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ABSTRACT

The simplest parafermi model is shown to be equivalent to a theory with two types of ordinary fermions that are dynamically indistinguishable. This model exhibits cluster properties as well as the usual analyticity and crossing properties, and hence parastatistics cannot be ruled out on any of these grounds. For a complete discussion of cluster properties it is necessary to establish what quantities are observable in systems containing identical particles that are not necessarily fermions or bosons. It is argued that particle permutations are observables and that it is consistent to assume that these are the only observables that depend on the ordering of variables. This theory of observables covers simultaneously both the first- and second-quantized treatments, which are equivalent only for fermions and bosons.

I. INTRODUCTION

A fundamental unresolved question is whether all particles of nature are necessarily either fermions or bosons. Theoretical investigations of other possibilities have followed one of two approaches. The first (1,2), formulated within the framework of first-quantized quantum theory, imposes the condition that particles of the same type be indistinguishable by requiring that all observables Q satisfy $PQP^{-1} = Q$, where P is any permutation on the order of the variables referring to the given type of particle. The second approach (3,4), formulated within the second-quantized framework, considers commutation relations more general than those leading to Fermi or Bose statistics. These two approaches are equivalent in the cases of Fermi and Bose statistics, but in general they are nonequivalent. Indeed, the basic equation $PQP^{-1} = Q$ of the first-quantized approach is generally ill-defined in the second-quantized framework. This is because the generalized commutation relations equate states with variables in different orders, and the results of the action of P upon equated states are not identical. The origin of the difficulty is that the generalized (parastatistics) commutation relations do not commute with the P's.

One aim of the present work is to provide a framework for the discussion of identical particles that encompasses both approaches. An important point, here, is a distinction between the above-mentioned place permutations P and particle permutations $\bar{P}$. We shall argue that particle permutations are observables. Indeed, the symmetry requirement $PQP^{-1} = Q$ of the first quantized approach is equivalent to the requirement that all observables that distinguish among states that differ only in order of variables are functions of the particle-permutations $\bar{P}$. The latter form of the indistinguishability requirement is equally applicable to the second-quantized approach.

We begin, in Section 2, by reviewing a particular case of modified commutation relations, namely that of parafermi statistics of order two. This case has been studied in perturbation theory by Volkov (4), using a particular Hamiltonian. We show that the Volkov model is in a certain sense trivial in that it is physically equivalent to a theory in which there are two different types of ordinary fermions that are statistically distinct but physically indistinguishable. An example of such particles would be the neutron and the proton in a model with an isospin-independent interaction. As this possibility does not violate any general requirements, such as cluster, crossing, or analyticity properties, it is evidently impossible to use such general principles to rule out the possibility that there exist particles obeying parastatistics. The example of the Volkov model suggests that all theories involving particles obeying parastatistics are reducible to theories involving only fermions and bosons, but we have not attempted to prove this.[3]

In Section 3 the question of what quantities are observable is considered and it is argued that particle permutations are observables. In Section 4 we derive the restrictions on the elastic

S-matrix arising from the requirement that all quantities measurable by means of elastic-scattering experiments be functions of place permutations. This provides a derivation, from general requirements, of the particular form of S that arose in the Volkov model from special field-theoretic properties.

In Section 5 we examine whether the possibility of inelastic interactions enables the class of observables to be extended beyond the set of particle permutations $\bar{\bar{P}}$. We show that the answer is no, first in a particular example of an inelastic interaction and then generally within the framework of the Volkov model. Cluster properties play an important role in this analysis.

II. THE VOLKOV MODEL

Parafermi statistics of order two corresponds to the following trilinear relations (1) among the particle creation and annihilation operators $a^\dagger$ and a:

$$\begin{aligned} a_k a_n^\dagger a_m + a_m a_n^\dagger a_k &= \delta_{kn} a_m + \delta_{mn} a_k \\ a_k a_n^\dagger a_m^\dagger + a_m^\dagger a_n^\dagger a_k &= \delta_{kn} a_m^\dagger \\ a_k^\dagger a_n^\dagger a_m^\dagger + a_m^\dagger a_n^\dagger a_k^\dagger &= 0 . \end{aligned} \quad (2.1)$$

There are also similar relations for the corresponding antiparticle operators $\bar{a}^\dagger$ and $\bar{a}$, as well as for mixtures of the particle and antiparticle operators. In the latter case a given Kronecker δ on the right-hand side occurs only when its two indices both refer either to particles or to antiparticles.

Volkov has shown (2) that if the Fermi fields of the ordinary electromagnetic interaction

$$H_{INT} = [\,\bar{\psi},\, \gamma_\mu \psi\,]A^\mu \tag{2.2}$$

are replaced by parafermi fields of order two, then the S-matrix elements can be expressed as a sum of functions corresponding to different Feynman graphs. The rules for computing the function corresponding to a given graph are identical to the usual rules, except for an extra factor of two for each ψ-particle closed loop.

We represent by $\widetilde{S}(q_1 q_2 \cdots q_N \mid p_1 p_2 \cdots p_N)$ the function obtained by summing the functions corresponding to all Feynman graphs in which the parafermion line that begins at p_i ends at q_i. The photon variables are unimportant in this discussion and are suppressed. By virtue of the symmetry of the Feynman rules with respect to the identification of the N parafermion lines, the function $\widetilde{S}$ has the symmetry property

$$\widetilde{S}(q_{P1}\, q_{P2} \cdots q_{PN} \mid p_{P1}, p_{P2}, \cdots p_{PN}) = \widetilde{S}(q_1 q_2 \cdots q_N \mid p_1 p_2 \cdots p_N)\ , \tag{2.3}$$

where P1, P2, ⋯ PN is any permutation of 1, 2, ⋯ N. The general significance of such a symmetry property in any theory of identical particles is discussed in Section 4.

The S-matrix elements themselves do not display the symmetry property (2.3). If we define

$$|p_1\, p_2 \cdots p_N\rangle = a^\dagger(p_1)\, a^\dagger(p_2) \cdots a^\dagger(p_N)\, |\, 0\, \rangle , \qquad (2.4)$$

then it follows from (2.1) that

$$|p_1\, p_2 \cdots p_{r-1}\, p_r\, p_{r+1} \cdots p_N\rangle = -|p_1\, p_2 \cdots p_{r+1}\, p_r\, p_{r-1} \cdots p_N\rangle , \qquad (2.5)$$

by virtue of which the three states

$$|p_1\, p_2\, p_3\rangle = -|p_3\, p_2\, p_1\rangle ,$$

$$|p_2\, p_3\, p_1\rangle = -|p_1\, p_3\, p_2\rangle ,$$

and

$$|p_3\, p_1\, p_2\rangle = -|p_2\, p_1\, p_3\rangle \qquad (2.6)$$

span the subspace of states corresponding to the momenta $p_1 p_2 p_3$. The S-matrix elements are given, according to Volkov's rules, as

$$\begin{aligned} \langle q_1 q_2 q_3\, |S|\, p_1 p_2 p_3\rangle &= \tilde{S}(q_1 q_2 q_3 \mid p_1 p_2 p_3) - \tilde{S}(q_3 q_2 q_1 \mid p_1 p_2 p_3) \\ &= \tilde{S}(q_1 q_2 q_3 \mid p_1 p_2 p_3) - \tilde{S}(q_1 q_2 q_3 \mid p_3 p_2 p_1) , \end{aligned} \qquad (2.7)$$

which do not satisfy (2.3).

The Volkov model is equivalent to a special case of a theory involving two distinct but dynamically indistinguishable fermions.[3] Let $\{\alpha_i^\dagger\}$ and $\{\beta_i^\dagger\}$ be two sets of ordinary fermion creation operators, such that each $\alpha^\dagger$ anticommutes with each $\beta^\dagger$, etc. Writing

$$\alpha_i = A_i \qquad\qquad \beta_i^\dagger = A_i^\dagger$$

$$\beta_i = B_i \qquad\qquad \alpha_i^\dagger = B_i^\dagger \qquad (2.8)$$

one obtains from the usual fermion anticommutation relations the equations

$$A_k B_n^\dagger A_m + A_m B_n^\dagger A_k = \delta_{kn} A_m + \delta_{mn} A_k ,$$

$$A_k B_n^\dagger A_m^\dagger + A_m^\dagger B_n^\dagger A_k = \delta_{kn} A_m^\dagger ,$$

and

$$A_k^\dagger B_n^\dagger A_m^\dagger + A_m^\dagger B_n^\dagger A_k^\dagger = 0 , \qquad (2.9)$$

and also a similar set of equations in which the A's and B's are interchanged. The similarity of (2.9) and (2.1) is apparent. The relations similar to (2.9) involving antiparticle operators are obtained by making the further identifications

$$\bar{\alpha}_i = \bar{B}_i \qquad\qquad \bar{\beta}_i^\dagger = B_i^\dagger$$

$$\bar{\beta}_i = \bar{A}_i \qquad\qquad \bar{\alpha}_i^\dagger = \bar{A}_i^\dagger . \qquad (2.10)$$

That is, if any product of parafermi operators is transcribed into an alternating sequence of A- and B-type operators, then the original parafermi commutation relations are still maintained. This ensures that any parafermi model of order two in which all observables are of even degree in the parafermi fields can be reduced to an equivalent model involving only ordinary Fermions. One transcribes the operators $a_i^\dagger$, a_i, $\bar{a}_i^\dagger$ and $\bar{a}_i$ occurring in the observables into corresponding A- or B-type operators, respectively, according to whether they occupy even or odd positions, counting from the left. The same rule is applied to the products of creation operators that create the basis states, so that (2.4), for example, becomes

$$|p_1 p_2 p_3 p_4 \cdots \rangle = \alpha^\dagger(p_1)\beta^\dagger(p_2)\ \alpha^\dagger(p_3)\beta^\dagger(p_4) \cdots |\ 0\rangle . \qquad (2.11)$$

The Volkov interaction Hamiltonian involves the parafields in the form of the commutator $[\bar{\psi}, \psi]$, which contains the terms

$$a_m^\dagger a_n + a_m^\dagger \bar{a}_n^\dagger + \bar{a}_m a_n + \bar{a}_m \bar{a}_n^\dagger$$
$$- a_n a_m^\dagger - \bar{a}_n^\dagger a_m^\dagger - a_n \bar{a}_m - \bar{a}_n^\dagger \bar{a}_m . \qquad (2.12)$$

According to the rules this sum is transcribed into

$$\alpha_m^\dagger \alpha_n + \alpha_m^\dagger \bar{\alpha}_n^\dagger + \bar{\alpha}_m \alpha_n + \bar{\alpha}_m \bar{\alpha}_n^\dagger$$
$$- \beta_n \beta_m^\dagger - \bar{\beta}_n^\dagger \beta_m^\dagger - \beta_n \bar{\beta}_m - \bar{\beta}_n^\dagger \bar{\beta}_m . \qquad (2.13)$$

Because the interaction Hamiltonian involves an even number of operators ψ, one can make the transcription (2.12) to (2.13) of H_{INT} before expanding the exponential in

$$S = T\{\exp[-i \int H_{INT}(x)\ d^4x]\ \} ,$$

and the required alternation of A- and B-type operators will persist after the expansion. After the transcription one may use the usual fermion anticommutation relations to reorder the α and β operators, thereby obtaining via Wick's theorem the usual Feynman rules. The factor of two associated with an internal closed loop evidently arises because the loop can represent either an α-particle or a β-particle, and the two possibilities contribute equally because the interaction is symmetrical in the operators α and β. (This can be seen from (2.13) by using the anticommutation relations for the β to reverse the order of the factors in each of the last four terms.) This symmetry of H_{INT} under interchange of α and β means that these two types of particle, though statistically distinct, are dynamically indistinguishable. This is necessary for the validity of the symmetry property (2.3), since a permutation of adjacent particles in a state interchanges α- and β-type particles.

The ordinary fermion reactions that arise from the parafermion reactions by the transcription just described appear to be special in that the numbers of the two kinds of fermions can differ at most by one. Though this restriction might apparently differentiate a world having two distinct indistinguishable types of fermions from a world involving parafermions, the cluster properties effectively nullify this distinction: the particles in any localized region could be preponderantly of one type or the other.

The fact that the two types of fermions are dynamically indistinguishable imposes limitations on what is observable in certain types of experiments. For example, elastic-scattering experiments cannot distinguish between the two-particle states $|p_1p_2\rangle$ and $|p_2p_1\rangle$, even though these states are independent. This is because (2.3) implies that

$$\langle q_1\, q_2\, |S|\, p_1\, p_2\rangle = \langle q_2\, q_1\, |S|\, p_2\, p_1\rangle , \tag{2.14}$$

so that an elastic-scattering experiment would enable one to distinguish between $|p_1\, p_2\rangle$ and $|p_2\, p_1\rangle$ only if one had some way of distinguishing between $|q_1\, q_2\rangle$ and $|q_2\, q_1\rangle$. On the other hand, the two states

$$|p_1\, p_2\rangle_\pm = \frac{1}{\sqrt{2}} \left(|p_1\, p_2\rangle \pm |p_2\, p_1\rangle\right)$$

are distinguishable, since the two matrix elements

$$_\pm\langle q_1\, q_2\, |S|\, p_1\, p_2\rangle_\pm \equiv \langle q_1\, q_2\, |S|\, p_1\, p_2\rangle \pm \langle q_2\, q_1\, |S|\, p_1\, p_2\rangle \tag{2.15}$$

are different, while the matrix-elements $_\mp\langle q_1\, q_2\, |S|\, p_1\, p_2\rangle_\pm$ vanish.

The general question of what quantities are observable in systems of identical particles is important both in its own right and for a discussion of cluster properties.

III. PERMUTATIONS AS OBSERVABLES

Single-particle experiments are experiments that establish only that certain free-particle wave functions are occupied whereas certain others are not. Each free-particle wave function corresponds to a certain type of particle, and an occupied wave function is regarded as a particle of the corresponding type.

Let $\phi_1 , \phi_2 , \cdots \phi_N$ be a set of N orthogonal

free-particle wave functions, all corresponding to a single type of particle, and let $|\phi_1, \phi_2, \cdots \phi_N\rangle$ be a pure state in which these wave functions are all occupied. Other wave functions may also be occupied, but these are irrelevant to our considerations. The wave functions ϕ_i, $i = 1, \cdots N$, can be gradually changed in such a way that $(\phi_1, \phi_2, \cdots \phi_N)$ becomes $(\phi_{\bar{P}1}, \phi_{\bar{P}2}, \cdots \phi_{\bar{P}N})$, where $(\bar{P}1, \bar{P}2, \cdots \bar{P}N)$ is a permutation of $(1, 2, \cdots N)$. The state thus obtained is $|\phi_{\bar{P}1}, \phi_{\bar{P}2}, \cdots \phi_{\bar{P}N}\rangle$, which will be abbreviated as $|\bar{P}\rangle$. The various states $|\bar{P}\rangle$ are indistinguishable by single-particle experiments, since these establish only that certain wave functions are occupied; they do not determine the position of the ϕ_i within the state vector.

The subspace spanned by the N! vectors $|\bar{P}\rangle$ will be denoted by $\mathcal{S}(\phi_1, \phi_2, \cdots \phi_N)$. If the N particles are identical fermions or bosons then the various $|\bar{P}\rangle$ are all multiples of any single one of them, and hence $\mathcal{S}$ is one-dimensional. To encompass more general cases we allow the $|\bar{P}\rangle$ to have relations of the form

$$P\,|\bar{P}\rangle = \sigma_P|\bar{P}\rangle \qquad\qquad \sigma_P = \pm 1 \tag{3.1}$$

valid for all $\bar{P}$, where P is some place permutation. That is,

$$\begin{aligned} P\,|\bar{P}\rangle &\equiv P\,|\phi_{\bar{P}1}, \phi_{\bar{P}2}, \cdots \phi_{\bar{P}N}\rangle \\ &= |\phi_{\bar{P}P1}, \phi_{\bar{P}P2}, \cdots \phi_{\bar{P}PN}\rangle, \end{aligned} \tag{3.2}$$

where $(P1, P2, \cdots PN)$ is a permutation of $(1, 2, \cdots N)$. The Fermi and Bose statistics are included as special cases, as are parastatistics [see, for example, (2.5)] and Boltzmann statistics (which is the case in which all the $|\bar{P}\rangle$ are linearly independent).

The various states of $\mathcal{S}$ are indistinguishable by single-particle experiments, but the question arises whether any operators in the space $\mathcal{S}$ are observable by means of other types of experiments.

In order to obtain some distinction among the states of $\mathcal{S}$, let the particles i and j corresponding to the wave functions ϕ_i and ϕ_j be caused to scatter on each other. For spinless particles the scattered wave at center-of-mass angle $\pi/2$ contains only the symmetric part of the wave function with respect to the operator $\bar{P}_{ij}$ that interchanges particles i and j. Alternatively, the energies can be adjusted so that some single partial wave dominates the scattering. This allows the system to be projected on either eigenstate of the particle-exchange operator $\bar{P}_{ij}$.

The operator $\bar{P}_{ij}$ exchanges the particles i and j. This means that wave functions ϕ_i and ϕ_j representing these particles are interchanged. If $\bar{P}_{ij}$ acts on a superposition of states $|\bar{P}\rangle$, then the positions of the interchanged indices i and j will be different in different terms. The particle-exchange operator $\bar{P}_{ij}$ is thus to be distinguished from a place-permutation operator P_{ab} that interchanges the occupants of positions a and b, regardless of their identity. The fact that the observable is a particle-permutation $\bar{P}$, and not a place-permutation P, is a consequence of the fact that particles are experimentally identified by their wave functions, not by the positions of indices. This distinction is crucial to our discussion. The relation between particle-permutations $\bar{P}$ and place-permutations P is examined in an appendix, where it is shown how the $\bar{P}$ and P play the reciprocal roles of observables and symmetry operators, respectively.

We now examine the possibility that all the observables in $\mathcal{S}$ are functions of the particle-permutations $\bar{P}$. Let the subspace $\mathcal{S}$ be decomposed into the smallest subspaces such that every $\bar{P}$ maps each subspace into itself. In other words, $\mathcal{S}$ is decomposed so that the $\bar{P}$'s act irreducibly within these various subspaces. The assumption that all observables are functions of the $\bar{P}$'s means that the observables have no matrix elements connecting different subspaces. Thus the relative phases of the components of the wave function lying in different subspaces cannot be measured.

Let the set of vectors

$$|\ell_\mu{}^r(\phi_1, \phi_2, \cdots \phi_N)\rangle = \sum_\alpha |\bar{P}_\alpha\rangle\, c(\alpha;r,\mu) \qquad (3.3)$$

be an orthonormal basis of the <u>r</u>th subspace, where $\mu = 1, 2, \cdots n_r$. The sum over α is such that the vectors $|\bar{P}_\alpha\rangle$ are linearly independent and span the subspace $\mathcal{S}$. We assume that they are also orthonormal. The application of the particle permutation $\bar{P}$ that replaces the set of particles $(1, 2, \cdots N)$ by the set

$(\bar{P}1, \bar{P}2, \cdots \bar{P}N)$ gives

$$\bar{P}|\ell_\mu{}^r(\phi_1, \phi_2, \cdots \phi_N)\rangle \equiv |\ell_\mu{}^r(\phi_{\bar{P}1}, \phi_{\bar{P}2}, \cdots \phi_{\bar{P}N})\rangle,$$

$$= \sum_\alpha |\bar{P}\bar{P}_\alpha\rangle\, c(\alpha;r,\mu),$$

$$= \sum_{\nu=1}^{n_r} |\ell_\nu{}^r(\phi_1, \phi_2, \cdots \phi_N)\rangle\, U_{\nu\mu}{}^r(\bar{P}). \tag{3.4}$$

The matrices $\underset{\sim}{U}^r(\bar{P})$ are unitary for each r and $\bar{P}$. This follows from the fact that for each fixed $\bar{P}$ the set of vectors $|\bar{P}\bar{P}_\alpha\rangle$ is orthonormal whenever the set of $|\bar{P}_\alpha\rangle$ is. This is true because each $|\bar{P}\bar{P}_\alpha\rangle$ is plus or minus one of the $|\bar{P}_\alpha\rangle$, and no two of the $|\bar{P}\bar{P}_\alpha\rangle$ can be equal to within a sign to a single one of the $|\bar{P}_\alpha\rangle$. This latter fact is a consequence of the invariance of relationships of the form (3.1) under the particle permutation $\bar{P}$.

The set of matrices $\underset{\sim}{U}^r(\bar{P})$ for each value of r is an irreducible representation of the group of permutations $\bar{P}$. When there are no linear relationships (3.1) among the various states $|\bar{P}\rangle$, these states can be used as basic vectors of the regular representation of the group of permutations $\bar{P}$. It follows from a fundamental theorem of group theory (5) that the irreducible representations fall into classes such that the representations within a class are equivalent and the number of representations in any class is equal to the dimension n_r of any one of the equivalent irreducible representations of that class. The basic vectors $|\ell_\mu{}^r\rangle$ can be chosen so that the various equivalent representations $U_{\nu\mu}{}^r(\bar{P})$ are identical. Then the general function of the $\bar{P}$ in $\mathcal{S}(\phi_1, \phi_2, \cdots \phi_N)$ takes the form

$$Q = \sum_r \sum_{\mu,\nu=1}^{n_r} |\ell_\mu{}^r\rangle\, Q_{\mu\nu}^r \,\langle \ell_\nu{}^r|, \tag{3.5a}$$

where the various $\underset{\sim}{Q}^r$ corresponding to equivalent irreducible representations are identical:

$$\underset{\sim}{Q}^r = \underset{\sim}{Q}^s \quad \text{for} \quad C_r = C_s. \tag{3.5b}$$

Here C_r is the class of equivalent irreducible representations containing the one specified by r.

When there is more than one equivalent irreducible representation within a class, the division into subspaces of the part of $\mathcal{S}$ corresponding to this class is nonunique; a new division can be defined by

$$|\ell_\mu{}^r\rangle' = \sum_s |\ell_\mu{}^s\rangle\, V^{sr}, \tag{3.6}$$

where the summation is over the indices s within the class, and $\underset{\sim}{V}$ is a unitary matrix. The form (3.5) is invariant under a transformation (3.6), and the state $|\ell_\mu{}^r\rangle'$ is thus experimentally indistinguishable from the state $|\ell_\mu{}^r\rangle$.

One way of inducing a transformation (3.6) is to apply a place permuation P to the vectors $|\bar{P}_\alpha\rangle$ of (3.3). This is discussed in the appendix.

We illustrate the above discussion by considering three-particle states. Suppose first that the order of the parastatistics is at least three, so that there are no linear relations of the type (2.6) among the basic state vectors $\bar{P}|\phi_1\ \phi_2\ \phi_3\rangle$. The orthonormal basic vectors $|\ell_\mu^{\ r}\rangle$ of $\mathcal{S}\ (\phi_1\ \phi_2\ \phi_3)$ may then be taken as:

$$|\ell_1^1\rangle = \tfrac{1}{2}\Big[|\phi_2\ \phi_3\ \phi_1\rangle - |\phi_3\ \phi_1\ \phi_2\rangle + |\phi_1\ \phi_3\ \phi_2\rangle - |\phi_2\ \phi_1\ \phi_3\rangle\Big]$$

$$|\ell_2^1\rangle = \frac{1}{2\sqrt{3}}\Big[2|\phi_1\ \phi_2\ \phi_3\rangle - |\phi_2\ \phi_3\ \phi_1\rangle - |\phi_3\ \phi_1\ \phi_2\rangle + 2|\phi_3\ \phi_2\ \phi_1\rangle - |\phi_1\ \phi_3\ \phi_2\rangle - |\phi_2\ \phi_1\ \phi_3\rangle\Big]$$

$$|\ell_1^2\rangle = \frac{-1}{2\sqrt{3}}\Big[2|\phi_1\ \phi_2\ \phi_3\rangle - |\phi_2\ \phi_3\ \phi_1\rangle - |\phi_3\ \phi_1\ \phi_2\rangle - 2|\phi_3\ \phi_2\ \phi_1\rangle + |\phi_1\ \phi_3\ \phi_2\rangle + |\phi_2\ \phi_1\ \phi_3\rangle\Big]$$

$$|\ell_2^2\rangle = \tfrac{1}{2}\Big[|\phi_2\ \phi_3\ \phi_1\rangle - |\phi_3\ \phi_1\ \phi_2\rangle - |\phi_1\ \phi_3\ \phi_2\rangle + |\phi_2\ \phi_1\ \phi_3\rangle\Big]$$

$$|\ell^3\rangle = \frac{1}{\sqrt{6}}\Big[|\phi_1\ \phi_2\ \phi_3\rangle + |\phi_2\ \phi_3\ \phi_1\rangle + |\phi_3\ \phi_1\ \phi_2\rangle + |\phi_3\ \phi_2\ \phi_1\rangle + |\phi_1\ \phi_3\ \phi_2\rangle + |\phi_2\ \phi_1\ \phi_3\rangle\Big]$$

$$|\ell^4\rangle = \frac{1}{\sqrt{6}}\Big[|\phi_1\ \phi_2\ \phi_3\rangle + |\phi_2\ \phi_3\ \phi_1\rangle + |\phi_3\ \phi_1\ \phi_2\rangle - |\phi_3\ \phi_2\ \phi_1\rangle - |\phi_1\ \phi_3\ \phi_2\rangle - |\phi_2\ \phi_1\ \phi_3\rangle\Big]. \tag{3.7}$$

The first and second subspaces correspond to the same class of equivalent irreducible representations, and hence our division into these subspaces is not unique. For example, if we applied a place permutation P to every state on the right-hand side of (3.7), we would have a new division of $\mathcal{A}$ into subspaces, in which $|\ell_1^1\rangle$ is generally mixed with $|\ell_1^2\rangle$, and $|\ell_2^1\rangle$ with $|\ell_2^2\rangle$, as in (3.6).

The states of a given subspace are mapped into one another by all permutations $\bar{P}$, and they all correspond to a single Young tableau. Since the first two subspaces in (3.7) correspond to equivalent irreducible representations, they also have the same Young tableau, namely the triangular tableau. It can be seen that the states $|\ell_1^1\rangle$ and $|\ell_1^2\rangle$ are eigenstates of $\bar{P}_{13}$ with eigenvalue -1, while the states $|\ell_2^1\rangle$ and $|\ell_2^2\rangle$ are eigenvalues of $\bar{P}_{13}$ with eigenvalue +1. Thus the three-particle system can be in a state corresponding to a triangular Young tableau and still have a definite symmetry with respect to any two of its particles. This result is to be contrasted with the remarks of Steinmann.(4)

When the order of the parastatistics is less than three (which includes the case of ordinary statistics), the number of basic subspaces in (3.7) is reduced. In the example of order-two parafermi statistics the relations like (2.6) imply that the first and third subspaces vanish.

To conclude this section we show that any Q of the form (3.5) can be expressed as a linear combination of permutation operators $\bar{P}$. This follows from the well-known relation (6)

$$\frac{1}{N!} \sum_{\alpha} \langle \ell_\mu{}^r|\bar{P}_\alpha|\ell_\nu{}^r\rangle \; \langle \ell_\rho{}^s|\bar{P}_\alpha{}^{-1}|\ell_\sigma{}^s\rangle = \delta_{\mu\sigma}\,\delta_{\nu\rho}\,\delta^{C_r C_s}\,(n_r)^{-1},$$

where C_r, C_s are the classes to which the rth and sth subspaces belong, and n_r is the dimension of the rth subspace. Hence, one can write

$$Q = \frac{1}{N!} \sum_{\alpha} \sum_{r} \sum_{\rho,\sigma} \bar{P}_\alpha \langle \ell_\rho{}^r|\bar{P}_\alpha{}^{-1}|\ell_\sigma{}^r\rangle \; \langle \ell_\sigma{}^r|\; Q \;|\ell_\rho{}^r\rangle. \tag{3.8}$$

A consequence of this is that if all (Hermitian) functions of the $\bar{P}$ are observable, then any (Hermitian) operator Q of the form (3.5) is observable.

IV. ELASTIC INTERACTIONS

To discuss elastic interactions, we must allow the possibility that the single-particle wave functions ϕ_i are changed. Although not all superpositions of the N-particle states are observable, we suppose that single-particle wave functions can be superposed, with a measurable relative phase. Then the form (3.5) implies that

$$\langle \ell_\mu{}^r(\phi_1+\lambda_1\psi_1,\ \phi_2+\lambda_2\psi_2,\cdots\phi_N+\lambda_N\psi_N)\,|Q|\,\ell_\nu{}^s(\phi_1+\lambda_1\psi_1,\ \phi_2+\lambda_2\psi_2,\cdots\phi_N+\lambda_N\psi_N)\rangle = 0 \quad (r \neq s)$$

for all wave functions ϕ_i, ψ_i and numbers λ_i. Consequently we have

$$\langle \ell_\mu{}^r(\phi_1,\ \phi_2,\cdots\phi_N)\ |Q|\ \ell_\nu{}^s(\psi_1,\ \psi_2,\cdots\psi_N)\rangle = 0, \quad (r \neq s)\ . \tag{4.1}$$

If before a scattering an observable is represented by the operator Q, then after the scattering it is represented by $Q' = S\,Q\,S^{\dagger}$. Alternatively, if after a scattering an observable is represented by Q, then before the scattering it will have been represented by $Q'' = S^{\dagger}QS$. The condition that both Q' and Q'' be observables for any observable Q evidently imposes certain constraints on the operator S. We now show that if all Hermitian functions of the $\bar{P}$ are observable, then these constraints can be expressed in the form

$$\langle \ell_{\mu}{}^{r}(\phi_1, \phi_2, \cdots \phi_N)\ |S|\ \ell_{\nu}{}^{s}(\psi_1, \psi_2, \cdots \psi_N)\rangle = 0 \qquad (r \neq s) \tag{4.2a}$$

and

$$\langle \ell_{\mu}{}^{r}(\phi_1, \phi_2, \cdots \phi_N)\ |S|\ \ell_{\nu}{}^{r}(\psi_1, \psi_2, \cdots \psi_N)\rangle$$

$$= \langle \ell_{\mu}{}^{s}(\phi_1, \phi_2, \cdots \phi_N)\ |S|\ \ell_{\nu}{}^{s}(\psi_1, \psi_2, \cdots \psi_N)\rangle\ , \quad (C_r = C_s) \tag{4.2b}$$

where C_r denotes the class of subspaces corresponding to the irreducible representations of the $\bar{P}$ equivalent to $\underset{\sim}{U}^{r}(\bar{P})$. According to the result at the end of the last section, Q can be any Hermitian operator satisfying (3.5). We take Q to be

$$Q = \sum_{s \in C} |\ell_{\nu}{}^{s}(\psi_1, \psi_2, \cdots \psi_N)\rangle\ \langle \ell_{\nu}{}^{s}(\psi_1, \psi_2, \cdots \psi_N)|\ . \tag{4.3}$$

If we could take a single term instead of the sum in (4.3), the proof of (4.2) would be simple. However, (3.5b) requires a summation over all subspaces s that belong to the same class C of equivalent irreducible representations of the $\bar{P}$. Let C' be another such class such that the number of subspaces in C' is at least equal to the number of subspaces in C. Then the requirement that $SQS^\dagger$ be an observable yields, in particular,

$$\sum_{s\epsilon C} \langle \ell_\mu{}^r(\phi_1,\phi_2,\cdots\phi_N)\ |S|\ \ell_\nu{}^s(\psi_1,\psi_2,\cdots\psi_N)\rangle$$

$$\times \langle \ell_\nu{}^s(\psi_1,\psi_2,\cdots\psi_N)\ |S^\dagger|\ \ell_\nu{}^{s_1}(\psi_1,\psi_2,\cdots\psi_N)\rangle = 0\ ,$$

$$(r \in C',\ s_1 \in C) \qquad (4.4a)$$

and

$$\sum_{s\epsilon C} \langle \ell_\mu{}^{r_1}(\phi_1,\phi_2,\cdots\phi_N)\ |S|\ \ell_\nu{}^s(\psi_1,\psi_2,\cdots\psi_N)\rangle$$

$$\times \langle \ell_\nu{}^s(\psi_1,\psi_2,\cdots\psi_N)\ |S^\dagger|\ \ell_\mu{}^{r_2}(\phi_1,\phi_2,\cdots\phi_N)\rangle = 0$$

$$(r_1,r_2 \in C',\ r_1 \neq r_2)\ . \qquad (4.4b)$$

If we write down (4.4a) for as many values of r as there are subspaces in the class C, and assume that because S contains a no-scattering part not all the matrix elements $\langle \ell_\nu{}^s(\psi_1,\psi_2,\cdots\psi_N\ |S^\dagger|\ \ell_\nu{}^{s_1}(\psi_1,\psi_2,\cdots\psi_N)\rangle$ can vanish, we obtain

$$\det\langle \ell_\mu{}^r(\phi_1,\phi_2,\cdots\phi_N)\ |S|\ \ell_\nu{}^s(\psi_1,\psi_2,\cdots\psi_N)\rangle = 0$$

$$(r \in C',\ s \in C). \qquad (4.5)$$

Here we are taking the determinant of a square matrix whose elements are labeled by r and s. Now the sum in (4.4b) can be thought of as a product of two such matrices, and (4.4b) says that all the offdiagonal elements of the product matrix are zero. Therefore its determinant is equal to the product of its diagonal elements. But this determinant vanishes because one of the matrices that form the product is the matrix in (4.5). Hence at least one of the diagonal elements of the product matrix vanishes. But (3.5b) requires that all these diagonal elements be equal. Hence we have proved (4.4b) also for $r_1 = r_2$, which implies (4.2a), for this case that r and s label subspaces belonging to different classes such that the number of subspaces in the first class is not less than the number in the second. To prove (4.2a) when the number in the first class is less than the number in the second, we argue similarly starting from the requirement that $S^\dagger QS$ be an observable.

Finally, we consider the case in which r and s in (4.2a) belong to the same class. When r_1 and r_2 are in the same class, the states $|\ell_\mu^{\,r_1}(\phi_1,\phi_2,\cdots\phi_N)\rangle$ and $|\ell_\mu^{\,r_2}(\phi_1,\phi_2,\cdots\phi_N)\rangle$ contribute equally to all observables by virtue of (3.5b). Hence a transition to one cannot be distinguished from a transition to the other. This means that we are free to adopt the convention that transitions between states of a given class do not mix different subspaces. That is, for r and s in the same class we can take (4.2a) to be true by definition. Then (3.5b) ensures (4.2b). (Of course in a theory where S is calculated by some specific rule, such as from a Lagrangian, this special form may not come out automatically, though it probably would in any natural theory.) Notice that once this definition is made, the previous proof that no transitions take place from one class of subspaces to another can be greatly simplified.

The result (4.2), together with (3.8), allows one to express the elastic part of S as a linear combination of the $\bar{P}$. To do this we introduce operators $\Delta(\psi_i \rightarrow \phi_i)$ that change a given wave function ψ_i to a new function ϕ_i. Then the part of S that corresponds to a transition from any state corresponding to the set of wave functions ψ to any state corresponding to the set of wave functions ϕ can be written

$$\sum_{\alpha} \xi_\alpha \bar{P}_\alpha \prod_{i=1} \Delta(\psi_i \rightarrow \phi_i), \tag{4.6}$$

where the coefficients ξ depend on the wave functions, but not on their ordering.

If N is not greater than the order of the parastatistics, so that any change in the order of the wave functions in the state $|\phi_1\phi_2\cdots\phi_N\rangle$ produces an orthogonal state, we have

$$P^{-1}\,\bar{P}\,P\,|\phi_1\phi_2\cdots\phi_N\rangle = \bar{P}\,|\phi_1\phi_2\cdots\phi_N\rangle \tag{4.7}$$

for any particle permutation $\bar{P}$ and any place permutation P. Hence the structure (4.6) for S corresponds to the property

$$\langle\phi_1\phi_2\cdots\phi_N|P^{-1}\,S\,P|\psi_1\psi_2\cdots\psi_N\rangle = \langle\phi_1\phi_2\cdots\phi_N\,|S|\,\psi_1\psi_2\cdots\psi_N\rangle \tag{4.8}$$

for any place permutation P.

However (4.8) does not hold when N is greater than the order of the parastatistics. We illustrate this for the case

$N = 3$ with order-two parafermi statistics, specializing now to the case where the wave functions represent momentum eigenstates. We may write

$$\bar{P}\,|p_1p_2p_3\rangle = \sum_{ijk} \widetilde{P}(p_ip_jp_k \mid p_1p_2p_3)\,|\,p_ip_jp_k\rangle, \qquad (4.9)$$

where i,j,k is a permuation of 1,2,3 and the function $\widetilde{P}(p_ip_jp_k \mid p_1p_2p_3)$ is unity if i,j,k is that permutation of 1,2,3 given by $\bar{P}$, and is zero otherwise. The functions $\widetilde{P}$ are invariant under place permutations:

$$\widetilde{P}(P\; p_ip_jp_k \mid P\; p_1p_2p_3) = \widetilde{P}(p_ip_jp_k \mid p_1p_2p_3)\,. \qquad (4.10)$$

But the matrix elements of the operator $\bar{P}$ take the form

$$\langle p_ip_jp_k \mid \bar{P} \mid p_1p_2p_3\rangle = \widetilde{P}(p_ip_jp_k \mid p_1p_2p_3) - \widetilde{P}(p_kp_jp_i \mid p_1p_2p_3), \qquad (4.11)$$

because of (2.6), and are not place-permutation invariant. Combining (4.9) and (4.6) we get a similar decomposition of the elastic S matrix, which in the N-particle case reads

$$S \mid p_1p_2\cdots p_N\rangle = \sum_{q_1q_2\cdots q_N} \widetilde{S}(q_1q_2\cdots q_N \mid p_1p_2\cdots p_N) \mid q_1q_2\cdots q_N\rangle \qquad (4.12)$$

with the functions $\widetilde{S}$ satisfying

$$\widetilde{S}(P\; q_1q_2\cdots q_N \mid P\; p_1p_2\cdots p_N) = \widetilde{S}(q_1q_2\cdots q_N \mid p_1p_2\cdots p_N) \qquad (4.13)$$

for all place permutations P. These results are of exactly the form obtained from the field-theory model of Volkov, in which $\tilde{S}$ represented the function corresponding to a sum of Feynman graphs, and property (4.13) was an expression of the dynamical indistinguishability of the particles.

In the general case of identical particles, it is natural to define dynamical indistinguishability as the property that there are functions $\tilde{S}$ that both satisfy (4.12) and possess the symmetry (4.13). If the various basis vectors $|q_1 q_2 \cdots q_N\rangle$ are not all orthonormal then $\tilde{S}$ is not defined by (4.12) alone, and the S-matrix elements themselves may not satisfy (4.13). On the other hand, if these states are orthonormal, as in the first-quantized theories, then the functions $\tilde{S}(q_1 q_2 \cdots q_N \mid p_1 p_2 \cdots p_N)$ are equal to the S-matrix elements, and (4.13) is simply the usual requirement $PSP^{-1} = S$, which is commonly taken as a basic expression of indistinguishability.

V. CLUSTER PROPERTIES AND INELASTIC PROCESSES

In Sections 3 and 4 we found that, as long as only elastic scatterings are considered, it is consistent to suppose that particle permutations are the only observables that distinguish among states corresponding to different orderings of the ϕ_i. This would mean that no elastic-scattering experiment can measure the relative phase of parts of a state vector belonging to different particle-permutation subspaces. It is natural to ask whether an inelastic reaction can be used to measure such a phase. In this section we examine first a simple production experiment in the Volkov model and find that these phases remain unobservable. It is then shown that this result remains true under more general conditions.

We first make some remarks concerning the cluster properties of elastic-scattering processes. For the theory to be acceptable it is necessary that observations on a set of N particles on the earth be essentially unaffected by the presence of particles on the moon. That is, it should be possible to describe all interactions among some certain N particles either in terms of states that contain only those N particles or, alternatively, in terms of states that contain also the particles on the moon. These descriptions should agree. It is evident that observables that are functions of the $\bar{P}$ do fulfill this requirement; no matter how the wave functions ϕ of the particles on the earth are ordered relative to the wave functions ψ of the particles on the moon, the N-particle permutations $\bar{P}$ permute the ϕ in a manner independent of the ψ. Elastic interactions of the form (4.6) also fulfill the requirement for essentially the same reason; all dependence upon the positions of the ϕ enters only through the $\bar{P}$.

Consider now a system of two particles described by wave functions ϕ_1 and ϕ_2. Disregarding the presence of other particles on the moon, we can use elastic-scattering experiments to establish that the system is in the state

$$\lambda|\phi_2\,\phi_1\rangle_+ \;+\; \mu|\phi_2\,\phi_1\rangle_- \;\equiv\; \frac{1}{\sqrt{2}}\left[(\lambda+\mu)|\phi_2\,\phi_1\rangle \;+\; (\lambda-\mu)|\phi_1\,\phi_2\rangle\right], \tag{5.1}$$

where the relative magnitudes of λ and μ are measurable, but not their relative phase. Let now a pair $(\phi_3, \bar{\phi}_4)$ be produced, and suppose that the production takes place in a region in space far removed from (ϕ_1, ϕ_2), so that the particles described by ϕ_1 and ϕ_2 are not involved in the production process. To obtain the new state we act on (5.1) with the appropriate term in the interaction Hamiltonian (2.2). According to (2.13) this yields a multiple of the state

$$\left[|\phi_3\bar{\phi}_4\phi_2\phi_1\rangle - |\bar{\phi}_4\phi_3\phi_2\phi_1\rangle\right](\lambda+\mu)/\sqrt{2} + \left[|\phi_3\bar{\phi}_4\phi_1\phi_2\rangle - |\bar{\phi}_4\phi_3\phi_1\phi_2\rangle\right](\lambda-\mu)/\sqrt{2}$$

$$\equiv |\bar{\phi}_4\phi_1\phi_2\phi_3\rangle(\lambda+\mu)/\sqrt{2} - |\bar{\phi}_4\phi_3\phi_1\phi_2\rangle(\lambda-\mu)/\sqrt{2} + |\phi_1\phi_2\phi_3\bar{\phi}_4\rangle(\lambda-\mu)/\sqrt{2} - |\phi_3\phi_1\phi_2\bar{\phi}_4\rangle(\lambda+\mu)/\sqrt{2} \tag{5.2}$$

where (2.5) is used to get the second version.

Elastic-scattering experiments on the three particles represented by ϕ_1, ϕ_2, and ϕ_3 allow measurements of the three-particle permutation operators $\bar{P}$, and an obvious question is whether knowledge of the values of these observables gives information on the relative phase of λ and μ. Direct calculation shows that the answer is no.

Yet it does seem that (5.2) implies that measurements on the initial two-particle system can establish phase relations among states in different three-particle subspaces. In particular, if the initial state is found to correspond to $\lambda = 0$, $\mu = 1$, then (5.2) becomes

$$\frac{1}{\sqrt{2}} \left\{ \left[|\bar{\phi}_4 \phi_1 \phi_2 \phi_3\rangle + |\bar{\phi}_4 \phi_3 \phi_1 \phi_2\rangle \right] - \left[|\phi_1 \phi_2 \phi_3 \bar{\phi}_4\rangle + |\phi_3 \phi_1 \phi_2 \bar{\phi}_4\rangle \right] \right\} . \tag{5.3}$$

Suppose now that the antiparticle $\bar{\phi}_4$ is removed to the moon. Then the cluster properties mentioned above imply that the description of the particles left on the earth is equivalent to that given by the three-particle state

$$|\phi_1 \phi_2 \phi_3\rangle + |\phi_3 \phi_1 \phi_2\rangle . \tag{5.4}$$

(The two quantities enclosed in square brackets in (5.3) do not interfere for interactions where $\bar{\phi}_4$ is removed to the moon, since in the first bracket the remaining particles are of types $\beta\alpha\beta$, while in the second they are $\alpha\beta\alpha$). Bearing in mind relations of the type (2.6), we see that (5.4) is a superposition with definite relative phase of states from two different subspaces of (3.7).

Specifically, it is

$$|\phi_-\rangle = (2\sqrt{3}\,|A\rangle + \sqrt{6}\,|B\rangle)/3, \qquad (5.5)$$

where

$$|A\rangle = \frac{1}{\sqrt{3}}\,(|\phi_1\phi_2\phi_3\rangle + |\phi_2\phi_3\phi_1\rangle + |\phi_3\phi_1\phi_2\rangle)$$

$$|B\rangle = \frac{1}{\sqrt{6}}\,(|\phi_1\phi_2\phi_3\rangle - 2|\phi_2\phi_3\phi_1\rangle + |\phi_3\phi_1\phi_2\rangle)\,. \qquad (5.6)$$

Thus it would appear that the relative phase of the components $|A\rangle$ and $|B\rangle$ of a three-particle state can in certain circumstances be measured; if on annihilating one of the three particles we find that the resulting two-particle state is antisymmetric, then we can apparently conclude that the relative phase was that of (5.5).

To understand why this is in fact not true we must again consider the cluster properties. The determination that the initial two-particle system has $\bar{P}_{12} = -1$ does not necessarily mean that it is represented by the state $|\phi_2\,\phi_1\rangle_-$. The possible presence of other particles in the universe allows it to be represented also by states where these other particles are introduced in various ways. One sees from (2.6) that any such state can be reduced to one of the three forms

$$|\phi_2\,\phi_1\rangle_-\;|\psi\psi'\cdots\rangle = |-\rangle$$

$$|\phi_2\,\psi\,\phi_1\rangle\,|\psi'\cdots\rangle = |a\rangle$$

$$|\psi\,\phi_2\,\psi'\,\phi_1\rangle\,|\psi''\cdots\rangle = |b\rangle. \qquad (5.7)$$

These states differ essentially only in whether the particles are of type α or β. This means that the initial system should properly be described by the density matrix

$$\rho = \sum_i w_i |i\rangle \langle i|, \tag{5.8}$$

where w_i is the statistical weight of the state i, and i runs over $-$, a, and b. The final three-particle system is described by the corresponding density matrix

$$\rho' = \sum_i w_i |\bar{\underline{\phi}}_i\rangle \langle \bar{\underline{\phi}}_i|, \tag{5.9}$$

where the calculation of $|\bar{\underline{\phi}}_a\rangle$ and $|\bar{\underline{\phi}}_b\rangle$ is analogous to the calculation of $|\bar{\underline{\phi}}_-\rangle$.

We have said that, in order that the cluster properties of the theory be acceptable, each state that appears in either of the sums in (5.8) and (5.9) must give equal expectation value of each observable. That is, the values of the weight w_i should be irrelevant. This is indeed the case for the operators $\bar{P}$.

On the other hand, the relative phase of the components $|A\rangle$ and $|B\rangle$ is not the same for all of the $\bar{\underline{\phi}}_i$. This means, in the first place, that if this phase were measurable, then the cluster property would be violated. Conversely, in any calculation depending on the relative phases of A and B, one must take into account all of the states $\bar{\underline{\phi}}_i$.

The relative phase of A and B in the state $\bar{\underline{\phi}}_i$ is the phase $\langle A|\bar{\underline{\phi}}_i\rangle \langle \bar{\underline{\phi}}_i|B\rangle$. The phase is well-defined unless this matrix element vanishes. When the various $\bar{\phi}_i$ are considered, the average value of the matrix element becomes $\langle A|\rho'|B\rangle$, which is

$$\frac{1}{3} \sum_i \langle A|\bar{\phi}_i\rangle \langle\bar{\phi}_i|B\rangle \qquad , \qquad (5.10)$$

if the w_i are all equal.

Calculation of (5.10) gives zero. This implies that the relative phase of A and B can be determined only if one knows that the weights w_i are unequal. To know that would entail having knowledge of the universe as a whole. But this is precluded by the cluster property, which says, in effect, that the possible presence of additional particles in the universe cannot be ruled out by measurements involving any finite group of particles. The cluster property is thus self-consistent.

This result requires some discussion. The Volkov model was shown in Section 2 to be equivalent to a theory with two dynamically indistinguishable fermions. This theory of course has certain cluster properties, which are the ones appropriate to a theory with two types of fermions. However, when one formulates the cluster property for parafermions, one requires specifically that connections between observable quantities be independent of the existence of unobserved particles. It is not clear that these two cluster properties are equivalent; indeed we have seen that the second requires a delicate cancellation among different states, while the first does not.

The vanishing of (5.10) is one of an infinite number of relations of the form

$$\langle C, r, \mu|\rho|D, s, \nu\rangle = \delta_{CD}\, \delta_{rs}\, \rho_{\mu\nu}^{Cr}. \qquad (5.11)$$

Here C and D label classes of irreducible representation of the $\bar{P}$, r and s label particular representations in these classes of equivalent representations, and μ and ν label particular basis vectors of the relevant subspaces. All of the equations (5.11) must be valid in order that all observables be functions of the $\bar{P}$ and in order that the parafermi cluster property be valid.

These equations (5.11) follow from the invariance of ρ under rotations R in the isotopic spin space of particles α and β:

$$R\, \rho\, R^{-1} = \rho. \qquad (5.12)$$

The invariance (5.12) will be assumed to apply for some original state of the system, as it was in our example when we took all the w_i to be equal. Then the invariance of H_{INT} under rotations R guarantees that (5.12) also applies to the final state.

We begin the proof of (5.11) by casting the theory into the isotopic-spin framework. The $(\phi_1, \phi_2, \cdots \phi_N)$ used previously is replaced by $(\phi_1^{\gamma_1}, \phi_2^{\gamma_2}, \cdots \phi_N^{\gamma_N})$, where γ_i is an index that denotes whether the particle is of type α or β. Thus, whereas originally the distinction between α- and β- type particles lay in the distinction between odd and even positions in the state vector, now this distinction lies in the value of the isotopic-spin index γ. Because of the anticommutation relations between the α- and β- operators, the various $\phi_i^{\gamma_i}$ in the state vector $|\phi_1^{\gamma_1}, \phi_2^{\gamma_2}, \cdots \phi_N^{\gamma_N}\rangle$ anticommute. Hence to eliminate redundant variables we bring all state vectors to the form in which the $\phi_i^{\gamma_i}$ appear in numerical sequence on the index i. An important feature of this transformation is that particle permutations $\bar{P}$ acting on the original α-β states becomes place-permutation operators P acting on the corresponding isotopic-spin states. This is because the indices 1, 2,$\cdots$N that identify particles in the α-β formalism specify the positions of the index in this isotopic-spin framework.

The transformation R in (5.12) takes the form of a tensor product $R = A \otimes A \otimes \cdots A = \otimes A^N$, where A represents a rotation in the two-dimensional isotopic-spin space. The enveloping algebra (7) of the set of R's is the commutator algebra $\mathcal{A}'$ of the algebra $\mathcal{A}$ of place permutations P. Equation (5.12) says that ρ lies in the commutator algebra $\mathcal{A}''$ of $\mathcal{A}'$. However, $\mathcal{A}''$ is just $\mathcal{A}$ itself.(8) Thus any ρ satisfying (5.12) is a linear function of the P's. In the α - β formalism it is therefore a function of the place-permutations $\bar{P}$.

We conclude by summarizing the paper with cluster properties as the focal point. The parafermi commutation relations (2.1) suggest that cluster properties may be violated in parastatistical theories. This is because the removal of the middle operators

$a_n^\dagger$ on the left-hand side leads to nonvalid relations. However, the proved equivalence of the parafermi theory to a theory involving only ordinary fermions show that certain cluster properties are in fact maintained. But because the particles associated with even and odd positions are nonequivalent, it is not immediately clear that one necessarily has the physically significant cluster property that all correlations among observables are independent of the existence of possible unobserved particles on the moon. To examine this question one must establish what quantities are observable in systems containing identical particles.

The usual rule is that every observable commutes with every place permutation. This requirement is not applicable in parastatistics models, since place permutations are ill-defined owing to their noncommutability with the basic commutation relations. Therefore we argue directly from physical considerations that particle permutations are observables. In case no relations such as (2.5) inhibit the use of place permutations, the usual rule that all observables commute with place permutations is equivalent to the statement that all observables are functions of particle permutations (insofar as dependence on positions of variables is concerned). This is shown in the appendix. Since particle permutations, unlike place permutations, are unaffected by relations such as (2.5), it is natural to take the statement that all observables are functions of particle permutations (insofar as dependence on positions is concerned) as the generalization of the usual statement that all observables commute with place permutations.

The presumption that all observables are functions of particle permutations, and moreover that all such functions are observables, implies that the elastic-scattering matrix is expressible as a function of the particle permutations. Moreover, this requirement on the elastic S guarantees that only functions of particle permutations are observable by means of elastic-scattering experiments alone. Thus this presumption concerning the observables is consistent with the Volkov model, insofar as only elastic reactions are considered. Within this framework we then find that the physical cluster properties are satisfied: correlations between observables are not affected by the presence or absence of unobserved particles. This result holds also when production reactions are considered provided the cluster properties are invoked to guarantee an "isotopic-spin" symmetry of the original density matrix.

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APPENDIX

Place Permutations and Particle Permutations

Suppose the set of $N!$ states

$$|\Pi\rangle = |\phi_{\Pi 1}\ \phi_{\Pi 2} \cdots \phi_{\Pi N}\rangle, \tag{A.1}$$

is orthonormal, where $(\Pi 1, \Pi 2, \cdots \Pi N)$ are permutations of $(1, 2, \cdots N)$. Particle permutations $\bar{P}$ are defined by

$$\bar{P}|\Pi\rangle = |\bar{P}\Pi\rangle = |\phi_{\bar{P}\Pi 1}\ \phi_{\bar{P}\Pi 2} \cdots \phi_{\bar{P}\Pi N}\rangle, \tag{A.2}$$

where $(\bar{P}1, \bar{P}2, \cdots \bar{P}N)$ is a permutation of $(1, 2, \cdots N)$. The effect of $\bar{P}$ is to replace ϕ_1 by $\phi_{\bar{P}1}$, ϕ_2 by $\phi_{\bar{P}2}$, etc. It was explained in the text that the $\bar{P}$ are relevant because from the experimental viewpoint the various particles are identified by their wave functions ϕ_i, rather than by the positions in the state vector occupied by these wave functions.

Place permutations P are defined by

$$P|\Pi\rangle = |\phi_{\Pi P1}\ \phi_{\Pi P2} \cdots \phi_{\Pi PN}\rangle = |\Pi P\rangle, \tag{A.3}$$

where $(P1, P2, \cdots PN)$ is a permutation of $(1,2,\cdots N)$. The effect of P is to replace the occupant of position 1 by the occupant of position $P1$, the occupant of position 2 by the occupant of position $P2$, etc. According to (A.3) the product of two place permutations P_1 and P_2 give

$$P_2\, P_1 \mid \Pi\rangle \;=\; |\Pi\, P_1\, P_2\rangle . \qquad (A.4)$$

It is clear that all place permutations commute with all particle permutations, for particle permutations act without regard to place, and place permutations act without regard to the identities of the wave functions. Formally, this commutability follows from the equation

$$\bar{P}\, P \mid \Pi\rangle \;=\; |\bar{P}\, \Pi\, P\rangle \;=\; P\, \bar{P}|\Pi\rangle . \qquad (A.5)$$

Not only do all $\bar{P}$'s commute with all P's, and vice versa, but in the space $\mathcal{S}$ spanned by the $|\Pi\rangle$ all operators that commute with the P's are linear combinations of the $\bar{P}$'s and vice versa. This result, which we refer to as the reciprocity of place permutations and particle permutations, follows directly from theorem (3.4A) of reference 6. However, it is useful to give a proof that exhibits the result in a more concrete form.

The set of vectors $|\Pi\rangle$ are labelled by the set of $N!$ elements of the group of permutations on N objects. These vectors form a basis (substratum) of the regular representations of this group. (This is the representation such that $U(\bar{P})_{\Pi',\Pi} = \delta_{\Pi',\bar{P}\,\Pi}$). A fundamental result from group theory is that the regular representation is completely reducible, that every irreducible representation of the group appears in this reduction, and that the number of times a given irreducible representation appears in the reduction is equal to the dimension of that representation. Here equivalent irreducible representations are identified.

The transformation from the original basis $|\Pi\rangle$ to the basis $|C, r, \mu\rangle$ is given by

$$|C, r, \mu\rangle = |\Pi\rangle \langle\Pi|C, r, \mu\rangle. \tag{A.6}$$

Here C labels the class of equivalent irreducible representations, r labels the particular one of the equivalent irreducible representations, and μ labels the particular basis vector of the subspace corresponding to this representation. The action of $\bar{P}$ on the $|C, r, \mu\rangle$ is given by

$$\bar{P}\, |C, r, \mu\rangle = |C, r, \nu\rangle\, U^{C}_{\nu\mu}(\bar{P}), \tag{A.7}$$

where the set of $\underset{\sim}{U}^{C}(P)$ form the irreducible representation specified by C.

The action of P is given, as we shall prove, by

$$P\, |C, r, \mu\rangle = |C, s, \mu\rangle\, V^{sr}_{C}(P). \tag{A.8}$$

That is, the P acts within the space associated with a given class C and acts there as a matrix on the indices r and s that label the various equivalent irreducible representations of class C in the reduction of $\bar{P}$. The $\underset{\sim}{U}^{C}(\bar{P})$ and $\underset{\sim}{V}_{C}(P)$ are matrices of the same dimension.

The result (A.8) is derived as follows: initially one has

$$P\, |C, r, \mu\rangle = |D, s, \nu\rangle\, \langle D, s, \nu|P|C, r, \mu\rangle. \tag{A.9}$$

Because $P\bar{P} = \bar{P}P$, we have

$$\langle D, s, \nu|P|C r \mu\rangle\, U_{\mu\lambda}^{C}(\bar{P}) = U_{\nu\eta}^{D}(\bar{P})\, \langle D, s, \eta|P|C, r, \lambda\rangle \quad \text{(A.10)}$$

for all $\bar{P}$. Schur's lemma then ensures that $\langle D, s, \eta|P|C, r, \lambda\rangle = 0$ for $D \neq C$, and that for $D = C$ it is a constant matrix in the Greek indices. This constant can depend on s, r, and C, however, so we obtain (A.8).

The set of matrices $\underset{\sim}{V}_C(P)$, for each C, form a representation of the group of permutations P. These representations are irreducible, because if any of the sets $\underset{\sim}{V}_C$ were reducible, then the decomposition of the group of P's would give more classes of irreducible representations than the decomposition of the group of $\bar{P}$'s. This is not possible, since these two groups are essentially identical. The same argument shows that the $\underset{\sim}{V}_C(P)$ for different values of C are inequivalent.

The above arguments show, in fact, that any operator A in $\mathcal{S}$ that commutes with all the $\bar{P}$'s is represented by $\langle C, r, \mu|A|D, s, \nu\rangle = \delta_{CD}\, \delta_{\mu\nu}\, A_F^{rs}$. From the completeness and orthogonality properties of the $V_C^{rs}(P)$ one obtains

$$A = \sum_P A(P)\, P, \quad \text{(A.11)}$$

where

$$A(P) = \frac{1}{N!}\, \mathrm{Tr}(AP^{-1})$$
$$= \frac{1}{N!} \sum \langle C, r, \mu|A|D, s, \nu\rangle\, \langle D, s, \nu|P^{-1}|C, r, \mu\rangle, \quad \text{(A.12)}$$

as was already discussed in the text. That is, any function in $\mathcal{S}$ that commutes with all the $\bar{P}$'s is a linear combination of P's. The converse is clearly true also.

FOOTNOTES

1. This work was done under the auspices of the U.S. Atomic Energy Commission.

2. On leave of absence during the Michaelmas term 1966 from the Department of Applied Mathematics and Theoretical Physics and Christ's College, University of Cambridge, England.

3. It should be noted that our formulation of the Volkov model in terms of ordinary fermion operators is different from the canonical formalism of Green (3).

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