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Abelian Subalgebras in Z2-graded Lie Algebras; Partitions, Young Diagrams and Ballot Numbers

A Dissertation submitted in partial satisfaction
of the requirements for the degree of

Doctor of Philosophy

in

Mathematics

by

Ronald Raymond Dolbin

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Dissertation Committee:

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The list of people responsible for making sure my tenure at UC Riverside wasn't an abject failure, and the ways in which they have saved me from myself, is too long to fit on a single page, so I will apologize up front to those who feel they should be on this list, but aren't.

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To Douglas Adams, for bistromathics; it made this dissertation possible.

ABSTRACT OF THE DISSERTATION

Abelian Subalgebras in Z2-graded Lie Algebras; Partitions, Young Diagrams and Ballot Numbers

by

Ronald Raymond Dolbin

Doctor of Philosophy, Graduate Program in Mathematics
University of California, Riverside, August 2010
Dr. Vyjayanthi Chari, Chairperson

The dissertation is broken into two parts. Part I deals with the following problem: suppose $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a simple $\mathbb{Z}_2$-graded Lie algebra and let $\mathfrak{b}_0$ be a fixed Borel subalgebra of $\mathfrak{g}_0$; describe and enumerate the abelian $\mathfrak{b}_0$-stable subalgebras of $\mathfrak{g}_1$. The original proof uses a geometric approach; in Part I, we utilize an algebraic method which better describes the corresponding subalgebras. Part II focuses on a generalization of a combinatorial problem related to the representation theory of affine Lie algebras; given an arbitrary partition, we describe an iterative algorithm which at every level generates a ballot number of partitions.

Contents

Introduction

In this document we look at generalizations of two different results related to the representation theory of Lie algebras. Both results are very combinatorial in nature, so [1] and [20] are excellent references for any related questions.

The genesis of the results in Part I is [12], in which Kostant credits Peterson with the following result: if $\mathfrak{g}$ is a finite dimensional simple Lie algebra of rank n, then the number of abelian ideals of the Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ is 2^n. Since this paper's publication, many authors have written multiple papers ([4], [14], [17], [18], and [21], to name a few) generalizing these results in various ways, including replacing the Borel subalgebra with a parabolic subalgebra, and replacing abelian ideals with k-nilpotent ideals.

In all of these papers, the proofs rely on constructing a bijection between the set of ideals and the set of miniscule elements of the affine Weyl group. This method serves to yield the number of ideals, but it does not describe what the ideals actually look like. In section 1 of [7], a more algebraic approach is taken, and suitable conditions on antichains in the set of positive roots of $\mathfrak{g}$ are developed which not only enumerate but describe the ideals of parabolic subalgebras.

In [16], Panyushev further generalized the problem: suppose $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a simple $\mathbb{Z}_2$-graded Lie algebra and let $\mathfrak{b}_0$ be a fixed Borel subalgebra of $\mathfrak{g}_0$; describe and enumerate the abelian $\mathfrak{b}_0$-stable abelian subalgebras of $\mathfrak{g}_1$. In this paper, Panyushev answered the question in one specific instance, and the question was fully answered in [3]; however, both answers relied on a modified version of the previous methods involving the affine Weyl group, and thus fall short in describing the structure of these subalgebras. Part I generalizes the algebraic approach used in [7] to again enumerate and better describe these $\mathfrak{b}_0$-stable abelian subalgebras of $\mathfrak{g}_1$.

Part II stems from [8] and [9], in which Chari and Greenstein develop theory in an effort to prove that $U(\mathfrak{g}[t])$ is quasi-hereditary. Toward that end, [2] uses certain Young diagrams as conjectured representatives of basis elements for the successive quotients of a filtration of a particular module. These Young diagrams propagate in an easily described manner, and it is shown that at each level, they are enumerated by Catalan numbers. The initial partitions used in [2] have certain restrictions on them; in Part II, we answer the natural, purely combinatorial question of modifying the propagation algorithm so that it can start with any arbitrary partition, and then apply similar methods to show that the number of partitions at each level of propagation are enumerated by ballot numbers, a generalization of Catalan numbers.

Part I

Ideals in Parabolic Subalgebras of $\mathbb{Z}_2$-Graded Lie Algebras

Chapter 1

Ideals in Parabolic Subalgebras

In this chapter we introduce the reader to the notation and preliminary results about Lie algebras that will be used throughout Part I. We will also state and prove the theorem that allows one to calculate the number of k-nilpotent ideals of an arbitrary parabolic subalgebra of a finite-dimensional simple Lie algebra.

The following notation will be used throughout the paper: $\mathbb{C}$ is the set of complex numbers; $\mathbb{Z}$ is the set of integers; $\mathbb{Z}_+$ is the set of non-negative integers; and $\mathbb{N}$ is the set of positive integers.

1.1 Finite-dimensional Simple Lie Algebras

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra of rank n over $\mathbb{C}$, $\mathfrak{h} \subseteq \mathfrak{g}$ a fixed Cartan subalgebra of $\mathfrak{g}$. Let $R \subset \mathfrak{h}^*$ be the corresponding set of roots of $\mathfrak{g}$, and fix $\Delta = \{\alpha_1, \cdots, \alpha_n\}$ a set of simple roots, with $[n] = \{1, \cdots, n\}$ the index set for Δ. Let $(,) : \mathfrak{h}^* \times \mathfrak{h}^* \longrightarrow \mathbb{C}$ be the positive definite symmetric bilinear form induced from restricting the Killing form of $\mathfrak{g}$ to $\mathfrak{h}$. Let R^+ and $R^- = -R^+$ be the sets of positive and negative roots, respectively, and let $\mathfrak{n}^{\pm} = \bigoplus_{\alpha \in R^{\pm}} \mathfrak{g}_{\alpha}$. This yields a triangular decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{n}^{-} \oplus \mathfrak{h} \oplus \mathfrak{n}^{+}.$$

Let Q^{+} be the $\mathbb{Z}_{+}$-linear span of Δ; we use this to define the standard partial ordering on R:

$$\alpha \leq \beta \Leftrightarrow \beta - \alpha \in Q^{+},$$

and denote by θ the highest root with respect to the partial ordering. Also, for every $i \in [n]$ define a homomorphism $d_i : R \longrightarrow \mathbb{Z}$ by

$$\alpha = \sum_{i=1}^{n} d_i(\alpha)\alpha_i;$$

in essence, d_i computes the coefficient of the simple root α_i in α.

We state here a result which will be useful throughout Part I of this paper:

Lemma. (i) Suppose $\mu_1, \cdots, \mu_k \in R$ are such that $\mu_1 + \cdots + \mu_k \in R \cup \{0\}$. Then there exists i, $1 < i \leq k$ such that $\mu_1 + \mu_i \in R \cup \{0\}$.

(ii) Suppose $\mu_1, \cdots, \mu_k \in R$ are such that $\mu_1 + \cdots + \mu_k \in R \cup \{0\}$. Then there exists a reordering $\mu_1, \mu_{i_2}, \cdots, \mu_{i_k}$ such that $\mu_1 + \mu_{i_2} + \cdots + \mu_{i_j} \in R \cup \{0\}$ for all j, $1 < j \leq k$.

Proof. (i) We use induction on k. Suppose $\mu_1, \mu_2, \mu_3 \in R$ such that $\mu_1 + \mu_2 + \mu_3 \in R \cup \{0\}$. If $(\mu_1, \mu_2 + \mu_3) < 0$, then either $(\mu_1, \mu_2) < 0$ or $(\mu_1, \mu_3) < 0$, which implies either $\mu_1 + \mu_2 \in R \cup \{0\}$ or $\mu_1 + \mu_3 \in R \cup \{0\}$. If $(\mu_1, \mu_2 + \mu_3) \geq 0$, then $(\mu_1 + \mu_2 + \mu_3, \mu_2 + \mu_3) > 0$, which implies either $(\mu_1 + \mu_2 + \mu_3, \mu_2) > 0$ or $(\mu_1 + \mu_2 + \mu_3, \mu_3) > 0$, and thus either $(\mu_1 + \mu_2 + \mu_3) - \mu_2 = \mu_1 + \mu_3 \in R \cup \{0\}$ or $(\mu_1 + \mu_2 + \mu_3) - \mu_3 = \mu_1 + \mu_2 \in R \cup \{0\}$. Now suppose the result holds for all $k < r$, and suppose $\mu_1, \cdots, \mu_r \in R$ are such that $\mu_1 + \cdots + \mu_r \in R \cup \{0\}$. If

$(\mu_1, \mu_2 + \cdots + \mu_r) < 0$, then $(\mu_1, \mu_i) < 0$ for some i, $1 < i \leq r$, which implies $\mu_1 + \mu_i \in R \cup \{0\}$. If $(\mu_1, \mu_2 + \cdots + \mu_r) \geq 0$, then $(\mu_1 + \cdots + \mu_r, \mu_2 + \cdots + \mu_r) > 0$, which implies $(\mu_1 + \cdots + \mu_r, \mu_i) > 0$ for some i, $1 < i \leq r$, and thus $\mu_1 + \cdots \mu_{i-1} + \mu_{i+1} + \cdots + \mu_r \in R \cup \{0\}$, which satisfies the induction hypothesis.

(ii) Assume $\mu_1 + \cdots + \mu_k \in R \cup \{0\}$. By (i), there's an i_2, $1 < i_2 \leq k$ such that $\mu_1 + \mu_{i_2} \in R \cup \{0\}$. If $\mu_1 + \mu_{i_2} = 0$, then we have $\mu_2 + \cdots + \mu_{i_2-1} + \mu_{i_2+1} + \cdots + \mu_k \in R \cup \{0\}$, and we just start the algorithm over with this collection of roots. If $\mu_1 + \mu_{i_2} \in R$, we get $(\mu_1 + \mu_{i_2}) + \mu_2 + \cdots + \mu_{i_2-1} + \mu_{i_2+1} + \cdots + \mu_k \in R \cup \{0\}$, which is a sum of $k-1$ roots. Again by (i), there exists an $i_3 \in \{2, \cdots, i_2 - 1, i_2 + 1, \cdots, k\}$ such that $\mu_1 + \mu_{i_2} + \mu_{i_3} \in R \cup \{0\}$. We can repeat this process until we have exhausted all of the roots, yielding a reordering $\mu_1 + \mu_{i_2} + \cdots + \mu_{i_k} \in R \cup \{0\}$, and at each step in the process we get $\mu_1 + \mu_{i_2} + \cdots + \mu_{i_j} \in R \cup \{0\}$.

□

The Borel subalgebra $\mathfrak{b}$ is defined by $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$. A subalgebra $\mathfrak{p}$ is parabolic if $\mathfrak{b} \subseteq \mathfrak{p}$. $\mathfrak{i} \subseteq \mathfrak{p}$ is an ideal of $\mathfrak{p}$ if $[\mathfrak{p}, \mathfrak{i}] \subseteq \mathfrak{i}$. An ideal $\mathfrak{i}$ is k-nilpotent if, given any $k+1$ elements of $\mathfrak{i}$, say $x_1, \cdots, x_{k+1}$ (not necessarily distinct), we get

$$[x_1, [x_2, [\cdots [x_k, x_{k+1}] \cdots]]] = 0;$$

$\mathfrak{i}$ is abelian if it is 1-nilpotent.

1.2 Ideals in Parabolic Subalgebras

Since any ad-nilpotent ideal $\mathfrak{i}$ of a parabolic subalgebra $\mathfrak{p}$ is contained in the unipotent radical of $\mathfrak{p}$, $\mathfrak{i}$ determines a subset of R^+. This allows one to consider the conditions on a subset of R^+ which will give rise to an ideal of $\mathfrak{p}$, as well as an ideal of a particular nilpotence. This is the content of the following:

Definition. Let $J \subseteq [n]$.

1. $R(J) = \{\alpha \in R : d_i(\alpha) = 0 \text{ if } i \notin J\}$, and $R^+(J) = R(J) \cap R^+$.
2. $\Psi \subset R^+$ is a *J-ideal* if $\Psi \cap R^+(J) = \emptyset$ and, if $\alpha \in \Psi$, $\beta \in R^+ \cup R(J)$ with $\alpha + \beta \in R^+$, then $\alpha + \beta \in \Psi$.
3. $\Psi \subset R^+$ is *k-nilpotent* if, given any $\beta_1,\ldots,\beta_{k+1} \in \Psi$, $\displaystyle\sum_{j=1}^{k+1} \beta_j \notin R^+$.
4. $\Psi \subset R^+$ is *abelian* if it is 1-nilpotent.

Note that the set of J-ideals are in one-to-one correspondence with the set of ideals of the parabolic subalgebra $\mathfrak{p}_J = \mathfrak{b} \displaystyle\bigoplus_{\alpha \in R^+(J)} \mathfrak{g}_{-\alpha}$, via $\Psi \longleftrightarrow \mathfrak{i} = \displaystyle\bigoplus_{\alpha \in \Psi} \mathfrak{g}_\alpha$. Similarly, an ideal $\mathfrak{i}$ of $\mathfrak{p}_J$ is k-nilpotent if and only if the corresponding J-ideal Ψ is k-nilpotent.

1.3 Antichains

Given any partially ordered set P, an antichain is a collection of pairwise-unrelated elements of P. In the case when the partially ordered set is a J-ideal Ψ, one can easily construct an antichain simply by taking the collection of minimal elements of Ψ. What we aim to show is that this relation is in fact a bijection - an antichain, under sufficient conditions, uniquely generates a J-ideal.

Definition. A subset A of R^+ is a *J-antichain* if $A \cap R^+(J) = \emptyset$, and for any distinct $\alpha, \beta \in A$ and $j \in J$, we have $\alpha \not\leq \beta$, $\beta \not\leq \alpha$, and $\alpha - \alpha_j \notin R$.

Notice that a J-antichain determines a minimal set of generators for an ideal in $\mathfrak{p}_J$. The aim of this section is to show that there is a bijection between the set of

J-antichains and the set of J-ideals; before we can prove this, we need a preliminary result.

Lemma. Suppose A is a J-antichain. Then for all $\alpha \in A$ and $\gamma \in R^+(J)$, we have $\alpha - \gamma \notin R$.

Proof. We proceed by induction on $\mathrm{ht}(\gamma)$, with induction beginning when $\mathrm{ht}(\gamma) = 1$ by definition of J-antichain. Suppose the result holds for all $\gamma' \in R^+(J)$ with $\mathrm{ht}(\gamma') < r$, and suppose $\gamma \in R^+(J)$ with $\mathrm{ht}(\gamma) = r$. Choose $j \in J$ such that $(\gamma, \alpha_j) > 0$ (such a j must exist; otherwise $(\gamma, \alpha_j) \leq 0$ for all $j \in J$, which implies $(\gamma, \gamma) \leq 0$, a contradiction). Then $\gamma - \alpha_j \in R^+(J)$. Now suppose there exists $\alpha \in A$ such that $\alpha - \gamma \in R$. Since $\alpha - \alpha_{j'} \notin R$ for all $j' \in J$ implies $(\alpha, \alpha_{j'}) \leq 0$ for all $j' \in J$, we get $(\alpha - \gamma, \alpha_j) < 0$, and thus $\alpha - (\gamma - \alpha_j) \in R$. But this contradicts the induction hypothesis, since $\mathrm{ht}(\gamma - \alpha_j) = r - 1$. □

We now have the necessary machinery to prove the main result of this section. Given a J-antichain A, define the set $\Psi(A) = \bigcup_{\beta \in A} \{\alpha \in R^+ : \alpha \geq \beta\}$.

Proposition. The assignment $A \longrightarrow \Psi(A)$ is a bijection between the set of J-antichains in R^+ and the set of J-ideals in R^+.

Proof. We need to show that $\Psi(A)$ satisfies the definition of J-ideal. First, suppose that $\beta \in A$ and that $\alpha \geq \beta$; then $d_i(\alpha) \geq d_i(\beta)$ for all $i \in [n]$. Since $\beta \notin R^+(J)$ by definition of J-antichain, there exists $i_0 \in [n] \backslash J$ such that $d_{i_0}(\alpha) \geq d_{i_0}(\beta) > 0$. Thus $\alpha \notin R^+(J)$, which implies $\Psi(A) \cap R^+(J) = \emptyset$. It remains to show that, if $\alpha \in \Psi(A)$, $\gamma \in R^+ \cup R(J)$ with $\alpha + \gamma \in R^+$, then $\alpha + \gamma \in \Psi(A)$. If $\gamma \in R^+$ is such that $\alpha + \gamma \in R^+$, then $\alpha + \gamma \in \Psi(A)$, since $\alpha + \gamma \geq \alpha \geq \beta$. Now suppose that $\gamma \in R^+(J)$ is such that $\alpha - \gamma \in R$. Obviously $\alpha \neq \beta$ since $\beta - \gamma \notin R$ for all $\beta \in A$ and $\gamma \in R^+(J)$ by Lemma 1.3, so $\alpha > \beta$. Suppose $\mathrm{ht}(\alpha-\beta) = r$; then $\alpha = \beta+\alpha_1+\cdots+\alpha_r$, with $\alpha_i \in \Delta$ (not necessarily distinct) for all i. Then $\alpha-\gamma = -\gamma+(\beta+\alpha_1+\cdots+\alpha_r) \in R$. By Lemma 1.1(ii), there is a reordering of the roots, say $-\gamma+\alpha_{i_1}+\cdots+\alpha_{i_k}+\beta+\alpha_{i_{k+1}}+\cdots+\alpha_{i_r}$, such that the sum of the first t roots is in $R\cup\{0\}$ for all t; in particular, $-\gamma+\alpha_{i_1}+\cdots+\alpha_{i_k} \in R\cup\{0\}$ and $-\gamma+\alpha_{i_1}+\cdots+\alpha_{i_k}+\beta \in R\cup\{0\}$. Since $A\cap R^+(J) = \emptyset$ and $\alpha_{i_j} \in \Delta$ for all j, we must have $-\gamma+\alpha_{i_1}+\cdots+\alpha_{i_k}+\beta \in R$. By Lemma 1.3, we must have $-\gamma+\alpha_{i_1}+\cdots+\alpha_{i_k} \in R^+\cup\{0\}$, which gives $\alpha-\gamma = -\gamma+(\beta+\alpha_1+\cdots+\alpha_r) \geq -\gamma+\alpha_{i_1}+\cdots+\alpha_{i_k}+\beta \geq \beta$, and thus $\alpha-\gamma \in \Psi(A)$. □

1.4 The Main Theorem - k-nilpotent Antichains

Thus far we have constructed the following bijections:

$$\{\text{Ideals in } \mathfrak{p}_J\} \longleftrightarrow \{J\text{-ideals in } R^+\} \longleftrightarrow \{J\text{-antichains in } R^+\}.$$

We also have a bijection between k-nilpotent ideals of $\mathfrak{p}_J$ and k-nilpotent J-ideals in R^+. We would like to determine the necessary and sufficient conditions for a J-antichain to determine a k-nilpotent J-ideal. To do that, we need the following preliminary result:

Lemma. Let $\beta_1,\cdots,\beta_r \in R^+$ be such that $\sum_{i=1}^{r}\beta_i \leq \gamma$, with $\gamma \in R^+$. Then there exists $\beta'_1,\cdots,\beta'_r \in R^+$ such that $\beta'_i \geq \beta_i$ for all i and $\sum_{i=1}^{r}\beta'_i = \gamma$.

Proof. We use induction on $\mathrm{ht}(\gamma-\sum_{i=1}^{r}\beta_i)$. If $\mathrm{ht}(\gamma-\sum_{i=1}^{r}\beta_i) = 0$, then $\sum_{i=1}^{r}\beta_i = \gamma$, so $\beta'_i = \beta_i$ for all i. If $\mathrm{ht}(\gamma-\sum_{i=1}^{r}\beta_i) = 1$, then $\gamma = \alpha+\beta_1+\cdots+\beta_r$, where $\alpha \in \Delta$. By Lemma 1.1(i), there exists i such that $\alpha+\beta_i \in R^+$. Then letting $\beta'_i = \alpha+\beta_i$ and $\beta'_j = \beta_j$ for all other j yields the necessary result. Now suppose the result holds for any $\mu_1,\cdots,\mu_r \in R^+$ such that $\sum_{i=1}^{r}\mu_i \leq \gamma'$ for some $\gamma' \in R^+$ with $\mathrm{ht}(\gamma'-\sum_{i=1}^{r}\mu_i) < k$, and assume $\beta_1,\cdots,\beta_r \in R^+$ are such that $\sum_{i=1}^{r}\beta_i \leq \gamma$ for some $\gamma \in R^+$ and $\mathrm{ht}(\gamma-\sum_{i=1}^{r}\beta_i) = k$.

Then we have $\gamma = \alpha_1 + \cdots + \alpha_k + \beta_1 + \cdots + \beta_r$, with $\alpha_i \in \Delta$ (not necessarily distinct) for all i. Let $\nu \in R^+$ be maximal (with respect to height) such that $\nu \leq \alpha_1 + \cdots + \alpha_k$; then $\mathrm{ht}(\nu) = s \leq k$. We can assume without loss of generality that $\nu = \alpha_1 + \cdots + \alpha_s$. Then we have $\gamma = \nu + \alpha_{s+1} + \cdots + \alpha_k + \beta_1 + \cdots + \beta_r$. Again by Lemma 1.1(i), there exists i such that $\nu + \beta_i \in R^+$ (we know we can't add α_i, $s+1 \leq i \leq k$ by maximality of ν). Let $\beta_i' = \nu + \beta_i$ and $\beta_j' = \beta_j$ for all $j \neq i$; then we have r roots $\beta_1', \cdots, \beta_r' \in R^+$ with $\sum\limits_{i=1}^{r} \beta_i' \leq \gamma$ and $\mathrm{ht}(\gamma - \sum\limits_{i=1}^{r} \beta_i') = r - s < r$, triggering the induction hypothesis and proving the Lemma. $\square$

We can now prove the main result of this chapter: the necessary and sufficient condition for a J-antichain to generate a k-nilpotent J-ideal. Recall that θ is the highest root of $\mathfrak{g}$.

Theorem. Suppose A is a J-antichain. Then $\Psi(A)$ is k-nilpotent if and only if, for any $\beta_1, \cdots, \beta_{k+1} \in A$ (not necessarily distinct), $\sum\limits_{i=1}^{k+1} \beta_i \not\leq \theta$.

Proof. The backward direction is obvious, since for all $\alpha \in \Psi(A)$, there is a $\beta \in A$ such that $\alpha \geq \beta$. For the forward direction we prove the contrapositive: supposing there exist $\beta_1, \cdots, \beta_{k+1} \in A$ (not necessarily distinct) such that $\sum\limits_{i=1}^{k+1} \beta_i \leq \theta$, we need to show $\Psi(A)$ is not k-nilpotent. But this is a straightforward result of the above Lemma: there exist $\beta_i' \geq \beta_i$ for all i (i.e. $\beta_i' \in \Psi(A)$ for all i) such that $\sum\limits_{i=1}^{k+1} \beta_i' = \theta$, and thus $\Psi(A)$ is not k-nilpotent. $\square$

A useful corollary to the above theorem is its specialization to the case when $k = 1$. Recall the functions $d_i : R \longrightarrow \mathbb{Z}$, defined by $\alpha = \sum\limits_{i=1}^{n} d_i(\alpha)\alpha_i$.

Corollary. Let $J \subseteq I$, and let $A \subset R^+$ be a J-antichain. Then A is abelian if and only if the following hold:

1. given $\alpha \in A$, there exists $i \in I$ such that $2d_i(\alpha) > d_i(\theta)$; and
2. given $\alpha, \beta \in A$ with $\alpha \neq \beta$, there exists $i \in I$ (depending on α, β) such that $d_i(\alpha) + d_i(\beta) > d_i(\theta)$; in particular $d_i(\alpha) \neq 0$ and $d_i(\beta) \neq 0$.

1.5 Enumerating k-nilpotent ideals - A_n

We now use Theorem 1.4 to enumerate k-nilpotent ideals in parabolic subalgebras of $\mathfrak{g} = \mathfrak{sl}_{n+1}$. First we use Corollary 1.5 to compute the case when $k = 1$. Let $\mathbf{A}_{s,J}$ be the set of abelian J-antichains with s elements, with the convention that $\mathbf{A}_{0,J} = \{\emptyset\}$. For $i, j \in [n]$, $i \leq j$, let $\alpha_{i,j} = \alpha_i + \cdots + \alpha_j$, where $\alpha_{i,i} = \alpha_i$. When $\mathfrak{g}$ is of type A_n, $R^+ = \{\alpha_{i,j} : i, j \in I,\ i \leq j\}$, and $\theta = \alpha_{1,n}$. To have $A \in \mathbf{A}_{s,J}$, we need to see how the elements of A behave with respect to the definition of a J-antichain, as well as the two conditions of Corollary 1.5.

For A to be a J-antichain, consider two roots $\alpha_{i,j}, \alpha_{k,\ell} \in A, \alpha_{i,j} \neq \alpha_{k,\ell}$. If $i = k$, then $\alpha_{i,\min(j,\ell)} \leq \alpha_{i,\max(j,\ell)}$, a contradiction. So assume without loss of generality that $i < k$. Then $j < \ell$, since otherwise $\alpha_{k,\ell} \leq \alpha_{i,j}$. Now let $\alpha_{k,\ell} \in A, j \in J$. We need $\alpha_{k,\ell} - \alpha_{j,j} \notin R$. Since $\alpha_{k,\ell} - \alpha_{j,j} \in R$ only when $j \in \{k, \ell\}$, $\alpha_{k,\ell} - \alpha_{j,j} \notin R \Leftrightarrow k, \ell \notin J$.

Now consider Corollary 1.5. Since $2d_i(\alpha_{i,j}) > d_i(\theta)$ for any $\alpha_{i,j} \in R^+$, there are no roots which are excluded a priori from A by the first condition. We already know from the definition of J-antichain that if $\alpha_{i,j}, \alpha_{k,\ell} \in A, \alpha_{i,j} \neq \alpha_{k,\ell}$, then $i < j$ and $k < \ell$. If $j < k$, then $d_p(\alpha_{i,j}) + d_p(\alpha_{k,\ell}) \leq d_p(\theta)$ for all p, contradicting the second condition. So $i < k \leq j < \ell$.

From this information, we can now construct a general abelian J-antichain of size s:

$$\{\alpha_{i_k,j_k} : 1 \leq k \leq s;\ i_k, j_k \in [n]\backslash J;\ i_k < i_{k+1}, i_s \leq j_1, j_k < j_{k+1}\}.$$

Given this description, $\mathbf{A}_{s,J}$ breaks into 2 cases: $i_s < j_1$, and $i_s = j_1$. These correspond to subsets of $[n]\backslash J$ of size $2s$ in the first case (2 endpoints for each of the s elements of A), and $2s-1$ in the second (one less than in case 1 since $i_s = j_1$). Thus $\#\mathbf{A}_{s,J} = \binom{n-\#J}{2s-1} + \binom{n-\#J}{2s}$, and the number of abelian J-antichains is

$$\sum_s \#\mathbf{A}_{s,J} = \sum_s \left(\binom{n-\#J}{2s-1} + \binom{n-\#J}{2s} \right) = \sum_p \binom{n-\#J}{p} = 2^{n-\#J}.$$

Notice that, when $\#J = 0$, we get that the total number of abelian ideals of $\mathfrak{p}_\emptyset = \mathfrak{b}$ is 2^n, thus recovering Peterson's result.

The following theorem describes the sets of antichains of type A_n with nilpotence p:

Theorem. Let $A = \{\{\alpha_{i_k,j_k}\}_{1\le k\le s} : \ i_k, j_k \in I, i_k < i_{k+1}, j_k < j_{k+1}\}$. Then $\Psi(A)$ is an ideal of nilpotence p and p is minimal with this property if and only if there exists $r_\ell \in \mathbb{Z}_+$, $1 \le \ell \le p$ with $r_1 + r_2 + \cdots + r_p = s$ such that

$$i_{r_1} \le j_1 < i_{r_1+1}, \ \ i_{r_1+r_2} \le j_{r_1+1} < i_{r_1+r_2+1}, \ \cdots, \ i_s \le j_{s-r_p+1}.$$

In essence, the theorem states that an ideal of nilpotence p is a disjoint union of p abelian antichains.

Chapter 2

Generalizing to $\mathbb{Z}_2$-graded Lie Algebras

In this chapter we look at a generalization of the results from Chapter 1 to the case when we have a $\mathbb{Z}_2$-graded Lie algebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$.

2.1 Involutions and $\mathbb{Z}_2$-graded Lie algebras

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra, and let σ be an involution, i.e. an automorphism of $\mathfrak{g}$ such that $\sigma^2 = 1$. Then the eigenvalues of σ are ± 1. This yields a $\mathbb{Z}_2$-gradation

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1,$$

where $\mathfrak{g}_i = \left\{ x \in \mathfrak{g} : \sigma(x) = (-1)^i x \right\}$. The converse is also true: given a $\mathbb{Z}_2$-gradation, there is a corresponding involution σ of $\mathfrak{g}$.

The following theorem from [11] classifies all possible $\mathbb{Z}_2$-gradings of $\mathfrak{g}$:

Theorem. Let $\mathbf{s} = (s_0, \cdots, s_n)$ be a sequence of nonnegative relatively prime integers such that $k\left(\sum_{i=0}^{n} a_i s_i\right) = 2$, where a_i are the labels of the diagram $X_n^{(k)}$.

(a) The relations $\sigma_{\mathbf{s},k}(E_j) = (-1)^{s_j} E_j$, $0 \leq j \leq n$, define (uniquely) an involution of $\mathfrak{g}$.

(b) Up to conjugation by an automorphism of $\mathfrak{g}$, the involutions $\sigma_{\mathbf{s};k}$ exhaust all involutions of $\mathfrak{g}$.

(c) $\sigma_{\mathbf{s};k}$ is conjugate to $\sigma_{\mathbf{s}';k'}$ by an automorphism of $\mathfrak{g}$ if and only if $k = k'$ and the sequence $\mathbf{s}$ can be transformed into $\mathbf{s}'$ by an automorphism of the diagram $X_n^{(k)}$.

By the above theorem, we can assume σ is of type $(\mathbf{s}, k)$, where $k\left(\sum_{i=0}^{n} a_i s_i\right) = 2$. Since all of the numbers in this equation must be non-negative integers, it only has 3 different types of solutions:

Type 1: $k = 1, s_p = 1, a_p = 2,$ and $s_j = 0$ for all $j \neq p$;

Type 2: $k = s_p = s_q = a_p = a_q = 1,$ and $s_j = 0$ for all $j \notin \{p, q\}$;

Type 3: $k = 2, s_p = 1, a_p = 1,$ and $s_j = 0$ for all $j \neq p$.

Notice that in all cases we have $s_i \in \{0, 1\}$ for all i. The following Proposition from [11] allows us to completely describe the $\mathbb{Z}_2$-grading of $\mathfrak{g}$.

Proposition. Let $\sigma_{\mathbf{s};k}$ be an involution.

(1) Let $i_1, \cdots, i_q$ be all indices such that $s_{i_1} = \cdots = s_{i_q} = 0$. Then $\mathfrak{g}_0(\mathbf{s}; k)$ is isomorphic to a direct sum of the $(n - q)$-dimensional center and a semisimple Lie algebra whose Dynkin diagram is the subdiagram of the affine diagram $X_n^{(k)}$ consisting of vertices $i_1, \cdots, i_q$.

(2) Let $j_1, \cdots, j_r$ be all indices such that $s_{j_1} = \cdots = s_{j_r} = 1$. Then, as a $\mathfrak{g}_0(\mathbf{s}; k)$-module, $\mathfrak{g}_1(\mathbf{s}; k)$ is isomorphic to a direct sum of r irreducible modules with highest weights $-\alpha_{j_1}, \cdots, -\alpha_{j_r}$.

In particular, this proposition, combined with the above cases, implies that $\mathfrak{g}_1$ is either a highest weight module or a direct sum of two highest weight modules.

2.2 $\mathfrak{b}_0$-stable Abelian Subalgebras of $\mathfrak{g}_1$

We would now like to generalize the results from Chapter 1 to the case of $\mathbb{Z}_2$-graded Lie algebras, but first, we must introduce some new notation. Let $\sigma = \sigma_{(s_0,\cdots,s_n;k)}$ be a fixed, order 2 automorphism corresponding to $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Let Δ_0 be the set of simple roots of $\mathfrak{g}_0$, R_0 the set of roots of $\mathfrak{g}_0$, and R_1 the set of weights of $\mathfrak{g}_1$. Define a new partial ordering on R_0 and R_1:

$$\alpha \prec \beta \Leftrightarrow \beta - \alpha \text{ is in the } \mathbb{Z}^+\text{-span of } \Delta_0.$$

Let A be an antichain in R_1 with respect to this partial ordering. For the same reason an antichain in R^+ determined an ideal of $\mathfrak{b}$, an antichain in R_1 determines a $\mathfrak{b}_0$-submodule of $\mathfrak{g}_1$. We want to figure out a condition on A that makes the corresponding submodule a $\mathfrak{b}_0$-stable abelian subalgebra.

Theorem. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Let $\{\theta_j\}_{j=1}^s$ be the highest roots of the connected subdiagrams of $\mathfrak{g}_0$, and let A be an antichain in R_1 with respect to $\prec$. Then A is abelian if and only if, for any $\alpha, \beta \in A$ (not necessarily distinct), $\alpha + \beta \not\prec \theta_j$ for all j.

Proof. The forward direction is obvious. To prove the backward direction, suppose there exist $\alpha, \beta \in A$ such that $\alpha + \beta \prec \theta_j$ for some j. If $\alpha + \beta \in R_0$, then we're done. If not, then we have $\alpha + \beta + \gamma_1 + \cdots + \gamma_r = \theta_j$, where each $\gamma_t \in \Delta_0$ (not necessarily distinct).

By Lemma 1.1(i), there exists a γ_{t_1} such that $\alpha + \gamma_{t_1} \in R_1$. If $\alpha + \gamma_{t_1} + \beta \in R_0$, we're done. If not, there exists a γ_{t_2} such that $\alpha + \gamma_{t_1} + \gamma_{t_2} \in R_1$. Since there are only finitely many roots, this process must terminate, either with $\alpha + \gamma_{t_1} + \cdots + \gamma_{t_d} + \beta \in R_0$ for some d, or with $\alpha + \beta + \gamma_1 + \cdots + \gamma_r = \theta_j \in R_0$, yielding the result. □

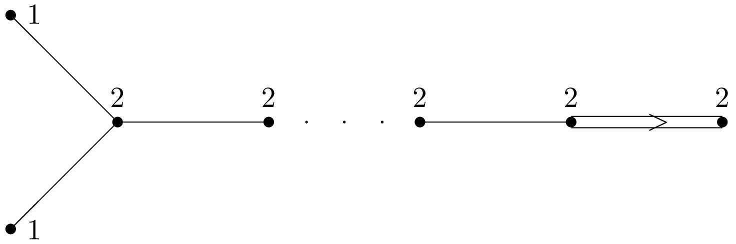
Notice that this theorem generalizes part of the results from [7]; if we take σ to be the involution of type $(1, 0, \cdots, 0; 1)$ for any untwisted affine Lie algebra of type $X_n^{(1)}$, then $\mathfrak{g}_0$ is a simple Lie algebra $\mathfrak{g}$ of type X_n, $\mathfrak{g}_1 = V(-\alpha_0) = V(\theta) \cong \mathfrak{g}$, and thus $\mathfrak{b}_0$-stable abelian subalgebras of $\mathfrak{g}_1$ are simply abelian ideals of $\mathfrak{b}$.

2.3 Counting $\mathfrak{b}_0$-stable Abelian Subalgebras - $B_n^{(1)}$

We will now do an example illustrating how to use Theorem 2.2 to enumerate the $\mathfrak{b}_0$-stable abelian subalgebras of $\mathfrak{g}_1$. Consider the case when $\mathfrak{g}$ is of type B_n. Recall that $\alpha_{i,j} = \alpha_i + \cdots + \alpha_j$, and let $\beta_{i,j} = \alpha_{i,n} + \alpha_{j,n}$. Then

$$R^+ = \{\alpha_{i,j} : 1 \le i \le j \le n\} \cup \{\beta_{i,j} : 1 \le i < j \le n\}.$$

We want to study the case when σ is of type 2 as described in Section 2.1, namely $k = s_p = s_q = a_p = a_q = 1$ and $s_i = 0$ for all $i \notin \{p, q\}$. By Theorem 2.1, we look at the diagram and labels of the affine Lie algebra of type $B_n^{(1)}$:



Since there are only two nodes with label 1, there is only one permutation of type 2 for B_n: the permutation σ of type $(1, 1, 0, \cdots, 0; 1)$. By Proposition 2.1, this implies $\mathfrak{g}_0$ is a simple Lie algebra of type B_{n-1} with indices from 2 to n and thus has highest root $\theta_0 = \beta_{2,3}$, and $\mathfrak{g}_1 = V(-\alpha_0) \oplus V(-\alpha_1)$. By the labelling of $B_n^{(1)}$, we know $\alpha_0 + \alpha_1 + 2(\alpha_2 + \cdots + \alpha_n) = 0$, or $\alpha_0 = -\theta$, where $\theta = \beta_{1,2}$ is the highest root of $\mathfrak{g}$. Thus $\mathfrak{g}_1 = V(\beta_{1,2}) \oplus V(-\alpha_{1,1})$. It is easy to see that $\mathfrak{g}_1$ has the following weight lattice:

$$\beta_{1,2} \geq \beta_{1,3} \geq \cdots \geq \beta_{1,n} \geq \alpha_{1,n} \geq \alpha_{1,n-1} \geq \cdots \geq \alpha_{1,1}$$

$$-\alpha_{1,1} \geq -\alpha_{1,2} \geq \cdots \geq -\alpha_{1,n} \geq -\beta_{1,n} \geq -\beta_{1,n-1} \geq \cdots \geq -\beta_{1,2}.$$

Notice that, for every weight μ in $\mathfrak{g}_1$, $d_1(\mu) \neq 0$. This implies that $2\mu \not\leq \theta_0$, since $d_1(\theta_0 - 2\mu) \neq 0$, and thus $\theta_0 - 2\mu \notin Q_0^+$ for all μ. Therefore, $\{\mu\}$ is an abelian antichain for all μ.

Since the weights of $V(\beta_{1,2})$ and $V(-\alpha_{1,1})$ are separately totally ordered, the only other possible nontrivial antichains must look like $\{\mu, \nu\}$, where μ is a weight of $V(\beta_{1,2})$ and ν is a weight of $V(-\alpha_{1,1})$. Since $d_1(\mu) = 1$ and $d_1(\nu) = -1$, $d_1(\mu + \nu) = 0$, which implies $\mu + \nu \in Q_0$. We need to find μ and ν such that $\mu + \nu \not\leq \theta_0 = \beta_2, 3$. Since $0 \leq d_i(\mu) \leq 2$ and $-2 \leq d_i(\nu) \leq 0$ for $i = 2, cdots, n$, we get $d_i(\mu + \nu) \leq 2$ for all corresponding i. Since $d_2(\beta_{2,3}) = 1$ and $d_i(\beta_{2,3}) = 2$ for all $i = 3, \cdots, n$, the only way for $\mu + \nu \not\leq \beta_2, 3$ is if $d_2(\mu + \nu) = 2$. This only occurs if $\mu = \beta_{1,2}$ and $\nu = -\alpha_{1,1}$, and thus $\{\beta_{1,2}, -\alpha_{1,1}\}$ is the only 2-element abelian antichain.

Therefore, the total number of abelian antichains is $4n$: the empty set, the $4n - 2$ weights as individual antichains, and $\{\beta_{1,2}, -\alpha_{1,1}\}$. Notice that this enumeration agrees with the results obtained in [3].

Part II

Partitions, Young Diagrams and Ballot Numbers

Chapter 3

Background Material

In this chapter we will familiarize the reader with the combinatorial objects, formulas and identities that will be used throughout Part II. We will also describe the preliminary results that will be generalized in the following chapter.

3.1 Catalan Numbers, Ballot Numbers and Identities

For $\ell \in \mathbb{Z}_+$, the ℓ^{th} Catalan number is

$$C_\ell = \binom{2\ell}{\ell} - \binom{2\ell}{\ell-1} = \frac{1}{\ell+1}\binom{2\ell}{\ell}.$$

These numbers were initially discovered by enumerating triangulations of a regular $\ell+2$-gon, but the most prominent objects used for enumerating the Catalan numbers are Dyck paths: paths from $(0,0)$ to $(2\ell,0)$ using ℓ copies of the vector $[1,1]$ ("ups") and ℓ copies of the vector $[1,-1]$ ("downs") that stay at or above the x-axis.

One can generalize the notion of Dyck paths by constructing paths from $(0,0)$ to $(2\ell+m,m)$ using $\ell+m$ ups and ℓ downs that stay at or above the x-axis. The numbers which enumerate such paths are called ballot numbers, and their formula is

$$b_{\ell,m} = \binom{2\ell+m}{\ell} - \binom{2\ell+m}{\ell-1} = \frac{m+1}{\ell+m+1}\binom{2\ell+m}{\ell}.$$

Notice that when $m = 0$ we obtain Dyck paths and Catalan numbers. These numbers are called ballot numbers because of the election-based interpretation of the corresponding paths: how many ways can votes be cast so that candidate A beats candidate B by m votes and candidate A was never behind in the election?

Here we will list combinatorial identities which will be used throughout the calculations in the rest of Part II. First is the standard Pascal's Triangle identity:

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}. \tag{3.1}$$

The second is proven in [2], though is probably a standard result:

$$\binom{m+\ell}{\ell} = \sum_{j\geq 0}(-1)^j\binom{\ell-j}{j} b_{\ell-j,m}. \tag{3.2}$$

3.2 Partitions and Young Diagrams

$\lambda \in \mathbb{N}^n$ is a partition with n parts if

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n).$$

We denote the set of all partitions by $\mathcal{P}$. Given $\lambda \in \mathcal{P}$ set

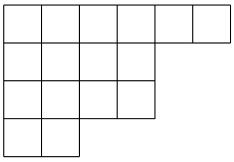
$$\lambda \setminus \{\lambda_n\} = (\lambda_1 \geq \cdots \geq \lambda_{n-1}),$$

and for $\lambda_{n+1} \in \mathbb{N}$ such that $\lambda_{n+1} \leq \lambda_n$, set

$$(\lambda : \lambda_{n+1}) = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq \lambda_{n+1}).$$

Given a partition with n parts, these operations allow us to construct partitions with $n-1$ and $n+1$ parts, respectively.

Corresponding to any partition is a combinatorial object called a Young diagram. For the partition $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathcal{P}$, the corresponding Young diagram is a collection of left-justified squares, with λ_1 squares in the first row, λ_2 in the second, and so on. For example, given the partition $\lambda = (6, 4, 4, 2)$, the corresponding Young diagram is:



3.3 The Set $\mathcal{P}^{n,k}$ and the Map τ_k

For $k, n \in \mathbb{N}$, let $\mathcal{P}^{n,k}$ be the set of partitions with exactly n parts where no part is bigger than k, i.e.

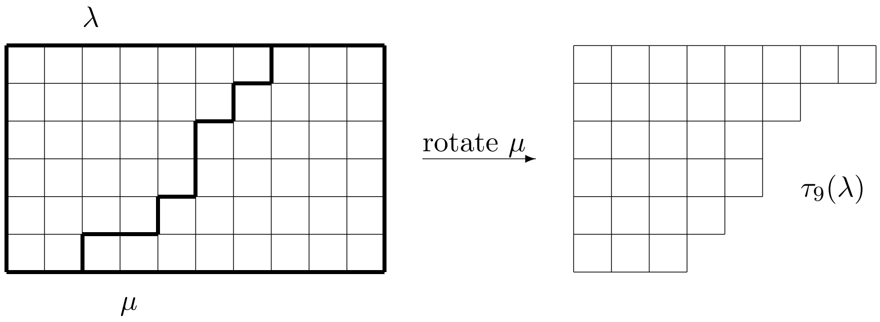
$$\mathcal{P}^{n,k} = \{\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n) : \lambda_1 \leq k, \lambda_n > 0\}.$$

Let $\tau_k : \mathcal{P}^{n,k} \to \mathcal{P}^{n,k}$ be defined by

$$\tau_k(\lambda_1 \geq \cdots \geq \lambda_n) = (k + 1 - \lambda_n \geq \cdots \geq k + 1 - \lambda_1).$$

Clearly τ_k is a bijection of order two. To understand the map τ_k in terms of Young diagrams, it is convenient to think of $\mathcal{P}^{n,k}$ as the set of partitions whose Young diagrams lie in an $n \times k$ rectangle and have exactly n rows. The Young diagram of $\tau_k(\lambda)$ is obtained by taking the skew diagram $(k + 1)^n \setminus \lambda$ and rotating it by one hundred eighty degrees.

As an example, we can regard the partition $\lambda = (7 \geq 6 \geq 5 \geq 5 \geq 4 \geq 2)$ as an element of $\mathcal{P}^{6,9}$ in which case we have $\tau_9(\lambda) = (8 \geq 6 \geq 5 \geq 5 \geq 4 \geq 3)$, and pictorially, we get the following diagram:



3.4 The Set $\mathcal{P}^\ell(\lambda)$

Set $\mathcal{P}^k = \mathcal{P}^{k,k}$. Given $\lambda \in \mathcal{P}^k$ and $\ell, k \in \mathbb{N}$ with $\ell \geq k$, define subsets $\mathcal{P}^\ell(\lambda)$ of $\mathcal{P}^\ell$ inductively, by

$$\mathcal{P}^k(\lambda) = \{\lambda\} \cup \{\tau_k\lambda\}, \quad \mathcal{P}^\ell(\lambda) = \mathcal{P}^\ell_{\mathrm{d}}(\lambda) \cup \mathcal{P}^\ell_\tau(\lambda),$$

where

$$\mathcal{P}^\ell_{\mathrm{d}}(\lambda) = \{\mu \in \mathcal{P}^\ell : \mu \setminus \{\mu_\ell\} \in \mathcal{P}^{\ell-1}(\lambda)\}, \text{ and}$$

$$\mathcal{P}^\ell_\tau(\lambda) = \{\mu \in \mathcal{P}^\ell : \tau_\ell\mu \setminus \{\ell + 1 - \mu_1\} \in \mathcal{P}^{\ell-1}(\lambda)\} = \tau_\ell\mathcal{P}^\ell_{\mathrm{d}}(\lambda).$$

Clearly $\mathcal{P}^\ell(\tau_k\lambda) = \tau_\ell\mathcal{P}^\ell(\lambda) = \mathcal{P}^\ell(\lambda)$.

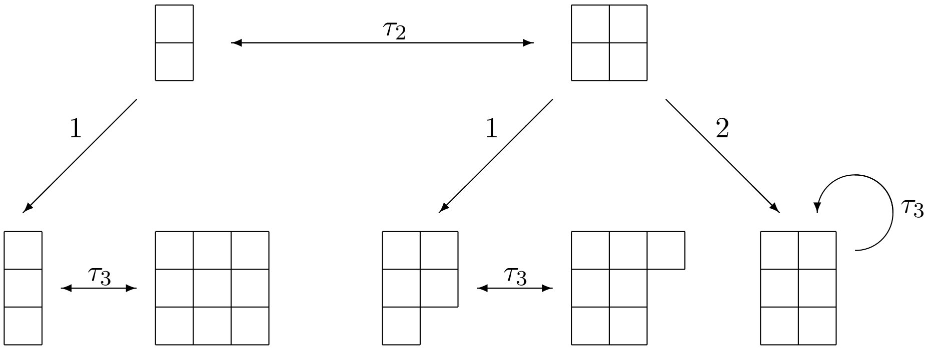
The following lemma, while elementary, is important for proving many of the results in the rest of this chapter:

Lemma. Let $\mu \in \mathcal{P}^\ell$. Then

$$\mu \in \mathcal{P}^\ell_{\mathrm{d}}(\lambda) \implies \mu_1 \leq \ell - 1,$$

$$\mu \in \mathcal{P}^\ell_\tau(\lambda) \implies \mu_\ell > 1.$$

We illustrate the recursive definition of $\mathcal{P}^\ell(\lambda)$ in a simple case using Young diagrams. Given $\lambda = (1 \geq 1) \in \mathcal{P}^2$, the elements of $\mathcal{P}^2(\lambda)$ and $\mathcal{P}^3(\lambda)$ are obtained as follows:



For $k \in \mathbb{N}$, we define

$$\mathcal{P}^k_{\text{sqb}} = \{\lambda \in \mathcal{P}^k : \lambda_1 = k, \quad \lambda_k = 1\}.$$

The goal of [2] is to compute $\#\mathcal{P}^\ell(\lambda)$ for $\lambda \in \mathcal{P}^k_{\text{sqb}}$. In order to do this, some preliminary results are required. The first tells us where $\mathcal{P}^\ell_{\text{d}}(\lambda)$ and $\mathcal{P}^\ell_\tau(\lambda)$ overlap:

Proposition. Let $\ell \geq k \geq 1$, $\lambda \in \mathcal{P}^k_{\text{sqb}}$. We have

$$\begin{aligned} \mathcal{P}^\ell_{\text{d}}(\lambda) \cap \mathcal{P}^\ell_\tau(\lambda) &= \{\mu \in \mathcal{P}^\ell_{\text{d}}(\lambda) : \ell - 1 \geq \mu_1 \geq \mu_\ell \geq 2\} \\ &= \{\mu \in \mathcal{P}^\ell_\tau(\lambda) : \ell - 1 \geq \mu_1 \geq \mu_\ell \geq 2\} \\ &= \{\mu \in \mathcal{P}^\ell(\lambda) : \ell - 1 \geq \mu_1 \geq \mu_\ell \geq 2\}. \end{aligned}$$

The following is an obvious extension of the Proposition:

Corollary. For all $\ell \geq k$, we have

$$\begin{aligned} \mathcal{P}^\ell(\lambda) &= \mathcal{P}^\ell_{\text{d}}(\lambda) \ \bigsqcup \ \{\mu \in \mathcal{P}^\ell_\tau(\lambda) : \mu_1 = \ell\} \\ &= \mathcal{P}^\ell_\tau(\lambda) \ \bigsqcup \ \{\mu \in \mathcal{P}^\ell_{\text{d}}(\lambda) : \mu_\ell = 1\}. \end{aligned}$$

What the Corollary tells us that we only need to apply τ_ℓ to elements in $\mathcal{P}^\ell_{\text{d}}(\lambda)$ whose ℓ^{th} entry is 1 in order to obtain all of $\mathcal{P}^\ell(\lambda)$. This gives us the necessary machinery to prove the following:

Theorem. (i) Let $\ell, k \in \mathbb{N}$ be such that $\ell \geq k$ and let $\lambda \in \mathcal{P}^k_{\mathrm{sqb}}$. Then,

$$\#\mathcal{P}^\ell(\lambda) = \begin{cases} c_{\ell-k+1}, & \lambda = \tau_k\lambda, \\ \\ 2c_{\ell-k+1}, & \lambda \neq \tau_k\lambda. \end{cases}$$

(ii) Let $\lambda \in \mathcal{P}^k_{\mathrm{sqb}}$, $\nu \in \mathcal{P}^s_{\mathrm{sqb}}$. For all $\ell \in \mathbb{Z}_+$ with $\ell \geq \max(k, s)$, we have

$$\mathcal{P}^\ell(\lambda) \cap \mathcal{P}^\ell(\nu) = \emptyset, \quad \text{if} \quad \nu \notin \{\lambda, \tau_k(\lambda)\}.$$

(iii) We have

$$\mathcal{P}^\ell = \bigsqcup_{\{\lambda \in \mathcal{P}^k_{\mathrm{sqb}}:\ \ell\ \geq\ k \geq\ 1\}} \mathcal{P}^\ell(\lambda).$$

What is remarkable about this result is the map τ. While many other combinatorial objects enumerated by the Catalan numbers have natural bijections (for example, taking a given Dyck path and flipping it about the line $x = \ell$), none of them are equivalent to τ.

Chapter 4

Counting $\mathcal{P}^{\ell}(\lambda)$ for Arbitrary λ

In this chapter we generalize many of the results from chapter 3 by allowing our starting partition to be of any size, rather than just an element of $\mathcal{P}^k_{\text{sqb}}$ for some k. The proofs of all results in this chapter can be used as proofs of the corresponding results in the previous chapter.

4.1 Constructing τ_ℓ and $\mathcal{P}^{\ell}(\lambda)$ for an Arbitrary λ

Let $\lambda = (\lambda_1, \cdots, \lambda_k) \in \mathcal{P}$ be arbitrary. We need to construct a map τ_ℓ for $\ell \geq k$ that reflects the operation from Chapter 3. The key fact about τ_k as it relates to $\lambda \in \mathcal{P}^k_{\text{sqb}}$ is that $(\tau_k(\lambda))_1 = \lambda_1$ and $(\tau_k(\lambda))_k = \lambda_k = 1$; in other words, λ and $\tau_k(\lambda)$ have the same first and last entries. To imitate that property, we define

$$\tau_k(\lambda_1, \lambda_2, \cdots, \lambda_{k-1}, \lambda_k) = (\lambda_1, \lambda_1 + \lambda_k - \lambda_{k-1}, \cdots, \lambda_1 + \lambda_k - \lambda_2, \lambda_k);$$

from the perspective of Young diagrams, $\tau_k(\lambda)$ is the skew diagram $(\lambda_1 + \lambda_k)^k \backslash \lambda$.

The second important piece of information is the relation between τ_ℓ and $\tau_{\ell+1}$. In Chapter 3, when viewing τ_ℓ as a skew diagram map, the rectangle encasing the initial partition had length $\ell+1$; so when the index of τ went up by 1, the size of the encasing rectangle went up by 1 as well. This allows us to define both τ_ℓ and $\mathcal{P}^\ell(\lambda)$ recursively:

$$\mathcal{P}^k(\lambda) = \{\lambda\} \cup \{\tau_k \lambda\}, \quad \mathcal{P}^\ell(\lambda) = \mathcal{P}^\ell_{\mathrm{d}}(\lambda) \cup \mathcal{P}^\ell_\tau(\lambda),$$

where

$$\mathcal{P}^\ell_{\mathrm{d}}(\lambda) = \{(\mu : \mu_\ell) : \mu \in \mathcal{P}^{\ell-1}(\lambda)\}, \quad \mathcal{P}^\ell_\tau(\lambda) = \tau_\ell\left(\mathcal{P}^\ell_{\mathrm{d}}(\lambda)\right),$$

and for $\mu \in \mathcal{P}^\ell(\lambda)$,

$$\tau_\ell(\mu_1, \cdots, \mu_\ell) = (\lambda_1 + \lambda_k + \ell - k - \mu_\ell, \cdots, \lambda_1 + \lambda_k + \ell - k - \mu_1).$$

The following is a generalization of Lemma 3.4:

Lemma. Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k)$ be a partition, and let $\mu \in \mathcal{P}^\ell(\lambda)$, $\ell \geq k$. Then $\lambda_1 \leq \mu_1 \leq \lambda_1 + \lambda_k + \ell - k - 1$, and $1 \leq \mu_\ell \leq \lambda_k + \ell - k$. In particular, $\mu \in \mathcal{P}^\ell_d(\lambda)$ implies $\lambda_1 \leq \mu_1 \leq \lambda_1 + \lambda_k + \ell - k - 2$ and $1 \leq \mu_\ell \leq \lambda_k + r - k - 1$; and $\mu \in \mathcal{P}^\ell_\tau(\lambda)$ implies $\lambda_1 + 1 \leq \mu_1 \leq \lambda_1 + \lambda_k + \ell - k - 1$ and $2 \leq \mu_\ell \leq \lambda_k + r - k$.

Proof. We use induction on ℓ, with the basis step trivially true for $\ell = k$ and easily checked for $\ell = k+1$. Now assume the Lemma is true for all $\ell < r$, and let $\mu \in \mathcal{P}^r(\lambda)$. If $\mu \in \mathcal{P}^r_d(\lambda)$, then $\mu \backslash \{\mu_r\} \in \mathcal{P}^{r-1}(\lambda)$. By induction, this means $\lambda_1 \leq \mu_1 \leq \lambda_1 + \lambda_k + r - k - 2$ and $1 \leq \mu_{r-1} \leq \lambda_k + r - k - 1$; since $\mu \in \mathcal{P}^r_d(\lambda)$, we get $1 \leq \mu_r \leq \lambda_k + r - k - 1$.

Now suppose $\mu \in \mathcal{P}^r_\tau(\lambda)$. This implies $\mu = \tau_r(\mu')$ for some $\mu' \in \mathcal{P}^r_d(\lambda)$. By above, we know this means $\lambda_1 \leq \mu'_1 \leq \lambda_1 + \lambda_k + r - k - 2$ and $1 \leq \mu'_r \leq \lambda_k + r - k - 1$. Since $\mu_1 = (\tau_r(\mu'))_r$ and $\mu_r = (\tau_r(\mu'))_1$, we get the new inequalities $2 \leq \mu_r \leq \lambda_k + r - k$ and $\lambda_1 + 1 \leq \mu_1 \leq \lambda_1 + \lambda_k + r - k - 1$. Combining these inequalities with the previous two yields the result. $\square$

In order to more easily compute $\#\mathcal{P}^\ell(\lambda)$, we need a result like Proposition 3.4 that tells us where $\mathcal{P}^\ell_{\mathrm{d}}(\lambda)$ and $\mathcal{P}^\ell_\tau(\lambda)$ intersect.

Proposition.

$$\mathcal{P}^\ell_d(\lambda) \cap \mathcal{P}^\ell_\tau(\lambda) = \left\{ \mu \in \mathcal{P}^\ell(\lambda) : \begin{array}{r} \lambda_1 + 1 \leq \mu_1 \leq \lambda_1 + \lambda_k + \ell - k - 2 \\ 2 \leq \mu_\ell \leq \lambda_k + \ell - k - 1 \end{array} \right\}.$$

Proof. The left side being contained in the right is an obvious consequence of the previous Lemma. For the other containment, we use induction on ℓ, with induction vacuously holding for $\ell = k$ and $\ell = k + 1$. For the inductive step, we need to show that if $\mu \in \mathcal{P}^r_d(\lambda)$ satisfies the right-hand inequalities, then $\tau_r(\mu) \in \mathcal{P}^r_d(\lambda)$, i.e. $\tau_r(\mu) \backslash \{\lambda_1 + \lambda_k + r - k - \mu_1\} \in \mathcal{P}^{r-1}(\lambda)$. This is true if and only if

$$\tau_{r-1}(\tau_r(\mu) \backslash \{\lambda_1 + \lambda_k + r - k - \mu_1\}) = (\mu_2 - 1 \geq \cdots \geq \mu_r - 1) \in \mathcal{P}^{r-1}(\lambda).$$

$\mu \in \mathcal{P}^r_d(\lambda)$ implies $\mu \backslash \{\mu_r\} \in \mathcal{P}^{r-1}(\lambda)$. If $\mu_1 \neq \lambda_1 + \lambda_k + \ell - k - 2$ and $\mu_{r-1} \neq \lambda_k + \ell - k - 1$, this element satisfies the requirements of the induction hypothesis, so $\tau_{r-1}(\mu \backslash \{\mu_r\}) = (\lambda_1 + \lambda_k + r - 1 - k - \mu_{r-1} \geq \cdots \geq \lambda_1 + \lambda_k + r - 1 - k - \mu_1) \in \mathcal{P}^{r-1}_d(\lambda)$, which implies $(\lambda_1 + \lambda_k + r - 1 - k - \mu_{r-1} \geq \cdots \geq \lambda_1 + \lambda_k + r - 1 - k - \mu_2) \in \mathcal{P}^{r-2}(\lambda)$, and thus $\tau_{r-2}(\lambda_1 + \lambda_k + r - 1 - k - \mu_{r-1} \geq \cdots \geq \lambda_1 + \lambda_k + r - 1 - k - \mu_2) = (\mu_2 - 1, \cdots, \mu_{r-1} - 1) \in \mathcal{P}^{r-2}(\lambda)$. Since $\mu_r \leq \mu_{r-1}$, we get $(\mu_2 - 1, \cdots, \mu_{r-1} - 1, \mu_r - 1) \in \mathcal{P}^{r-1}(\lambda)$ as required.

Now suppose $\mu_1 = \lambda_1 + \lambda_k + \ell - k - 2$. Then $(\tau_{r-1}(\mu \backslash \{\mu_r\}))_{r-1} = 1$, and thus $\tau_{r-1}(\mu \backslash \{\mu_r\}) \in \mathcal{P}^{r-1}_d(\lambda)$. This implies $\tau_{r-1}(\mu \backslash \{\mu_r\}) \backslash \{1\} \in \mathcal{P}^{r-2}(\lambda)$, and so $\tau_{r-2}(\tau_{r-1}(\mu \backslash \{\mu_r\}) \backslash \{1\}) = (\mu_2 - 1, \cdots, \mu_{r-1} - 1) \in \mathcal{P}^{r-2}(\lambda)$, which yields the necessary result. Similarly, suppose $\mu_{r-1} = \lambda_k + r - k - 1$. Then $(\tau_{r-1}(\mu \backslash \{\mu_r\}))_1 = \lambda_1$, and thus $\tau_{r-1}(\mu \backslash \{\mu_r\}) \in \mathcal{P}^{r-1}_d(\lambda)$. This implies $\tau_{r-1}(\mu \backslash \{\mu_r\}) \backslash \{\lambda_1 + \lambda_k - \ell - k - 1\mu_1\} \in \mathcal{P}^{r-2}(\lambda)$, and so $\tau_{r-2}(\tau_{r-1}(\mu \backslash \{\mu_r\}) \backslash \{\lambda_1 + \lambda_k - \ell - k - 1\mu_1\}) = (\mu_2 - 1, \cdots, \mu_{r-1} - 1) \in \mathcal{P}^{r-2}(\lambda)$, again yielding the required result. $\square$

From the Proposition we get the obvious result which parallels Corollary 3.4:

Corollary. $\mathcal{P}^\ell(\lambda) = \mathcal{P}^\ell_d(\lambda) \sqcup \tau\left(\left\{\mu \in \mathcal{P}^\ell_d(\lambda) : \mu_1 = \lambda_1\right\} \cup \left\{\mu \in \mathcal{P}^\ell_d(\lambda) : \mu_\ell = 1\right\}\right)$
$= \mathcal{P}^\ell_d(\lambda) \sqcup \left(\left\{\nu \in \mathcal{P}^\ell_\tau(\lambda) : \nu_\ell = \lambda_k + \ell - k\right\} \cup \left\{\nu \in \mathcal{P}^\ell_\tau(\lambda) : \nu_1 = \lambda_1 + \lambda_k + \ell - k - 1\right\}\right)$.

What this corollary tells us is that it is only necessary to apply τ_ℓ to a partition μ in $\mathcal{P}^\ell_d(\lambda)$ if $\mu_1 = \lambda_1$ or $\mu_\ell = 1$. The reason applying τ_ℓ to partitions such that $mu_1 = \lambda_1$ did not appear in Corollary 3.4 is because it is redundant: if $\mu \in \mathcal{P}^\ell_d(\lambda)$, $\lambda \in \mathcal{P}^k_{\mathrm{sqb}}$, is such that $\mu_1 = \lambda_1$, then it must be the case that $\mu_\ell = 1$.

4.2 Computing Recurrence Relations and Initial Conditions

Let $\mathcal{P}^\ell(\lambda, r) = \left\{\mu \in \mathcal{P}^\ell(\lambda) : \mu_\ell \geq r\right\}$, and let $e_{\ell,r}(\lambda) = \#\mathcal{P}^\ell(\lambda, r)$. By Lemma 4.1, we know $1 \leq r \leq \lambda_k + \ell - k$. We want to compute $\#\mathcal{P}^\ell(\lambda) = e_{\ell,1}(\lambda)$.

When $r = \lambda_k + \ell - k$, we get

$$e_{\ell,\lambda_k+\ell-k}(\lambda) = \#\{\mu \in \mathcal{P}^\ell(\lambda) : \mu_\ell = \lambda_k + \ell - k\}.$$

Since τ_ℓ is an involution for all ℓ, $\#\tau(S) = \#S$ for all S. This yields

$$e_{\ell,\lambda_k+\ell-k}(\lambda) = \#\left(\tau\left(\{\mu \in \mathcal{P}^\ell(\lambda) : \mu_\ell = \lambda_k + \ell - k\}\right)\right) = \#\{\nu \in \mathcal{P}^\ell(\lambda) : \nu_1 = \lambda_1\}.$$

Since $\nu_1 = \lambda_1$ implies $\nu \in \mathcal{P}^\ell_{\mathrm{d}}(\lambda)$ for all ℓ by Lemma 4.1, ν is a "pure" direct descendant: to get ν from λ, we add a partition with $\ell - k$ parts, whose first part is at most λ_k, to the bottom of λ. Thus we get

$$e_{\ell,\lambda_k+\ell-k}(\lambda) = \#\mathcal{P}^{\ell-k,\lambda_k} = \binom{\lambda_k - 1 + \ell - k}{\lambda_k - 1}. \tag{4.1}$$

From Corollary 4.1, we get

$$\begin{aligned}
\mathcal{P}^\ell(\lambda,r) &= \mathcal{P}^\ell_d(\lambda,r)\sqcup\left(\begin{array}{c}\left\{\nu\in\mathcal{P}^\ell_\tau(\lambda):\nu_\ell=\lambda_k+\ell-k\geq r\right\}\cup\\ \left\{\nu\in\mathcal{P}^\ell_\tau(\lambda):\nu_1=\lambda_1+\lambda_k+\ell-k-1,\,\nu_\ell\geq r\right\}\end{array}\right)\\
&= \mathcal{P}^\ell_d(\lambda,r)\sqcup\tau_\ell\left(\begin{array}{c}\left\{\mu\in\mathcal{P}^\ell_d(\lambda):\mu_1=\lambda_1\leq\lambda_1+\lambda_k+\ell-k-r\right\}\cup\\ \left\{\mu\in\mathcal{P}^\ell_d(\lambda):\mu_\ell=1,\,\mu_1\leq\lambda_1+\lambda_k+\ell-k-r\right\}\end{array}\right).
\end{aligned}$$

For simplicity, let

$$A=\left\{\mu\in\mathcal{P}^\ell_d(\lambda):\mu_1=\lambda_1\leq\lambda_1+\lambda_k+\ell-k-r\right\},$$
$$B=\left\{\mu\in\mathcal{P}^\ell_d(\lambda):\mu_\ell=1,\,\mu_1\leq\lambda_1+\lambda_k+\ell-k-r\right\}.$$

Again, τ_ℓ being an involution gives

$$\#\mathcal{P}^\ell(\lambda,r)=\#\mathcal{P}^\ell_d(\lambda,r)+\#(A\cup B)=\#\mathcal{P}^\ell_d(\lambda,r)+\#A+\#B-\#(A\cap B). \qquad (4.2)$$

To compute $\#\mathcal{P}^\ell_{\mathrm{d}}(\lambda,r)$, notice that

$$\mathcal{P}^\ell_{\mathrm{d}}(\lambda,r)=\bigsqcup_{s\geq r}\{(\mu:s):\mu\in\mathcal{P}^{\ell-1}(\lambda,s)\},$$

and hence

$$\#\mathcal{P}^\ell_{\mathrm{d}}(\lambda,r)=\sum_{s\geq r}e_{\ell-1,s}(\lambda).$$

To compute $\#A$, notice that the inequality condition is always true because $r\leq\lambda_k+\ell-k$. Thus by (4.1) we get

$$\#A=\#\left\{\mu\in\mathcal{P}^\ell_d(\lambda):\mu_1=\lambda_1\right\}=\binom{\lambda_k-1+\ell-k}{\lambda_k-1}.$$

Next, we calculate $\#B$:

$$\begin{aligned}
\#B&=\#\left\{\mu\in\mathcal{P}^\ell_d(\lambda):\mu_\ell=1,\,\mu_1\leq\lambda_1+\lambda_k+\ell-k-r\right\}\\
&=\#\left\{\mu\in\mathcal{P}^{\ell-1}(\lambda):\mu_1\leq\lambda_1+\lambda_k+\ell-k-r\right\}\\
&=\#\left\{\nu\in\mathcal{P}^{\ell-1}(\lambda):\nu_{\ell-1}\geq r-1\right\}\\
&=e_{\ell-1,r-1}(\lambda).
\end{aligned}$$

Finally, using (4.1) at $\ell - 1$ we compute $\#(A \cap B)$:

$$\begin{aligned}\#(A \cap B) &= \#\left\{\mu \in \mathcal{P}^{\ell}_{d}(\lambda) : \mu_\ell = 1,\, \mu_1 = \lambda_1\right\} \\ &= \#\left\{\mu \in \mathcal{P}^{\ell-1}(\lambda) : \mu_1 = \lambda_1\right\} \\ &= \binom{\lambda_k - 1 + \ell - 1 - k}{\lambda_k - 1}.\end{aligned}$$

Plugging all of these values into (4.2), we get

$$\begin{aligned}e_{\ell,r}(\lambda) &= \left(\sum_{s \geq r} e_{\ell-1,s}(\lambda)\right) + \binom{\lambda_k - 1 + \ell - k}{\lambda_k - 1} + e_{\ell-1,r-1}(\lambda) - \binom{\lambda_k - 2 + \ell - k}{\lambda_k - 1} \\ &= \left(\sum_{s \geq r-1} e_{\ell-1,s}(\lambda)\right) + \binom{\lambda_k - 2 + \ell - k}{\lambda_k - 2}.\end{aligned}$$

Notice that when $r = \lambda_k + \ell - k$, the formula yields

$$e_{\ell,\lambda_k+\ell-k}(\lambda) = e_{\ell,\lambda_k+\ell-k}(\lambda) + \binom{\lambda_k - 2 + \ell - k}{\lambda_k - 2},$$

which (4.1) satisfies. Also notice that when $r < \lambda_k + \ell - k$, we get

$$e_{\ell,r}(\lambda) = e_{\ell-1,r-1}(\lambda) + e_{\ell,r+1}(\lambda).$$

Finally, we need to keep in mind that all of these calculations have been with respect to a single initial partition, i.e. if $\tau_k(\lambda) = \lambda$. If $\tau_k(\lambda) \neq \lambda$, we simply get twice the number as when they are equal, so we will focus on the equal case.

We can now state the final recurrence relation and initial conditions:

$$e_{\ell,r}(\lambda) = 0 \text{ for } r > \lambda_k - \ell - k, \tag{4.3}$$

$$e_{\ell,\lambda_k-\ell-k}(\lambda) = \binom{\lambda_k - 1 + \ell - k}{\lambda_k - 1}, \tag{4.4}$$

$$e_{\ell,r}(\lambda) = e_{\ell-1,r-1}(\lambda) + e_{\ell,r+1}(\lambda) \text{ for } r < \lambda_k - \ell - k. \tag{4.5}$$

4.3 Solving the Recurrence

We now solve the recurrence relation from Section 4.2.

Proposition. For $\ell \geq k$ and $r \geq 0$,

$$e_{\ell,r}(\lambda) = \begin{cases} \binom{2\lambda_k - 1 - r + 2(\ell - k)}{2\lambda_k - 1 - r + \ell - k}, & r \geq \ell - k \\ \sum_{s\geq 0} (-1)^s \binom{r-s}{s} b_{\ell-k-s,2\lambda_k-1}, & r < \ell - k. \end{cases}$$

Proof. We need to show the solution satisfies (4.3), (4.4) and (4.5). (4.3) is an obvious consequence of Lemma 4.1. Since $\lambda_k + \ell - k > \ell - k$, by the formula we get

$$e_{\ell,\lambda_k+\ell-k}(\lambda) = \binom{2\lambda_k - 1 - (\lambda_k + \ell - k) + 2(\ell - k)}{2\lambda_k - 1 - (\lambda_k + \ell - k) + \ell - k} = \binom{\lambda_k - 1 + \ell - k}{\lambda_k - 1},$$

which agrees with (4.1).

Now we need to verify (4.5). First suppose $r < \ell - k - 1$. Then

$$\begin{aligned}
&e_{\ell-1,r-1}(\lambda) + e_{\ell,r+1}(\lambda) \\
=&\sum_{s\geq 0}(-1)^s \binom{(r-1)-s}{s} b_{(\ell-1)-k-s,2\lambda_k-1} + \sum_{s\geq 0}(-1)^s \binom{(r+1)-s}{s} b_{\ell-k-s,2\lambda_k-1} \\
=&\sum_{s\geq 1}(-1)^{s-1} \binom{r-s}{s-1} b_{\ell-k-s,2\lambda_k-1} + \sum_{s\geq 0}(-1)^s \binom{r+1-s}{s} b_{\ell-k-s,2\lambda_k-1} \\
=&b_{\ell-k,2\lambda_k-1} + \sum_{s\geq 1}(-1)^{s-1} b_{\ell-k-s,2\lambda_k-1} \left(\binom{r-s}{s-1} - \binom{r+1-s}{s} \right) \\
=&b_{\ell-k,2\lambda_k-1} + \sum_{s\geq 1}(-1)^{s-1} b_{\ell-k-s,2\lambda_k-1} \left(-\binom{r-s}{s} \right) \quad \text{(by (3.1))} \\
=&b_{\ell-k,2\lambda_k-1} + \sum_{s\geq 1}(-1)^s \binom{r-s}{s} b_{\ell-k-s,2\lambda_k-1} \\
=&\sum_{s\geq 0}(-1)^s \binom{r-s}{s} b_{\ell-k-s,2\lambda_k-1} \\
=&e_{\ell,r}(\lambda).
\end{aligned}$$

Next, suppose $r = \ell - k - 1$. Then

$$\begin{aligned}
&e_{\ell-1,\ell-k-2}(\lambda) + e_{\ell,\ell-k}(\lambda)\\
=&\sum_{s\geq 0}(-1)^s\binom{\ell-k-2-s}{s}b_{(\ell-1)-k-s,2\lambda_k-1} + \binom{2\lambda_k-1-(\ell-k)+2(\ell-k)}{2\lambda_k-1-(\ell-k)+\ell-k}\\
=&\sum_{s\geq 0}(-1)^s\binom{\ell-k-s-2}{s}b_{\ell-k-s-1,2\lambda_k-1} + \binom{2\lambda_k-1+\ell-k}{2\lambda_k-1}\\
=&\sum_{s\geq 0}(-1)^s\binom{\ell-k-s-2}{s}b_{\ell-k-s-1,2\lambda_k-1} + \sum_{s\geq 0}(-1)^s\binom{\ell-k-s}{s}b_{\ell-k-s,2\lambda_k-1}\\
&\text{(by (3.2))}\\
=&e_{\ell,\ell-k-1}(\lambda),
\end{aligned}$$

by the same algebraic manipulations as in the last case. Finally, suppose $r \geq \ell - k$. Then

$$\begin{aligned}
&e_{\ell-1,r-1}(\lambda) + e_{\ell,r+1}(\lambda)\\
=&\binom{2\lambda_k-1-(r-1)+2((\ell-1)-k)}{2\lambda_k-1-(r-1)+(\ell-1)-k} + \binom{2\lambda_k-1-(r+1)+2(\ell-k)}{2\lambda_k-1-(r+1)+\ell-k}\\
=&\binom{2\lambda_k-2-r+2(\ell-k)}{2\lambda_k-1-r+\ell-k} + \binom{2\lambda_k-2-r+2(\ell-k)}{2\lambda_k-2-r+\ell-k}\\
=&\binom{2\lambda_k-1-r+2(\ell-k)}{2\lambda_k-1-r+\ell-k} \text{ (by (3.1))}\\
=&e_{\ell,r}(\lambda).
\end{aligned}$$

□

We can now compute $\#\mathcal{P}^\ell(\lambda)$:

$$\#\mathcal{P}^\ell(\lambda) = \#\mathcal{P}^\ell(\lambda,1) = \sum_{s\geq 0}(-1)^s\binom{1-s}{s}b_{\ell-k-s,2\lambda_k-1} = b_{\ell-k,2\lambda_k-1}.$$

Notice that when $\lambda_k = 1$, we get

$$\#\mathcal{P}^\ell(\lambda) = b_{\ell-k,1} = c_{\ell-k+1},$$

which agrees with Theorem 3.4(i).

Conclusions

In Part I, we developed the theory necessary to enumerate k-nilpotent ideals of a parabolic subalgebra of a simple Lie algebra $\mathfrak{g}$ using the partial order on the set of roots. We then generalized this approach to the case when $\mathfrak{g}$ has a $\mathbb{Z}_2$-grading $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, and enumerated the $\mathfrak{b}_0$-stable abelian subalgebras of $\mathfrak{g}_1$. Both of these approaches provide more information about the structure of the enumerated objects than the previous methods of proof. The examples computed in both scenarios illustrate how this algebraic method makes describing the structure of the k-nilpotent ideals and $\mathfrak{b}_0$-stable abelian subalgebras very straightforward.

In Part II, we described an algorithm for generating partitions from a so-called square-bounded partition such that the number of partitions generated at a given iteration is a Catalan number. This process is unique from other methods of generating objects of quantity equal to a Catalan number because the map τ_ℓ is an involution which occurs naturally in our case, but is not reflected in any other known collection of objects in a natural way. We then modified the algorithm and the map τ_ℓ so we could start with any partition, and found that the number of partitions generated at any given iteration was a ballot number, a generalization of the Catalan numbers.

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