A Problem of Minimax Estimation with Directional Information*

12/07/94

Thomas S. Ferguson, UCLA

Dedicated with great respect to Paul G. Hoel

Abstract. This problem is in the area of minimax selection of experiments. Nature chooses a number θ in the closed interval $[-1, 1]$. The statistician chooses a number y in the same interval (an experiment) and is informed whether $\theta < y$, $\theta = y$, or $\theta > y$. Based on this information, the statistician then estimates θ with squared error loss. The minimax solution of this problem is found. The minimax value is $1/(2e)$. The least favorable distribution involves the truncated t-distribution with two degrees of freedom. The minimax choice of experiment involves the truncated t-distribution with zero degrees of freedom.

Key Words: Selection of experiments, optimal design, search games, minimum variance partitions.

* Delivered August 1994 at the Joint Statistical Meetings, Toronto, in the Session in Honor of Paul G. Hoel's 90th Birthday.

1. Introduction.

Paul Hoel has done important work on many problems. There is for example his early work on approximation, factor analysis, and indices of dispersion. Or the more mathematical work on unbiased estimation, similar regions, optimum Bayesian classification, confidence bounds for polynomial regression, and the sequential F-test. But it is probably his mature work on optimal design in regression problems that has had the most impact in the statistical world. This work appeared in a series of 6 or 7 papers in the Annals of Mathematical Statistics starting in 1958. In this work he solves some important and difficult problems in minimax design. In this paper I am going to emphasize how difficult these problems are by looking at one of the simplest nontrivial problems in minimax selection of experiment. This little result is not in any way comparable to the really fine work Hoel has done on experimental design and is only offered as a tidbit that may be amusing.

The problem treated is the estimation of a parameter based on information about whether the parameter is greater or less than selected points (design points). When the minimax criterion is used, these problems are called search games with directional information. The discrete version, generalizing "twenty questions", has been studied by Gilbert (1962), Johnson (1964) and Gal (1974). A generalization to sequential selection of an infinite sequence of design points in the unit interval, with loss equal to the sum of the distances to the true point, has been studied by Baston and Bostock (1985) and Alpern (1985). Problems using a Bayes criterion with respect to a uniform prior have been considered in Cameron and Narayanamurthy (1964), Murakami (1971,1976) and Hinderer (1990). If lying is allowed, this has been called Ulam's game and treated in Spencer (1984,1992) and Czyzowicz et al. (1989). Berry and Mensch (1986) consider a related Bayes problem allowing false directional information to be given with known probabilities. A search game with a continuous parameter and with possibly false directional information has been treated in an interesting paper of Gal (1978).

Here we treat a continuous parameter version of a search game with directional information, but restrict the searcher to one question, after which he must estimate the parameter with squared error loss.

2. Statement of the problem.

The rules of the game are as follows. There are two players. Player I, the hider, chooses a number θ in the interval $[-1,1]$. Player II, the searcher, not knowing θ, chooses a number y in the interval $[-1,1]$. Then player II is informed whether $\theta < y$, $\theta = y$, or $\theta > y$ and estimates θ by a number z in the interval $[-1,1]$. The payoff is $(\theta - z)^2$ given by player II to player I. When player II is told that $\theta = y$, it is clearly optimal for him to choose $z = y$. With this understanding, a pure strategy for II is a triplet, (y, z_0, z_1) where z_0 denotes his estimate of θ when he is told $\theta < y$ and z_1 is his estimate of θ when he is told $\theta > y$. We may assume that z_0 is in the interval $[-1, y]$ and z_1 is in the interval $[y, 1]$. Player II's loss is then

$$L(\theta, (y, z_0, z_1)) = (\theta - z_0)^2 \mathrm{I}(\theta < y) + (\theta - z_1)^2 \mathrm{I}(\theta > y), \tag{1}$$

where $\mathrm{I}(A)$ represents the indicator function of the set A.

A mixed strategy for player I is a probability distribution, $F(\theta)$, on the interval $[-1,1]$. A mixed strategy for player II is a probability distribution on the three quantities (y, z_0, z_1), but since the loss to player II is a convex function of z_0 and z_1, he may restrict attention to strategies that are nonrandomized in the choice of z_0 and z_1. With this restriction, II's mixed strategies consist of a distribution $G(y)$ on the interval $[-1,1]$, and two functions, $z_0(y)$ and $z_1(y)$, with the understanding that if y is the value chosen from G and II is told that $\theta < y$ (resp. $\theta > y$), he will estimate y to be $z_0(y)$ (resp. $z_1(y)$).

3. Solution.

The minimax value of the game is $v = (2e)^{-1} = .184\ldots$. The unique optimal strategy for I is to choose θ in $[-1,1]$ according to the distribution $F(\theta)$ that has

$$\text{mass} \quad \frac{1}{(e+1)} = .269\ldots \quad \text{at} \quad \theta = -1$$

$$\text{mass} \quad \frac{1}{(e+1)} = .269\ldots \quad \text{at} \quad \theta = 1, \quad \text{and} \tag{2}$$

$$\text{density} \quad f(\theta) = \frac{1}{2e(e^{-1}+\theta^2)^{3/2}} \quad \text{for} \quad |\theta| < .5(1-e^{-1}) = .316\ldots$$

The unique optimal strategy for II is to choose y in $[-1, 1]$ according to the distribution $G(y)$ that has

$$\text{mass} \quad .25 \quad \text{at} \quad y = .5(1-e^{-1})$$

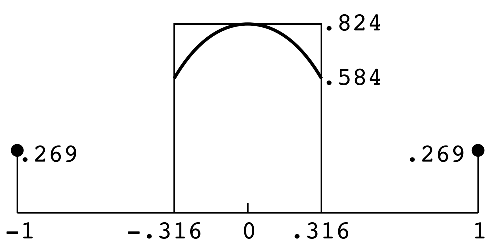
$$\text{mass} \quad .25 \quad \text{at} \quad y = -.5(1-e^{-1}) \quad \text{and}$$

$$\text{density} \quad g(y) = \frac{1}{2(e^{-1}+y^2)^{1/2}} \quad \text{for} \quad |y| < .5(1-e^{-1}). \tag{3}$$

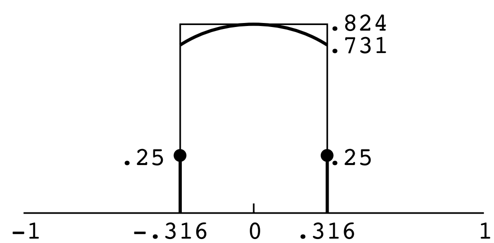
$$\text{If} \quad \theta < y, \quad \text{choose} \quad z = z_0(y) = y - \sqrt{e^{-1}+y^2}.$$

$$\text{If} \quad \theta > y, \quad \text{choose} \quad z = z_1(y) = y + \sqrt{e^{-1}+y^2}.$$

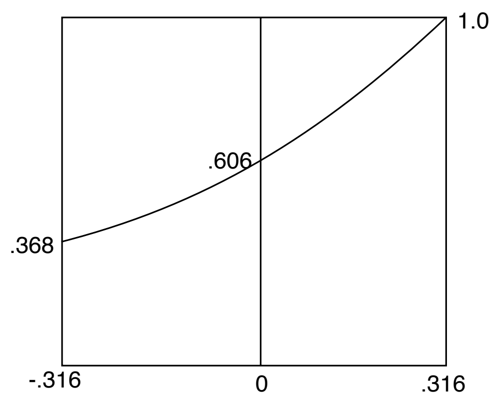
$$\text{If} \quad \theta = y, \quad \text{choose} \quad z = y.$$



$f(\theta)$



$g(y)$



$z_1(y)$

4. Proof.

It is convenient to introduce a generalization of the distribution (2). The scaled version of the t-distribution with 2 degrees of freedom has density

$$f(\theta|\sigma) = \frac{1}{2\sigma}(1 + \frac{\theta^2}{\sigma^2})^{-3/2}.$$

(The t_2-distribution has $\sigma = \sqrt{2}$.) For $0 < \sigma < 1$, let $F(\theta|\sigma)$ denote the distribution of a random variable, Θ, that has

$$\begin{array}{llll} \text{density} & f(\theta|\sigma) & \text{for} & |\theta| < \gamma \quad \text{and} \\ \text{mass} & p(\sigma) = \int_\gamma^\infty f(\theta|\sigma)\,d\theta & \text{at} & \theta = \pm 1 \end{array} \tag{4}$$

where

$$\gamma = \frac{1 - \sigma^2}{2}.$$

The distribution (2) has $\sigma = e^{-1/2}$. We use the following indefinite integral formulas.

$$\int f(\theta|\sigma)\,d\theta = \frac{\theta}{2\sigma}(1 + \frac{\theta^2}{\sigma^2})^{-1/2} \tag{5}$$

and

$$\int \theta\, f(\theta|\sigma)\,d\theta = -\frac{\sigma}{2}(1 + \frac{\theta^2}{\sigma^2})^{-1/2}$$

and

$$\int \theta^2 f(\theta|\sigma)\,d\theta = \frac{\sigma^2}{2}[\log(\frac{\theta}{\sigma} + (1 + \frac{\theta^2}{\sigma^2})^{1/2}) - \frac{\theta}{\sigma}(1 + \frac{\theta^2}{\sigma^2})^{-1/2}].$$

The mass $p(\sigma)$ may be found from (5) as

$$p(\sigma) = \frac{1}{2}[1 - \frac{\gamma}{\sigma}(1 + \frac{\gamma^2}{\sigma^2})^{-1/2}] = \frac{1}{2}[1 - \frac{1}{2}\frac{1 - \sigma^2}{1 + \sigma^2}].$$

Against the strategy where I chooses Θ according to the distribution (4), player II chooses $y \in [-\gamma, \gamma]$ and estimates θ to be $\mathrm{E}(\Theta|\Theta < y)$ if he learns $\Theta < y$, and $\mathrm{E}(\Theta|\Theta > y)$ if he learns $\Theta > y$. The expected payoff is

$$\begin{aligned} V(y) &= \mathrm{P}(\Theta < y)\mathrm{Var}(\Theta|\Theta < y) + \mathrm{P}(\Theta > y)\mathrm{Var}(\Theta|\Theta > y) \\ &= \mathrm{E}(\Theta^2) - \mathrm{P}(\Theta < y)\mathrm{E}(\Theta|\Theta < y)^2 - \mathrm{P}(\Theta > y)\mathrm{E}(\Theta|\Theta > y)^2 \\ &= \mathrm{E}(\Theta^2) - \frac{(\mathrm{E}(\Theta \mathrm{I}(\Theta > y)))^2}{\mathrm{P}(\Theta > y)\mathrm{P}(\Theta < y)}. \end{aligned} \tag{6}$$

Here we have used the formula $\mathrm{E}(\Theta \mathrm{I}(\Theta > y)) = -\mathrm{E}(\Theta \mathrm{I}(\Theta < y))$, which follows from the symmetry of the distribution of Θ. For $|y| < \gamma$,

$$\begin{aligned}
\mathrm{E}(\Theta \mathrm{I}(\Theta > y)) &= \int_y^\gamma \theta\, f(\theta|\sigma)\, d\theta + p(\sigma) \\
&= \frac{1}{2}[1 - (\sigma + \frac{\gamma}{\sigma})(1 + \frac{\gamma^2}{\sigma^2})^{-1/2} + \sigma(1 + \frac{y^2}{\sigma^2})^{-1/2}] \\
&= \frac{\sigma}{2}(1 + \frac{y^2}{\sigma^2})^{-1/2}.
\end{aligned}$$

The first term of (6) is independent of y.

$$\begin{aligned}
\mathrm{E}(\Theta^2) &= 2[\int_0^\gamma \theta^2 f(\theta|\sigma)\, d\theta + p(\sigma)] \\
&= 1 + \sigma^2 \log(\frac{\gamma}{\sigma} + (1 + \frac{\gamma^2}{\sigma^2})^{1/2}) - (\sigma^2 + 1)\frac{\gamma}{\sigma}(1 + \frac{\gamma^2}{\sigma^2})^{-1/2} \\
&= \sigma^2(1 - \log(\sigma)).
\end{aligned}$$

Hence,

$$V(y) = \sigma^2(1 - \log(\sigma)) - \sigma^2 = -\sigma^2 \log(\sigma)$$

is also independent of y. This is maximized (by player I) at $\sigma = e^{-1/2}$, giving a value $V(e^{-1/2}) = 1/(2e)$.

Now, we show that the strategy (3) for player II guarantees a value of at most $1/2e$. Again we generalize the strategy (3) for purposes of analysis. For $e^{-1} < \sigma < 1$, let Y be chosen from the distribution $G(y|\sigma)$ that has

$$\begin{array}{llll}
\text{density} & g(y|\sigma) = \frac{1}{2}(\sigma^2 + y^2)^{-1/2} & \text{for} \quad |y| < \gamma & \text{and} \\
\text{mass} & q(\sigma) = \frac{1}{2} - \int_0^\gamma g(y|\sigma)\, dy & \text{at} \quad y = \pm\gamma & \\
\text{choose} & z = z_0(y) = y - \sqrt{\sigma^2 + y^2} & \text{if} \quad \theta < y & \\
\text{choose} & z = z_1(y) = y + \sqrt{\sigma^2 + y^2} & \text{if} \quad \theta > y & \\
\text{choose} & z = y & \text{if} \quad \theta = y &
\end{array} \qquad (7)$$

where, again, $\gamma = (1 - \sigma^2)/2$. For $\sigma = e^{-1/2}$, this reduces to (3). Here are some useful indefinite integral formulas

$$\int g(y|\sigma)\,dy = \frac{1}{2}\log(y+\sqrt{\sigma^2+y^2})$$
$$\int z_1(y)g(y|\sigma)\,dy = \frac{1}{2}(y+\sqrt{\sigma^2+y^2})$$
$$\int z_0(y)g(y|\sigma)\,dy = -\frac{1}{2}(y-\sqrt{\sigma^2+y^2})$$
$$\int z_1(y)^2 g(y|\sigma)\,dy = \frac{y}{2}(y+\sqrt{\sigma^2+y^2})$$
$$\int z_0(y)^2 g(y|\sigma)\,dy = -\frac{y}{2}(y-\sqrt{\sigma^2+y^2}).$$

The expected payoff to I using θ against (7) is

$$\begin{aligned} W(\theta) &= \mathrm{E}(\theta - z_1(Y))^2\mathrm{I}(Y<\theta) + \mathrm{E}(\theta - z_0(Y))^2\mathrm{I}(Y>\theta) \\ &= \theta^2 - 2\theta[\mathrm{E}z_1(Y)\mathrm{I}(Y<\theta) + \mathrm{E}z_0(Y)\mathrm{I}(Y>\theta)] \\ &\quad + \mathrm{E}z_1(Y)^2\mathrm{I}(Y<\theta) + \mathrm{E}z_0(Y)^2\mathrm{I}(Y>\theta). \end{aligned}$$

We consider two cases.

Case 1. $|\theta| < \gamma$. Then

$$\begin{aligned} \mathrm{E}z_1(Y)\mathrm{I}(Y<\theta) &= \int_{-\gamma}^{\theta} z_1(y)g(y|\sigma)\,dy + q(\sigma)z_1(-\gamma) \\ &= \frac{1}{2}[(\theta+\sqrt{\sigma^2+\theta^2}) - (-\gamma+\sqrt{\sigma^2+\gamma^2})] + q(\sigma)z_1(-\gamma). \end{aligned}$$

Similarly,

$$\mathrm{E}z_0(Y)\mathrm{I}(Y>\theta) = \frac{1}{2}[(\theta-\sqrt{\sigma^2+\theta^2}) - (\gamma-\sqrt{\sigma^2+\gamma^2})] + q(\sigma)z_0(\gamma),$$

so that using $z_1(-\gamma) = -z_0(\gamma)$, one obtains

$$\mathrm{E}z_1(Y)\mathrm{I}(Y<\theta) + \mathrm{E}z_0(Y)\mathrm{I}(Y>\theta) = \theta.$$

Also,

$$\begin{aligned} \mathrm{E}z_1(Y)^2\mathrm{I}(Y<\theta) &= \int_{-\gamma}^{\theta} z_1(y)^2 g(y|\sigma)\,dy + q(\sigma)z_1(-\gamma)^2 \\ &= \frac{\theta}{2}(\theta+\sqrt{\sigma^2+\theta^2}) + \frac{\gamma}{2}(-\gamma+\sqrt{\sigma^2+\gamma^2}) + q(\sigma)z_1(-\gamma)^2 \end{aligned}$$

and similarly,

$$\mathrm{E}z_0(Y)^2\mathrm{I}(Y > \theta) = \frac{\theta}{2}(\theta + \sqrt{\sigma^2 + \theta^2}) + \frac{\gamma}{2}(-\gamma + \sqrt{\sigma^2 + \gamma^2})] + q(\sigma)z_0(-\gamma)^2.$$

Now, using $-\gamma + \sqrt{\sigma^2 + \gamma^2} = \sigma^2$ and $z_0(\gamma)^2 = z_1(-\gamma)^2 = \sigma^4$, we have

$$\mathrm{E}z_1(Y)^2\mathrm{I}(Y < \theta) + \mathrm{E}z_0(Y)^2\mathrm{I}(Y > \theta) = \theta^2 + \gamma\sigma^2 + 2\sigma^4 q(\sigma).$$

Combining these, we have

$$W(\theta) = \theta^2 - 2\theta^2 + \theta^2 + \gamma\sigma^2 + 2\sigma^4 q(\sigma) = \gamma\sigma^2 + 2\sigma^4 q(\sigma)$$

independent of θ.

Case 2. $\gamma < |\theta| < 1$. Assume without loss of generality that $\gamma < \theta < 1$. Then $Y < \theta$ with probability one, and

$$W(\theta) = \theta^2 - 2\theta \mathrm{E}z_1(Y) + \mathrm{E}z_1(Y)^2.$$

But

$$\begin{aligned}\mathrm{E}z_1(Y) &= q(\sigma)z_1(-\gamma) + \int_{-\gamma}^{\gamma} z_1(y)g(y|\sigma)\,dy + q(\sigma)z_1(\gamma) \\ &= q(\sigma)(1+\sigma^2) + (1-\sigma^2)/2\end{aligned}$$

and

$$\begin{aligned}\mathrm{E}z_1(Y)^2 &= q(\sigma)[z_1(-\gamma)^2 + z_1(\gamma)^2] + \int_{-\gamma}^{\gamma} z_1(y)^2 g(y|\sigma)\,dy \\ &= q(\sigma)(\sigma^4 + 1) + (1-\sigma^4)/4.\end{aligned}$$

At $\sigma = e^{-1/2}$, we have $\mathrm{E}z_1(Y) = (3 - e{-}1)/4$ and $\mathrm{E}z_1(Y)^2 = 1/2$, so

$$W(\theta) = \theta^2 - 2\theta(3 - e^{-1})/4 + 1/2.$$

This is a quadratic in θ with minimum at the point $(3 - e^{-1})/4$, which is the midpoint of the interval $(\gamma, 1)$. Hence, on this interval, $W(\theta)$ on this interval is less than the value at its endpoints, that is,

$$W(\theta) \le W(1) = 1 - 2(3 - e^{-1})/4 + 1/2 = e^{-1}/2 \qquad \text{for} \qquad \gamma < \theta \le 1.$$

The third case is $\theta = \gamma$, but this allows II to estimate θ exactly when $y = \gamma$, which happens with probabilty 1/4, so $W(\gamma) < \lim_{\theta\to\gamma} W(\theta) = e^{-1}/2$. Combining this with the above, we have

$$W(\theta) = e^{-1}/2 \qquad \text{for} \qquad |\theta| < \gamma \qquad \text{and} \qquad \theta = \pm 1$$

$$W(\theta) < e^{-1}/2 \qquad \text{for} \qquad \gamma \le |\theta| < 1.$$

5. Remark.

Jim MacQueen points out that this result has an interesting implication in the area of minimum variance partitions. We are given the distribution of a random variable, X, and we desire to choose a partition of the sample space into k subsets for fixed k, say disjoint sets $A_1, \ldots, A_k$ whose union is the whole space, such that the conditional variances averaged over the partitions is as small as possible; that is we desire to have $\sum \mathrm{P}(X \in A_i)\mathrm{Var}(X|X \in A_i)$ as small as possible. Such a *minimum variance partition* may be considered as an efficient mapping of the observation into a k-valued random variable. When $k = 2$ and when the distribution of X is $F(x)$, where $F(\theta)$ is the optimal strategy of the hider given above, then it doesn't matter how you partition the interval $[-1, +1]$ into two intervals. All such partitions are minimum variance partitions.

References.

Alpern, S. (1985) "Search for a point in interval, with high-low feedback" *Math. Proc. Camb. Phil. Soc.* **98**, 569-578.

Baston, V. J. and Bostock, F. A. (1985) "A high-low search game on the unit interval" *Math. Proc. Camb. Phil. Soc.* **97**, 345-348.

Berry, D. A. and Mensch, R. F. (1986) "Discrete search with directional information" *Op. Res.* **34**, 470-477.

Cameron, S. H. and Narayanamurthy, S. G. (1964) "A search problem" *Op. Res.* **12**, 623-629.

Czyzowicz, J., Mundici, D. and Pelc, A. (1989) "Ulam's game with lies" *J. Comb. Theory A* **52**, 62-76.

Gal, S. (1974) "A discrete search game" *SIAM J. Appl Math.* **27**, 641-648.

Gal, S. (1978) "A stochastic search game" *SIAM J. Appl. Math.* **34**, 205-210.

Gilbert, E. N. (1962) "Games of identification or convergence" *SIAM Rev.* **4**, 16-24.

Hinderer, K. (1990) "On dichotomous search with direction-dependent costs for a uniformly hidden object" *Optimization* **21**, 215-229.

Johnson, S. M. (1964) "A search game" *Advances in Game Theory* AMS Study #52 39-48.

Murakami, S. (1971) "A dichotomous search" *J. Oper. Res. Soc Japan* **14**, 127-142.

Murakami, S. (1976) "A dichotomous search with travel cost" *J. Oper. Res. Soc. Japan* **19**, 245-254.

Spencer, J. (1984) "Guess a number — with lying" *Math. Mag* **57**, 105-108.

Spencer, J. (1992) "Ulam's searching game with a fixed number of lies" *Theoretical Comp. Sci.* **95**, 307-321.