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Questions of Connectedness of the Hilbert Scheme of Curves in \mathbb{P}^3

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Dedicated to S. Abhyankar on the occasion of his 70th birthday.

We review the present state of the problem, for each degree d and genus g , is the Hilbert scheme of locally Cohen–Macaulay curves in \mathbb{P}^3 connected?

1 Introduction

In studying algebraic curves in projective spaces, our forefathers in the 19th century noted that curves naturally move in algebraic families. In the projective plane, this is a simple matter. A curve of degree d is defined by a single homogeneous polynomial in the homogeneous coordinates x_0, x_1, x_2 . The coefficients of this polynomial give a point in another projective space, and in this way curves of degree d in the plane are parametrized by the points of a \mathbb{P}^N with $N = \frac{1}{2}d(d+3)$. For an open set of \mathbb{P}^N , the corresponding curve is irreducible and nonsingular. The remaining points of \mathbb{P}^N correspond to curves that are singular, or reducible, or have multiple components. In particular, the nonsingular curves of degree d in \mathbb{P}^2 form a single irreducible family.

In \mathbb{P}^3 , the situation is more complicated. For a given degree d , there may be curves with several different values of the genus g . Even for fixed d, g , the family of curves with given d, g may not be irreducible. An early example, noted by Halphen and Weyr in 1874 is the case $d = 9$ and $g = 10$. One type consists of curves C_1 of bidegree $(3,6)$ on a nonsingular quadric surface Q . The other type consists of curves C_2 that are the complete intersection of two cubic surfaces F_3 and F'_3 . These curves form two irreducible components of the Hilbert scheme $H_{d,g}^{sm}$ of smooth curves of degree d and genus g in \mathbb{P}^3 . Furthermore, it is not hard to see that every curve in $H_{9,10}^{sm}$ belongs to one of these two types, and that there is no

flat family of curves whose general member belongs to one type and whose special member belongs to the other type [7, Ch.IV, 6.5.4]. Thus the Hilbert scheme of smooth curves of given degree and genus in \mathbb{P}^3 need not be connected.

On the other hand, if one phrases the question more generally, by letting a “curve” mean an arbitrary closed subscheme of dimension 1 in \mathbb{P}^3 , then the Hilbert scheme for each degree d and arithmetic genus g is connected. In fact, in my thesis [6], I showed that the Hilbert scheme of closed subschemes of \mathbb{P}^n with Hilbert polynomial P is connected (provided it is nonempty) for any n and any P . In the proof, non-reduced schemes play an essential role. Here is the main idea of the proof for the case of curves in \mathbb{P}^3 . Suppose, for example, that we start with a nonsingular curve C . Its general projection to \mathbb{P}^2 will be a plane curve C_0 with nodes. Using the projection we can construct a flat family whose general member is C and whose special member C_1 is a curve with support C_0 , having embedded points at the nodes (see [7, III, 9.8.4]) for an example showing how these embedded points arise). Then we can make another flat family, pulling the embedded points off C_1 , to get C_0 union a number of points in \mathbb{P}^2 . Finally, we move C_0 in a flat family of plane curves to a union of lines in \mathbb{P}^2 meeting at a single point. If the original curve C had degree d and genus g , we obtain in this way a “fan” of d lines in the plane together with $k = \frac{1}{2}(d-1)(d-2) - g$ isolated points in the plane. Any other curve C' with the same d, g can be connected by a sequence of flat specializations and generalizations to the same configuration, so C and C' are connected within the Hilbert scheme $H_{d,g}$, of all closed subschemes of \mathbb{P}^3 of dimension 1, degree d , and arithmetic genus g . For nonreduced curves a slightly more complicated, but similar method applies.

Thus we have a connectedness theorem for the Hilbert scheme of curves in \mathbb{P}^3 , but it is unsatisfactory in that, even if we want to connect one smooth curve to another, we must pass by way of schemes with embedded points and isolated points, which one can argue should not really count as “curves”.

With the development of liaison theory in recent years [20],[10], an intermediate class of curves has received much attention, the locally Cohen-Macaulay curves. We say a curve is locally Cohen-Macaulay if it is a scheme of equidimension 1, and all its local rings are Cohen-Macaulay rings. Equivalently it is a 1-dimensional scheme with no embedded points or isolated points. It is clear that this class of curves is the natural class in which to do liaison: even if one is primarily interested in nonsingular curves, the minimal curves in a biliaison class may be reducible and non-reduced. So we pose the question: Is the Hilbert scheme $H_{d,g}^{CM}$ of locally Cohen-Macaulay curves of degree d and arithmetic genus g in \mathbb{P}^3 connected? The answer is unknown at present, so we devote this paper to a survey of the current state of this question.

2 Known results

2.1 When is $H_{d,g}$ nonempty?

Before discussing whether a given Hilbert scheme is connected, one should at least know when it is nonempty.

For smooth curves in \mathbb{P}^3 , the result was stated by Halphen [5] with an incorrect proof, and proved one hundred years later by Gruson and Peskine [4],[8]. There exists an irreducible smooth curve C of degree d and genus g in \mathbb{P}^3 if and only if either

- a) $d \geq 1$ and $g = \frac{1}{2}(d-1)(d-2)$ (these are the plane curves), or
- b) there exist $a, b > 0$ with $d = a + b$ and $g = (a-1)(b-1)$
(these are curves on quadric surfaces), or
- c) $d \geq 3$ and $0 \leq g \leq \frac{1}{6}d(d-3) + 1$.

The hardest part of the proof is the existence of curves for all (d, g) in the range c), which they construct on suitable cubic and quartic surfaces in \mathbb{P}^3 .

If one considers all one-dimensional closed subschemes of \mathbb{P}^3 , the answer was known to Macaulay [19], and rediscovered in [6]. Then $H_{d,g}$ is nonempty for all $d \geq 1$ and all arithmetic genus $g \leq \frac{1}{2}(d-1)(d-2)$. The existence is simple. Just take a plane curve of degree d and add lots of isolated points. Note that the arithmetic genus g can become arbitrarily negative.

For locally Cohen-Macaulay curves, the answer is slightly more complicated, but not too difficult [9]. A locally Cohen-Macaulay curve with given d, g exists if and only if either

- a) $d \geq 1$, $g = \frac{1}{2}(d-1)(d-2)$ (a plane curves), or
- b) $d \geq 2$, $g \leq \frac{1}{2}(d-2)(d-3)$

For $d = 2$ one can exhibit a multiplicity two structure on a line with any given arithmetic genus $g \leq 0$. For example, the scheme in \mathbb{P}^3 defined by the homogeneous ideal $(x^2, xy, y^2, xz^r - yw^r)$, for any $r \geq 0$, has $g = -r$. Then one can construct curves for all (d, g) in case b) above by taking a plane curve of degree $d-1$ containing a line, and putting a suitable multiplicity two structure on the line.

2.2 When is $H_{d,g}$ irreducible?

From now on, we will consider only locally Cohen-Macaulay curves, and denote $H_{d,g}^{CM}$ by $H_{d,g}$.

There are some values of (d, g) for which $H_{d,g}$ is irreducible, and hence trivially connected [22]. These are

- a) $d \geq 1$, $g = \frac{1}{2}(d-1)(d-2)$, the plane curves
- b) $d = 2$, $g \leq 0$. For $g = 0$ we have a plane curve; for $g = -1$, two disjoint lines or a double line on a quadric; and for $g \leq -2$, double structures on a line.
- c) Some special values of g for higher degree, namely $(d, g) = (3, 0), (3, -1), (4, 1)$, and

$$d \geq 5, \quad \frac{1}{2}(d-3)(d-4) + 1 < g \leq \frac{1}{2}(d-2)(d-3).$$

For all other (d, g) , namely $d = 3$, $g \leq -2$; $d = 4$, $g \leq 0$; and $d \geq 5$, $g \leq \frac{1}{2}(d-3)(d-4) + 1$, $H_{d,g}$ has two or more irreducible components.

2.3 Extremal curves

For any curve $C \subseteq \mathbb{P}^3$, an important invariant is the *Rao module* $M(C) = \bigoplus_{n \in \mathbb{Z}} H^1(\mathcal{I}_C(n))$. The dimensions of the graded components of the Rao module are the *Rao function* $\rho_C(n) = \dim H^1(\mathcal{I}_C(n))$. A curve is arithmetically Cohen-Macaulay (ACM) if and only if its Rao module is 0.

For non-plane curves, there are explicit bounds on the Rao function in terms of d and g [21]. In particular, for all n we have

$$\rho(n) \leq \frac{1}{2}(d-2)(d-3) - g.$$

Thus, if $g = \frac{1}{2}(d-2)(d-3)$, the Rao function is 0, so the curve is necessarily ACM, and one knows in this case that the Hilbert scheme is irreducible [3].

If $g < \frac{1}{2}(d-2)(d-3)$, then one has the more precise result that $\rho(n)$ is bounded by a function that is equal to $\frac{1}{2}(d-2)(d-3) - g$ for $0 \leq n \leq d-2$, and decreases with slope 1 (resp. -1) to zero on both ends of this range.

In their paper [22], Martin-Deschamps and Perrin define an *extremal* curve to be a non-ACM curve whose Rao function is equal to this bound for all n . For any $g < \frac{1}{2}(d-2)(d-3)$, they show the existence of extremal curves, and show that they form an irreducible component of the Hilbert scheme.

For curves that are not extremal, Nollet has established a stronger bound on the Rao function [24]. If $d \geq 5$ and the curve is neither ACM nor extremal, then $\rho(n) \leq \frac{1}{2}(d-3)(d-4) + 1 - g$. In particular this implies $g \leq \frac{1}{2}(d-3)(d-4) + 1$. Thus any curve of $d \geq 5$ and $g > \frac{1}{2}(d-3)(d-4) + 1$ must be extremal, and we conclude that the Hilbert scheme is irreducible in that range. Curves satisfying Nollet's stronger bounds are called *subextremal*.

2.4 When is $H_{d,g}$ connected?

If the Hilbert scheme has two or more irreducible components, which happens for $d = 3$, $g \leq -2$; $d = 4$, $g \leq 0$; and $d \geq 5$, $g > \frac{1}{2}(d-3)(d-4) + 1$ we can ask if it is connected. Here are some cases in which it is known to be connected.

1. If $d = 3$, $g \leq -2$, then $H_{d,g}$ is connected [25], and has approximately $\frac{1}{3}|g|$ irreducible components.
2. If $d = 4$, $g \leq 0$, then $H_{d,g}$ is connected [27].
3. If $d \geq 5$ and $g > \frac{1}{2}(d-3)(d-4) + 1$, then $H_{d,g}$ has two irreducible components, consisting of the extremal curves in one and the ACM curves in the other, and is connected [26].
4. If $d \geq 4$ and $g = \frac{1}{2}(d-3)(d-4)$, then $H_{d,g}$ has 2, 3, or 4 irreducible components, and is connected [1].
5. If $d = 5$, $g = 0$, $H_{d,g}$ has four irreducible components and is connected [14].
6. If $d \geq 6$ and $g = \frac{1}{2}(d-3)(d-4) - 1$, then $H_{d,g}$ has four or five irreducible components, and is connected [29].

These cases, together with the cases of $H_{d,g}$ irreducible listed above, are the only cases in which it is known that $H_{d,g}$ is connected at present. The problem falls into two halves. The first is to list the irreducible components of $H_{d,g}$, and the second is to show the existence of flat families of curves connecting the different components. It is the first of these that is blocking further progress at the moment, because it requires a classification of all curves of the given d, g . The most difficult part is to understand all the possible nonreduced structures on a curve of lesser degree. Thus already the case of multiplicity four structures on a line is extremely complicated.

To avoid the first problem, we formulate the question differently.

2.5 Curves connected to extremal curves

For a given d, g , there is always one irreducible component of $H_{d,g}$ consisting of the extremal curves. So we ask, which classes of curves can be connected by flat families in $H_{d,g}$ to an extremal curve? If every curve with the given (d, g) is connected to an extremal curve, then $H_{d,g}$ is connected.

The advantage of this question is that we do not have to classify all curves of type (d, g) . Here are some cases that are known, namely curves that can be connected within $H_{d,g}$ to an extremal curve of the same degree and genus.

1. Any disjoint union of lines [14].
2. Any smooth curve with $d \geq g + 3$ [14].
3. Any ACM curve [14].
4. Any curve in the biliaison equivalence class of an extremal curve [30].
5. Any curve whose Rao module is a complete intersection (also called a Koszul module). [28].

3 Techniques used

The classification part of the problem uses standard methods. What is new in studying connectedness questions is to prove the existence of flat families of curves, whose general member lies in one irreducible component, and whose special member lies in another irreducible component. We discuss here the different methods used to construct such families.

3.1 Explicit equations

If one knows the equations of the two types of curves, one can attempt to make a flat family by writing equations depending on a parameter t . A simple example of this is the family of twisted cubic curves having as limit a plane nodal curve with an embedded point [7, III, 9.8.4]. Explicit equations are used in the papers [25],[26],[15]. This technique is obviously limited to situations where one has only to deal with very explicit examples of curves.

3.2 Line drawings

This method, used in [14], is an extension of the first. Some families of multiple structures on lines are proved by explicit equations. These are then used as lemmas in drawings of much more complicated curves, supported on unions of lines. Combined with the complete description of curves contained in a double plane [15], this allows one to show existence of families for many types of smooth curves that specialize to stick figures, such as the nonspecial curves with $d \geq g + 3$.

3.3 Triades

This method is the most sophisticated, and potentially the most powerful, but also the most technically difficult. This method is developed in the three papers [16],[18],[17] and applied in the papers [1],[2],[28].

The idea is to develop an algebraic theory of flat families, biliaison, and Rao modules similar to the well known theory of biliaison and Rao module for individual curves [20].

In a flat family C_t of curves in \mathbb{P}^3 parametrized by a parameter scheme T , the Rao module $M(C_t)$ is not in general constant in the family. Also the sheaf analogue $\bigoplus_{n \in \mathbb{Z}} R^1 f_* (\mathcal{I}_C(n))$ as a sheaf of graded S -modules over T , does not commute with base extension. So, for example, if $T = \text{Spec } A$ is affine, one is led to consider the functor on A -modules

$$M \mapsto \bigoplus_{n \in \mathbb{Z}} H^1(\mathcal{I}_C(n) \otimes_A M) .$$

This is a coherent functor in the sense of Auslander [13], but it is still not a fine enough invariant to play the role of the Rao module for a family. So instead we consider the *triad* associated to the family C : it is a 3-term complex $L_1 \rightarrow L_0 \rightarrow L_{-1}$ of graded S_A -modules, where $S_A = A[x_0, x_1, x_2, x_3]$ is the homogeneous coordinate ring of \mathbb{P}_A^3 , whose middle cohomology retrieves the cohomology $H^1(\mathcal{I}_C(n))$ of the family, and which satisfies certain other technical conditions (see [16, 1.10] for the precise definition). There is a notion of pseudoisomorphism for triades [16, 1.7], and then one obtains the analogue of Rao's theorem, that two families of curves are in the same biliaison equivalence class if and only if their triades are pseudoisomorphic up to shift in degrees [16, 3.9].

There is also an algorithmic method of constructing the universal family of curves associated to a triad [17], and this becomes the basic method of constructing flat families of curves. The difficulty is that the triad is not determined simply by knowing the Rao modules of the general curve and the special curve: there are other choices to be made to determine the triad. Thus to show the existence of a family connecting curves of particular types, one has to choose carefully a suitable triad to give the family. This means also that while the method of triads is good for making families, it is more difficult to prove the non-existence of families between given types of curves.

See also [11] for a slightly less brief introduction to the theory of triades.

4 An example

Here we describe an example for which it is not yet known whether $H_{d,g}$ is connected or not.

We consider smooth curves C of bidegree $(3,7)$ on a smooth quadric surface Q in \mathbb{P}^3 . Then $d = 10$, $g = 12$. We do not know if these curves can be connected to extremal curves. Because of semicontinuity, these curves cannot be specializations of a family of curves not contained in quadric surfaces. So these curves form an open subset of an irreducible component of $H_{10,12}$. The only possibility for connecting them to other curves requires specializing the quadric surface Q to a quadric cone, the union of two planes, or a double plane. One can show that if Q specializes to a cone or to a union of two planes, the curves must necessarily acquire embedded points [12, §2]. So the only case remaining is when Q specializes to a double plane. Since one knows all about curves in the double plane [15], it would be sufficient to show the existence of a flat family going from the curves C to a locally Cohen-Macaulay curve in the double plane, but this question has so far resisted analysis.

Another approach is to use biliaison. If one has a flat family going from a curve C_0 to an extremal curve E_0 , then by biliaison of the family one obtains a flat family from C_1 to E_1 , where C_1 and E_1 are in the biliaison classes of C_0 and E_0 , respectively. Schlesinger's result [30] shows that E_1 can be connected to an extremal curve with the same degree and genus.

Now our curve C of bidegree $(3,7)$ on Q is in the biliaison class of a curve C_0 consisting of four skew lines, and one knows that four skew lines can be connected to an extremal curve (cf. §2E above). The catch is that in order to perform a biliaison of the family on the quadric surface, the entire family must be contained in quadric surfaces. In the case of four skew lines on Q , we do not know if they can be specialized *on a quadric surface* to an extremal curve. The way we know they are connected to an extremal curve is to pull them off the quadric surface, giving more room to move around and then specialize.

So it seems that this example is a test case for the connectedness question, and might possibly lead to a counterexample.

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