

and so by the definition of M_2^-, we have $M_2^- \models r(\gamma^{\text{tuple}}(\bar{a}))$ if and only if $\beta_0(\gamma(r)) = (\beta_0 \circ \gamma)(r) \in H$. By (I), this is equivalent to $\gamma(r) \in \mathfrak{p}(\bar{a})$, i.e. $M_1 \models (\gamma(r))(\bar{a})$. Since M_1 itself satisfies the almost morphic conditions, this is equivalent to $M_1 \models r(\gamma^{\text{tuple}}(\bar{a}))$, as desired.

Now we show that we can get such an H. We use an argument similar to that of Lemma 21 or Lemma 32. Let A be the distributive lattice which is sort $m+n$ of L. Define

$$H_F := \{z \in A \mid z \geq \beta_0(r) \wedge \bigwedge_{i=1}^{n} \beta_i(r_i) \text{ for some } r \in \mathfrak{p}(\bar{a}) \text{ and } r_i \in G_i\}$$

and

$$H_I := \{z \in A \mid \beta_0(r) \geq z \text{ for some } r \notin \mathfrak{p}(\bar{a})\}$$

Then H_F is a filter, H_I is an ideal, and they are disjoint. The main thing to check is the disjointness. Suppose, to get a contradiction, that

$$\beta_0(s) \geq z \geq \beta_0(r) \wedge \bigwedge_{i=1}^{n} \beta_i(r_i)$$

where $s \notin \mathfrak{p}(\bar{a})$, $r \in \mathfrak{p}(\bar{a})$, and $r_i \in G_i$ for each i. Observe that $c^{(n)} = \beta_0$, and so by repeated use of the facts that $\exists(c(t)) \leq t$ and $\exists$ is increasing, we get

$$s \geq \exists^{(n)}(\beta_0(s)) \geq \exists^{(n)}(\beta_0(r) \wedge \bigwedge_{i=1}^{n} \beta_i(r_i))$$

By repeated use of axiom (9), the right hand side becomes

$$r \wedge \exists^{(n)}(\bigwedge_{i=1}^{n} \beta_i(r_i))$$

Putting this all together with axiom (10), we see that

$$s \geq r \wedge \bigwedge_{i=1}^{n} \alpha_i(\exists(r_i))$$

To get that $s \in \mathfrak{p}(\bar{a})$, a contradiction, we will show that $\alpha_i(\exists(r_i)) \in \mathfrak{p}(\bar{a})$ for each i. By axiom (8), $c(\exists(r_i)) \geq r_i \in G_i$, so $\exists(r_i) \in \mathfrak{p}(\bar{a}_i)$ (recall that $c^{-1}(G_i) = \mathfrak{p}(\bar{a}_i)$ by assumption). Then, as M_1 satisfies the almost morphic conditions, and $\alpha_i^{\text{tuple}}(\bar{a}) = \bar{a}_i$, we get that $\alpha_i(\exists(r_i)) \in \mathfrak{p}(\bar{a})$.

A prime filter H which extends H_F and is disjoint from H_I is as desired. □

If $f\colon A \to B$ is a function, we use $\ker(f)$ to denote the relation $\{(a, a') \in A^2 \mid f(a) = f(a')\}$.

Lemma 35. *Let L be an algebra that satisfies axioms (1)-(3) and (7)-(10). Let φ be an almost morphism from L to some positive existential algebra. Then there is a morphism φ^+ from L to some positive existential algebra such that $\ker(\varphi^+) \subseteq \ker(\varphi)$. In particular, if φ is 1-1, then φ^+ is an embedding.*

Proof. The almost morphism φ gives rise to a model M_1 which satisfies the almost morphic conditions. By Lemma 34 there is a model $M_2 \supseteq M_1$ of the almost morphic conditions which has witnesses over M_1. Continuing in this way, we get a sequence

$$M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$$

of length ω where M_{n+1} has witnesses over M_n. Let M^+ be the union of this chain of models. Since the almost morphic conditions are of a form preserved by unions of chains, we get that M^+ models them too.

Further, if $M^+ \models (\exists(r))(\bar{a})$, then $M_n \models (\exists(r))(\bar{a})$ for some n. By Lemma 32, there is a prime filter G such that $\mathfrak{p}(\bar{a}) = c^{-1}(G)$ and $r \in G$. Since M_{n+1} has witnesses over M_n, there is some element $b \in M_{n+1}$ such that $M_{n+1} \models r(\bar{a}b)$. Thus, $M^+ \models r(\bar{a}b)$. In summary, $M^+ \models (\exists(r))(\bar{a})$ implies $M^+ \models \exists y(r(\bar{a}y))$. Thus, the function given by $\varphi^+(r) := r^{M^+}$ is a morphism from L to the positive existential algebra of relations on the underlying set of M^+.

Finally, suppose $\varphi^+(r) = \varphi^+(s)$. We show that $\varphi(r) = \varphi(s)$. If there were $\bar{a}$ from M_1 with $\bar{a} \in \varphi(r) - \varphi(s)$, then $\bar{a} \in \varphi^+(r) - \varphi^+(s)$ as well, because $M_1 \subseteq M^+$. □

From Lemma 30 and Lemma 35 we get the following proposition:

Proposition 36. *Axioms (1)-(4) and (7)-(10) equationally axiomatize the subalgebras of products of the positive existential algebras.*

Theorem 25 follows immediately from Lemma 31 and Lemma 35.

Equality

If we wish to add equality, we may do so (modularly) with the following axioms. In this section we do not provide a detailed analysis, but rather just indicate briefly how the above argument changes.

Axioms for Equality

(11) a) $\Delta^n_{i,i} = 1$

b) $\Delta^n_{i,j} = \Delta^n_{j,i}$

c) $\Delta^n_{i,j} \wedge \Delta^n_{j,k} \leq \Delta^n_{i,k}$

(12) When $\alpha, \beta\colon k \to n$ are substitutions of matching arities we have the axiom:

$$\alpha(r) \wedge \bigwedge_{l=1}^{k} \Delta^n_{\alpha(l),\beta(l)} = \beta(r) \wedge \bigwedge_{l=1}^{k} \Delta^n_{\alpha(l),\beta(l)}$$

(13) For each substitution $\alpha\colon k \to n$ we have the axiom:

$$\alpha(\Delta^k_{i,j}) = \Delta^n_{\alpha(i),\alpha(j)}$$

It is straightforward to check that these equational axioms are all true in the concrete algebras where $\Delta^n_{i,j}$ is interpreted as the n-ary relation which holds of an n-tuple if and only if the i^{th} and j^{th} coordinates are equal. Axiom (11) corresponds to the usual properties of an equivalence relation. Axiom (12) is algebraically saying the obvious fact that

$$\{\bar{x} \mid \alpha^{\text{tuple}}(\bar{x}) \in r \text{ and } \alpha^{\text{tuple}}(\bar{x}) = \beta^{\text{tuple}}(\bar{x})\} = \{\bar{x} \mid \beta^{\text{tuple}}(\bar{x}) \in r \text{ and } \alpha^{\text{tuple}}(\bar{x}) = \beta^{\text{tuple}}(\bar{x})\}$$

Finally, to make sense of axiom (13), recall that the i^{th} coordinate of $\alpha^{\text{tuple}}(\bar{x})$ is $x_{\alpha(i)}$. So, $\alpha^{\text{tuple}}(\bar{x}) \in \Delta^k_{i,j}$ if and only if $x_{\alpha(i)} = x_{\alpha(j)}$.

Let F be a prime filter on sort n. Our basic strategy is the same – get a modified version of Lemma 19 by having F correspond to a tuple $(F_1, \ldots, F_n)$ – except that now we may have to identify certain of the F_i. By axiom (11), we may put

$$F_i = F_j \iff \Delta^n_{i,j} \in F$$

That is, the relation $\{(i,j) \mid \Delta_{i,j} \in F\}$ is an equivalence relation. In detail, it is reflexive by axiom (11) part (a). It is symmetric by axiom (11) part (b). And it is transitive by axiom (11) part (c). But then we may have $\alpha^{\text{tuple}}(\bar{F}) = \beta^{\text{tuple}}(\bar{F})$ for distinct substitutions α and β. The upshot of axiom (12) is that this won't matter: If $\alpha^{\text{tuple}}(\bar{F}) = \beta^{\text{tuple}}(\bar{F})$ then $\alpha(r) \in F \iff \beta(r) \in F$. To see this, observe that

$$\begin{aligned}\alpha^{\text{tuple}}(\bar{F}) = \beta^{\text{tuple}}(\bar{F}) &\iff F_{\alpha(l)} = F_{\beta(l)} \text{ for each } l = 1, \ldots, k \\ &\iff \bigwedge_{l=1}^{k} \Delta^n_{\alpha(l),\beta(l)} \in F\end{aligned}$$

So if $\alpha(r) \in F$ and $\alpha^{\text{tuple}}(\bar{F}) = \beta^{\text{tuple}}(\bar{F})$, then we get

$$\begin{aligned}\beta(r) &\geq \beta(r) \wedge \bigwedge_{l=1}^{k} \Delta^n_{\alpha(l),\beta(l)} \\ &= \alpha(r) \wedge \bigwedge_{l=1}^{k} \Delta^n_{\alpha(l),\beta(l)} \\ &\in F\end{aligned}$$

and so $\beta(r) \in F$.

Now we are in a position to obtain the with-equality version of Lemma 19, using axiom (13) for the preservation of the $\Delta^n_{i,j}$. For definiteness, we state the lemma for positive quantifier-free algebras with equality.

Lemma 37. *Let L be an algebra that satisfies axioms (1)-(3), (11)-(13). Let F be a prime filter on sort n. Let $F_1, \ldots, F_n$ be symbols such that $F_i = F_j$ if and only if $\Delta^n_{i,j} \in F$. Let $W = \{F_1, \ldots, F_n\}$. Let $A(W)$ be the positive quantifier-free algebra with equality on the relations of W. Then $\varphi \colon L \to A(W)$ defined by*

$$\alpha^{\text{tuple}}(\bar{F}) \in \varphi(r) \iff \alpha(r) \in F$$

is a morphism.

Proof. As observed above, this definition of φ is unambiguous by axiom (12).

The new thing we have to check is that

$$\varphi(\Delta^k_{i,j}) = \{\alpha^{\text{tuple}}(\bar{F}) \mid F_{\alpha(i)} = F_{\alpha(j)}\}$$

Well,

$$\begin{aligned}\varphi(\Delta^k_{i,j}) &= \{\alpha^{\text{tuple}}(\bar{F}) \mid \alpha(\Delta^k_{i,j}) \in F\} \\ &= \{\alpha^{\text{tuple}}(\bar{F}) \mid \Delta^n_{\alpha(i),\alpha(j)} \in F\} \\ &= \{\alpha^{\text{tuple}}(\bar{F}) \mid F_{\alpha(i)} = F_{\alpha(j)}\}\end{aligned}$$

□

2.5 Theories

We now consider how formulas and theories may be understood in the context of our multisorted algebraic approach. Thinking about first order theories in an algebraic way is not new, and it was discussed even in our multisorted formalism by Börner in Section 3.7 of [3]. However, our discussion here will help explain the value of axiom (0) in letting us have a uniform argument for the various reducts.

Let us say a first order **formula** in some relational signature σ is an element of the free algebra (in the first order algebra signature) generated by the symbols of σ (which are relation symbols of various fixed finite arities). Let us use the notation F_σ to refer to this free algebra. Positive existential formulas, quantifier-free formulas, etc. are defined correspondingly. For an example, let σ consist of a unary relation symbol R and a binary relation symbol S. Let $\alpha \colon 2 \to 2$ be the substitution $\alpha^{\text{tuple}}(x, y) = (y, x)$. Then

$$R,\ S,\ \alpha(S),\ \exists(\alpha(S)),\ R \wedge \exists(\alpha(S))$$

are some formulas.

This way of viewing formulas does away with bound/free variables and the associated "alphabetic variants", but of course a formula up to logical equivalence may have more than one syntactic representation in this formalism as well (e.g. $\alpha(\alpha(S))$ and S are logically equivalent). Also note that the variable context has now become an intrinsic part of the formula (its arity).

A first order σ-**structure** is a morphism from F_σ to some first order algebra M. This is the same as a function which assigns to every relation symbol of σ a relation on the underlying set of M. Let us use K to denote the class of concrete algebras for the kind of logic under consideration (i.e. K could be the first order algebras, or the positive existential algebras, etc.). Then a σ-structure for whatever logic is under consideration is a morphism from F_σ to an algebra $M \in K$.

Given any collection T of identities of formulas (i.e. pairs of formulas from the same sort), the statement that a structure $f \colon F_\sigma \to M$ is a **model** of T means that $f(r) = f(s)$ for each pair $(r, s) \in T$. A (partial) **theory** T is an "implicationally closed" collection of identities in the sense that if every model f of T satisfies $f(r) = f(s)$, then also $(r, s) \in T$. Every theory is in particular a congruence relation on F_σ. Thus, we have an associated quotient F_σ/T, which could be called the theory too. A morphism $F_\sigma/T \to M \in K$ is the same thing as a model of T. The algebras that arise as quotients in this way (i.e., are of the form F_σ/T for some signature σ and some σ-theory T) are exactly the subalgebras of the products of the concrete algebras, i.e. $SP(K)$.

We include the easy verification of this fact for illustrative purposes. First let $Q = F_\sigma/T$ be such a quotient. We now prove that Q must satisfy the equational theory of K (which is equivalent to the Horn clause theory for the K of present interest). Let $\varphi(\bar{r}) = \chi(\bar{r})$ be an equation true in all members of K. Let $\bar{r}$ be some tuple from Q. Then let $f \colon Q \to M \in K$ be any model of T. Of course we must have $\varphi(f\bar{r}) = \chi(f\bar{r})$. Since f is a morphism, this yields $f(\varphi(\bar{r})) = f(\chi(\bar{r}))$. This works for any model f, and so by the assumption that T is implicationally closed, we get that $Q \models \varphi(\bar{r}) = \chi(\bar{r})$ too. Since Q satisfies the equational theory of K, by Proposition 20 or its analogue, we get that $Q \in SP(K)$.

Conversely, let $Q \in SP(K)$. Specifically let $Q \subseteq \prod_{i \in I} M_i$ where the M_i are in K. Introduce a signature σ with a symbol for each element of Q. Then $F_\sigma/T = Q$ for some congruence T. We claim T is a theory, i.e. is implicationally closed. Suppose $r, s \in Q$ with $f(r) = f(s)$ for all morphisms $f \colon Q \to M \in K$. Then in particular for the projections $\pi_i \colon Q \to M_i$ $(i \in I)$ we have $\pi_i(r) = \pi_i(s)$. So $r = s$.

We may thus say that theories are simply subalgebras of products of the concrete algebras in question (with specified generators). The usual notion of first order theory is an (implicationally closed) collection of sentences (identities of the form $\varphi = 1$ in sort zero). In the first order case, where universal quantification and the biconditional are present, this agrees with the notion of theory described above, essentially because $r = s$ in a first order algebra if and only if $\forall^{(n)}(r \leftrightarrow s) = 1$ (where r and s are in sort n). Intuitively speaking, in the first order signature, the zero sort controls all the sorts. For the reducts this is not true. To illustrate this point, and to help explain the value of axiom (0), we now show that first order theories with exactly two elements in sort zero are the ones in $S(K)$, but importantly that this characterization does not hold for the reducts.

When considered as a collection of sentences, a first order theory is said to be **complete** when every sentence or its negation (but not both) is in the theory. Translating this to the quotient view of theories, this says that there are exactly two elements in sort zero. Of course any subalgebra of a first order algebra is going to be a theory with exactly two elements of sort zero. But the converse is true as well, *when negation and projection are present*. We check that axiom (0) follows from the Horn clause theory of first order algebras together with the assumption that there are exactly two elements of sort zero.

First we observe that in any algebra satisfying the Horn clause theory of first order algebras, for any element t of sort k we have $t = 0 \iff \exists^{(k)}(t) = 0$, because the two directions of this bi-implication are both Horn clauses true of first order algebras (let us say are "true Horn clauses"). If additionally we have an algebra with exactly two elements in sort zero (0 and 1), then $t \neq 0$ if and only if $\exists^{(k)}(t) = 1$.

We now prove axiom (0) in contrapositive form. Let $s_i \not\geq r_i$ for each $i = 1, \dots, m$, where $r_i, s_i \colon k_i$, and let $c_i \colon k_i \to (k_1 + \cdots + k_m) = n$ be partitioning cylindrifications. Since $t \wedge \neg u = 0 \implies u \geq t$ is a true Horn clause, we get that $r_i \wedge \neg s_i \neq 0$ for each i. Thus, $\exists^{(k_i)}(r_i \wedge \neg s_i) = 1$. Another true Horn clause is

$$\bigwedge_{i=1}^{m} \exists^{(k_i)}(t_i) = 1 \implies \exists^{(n)} \bigwedge_{i=1}^{m} c_i(t_i) = 1$$

Thus, we get in our case

$$\exists^{(n)} \bigwedge_{i=1}^{m} c_i(r_i \wedge \neg s_i) = 1$$

So

$$\bigwedge_{i=1}^{m} c_i(r_i \wedge \neg s_i) \neq 0$$

which simplifies to $\bigwedge_{i=1}^{m} c_i(r_i) \wedge \neg \bigvee_{i=1}^{m} c_i(s_i) \neq 0$. So $\bigwedge_{i=1}^{m} c_i(r_i) \not\leq \bigvee_{i=1}^{m} c_i(s_i)$.

So, we could have presented an axiomatization of the universal theory of first order algebras by just taking the Horn clause theory and adding to it the axiom that there are exactly two elements in sort zero. However, this would not have yielded results uniformly for the reducts as well. There is a model of the Horn clause theory of positive existential algebras which has exactly two elements of sort zero, but fails to satisfy axiom (0). To see this, consider the (partial) first order theory (presently we will be taking a positive existential reduct) in a language with three unary relation symbols R, A, and B generated by the following sentences:

(i) $\exists x(A(x) \wedge B(x))$

(ii) $\forall x(R(x) \iff A(x)) \vee \forall x(R(x) \iff B(x))$

Let Q be the associated subalgebra of a product of first order algebras. Consider the positive existential reduct of Q, and then consider the subalgebra generated by R, A, and B. Call it Q_0, and note that Q_0 is itself a subalgebra of a product of positive existential algebras. Note that $X \in Q_0$ if and only if there is some positive existential formula φ such that $\varphi(R, A, B) = X$. Because we're dealing with unary relation symbols, and projections of conjunctions of some of R, A, B are predictably 1, we in fact may assume that φ is positive quantifier-free. Every element of sort zero in Q_0 is obtained by projecting an element of sort one. One can check the only possible values are 0 and 1 (and $0 \neq 1$ because our theory has a model). On the other hand, letting $c_1^{\text{tuple}}(x, y) = x$ and $c_2^{\text{tuple}}(x, y) = y$, we have $c_1(A) \wedge c_2(B) \leq c_1(R) \vee c_2(R)$ (i.e. $A(x) \wedge B(y) \models R(x) \vee R(y)$), but $A \not\leq R$ and $B \not\leq R$, violating axiom (0).

It is easy to give a theory in the quantifier-free signature which does not satisfy axiom (0) and still has exactly two elements in sort zero, because there are no functions going from the higher sorts to sort zero in this case. So any violation of axiom (0) not involving sort zero yields an example. For instance, consider the quantifier-free algebra $A(W)$ of relations on a set W of one element. Then the product $L := A(W) \times A(W)$ has a "diamond" for each sort. Let us use 0, a, b, and 1 to denote the elements of L in sort one. Let $c_1^{\text{tuple}}(x, y) = x$ and $c_2^{\text{tuple}}(x, y) = y$ be partitioning cylindrifications. Then $c_1(a) \wedge c_2(b)$ is the bottom element in sort two. Thus, $c_1(b) \vee c_2(0) \geq c_1(a) \wedge c_2(b)$. However, $b \not\geq a$ and $0 \not\geq b$, violating axiom (0).

2.6 Dealing with Function Symbols

We briefly indicate how to deal with function symbols. Let π be a fixed functional signature. We have terms $\alpha(x_1, \ldots, x_n)$ defined as usual (elements of the free $\pi(\bar{x})$-algebra where the $\bar{x}$

are extra constant symbols). From these we obtain "term-tuples"

$$\alpha(\bar{x}) = (\alpha_1(\bar{x}), \ldots, \alpha_k(\bar{x}))$$

where each $\alpha_i(\bar{x})$ is a term.

The term-tuples induce operations on tuples of a π-algebra W in the obvious way. Given a tuple $\bar{x} \in W^n$, we get $\alpha(\bar{x}) = (\alpha_1(\bar{x}), \ldots, \alpha_k(\bar{x})) \in W^k$. The inverse images of these are operations on relations going in the reverse direction $\alpha\colon \mathcal{P}(W^k) \to \mathcal{P}(W^n)$. The substitutions are obtained as a special case for any signature π, and when π is the empty signature, they are the only term-tuples. The multisorted signature of interest to us now has an operation of arity $\alpha\colon k \to n$ for each such term-tuple, and the concrete algebras of interest are the ones that arise from considering the relations on a π-algebra.

With respect to axiomatization, if equality is not present, we need only change axioms (2), (3), and (5) by expanding their scope to include all term-tuples (not just substitutions).

As for equality, let us have a constant $\Delta_{\alpha,\beta}$ of sort n for each pair of term-tuples $\alpha, \beta\colon k \to n$. The intended interpretation is $\Delta_{\alpha,\beta} := \{\bar{x} \mid \alpha(\bar{x}) = \beta(\bar{x})\}$. In the special case $\alpha(\bar{x}) = x_i$ and $\beta(\bar{x}) = x_j$, we get $\Delta_{\alpha,\beta} = \Delta^n_{i,j}$.

Then we rewrite axioms (11)-(13) as follows:

- · $\Delta_{\alpha,\alpha} = 1$
  · $\Delta_{\alpha,\beta} = \Delta_{\beta,\alpha}$
  · $\Delta_{\alpha,\beta} \wedge \Delta_{\beta,\gamma} \leq \Delta_{\alpha,\gamma}$
- $\alpha(r) \wedge \Delta_{\alpha,\beta} = \beta(r) \wedge \Delta_{\alpha,\beta}$
- $\gamma(\Delta_{\alpha,\beta}) = \Delta_{\gamma\circ\alpha,\gamma\circ\beta}$

To these we also add

- $\Delta_{\alpha,\beta} = \bigwedge_{i=1}^{k} \Delta_{\alpha_i,\beta_i}$ where $\alpha = (\alpha_1, \ldots, \alpha_k)$ and $\beta = (\beta_1, \ldots, \beta_k)$
- $\Delta_{\alpha,\beta} \leq \Delta_{\alpha\circ\gamma,\beta\circ\gamma}$

So how do the proofs get modified? The only essential change is with the analogues of Lemma 19. We want to have a prime filter F on sort n of an abstract algebra satisfying the axioms give rise to a morphism to a concrete algebra. When equality isn't present, instead of letting $W = \{F_1, \ldots, F_n\}$, we let W be the free π-algebra with $F_1, \ldots, F_n$ as generators. In the special case where we have no function symbols, i.e. π is empty, we get back the old W. We define the morphism as before: $\alpha(\bar{F}) \in \varphi(r) \iff \alpha(r) \in F$, except that now α may range over all the term-tuples, not just the substitutions.

When equality is present, we additionally identify certain elements of this free algebra by saying $\alpha(\bar{F}) = \beta(\bar{F})$ if and only if $\Delta_{\alpha,\beta} \in F$. The additional axioms ensure that this makes sense and in fact gives a congruence relation.

Finally, the fixed functional signature π can also be taken to be multisorted, with only minor modifications to our argument.

2.7 The Axioms

For ease of reference, here is a list of the main axioms considered:

(0) When $c_1, \dots, c_m$ are partitioning cylindrifications we have the axiom:

$$\text{If } \bigvee_{i=1}^{m} c_i(s_i) \geq \bigwedge_{i=1}^{m} c_i(r_i), \text{ then } s_i \geq r_i \text{ for some } i = 1, \dots, m.$$

(1) $0, 1, \vee, \wedge$ form a (bounded) distributive lattice in each sort.

(2) Substitutions preserve $0, 1, \vee, \wedge$

(3) $(\beta \circ \alpha)(r) = \beta(\alpha(r))$

(4) $\mathrm{id}(r) = r$

(5) $\alpha(\neg r) = \neg\alpha(r)$

(6) $r \vee \neg r = 1$, $r \wedge \neg r = 0$

(7) $\exists$ preserves 0 and $\vee$

(8) $r \leq c(\exists(r))$

(9) $\exists(r \wedge c(s)) = \exists(r) \wedge s$

(10) Let $\alpha_i(\bar{x})$ be substitutions for $i = 1, \dots, n$. Define $\beta_i(\bar{x}y_1 \cdots y_n) = \alpha_i(\bar{x})y_i$. Then we have the axiom

$$\exists^{(n)}(\bigwedge_{i=1}^{n} \beta_i(r_i)) = \bigwedge_{i=1}^{n} \alpha_i(\exists(r_i))$$

where $\exists^{(n)}$ means we apply projection n times.

(11) a) $\Delta^n_{i,i} = 1$
b) $\Delta^n_{i,j} = \Delta^n_{j,i}$
c) $\Delta^n_{i,j} \wedge \Delta^n_{j,k} \leq \Delta^n_{i,k}$

(12) When $\alpha, \beta \colon k \to n$ are substitutions of matching arities we have the axiom:

$$\alpha(r) \wedge \bigwedge_{l=1}^{k} \Delta^n_{\alpha(l),\beta(l)} = \beta(r) \wedge \bigwedge_{l=1}^{k} \Delta^n_{\alpha(l),\beta(l)}$$

(13) For each substitution $\alpha \colon k \to n$ we have the axiom:

$$\alpha(\Delta^k_{i,j}) = \Delta^n_{\alpha(i),\alpha(j)}$$

Chapter 3

Actions Arising from Intersection and Union

In this chapter we investigate actions arising from intersection and union and posets arising from actions arising from intersection. All sections in this chapter except Section 3.5 are joint work with Alex Kruckman ([12]).

3.1 Introduction

An *action* (of S on C on the right) is a pair of sets, C and S, and a function $f\colon C \times S \to C$. We denote by S^* the set of words in S (i.e. finite sequences of elements of S, including the empty sequence). For brevity, we write $f(c, s)$ as cs, and similarly given $c \in C$ and $w \in S^*$, cw is an element of C.

One intuitive interpretation of actions has been given by philosophers studying conversational dynamics (as in [14], for example). Given an action (C, S), we can think of C as the contexts that a conversation can have, and S as the sentences which, when asserted, change the context. A natural class of concrete models can be described by taking both the contexts $c \in C$ and the sentences $s \in S$ to be sets of possible worlds. Then asserting s in context c corresponds to cutting down the set of possible worlds by intersection $c \cap s$.

With this motivation, Rothschild and Yalcin pointed out in [14] that the actions which can be expressed using set intersection in this way are exactly the idempotent, commutative actions. In detail, an action (C, S) is called *idempotent* when $css = cs$ and *commutative* when $cs_1s_2 = cs_2s_1$ (for convenience we have the convention that when we assert equations we mean them to be universally quantified, unless it is clear from context that we are discussing particular elements). When the elements of C and S can be identified with subsets of some set in such a way that $cs = c \cap s$, then we say that the action is a $\downarrow$-*action*. The idea behind the name – to be read "down-action" – is that going from c to cs involves taking some things away from c. Later we will introduce $\uparrow\!\downarrow$-actions ("up-down actions") which will allow for both taking things away and adding things in.

Theorem 38 (Rothschild and Yalcin)**.** *An action is a* $\downarrow$*-action if and only if it is idempotent and commutative.*

Proof. It is easy to check that any action expressible using set intersection is idempotent and commutative. To see the other direction, one may identify an element $c \in C$ with $O(c) = \{cw \mid w \in S^*\}$, the orbit of c, and identify $s \in S$ with $F(s) = \{c \mid cs = c\} \cup \{s\}$, the fixed points of s together with the tag "s" to ensure F is 1-1. From idempotence and commutativity it follows that if $w \in S^*$ and $dw = c$, then every element of $O(c)$ is fixed by w. We claim that O is 1-1. If $O(c) = O(d)$ then $d \in O(c)$ and $c \in O(d)$ and so there is $w \in S^*$ with $dw = c$. By our earlier observation it follows that d is fixed by w, so $d = dw = c$. Finally we can check that $O(cs) = O(c) \cap F(s)$. Let $csw \in O(cs)$ (where $w\colon S^*$). Of course $csw \in O(c)$. Further $csws = cssw = csw$, so $csw \in F(s)$. Now let $cw \in O(c) \cap F(s)$ (where again $w\colon S^*$). Then $cws = cw$, and so $csw = cw$. So $cw \in O(cs)$. □

Because of the duality between $\cap$ and $\cup$, $\downarrow$-actions could equally be called $\uparrow$-actions, but we use $\downarrow$ because of the linguistic motivation for these actions given above.

Seeing that such a tidy characterization came about looking at intersection, a natural question arises: What happens if we also throw union into the mix? From the conversational dynamics perspective described above, in the purely intersective case, sentences can only rule out possibilities. Allowing union could capture situations in which some sentences rule out possibilities, while others rule possibilities back in.

We address this question in two ways. First, in Section 3.2, we consider actions in which each element of S acts by both intersection and union. We say an action (C, S) is an $\updownarrow$*-action* if each element of C can be identified with a set, and each element of S can be identified with a pair of sets (s^-, s^+), such that $s^+ \subseteq s^-$, in such a way that the action of s on c is given by $(c \cap s^-) \cup s^+$.

An alternative way of adding in union is to label each element of S as an intersection element or a union element. In this setup, sentences can no longer rule out and rule in possibilities simultaneously; instead, each sentence can only do one or the other. In Section 3.3, we introduce the class of $\updownarrow$*-biactions*, so named because they are 3-sorted algebras (C, S^-, S^+) with an action of S^- on C by intersection and an action of S^+ on C by union.

The main results of Sections 3.2 and 3.3, Theorem 47 and Theorem 54, are axiomatizations of the classes of $\updownarrow$-actions and $\updownarrow$-biactions. Unlike in the case of actions expressed using set intersection, these classes do not admit equational axiomatizations. However, each class is a quasivariety, axiomatized by finitely many equational axioms (which give the equational theory of the class - see Propositions 43 and 49) together with a single infinite Horn clause schema. The axioms will be explained later on in the chapter, but we will write them down here for reference.

The $\updownarrow$-actions are axiomatized by idempotence (I), previous redundant (PR), and the strong links axioms (SL). Below, c, d, and the a_i are variables of sort C, s and t are variables of sort S, and the w_i are words of sort S^* (arbitrary sequences of variables from S).

$$\begin{aligned} &\text{(I)} && css = cs \\ &\text{(PR)} && csts = cts \\ &\text{(SL)} && \left(\left(\bigwedge_{i=1}^{n} cw_i = dw_i \right) \wedge c = a_0 \wedge d = a_n \wedge \right. \\ &&& \left. \left(\bigwedge_{i=1}^{n} a_{i-1}w_i = a_{i-1} \wedge a_i w_i = a_i \right) \right) \rightarrow (c = d) \end{aligned}$$

The $\updownarrow$-biactions are axiomatized by idempotence (I), previous redundant (PR), commutativity (C) in S^+ and S^-, and the subset axioms (S). Below, c, d, and e are variables of sort C, s and u are variables of sort S^-, t and v are variables of sort S^+, and w is a word of sort $(S^- \cup S^+)^*$ (an arbitrary sequence of variables from S^- and S^+).

$$\begin{aligned} &\text{(I)} && css = cs \quad ctt = ct \\ &\text{(PR)} && csts = cts \quad ctst = cst \\ &\text{(C)} && csu = cus \quad ctv = cvt \\ &\text{(S)} && (csw = dsw \wedge ctw = dtw \wedge esw = etw) \rightarrow (cw = dw) \end{aligned}$$

The class of $\updownarrow$-actions is closely related to a certain class of single-sorted algebras we call set bands, which we discuss in Section 3.4. It turns out that the class of set bands is exactly the quasivariety generated by a certain 3-element semigroup. This quasivariety was studied and axiomatized in [13], but with the motivation coming from hyperplane arrangements. The connection to $\updownarrow$-actions provides an additional motivation for studying this quasivariety.

Our solution to the $\updownarrow$-action axiomatization problem was obtained after observing the connection with the single-sorted set bands axiomatization problem, and adapting the solution of Margolis et al. in [13] to the action case. The argument in the case of $\updownarrow$-biactions is different than in the case of $\updownarrow$-actions, but it shares the same basic structure.

The last section of this chapter has to do with posets that arise from $\downarrow$-actions. When (C, S) is a $\downarrow$-action, for $c, d \in C$ we put $c \leq d$ iff there is some finite sequence of elements $s_1, \ldots, s_n$ of S such that $ds_1 \cdots s_n = c$. We give a reasonable second order characterization of these posets, and show that they are not first order axiomatizable. Along the way we give some necessary, and some sufficient, first order conditions.

3.2 $\updownarrow$-actions

We begin by reviewing our notational conventions for actions. We view an action (C, S) as an algebra in a two-sorted signature with a single function symbol $f\colon C \times S \to C$. When c and s are elements or variables of sorts C and S, respectively, we write cs for $f(c, s)$. We denote by S^* the set of words in S. Given $c \in C$ and $w \in S^*$, cw is an element of sort C.

For all $w \in S^*$, let $f_w \colon C \to C$ be the function $c \mapsto cw$. We say w is an *identity operation* if f_w is the identity function, and w is a *constant operation* (with value d) if f_w is the constant function $f_w(c) = d$ for all $c \in C$.

Given a set X, we form an action $F(X)$ called the *full* $\updownarrows$*-action on* X by setting

$$\begin{aligned} C &= \{c \mid c \subseteq X\} \\ S &= \{(s^-, s^+) \mid s^-, s^+ \subseteq X \text{ and } s^+ \subseteq s^-\} \\ f(c, (s^-, s^+)) &= (c \cap s^-) \cup s^+. \end{aligned}$$

An action is a $\updownarrows$*-action* if it is isomorphic to a subalgebra of the full $\updownarrows$-action on some set X. In other words, a $\updownarrows$-action is an action (C, S) where each element of C can be identified with a subset c of some set X and each element of S can be identified with a pair $s = (s^-, s^+)$ of subsets of X with $s^+ \subseteq s^-$, such that the action of s on c is given by intersection with s^- and union with s^+.

Note that the condition that $s^+ \subseteq s^-$ implies that $(c \cap s^-) \cup s+ = (c \cup s^+) \cap s^-$, so the order of operations in the definition doesn't matter. This restriction is convenient but not important; in Proposition 48 below, we show that if we allow all pairs of subsets of X in the S sort, we get the same class of algebras up to isomorphism.

Proposition 39. *The class of* $\updownarrows$*-actions is pseudo-elementary.*

Proof. Expand the signature by an additional sort W and additional binary relations $\in \colon W \times C$, $\in^- \colon W \times S$, and $\in^+ \colon W \times S$. Then let T be the theory which asserts extensionality:

$$(\forall c, d : C\,)((\forall w \colon W\, w \in c \leftrightarrow w \in d) \to c = d)$$

and

$$(\forall s, t : S\,)((\forall w \colon W\, (w \in^- s \leftrightarrow w \in^- t) \wedge (w \in^+ s \leftrightarrow w \in^+ t)) \to s = t),$$

the subset condition on S:

$$(\forall s : S\,)(\forall w : W\, (w \in^+ s \to w \in^- s)),$$

and the way S acts on C:

$$(\forall w : W\,)(\forall c : C\,)(\forall s : S\,)(w \in cs \leftrightarrow ((w \in c \wedge w \in^- s) \vee w \in^+ s)).$$

Now, every $\updownarrows$-action can clearly be expanded to become a model of T. Conversely, given a model of T, we may embed its reduct into the full $\updownarrows$-action on W by associating to $c \in C$ the set $\{w \in W \mid w \in c\}$ and to $s \in S$ the pair $(\{w \in W \mid w \in^- s\}, \{w \in W \mid w \in^+ s\})$. This is 1-1 by extensionality and is a homomorphism by the fourth sentence in T. So T witnesses that the class of $\updownarrows$-actions is pseudo-elementary. ☐

It is straightforward to verify that the operation F which takes a set X to the full $\updownarrows$-action on X turns disjoint unions of sets into products of algebras. Thus Proposition 13 applies and we have:

Corollary 40. *The class of* $\uparrow\!\!\downarrow$*-actions is axiomatized by the Horn clause theory of* $F(1)$.

It's worth writing down $F(1)$ explicitly: $F(1) = (C(1), S(1))$, where, naming $0 = \emptyset$, we have $C(1) = \{0, 1\}$ and $S(1) = \{(1, 0), (0, 0), (1, 1)\}$. On C, $(1, 0)$ acts as an identity operation, $(0, 0)$ as a constant operation with value 0, and $(1, 1)$ as a constant operation with value 1.

Now our goal is to characterize the class of $\uparrow\!\!\downarrow$-actions by conditions which translate to Horn clause axioms.

Recall that an action is *idempotent* if $css = cs$ for all $c \in C$ and $s \in S$. We say an action is *previous redundant* if $csts = cts$ for all $c \in C$ and $s, t \in S$. Further, an action is *fully previous redundant* if $csws = cws$ for $c \in C$, $s \in S$, and $w \in S^*$. Previous redundant is so called because from the point of view of the second s, the previous s is redundant and can be removed.

Lemma 41. *Any action which is idempotent and previous redundant is fully previous redundant.*

Proof. By induction on the length of the word $w \in S^*$. The cases when w has length 0 and 1 are covered by idempotence and previous redundance.

Now suppose that the length of w is $n + 1 \geq 2$, and write w as $w't$, where w' is a word of length n. Then $csws = (csw')ts = (csw')sts$ by previous redundant. Applying the induction hypothesis to $csw's$, this is equal to $cw'sts = cw'ts = cws$, by another application of previous redundant. $\square$

An *n-step link* between c and d is a sequence $c = a_0, a_1, \ldots, a_{n-1}, a_n = d$ of elements of C and a sequence $w_1, \ldots, w_n$ of words in S^* such that for each $i = 1, \ldots, n$, a_{i-1} and a_i are fixed points of w_i, i.e. $a_{i-1}w_i = a_{i-1}$ and $a_iw_i = a_i$. A *strong link* between c and d is an n-step link, for some $n \geq 0$, such that additionally $cw_i = dw_i$ for all $i = 1, \ldots, n$. Every $c \in C$ is trivially strongly linked to itself (by a 0-step link). A strong link between c and d is *nontrivial* if $c \neq d$.

Note that there is a 1-step link between any two elements c and d, taking w_1 to be the empty word (or any identity operation). However, any nontrivial strong link must be at least two steps. Indeed, a 1-step link between c and d is witnessed by $w \in S^*$ such that $cw = c$ and $dw = d$. But if this link is strong, then $c = cw = dw = d$. Similarly, no identity operation can appear in a nontrivial strong link.

The condition that all strong links are trivial is expressed by infinitely many Horn clauses, obtained by varying the natural number n (the length of the n-step link) and the lengths of the sequences of variables w_i of sort S in the schema below.

$$\left(c = a_0 \wedge d = a_n \wedge \left(\bigwedge_{i=1}^{n} cw_i = dw_i \wedge a_{i-1}w_i = a_{i-1} \wedge a_iw_i = a_i\right)\right) \to (c = d)$$

We can now establish one half of our characterization.

Proposition 42. *The action $F(1)$ is idempotent, previous redundant, and has no nontrivial strong links. By Corollary 40, these conditions are true in every $\upharpoonleft\downharpoonright$-action.*

Proof. Idempotence is clear, since each element of S acts as an identity or a constant operation on C. To check previous redundant, let $c \in C$, $s, t \in S$. If $s = (1, 0)$, then $csts = ct = cts$, since s acts as an identity on C. If $s = (0, 0)$, then $csts = 0 = cts$, and if $s = (1, 1)$, then $csts = 1 = cts$.

To check that all strong links are trivial, we just need to see that 0 and 1 are not strongly linked in $F(1)$. If they were, then in particular there would be a 1-step strong link between them, but we have already seen that all 1-step strong links are trivial. □

Next, we pin down the equational theory of $\upharpoonleft\downharpoonright$-actions.

Proposition 43. *The idempotent and previous redundant equations axiomatize the equational theory of $\upharpoonleft\downharpoonright$-actions.*

Proof. That the $\upharpoonleft\downharpoonright$-actions are idempotent and previous redundant follows from Proposition 42.

In the other direction, first note that the only terms in sort S are single variables, and since there are $\upharpoonleft\downharpoonright$-actions in which $|S| > 1$, the only equation in sort S which is universally true on $\upharpoonleft\downharpoonright$-actions is the tautology $s = s$.

So let $cs_1 \cdots s_n = dt_1 \cdots t_m$ be some equation in sort C that is universally true in $\upharpoonleft\downharpoonright$-actions. First we note that c must be the same variable as d. Otherwise, in $F(1)$, put $c = 0$, $d = 1$, and put all S-variables equal to $(1, 0)$. Then the two sides are different.

By repeatedly applying idempotence and previous redundant on each side, we may assume that among the s_i each variable occurs only once, and similarly for the t_j.

Next, we observe that the two sides must have the same S-variables and hence the same length. Otherwise, without loss of generality, let s_i be a variable that doesn't occur among the t_j. Again in $F(1)$, put $s_i = (0, 0)$, put all other S-variables equal to $(1, 0)$, and put $c = d = 1$. Then the two sides are different.

So we are looking at an equation like $cs_1 \cdots s_n = ct_1 \cdots t_n$. We now show that $s_n = t_n$, then $s_{n-1} = t_{n-1}$, and so on down to $s_1 = t_1$.

If $s_n \neq t_n$, then we could put $s_n = (0, 0)$ and $t_n = (1, 1)$ and the two sides would be different. By induction, assume $s_i = t_i$ for $i > k$, and suppose for contradiction that $s_k \neq t_k$. We can put $s_i = t_i = (1, 0)$ for $i > k$ and put $s_k = (0, 0)$ and $t_k = (1, 1)$. Then $cs_1 \cdots s_n = 0 \neq 1 = ct_1 \cdots t_m$.

Hence the equation $cs_1 \cdots s_n = dt_1 \cdots t_n$ is a tautology, from which the original equation follows by applications of idempotence and previous redundant. □

Unlike actions expressed using set intersection (Theorem 38), the class of $\upharpoonleft\downharpoonright$-actions does not have an equational axiomatization. This is demonstrated by the following example, which shows that the condition that all strong links are trivial does not follow from the equational theory.

Example 44. Let $C = \{c, d, e\}$, let $S = \{s, t\}$, and put $cs = ds = c$, $ct = dt = d$, and $es = et = e$. Letting $a_0 = c$, $a_1 = e$, $a_2 = d$, and $w_1 = s$, $w_2 = t$ we get a 2-step link between c and d, and in fact this is a nontrivial strong link, since $cs = ds$ and $ct = dt$. Hence (C, S) is not an $\updownarrows$-action.

To see that this action is fully previous redundant, consider the equation $xywy = xwy$ with $x \in C$, $y \in S$, and $w \in S^*$. If $x = e$, then both sides are e. Otherwise, both sides are c or d in accordance with whether y is t or s.

In the proof of Theorem 47, we will use two auxiliary actions, (C, S^*), and $(\overline{C}, S)$, constructed from an action (C, S).

Recall that S^* is the set of words in S. Note that there is a natural action of S^* on C, and that the action (C, S) embeds into the action (C, S^*).

Lemma 45. *If (C, S) is an idempotent and previous redundant action in which all strong links are trivial, then so is (C, S^*).*

Proof. Any word $w \in (S^*)^*$ is equivalent to a word $w' \in S^*$. Then any pair of elements in C which are strongly linked in (C, S^*) are also strongly linked in (C, S), and hence all strong links are trivial in (C, S^*).

For the other axioms, we show that (C, S^*) is fully previous redundant. If $c \in C$, $s, t \in S^*$, then $csts = cts$ by n applications of fully previous redundant in (C, S), where n is the length of the word s. □

Define a binary relation $\sim$ on C by $c \sim d$ if and only if there exists $s \in S$ such that s is not an identity operation and $c = cs$ and $d = ds$. When the action is idempotent, this is equivalent to putting $c \sim d$ when both c and d are in the image of a common non-identity operation. $\sim$ is a symmetric relation, so its reflexive and transitive closure $\approx$ is an equivalence relation. Explicitly, we have $c \approx d$ if and only if for some $n \geq 0$ there exist $a_0, \ldots, a_n \in C$ and non-identity operations $s_1, \ldots, s_n \in S$ such that $c = a_0$, $d = a_n$, $a_{i-1}s_i = a_{i-1}$, and $a_i s_i = a_i$ for $i = 1, \ldots, n$. Let $\overline{C} = C/\approx$.

This definition is very similar to the definition of an n-step link, but here we require the witnesses s_i to be in S, not S^*, and we exclude identity operations.

Lemma 46. *For any fully previous redundant action (C, S), $\approx$ is a congruence on C, i.e. $(\overline{C}, S)$ inherits the structure of an action. Moreover, $(\overline{C}, S)$ is an $\updownarrows$-action.*

Proof. We must check that for all $c, d \in C$ and $s \in S$, if $c \approx d$, then $cs \approx ds$. If s is an identity operation, then $cs = c \approx d = ds$. If s is not an identity operation, then in fact $cs \sim ds$ by idempotence.

To show that $(\overline{C}, S)$ is an $\updownarrows$-action, we embed it in an $\updownarrows$-action. Note that for all $s \in S$, s is either an identity operation or a constant operation on $\overline{C}$. Indeed, if s is an identity operation on C, then the same is true on $\overline{C}$. If s is not an identity operation on C, then for all $a, b \in C$, $as \sim bs$ by idempotence, so $as = bs$ in $\overline{C}$, and s is a constant operation on $\overline{C}$.

We define an embedding $\psi\colon (\overline{C}, S) \to F(\overline{C} \coprod S)$ as follows:

$$c \mapsto \{c\} \text{ if } c \in \overline{C}$$

$$s \mapsto \begin{cases} (\overline{C} \cup \{s\}, \emptyset) \text{ if } s \in S \text{ and } s \text{ is an identity operation} \\ (\{d, s\}, \{d\}) \text{ if } s \in S \text{ and } s \text{ is a constant operation with value } d \end{cases}$$

This map is clearly injective on $\overline{C}$, and the dummy element s is included in $\psi(s)$ for all s to ensure that it is injective on S.

Now if $c \in \overline{C}$ and $s \in S$ is an identity operation, then $\psi(c)\psi(s) = (\{c\} \cap (\overline{C} \cup \{s\})) \cup \emptyset = \{c\} = \psi(c) = \psi(cs)$. If $s \in S$ is a constant operation with value d, then $\psi(c)\psi(s) = (\{c\} \cap \{d, s\}) \cup \{d\} = \{d\} = \psi(d) = \psi(cs)$. □

Theorem 47. *An action is an $\uparrow\!\!\downarrow$-action if and only if it is idempotent and previous redundant and all strong links are trivial.*

Proof. We established in Proposition 42 that all $\uparrow\downarrow$-actions are idempotent and previous redundant and have no nontrivial strong links. It remains to show the converse.

By Propositions 39 and 5, it suffices to consider finitely generated actions. But any finitely generated fully previous redundant action is actually finite, because any term in the generators is equivalent to one in which no generator appears more than once. We may thus proceed by induction on $|C|$.

Our plan is to embed (C, S) into a product of $\uparrow\downarrow$-actions, from which it follows by Proposition 13 that it is an $\uparrow\downarrow$-action. To do this, we observe that if, for every pair of distinct elements in the same sort of (C, S), there is a homomorphism to some $\uparrow\downarrow$-action separating these elements, then the product of all these maps is an injective map to the product of these $\uparrow\!\!\downarrow$-actions.

To separate elements of the S sort, define a map $\varphi\colon (C, S) \to F(S)$ by $c \mapsto \emptyset$ for all $c \in C$ and $s \mapsto (\{s\}, \emptyset)$ for all $s \in S$. Then for all $c \in C$ and $s \in S$, $\varphi(c)\varphi(s) = \emptyset = \varphi(cs)$, so φ is a homomorphism, and φ is injective on S.

In the base case, when $|C| = 1$, the map described above is injective on all of (C, S), and we're done. So let $|C| > 1$ and let $c \neq d$ in C be two elements to separate.

Case 1: There exists $t \in S$ such that $ct \neq dt$, and t is not an identity operation.

We define a map $\varphi\colon (C, S) \to (C, S^*)$ by $c \mapsto ct$ for $c \in C$ and $s \mapsto st$ for $s \in S$. This is a homomorphism, since for all $c \in C$ and $s \in S$, $\varphi(c)\varphi(s) = ctst = cst = \varphi(cs)$ by previous redundant. Since $ct \neq dt$, $\varphi(c) \neq \varphi(d)$.

By Lemma 45, (C, S^*) is a previous redundant action in which all strong links are trivial, and the image of φ is a subalgebra $(Ct, St) \subseteq (C, S^*)$, so the same is true of (Ct, St).

We claim that $|Ct| < |C|$. Then we are done by induction, since (Ct, St) is an $\uparrow\!\!\downarrow$-action. By definition $Ct \subseteq C$. Suppose for contradiction it were all of C. Then for all $c \in C$, $c = dt$ for some $d \in C$, so $ct = dtt = dt = c$, and t is an identity operation on C, contradiction.

Case 2: For all $t \in S$, either $ct = dt$, or t is an identity operation.

By Lemma 46, the map $q\colon (C,S) \to (\overline{C}, S)$ is a homomorphism to an $\updownarrows$-action. We'll be done if we show that q separates c and d, i.e. that $c \not\approx d$.

Suppose for contradiction that $c \approx d$. This is witnessed by sequences $c = a_0, a_1, \ldots, a_n = d$ in C and $s_1, \ldots, s_n$ in S such that for all i, $a_{i-1}s_i = a_{i-1}$, $a_i s_i = a_i$, and s_i is not an identity operation. But then $cs_i = ds_i$, so this data would also witness that c and d are strongly linked, contradicting the assumption that (C,S) has no nontrivial strong links. □

We conclude this section by considering the question of what changes if, in the definition of the full $\updownarrows$-action, the requirement that $s^+ \subseteq s^-$ is dropped. Formally, we have a new construction F' of actions from sets, defined by $F'(X) = (C', S')$ where

$$\begin{aligned} C' &= \{c \mid c \subseteq X\} \\ S' &= \{(s^-, s^+) \mid s^-, s^+ \subseteq X\} \\ f(c, (s^-, s^+)) &= (c \cap s^-) \cup s^+. \end{aligned}$$

Say an action is a $\updownarrows'$-action if it is isomorphic to a subalgebra of $F'(X)$ for some set X. It is easy to check once again that the class of $\updownarrows'$-actions is pseudo-elementary and that F' turns disjoint unions of sets into products of algebras, so Proposition 13 applies.

Intuitively, if an element x is in s^+, it doesn't matter whether it is in s^-: if intersection with s^- removes it, it will just get added in again by union with s^+. So in moving from $F(X)$ to $F'(X)$, we haven't made a substantial change; we have only added some extra elements of the S sort of $F'(X)$ which have the same action on C as elements that were already in $F(X)$. The following proposition makes this precise.

Proposition 48. *Every $\updownarrows'$-action is a $\updownarrows$-action, and vice versa.*

Proof. By Proposition 13, the $\updownarrows$-actions and $\updownarrows'$-actions are the classes of structures generated under product and substructure by $F(1)$ and $F'(1)$, respectively, so it suffices to show that $F(1)$ is an $\updownarrows'$-action and $F'(1)$ is an $\updownarrows$-action.

We have $F(1) = (C, S)$ and $F'(1) = (C', S')$, where

$$\begin{aligned} C = C' &= \{0, 1\} \\ S &= \{(1,0), (0,0), (1,1)\} \\ S' &= \{(1,0), (0,0), (1,1), (0,1)\} \end{aligned}$$

Now clearly $F(1)$ is an $\updownarrows'$-action, since it embeds in $F'(1)$. In the other direction, since $(0,1)$ and $(1,1)$ act on C in the same way, we can embed $F'(1)$ into an $\updownarrows$-action in a way that separates them with a dummy element x. Define a map $F'(1) \to F(1 \cup \{x\})$ which is the identity on C and acts as follows on S':

$$\begin{aligned} (1,0) &\mapsto (1,0) \\ (0,0) &\mapsto (0,0) \\ (1,1) &\mapsto (1,1) \\ (0,1) &\mapsto (1 \cup \{x\}, 1). \end{aligned}$$

□

3.3 ⇅-biactions

A *biaction* (C, S^-, S^+) is a pair of functions $f\colon C \times S^- \to C$ and $g\colon C \times S^+ \to C$. We write $f(c, s)$ as cs and $g(c, t)$ as ct.

Given a set X, we form a biaction $F(X)$ called the *full ↑↓-biaction on* X by setting $C = S^- = S^+ = \mathcal{P}(X)$, and for $c \in C$, $s \in S^-$, and $t \in S^+$, we put $cs = c \cap s$ and $ct = c \cup t$.

A biaction is an *⇅-biaction* if it is isomorphic to a subalgebra of the full ⇅-biaction on some set X. In other words, an ⇅-biaction is a biaction where the elements of C, S^-, and S^+ can be identified with sets in such a way that $cs = c \cap s$ when $s \in S^-$ and $ct = c \cup t$ when $t \in S^+$.

Every ↑↓-biaction gives rise to an ↑↓-action by combining S^- and S^+ into one sort. Formally, if (C, S^-, S^+) is a subalgebra of the full ↑↓-biaction on X, we can identify the element $s \in S^-$ with $(s, \emptyset)$ and $t \in S^+$ with (X, t) in the full ⇅-action on X. However, we can not in general go the other direction. That is, given an ⇅-action (C, S) we can not in general divide S into two parts S^- and S^+ so as to have an ⇅-biaction (see Example 50). In this sense there are more ⇅-actions than ⇅-biactions.

We now present axioms for ⇅-biactions. First we note that ⇅-biactions are commutative in both the S^- and S^+ sorts in the sense that $cst = cts$ whenever s and t are both in S^- or both in S^+. This is obvious from the definition of ⇅-biactions because intersection and union are associative and commutative. Of course, elements of S^- do not commute with elements of S^+ in general.

Next we note that ⇅-biactions are idempotent and previous redundant. That is, for all $c \in C$, $s \in S^-$ and $t \in S^+$, we have $css = cs$, $ctt = ct$, $csts = cts$, and $ctst = cst$. This is because the action obtained by combining S^- and S^+ into one sort is an ⇅-action, and we've already observed that ⇅-actions are idempotent and previous redundant.

We have only stated previous redundant for variables s and t of different sorts. This is because if s and t are in the same sort, $csts = cts$ follows from commutativity and idempotence. Just as in Lemma 41, idempotence and previous redundant are enough to imply fully previous redundant: for all $c \in C$, $s \in (S^- \cup S^+)$, and $w \in (S^- \cup S^+)^*$, $csws = cws$.

We have already introduced enough axioms to describe the equational theory of ↑↓-biactions.

Proposition 49. *The equations expressing idempotence, previous redundant, and commutativity in S^- and S^+ axiomatize the equational theory of ⇅-biactions.*

Proof. Similar to the proof of Proposition 43. □

With the equational theory under our belt, we may now more easily present an example of an ⇅-action which can't be reinterpreted as an ⇅-biaction.

Example 50. Let $C = \{c, d, e\}$ where $c = \{1\}$, $d = \{2\}$, and $e = \{3\}$. Let $S = \{s_c, s_d, s_e\}$ where $s_c = (\{1\}, \{1\})$, $s_d = (\{2\}, \{2\})$, and $s_e = (\{3\}, \{3\})$. Each s_x acts as the constant function with value x. Clearly (C, S) is an ↑↓-action. The question under consideration is whether we can divide S into two parts S^+ and S^- so that the resulting biaction is an $\uparrow\downarrow$-biaction. Any way of doing this will involve putting two elements of S into the same sort, say s_x and s_y (where $x \neq y$). If we indeed have an $\uparrow\downarrow$-biaction we should have $y = cs_x s_y = cs_y s_x = x$, which is a contradiction.

Once again, the equational theory is not enough to axiomatize the class in question, as the following example illustrates.

Example 51. Let $C = \{c, d\}$, $S^- = \{s\}$, and $S^+ = \{t\}$. Define $cs = ct = ds = dt = d$. This biaction is idempotent, previous redundant, and commutative in S^- and S^+. However, it can't be an $\uparrow\downarrow$-biaction because we can't go from c to d by subtracting elements by s on the one hand and adding elements by t on the other ($cs = d$ implies $d \subseteq c$ and $ct = d$ implies $c \subseteq d$).

So we need to add Horn clause axioms to supplement our equational ones. The first axiom is called the *basic subset axiom.* Let $s \in S^-$ and $t \in S^+$. If $cs = ds$ and $ct = dt$ and $es = et$, then $c = d$. Let's see why this axiom is true for the $\uparrow\downarrow$-biactions. First note that $t \subseteq et = es \subseteq s$, and so $t \subseteq s$. Next, $cs = ds$ implies c and d agree inside s, and $ct = dt$ implies c and d agree outside t. Since t is a subset of s, we get that c and d agree everywhere.

Next we add a series of modified versions of the basic subset axiom. For each word w consisting of variables of sorts S^- and S^+, we add w to the end of each term that occurs in the Subset axiom to form a new axiom. That is, we get an axiom

$$csw = dsw \wedge ctw = dtw \wedge esw = etw \to cw = dw$$

for each word w. Let's call all these axioms the *extra subset axioms.*

Let's check that the extra subset axioms are true in $\uparrow\downarrow$-biactions. We will do this by induction as follows. Suppose that we have a Horn clause

$$(\star) \qquad (x_1 = y_1 \wedge \cdots \wedge x_n = y_n) \to (x_{n+1} = y_{n+1}),$$

where x_i, y_i are terms, which is universally true in $\uparrow\downarrow$-biactions. Let s be a variable of sort S^- or S^+. We wish to show that

$$(x_1 s = y_1 s \wedge \cdots \wedge x_n s = y_n s) \to (x_{n+1} s = y_{n+1} s)$$

is also universally true in $\uparrow\downarrow$-biactions. This will be enough, since each extra subset axiom can be built up from the basic subset axiom adding one variable at a time. Consider an $\uparrow\downarrow$-biaction (C, S^-, S^+) and assignment of variables so that $x_i s = y_i s$ for all $1 \leq i \leq n$.

In the case s is of sort S^-, form a new $\uparrow\downarrow$-biaction $(C \cap s, S^- \cap s, S^+ \cap s)$ which is the restriction of the original $\uparrow\downarrow$-biaction to s. In detail, $C \cap s = \{c \cap s \mid c \in C\}$, $S^- \cap s = \{t \cap s \mid t \in S^-\}$, and $S^+ \cap s = \{t \cap s \mid t \in S^+\}$. There is an obvious homomorphism φ from the original to the restriction given by intersection by s on each sort. Since $\varphi(x_1) = x_1 s$ and $\varphi(y_1) = y_1 s$ and so on, we have by assumption

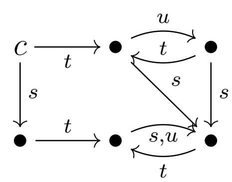
$$\varphi(x_1) = \varphi(y_1) \wedge \cdots \wedge \varphi(x_n) = \varphi(y_n)$$

and, since φ is a homomorphism, these equations are an instance of the premises of ($\star$) in the restriction. Since the restriction is an $\uparrow\downarrow$-biaction, we get the conclusion of ($\star$), $\varphi(x_{n+1}) = \varphi(y_{n+1})$. Hence $x_{n+1}s = y_{n+1}s$ in (C, S^-, S^+), as desired.

In the case s is of sort S^+, we form a new $\Updownarrow$-biaction $(C \cup s, S^- \cup s, S^+ \cup s)$, which is essentially the restriction of the original biaction to the complement of s. The argument goes just as in the S^- case. Alternatively, this case follows from the duality of $\Updownarrow$-biactions.

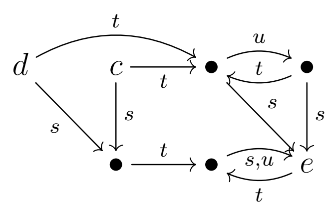
Example 52. We provide an example showing that the extra subset axioms do not follow from previous redundant, commutativity in S^- and S^+, and the basic subset axiom.

Consider the $\Updownarrow$-biaction given by the diagram below. Here, $s, u \in S^-$ and $t \in S^+$. If there is no arrow out of a node labeled by an element of S^+ or S^-, then that node is fixed by that element. For example, $cu = c$.



To see that this is an $\Updownarrow$-biaction, it suffices to check that is is the subalgebra of the full $\Updownarrow$-biaction on three elements $\{1, 2, 3\}$ generated by $c = \{1\}$ in C, $u = \{1, 2\}$ and $s = \{2\}$ in S^-, and $t = \{2, 3\}$ in S^+.

Now we replace the element c with two elements, c and d, obtaining the following biaction:



Since there are no arrows into c and d from any vertices other than themselves, it is easy to see that this biaction is still idempotent, previous redundant, and commutative in S^- and S^+. Moreover, since no element is sent to the same place by both s and t or both u and t, there are no instances of the basic subset axiom to check. However, we have $csu = dsu$ and $ctu = dtu$ and $esu = etu$, but $cu = c \neq d = du$, so this biaction fails to satisfy the extra subset axioms.

Lemma 53. *Let $B = (C, S^-, S^+)$ be a biaction satisfying the axioms (idempotence, previous redundant, commutativity in S^- and S^+, and the (basic and extra) subset axioms). Let t be an element of S^- or S^+. Define a biaction $B_t = (C_t, S_t^-, S_t^+)$ by putting*

$$C_t = \{ct \in C \mid c \in C\}$$
$$S_t^- = S^-$$
$$S_t^+ = S^+,$$

and the action is defined as follows. Given $s \in S_t^-$ or $s \in S_t^+$, and $c \in C_t$ we define the B_t-action of s on c to be cst. Then B_t also satisfies the axioms.

Proof. The equational axioms are easy to check. For example, let $c \in C_t$, and $s, u \in S^- t$. Then commutativity of s and u on c in B_t amounts to the equation $cstut = cutst$ in B, which is equivalent to $csut = cust$ in B by previous redundant, and this last equation is true by commutativity in B.

It remains to check B_t satisfies the subset axioms. Let one of the subset axioms be given, written as

$$(\star) \qquad (x_1 = y_1 \wedge \cdots \wedge x_n = y_n) \to (x_{n+1} = y_{n+1}).$$

Suppose variables are given assignments in such a way that $x_1 = y_1, \ldots, x_n = y_n$ in B_t. Then, applying previous redundant to remove all intermediate t's, we have $x_i t = y_i t$ in B for all $1 \leq i \leq n$. And so we may cite the extra subset axiom obtained from $(\star)$ by adding t to every term to conclude that $x_{n+1}t = y_{n+1}t$ in B, i.e. $x_{n+1} = y_{n+1}$ in B_t. □

Theorem 54. *A biaction is an $\uparrow\!\downarrow$-biaction if and only if it is idempotent, previous redundant, and commutative in S^- and S^+, and it satisfies the (basic and extra) subset axioms.*

Proof. When introducing the axioms, we proved that $\uparrow\!\downarrow$-biactions satisfy these axioms. So it remains to show that a biaction satisfying these axioms is an $\uparrow\!\downarrow$-biaction.

We can do the same tricks we did in the case of $\uparrow\!\downarrow$-actions: $\uparrow\!\downarrow$-biactions form a pseudo-elementary class by essentially the same argument as in Proposition 39 for $\uparrow\!\downarrow$-actions. Also, $\uparrow\!\downarrow$-biactions are the subalgebras of full $\uparrow\!\downarrow$-biactions of the form $F(X)$, and F is a function which turns disjoint unions of sets into products of algebras, so the class of $\uparrow\!\downarrow$-biactions is closed under substructure and product (Proposition 13). So given a biaction $B = (C, S^-, S^+)$ satisfying the axioms, it suffices to find, for each pair of distinct elements in the same sort, a homomorphism to an $\uparrow\!\downarrow$-biaction separating these two elements.

Our axioms are universal (in fact they are Horn clauses) and so by Proposition 5, we need only check that every finitely generated model of the axioms is an $\uparrow\!\downarrow$-biaction. But, once again, fully previous redundant implies that every finitely generated model is finite, and we can do induction on $|C|$.

First we show that no matter what the size of C, we can separate elements in sort S^- and in sort S^+. Let's consider S^-. Define a map $\varphi\colon (C, S^-, S^+) \to F(S^-)$ by $c \mapsto \emptyset$ for all $c \in C$, $s \mapsto \{s\}$ for all $s \in S^-$, and $t \mapsto \emptyset$ for all $t \in S^+$. It's easy to check that φ is a homomorphism and it is injective on S^-. S^+ works dually.

We turn now to C. In the base case, when $|C| = 1$, there is no pair of distinct elements to separate in sort C, and so we're done. So let $|C| > 1$ and let $c \neq d$ in C.

Case 1: There is a $t \in S^-$ or a $t \in S^+$ such that $ct \neq dt$ and t is not an identity operation. Consider the biaction $B_t = (C_t, S_t^-, S_t^+)$ defined as in Lemma 53.

We will show that $|C_t| < |C|$ and that B_t satisfies the axioms, and so by induction we can conclude that B_t is an $\uparrow\!\downarrow$-biaction. Consider the map $\varphi\colon B \to B_t$ defined by $\varphi(c) = ct$ for $c \in C$ and $\varphi(s) = s$ for s in either S^- or S^+. This is a homomorphism, since $\varphi(cs) = cst =$

$ctst = \varphi(c)\varphi(s)$, and it has $\varphi(c) \neq \varphi(d)$ by assumption. Hence we've found a separating homomorphism to an $\uparrow\!\!\downarrow$-biaction.

Of course $|C_t| \leq |C|$. If $|C_t| = |C|$, then for every $c \in C$ there is $d \in C$ such that $dt = c$. Then $ct = dtt = dt = c$ by idempotence and so t is an identity operation, contrary to assumption.

Case 2: For every $t \in S^- \cup S^+$, either $ct = dt$ or t is an identity operation.

We form a quotient $\bar{B} = (\bar{C}, S^-, S^+)$ of $B = (C, S^-, S^+)$ as follows. For $a, b \in C$, we put $a \approx b$ when $as = a$ and $bt = b$ for some non-identity operations $s, t \in S^-$ or some non-identity operations $s, t \in S^+$, or $a = b$. In other words, we identify all the elements of C which are fixed by any non-identity operation in S^-, and similarly we identify all the elements of C which are fixed by any non-identity operation in S^+. To show this is transitive, it suffices to show that there is no element which is fixed by both a non-identity operation in S^- and a non-identity operation in S^+. Suppose that for some $e \in C$, $es = e = et$ where $s \in S^-$ and $t \in S^+$ and s and t are not identity operations. Then also $cs = ds$ and $ct = dt$ by assumption, and so the premises of the basic subset axiom are satisfied. We conclude that $c = d$, which is a contradiction.

Now let's check that $\approx$ is a congruence. If $a \approx b$ and s is in S^- or S^+, then $as \approx bs$ because either s is an identity operation and $as = a \approx b = bs$, or $ass = as$ and $bss = bs$ witness that $as \approx bs$.

Next we show $c \not\approx d$. Suppose for contradiction that $c \approx d$, and suppose that this is witnessed by $s, t \in S^-$ non-identity operations such that $cs = c$ and $dt = d$ (the case $s, t \in S^+$ is the same). By our assumptions, we get $c = cs = ds$ and $d = dt = ct$. But then $c = ds = cts = cst = ct = d$, a contradiction.

So the quotient map is a homomorphism from B to $\bar{B}$ that separates c and d. It remains to show that $\bar{B}$ is an $\uparrow\downarrow$-biaction. Observe that every non-identity operation $s \in S^-$ is a constant operation with the same constant in each case, since if $a, b \in C$ and $s, t \in S^-$ are non-identity operations, $ass = as$ and $btt = bt$ witnesses that $as \approx bt$. The same is true for S^+. The argument for transitivity above showed also that these constants must be different. Let's call them a_- and a_+. We define a map $\varphi\colon \bar{B} \to F(\bar{C} \cup S^- \cup S^+)$ as follows.

$$a \mapsto \begin{cases} \{a\} \cup S^+ & \text{if } a \neq a_-, a_+ \\ S^+ & \text{if } a = a_- \\ \bar{C} \cup S^+ & \text{if } a = a_+ \end{cases}$$

$$s \in S^- \mapsto \begin{cases} \bar{C} \cup \{s\} \cup S^+ & \text{if } s \text{ is an identity} \\ \{s\} \cup S^+ & \text{if } s \text{ is constant} \end{cases}$$

$$s \in S^+ \mapsto \begin{cases} \{s\} & \text{if } s \text{ is an identity} \\ \bar{C} \cup \{s\} & \text{if } s \text{ is constant} \end{cases}$$

It is easily checked that this is a homomorphism and it is 1-1 on each sort. □

One question for further study is to understand what kind of biactions you get if instead of starting with distributive lattices (powersets) you started with some other kind of lattice.

3.4 Set bands

Let (C, S) be the full $\uparrow\downarrow$-action on some set. What product $\cdot$ can we put on S so that $(cs)t = c(s \cdot t)$? The following calculation gives an answer. Let $s = (s^-, s^+)$ and $t = (t^-, t^+)$. Then,

$$\begin{aligned}(cs)t &= (((c \cap s^-) \cup s^+) \cap t^-) \cup t^+ \\ &= (c \cap s^- \cap t^-) \cup (s^+ \cap t^-) \cup t^+ \\ &= (c \cap (s^- \cap t^-)) \cup (c \cap t^+) \cup (s^+ \cap t^-) \cup t^+ \\ &= (c \cap ((s^- \cap t^-) \cup t^+)) \cup ((s^+ \cap t^-) \cup t^+).\end{aligned}$$

This motivates the following definition. Given a set X, we form an algebra $(F(X), \cdot)$, called the *full set band on* X, by setting

$$\begin{aligned}F(X) &= \{(s^-, s^+) \mid s^-, s^+ \subseteq X \text{ and } s^+ \subseteq s^-\} \\ (s^-, s^+) \cdot (t^-, t^+) &= ((s^- \cap t^-) \cup t^+, (s^+ \cap t^-) \cup t^+)\end{aligned}$$

In general, an algebra is called a *set band* if it isomorphic to a subalgebra of the full set band on some set X.

Set bands are indeed bands (idempotent semigroups), and their definition involves intersection and union, hence the name "set bands". Further, set bands are right regular ($xyx = yx$). One way to check this is to observe that every set band is the semigroup of operations for some $\uparrow\downarrow$-action (C, S), i.e. the semigroup of functions on C generated by $\{f_s : C \to C \mid s \in S\}$, and right regularity follows from previous redundant. That conversely every semigroup of operations of an $\uparrow\downarrow$-action (which is, a priori, a quotient of a set band) is a set band follows from Theorem 56.

We state down without proof (due to the similarity with $\uparrow\downarrow$-actions) a few facts about set bands.

Proposition 55. *Associativity, idempotence, and right regularity axiomatize the equational theory of set bands.*

The class of set bands is pseudo-elementary, and F turns disjoint unions into products, and so the set bands are the subalgebras of the products of the algebra $F(1)$ and admit a Horn clause axiomatization.

In [13], Margolis et al. study the class of subsemigroups of the "hyperplane face monoids", which they identify as the quasivariety of algebras generated (under subalgebra and product) by a certain three-element algebra. This algebra is essentially $F(1)$, with the superficial difference that the order of multiplication is reversed (e.g. it is left regular instead of right regular), and so their algebras are exactly the set bands, after reversing the multiplication. They show that this quasivariety is axiomatized by associativity, idempotence, left regularity, and a schema of Horn clauses which is called (CC) in [13], and which is very similar to our condition on $\uparrow\downarrow$-actions that all strong links are trivial. Their method led directly to the proof of Theorem 47, and inspired the proof of Theorem 54. We think it is interesting that the same class of algebras arose in these two ways, with such different motivations.

For completeness, we'll state a version of the theorem characterizing set bands, adapted to our vocabulary. We say that two elements c, d of an algebra $(S, \cdot)$ are *strongly linked* when for some natural number n there exist $a_0, \ldots, a_n$ and $s_1, \ldots, s_n$ in S such that $c = a_0$, $d = a_n$, and for $i = 1, \ldots, n$, $a_{i-1}s_i = a_{i-1}$, $a_is_i = a_i$, and $cs_i = ds_i$, and we say that the strong link between c and d is *trivial* when $c = d$.

Theorem 56. *Set bands are axiomatized by associativity, idempotence, right regularity, and the condition that all strong links are trivial.*

3.5 Orders from Intersection

Let us return again to the situation where we only allow intersection. That is, we look at $\downarrow$-actions. Now, each $\downarrow$-action $f\colon C \times S \to C$ gives rise to a partial order $\leq_f$ on C as follows. We put $c \leq_f d$ iff $c \in O(d) := \{ds \mid s \in S^*\}$. Intuitively, $c \leq d$ when you can be led from d to c by asserting some number of sentences. It's easy to check this is reflexive and transitive, and we already checked it's antisymmetric in the proof of Theorem 38, so it is indeed a partial order. Let's call a poset that arises in this way a *$\downarrow$-poset.*

Our goal in this section is to give a reasonable second order characterization of the class of $\downarrow$-posets, and show that this class is not first order axiomatizable. Along the way we will give some necessary first order conditions, and some sufficient first order conditions.

Before we move forward to some axioms, let me point out a minor simplification. Observe that the $\downarrow$-actions (C, S) and (C, S^*) give rise to the same partial order on C. Thus, a poset $(C, \leq)$ is a $\downarrow$-poset iff there is some set S and some idempotent and commutative action $f\colon C \times S \to C$ such that $(C, \leq) \cong (C, \leq_f)$ and $c \leq_f d$ iff there is an $s \in S$ with $ds = c$, where $\leq_f$ is defined as above.

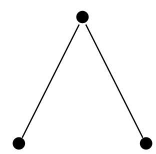
Another alternative definition of $\downarrow$-poset, for the reader's reference and comparison, are the posets that arise in the following way. Let S be a semilattice with identity that acts on a set C. For $c, d \in C$, put $c \leq d$ iff there is some $s \in S$ such that $ds = c$.

Common Upper Bound Implies Meet

First we note that not every partial order is a $\downarrow$-poset. For example, consider Figure 3.1. This poset fails to satisfy a necessary property of $\downarrow$-posets given in the following proposition.

Proposition 57. *If $(C, \leq)$ is a $\downarrow$-poset, then for all $y, a \in C$, if there is $b \in C$ with $b \geq a, y$, then $y \wedge a$ exists.*

Figure 3.1: A poset which is not a $\downarrow$-poset



Proof. According to our simplification, there is some set S and some idempotent and commutative action $f\colon C \times S \to C$ such that $(C, \leq) \cong (C, \leq_f)$ and $c \leq_f d$ iff there is an $s \in S$ with $ds = c$. Let $s, s' \in S$ such that $bs = a$ and $bs' = y$. Then we claim $bss' = y \wedge a$. Of course $bss' \leq a, y$. Let $x \leq a, y$. We want to show $bss' \geq x$. Let $at = x = yt'$. Then

$$\begin{aligned} bss'tt' &= bsts't' \\ &= ats't' \\ &= xs't' \\ &= bs't's't' \\ &= bs't' \\ &= x \end{aligned}$$

□

So if C is a $\downarrow$-poset, then there must be certain meets. In fact, if all meets exist, then C must be $\downarrow$-poset.

Proposition 58. *If C is a meet semilattice, i.e. C is a partial order with finite meets, then C is a $\downarrow$-poset.*

Proof. Note that $\wedge\colon C \times C \to C$ is idempotent and commutative. We claim that $\leq = \leq_\wedge$. If $c \leq d$ then $d \wedge c = c$ and so $c \leq_\wedge d$. If $c \leq_\wedge d$ then there is $(s_1 \cdots s_n) \in C^*$ such that $d \wedge s_1 \wedge \cdots \wedge s_n = c$ and so $c \leq d$. □

Not all $\downarrow$-posets are meet semilattices. For example, the poset in Figure 3.2 lacks meets but is a $\downarrow$-poset. Of course, any time two elements have a common upper bound then there is a meet.

A Further Necessary Condition

A poset satisfying the condition "common upper bounds implies meets" is not enough to ensure that it is a $\downarrow$-poset. For example, consider Figure 3.3. The issue with Figure 3.3 is that it doesn't satisfy a more complicated necessary condition explained by the following proposition. There must exist a sort of "relative meet".

Figure 3.2: A ↓-poset which is not a meet semilattice

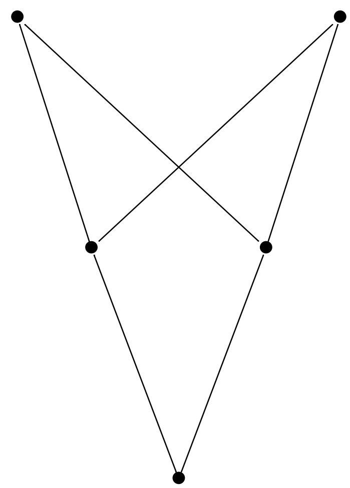


Figure 3.3: Not a ↓-poset but satisfies "common upper bound implies meet"

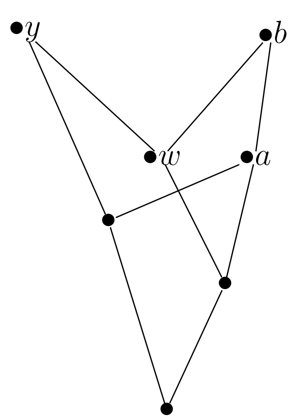
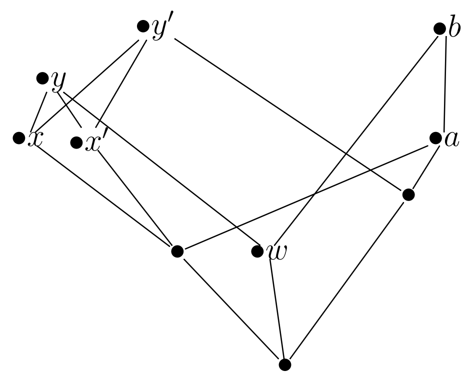


Figure 3.4: Not quite a $\downarrow$-poset



Proposition 59. *If C is a $\downarrow$-poset and $y, a, b \in C$ with $b \geq a$ then there is a $y_a^b \in C$ such that*

1. $y_a^b \leq y$
2. *For all $n \leq y, a$ we have $n \leq y_a^b$*
3. *For all $w \leq y, b$ we have $y_a^b \wedge w \leq a$ ($y_a^b, w \leq y$ so $y_a^b \wedge w$ must exist by Proposition 57)*

Proof. By our minor simplification mentioned at the beginning of this section, there is a set S and an idempotent, commutative action $f : C \times S \to C$ such that $(C, \leq) \cong (C, \leq_f)$ and $c \leq_f d$ iff there is an $s \in S$ with $ds = c$. As $b \geq a$ there is an $s \in S$ such that $bs = a$. We claim that ys works for y_a^b. Of course $ys \leq y$. Let $n \leq y, a$. Since $n \leq bs = a$, it follows that n is fixed by s. Let $ys' = n$. Then $n = ys's = yss' \leq ys$.

Now let $w \leq y, b$. Let $z = ys \wedge w$. We wish to show that $z \leq a$. Since $z \leq ys$, it follows that z is fixed by s. Since $z \leq w$ and $w \leq b$, it follows that $z \leq b$ and so $z = zs \leq bs = a$ as desired. □

We now have a stronger necessary condition, but still what we have is not sufficient. An example is given in Figure 3.4, but it will become clear only later (with Proposition 62) exactly why it doesn't work. Of course, one can check that it does satisfy what we have so far.

Selector Functions

We begin by observing how in the case where C is finite, the foregoing proposition (Proposition 59) actually can yield something a bit stronger.

Proposition 60. *Suppose C is a finite $\downarrow$-poset. Let $y, a \in C$. Then there is an element $y_a \in C$ such that*

1. $y_a \leq y$

2. *For all* $n \leq y, a$ *we have* $n \leq y_a$

3. *For all* $b \geq a$ *and all* $w \leq y, b$ *we have* $y_a \wedge w \leq a$

Proof. For each $b \geq a$, there is a y_a^b as in the previous proposition. Then we set

$$y_a := \bigwedge_{b \geq a} y_a^b$$

This works as long as it exists, and since A is finite and all the y_a^b are below y, it does exist. □

Example 61. The above proposition is not true if we forego the finiteness assumption, as this example shows. Let C consist of the following sets:

- $n = \{0, 1, \ldots, n-1\}$, for each $n \in \mathbb{N}$.
- $a = \mathbb{N} \cup \{A\}$ where A is a new symbol.
- $w_n = n \cup \{W_n\}$ where W_n is a new symbol, for each $n \in \mathbb{N}$.
- $b_n = \mathbb{N} \cup \{A, W_n\}$, for each $n \in \mathbb{N}$.
- $u_F = \mathbb{N} \cup \{W_i \mid i \notin F\}$, for each finite $F \subseteq \mathbb{N}$.

Let S consist of the following sets:

- n, for each $n \in \mathbb{N}$.
- $\mathbb{N} \cup \{A\} \cup \{W_i \mid i \neq n\}$, for $n \in \mathbb{N}$.
- $n \cup \{W_n\}$, for $n \in \mathbb{N}$.

One can check that for any $c \in C$ and any $s \in S$ we have $c \cap s \in C$. Hence, S induces a $\downarrow$-poset on C. However, this $\downarrow$-poset does not satisfy the conclusion of Proposition 60. Let $y = u_\emptyset$. Now, if y_a exists, it must be above all the natural numbers because the natural numbers are below both y and a. y_a must also be below (or equal to) y. This leaves as possibilities the u_F. So let $y_a = u_F$ for some finite $F \subseteq \mathbb{N}$ to get a contradiction. Then let n be some natural not in F, so that $W_n \in u_F$. Then $u_F \wedge w_n = w_n \not\leq a$ yet $w_n \leq y, b_n$.

In the finite case we can understand the existence of the y_a as the existence for each $a \in C$ of a certain function $h_a \colon C \to C$ (i.e. $h_a(y) = y_a$). But since we are interested in the infinite $\downarrow$-posets as well, let's consider functions $h_{ab} \colon C \to C$ that exist for each pair $a \leq b$ in C, according to Proposition 59. Such a function can also be assumed to cohere in a certain way, as explained in the following proposition.

Proposition 62. *Let C be a $\downarrow$-poset. Then for every pair $a, b \in C$ with $a \leq b$ there is a function $h_{ab}\colon C \to C$ such that*

1. *For all $y \in C$, we have $h_{ab}(y) \leq y$*
2. *For all $n \leq y, a$ we have $n \leq h_{ab}(y)$*
3. *For all $w \leq y, b$ we have $h_{ab}(y) \wedge w \leq a$*
4. *(coherence) For all $y, y', x \in C$ with $x \leq y, h_{ab}(y')$ we have $x \leq h_{ab}(y)$*

Such a function h_{ab} shall be called a "selector" function for $a \leq b$.

Proof. Let $a \leq b$ be given. We already know how to get a function that satisfies the first three items. That is, in Proposition 59 we let $bs = a$ and put $h_{ab}(y) = ys$. We just need to check the fourth. Let $x \leq y, y's$. We want to show $x \leq ys$. Since $x \leq y's$, we get that x is fixed by s, and so from $x \leq y$ we get $x = xs \leq ys$. So $x \leq ys$. □

Let's return to Figure 3.4 and check that it is not a $\downarrow$-poset. Observe that $h_{ab}(y')$ must equal y' (e.g. apply Proposition 62 item (2) with n being the two nodes immediately below a). Also note that $h_{ab}(y)$ cannot be y because of w. Finally, as $x, x' \leq y, h_{ab}(y')$ we have $x, x' \leq h_{ab}(y)$, a contradiction.

We may simplify these four assumptions as follows.

Proposition 63. *Let C be a poset. In the definition of selector function $h_{ab}\colon C \to C$ for $a \leq b$ in C (see Proposition 62) we may replace the second and third assumptions by the assumption that we have $h_{ab}(b) = a$. I.e., h_{ab} is a selector function iff the following three conditions hold:*

1. *For all $y \in C$, we have $h_{ab}(y) \leq y$*
2. *$h_{ab}(b) = a$*
3. *(coherence) For all $y, y', x \in C$ with $x \leq y, h_{ab}(y')$ we have $x \leq h_{ab}(y)$*

Proof. Suppose first that h_{ab} is a selector function according to the original definition. Then $b \leq b, b$ and so we have $h_{ab}(b) \wedge b \leq a$ by the old third condition. By the first condition we have $h_{ab}(b) \leq b$ so $h_{ab}(b) \wedge b = h_{ab}(b)$. Thus $h_{ab}(b) \leq a$. Also, we have by the old second condition $a \leq h_{ab}(b)$ since $a \leq b, a$. Thus $h_{ab}(b) = a$.

Now suppose h_{ab} is a selector function in the new sense. First we check the old second condition. Let $n \leq y, a$. Then $n \leq y, h_{ab}(b) = a$ and applying coherence we obtain $n \leq h_{ab}(y)$.

Finally let's check the old third condition. Let $w \leq y, b$. We want to show $h_{ab}(y) \wedge w \leq a$. First we explain that $h_{ab}(y) \wedge w$ exists: $h_{ab}(w) \leq w$, and $h_{ab}(w) \leq h_{ab}(y)$ because of coherence applied to $h_{ab}(w) \leq y, h_{ab}(w)$. Also, if $x \leq w, h_{ab}(y)$, then $x \leq h_{ab}(w)$ exactly by coherence. Now back to the main verification. Since $h_{ab}(y) \wedge w \leq b, h_{ab}(y)$, we have by the coherence condition that $h_{ab}(y) \wedge w \leq h_{ab}(b) = a$. □

The existence of a selector function for each pair $a \leq b$ is a second order characterization of $\downarrow$-posets.

Proposition 64. *Any poset $(C, \leq)$ with a selector function h_{ab} for each pair $a \leq b$ in C is a $\downarrow$-poset.*

Proof. Let $S = \{(a, b) \in C^2 \mid a \leq b\}$. Define $h \colon C \times S \to C$ by $h(y, (a, b)) = h_{ab}(y)$. We check that h is idempotent, commutative, and gives rise to the same order, i.e. $\leq = \leq_h$.

Idempotence says that $h_{ab}(h_{ab}(y)) = h_{ab}(y)$ for all $a, b, y \in C$ with $a \leq b$. Well, $h_{ab}(y) \leq h_{ab}(y), h_{ab}(y)$ so we have $h_{ab}(y) \leq h_{ab}(h_{ab}(y))$ by coherence. Also, $h_{ab}(h_{ab}(y)) \leq h_{ab}(y)$ by the first condition on a selector function.

Commutativity says that $h_{ab}(h_{a'b'}(y)) = h_{a'b'}(h_{ab}(y))$. By symmetry it suffices to show that $h_{ab}(h_{a'b'}(y)) \leq h_{a'b'}(h_{ab}(y))$. By coherence, then, it suffices to show $h_{ab}(h_{a'b'}(y)) \leq h_{ab}(y), h_{a'b'}(y)$. Of course $h_{ab}(h_{a'b'}(y)) \leq h_{a'b'}(y)$ (by the first condition on a selector function). To show $h_{ab}(h_{a'b'}(y)) \leq h_{ab}(y)$ it suffices, by coherence again, to show that $h_{ab}(h_{a'b'}(y)) \leq y, h_{ab}(h_{a'b'}(y))$, which is true by the first condition.

Now we check that $a \leq b$ iff $a \leq_h b$. Let $a \leq b$. Then $h_{ab}(b) = a$ by the second condition on a selector function given in Proposition 63. So $a \leq_h b$. Now let $a \leq_h b$, i.e. let $s_1, \ldots, s_n \in S$ with $bs_1 \cdots s_n = a$. Then by repeated application of the first condition on a selector function, we have $a \leq b$. □

Theorem 65. *A poset $(C, \leq)$ is a $\downarrow$-poset iff for every pair $a \leq b$ in C there is a function $h \colon C \to C$ such that*

- $h(b) = a$
- *For all $y \in C$ we have $h(y) \leq y$*
- *For all $x, y, z \in C$ we have $x \leq y, h(z)$ implies $x \leq h(y)$*

Proof. This theorem is just a summary of Propositions 62, 63, and 64. □

The condition on a function h from a poset $(C, \leq)$ to itself that it satisfy the latter two conditions in the above theorem (namely $h(y) \leq y$, and $x \leq y, h(z)$ implies $x \leq h(y)$) can be re-expressed as the conjunction of the following three conditions:

- $h(y) \leq y$
- $x \leq y$ implies $h(x) \leq h(y)$
- $x \leq h(y)$ implies $h(x) = x$

Let's call a function that satisfies these three conditions a **$\downarrow$-function**. George Bergman pointed out the following way of thinking about Theorem 65. One can check that the collection of $\downarrow$-functions on any poset form a semilattice with identity (under composition). So a poset is a $\downarrow$-poset iff there are "enough" $\downarrow$-functions in the sense that if $c \leq d$ then there is some $\downarrow$-function h with $h(d) = c$. Furthermore, every poset has a canonical $\downarrow$-poset "sitting inside" it. In detail, if $(C, \leq)$ is a poset, and $c, d \in C$, then we put $c \leq' d$ iff there is some $\downarrow$-function h with $h(d) = c$. It follows that $(C, \leq')$ is a $\downarrow$-poset, $(\leq') \subseteq (\leq)$, and $(\leq') = (\leq)$ when $(C, \leq)$ is already a $\downarrow$-poset. It is natural to wonder about the relationship between $\leq'$ and $\leq$, and here are a few initial observations. $\leq'$ is not necessarily the largest subset of $\leq$ which makes C into a $\downarrow$-poset. For example, the only $\downarrow$-function on the poset in Figure 3.1 is the identity function. So $\leq'$ will just be the diagonal binary relation. But alternatively just removing one of the line segments in the figure results in a $\downarrow$-poset as well. Another observation is that although every $\leq$-$\downarrow$-function will be a $\leq'$-$\downarrow$-function, the reverse is not true. An example is given by taking the poset of Figure 3.1 – let's specifically call the elements a, b, x with $a < x > b$ – and adding elements $a' < x' > b'$, and putting $a' > a$, $x' > x$, and $b' > b$. One can check that the only $\leq$-$\downarrow$-functions are the identity and the function determined by $a' \mapsto a$, $x' \mapsto x$, and $b' \mapsto b$. This makes the $\leq'$ poset into three disconnected line segments, and so there are eight $\leq'$-$\downarrow$-functions.

Instead of thinking about $\downarrow$-functions, one can also think about $\downarrow$-sets. Let's call a subset H of a poset $(C, \leq)$ a **$\downarrow$-set** when H is downward-closed (i.e. $x \leq y$ and $y \in H$ imply $x \in H$) and for every element $x \in C$ there is a largest element y in H which has $y \leq x$. There is a 1-1 correspondence between $\downarrow$-functions and $\downarrow$-sets. Given a $\downarrow$-function, its image is a $\downarrow$-set. The inverse of this operation is as follows. Given a $\downarrow$-set H, we obtain a $\downarrow$-function by sending $x \in C$ to the largest element y in H which has $y \leq x$.

For finite posets, we can express the existence of selector functions in a first order way using the following theory. For each $n = 1, 2, 3, \ldots$, put φ_n equal to the first order sentence which asserts:

$$\forall a \leq b \forall y_1 \cdots y_n \exists z_1 \cdots z_n \text{ such that}$$

- $z_i \leq y_i$ for $i = 1, \ldots, n$
- if $y_i = b$ then $z_i = a$ for $i = 1, \ldots, n$
- for all $x \leq y_i, z_j$ we have $x \leq z_i$, for i, j among $1, \ldots, n$

Put $T = \{\varphi_n \mid n = 1, 2, \ldots\}$. A finite poset models T iff it has selector functions (iff it is a $\downarrow$-poset). Every infinite $\downarrow$-poset satisfies T as well, but not every infinite poset modeling T is a $\downarrow$-poset.

The theory T is not finitely axiomatizable. Let me cursorily describe for every $n \geq 2$ a poset which satisfies $\{\varphi_1, \ldots, \varphi_n\}$ but not φ_{n+1}. The elements of the poset shall be

$$\begin{array}{l} y_1, \ldots, y_{n-1} \\ x_1, x'_1, \ldots, x_{n-1}, x'_{n-1} \\ 0, 1, \ldots, n-1 \\ w, q, z, a, b \end{array}$$

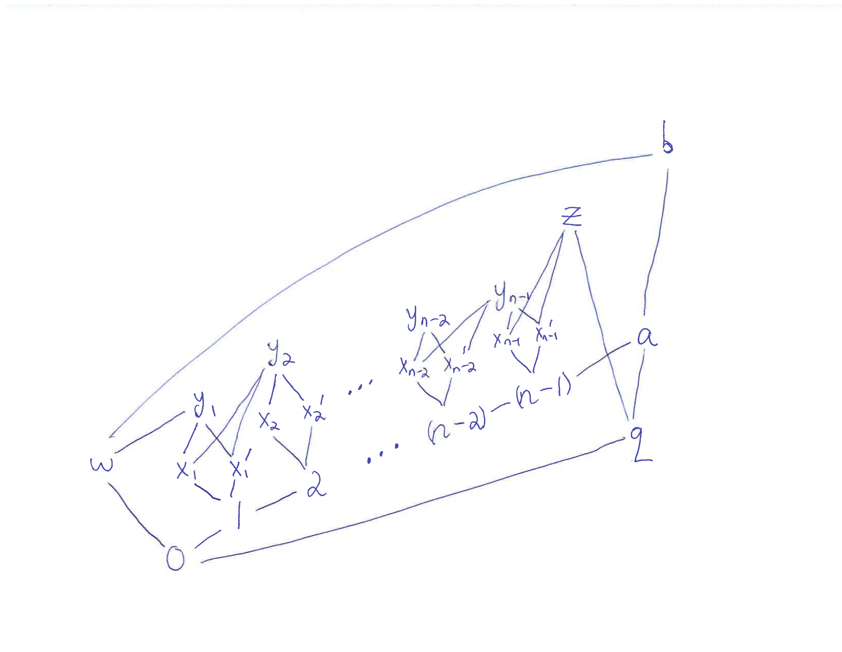


Figure 3.5: Examples showing T is not finitely axiomatizable

We stipulate that $y_1 \geq x_1, x_1', w$, that $y_i \geq x_i, x_i', x_{i-1}, x_{i-1}'$ for $i > 1$, that $x_i, x_i' \geq i$, that $i+1 \geq i$, that $w, q \geq 0$, that $z \geq x_{n-1}, x_{n-1}', q$, that $a \geq n-1, q$, and finally that $b \geq a, w$. Then we take the partial order generated by these stipulations (there's no loop in the above). See Figure 3.5 for an illustration. It can be checked it doesn't satisfy φ_{n+1} because there is no partial selector function for $a \leq b$ that works for $y_1, \ldots, y_{n-1}, z, b$, by an iterated argument similar to that involved in showing Figure 3.4 was not a $\downarrow$-poset. On the other hand, if one of the y_i's or z or b is not present in the list of elements the partial selector function is supposed to deal with, then a partial selector function can be found. Also, selector functions for the pairs other than $a \leq b$ can be found. (The details here are omitted, because they are similar to the ideas used in the next section to show non-first order axiomatizability.) Thus, the poset satisfies $\varphi_1, \ldots, \varphi_n$.

It is unclear to me at present whether there is a first order sentence θ such that a finite poset satisfies θ iff it is a $\downarrow$-poset. It is also not clear to me whether there is a polynomial time algorithm to decide whether a finite poset is a $\downarrow$-poset. (There is a non-deterministic polynomial time algorithm because we have essentially given an existential second order sentence characterizing $\downarrow$-posets. For a reference on logical characterizations of complexity classes, see [11].)

The theory T does not axiomatize $\downarrow$-posets. Indeed, no first order theory can (as we show in the next section), but we give here an extra example of how T fails (for possible future reference). Let the poset M consist of the elements a, b, w, y, q, n for each natural number

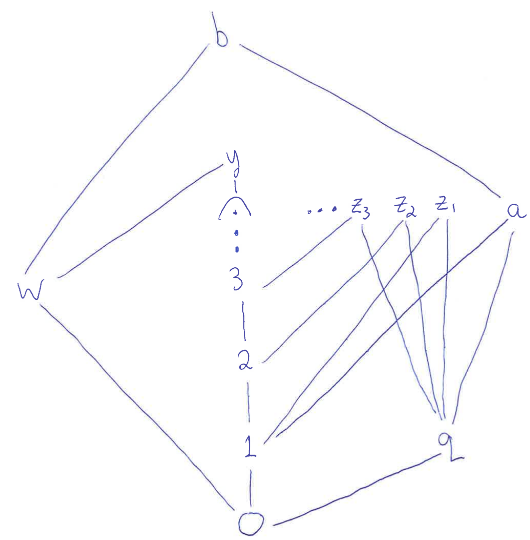


Figure 3.6: Poset modeling T which is not a $\downarrow$-poset

$n \in \mathbb{N}$, and z_n for each $n \geq 1$. We declare the usual ordering on $\mathbb{N}$, and we put $0 \leq w, q$, and $1, q \leq a$, and $a, w \leq b$, and $w \leq y$, and $n \leq y$ for each n, and $q, n \leq z_n$ for each $n \geq 1$, and we take the reflexive, transitive closure to get the poset $(M, \leq)$. See Figure 3.6 for an illustration. This is not a $\downarrow$-poset because $h_{ab}(y)$ would have to be less than y because of w, but $h_{ab}(y) \geq n$ for each n because of z_n. At the same time, it can be checked that M satisfies T.

Not First Order Axiomatizable

Theorem 66. *The class of $\downarrow$-posets is not first order axiomatizable.*

Proof. We give an example of a poset M which is not a $\downarrow$-poset, but such that any $\aleph_1$-saturated elementary extension of M is on the other hand a $\downarrow$-poset. The elements of M include all the natural numbers, which are ordered in the usual way, as well as elements a, b, w, y, q, which satisfy $0 \leq w, q$, and $1, q \leq a \leq b$, and $w \leq y, b$, and $n \leq y$ for all n. For each $n \geq 2$ there are also additional elements $u_{n1}, u_{n2}, \ldots, u_{nn}$, $x_{n1}, x_{n2}, \ldots, x_{nn}$, $x'_{n1}, x'_{n2}, \ldots, x'_{nn}$, and z_n. For each $n \geq 2$ we put $n, x_{n1}, x'_{n1} \leq u_{n1}$, and $q, x_{nn}, x'_{nn} \leq z_n$, and $1 \leq x_{ni}, x'_{ni}, x_{n(i+1)}, x'_{n(i+1)} \leq u_{n(i+1)}$ for $i = 1, 2, \ldots, n-1$. We take the reflexive, transitive closure of all of these stipulations to get the poset $(M, \leq)$. (It can be checked that there's no loop in the stipulations.) See Figure 3.7 for an illustration.

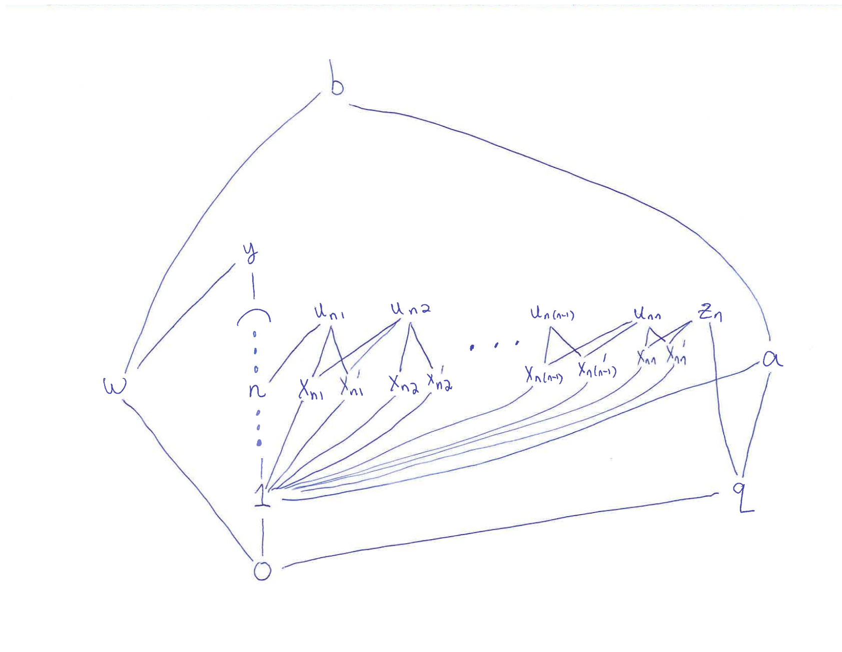


Figure 3.7: Example showing $\downarrow$-posets are not first order axiomatizable

This poset M is not a $\downarrow$-poset for the following reason. If it were, then there would be a selector function h for the pair $a \leq b$. Since 1 and q are both below a and z_n but the only thing below z_n and above both 1 and q is z_n itself, we have $h(z_n) = z_n$. From this it follows that $h(x_{nn}) = x_{nn}$ and $h(x'_{nn}) = x'_{nn}$. From this it follows that $h(u_{nn}) \geq x_{nn}, x'_{nn}$ and so $h(u_{nn}) = u_{nn}$. Similar reasoning shows that for each $i = n, n-1, \ldots, 2, 1$ we have $h(u_{ni}) = u_{ni}$, and so in fact $h(n) = n$. This implies that $h(y) \geq n$ for each n, and so $h(y) = y$. But then $h(w) = w$, and this contradicts the fact that $w \leq b$ and $w \not\leq a$.

Let M^+ be any $\aleph_1$-saturated elementary extension of M. We will show that M^+ is a $\downarrow$-poset, and so the class of $\downarrow$-posets is not first order axiomatizable. We will show that for every pair of elements $c \leq d$ in M^+ there is a selector function $h\colon M^+ \to M^+$ for them. Note that if c is a maximal element, then we can take h to be the identity function, and if c has a meet with every other element in the poset, then we can take $h(y) = c \wedge y$. We observe that every element of M is either a, or maximal, or has meets with every other element. This can be expressed in a first order way, and so the same is true of M^+. Thus, in order to check that every pair $c \leq d$ has a selector function, we need only check for $c = a$. If $d = a$ then again we can let h be the identity function, so we need only find a selector function for the pair $a \leq b$. As M^+ is $\aleph_1$-saturated, there is an element $N < y$ for which $n < N$ for all natural numbers n. We define $h(b) = a$, $h(w) = 0$, $h(c) = N$ if $c \geq N$, $h(c) = 1$ if there is a finite sequence of elements

$$N < c_1 > c_2 < c_3 \cdots c_n$$

with $n \geq 2$ and $c_i \not\leq y$ and $c = c_n \neq c_1$ (we will refer to this condition by saying that c is "connected" to N), and finally h is the identity function elsewhere. We now verify that this defines a selector function. Of course $h(b) = a$ by definition. To check that $h(c) \leq c$, we need only verify that if $N < c_1 > c_2 < c_3 \cdots c_n$ and $n \geq 2$ and $c_i \not\leq y$ then $c_n \geq 1$. Observe that in the structure M we have:

For all $d, d_1, d_2, \ldots, d_n$, if $d \geq n$ and $d < y$ and $d < d_1 > d_2 < d_3 \cdots d_n$ and $d_i \not\leq y$ for each $i = 1, \ldots, n$ then $d_n \geq 1$

This is first order, and so is true in M^+ and so $c_n \geq 1$ as desired.

Now we check the condition that $c \leq d, h(e)$ implies $c \leq h(d)$. We need only concern ourselves with d for which $h(d) < d$. Such d are: w, b, those bigger than N, and those "connected" to N. Recall that we put $h(w) = 0$. The only things above w in M are y and b, and the only thing below w is 0, and so this is true in M^+ too. To check the condition for $d = w$, we just need to check that $w \not\leq h(e)$ for any e. Since $h(e) \leq e$ and we have put $h(b) = a$ and $h(y) = N \neq b, y, w$, this is true. Now consider the condition for $d = b$. The elements in M^+ less than b are b, a, w, q, 1, and 0 (because this is true in M). Each of these is less than $h(b) = a$ except for b and w. But neither of these is less than some $h(e)$, as was already observed.

Now we check the condition for $d > N$. Recall we put $h(d) = N$. Note that the elements other than w less than y form a linearly ordered set (in M and so too in M^+). If $N < d < y$ then $c \leq d$ implies that $c \leq N$ or $c > N$. The case $c \leq N = h(d)$ is no problem. In the case $c > N$, if $c \leq h(e)$, then $N < c \leq e$ too and so $h(e) = N$ and thus $c \leq N$, a contradiction. Now let $N < d \not\leq y$. Let $c \leq d$. If $c = d$, then observe that c is maximal (because it's true in M that any element c which has $m < c \not\leq y$ for some m with $1 < m < y$ is maximal) and observe that $c \not\leq h(c) = N$ so $c \not\leq h(e)$ for any e. If $c < d$ and $c \leq y$, then $c \neq w$ and so $c \leq N$ or $c > N$ (again using first order properties of M). We already dealt with this situation. If $c < d$ and $c \not\leq y$, then any element e above c also has $e \not\leq y$ (a first order property of M) and so we get $h(e) = 1$ because such an e is "connected" to N (including the case $e = c$) unless $e = d$ and then $h(e) = N$. We know that $c \not\leq 1 < y$ and $c \not\leq N < y$. Now let $N < d = y$. If $c \leq d = y$ then $c = w$ or $c \leq N$ or $c > N$. We've dealt with these cases already.

Finally, suppose that d is "connected" to N. Let $N < c_1 > c_2 \cdots c_n$ with $c_i \not\leq y$ and $c_n = d \neq c_1$. Let $c \leq d$. Then either $c \leq 1$ or $c \not\leq y$ (a first order property of M). In the latter case it follows (again a first order property of M) that c is "connected" to N and in fact that everything above c, except possibly c_1 (which has $h(c_1) = N$), is "connected" to N. So $c \not\leq h(e)$ for any e. $\square$

Chapter 4

Finitely Determined Morphisms

In this chapter we investigate under what conditions morphisms may be pinned down by knowing finitely many of their values. We introduce a few related definitions. Proposition 92 summarizes some facts about how these definitions relate to each other. We end with some questions about generalizing the concept of finitely determined to other situations. This chapter is motivated by the observation that if we think of a structure A as the possible sentences of some language, and another structure B as the possible meanings, then a morphism $f\colon A \to B$ is a way of compositionally assigning meanings to sentences.

Definition 67. Let A and B be structures in the same signature. We say (A, B) is **finitely determined** (FD) when there is a finite subset A_0 of A such that for all morphisms $f, g\colon A \to B$ we have $f \restriction A_0 = g \restriction A_0$ implies $f = g$. We say that $(A, B, f\colon A \to B)$ is **relatively finitely determined** when there is a finite subset A_0 of A such that for all $g\colon A \to B$ we have $f \restriction A_0 = g \restriction A_0$ implies $f = g$. We say that (A, B) is **always relatively finitely determined** (ARFD) when (A, B, f) is relatively finitely determined for each f.

In terms of the intuition that $f\colon A \to B$ assigns meanings to sentences, if (A, B, f) is relatively finitely determined, then f has a shot of being learnable by a finite agent.

Example 68. Let $A = (\mathbb{Z}, S)$, the integers with the usual successor funciton, and let B also be the integers with successor except that B has an extra copy of the negative integers, so that there are two predecessors of 0. Now, (A, B) is not finitely determined, because for any finite subset A_0 of A we can find a function f such that $f(A_0)$ is contained in the non-negative integers, and so there are two different functions that match f's behavior on A_0 depending on where its tail is. On the other hand, (A, B) is always relatively finitely determined (ARFD): given $f\colon A \to B$, let $A_0 = f^{-1}(\{-1, (-1)'\})$. If we change B to $(2^{<\omega}, S)$, where $S(\emptyset) = \emptyset$, $S(1011) = 101$, etc., then (A, B) also isn't ARFD. The idea of course is that now a morphism $A \to B$ has to make infinitely many choices. The intuitive moral of this example is that we expect a morphism to be relatively finitely determined when it only has to make "finitely many decisions".

Let's investigate under what kinds of conditions we have (A, B) finitely determined. One simple observation is that if A is finitely generated then (A, B) is finitely determined. However, the converse is not true.

Example 69. Let $A = B = (\mathbb{Q}^{>0}, +)$. Note A is not finitely generated. Let $A_0 \subseteq_{fin} A$ and let $b < \min(A_0)$ with $b \in A$. Then $b \notin \langle A_0 \rangle$. However, (A, B) is finitely determined by $A_0 = \{1\}$. Note that for all $b \in A$ there are terms $t_1(x)$ and $t_2(y)$ such that $t_1(b) = t_2(1)$ and $t_1 \colon B \to B$ is 1-1. In detail, if $b = \frac{m}{n}$ then

$$\underbrace{b + \cdots + b}_{n \text{ times}} = \underbrace{1 + \cdots + 1}_{m \text{ times}}$$

As the following proposition shows, it follows that (A, B) is finitely determined.

Proposition 70. *Let A and B be structures. Let $\bar{a}$ be finitely many elements from A such that for all $b \in A$ there are terms $t_1(x)$ and $t_2(\bar{y})$ with $t_1 \colon B \to B$ 1-1 and $t_1(b) = t_2(\bar{a})$. Then (A, B) is finitely determined.*

Proof. Let $f, g \colon A \to B$ be morphisms that agree on $\bar{a}$. Then for any $b \in A$ we can, by assumption, choose terms t_1 and t_2 as in the statement of this proposition. Then

$$\begin{aligned} t_1(fb) &= t_2(f\bar{a}) \\ &= t_2(g\bar{a}) \\ &= t_1(gb) \end{aligned}$$

and so $fb = gb$. □

However, this weakening of finitely generated is not necessary either. Recall that a formula is said to be positive existential if it is atomic or built up from atomic formulas by means of $\vee$, $\wedge$, and/or $\exists$. We may also call such a formula $\exists_1^+$ or morphic. It's easy to check that every positive existential formula is preserved by morphisms. In fact, one can prove that every formula preserved by all morphisms is logically equivalent to a positive existential formula (see e.g. Exercise 5.2.6 in [5]). I'll use $\mathrm{tp}^+(b/A_0)$ to denote the positive existential type of b over parameters A_0. If $p(y)$ is a type with parameters from A and $f \colon A \to B$ then $f_* p(y)$ is a type with parameters from B where we replace each parameter $a \in A$ appearing in p by $f(a) \in B$.

Example 71. Let $A = B = (\mathbb{R}^{>0}, +)$. There are continuum many elements b in A, but only countably many pairs of terms $(t_1(x), t_2(\bar{y}))$. So, if (A, B) satisfied the hypotheses of the previous proposition, then there would be two elements $b, b' \in A$ assigned to the same pair (in fact continuum many assigned to some pair). Then $t_1(b) = t_2(\bar{a}) = t_1(b')$, but t_1 was supposed to be 1-1 on B (which is also A in this case). However, (A, B) is nevertheless finitely determined. Let $A_0 = \{1\}$. Let $b \in A$. Let $p(y)$ be the positive existential type of b/A_0. This includes:

$$\{\text{“}y > q\text{”} \mid q \in \mathbb{Q}^{>0}, b > q\} \cup \{\text{“}y < q\text{”} \mid q \in \mathbb{Q}^{>0}, b < q\}$$

where "$y > q$" really means, writing $q = \frac{m}{n}$,

$$\exists x \exists q (nq = m1 \wedge q + x = y)$$

and similarly for $y < q$. As for any morphism $f\colon A \to B$ we have $qf(1) = f(q)$, at least the following formulas (or their equivalents) are in $f_* p(y)$:

$$\{y > fq \mid q \in \mathbb{Q}^{>0}, b > q\} \cup \{y < fq \mid q \in \mathbb{Q}^{>0}, b < q\}$$

Then since $f(\mathbb{Q}^{>0})$ is dense in $\mathbb{R}^{>0}$, we get fb as the unique realization of $f_* p(y)$. To see that $f(\mathbb{Q}^{>0})$ is dense, let $c, d \in \mathbb{R}^{>0}$ with $c < d$. Let $\frac{c}{f(1)} < q < \frac{d}{f(1)}$ with $q \in \mathbb{Q}^{>0}$ as this is dense in $\mathbb{R}^{>0}$. Then $c < f(1)q = f(q) < d$. Finally, if w were another realization of $f_* p(y)$, WLOG $w < fb$, then by denseness there is fq with $w < fq < fb$. It follows that $b > q$ and so $y > fq$ is in $f_* p(y)$, a contradiction. The next proposition shows that (A, B) must be finitely determined.

Definition 72. Let A and B be structures. (A, B) is said to have the **strong morphic types property** when there is $A_0 \subseteq_{fin} A$ so that for all morphisms $f\colon A \to B$ and all $b \in A$ only fb realizes $f_* \operatorname{tp}^+(b/A_0)$. For the **weak morphic types property** we just change the type to $\operatorname{tp}^+(fb/fA_0)$.

Proposition 73. *If (A, B) has the strong morphic types property, then (A, B) is finitely determined.*

Proof. Let A_0 be the finite subset of A guaranteed to exist in the definition of the strong morphic types property. Let $f, g\colon A \to B$ be morphisms with $f \restriction A_0 = g \restriction A_0$. Let $b \in A$. Then fb and gb both realize $f_* \operatorname{tp}^+(b/A_0) = g_* \operatorname{tp}^+(b/A_0)$ and so $fb = gb$. □

The reader should be careful to note the distinction between the strong and weak morphic types property. It is true that $f_* \operatorname{tp}^+(b/A_0) \subseteq \operatorname{tp}^+(fb/fA_0)$ but in general equality doesn't hold. As a simple example, consider the morphism f from $(\mathbb{N}, S)$ to the trivial unary algebra with one element. Let $b \in \mathbb{N}$ and let $A_0 = \emptyset$. Then $S(y) = y$ is not in $f_* \operatorname{tp}^+(b)$ yet it is in $\operatorname{tp}^+(fb)$. More positive existential things can become true as we pass to the codomain. The adjectives strong and weak make sense because in the morphic types properties we are asking that there be a unique realization of the type. If p and q are two types with $p \subseteq q$, and there is at most one realization of p, then of course there is at most one realization of q.

The next example shows that the weak morphic types property does not imply the strong, nor does it imply finitely determined.

Example 74. Let the signature τ consist of ω-many unary relation symbols $R_0, R_1, R_2, \ldots$. Let A be the free τ-structure on ω-many generators. Of course this means that the R_i are all empty. Let B be the natural numbers, and let $R_i^B := \{i\}$. I claim (A, B) has the weak morphic types property. Define $A_0 = \emptyset$. Let $f\colon A \to B$ be a morphism. Let $b \in A$. Only fb realizes $\operatorname{tp}^+(fb)$ because this type includes $R_{fb}(y)$. However, because A is a free structure on infinitely many generators, (A, B) cannot be finitely determined, and hence also does not have the strong morphic types property.

Though as we saw, the strong morphic types property implies finitely determined, the converse is not true, as the following example shows. This example further shows that finitely determined does not even imply the weak morphic types property.

Example 75. Let A be $(\mathbb{Z}, S)$. We define B as follows. First, $A \subseteq B$. To A we add an additional predecessor w of 0. And before w we add a "fan" of finite chains of increasing length. I.e. for each $n \in \omega$ we add a distinct chain $x_{11}, x_{2n}, \ldots, x_{nn}$ where $S(x_{in}) = x_{(i+1)n}$ and $S(x_{nn}) = w$. Now, (A, B) is finitely determined by $A_0 = \{0\}$. However, (A, B) does not have the weak morphic types property. Given a finite subset A_0 of A, there is a morphism $f\colon A \to B$ with $f(A_0) \geq 0$. Let $b \in A$ with $fb = -1$. I claim that w realizes $\operatorname{tp}^+(fb/fA_0)$. It suffices to show that there is an elementary extension B' of B and an endomorphism $\alpha\colon B' \to B'$ which fixes fA_0 and sends -1 to w. Let B' be an ω^+-saturated elementary extension of B. Since the type $p(y_i, i \in \omega) := \{S(y_{i+1}) = y_i \mid i \in \omega\} \cup \{S(y_0) = w\}$ is finitely realizable, there are such elements y_i in B'. Then define an endomorphism to be constant everywhere except on the negative integers. Send -1 to w, -2 to y_0, -3 to y_1, and so on. It can be checked this is a morphism.

Since finitely determined doesn't imply the strong morphic types property, we might try to weaken this property a bit.

Definition 76. We say that (A, B) is **type determined** if there exists a finite subset A_0 of A so that the following ordinal-indexed sequence of increasing subsets of A converges to A:

$$A_\lambda := \bigcup_{i<\lambda} A_i$$
$$A_{i+1} := \{a \in A \mid \forall f\colon A \to B \text{ only } fa \text{ realizes } f_* \operatorname{tp}^+(a/A_i)\}$$

Proposition 77. *If (A, B) is type determined, then (A, B) is finitely determined.*

Proof. Let $f, g\colon A \to B$ with $f \restriction A_0 = g \restriction A_0$. By induction we can show that for each i, if $a \in A_i$, then $fa = ga$. This is obvious for limit steps. As for the successor step, we know that $f_* \operatorname{tp}^+(a/A_i) = g_* \operatorname{tp}^+(a/A_i)$ and as fa and ga are both the unique thing that realizes this type, we have $fa = ga$. □

However, once again, we have a condition which implies but is not implied by finitely determined. Example 75 above is not type determined. Given any finite subset A_0 of the integers, $A_\infty = [\min(A_0), \infty)$ in that example. However, as the following example shows, type determined is in general a bit weaker than the strong morphic types property.

Example 78. Let A and B be the multisorted structure $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mathcal{P}\mathcal{P}(\mathbb{N}), S, 0, \in, \notin)$. S and 0 are the usual successor and 0 of the natural numbers. $\in$ is the usual "is an element of" relation for $\mathbb{N} \times \mathcal{P}(\mathbb{N})$ and $\mathcal{P}(\mathbb{N}) \times \mathcal{P}\mathcal{P}(\mathbb{N})$ (if you like we could have two different relations), and $\notin$ is the usual "is not an element of" relation. (A, B) is trivially finitely determined because there is only one morphism from A to B. Of course, by the presence of S and 0, a morphism $f\colon A \to B$ must fix $\mathbb{N}$. Given $X \in P(\mathbb{N})$, and $n \in \mathbb{N}$, $n \in X$ iff $n = f(n) \in f(X)$, by the presence of $\in$ and $\notin$, and so $X = f(X)$. Now knowing that $\mathcal{P}(\mathbb{N})$ is fixed, the same argument gives that $\mathcal{P}\mathcal{P}(\mathbb{N})$ is fixed. If desired, we could attach another sort to allow more than one morphism, yet still keeping finitely determined. Earlier we saw an example where (A, B) is finitely determined, and both sets had size continuum. Here the sizes are $\beth_2$. It should be clear that we could modify the example to make the sets even bigger. This example also shows how the strong morphic types property and type determined come apart. (A, B) can't have the strong morphic types property, because there just aren't enough types. Given a finite subset A_0 of A, the number of positive existential types over A_0 is at most continuum because the language is countable. But there are more than continuum many elements in $A = B$, and so there must exist some $fb = b \neq fc = c$ realizing the same positive existential type over $fA_0 = A_0$. (A, B) doesn't even have the weak morphic types property. On the other hand, letting $A_0 = \emptyset$, we get $A_1 \supseteq \mathbb{N} \cup \mathcal{P}(\mathbb{N})$, and so $A_2 = A$. Thus, (A, B) is type determined.

One might object to Example 75 as a counterexample to the idea that finitely determined implies type determined, because of some seemingly peculiar properties of B. In that example, -1 and w have the same positive existential type, yet there's no endomorphism of B sending -1 to w. This motivates the following definition.

Definition 79. Let M be a structure and α a cardinal. M is said to be **α-endo-homogenous** when for every pair of tuples $\bar{a}, \bar{b}$ of length less than α, if $\mathrm{tp}^+(\bar{a}) \subseteq \mathrm{tp}^+(\bar{b})$, then there is an endomorphism of M sending $\bar{a}$ to $\bar{b}$.

Example 80. The structure $(\mathbb{N}, S)$ is ω-endo-homogenous. If $\mathrm{tp}^+(\bar{a}) \subseteq \mathrm{tp}^+(\bar{b})$, then the relative positions between the elements of $\bar{a}$ match the relative positions between the elements of $\bar{b}$. Further, $\min(\bar{a}) \leq \min(\bar{b})$ because we can say $\exists x(S^{\min(\bar{a})}(x) = \min(\bar{a}))$. It follows that there is an endomorphism taking $\bar{a}$ to $\bar{b}$ because any forward translation is an endomorphism.

Proposition 81. *If B is ω-endo-homogenous and (A, B) is finitely determined, then (A, B) has the weak morphic types property.*

Proof. Let A_0 be a finite subset of A that separates morphisms from A to B. Let $a \in A$ and let $f\colon A \to B$ be a morphism. Suppose w realizes $\mathrm{tp}^+(fa/fA_0)$. Then there is an endomorphism $e\colon B \to B$ fixing fA_0 and sending fa to w, by the ω-endo-homogenity of B. Then define $g\colon A \to B$ by $g := e \circ f$. Then $g \upharpoonright A_0 = f \upharpoonright A_0$ and so $f = g$. Thus, $fa = ga = w$ as desired. □

Question 82. Is it possible to have A and B where B is ω-endo-homogenous, (A, B) is finitely determined, and yet (A, B) is not type determined?

Obviously, if $f\colon A \to B$ commutes with tp^+ for each morphism from A to B, then the strong morphic types property is implied by the weak version. So there is some interest in the following question.

Question 83. Which morphisms $f\colon A \to B$ commute with tp^+ in the sense that for all $a \in A$ and all (finite) subsets A_0 of A we have $f_* \mathrm{tp}^+(a/A_0) = \mathrm{tp}^+(fa/fA_0)$?

The reader may be wondering if there are α-endo-homogenous elementary extensions of any structure, just as there are α-(strongly)-homogenous elementary extensions. The answer is yes, and a common argument for the existence of the latter can be adapted with little change.

Definition 84. A structure M is said to be **ω-weakly-endo-homogenous** when for all tuples $\bar{a}$ and $\bar{b}$ from M of finite length and all elements $c \in M$, if $\mathrm{tp}^+(\bar{a}) \subseteq \mathrm{tp}^+(\bar{b})$, then there is some $d \in M$ such that $\mathrm{tp}^+(\bar{a}c) \subseteq \mathrm{tp}^+(\bar{b}d)$.

Lemma 85. *Let L be a countable language, and let M be a countable L-structure. Then there is a countable ω-weakly-endo-homogenous elementary extension of M.*

Proof. We shall define an elementary chain

$$M = M_0 \preceq M_1 \preceq \cdots \preceq M_i \preceq \cdots$$

of length ω of countable L-structures with the property that if $\bar{a}, \bar{b}, c \in M_i$ and $\mathrm{tp}^+(\bar{a}) \subseteq \mathrm{tp}^+(\bar{b})$, then there is a $d \in M_{i+1}$ such that $\mathrm{tp}^+(\bar{a}c) \subseteq \mathrm{tp}^+(\bar{b}d)$. Then $M' := \bigcup_{i\in\omega} M_i$ will be the desired elementary extension.

So, to get this chain, assuming that M_i is defined, for each triple $(\bar{a}, \bar{b}, c)$ from M_i with $\mathrm{tp}^+(\bar{a}) \subseteq \mathrm{tp}^+(\bar{b})$, we introduce a new constant symbol $d_{\bar{a}\bar{b}c}$ and consider the theory T consisting of the elementary diagram of M_i together with the following sentences:

$$\{\varphi(\bar{b}, d_{\bar{a}\bar{b}c}) \mid \varphi(\bar{x}, y) \in \mathrm{tp}^+(\bar{a}c)\}$$

This theory T is consistent because given $\varphi_1(\bar{x}, y), \ldots, \varphi_n(\bar{x}, y) \in \mathrm{tp}^+(\bar{a}c)$, we have $M_i \models \exists y \bigwedge_j \varphi_j(\bar{b}, y)$. Thus there is a model M_{i+1} of T. □

Lemma 86. *Let M be a countable structure. Then M is ω-weakly-endo-homogenous iff M is ω-endo-homogenous.*

Proof. Of course endo-homogenous implies weakly-endo-homogenous, because if we have an endomorphism $f\colon M \to M$ which sends $\bar{a}$ to $\bar{b}$, then $\mathrm{tp}^+(\bar{a}c) \subseteq \mathrm{tp}^+(f\bar{a}fc) = \mathrm{tp}^+(\bar{b}fc)$ and we can let $d := fc$.

Now assume that M is countable and ω-weakly-endo-homogenous. Let $M = (m_1, m_2, \ldots)$. Let $\mathrm{tp}^+(\bar{a}) \subseteq \mathrm{tp}^+(\bar{b})$. By repeated application of weak-endo-homogeneity, we get that $\mathrm{tp}^+(\bar{a}m_1m_2\cdots) \subseteq \mathrm{tp}^+(\bar{b}n_1n_2\cdots)$ for some n_i. Then $m_i \mapsto n_i$ is a well-defined endomorphism of M taking $\bar{a}$ to $\bar{b}$. □

Proposition 87. *Let α be an infinite cardinal. Let M be an L-structure with $|M|, |L| \leq 2^\alpha$. Then there is some elementary extension M' of M with $|M'| \leq 2^\alpha$ and M' α^+-endo-homogenous.*

Proof. We construct an expanded elementary chain

$$M_0 \preceq \cdots \preceq M_i \preceq \cdots$$

of length 2^α. Each structure M_i is an L_i-structure where

1. $M_0 = M$ and $L_0 = L$
2. $|M_i| \leq 2^\alpha$ for each i
3. $M_\lambda = \bigcup_{i<\lambda} M_i$ for nonzero limit ordinals λ
4. $L_\lambda = \bigcup_{i<\lambda} L_i$ for nonzero limit ordinals λ
5. $M_i \preceq M_j \upharpoonright L_i$ for all $i < j$
6. $L_{i+1} = L_i \cup \{f_{\bar{a}\bar{b}}$ a unary function symbol $\mid \bar{a}, \bar{b}$ from M_i with length $\leq \alpha$ and $\mathrm{tp}^+_L(\bar{a}) \subseteq \mathrm{tp}^+_L(\bar{b})\}$
7. For each appropriate $f_{\bar{a}\bar{b}}$,

$$M_{i+1} \models \text{“}f_{\bar{a}\bar{b}} \text{ is an } L\text{-endomorphism sending } \bar{a} \text{ to } \bar{b}\text{”}$$

Once constructed, we put $M' := \bigcup_{i<2^\alpha} M_i \upharpoonright L$. Given tuples $\bar{a}$ and $\bar{b}$ of length $\leq \alpha$, since the cofinality of 2^α is bigger than α, we get that the tuples occur in some M_i, and so M_{i+1} thinks $f_{\bar{a}\bar{b}}$ is an L-endomorphism taking $\bar{a}$ to $\bar{b}$, and so M' must think so too. Also, M' has size at most 2^α since each M_i has size at most 2^α and the chain has length 2^α.

As the limit steps are easy, let's verify that the successor steps work. Suppose we are given M_i and we want to define the L_{i+1}-structure M_{i+1}. Observe that L_{i+1} is L_i plus unary function symbols $f_{\bar{a}\bar{b}}$ for all pairs of tuples of length at most α taken from M_i with $\mathrm{tp}^+_L(\bar{a}) \subseteq \mathrm{tp}^+_L(\bar{b})$. Note that there are at most 2^α many symbols added in this way, so $|L_{i+1}|$ remains at most 2^α. We want to show that the $L_{i+1}(M_i)$-theory consisting of the L_i-elementary diagram of M_i and sentences that assert each $f_{\bar{a}\bar{b}}$ is an L-endomorphism taking $\bar{a}$ to $\bar{b}$ is consistent. By the Robinson joint consistency theorem, we may deal with one function symbol $f_{\bar{a}\bar{b}}$ at a time (and even forget momentarily about the other unary function symbols that were already dealt with). Let T_0 be a finite collection of the sentences to be shown consistent. Note that the language $L' \cup \{f_{\bar{a}\bar{b}}\}$ of T_0 can be taken to be countable, and we can assume that the portions $\bar{a}'$ and $\bar{b}'$ of the tuples $\bar{a}$ and $\bar{b}$ present in L' are the same finite length and correspond. As only finitely many constants are present, there is a countable L'-elementary submodel N of M_i. By our lemmas, there is an $(L' \upharpoonright L)$-elementary extension N' of N which is ω-endo-homogenous. We know that $\mathrm{tp}^+_{L'\upharpoonright L}(\bar{a}') \subseteq \mathrm{tp}^+_{L'\upharpoonright L}(\bar{b}')$ and so there is some $(L' \upharpoonright L)$-endomorphism $f\colon N' \to N'$ sending $\bar{a}'$ to $\bar{b}'$. Since $L' - L$ contains only constants, N' is also an L'-elementary extension of N. Thus, $N' \models T_0$. □

Corollary 88. *Let α be a cardinal. Let M be a structure. Then there is an α-endo-homogenous elementary extension of M.*

Perhaps a more finite take on the problem of determining morphisms would be to consider partial morphisms.

Definition 89. Let A and B be structures. Let's say that a **partial morphism** $f\colon A \rightsquigarrow B$ is a partial function with the property that if $\bar{a} \in \mathrm{dom}(f)$ and $A \models Rt_1(\bar{a}) \cdots t_n(\bar{a})$ then $B \models Rt_1(f\bar{a}) \cdots t_n(f\bar{a})$ where R is a relation symbol (or equality) and the t_i are terms.

Let $A_0 \subseteq A$. We say that A_0 is **uniquely extendable** with respect to (A, B) when for every partial morphism $f\colon A \rightsquigarrow B$ with $\mathrm{dom}(f) = A_0$ and for all subsets X of A there is at most one partial morphism $f'\colon A \rightsquigarrow B$ such that $\mathrm{dom}(f') = A_0 \cup X$ and $f' \restriction A_0 = f$.

We say that (A, B) has the **unique extensions property** when there is a finite A_0 that is uniquely extendable.

Note that in the definition of uniquely extendable we could have taken the sets X to be singletons without changing anything, because the restriction of a partial morphism is a partial morphism.

The unique extensions property provides a more stringent form of determining. Indeed, it is even strictly stronger than type determined.

Proposition 90. *The unique extensions property implies the strong morphic types property (and hence type determined and finitely determined).*

Proof. Let (A, B) have the unique extensions property, and let $A_0 = \bar{a}$ be a finite subset of A that is uniquely extendable. Let $a \in A$ and let $f\colon A \to B$ be a morphism. Let w realize $f\,\mathrm{tp}^+(a/\bar{a})$. I claim that $(f \restriction A_0) \cup \{(a, w)\}$ is a partial morphism, from which it follows that $fa = w$ as desired. First of all, it is a partial function because, if $a \in A_0$, say $a = a_i$, then $y = fa_i$ is in the type $f\,\mathrm{tp}^+(a/\bar{a})$ and so $w = fa_i$. As for the partial morphism business, suppose $A \models R\bar{t}(a, \bar{a})$. Then $B \models R\bar{t}(w, f\bar{a})$ because w realizes $f\,\mathrm{tp}^+(a/\bar{a})$. □

Example 91. This example shows that the strong morphic types property does not imply the unique extensions property. Let $A = (\mathbb{Z}, S)$ with S the usual successor function, but we express it using a binary relation symbol. We let $B = (2^{<\omega}, S)$ where $S = \{(x, x^\frown i) \mid x \in 2^{<\omega}, i \in 2\}$. There is no morphism from A to B, so trivially (A, B) has the strong morphic types property. However, given any finite subset A_0 of A, we can find a partial morphism with domain A_0 which has different extensions. Simply take an element $a > A_0$ and we have choices where to send it.

For the sake of convenience, let's record some of the observations we've made in one place:

Proposition 92. *The following are strict implications:*

$$\begin{aligned} \textit{unique extensions property} &\implies \textit{strong morphic types property} \\ &\implies \textit{type determined} \\ &\implies \textit{finitely determined} \\ &\implies \textit{always relatively finitely determined} \end{aligned}$$

Also, the strong morphic types property strictly implies the weak morphic types property. The weak morphic types property does not imply finitely determined, but it implies the strong morphic types property under the assumption of commuting types. Finitely determined does not imply the weak morphic types property, but it does under the assumption that the codomain is ω-endo-homogenous. Type determined does not imply the weak morphic types property.

One question that remains is Question 82.

Now we turn to a particular kind of situation where (A, B) is finitely determined, but we additionally ask which finite subsets A_0 of A work to determine the morphisms. The proofs of the following propositions are easy and are therefore omitted.

Proposition 93. *Let τ be a signature. Let A be the free τ-structure on α generators. Let B be a τ-structure with $|B| \geq 2$. Then (A, B) is finitely determined iff α is finite.*

Proposition 94. *Let A be the free τ-structure on finitely many generators $\bar{a}$. Let $A_0 = \{t_1(\bar{a}), \ldots, t_n(\bar{a})\}$ be a finite subset of A. Then A_0 determines morphisms from A to B iff for all $\bar{c}, \bar{d} \in B$ we have*

$$(t_i(\bar{c}) = t_i(\bar{d}) \text{ each } i) \implies \bar{c} = \bar{d}$$

One may be interested in presentations of free algebras instead of free algebras themselves, as exemplified by Question 95. We now introduce some definitions and examples relevant to this question.

For this question we suppose that A is not an algebra, but instead simply some subset of Σ^* for some finite set Σ, i.e., A is a collection of words (a "language") where the alphabet is Σ. We continue to suppose that B is some algebra, and we look at functions from A to B, but only ones that "factor through a free algebra" in some sense. There should be a finitely generated free algebra F and functions $f : A \to F$ and $g : F \to B$ such that g is an algebra morphism and f is a bijection. Of course, without further constraints, the condition that $h : A \to B$ is the composition $g \circ f$ of some such g and f means nothing. There are two (related) kinds of restrictions we might add: one on the kind of language A itself (e.g. context-free), the other on the function f assigning elements of the free algebra to the sentences (e.g. some kind of monotonicity or agreement with how the language is generated). Let's formulate some reasonable restrictions motivated by the propositional logic examples involving Polish notation and parenthesis notation.

Recall that a context-free language is a language constructed as follows. Let Σ be a finite set (whose elements are called terminals) and let V be another finite set (which is disjoint from Σ and whose elements are called non-terminals, and includes one element S called the start symbol), and let $R \subseteq V \times (V \cup \Sigma)^*$ be a relation (whose elements are called rules). A derivation is a sequence $u_1 \to \cdots \to u_n$ such that $u_1 = S$, $u_n \in \Sigma^*$ and each $u_i \to u_{i+1}$ is the application of a rule in the sense that a symbol $T \in V$ occurring in u_i is replaced by an image of T under R. Something is in the language iff it is obtained by some derivation.

For example, let $\Sigma = \{p_1, \cdots, p_n, \neg, \wedge, (,)\}$ and let $V = \{S\}$ and let $R = \{S \to (S \wedge S), S \to (\neg S), S \to p_i\}$. Then the language obtained is the usual parenthesis notation propositional logic. Or, letting $\Sigma = \{p_1, \ldots, p_n, \neg, \wedge\}$ and $V = \{S\}$ and $R = \{S \to \wedge SS, S \to \neg S, S \to p_i\}$ we get Polish notation.

Now let's suppose that $A \subseteq \Sigma^*$ is some context-free language (with associated non-terminals and rules). Given a finitely generated free algebra F (this is a free algebra for the signature, not for some variety), a bijection $f\colon A \to F$ is called "nice" if for every n-ary function symbol $\triangle$ in the signature there is a rule $S \to \beta(S_1, \cdots, S_n)$ (where each S_i is a distinct occurrence of S in β) such that for all $\varphi_1, \ldots, \varphi_n \in F$,

$$\beta(f^{-1}(\varphi_1), \ldots, f^{-1}(\varphi_n)) = f^{-1}(\triangle\varphi_1 \cdots \varphi_n)$$

Here $\beta(f^{-1}(\varphi_1), \ldots, f^{-1}(\varphi_n))$ denotes the result of replacing each S_i in β by the finite sequence $f^{-1}(\varphi_i)$. Note in particular that β has no non-terminals other than $S_1, \ldots, S_n$. For both the Polish notation and parenthesis notation, for example, we have a nice bijection.

Question 95. Suppose that A is some context-free language and B is some algebra. Does there exist a finite subset A_0 of A (and if so, which A_0 work) so that for any function $h_0\colon A_0 \to B$ there is at most one extension $h_0 \subset h\colon A \to B$ such that there is a finitely generated free algebra F and a nice bijection $f\colon A \to F$ and a morphism $g\colon F \to B$ such that $h = gf$?

Ehrenfeucht's conjecture, which was proven independently by Albert and Lawrence [1], and Guba in 1985, is a result which may be related to these notions of determination. It states that if Σ is a finite set and $L \subseteq \Sigma^*$ (recall Σ^* is the free monoid generated by Σ) then there is some finite subset T of L such that for all finite sets Δ and for all morphisms $g, h\colon \Sigma^* \to \Delta^*$ we have

$$g \restriction T = h \restriction T \implies g \restriction L = h \restriction L$$

We first give an intuitive interpretation of the content of this theorem. We conceive of Σ as a collection of words, and Σ^* is of course then the collection of all finite sequences of words. $L \subseteq \Sigma^*$ is thought of as all the meaningful sentences. A morphism $g\colon \Sigma^* \to \Delta^*$ is a dictionary for the words Σ. I.e., each word $\sigma \in \Sigma$ gets assigned a definition $g(\sigma)$. However, sometimes it's difficult to give definitions for little sentences (or just words by themselves), e.g. maybe it's difficult to give meaning to the word "the" and it's easier to give meaning to "the food is ready". This gives reason to restrict the domain to L, the meaningful sentences. The goal is to find a finite subset T of L whose definitions suffice to determine the definitions of everything in L. It may happen that $\Sigma \subseteq L$, in which case we can let $T = \Sigma$. I.e., if we have definitions for all the words, then we have definitions for all the sentences. However, it's not immediately clear that we can get a finite collection of *sentences* that determines the definitions of all sentences. This is the content of the theorem. It should be remarked however, that the theorem only allows "definitions" in finitely generated free monoids.

One question is how to relate Ehrenfeucht's conjecture to the discussion of finitely determined pairs. So we ask the question how we may understand the morphisms $h \restriction L$. We introduce a new signature $\mathcal{L}$ for describing such morphisms. For each natural number n, and for each pair of finite sequences $i_1, \ldots, i_k, j_1, \ldots, j_m$ from $\{1, 2, \ldots, n\}$ we introduce an n-ary relation symbol $R_{n\bar{i}\bar{j}}$. $\mathcal{L}$ is the signature that consists just of these relation symbols.

We may make $L \subseteq \Sigma^*$ into an $\mathcal{L}$-structure as follows. We set

$$R_{n\bar{i}\bar{j}}(a_1, \ldots, a_n) \iff a_{i_1} \cdots a_{i_k} = a_{j_1} \cdots a_{j_m}$$

We note that if $k = 0$ or $m = 0$ then the product is interpreted as the identity $\emptyset$.

Suppose that L and L' are concrete $\mathcal{L}$-structures with $L \subseteq \Sigma^*$ and $L' \subseteq \Delta^*$. Let $S(L)$ denote the submonoid of Σ^* generated by L, and similarly $S(L')$ denotes the submonoid of Δ^* generated by L'. Then $\mathcal{L}$-morphisms from L to L' are correspond to monoid-morphisms from $S(L)$ to $S(L')$ that send elements of L to elements of L'.

Question 96. Under what conditions do we have a pair of $\mathcal{L}$-structures (L, L') finitely determined?

Suppose that L and L' are concrete $\mathcal{L}$-structures with $L \subseteq \Sigma^*$ and $L' \subseteq \Delta^*$. If $\Sigma \subseteq L$, then an $\mathcal{L}$-morphism $L \to L'$ is pretty much the same thing as a monoid-morphism $\Sigma^* \to \Delta^*$. However, if $\Sigma \not\subseteq L$ then it might not work out the same way. Take, for instance $\Sigma = \{a\} = \Delta$ and let $L = \{aa, aaaa, a^6, \ldots\}$ and $L' = \Delta^*$. Define $h\colon L \to L'$ by sending a^{2n} to a^n. This is an $\mathcal{L}$-morphism but not the restriction of any monoid-morphism. Yet, in this example, if we redescribe Δ^* as $(\Delta')^*$ where $\Delta' = \{b\}$ and $bb = a$, then it is a restriction of a monoid-morphism. More generally we may phrase this idea as follows:

Question 97. Given any $\mathcal{L}$-morphism $h\colon L \to L'$ with $L \subseteq \Sigma^*$ and $L' \subseteq \Delta^*$ with Σ and Δ both finite, then is there some finite Θ and $L'' \subseteq \Theta^*$ such that

1. There is some $\mathcal{L}$-isomorphism $\varphi\colon L' \cong L''$, and
2. $\varphi \circ h$ is the restriction of some monoid-morphism from Σ^* to Θ^*.

The answer to this question is "no", as pointed out to me by George Bergman. Let $\Sigma = \{x, y\} = \Delta$ and $L = \{x, xy\}$. Then the submonoid of Σ^* generated by L, which we denote $S(L)$, is isomorphic to the free monoid on two generators. So there is a monoid-morphism $H\colon S(L) \to \Delta^*$ which sends x to x and xy to the identity element of Δ^*. Let $L' = \Delta^*$. The restriction $h := H \restriction L$ of this monoid-morphism to L is an $\mathcal{L}$-morphism of L into L'. If $\varphi\colon L' \to L''$ is any $\mathcal{L}$-isomorphism from L' to some subset L'' of some free monoid Θ^*, then, in particular, for any $z \in L'$ we have z is the identity element iff $\varphi(z)$ is the identity element. So, if there is some such φ, we get that $(\varphi \circ h)(x)$ is not the identity element, while $(\varphi \circ h)(xy)$ is the identity element. No monoid-morphism from Σ^* to Θ^* can have this property.

At this point I'd like to raise some more general questions about $\mathcal{L}$-structures. Of course not every $\mathcal{L}$-structure arises as a subset of a finitely generated free monoid. For example, consider the $\mathcal{L}$-structure obtained in a natural way from $(\mathbb{Z}, +)$. I.e., $R_{n\bar{i}\bar{j}}(\bar{a})$ iff $a_{i_1} + \cdots + a_{i_m} = a_{j_1} + \cdots + a_{j_k}$. Here we have $2 + 2 + (-2) = 2$ and $2 + 2 \neq 2$, yet in any $\mathcal{L}$-structure obtained as a subset of a free monoid, we have $aab = a$ implies $aa = a$. But this leads us to the question of how to characterize the $\mathcal{L}$-structures that do arise in this way.

Question 98. Which $\mathcal{L}$-structures arise as subsets of finitely generated free monoids? Is the theory of such $\mathcal{L}$-structures axiomatizable? Decidable?

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