Title
Insertion Device Design

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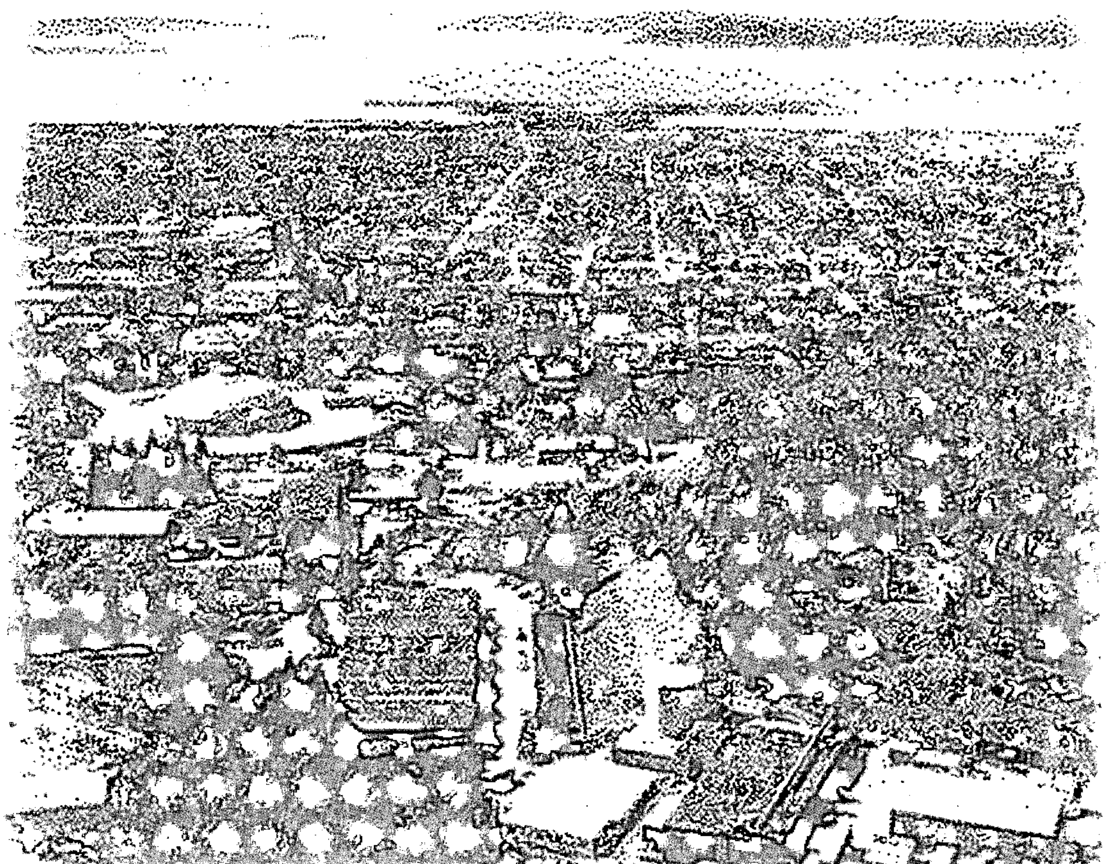
ERNEST ORLANDO LAWRENCE BERKELEY NATIONAL LABORATORY

Insertion Device Design

Sixteen Lectures Presented from October 1988 to March 1989

K. Halbach

March 1989



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V-8811-1.1-16

Insertion Device Design

Sixteen Lectures presented from October 1988 to March 1989

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Table of Contents of Insertion Device Lectures, by K.Halbach

Each lecture lasts about 2 hours and starts with a summary of the previous lecture. In this summary, topics are often formulated somewhat differently than in the original lecture in order to enhance clarity, or to illuminate the subject from a different perspective. For a review of a particular topic, it may therefore be useful to look at the viewgraphs/tapes of both the original lecture as well as the following lecture.

of non-dipole in dipole geometry; complete, detailed description of procedure for design of non-dipole in dipole geometry; application to design of hybrid ID pole, and to sextupole. "Exotic" non-dipoles are discussed in lecture #16.

#6;Dec.2.1988. Very detailed summary and re-formulation of 3D hybrid design procedure, and of design of non-dipole; details of hybrid ID pole design, and effect of changing the gap of hybrid ID on field distribution, viewed in dipole geometry; more on sextupole pole shape design; conformal mapping as a "thinking tool" (i.e. using the concepts without formulae): electrostatic extraction from the 88" cyclotron; solution to Dirichlet problem in a circle; mapping of interior of ideal multipole onto circular disk with Physics-information/understanding; flux between non-immediate-neighbor-poles of multipoles or hybrid ID is only symmetry dependent, not geometry dependent.

#7;Dec.21.1988. Field at edge of 2D CSEM without iron; simple way to evaluate/"see" value of LN((z0-z2)/(z0-z1)); design of Stanford Linear Collider arc magnets with POISSON in dipole geometry; POISSON-mesh; effect of saturation on field distribution in windowframe magnet: incorrect and correct analysis; Schwarz-Christoffel transformation: general recipe, removal of one corner from formula, and "arbitrary" placement of two other corners; application #1: field from dipole with zero pole width.

#8;Jan.6.1989. Relationship between curvature of V=const. and A=const. surfaces, and magnetic field properties. Rogowski surface derived from semi-infinite capacitor, and from first principles; proper and improper use of Rogowski contour. 2D needle with |E|=const. on tip. Analytical 2. order shim for semi-infinite dipole.

#9;Jan13.1989. S-C map of infinite array of ID poles. Excess flux and excess potential drop in Geometry 1 (G1). (An application is described in lecture #16). Excess flux in G2. Expansion of complex potential in G1 into exponentials.

#10;Jan19.1989. Taylor series T(T-S) manipulation algorithms: expansion coefficients for (1+a*z)^e, for a product of 2 T-S, for the inverse of a T-S, and when a T-S is used as a variable for another T-S, and for one T-S divided by another (given as homework, with solution in lecture #11). BASIC-program with these algorithms. Method to expand F' into exponentials when dz/dt cannot be integrated in closed form, with a program for G2.

#11;Febr.3.1989. Expansion of field errors in exponentials for finite width dipole. Summary of T-S-manipulation algorithms. S-C transformation of polygon onto circle. General 3D hybrid theory with many iron blocks.

Capacities; equivalent circuit diagram. Capacities for ID. "Invisible" flux.

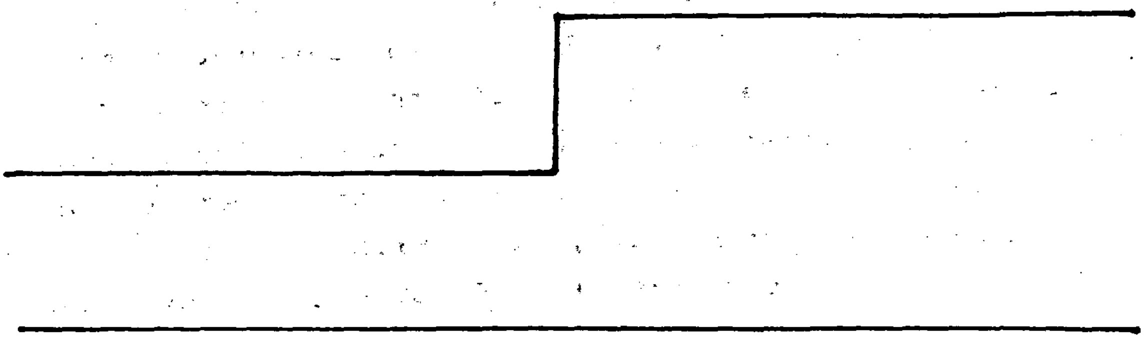
#12;Febr.10.1989. Design of entrance/exit excitation for straight (average) trajectories. Capacity between non-adjacent poles of ID, except for contribution in region close to midplane. Program for calculation of capacities of ID. A subtle point about ID capacities. Application of capacitor concept to a particle-spectrometer-like magnet. Propagation of errors/perturbations in a 2-capacitor-ladder network that describes an ID. Line integral errors due to gap error, easy axis orientation error, pole thicknes error, taking into account partial self-compensation of these errors.

#13;Febr.17.1989. Calculation of an integral needed for error assessment with information provided by POISSON. Capacity between non-adjacent poles close to midplane. CSEM-placement for a third order entry/exit system. Details about properties of symmetric/antisymmetric error fields. An ID that is antisymmetric with respect to midplane. Propagation of perturbation in a 3-capacitor model of an ID. Solution of the 2D equation of motion in Schwarz-Christoffel geometry.

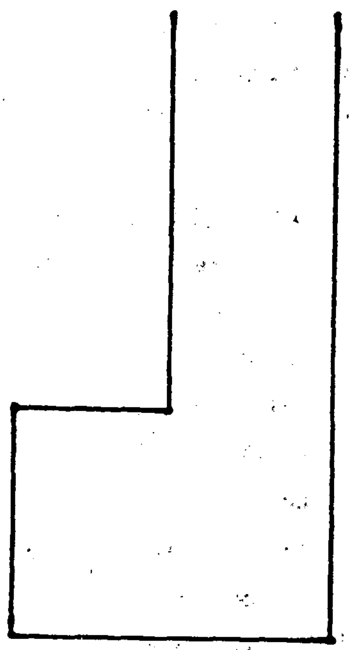
#14+15;March 3+10.1989. Line integral errors from easy axis orientation error in 3 side by side CSEM blocks. Analysis of device to measure easy axis orientation errors along one side of a CSEM block. Formulation of analysis of G3 with two different excitation patterns. Discussion of the following major details needed for analysis of G3: multidimensional secant equation solver; method to remove singularities from the limits of integrals to be evaluated numerically; some properties of constants entering into this problem, and using these properties to force smooth but firm bounds on the range of values these parameters can assume; derive formulae for calculation of flux and excess flux; procedure to do a Fourier expansion of the ID-fields. Line integral errors from gap between CSEM and pole, and CSEM blocks of different strengths. The Orthogonal Analog Model, with some applications.

#16;March 17.1989. Design of a very "exotic" 2D magnet in dipole geometry, with strong emphasis on difficulties and pitfalls that can occur. Application of the excess potential drop concept to the calculation of capacities of ID. Derivation of a closed expression for an integral, demonstrating some very important and useful mathematical techniques.

G1



G2



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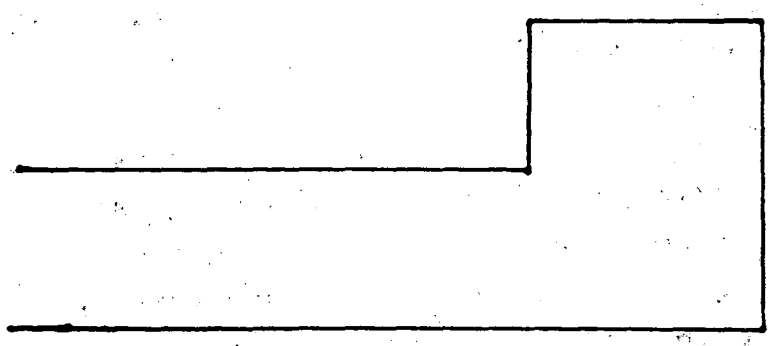
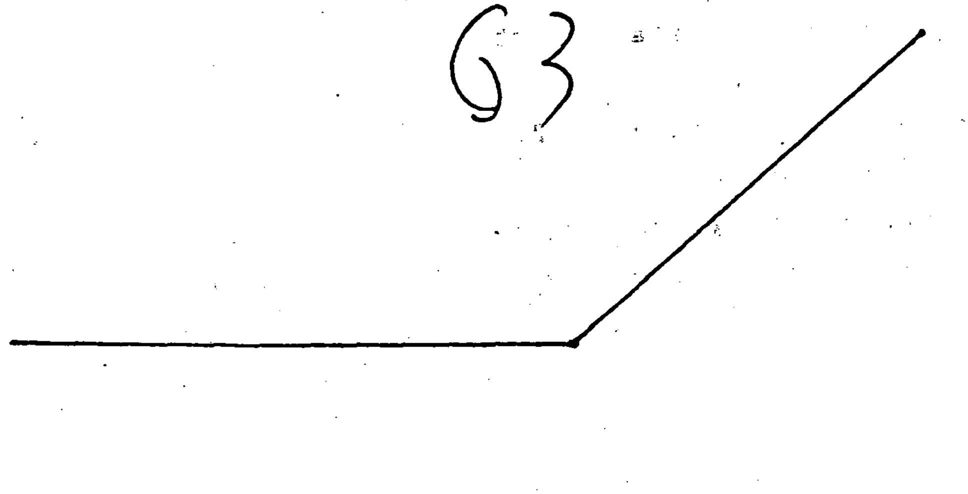
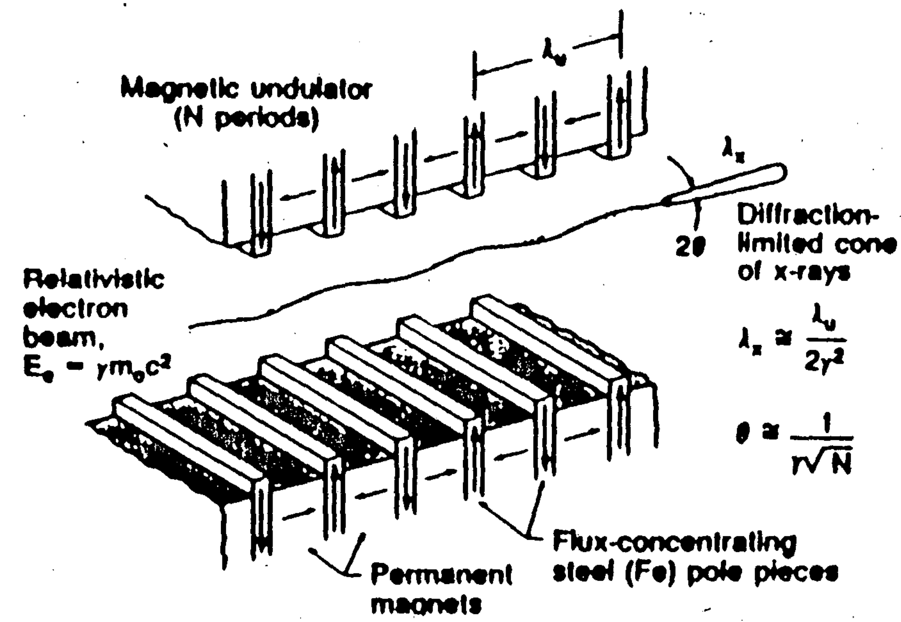


G3



Insertion Device Design

Klaus Halbach



Lecture 1.

October 21, 1988

6

(1.1)

ID - Design
K. Halbach.

A) "Maxwell".

$\oint \vec{H} \cdot d\vec{s} = \mathcal{I} = \int \vec{j} \cdot d\vec{a} \longleftrightarrow \text{curl } \vec{H} = \vec{j}$

(arrow under $\mathcal{I}$: A)

$V_{ind} = \oint \vec{E} \cdot d\vec{s} = -\int \dot{\vec{B}} \, d\vec{a} = -\dot{\Phi} \longleftrightarrow \text{curl } \vec{E} = -\dot{\vec{B}}$

(arrow under V_{ind}: V)

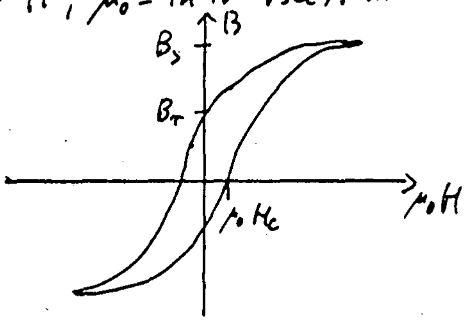
$\hookrightarrow \text{div } \vec{B} = (\rho =) 0$

Vacuum: $\vec{B} = \mu_0 \cdot \vec{H} = \tilde{\vec{H}}$; $\mu_0 = 4\pi \cdot 10^{-7}$ Vsec $A^{-1} m^{-1}$

$\vec{B} = \vec{B}(\vec{H})$

"isotropic" iron

not really isotropic.



Typical values: $B_s = 2T$

$B_r = 1T$

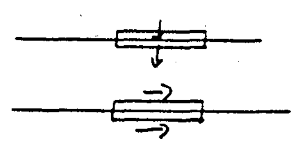
$\tilde{H}_c = 10^{-4} T$

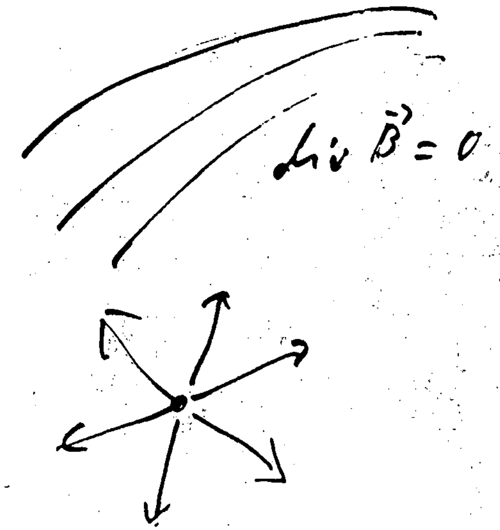
$B = \mu_0 \mu H$, μ of order 10^3 (can be as large as 10^5)

Continuity across interface

$\text{div } \vec{B} = 0 \rightarrow \Delta B_\perp = 0$

$\text{curl } \vec{H} = 0 \rightarrow \Delta H_\parallel = 0$





1.1a

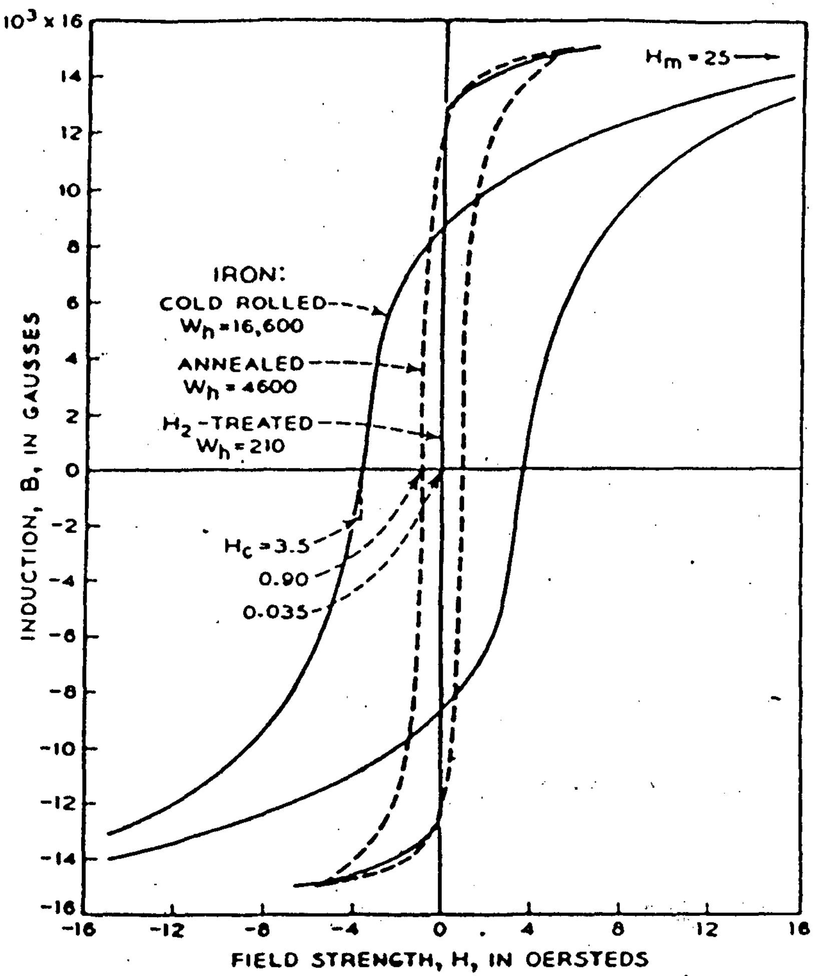
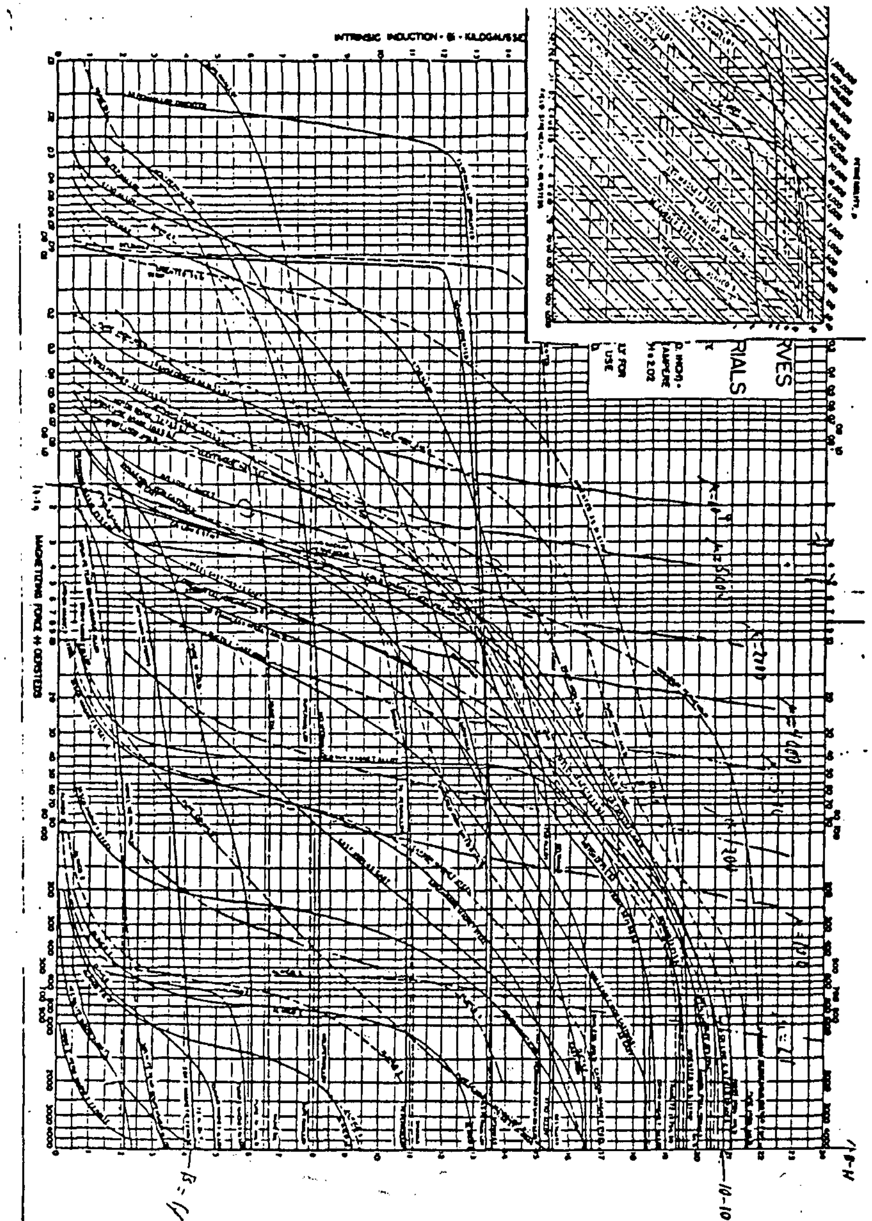


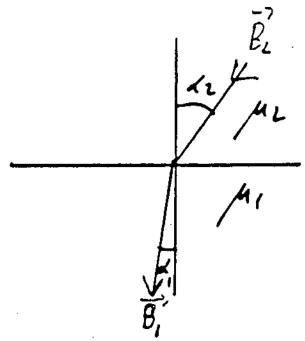
Fig. 11–28. Effect of treatment of specimen on the hysteresis of ir

W_h = 16 600 for B_m = 15 000. After annealing in the usua



(1.2)

"Isotropic" Medium



$\operatorname{tg}\alpha_2 / \operatorname{tg}\alpha_1 = \mu_2/\mu_1$

$\mu_1/\mu_2 = 0 \rightarrow \alpha_1 = 0$

PM-material later.

$\vec{j}=0$; $\frac{\partial}{\partial t}=0$; $\rightarrow$ curl $\vec{H}=0$; div $\vec{B}=0$; $\vec{B}=\vec{B}(\vec{H})$

1) $\vec{H} = -\text{grad}\, V \rightarrow \text{curl}\, \vec{H} \equiv 0$

$\vec{B} = \mu_0 \vec{H}$: div $\vec{B} = 0 \rightarrow$ div grad $V = \nabla^2 V = 0$ ← Laplace equ.

$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

$H_x = -\partial V/\partial x \Longrightarrow \nabla^2 H_{x,y,z} = 0$; $(\nabla^2 H_r \neq 0 !!)$

↑ no max, min, inside volume, max, min always on surface!!

$H_{x,ideal}$; $H_{x,real}$; $\rightarrow \Delta H_{x\,error}$ satisfy Laplace equ.

Specify, measure, e.t.c. fields on surface of volume of interest!!

(1.3)

In vacuum

$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = 0$; $\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \underbrace{\frac{\partial H_z}{\partial z}}_{=0 \text{ in 2D case}} = 0$

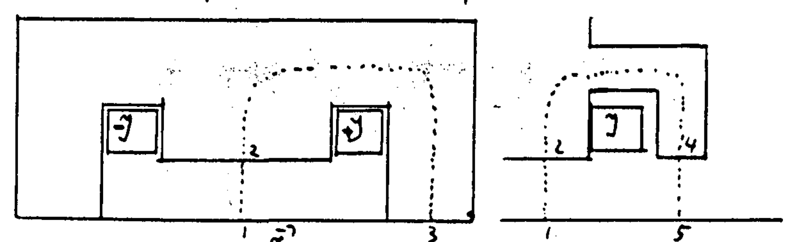
$\int_{z_1}^{z_2} H_{x,y,z}(x,y,z)\, dz = \mathcal{H}_{x,y,z}(x,y)$

$\frac{\partial \mathcal{H}_y}{\partial x} - \frac{\partial \mathcal{H}_x}{\partial y} = 0$;

$\frac{\partial \mathcal{H}_x}{\partial x} + \frac{\partial \mathcal{H}_y}{\partial y} = H_z(x,y,z_1) - H_z(x,y,z_2)$

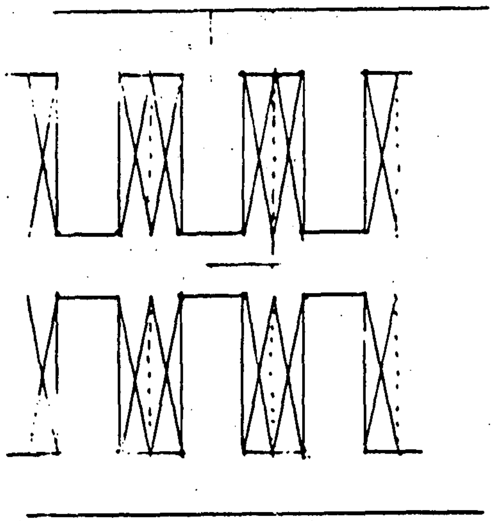
If $H_z(x,y,z_1) = H_z(x,y,z_2)$, $\mathcal{H}_x, \mathcal{H}_y$ obey 2D diff. equ's!!!

Problem with V: often, there are, somewhere, currents in system.



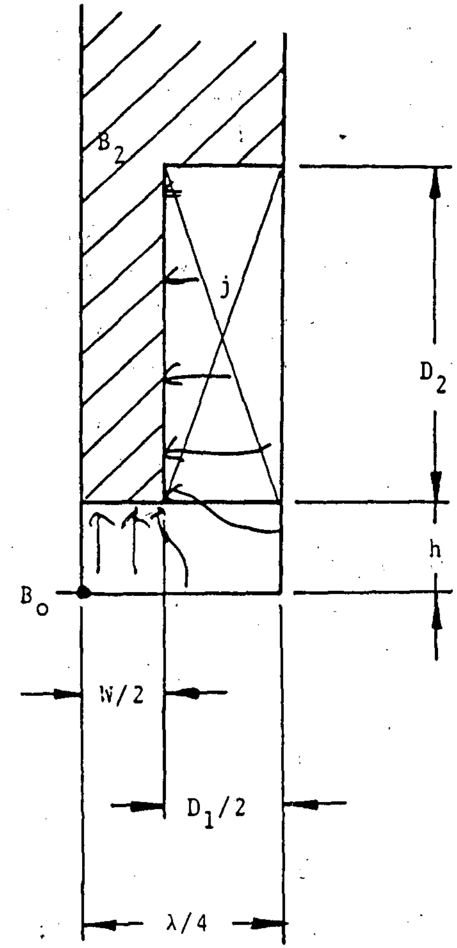
$\Delta V = V_n - V_m = \int_{\vec{r}_n}^{\vec{r}_m} \vec{H}\cdot d\vec{s}$ requires definition of path!

λ/4 section of em U/W



$$\overline{H} \cdot h = j \cdot D_2 \cdot D_1/2 - \int_{iron} \vec{H} \cdot d\vec{s}$$

$$D_2 = \frac{\overline{H}}{j} \cdot \frac{2h}{D_1}$$



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(1.5)

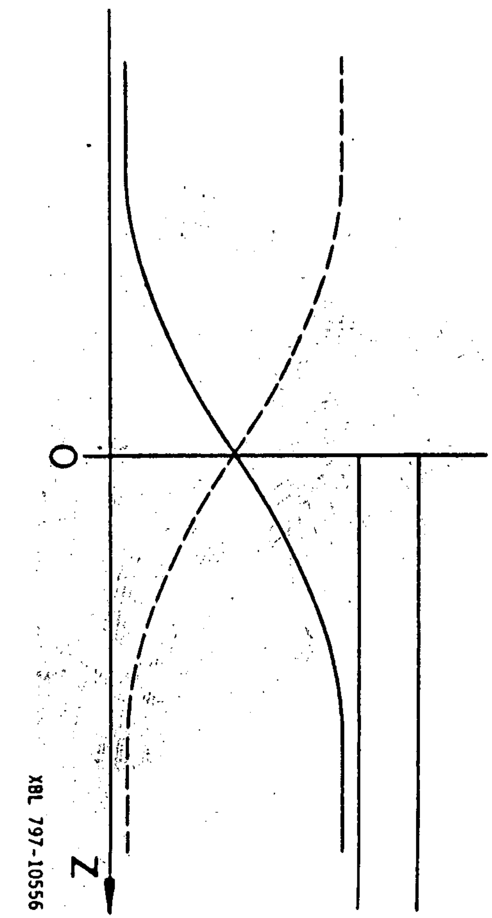
2 D QUADRUPOLE FIELD

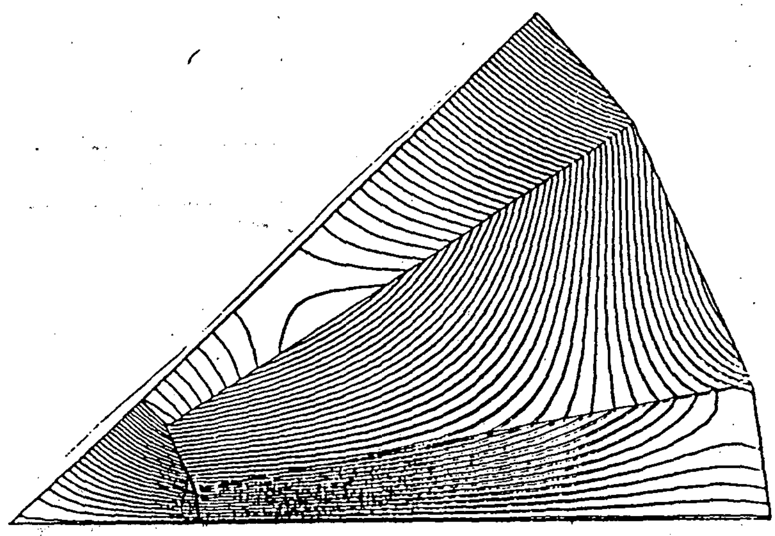
$$B_x - i\,B_y = B_r \cdot \frac{X+iY}{r_1} \cdot 2 \cdot \left(1 - \frac{r_1}{r_2}\right) \cdot \frac{\sin(2\pi/M)}{2\pi/M} \cdot \cos^2(\pi/M)$$

Possible Harmonics: $n = 2 + \nu \cdot M;\ \nu = 0, (1), 2 \ldots$

2 D dipole

$$B = B_r \cdot \ln(r_2/r_1) \cdot \frac{\sin 2\pi/M}{2\pi/M}$$





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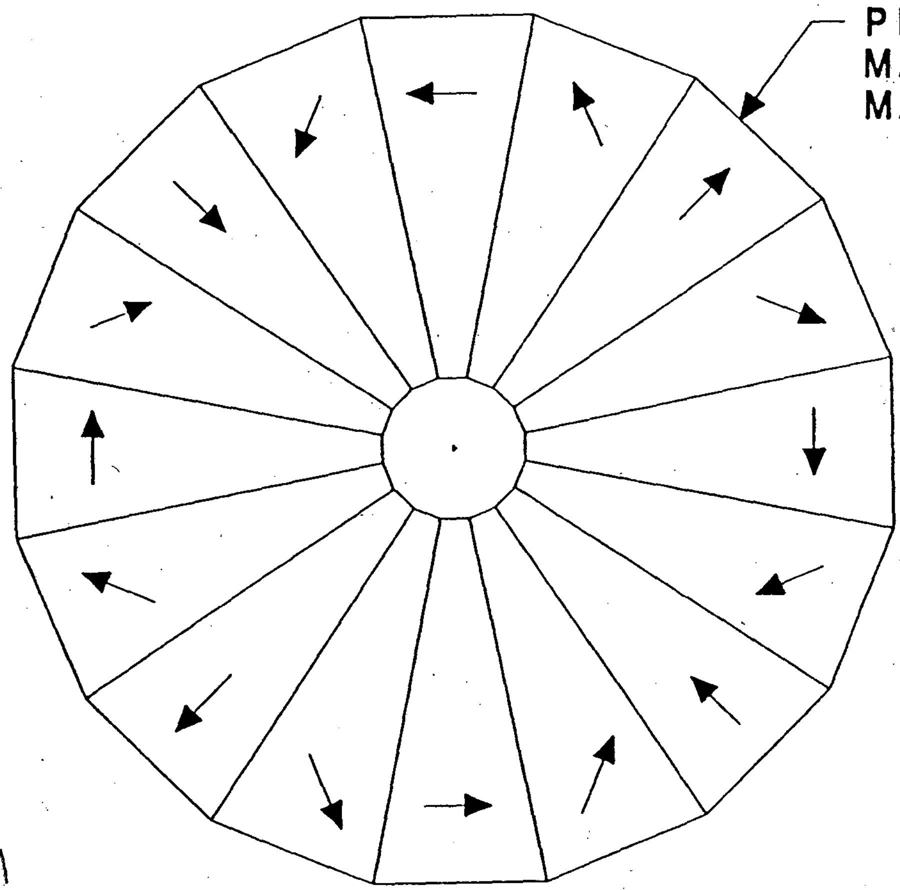
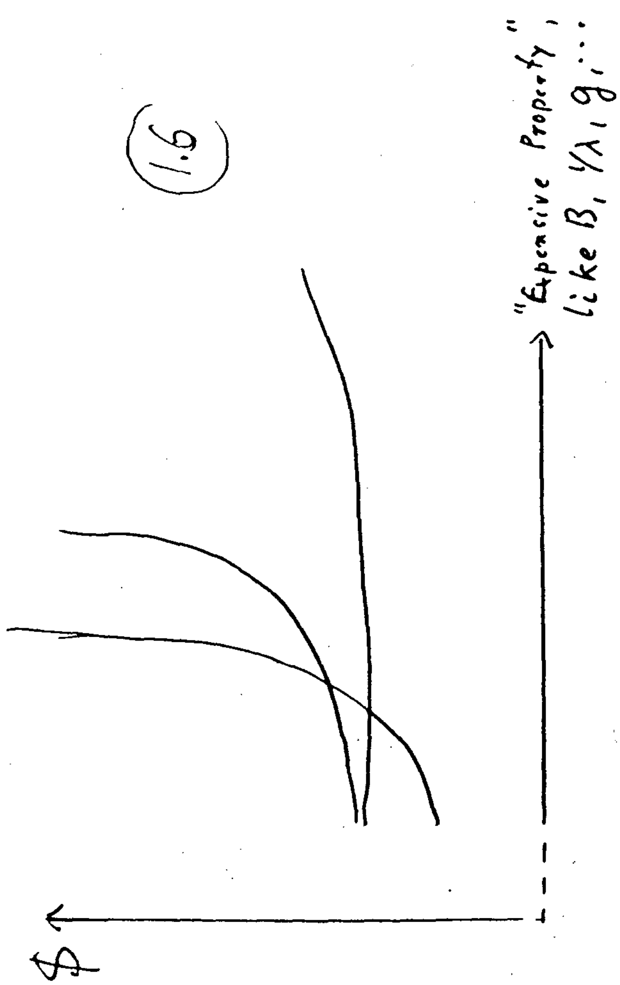


ADVANTAGES OF PM SYSTEMS

- Strongest fields when small
- Compact
- Immersible in other fields
- "Analytical" material
- No power supplies
- No cooling
- No power bill

} • Reliability • Convenience

(1.6)



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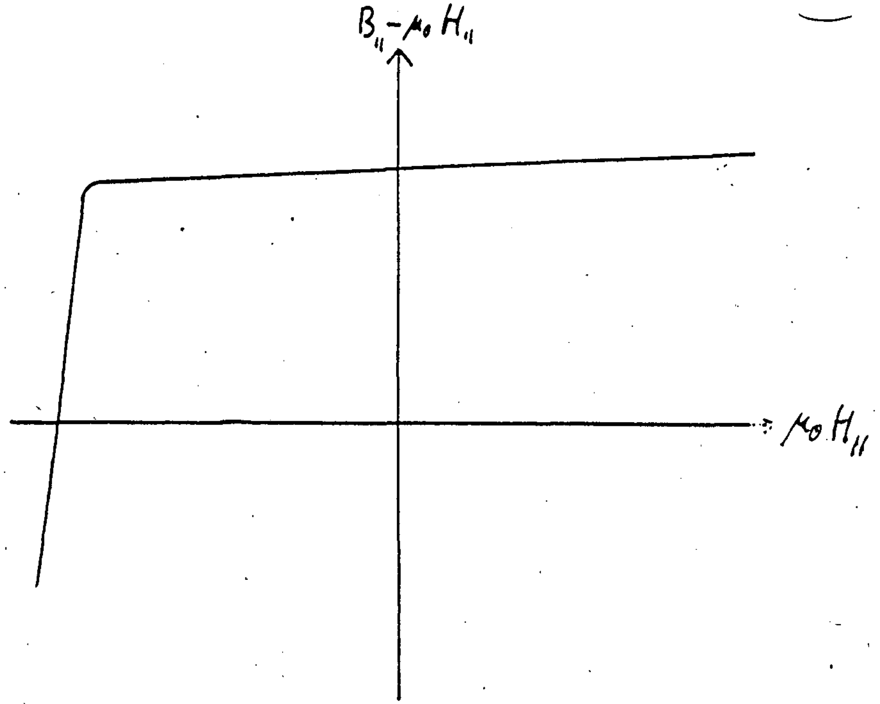
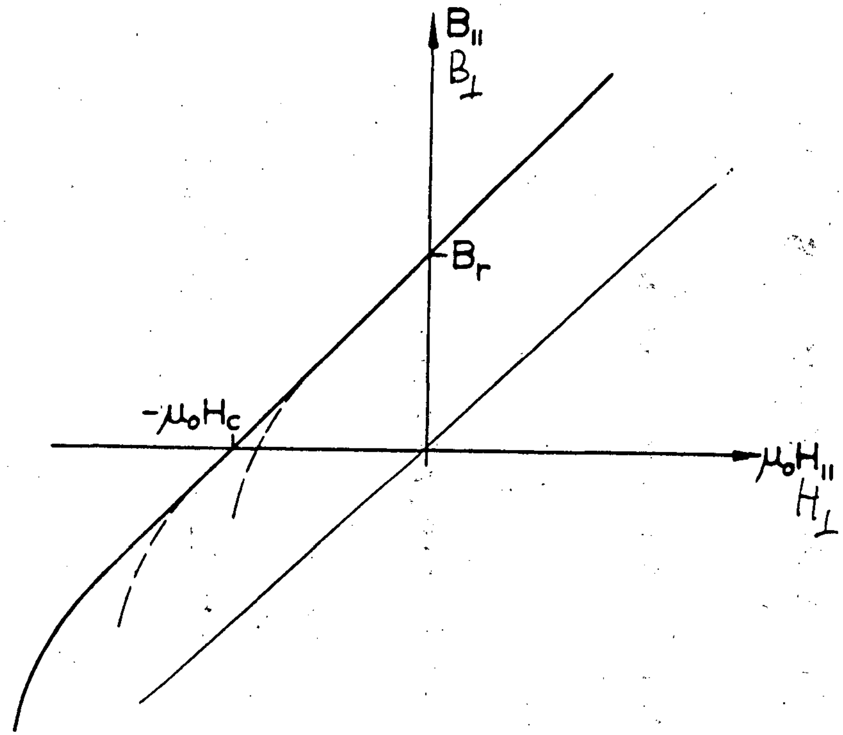


(8)

(1.9)



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(1.10)

$$B_{\parallel} = \mu_0 \mu_{\parallel} H_{\parallel} + B_r, \quad B_{\perp} = \mu_0 \mu_{\perp} H_{\perp} \quad \Big\} \quad \vec{B} = \mu_0 \hat{\mu} \times \vec{H} + \vec{B}_r, \quad \vec{H} = \hat{\gamma} \times \vec{B} - \vec{H}_c$$

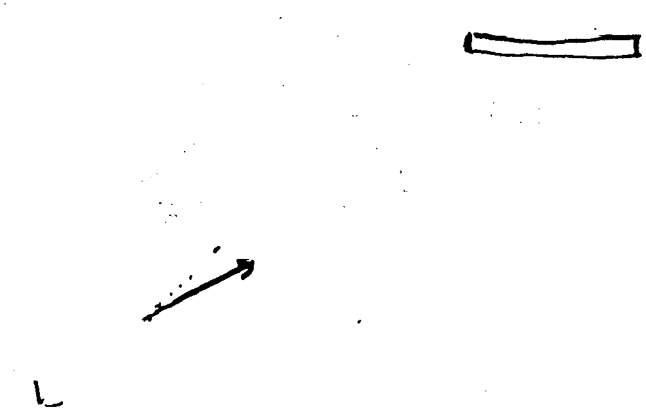
this $\rightarrow$ into $\quad \text{curl}\, \vec{H} = 0 \quad$ or $\quad \text{div}\, \vec{B} = 0$:

$$\text{curl}\,(\hat{\gamma} \times \vec{B}) = \text{curl}\, \vec{H}_c = \vec{j}_{eq}, \quad \text{div}\, \vec{B} = 0 \quad \Big\| \quad \text{or} \quad \text{div}\,(\mu_0 \hat{\mu} \times \vec{H}) = -\text{div}\, \vec{B}_r = \rho_{eq}, \quad \text{curl}\, \vec{H} = 0$$

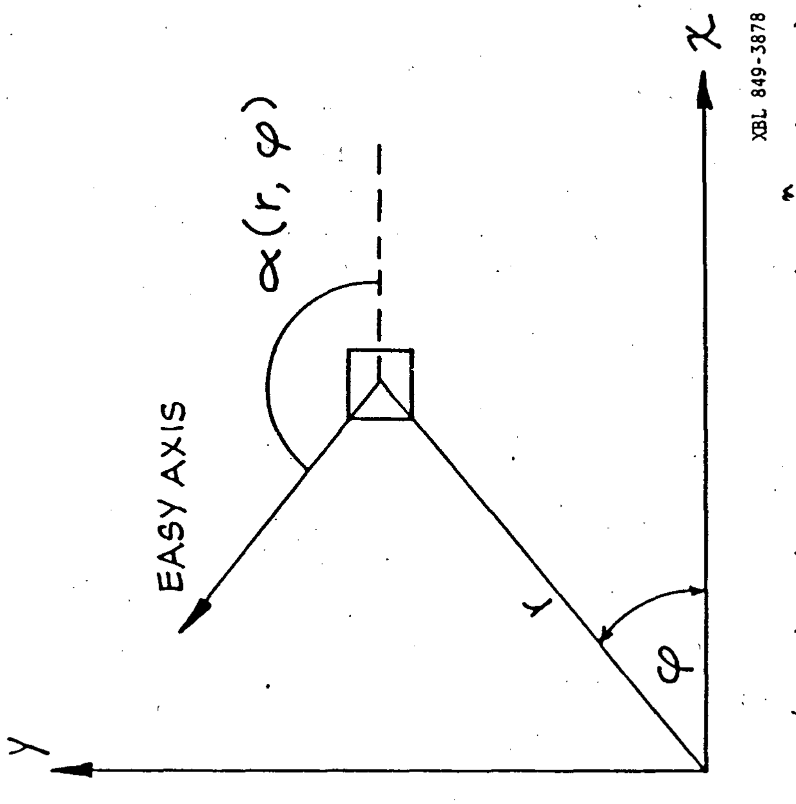
This represents passive material $(\hat{\gamma}, \hat{\mu})$ with active terms/properties $(\vec{j}_{eq}, \rho_{eq})$.

Homogeneous magnetization:

$\vec{j}_{eq} \rightarrow$ current sheet $\searrow$
$\rho_{eq} \rightarrow$ charge sheet $\nearrow$ CSEM



(1.12)

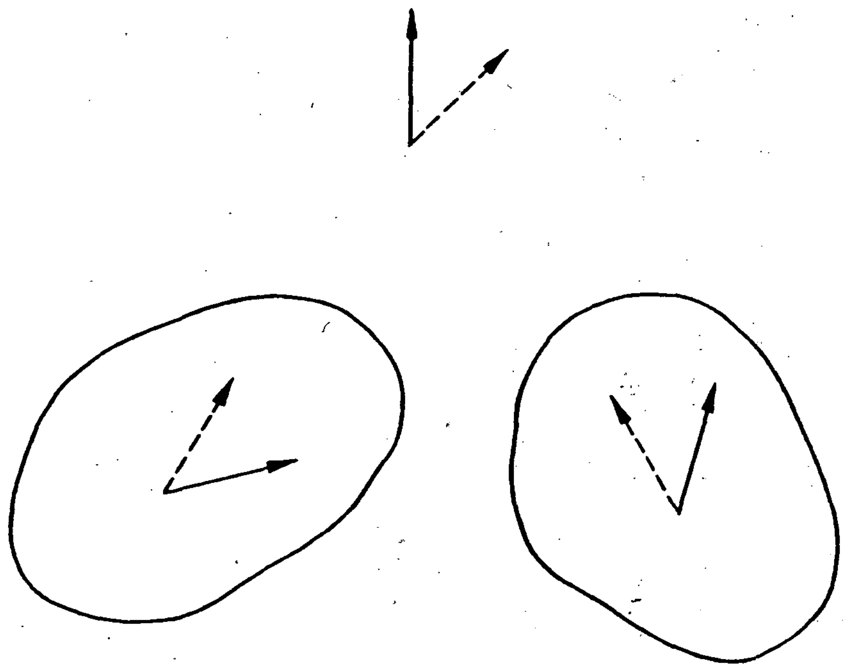


$\chi(r,\varphi) = (n+1)\cdot\varphi$ for $V \sim r^n \sin(n\varphi+\beta)$

Figure 2

(10)

2D; no te



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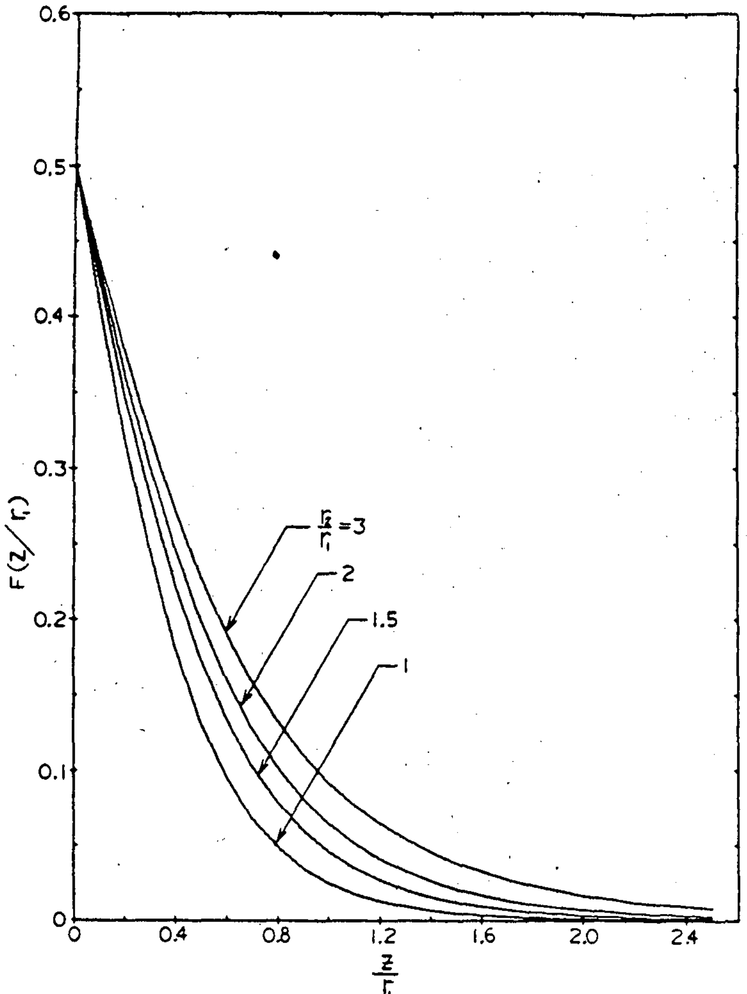
(1.17)



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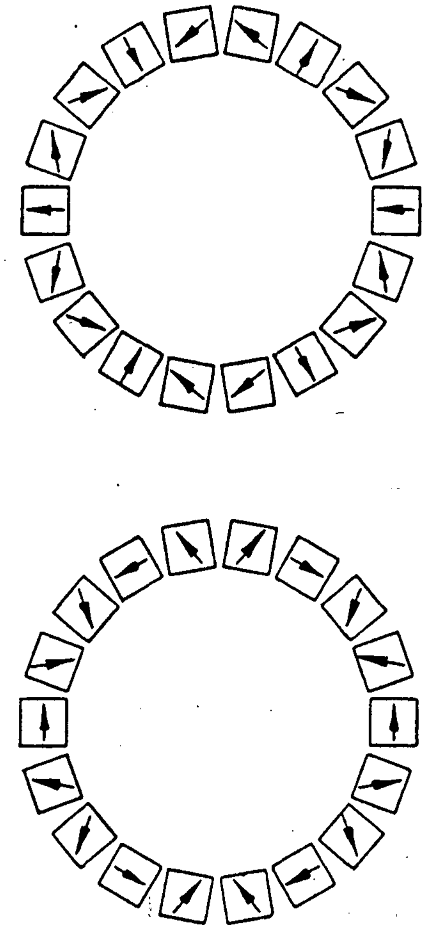
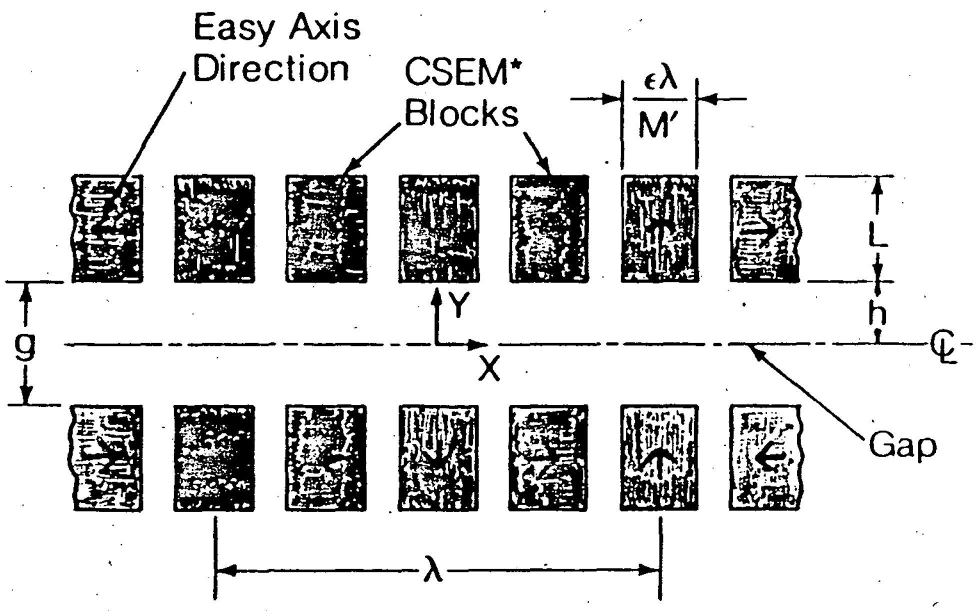
(1.16)





PURE CSEM* W / U
CROSS SECTION

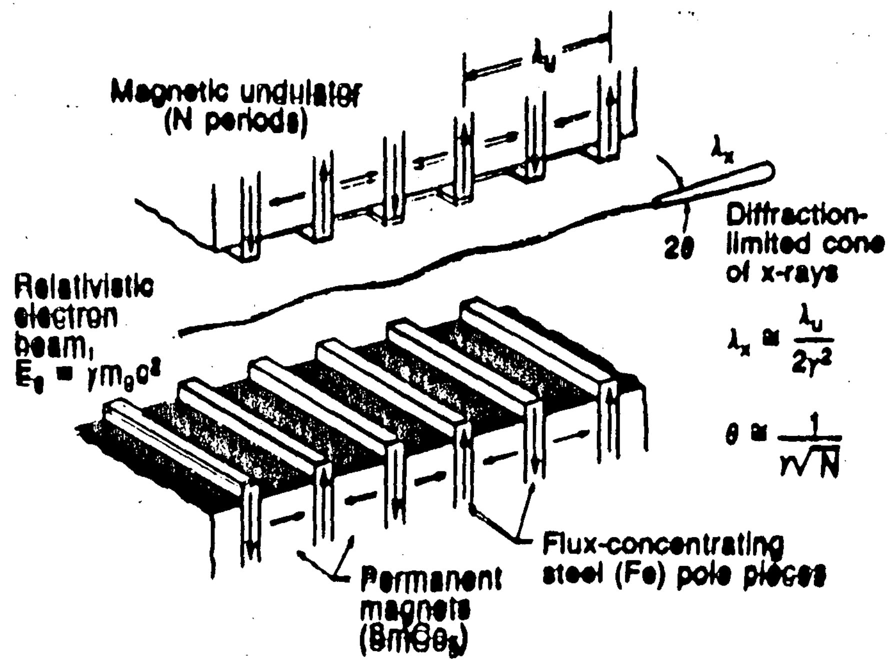
*Current Sheet Equivalent Material - e.g. REC

Insertion Device Design

Klaus Halbach

Lecture 2.

October 28, 1988



Literature

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- McCaig: Permanent Magnets in Theory and Prac[tice]
  John Wiley + S., 1977
- NIM 169, 1 (1980) (Theory, no iron)
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- JAP 57, 3605 (1985) (Review)
- Proc. 1986 Linac Conf. (Review)
- Specialty Magnets, Proc. 1985 US Acc. Sch. (LBL 21945)

Summary of lecture #1, 10/21/88

$\int \vec{H}\cdot d\vec{s} = \int \vec{j}\cdot d\vec{a} = I \longleftrightarrow \text{curl}\,\vec{H} = \vec{j}$

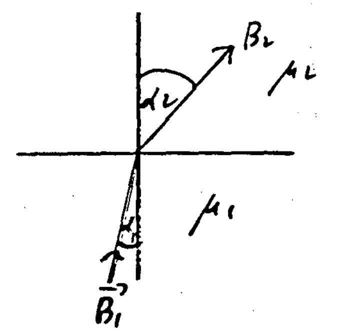
$V_{ind} = \oint \vec{E}\cdot d\vec{s} = -\dot{\phi};\ \phi = \int \vec{B}\cdot d\vec{a} \longleftrightarrow \text{curl}\,\vec{E} = -\dot{\vec{B}}$

$\text{div}\,\vec{B} = \rho = 0$

Continuity: $\Delta B_\perp = 0;\ \Delta H_\parallel = 0$

$\vec{B} = \vec{B}(\vec{H})$; soft iron: $\mu_0 H_c = -1G$; $\vec{B} = \mu_0 \mu \vec{H}$

μ of order $10^3 - 10^5$



$B_2 \cos\alpha_2 = B_1 \cos\alpha_1$
$H_2 \sin\alpha_2 = H_1 \sin\alpha_1$
$\tan\alpha_2/\mu_2 = \tan\alpha_1/\mu_1$

} For isotropic medium.

$\vec{j} = 0$: can use $\vec{H} = -\text{grad}\,V$; vacuum: $\nabla^2 V = 0$; $\nabla^2 H_{x,y,z} = 0$

but: V not single valued if $\vec{j} \neq 0$ somewhere in system, because $\oint \vec{H}\cdot d\vec{s} = -\Delta V = I$

Because of limits on j, B_{sat}, for small devices PM-systems give more fields than EM systems.

Over large range of $H_{\parallel}$

4 ways to describe CSEM:

$$\left\{\begin{array}{l} B_{\parallel} - \mu_0 H_{\parallel} \approx \text{const} = B_r \; (.8\text{-}1.2\,T) \\ B_{\parallel} = \mu_0 H_{\parallel} + B_r \\ \text{or: vacuum + either } \vec{j}_{eq} = \text{curl } \vec{H}_c \\ \qquad\qquad \text{or } \; \rho_{eq} = -\text{div } \vec{B}_r \end{array}\right\} \quad B_{\perp} \approx \mu_0 H_{\perp}$$

For homogeneously magnetized material

$\vec{j}_{eq}$ = current sheet; ρ_{eq} = charge sheet

Application of ↑: "normal" solenoid = homogeneous field inside, no field outside, + fields from charge sheets at end.

Easy axis rotation theorem (only for 2D, no iron)

Basic CSEM system optimization: determine optimum easy axis orientation everywhere.

Iron-free CSEM quad, sextupole, undulator.

End of summary, except for illustration graphs

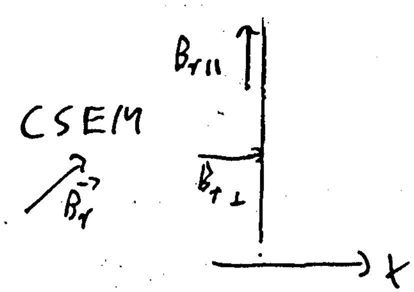
$j = 0$ everywhere:

$$\int \vec{B}\cdot\vec{H}\,dv = -\int \vec{B}\cdot \text{grad}V\,dv = -\int \text{div}\,V\vec{B}\,dv = -\int V\vec{B}\cdot d\vec{a} = 0$$

$$\text{div}\,V\vec{B} = \vec{B}\cdot\text{grad}V + V\,\text{div}\,\vec{B}$$

$$\int \vec{B}\cdot\vec{H}\,dv = \int_{vac} + \underbrace{\int_{iron}}_{\text{very small compared to } \int_{vac}} + \int_{CSEM} = 0$$

$$\left(\int \vec{B}\cdot\vec{H}\,dv\right)_{vac} = -\left(\int \vec{B}\cdot\vec{H}\,dv\right)_{CSEM}$$



$$q = -\int \text{div}\,\vec{B}_r\,dv = -\int \text{div}\,\vec{B}_r\,dy\,dz\,dx$$

$$q = -a\,B_{r\perp}\Big|_{\ell}^{r} = a\cdot B_{r\perp} = a\cdot\sigma$$

↑ charge density on surface

21

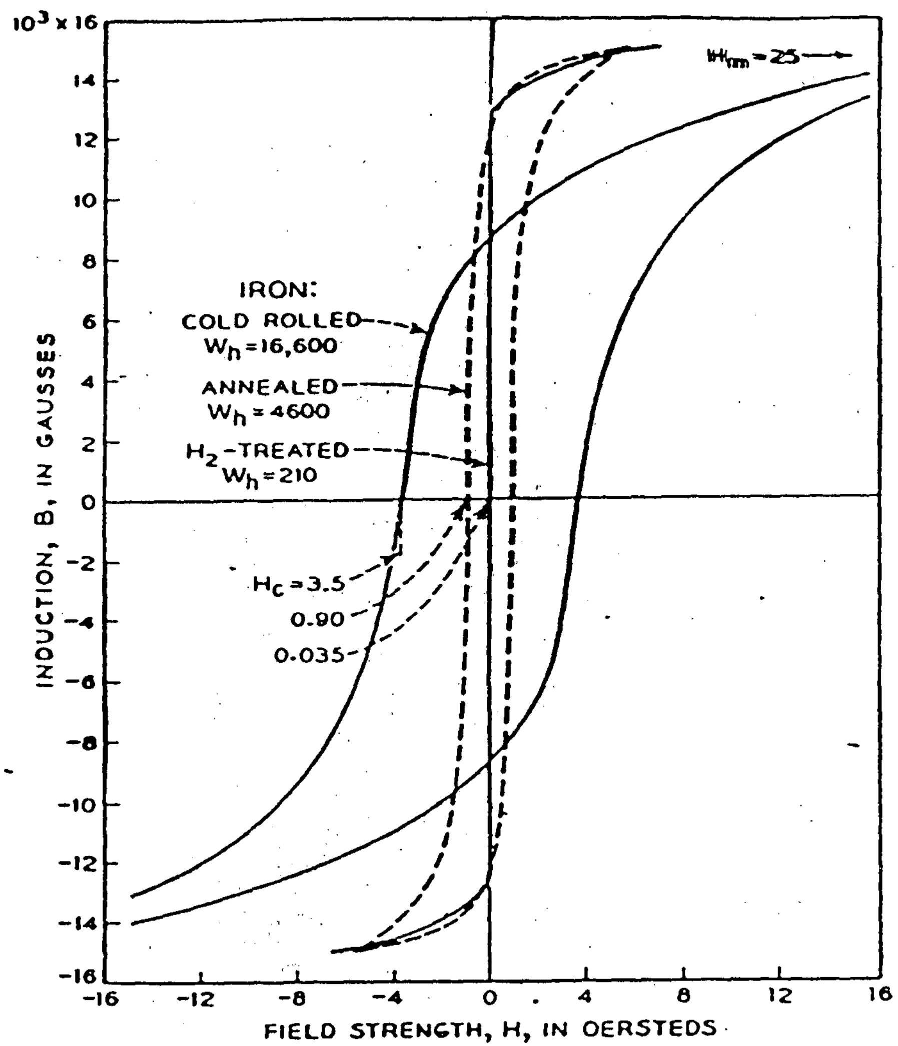
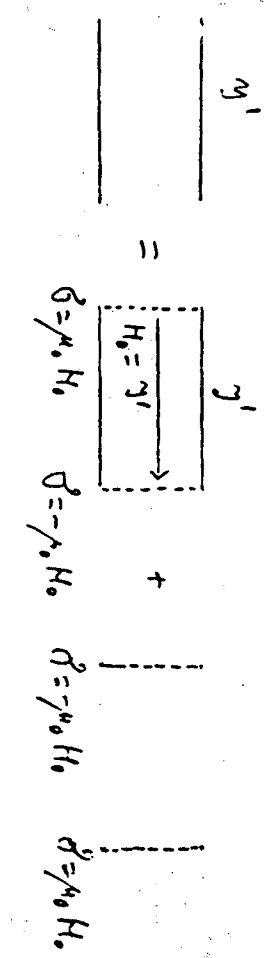
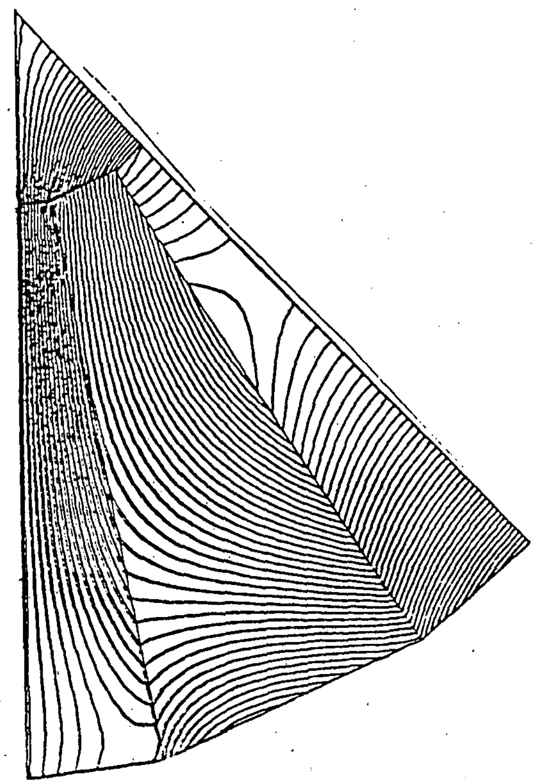


FIG. 11–28. Effect of treatment of specimen on the hysteresis of i

W_h = 16 600 for B_m = 15 000. After annealing in the usu





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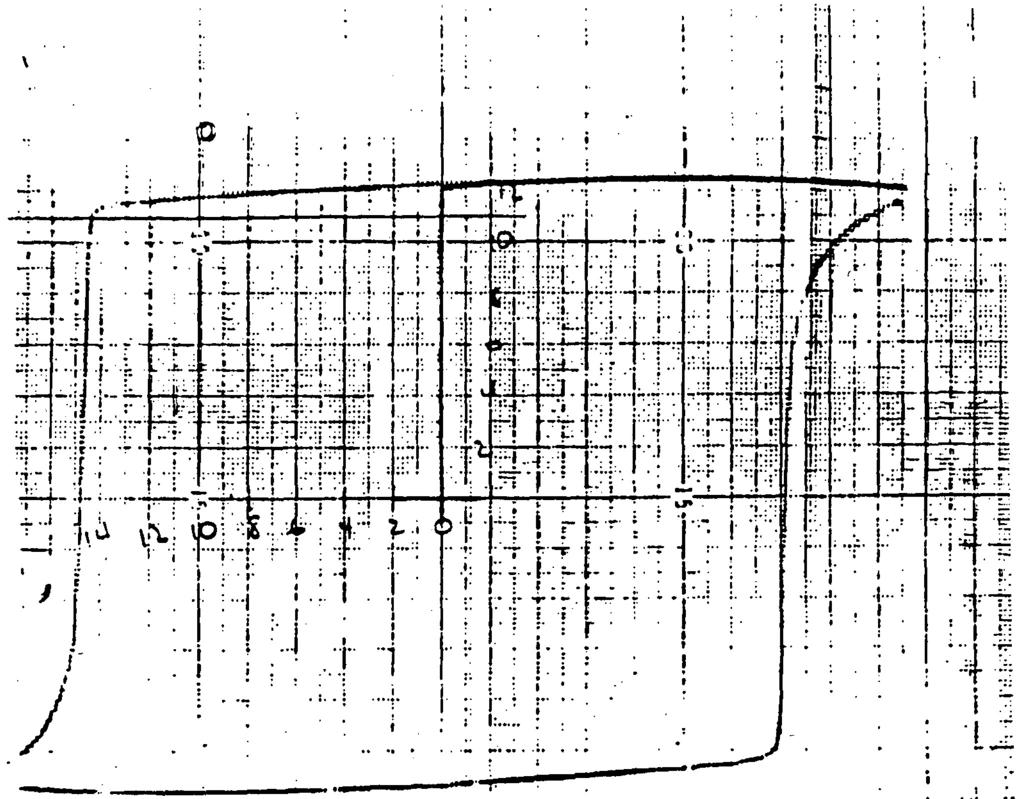


A-1

Br	12300 G
iHc	14750 Oe
bHc	11700 Oe
(B·H)max	35.8 MGOe
iHk	14400 Oe

JAN 6 1986

Shin-Etsu Chemical Co.,Ltd.

Last of illustration graphs for summary

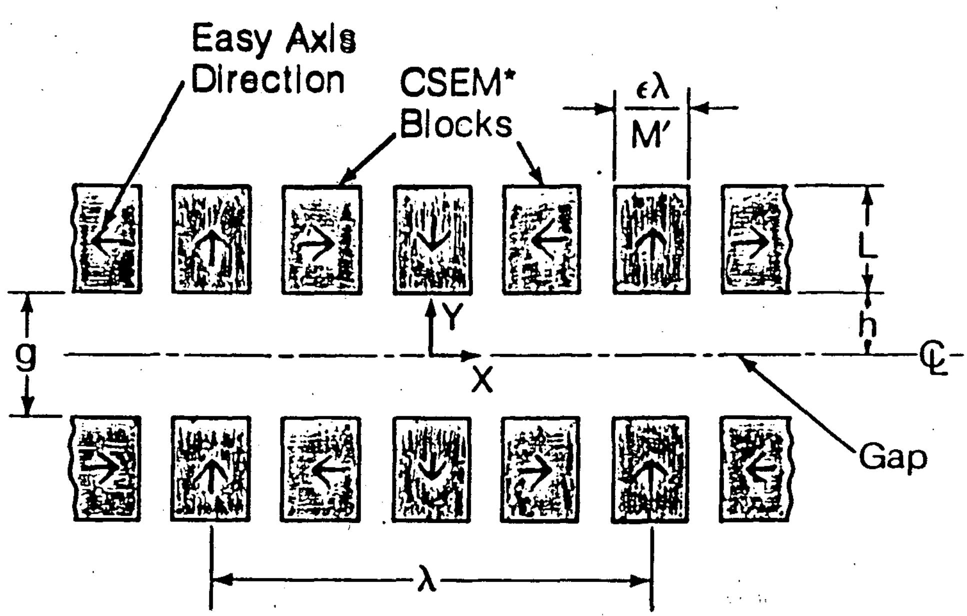
PURE CSEM CONFIGURATION PERFORMANCE

$$B^* = i \cdot 2 \cdot B_r \sum_{\mu=0} \cos(n\,k\,z) \cdot e^{-nkh} \cdot \frac{\sin(n\,\epsilon\,\pi/M')}{(n\pi/M')} \cdot (1 - e^{-nkL})$$

$n = 1 + \mu M'$
$k = 2\pi/\lambda$
$z = x + iy$
$B^* = B_x - i B_y$

Example:
for: $L = \lambda/2$
$M' = 4$
$B_r = 0.9$ Teslas (REC)

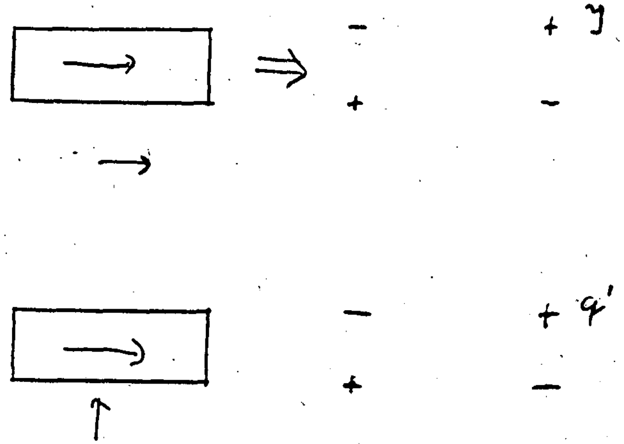
$$B^*_{\mu=0}(\text{Teslas}) = i \cdot 1.55 e^{-kh} \cdot \cos(kz)$$



PURE CSEM* W / U CROSS SECTION

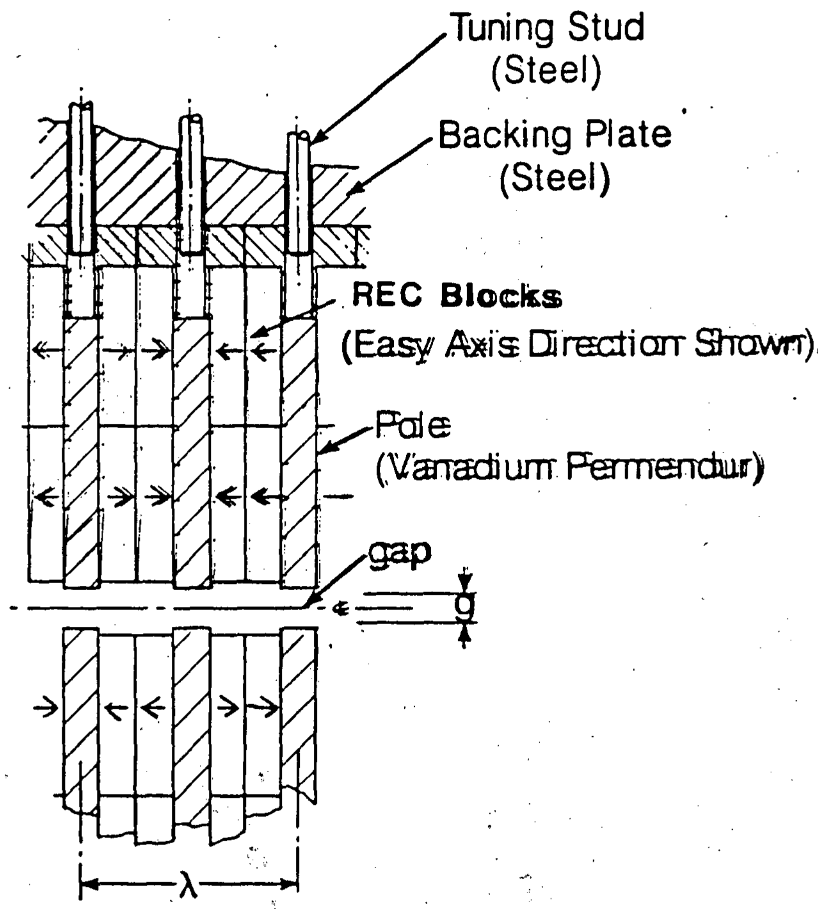
*Current Sheet Equivalent Material - e.g. REC

Effect of movement of CSEM block

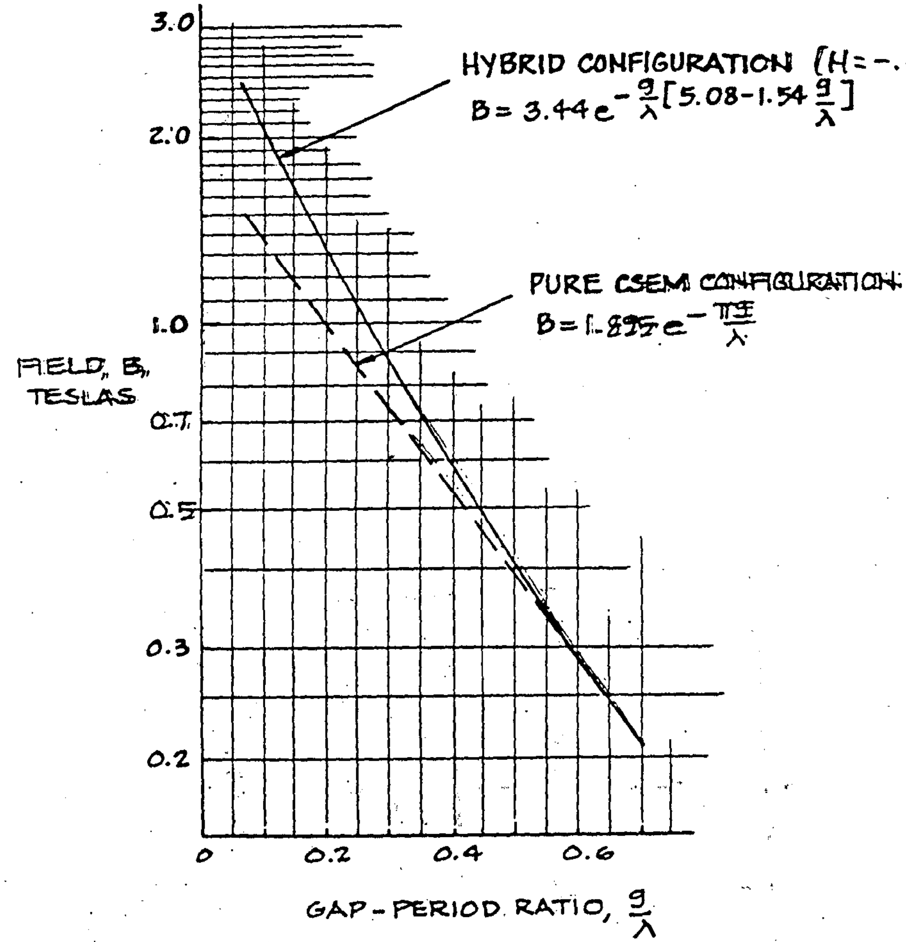


Same representation of perturbation effect can be used for <u>em</u> wigglers, i.e. ELF-W and SC-W!!

Hybrid Insertion Device configuration with field tuning capability.



PURE CSEM AND HYBRID UNDULATOR / WIGGLER PERFORMANCE FOR NdFe (B_r = 1.1 TESLAS)



Focusing

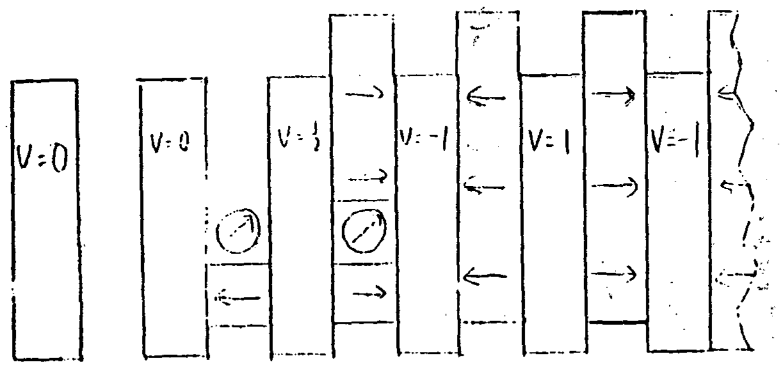
1) Curved poles

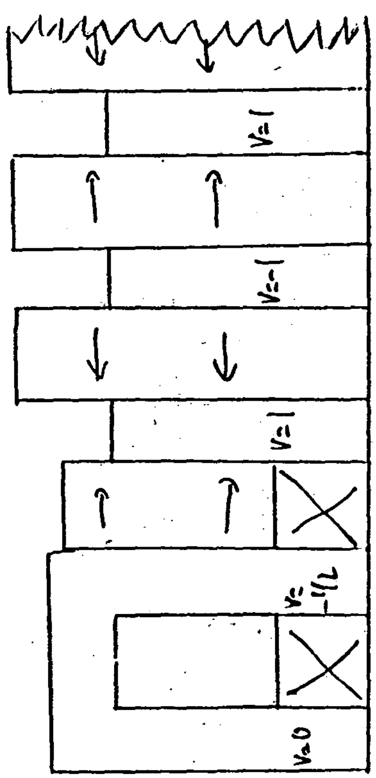
2) Superimposed quadrupole field

2.1) Imbed iron free U in a quadrupole

2.2) Canted poles

2.3) Quad windings inside U (possible even in Hybrid U!)


V=0
V=0
V=½
V=-1
V=1
V=-1

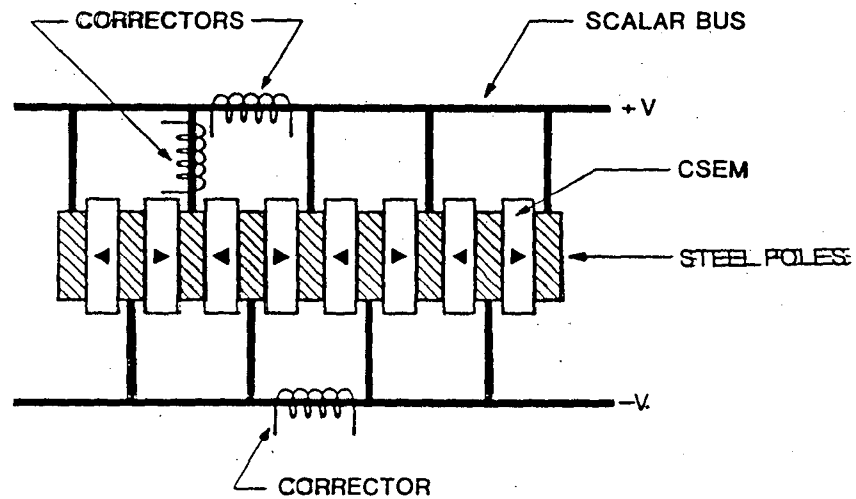

V=1
V=-1
V=1
V=-1/2
V=0

Shield against environmental fields
(Earth's field, crane, magnets, power supplies e.t.c)

ΔB ∥ midplane, ∥ traj. → "no effect"

ΔB ∥ midplane, ⊥ traj → "not possible" in hybrid (v. steering)

ΔB ⊥ midplane → displacement → "harmless"
→ steering → damaging..



XBL 858-3712

Excitation Errors

V- bus

Measure, sort, assign PM blocks

$\int \Delta B(z)\,dz = 0 \rightarrow$ no steering.

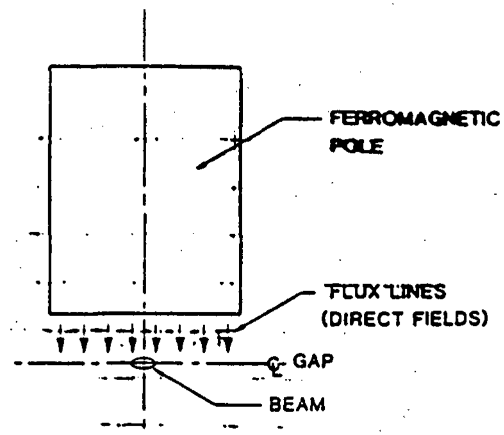
Gap Errors

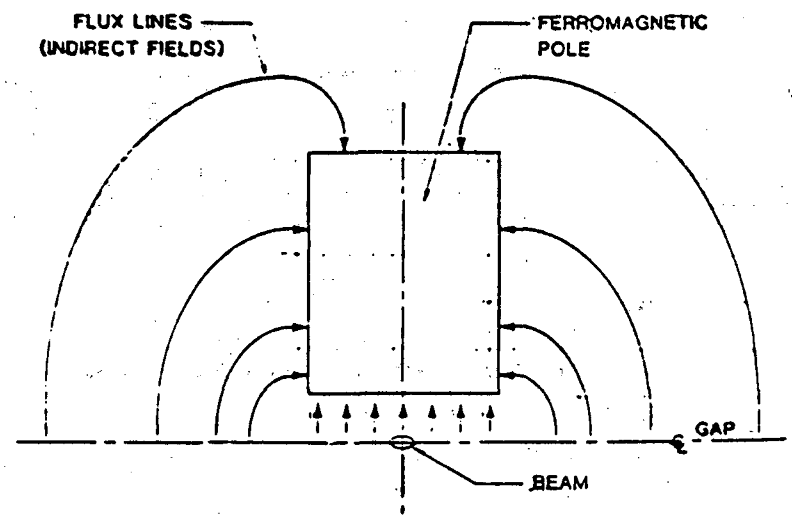
$\Delta B(z) =$ even $\rightarrow \left|\int \Delta B(z)\,dz\right| > 0$ with V bus.

$\left|\int \Delta B(z)\,dz\right| > 0$ without V-bus only because of 3D effects!

Iron properties

$\mu >> 1 \rightarrow$ iron properties "immaterial"





XBL 858 3716

Fig. 5

Easy Axis Orientation Error

$\Delta B(z) = \text{even}$

Important only close to midplane.

$\left|\int \Delta B(z)\,dz\right| > 0$ only because of 3D effects.

Measure orientation, correct block before assembly with grinder.

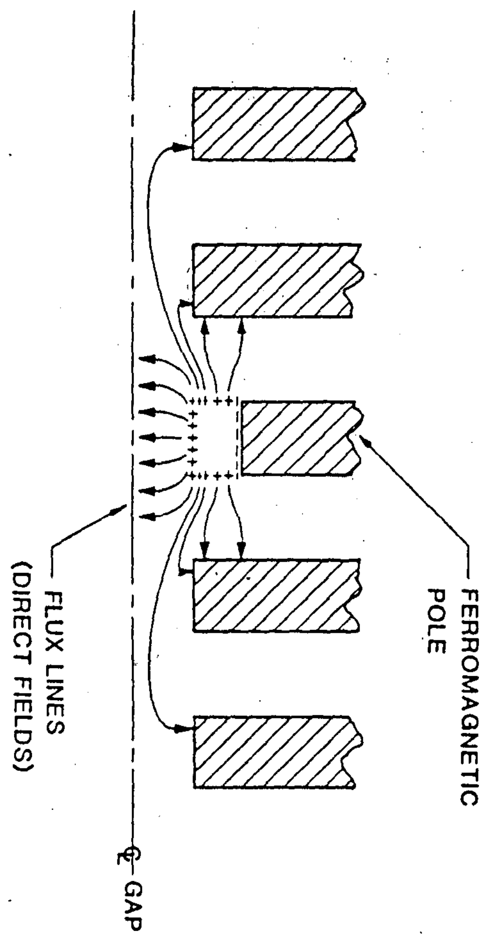


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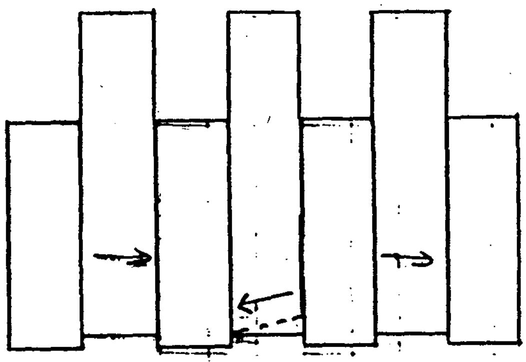
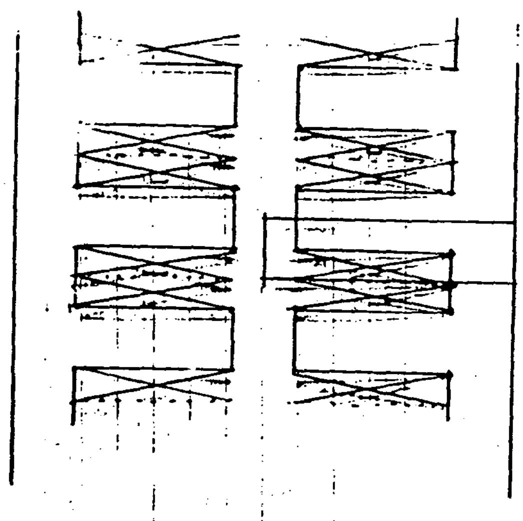
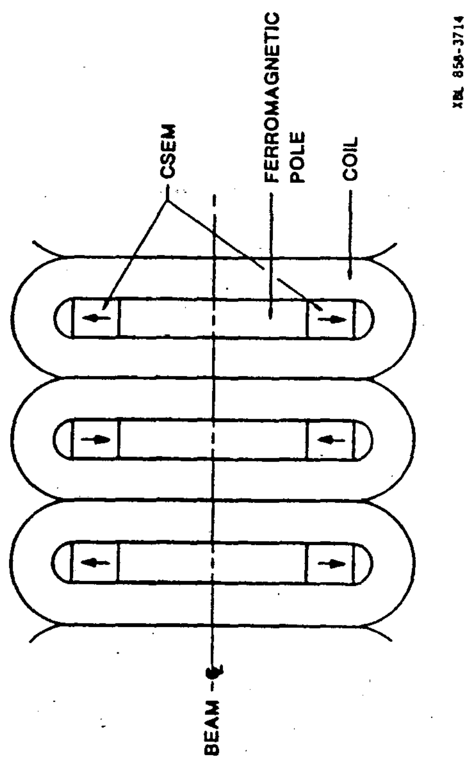
Fig. 4

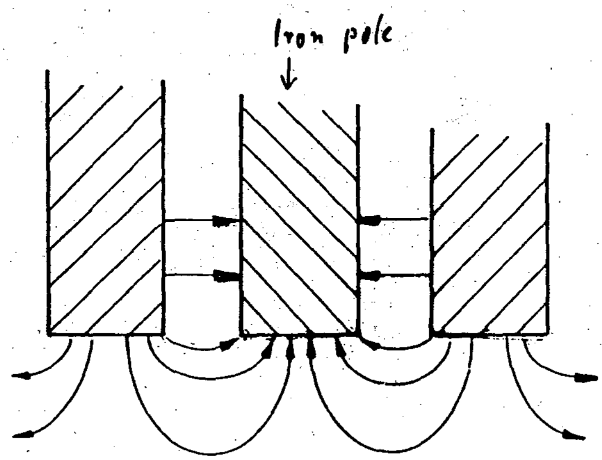




Plan view of PM assisted em U/W



Plan view of iron poles of U/W



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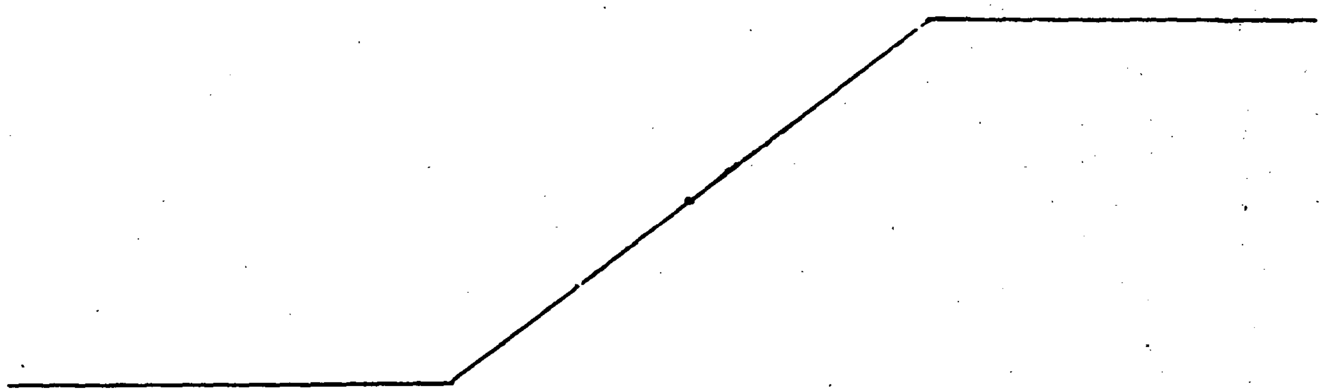
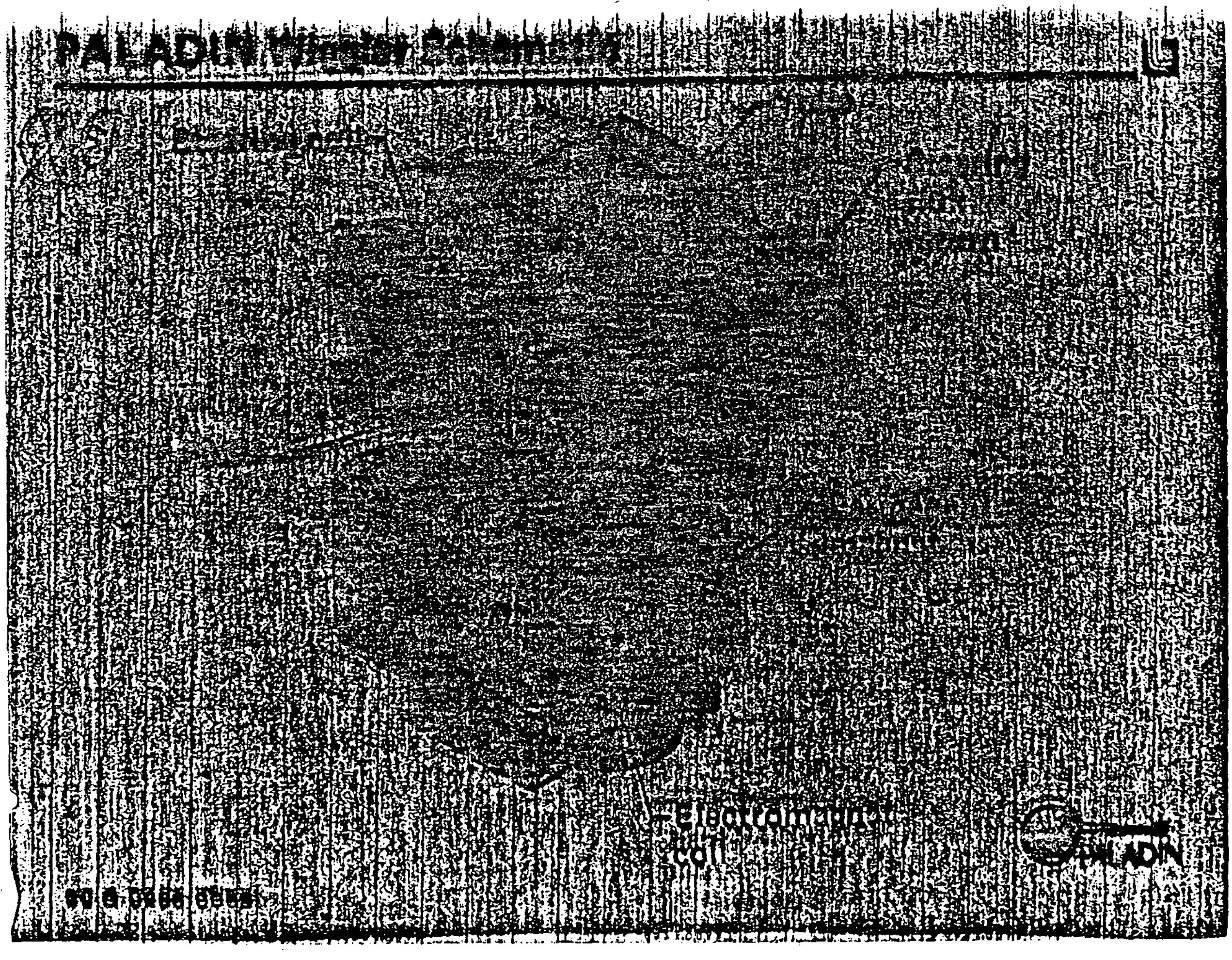



PALADIN

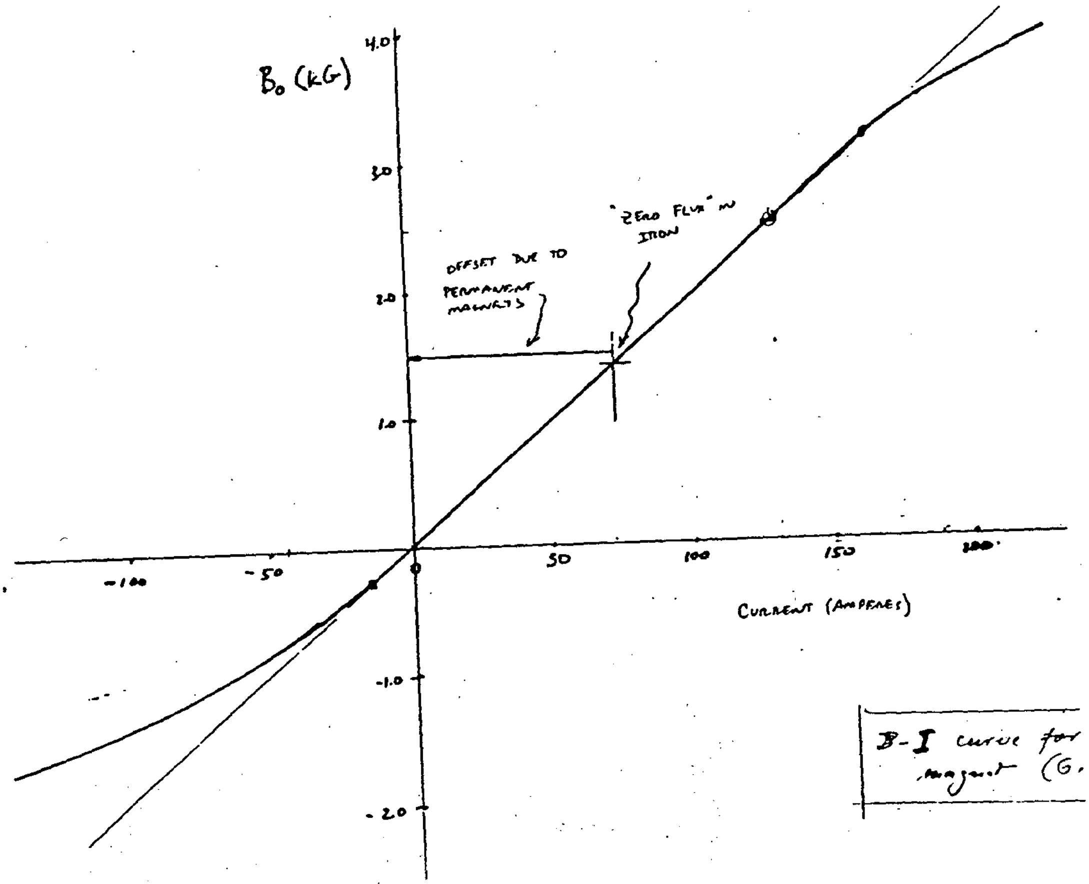
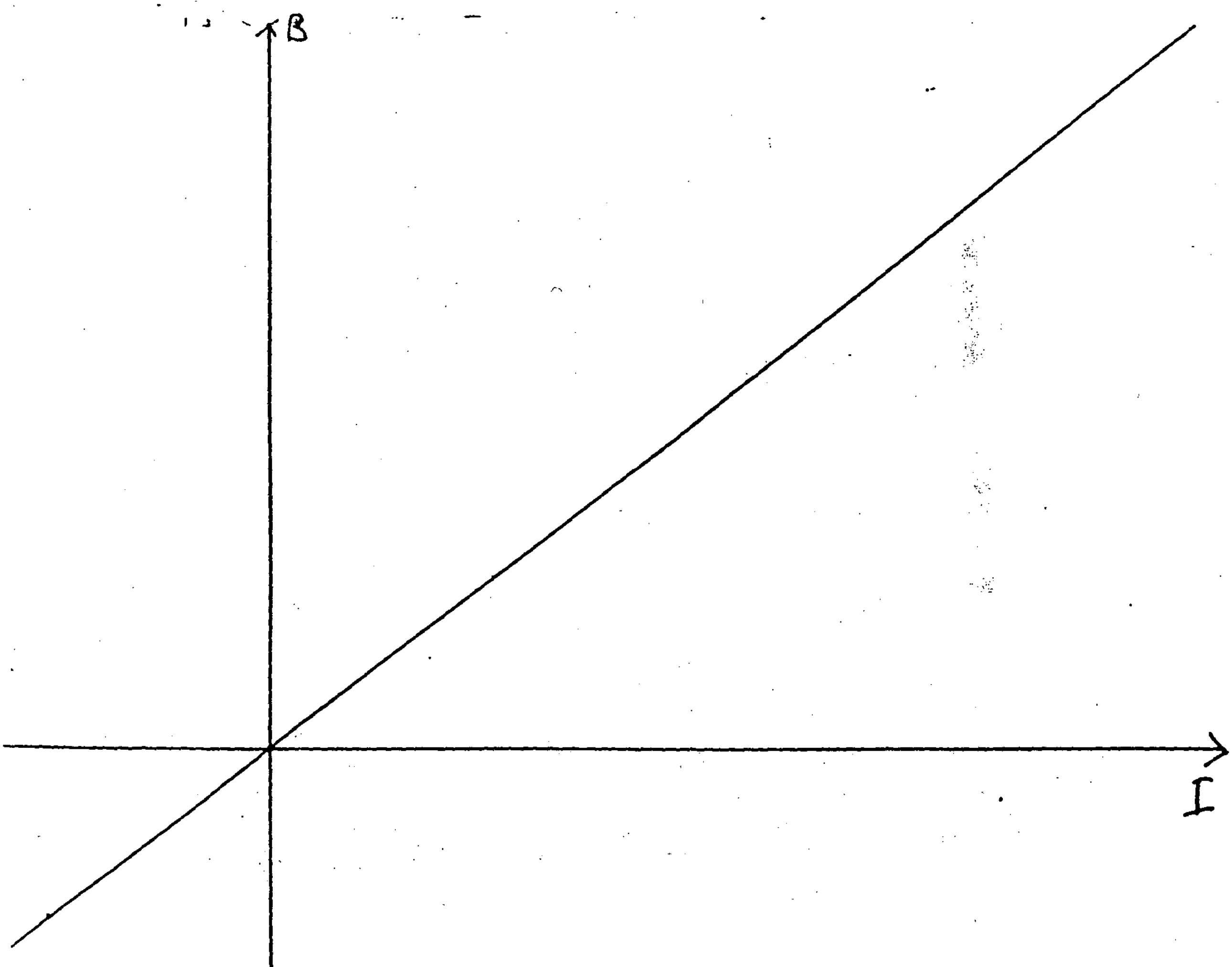
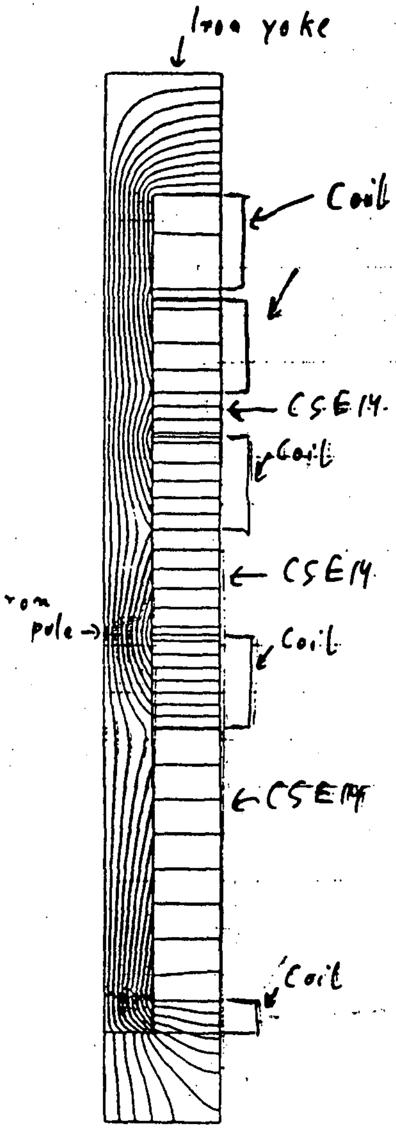


Fig.1. B-I curve for the Paladin wiggler prototype magnet
(Data courtesy of G. Deis)



λ/4 of Laced U/W

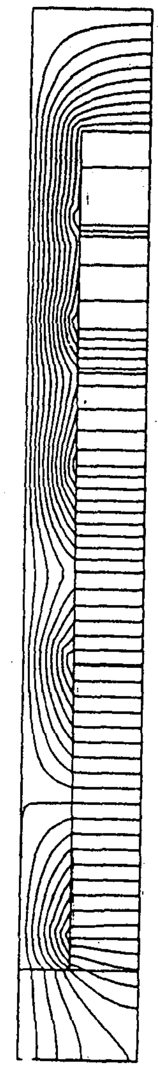


$j = 1.6\ \mathrm{A/cm^2}$

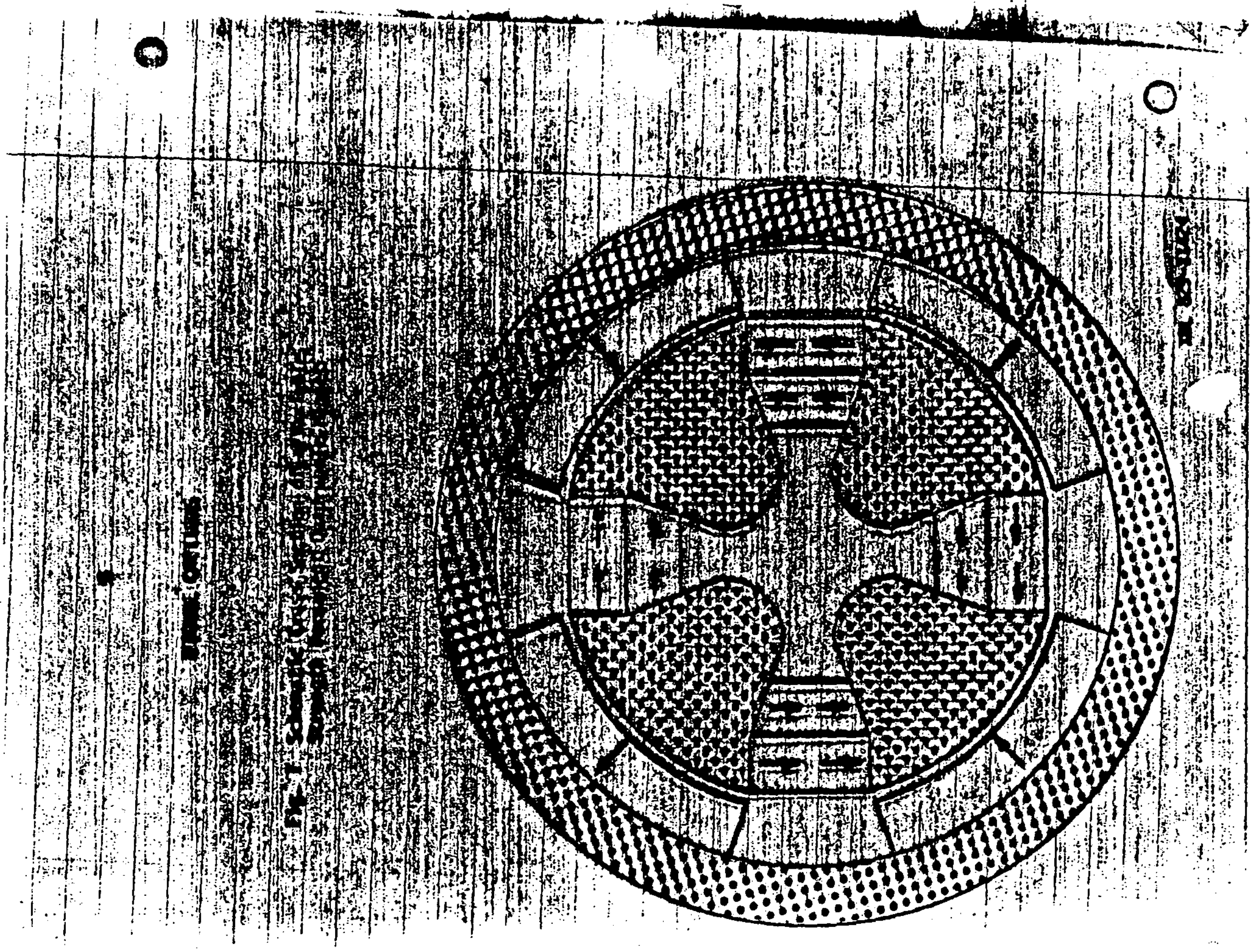
$B_0 = 5.34\ \mathrm{kG}$

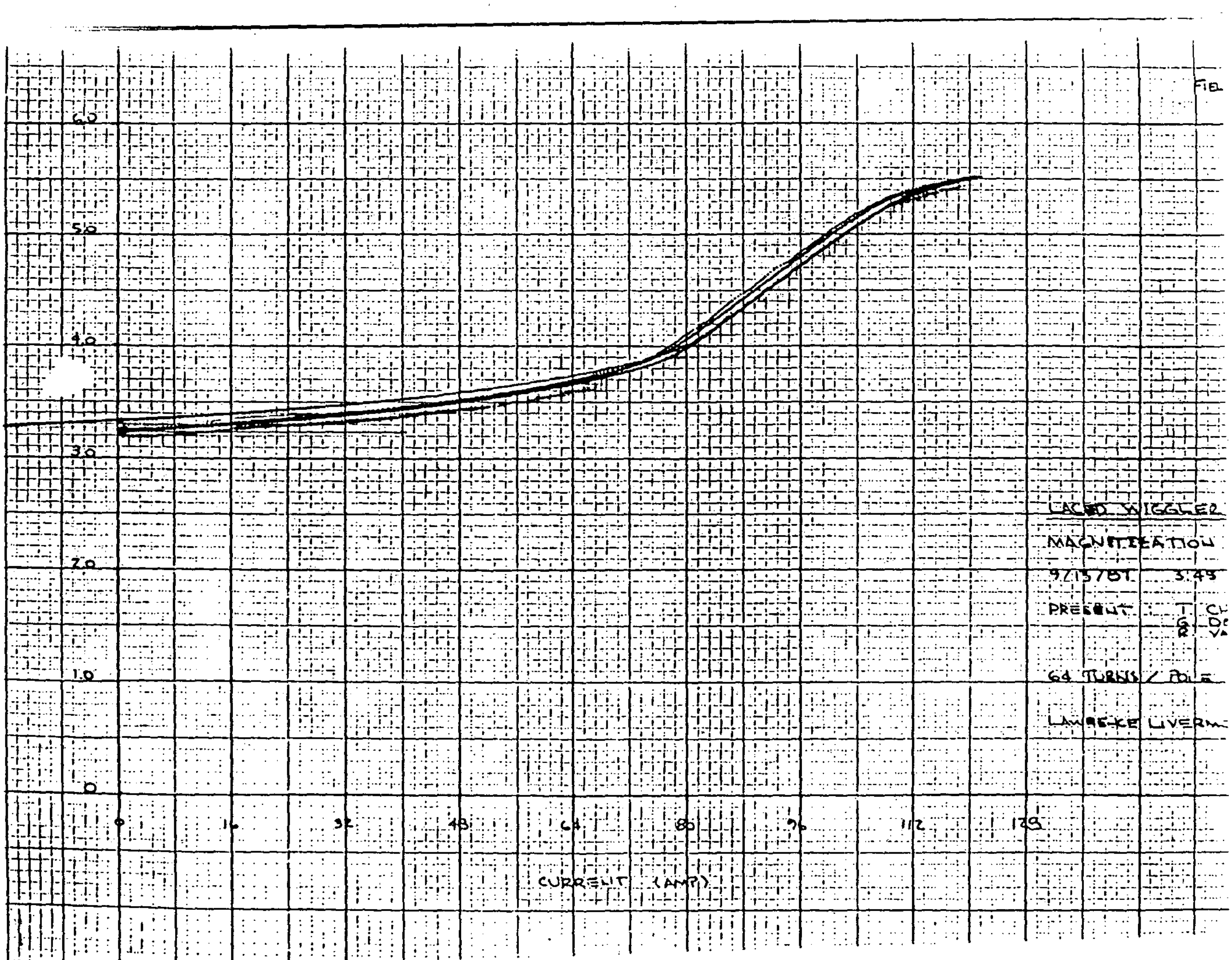
$$\left(\int \vec{H}\cdot d\vec{l}\right)_{gap} = I - \left(\int \vec{H}\cdot d\vec{l}\right)_{iron}$$

PROB. NAME - LACED WIGGLER, FULL EXCITATION CYCLE - 12



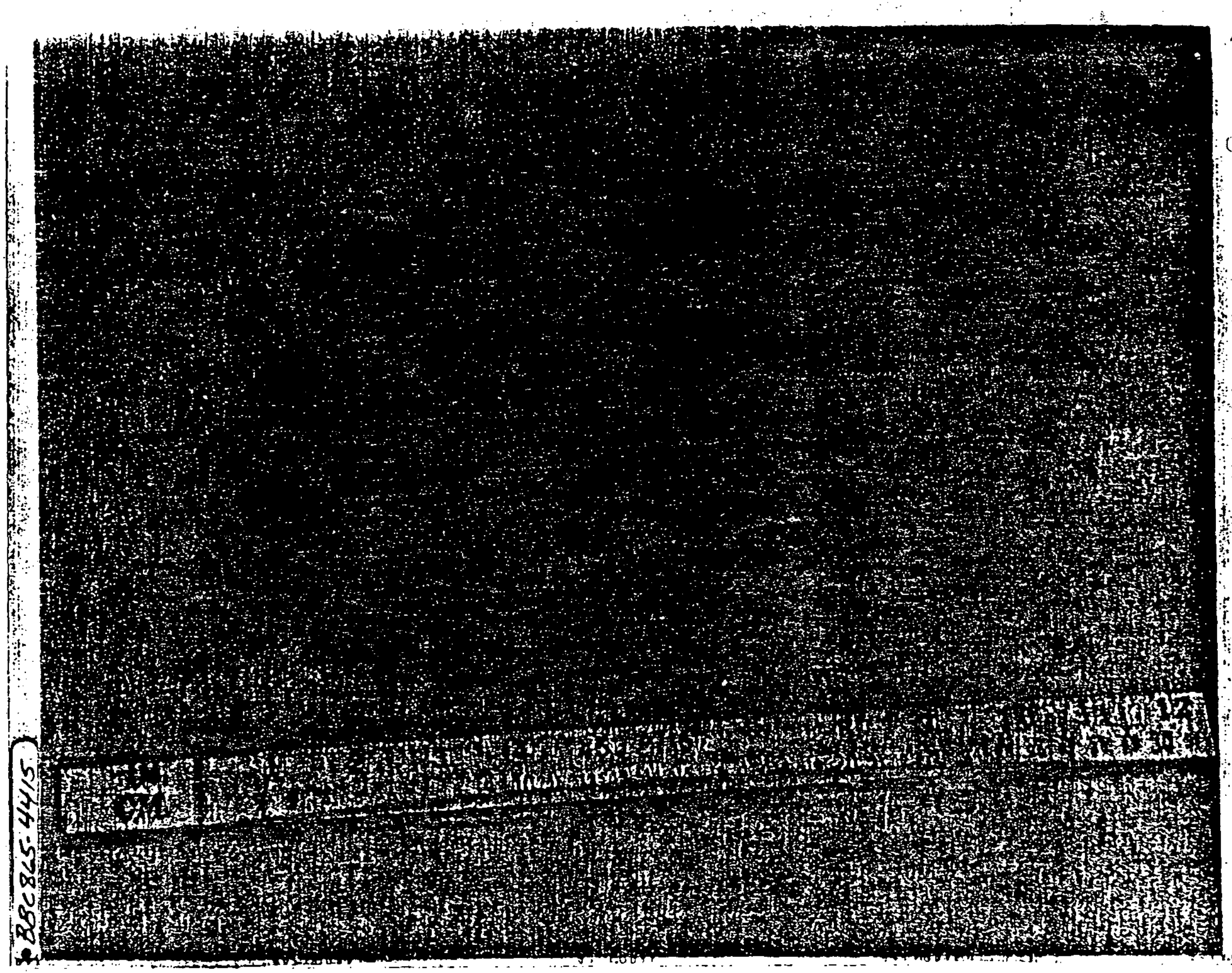
B. NAME - LACED WIGGLER, 68% EXCITATION CYCLE - 17

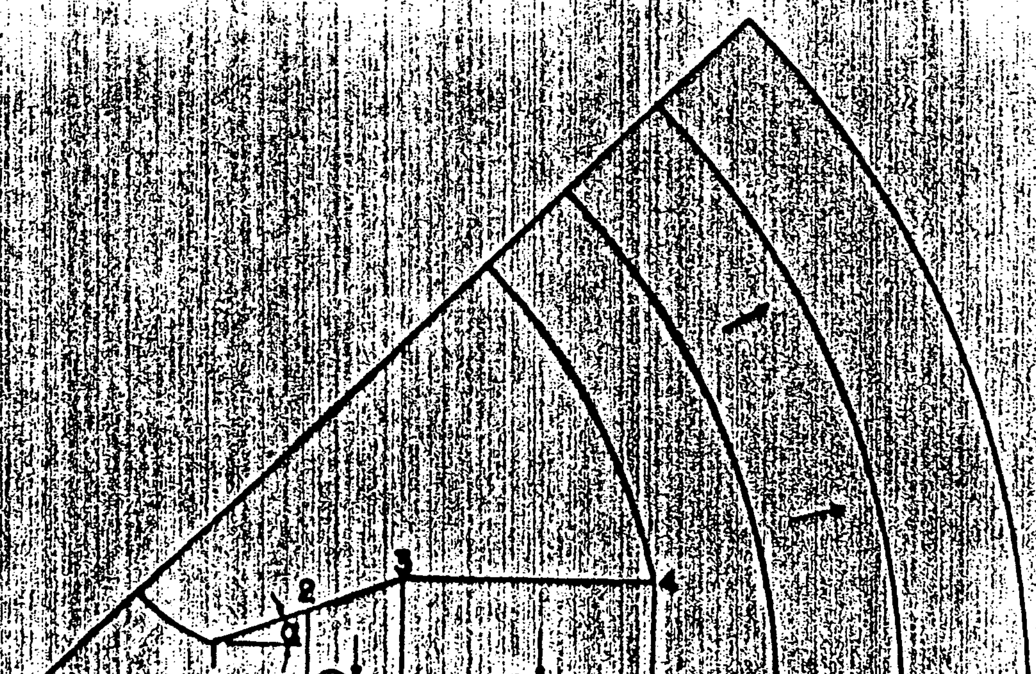


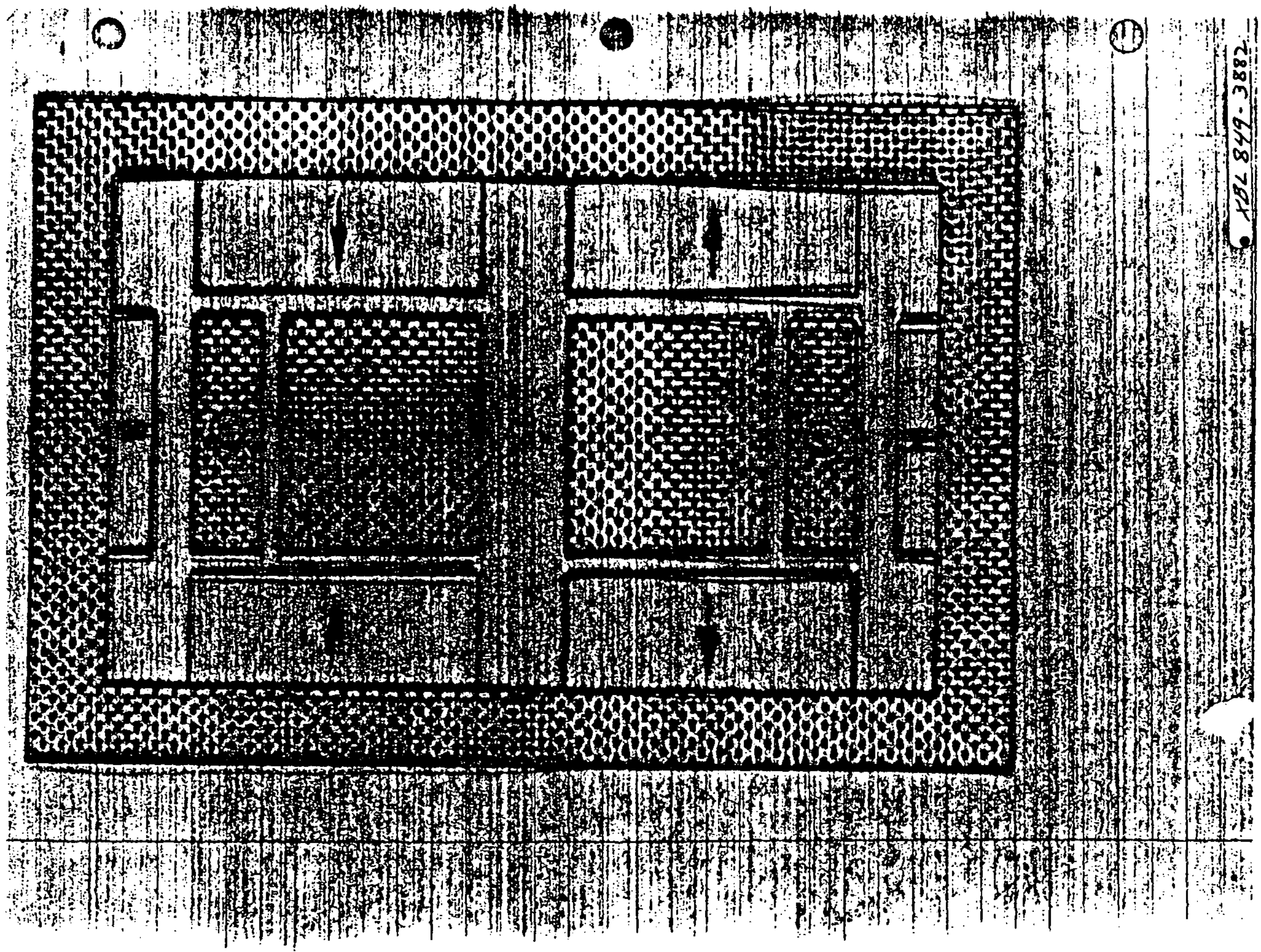


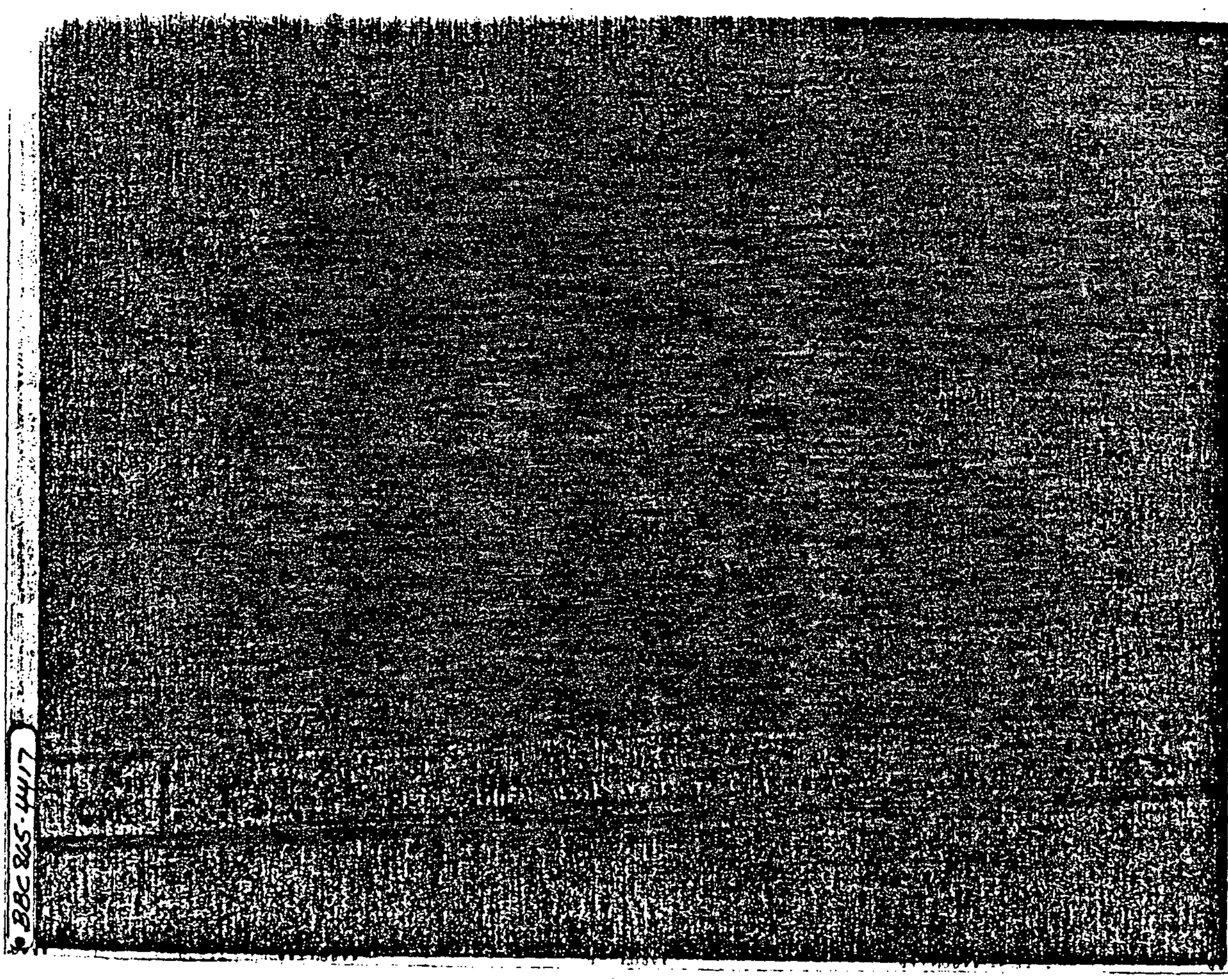
6.0
5.0
4.0
3.0
2.0
1.0
0
0
16
32
48
64
80
96
112
128
CURRENT (AMP)
LACED WIGGLER
PRESENT
64 TURNS / POLE

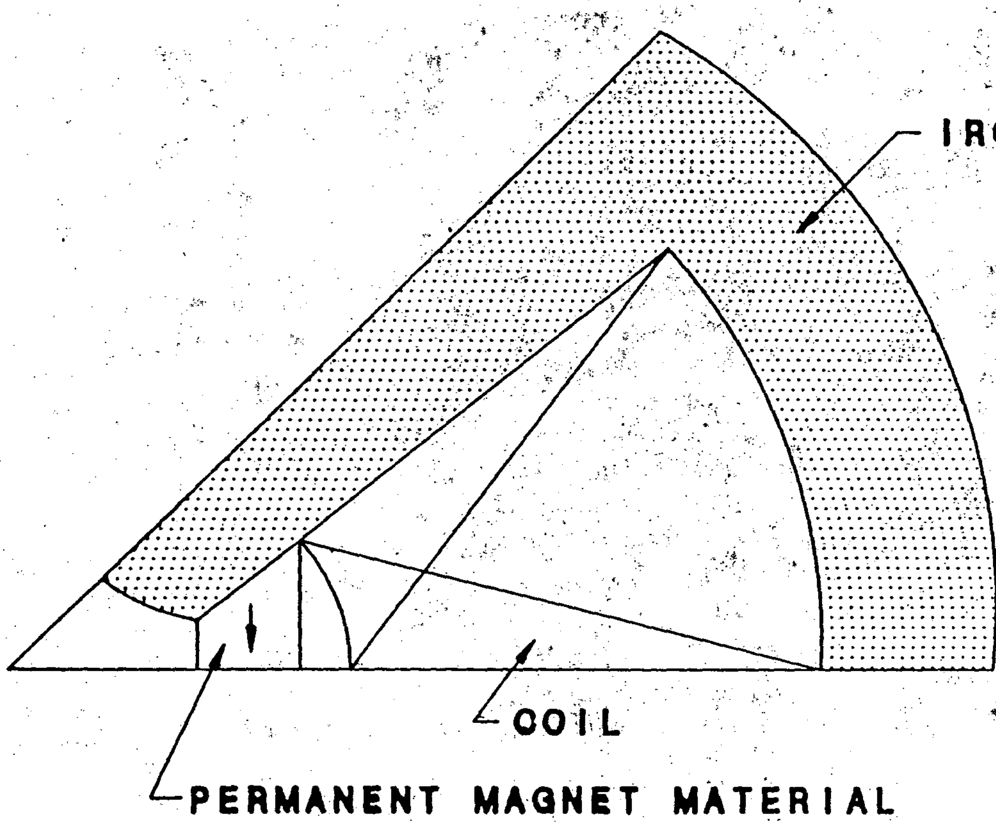
37

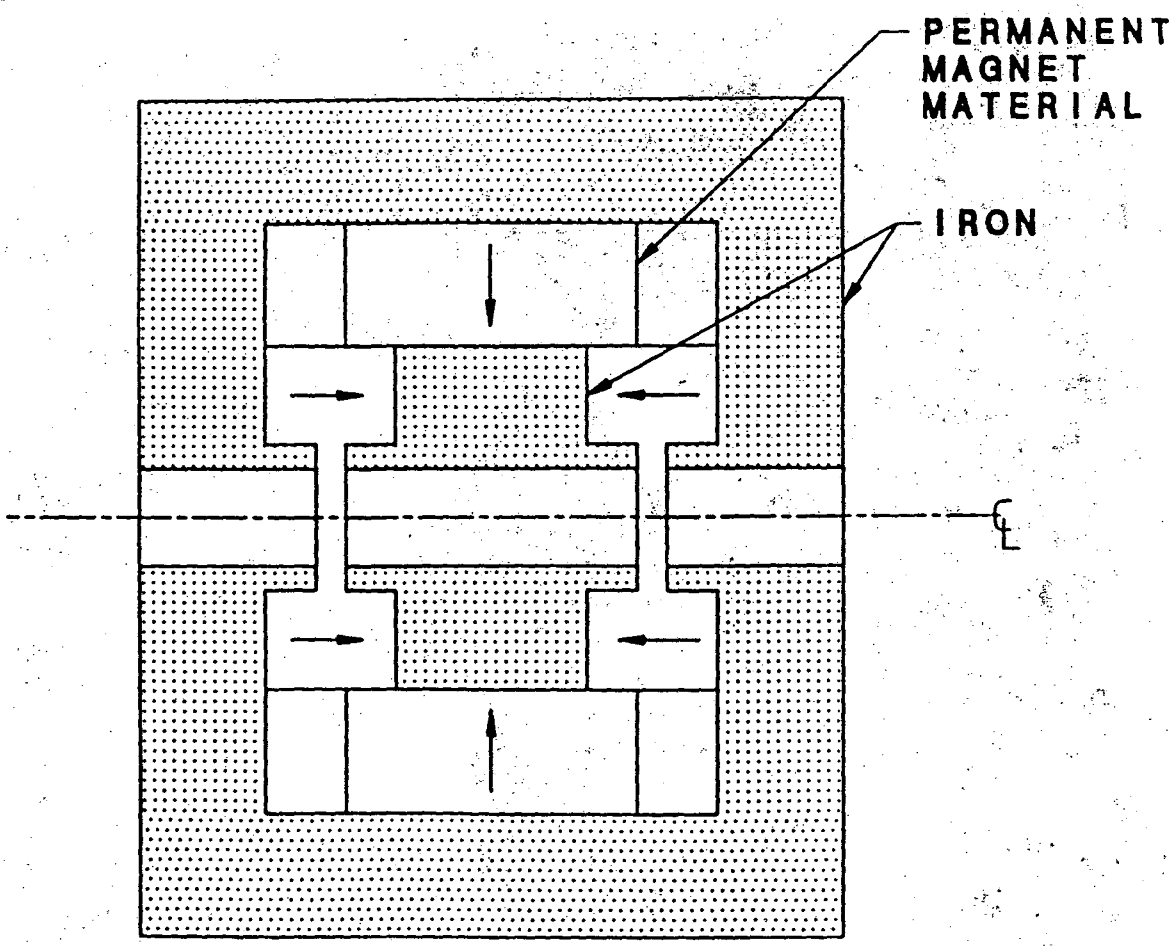




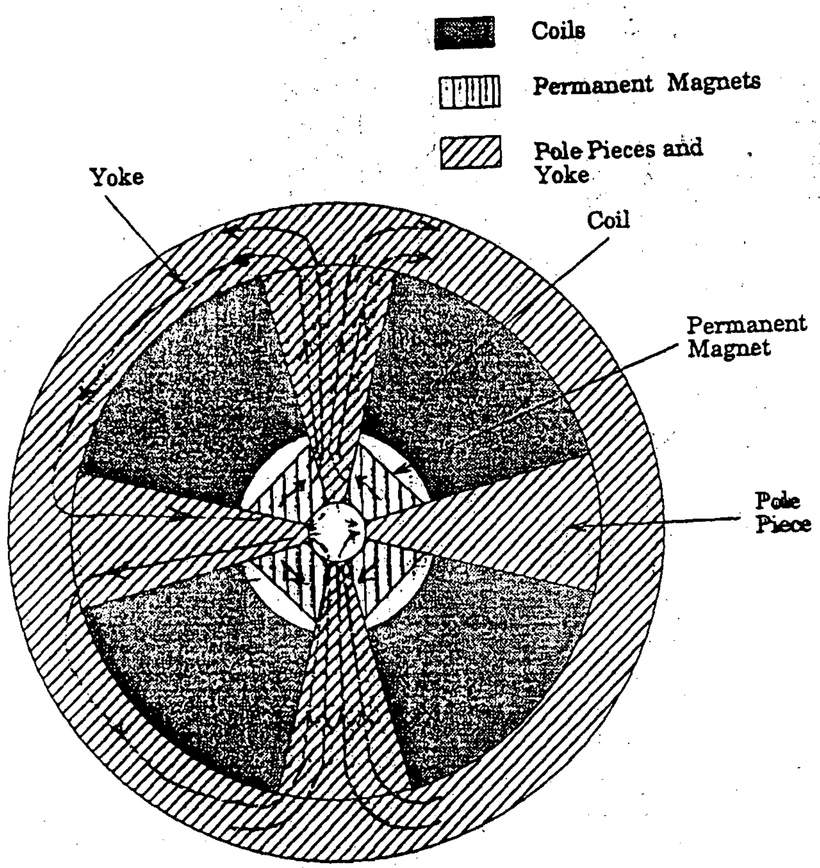

KOL 849-3882

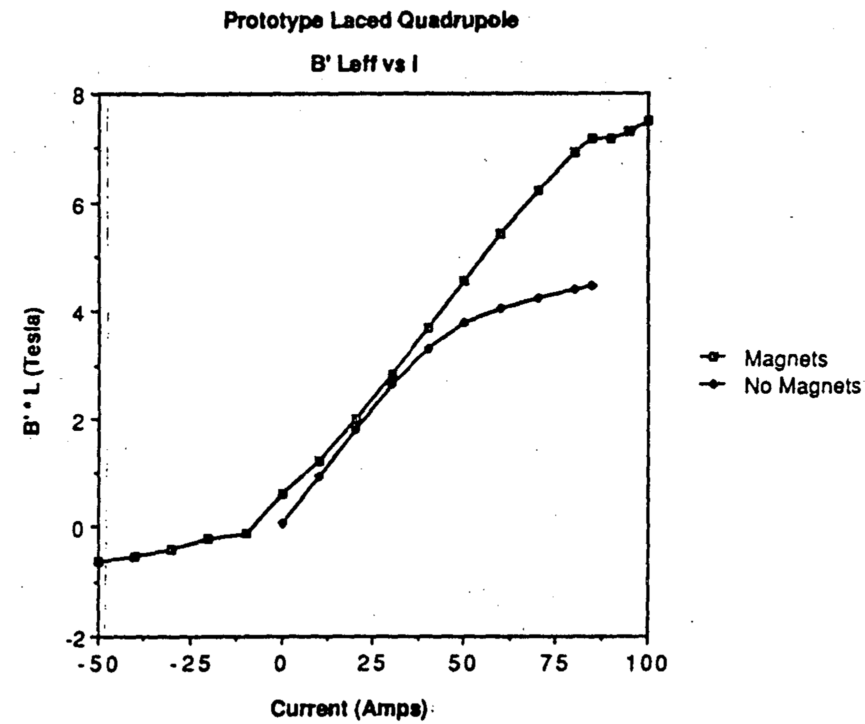

BBC 865-4417


IRON
COIL
PERMANENT MAGNET MATERIAL


PERMANENT MAGNET MATERIAL
IRON
℄

Laced Quadrupole





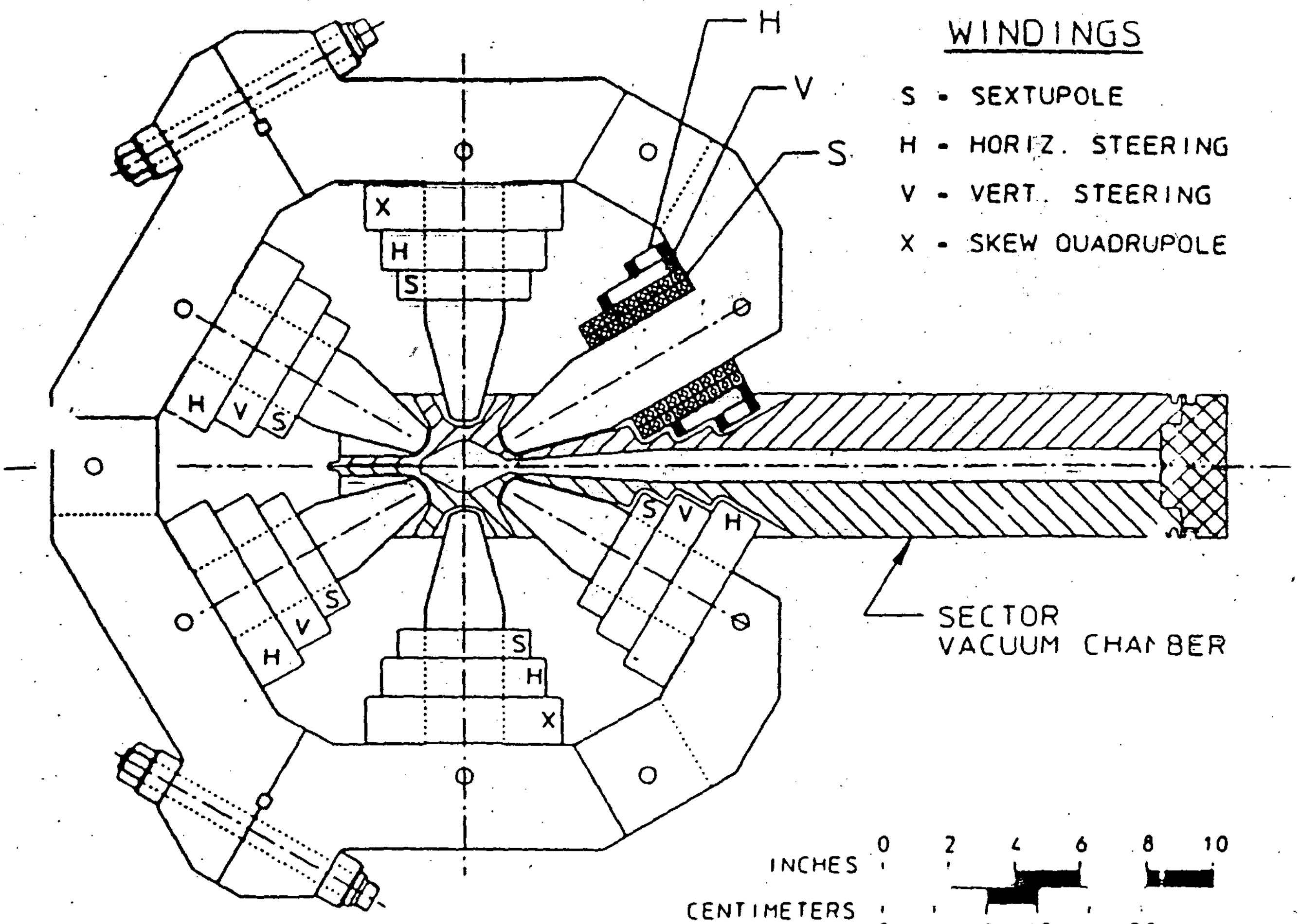


Fig. 3-37. Storage ring sextupole magnet cross section

This is the return to Maxwell's equ's.

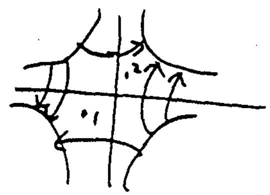
2) $\vec{B} = \text{curl}\,\vec{A} \rightarrow \text{div}\,\vec{B} \equiv 0$

$\vec{B} = \mu_0 \vec{H}$: $\text{curl}\,\vec{H} \sim \text{curl}\,\text{curl}\,\vec{A} = \vec{j}$

$\vec{A}$ has in general case 3 components $\rightarrow$ more complicated than V. I will use it rarely, except

2D : $\partial/\partial z = 0$: need only $A_z \neq 0$, i.e.

$\vec{A} = \vec{e}_z A$.



In general

$\Phi = \int \vec{B} \cdot d\vec{a} = \int \text{curl}\,\vec{A} \cdot d\vec{a}$

$\Phi = \oint \vec{A} \cdot d\vec{s}$

For this 2D case: $\Phi = L\,(A_2 - A_1)$

$A = \text{const} = \text{field line}$.

$B_x = \partial A/\partial y = A'_y = -\tilde{V}'_x$

$B_y = -A'_x = -\tilde{V}'_y$

$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = -\nabla^2 \tilde{V} = 0$; (satisfied by A "automatically")

$\frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial x} = \nabla^2 A = 0$; (satisfied by V "automatically")

B) Fct. of a complex variable

$z = x + iy$; $F(z) = A(x,y) + i\,V(x,y)$

Only allowed operations to define F: $+, -, \times, \div$

Not allowed: take complex conjugate of z, which would be $z^* = x - iy$. Will use this operation many times, but it is illegal in definition of a function of the complex variable z.

$\frac{\partial F}{\partial x} = \frac{dF}{dz} \cdot \frac{\partial z}{\partial x} = \frac{dF}{dz} = A'_x + i\,V'_x = V'_y - i\,A'_y$

$\frac{\partial F}{\partial y} = \frac{dF}{dz} \cdot \frac{\partial z}{\partial y} = i\,\frac{dF}{dz} = A'_y + i\,V'_y$

$A'_x = V'_y$; $V'_x = -A'_y$ C-R

$\nabla^2 F = 0 \rightarrow \nabla^2 A = 0$; $\nabla^2 V = 0$

$\uparrow$ = Math. Connection to physics:

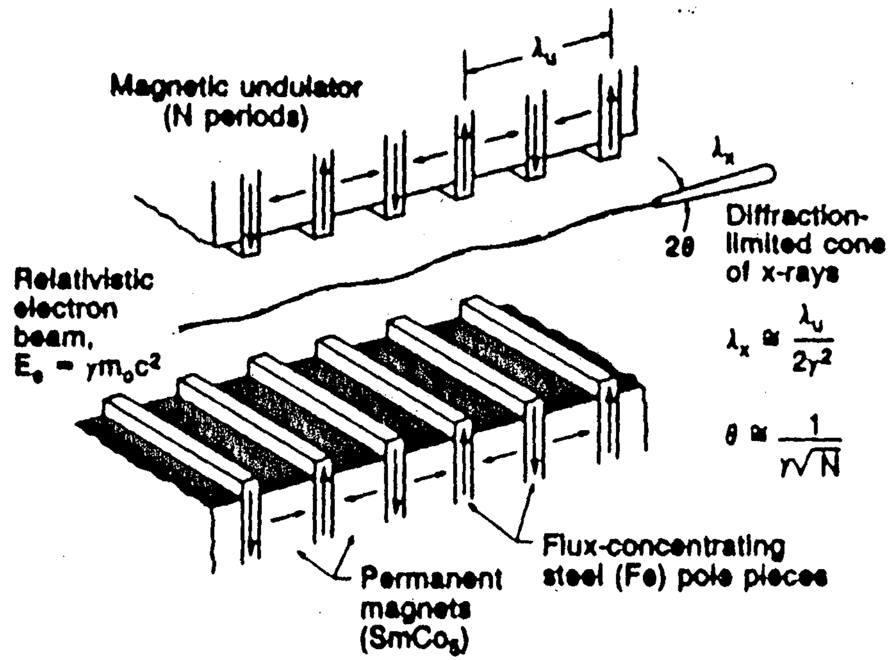
A, V satisfy same equ's. that vectorpot. A and scalar pot. $\tilde{V}$, describing fields B_x, B_y, did. Drop $\sim$ V;

Insertion Device Design

Klaus Halbach

Lecture 3.

November 4, 1988



43

(3.1)

Summary of lecture #2

- $\int \vec{B}\cdot\vec{H}\,dv = 0$ if $\vec{j} = 0$ everywhere.
- Error fields caused by perturbations / material flaws in iron-free ID
- Hybrid ID.
- Focusing in ID
- Design options for entrance/exit region of hybrid ID.
- Perturbation - consequences in hybrid ID.

  Most damaging: ΔB giving steering $\to \Delta B_\perp$ midp.
  Steering strongly associated with fields betwee sides of ID and midplane.
- Survey of other devices

  PM assisted EM: move operating • on $B(I)$-curv.
- Return to summary of Maxwell's equ's.

  Vector potential $\vec{A}$ in 3D, 2D
- 2D fields derived from A, V:

  $B_x = A'_y = -\tilde{V}'_x$; $B_y = -A'_x = -V'_y$
- Review of theory of a function of a complex variab.

End of summary of lecture #2

(3.2)

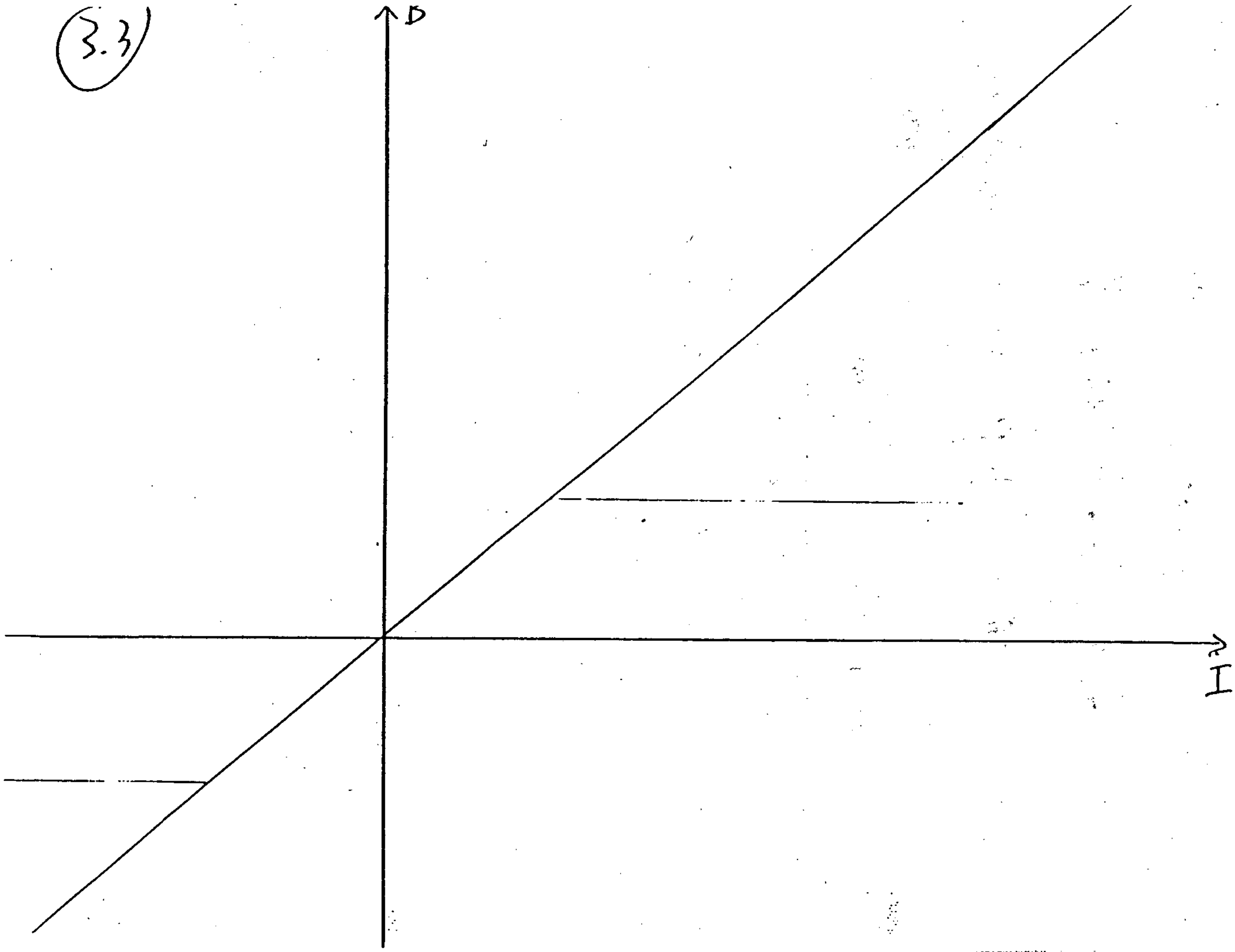
Stored energy density in CSEM.

$$\Delta \mathcal{E} = \int_1^2 \vec{H}\,d\vec{B} = \int_1^2 (\vec{H}_\parallel + \vec{H}_\perp)(d\vec{B}_\parallel + d\vec{B}_\perp)$$

$$\Delta \mathcal{E} = \int_1^2 (H_\parallel\,dB_\parallel + H_\perp\,dB_\perp) = \int_1^2 \Big(H_\parallel \cdot \underbrace{\frac{dB_\parallel}{dH_\parallel}}_{\mu_0\mu_\parallel} \cdot dH_\parallel + H_\perp \cdot \underbrace{\frac{dB_\perp}{dH_\perp}}_{\mu_0\mu_\perp} \cdot dH_\perp\Big)$$

$$\Delta \mathcal{E} = \frac{\mu_0}{2}\cdot\left(\mu_\parallel H_\parallel^2 + \mu_\perp H_\perp^2\right)\Big|_1^2$$

(3.3)



(3.4)

B) Fct. of a complex variable

$z = x + iy$; $F(z) = A(x,y) + i\,V(x,y)$

Only allowed operations to define F: $+, -, \times, \div$

<u>Not allowed:</u> take complex conjugate of z, which would be $z^* = x - iy$. Will use this operation many times, but it is illegal in definition of a function of the complex variable z.

$$\frac{\partial F}{\partial x} = \frac{dF}{dz}\cdot\frac{\partial z}{\partial x} = \frac{dF}{dz} = A'_x + i V'_x = V'_y - i A'_y$$

$$\frac{\partial F}{\partial y} = \frac{dF}{dz}\cdot\frac{\partial z}{\partial y} = i\frac{dF}{dz} = A'_y + i V'_y$$

$A'_x = V'_y$; $V'_x = -A'_y$ C-R

$\nabla^2 F = 0 \longrightarrow \nabla^2 A = 0;\ \nabla^2 V = 0$

- - - - - - - - - - - - - - -

$\uparrow$ = Math. Connection to physics:

A, V satisfy same equ's. that vectorpot. A and scalar pot. $\tilde{V}$, describing fields B_x, B_y, did. Drop $\tilde{V}$;

(3.5) Continuation of M-equ's.

$F = A + i\,V$ = complex potential.

$B_x - i\,B_y = B^* = i\,F'(z)_{tr}$
$H_x - i\,H_y = H^* = i\,F'(z)$
} Choice (H^*, B^*) determined by problem, prejudice;

Notation: When representing 2D vector by complex number, <u>always</u> use:

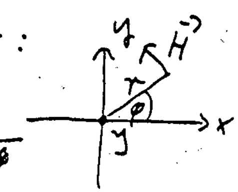
$a = a_x + i\,a_y$ ← a_y: y-component of vector $\vec{a}$; a_x: x-component of vector $\vec{a}$

$\uparrow$ Compl. number that represents 2D vector

Then, it is always true that

$$b\,a^* = \vec{a}\cdot\vec{b} + i(\vec{a}\times\vec{b})_z$$

Physics perspective on this:

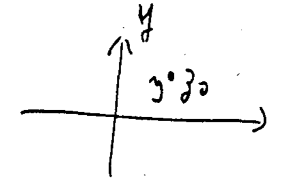


$$H = \frac{\eta}{2\pi r}\cdot e^{i\varphi}\cdot i = -\frac{\eta}{2\pi i\, r e^{-i\varphi}}$$

$r e^{i\varphi} = z$

$$H^*(z) = \frac{\eta}{2\pi i\cdot z}$$

$$H^*(z) = \frac{\eta}{2\pi i(z - z_0)} = i\,F'$$

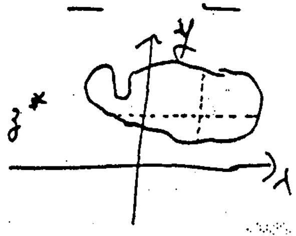


(3.6)

$H(z)^* = \frac{\gamma}{2\pi i(z-z_0)}$; $F = -\frac{\gamma}{2\pi}\ln(z-z_0)$ $\left(= \frac{q'}{2\pi i}\ln(z-z_0)\right.$ for line charges

---

More math.



G = general fct. of x, y; or z, z^*

$$\int \frac{\partial G}{\partial x} da = \int \frac{\partial G}{\partial x} dx\, dy = \oint G\, dy$$

$$\int \frac{\partial G}{\partial y} da = -\oint G\, dx$$

$x = (z + z^*)/2$; $y = (z - z^*)/2i$; $\frac{\partial G}{\partial z^*} = \frac{1}{2}\left(\frac{\partial G}{\partial x} + i\frac{\partial G}{\partial y}\right)$

$$\int \frac{\partial G}{\partial z^*} da = \frac{1}{2}\cdot\left(\oint G\, dy - i\oint G\, dx\right) = \frac{1}{2i}\cdot\oint G\, dz$$

similarly: $\int \frac{\partial G}{\partial z} da = -\frac{1}{2i}\cdot\oint G\, dz^*$

$G = A + iV$

$$\frac{\partial G}{\partial z^*} = \frac{1}{2}\left(A'_x - V'_y + i\left(V'_x + A'_y\right)\right)$$

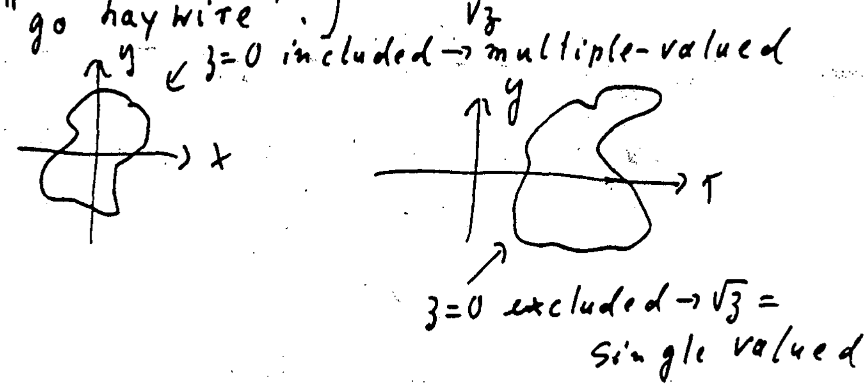
$\frac{\partial G}{\partial z^*} = 0$ when $A'_x = V'_y$; $V'_x = -A'_y$ = different way to state C–R.

When $\frac{\partial G}{\partial z^*} = 0$, and G = single valued (and no singularities) in area over which one integrates: $\oint G\, dz = 0$

(When G = multiple-valued, like $\sqrt{z}$ when

(3.7)

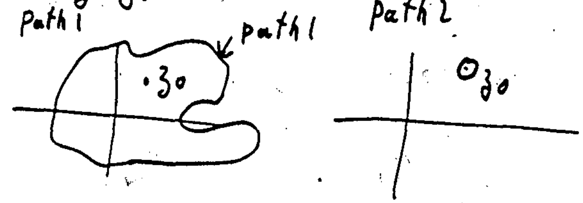
$z = 0$ included in area, "don't know" what value of G to take, except when I make a branch cut. But there, derivatives "go haywire".)



G = single valued, no singularities in region.

$$\oint \frac{G(z)}{z-z_0} dz = \oint \frac{G(z)}{z-z_0} dz = 2\pi i\, G(z_0)$$

path 1   path 2

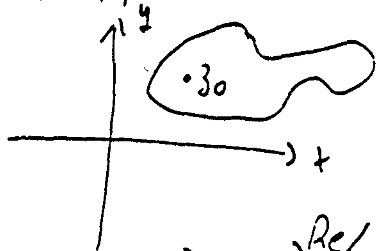


$z = z_0 + \rho e^{i\varphi}$; $dz = i\rho e^{i\varphi} d\varphi$

$$\oint \frac{G(z)}{(z-z_0)^n} dz = 2\pi i \cdot \frac{G^{(n-1)}(z_0)}{(n-1)!}$$ ← Cauchy's Integral Theor.

(3.8)

Application to H^*:



$$\oint \vec{H}\cdot d\vec{s} = \mathrm{Re}\oint H^* dz = \mathrm{Re}\int \frac{y\, dz}{2\pi i(z-z_0)} = y =$$

Ampère's theorem.

(3.9)

Two illustrative applications of C-S-theorem.

1) $I_1 = \int_0^{2\pi} \frac{d\varphi}{a+\cos\varphi}$; $a = \text{real}, > 1$

$e^{i\varphi} = z$; $d\varphi = \frac{dz}{iz}$; $I_1 = 2\cdot\oint \frac{dz/i}{z^2+2za+1}$

$z^2+2za+1=0$; $z_2 = -a \pm \sqrt{a^2-1}$; $z_2\cdot z_1 = 1$; $|z_2|<1$

$|z_1|>1$; $I_1 = 2\cdot\oint \frac{dz/i}{(z-z_2)(z-z_1)} = 2\cdot\frac{2\pi}{z_2-z_1} = \frac{2\pi}{\sqrt{a^2-1}}$

2) $I_2 = \int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1}\,dx = \mathrm{Re} \int \frac{e^{iaz}}{z^2+1}\,dz$; $a = \text{real}, \geq 0$

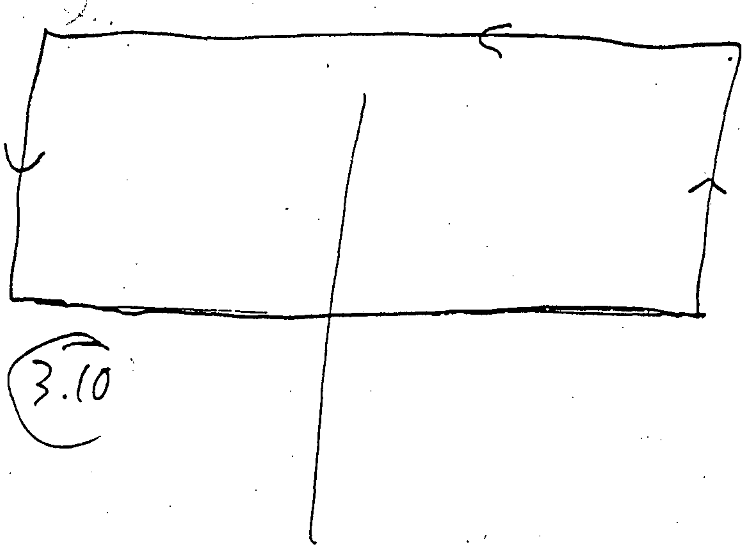
Close in upper ½ plane: $|e^{iaz}| = e^{-ay}$

$I_2 = \mathrm{Re}\oint \frac{e^{iaz}}{(z-i)(z+i)}\,dz = \mathrm{Re}\, 2\pi i\cdot\frac{e^{-a}}{2i} = \pi e^{-a}$

Many beautiful examples + sophisticated methods (tricks) in: Functions of a Complex Variable, theory and techique. Carrier, Krook, Pearson. McGraw Hill 1966.

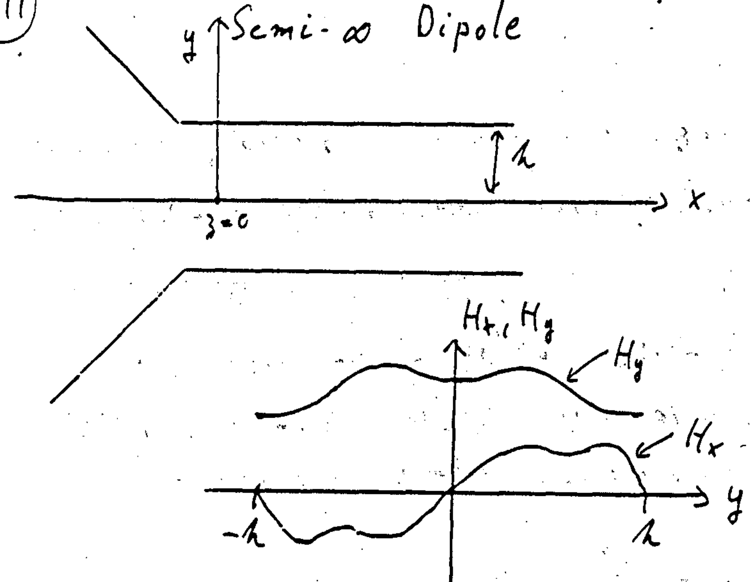
"Best" Introduction simpler level: Introduction to Complex Analysis. Z. Nehari, Allyn + Bacon, 1968

(3.10)



$$\frac{\partial H_y}{\partial y} + \frac{\partial H_x}{\partial x} = 0$$

(3.11) Semi-∞ Dipole



$H_x(-y) = -H_x(y)$; $H_y(-y) = H_y(y)$; $\frac{\partial H_y}{\partial y} = -\frac{\partial H_x}{\partial x}$

H_x, H_y = periodic with period $2h$

$$H_x - iH_y = \sum C_n e^{2\pi n i y/2h} \rightarrow \sum C_n e^{n\pi z/h}$$

C_n = imagin.; $C_n = 0$ for $n > 0$ $\left(\left| e^{n\pi z/h} \right| = e^{n\pi x/h} \right)$

$$H_x - iH_y = H^* = i \sum_{n=0}^{\infty} b_n e^{-n\pi z/h}$$

b_0 = field deep inside

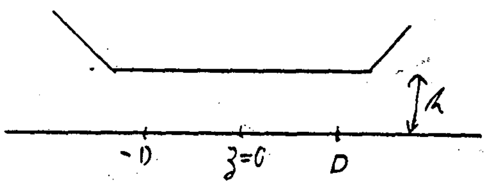
Field errors decay exponentially!!

Antisymm. fields

$$H^* = \sum_{n=0}^{\infty} a_n e^{-(n+1/2)\pi z/h}$$

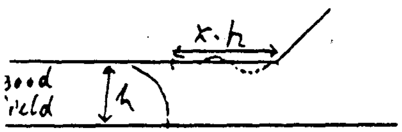
(3.12)

Symmetrical magnet



$$H^* = i \sum_{n=0}^{\infty} b_n \frac{\cosh(n\pi z/h)}{\cosh(n\pi D/h)}$$

Field quality in dipole with/without shims.



no shim: $\Delta B/B \approx \exp(-2.77(x+.9))$

shim : $\Delta B/B \approx \exp(-7.14(x+.25))$

↑ applicable to all 2D magnets with conformal mapping → details later.

(3.13)

Calculation of fields in, and design of, iron-fre[e] CSEM systems, following closely NIM 169, 1 (198[...]

Tools

Use throughout $dB_{\parallel}/d\mu_0 H_{\parallel} = dB_{\perp}/d\mu_0 H_{\perp} = 1$

3D: $V(\vec{r}_0) = \frac{q}{4\pi\mu_0 |\vec{r}_0 - \vec{r}|} \longrightarrow \int \frac{\rho(\vec{r})\, dv}{4\pi\mu_0 |\vec{r}_0 - \vec{r}|}$

$$4\pi V(\vec{r}_0) = \int \frac{-\operatorname{div} \vec{H}_c}{|\vec{r}_0 - \vec{r}|}\, dv$$

1) Homogeneously magnetized material → charge sheets on surface

$$4\pi V(\vec{r}_0) = \int \frac{\vec{H}_c \cdot d\vec{a}}{|\vec{r}_0 - \vec{r}|} = \vec{H}_c \cdot \oint \frac{d\vec{a}}{|\vec{r}_0 - \vec{r}|}$$

2) General case

$K(\vec{r}) = \frac{1}{|\vec{r}_0 - \vec{r}|}$; $4\pi V = \int -K \operatorname{div} \vec{H}_c\, dv$

$$\operatorname{div}(K \vec{H}_c) = K \operatorname{div} \vec{H}_c + \vec{H}_c \cdot \operatorname{grad} K$$

$$\int \operatorname{div}(K \vec{H}_c)\, dv = \oint K \vec{H}_c \cdot d\vec{a} = 0$$

$$4\pi V = \int \vec{H}_c \operatorname{grad} K\, dv = \int \vec{H}_c \cdot \frac{\vec{r}_0 - \vec{r}}{|\vec{r}_0 - \vec{r}|^3}\, dv$$

(3.14)

2D $\vec{B}_r{}^{D_1} = \Delta z$: $q' = |B_r| \cdot D_1$ at $z + \Delta z$, $q' = -|B_r| \cdot D_1$ at z

$$B^*(z_0) = \frac{1}{2\pi} |B_r| D_1 \left(\frac{1}{z_0 - (z + \Delta z)} - \frac{1}{z_0 - z} \right)$$

$$B^*(z_0) = \frac{|B_r| \Delta z \cdot D_1}{2\pi (z_0 - z)^2}$$

$$B^*(z_0) = \frac{1}{2\pi} \cdot \int \frac{B_r \, da}{(z_0 - z)^2}$$

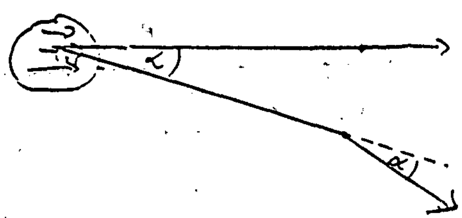
$B_r = B_{rx} + i B_{ry}$
$da = dx\,dy$

↑ Starting equ. for "all" 2D calculations.

Easy axis rotation theorem:

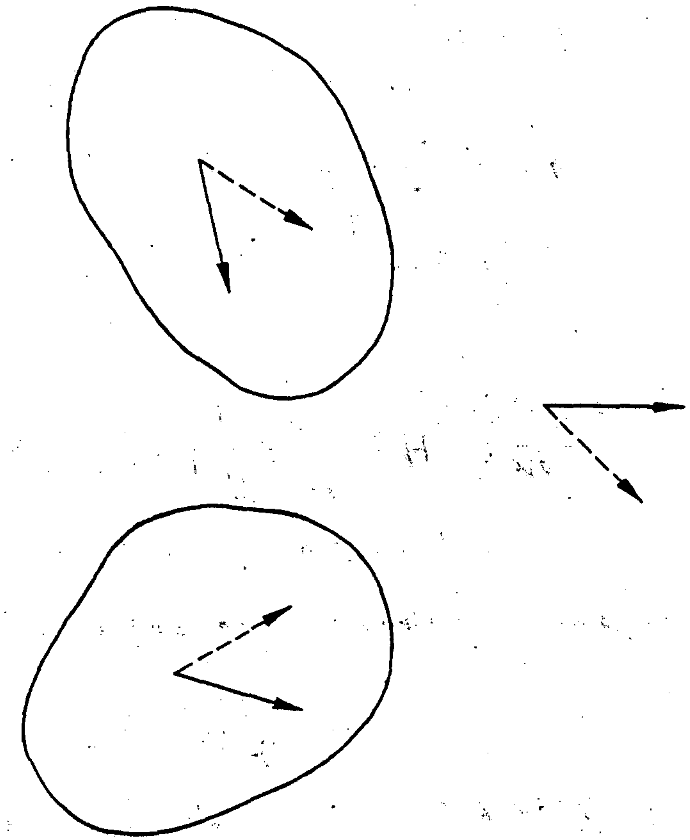
$$B_{r2} = B_{r1} \cdot e^{i\alpha} \longrightarrow B_2 = B_1 \cdot e^{-i\alpha}$$

Qualitative explanation



(10)

2D; no re



XBL 797-10558

(11)

51

(3.15)

Homogeneously magnetized block:

$$B^*(z_0) = \frac{B_r}{2\pi} \cdot \int \frac{dx\,dy}{(z_0 - z)^2}$$

$$B^*(z_0) = \frac{B_r}{2\pi} \cdot \oint \frac{dy}{z_0 - z} = -\frac{B_r}{2\pi i} \oint \frac{dx}{z_0 - z}$$

$$B^*(z_0) = -\frac{B_r}{4\pi i} \cdot \oint \frac{dz^*}{z_0 - z}$$

Applications

Multipole magnets

Notation: $F(z_0) = \sum_1 a_n z_0^n$

$n = 1$ = dipole; $n = 2$ = quadrupole; $n = 3$ = sextupole; ...

$B^* = iF' = \sum b_n z_0^{n-1}$; $b_n = i a_n$

Optimum easy axis orientation to produce multipole of order N

$$\frac{1}{z - z_0} = \sum_0 \frac{z_0^n}{z^{n+1}} \; ; \; \frac{1}{(z - z_0)^2} = \sum_1 \frac{n z_0^{n-1}}{z^{n+1}}$$

$$B^*(z_0) = \sum_1 z_0^{n-1} \cdot \underbrace{\frac{n}{2\pi} \cdot \int \frac{B_r}{z^{n+1}} da}_{\prime}$$

not homogeneously magnetized

(3.16)

With $z = r e^{i\varphi}$; $B_r = |B_r| \cdot e^{i\beta(r,\varphi)}$,

b_N optimized for $\beta(r,\varphi) - (N+1)\varphi = \text{const}$.

$\beta(r,\varphi) = (N+1)\varphi + \text{const}$.

Material between r_1, r_2 with $\beta = (N+1)\varphi$,

$b_n = 0$ for $n \neq N$; for $n = N \geq 2$

$$B^*(z_0) = \left(\frac{z_0}{r_1}\right)^{N-1} \cdot B_r \cdot \frac{N}{N-1} \left(1 - \left(\frac{r_1}{r_2}\right)^{N-1}\right)$$

$B^* = B_r \ln(r_2/r_1)$ for $N = 1$

segmented multipole, assembled from homogeneously magnetized blocks.

Reference block

$$B^*(z_0) = \sum_1 z_0^{n-1} \cdot \underbrace{\frac{B_{r0}}{4\pi i} \oint \frac{dz^*}{z^n}}_{C_{n0}}$$

Blocks $0, 1, 2 \dots M-1$; block m with

$B_r = B_{r0} \cdot \exp(i(N+1) \cdot m \cdot 2\pi/M)$

$C_{nm} = C_{n0} \cdot \exp(i(N+1-(n+1))\, m \cdot 2\pi/M)$

(3.17)

$$b_n = C_{n0} \cdot \sum_{m=0}^{M-1} \exp(i \cdot 2\pi \cdot m (N-n)/M) \qquad \sum_0^{M-1} q^n = \frac{1-q^M}{1-q}$$

$b_n \neq 0$ only for $n = N + \nu \cdot M$; $\nu = 0, 1 \ldots$

$$B^*(z_0) = \sum_{\nu=0} b_n z_0^{n-1} \qquad n = N + \nu \cdot M$$

$$b_n = M \cdot \frac{B_{r0}}{4\pi i} \cdot \oint \frac{dz^*}{z^n}$$

Refer. block geometry: CSEM with $r_1 < r < r_2$, within $\varphi = \pm \varepsilon \cdot \frac{\pi}{M}$

$$B^*(z_0) = B_r \sum_0 \left(\frac{z}{r_1}\right)^{n-1} \cdot \frac{n}{n-1} \left(1 - \left(\frac{r_1}{r_2}\right)^{n-1}\right) \cdot K_n$$

$$K_n = \frac{\sin(\varepsilon (n+1)\pi/M)}{(n+1)\pi/M} \qquad n = N + \nu \cdot M,\ \nu = 0, 1, \ldots$$

Linear array of CSEM:

$z = r_1 + W$ (change of coordinate origin)

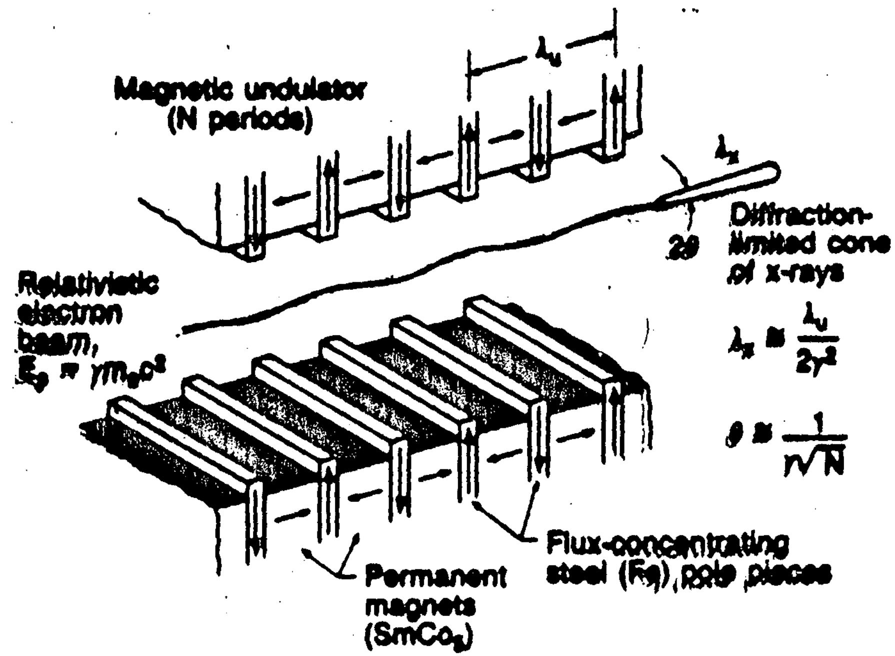
$r_2 = r_1 + D$ D = radial thickness of block; fixed.

$2\pi r_1 / N = \lambda$ = period length; fixed

$2\pi/\lambda = k$; $\rightarrow$ $N = k r_1$

Insertion Device Design

Klaus Halbach



Lecture 4.

November 11, 1988

(4.1)

Summary of lecture # 3

Fct. of complex variable $z = x + iy$: Relations between x, y-derivatives of Re, Im part of analytical fct. of z = same as between derivatives of vector/scalar potentials A, V.

$F = A + iV$ = fct. of z ⇒ automatically: $\nabla^2 A = \nabla^2 V = 0$

$H^* = H_x - iH_y = iF'$ = fct. of z (only) also.

↑ notation: $a = a_x + i a_y$.

Found H^* = fct. of z also by calculating fields from currents/charges.

More math: line integrals; integrals over areas → Cauchy's integral theorem

$$\int \frac{G(z)}{(z - z_0)^n} = 2\pi i \cdot G^{(n-1)}(z_0)/(n-1)!$$

Applications: integration techniques;

Decay of error fields in semi-∞ + finite width dipole: error fields ~ $\exp(-n\pi x/h)$

!!!!

(4.2)

Performance of dipole with/without shims.

Iron free CSEM systems

3D

$$4\pi V(\vec{r}_0) = \int \frac{-\operatorname{div}(\vec{H}_c)}{|\vec{r}_0 - \vec{r}|} dv \quad \text{general}$$

$$= \int \vec{H}_c \cdot \frac{\vec{r}_0 - \vec{r}}{|\vec{r}_0 - \vec{r}|^3} dv \quad \text{general}$$

$$= \vec{H}_c \cdot \int \frac{d\vec{a}}{|\vec{r}_0 - \vec{r}|} \quad \vec{H}_c = \text{const.}$$

2D

$$B^*(z_0) = \frac{1}{2\pi} \int \frac{B_r \, da}{(z_0 - z)^2}$$

Easy axis rotation theorem

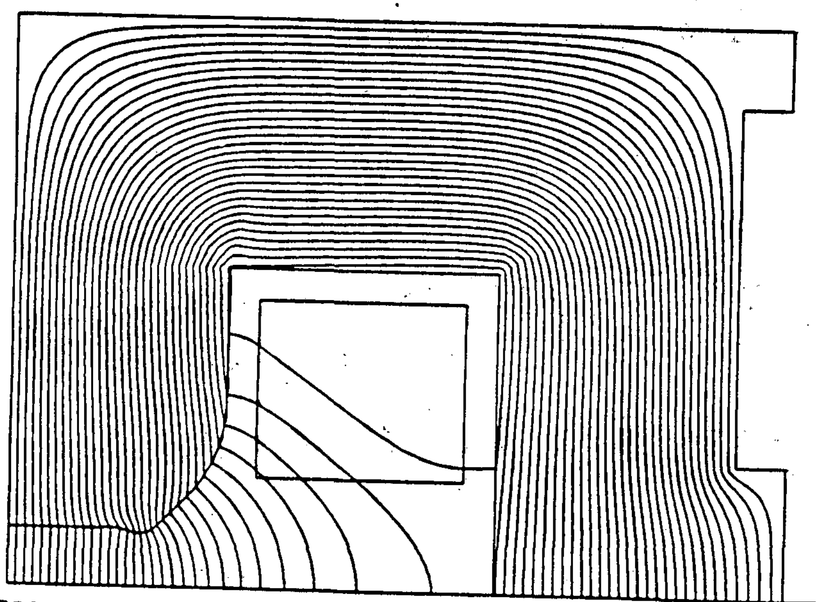
Different forms of ↑ for B_r = general/constant, in particular for multipole coefficients

$F(z_0) = \sum_1 a_n z_0^n$; $B^* = iF' = \sum b_n z_0^{n-1}$; $b_n = i a_n$

Ideal easy axis orientation to produce ideal multipole of order N: $\beta(r, \varphi) = (N+1) \cdot \varphi + \text{const}$

3

(5.7) 4.4



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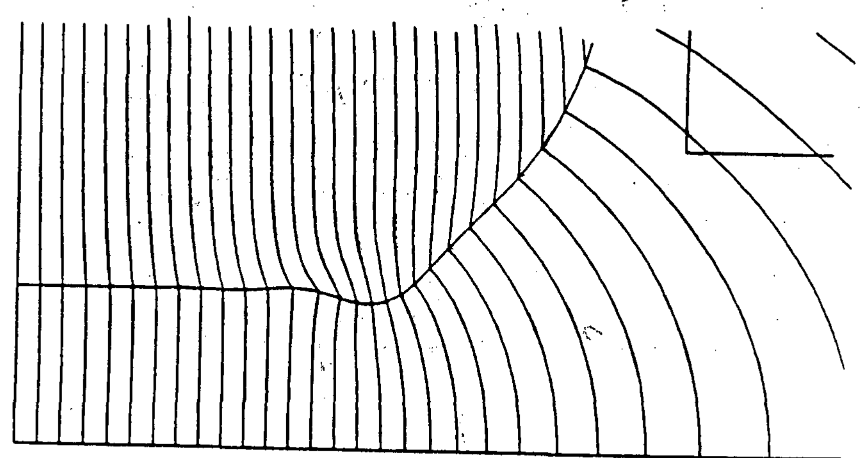

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56

(4.5)

Segmented multipole

$$B^*(z_0) = B_r \cdot \sum_{\nu=0} \left(\frac{z}{r_1}\right)^{n-1} \frac{n}{n-1} \left(1-\left(\frac{r_1}{r_2}\right)^{n-1}\right) K_n$$

$$K_n = \frac{\sin(\varepsilon (n+1)\pi/M)}{(n+1)\pi/M} \; ; \; n = N + \nu \cdot M$$

$$\nu = 0, (1), 2, 3 \cdots$$

Forbidden harmonics forbidden only because of compensation of harmonics produced by different blocks. N+M can be made to vanish "at source" with $\varepsilon = \frac{M}{N+1+M}$

Tolerances: reference block: $B^* = \sum z_0^{n-1} C_{n0}$

$$C_{n0} = \frac{B_{r0}}{4\pi i} \oint \frac{dz^*}{z^n} \; ; \; C_{nm} = C_{n0} \cdot \exp(2\pi i m (N-n)/M)$$

$$B_{r0} = B_r \cdot e^{i\beta} \quad B_r = |B_{r0}|$$

$$\Delta C_{n0} = \frac{\Delta B_r}{B_r} \cdot C_{n0}$$

$$\Delta C_{n0} = i \Delta \beta \cdot C_{n0}$$

$$\Delta C_{n0} = -n \Delta z \cdot C_{n+1\,0}$$

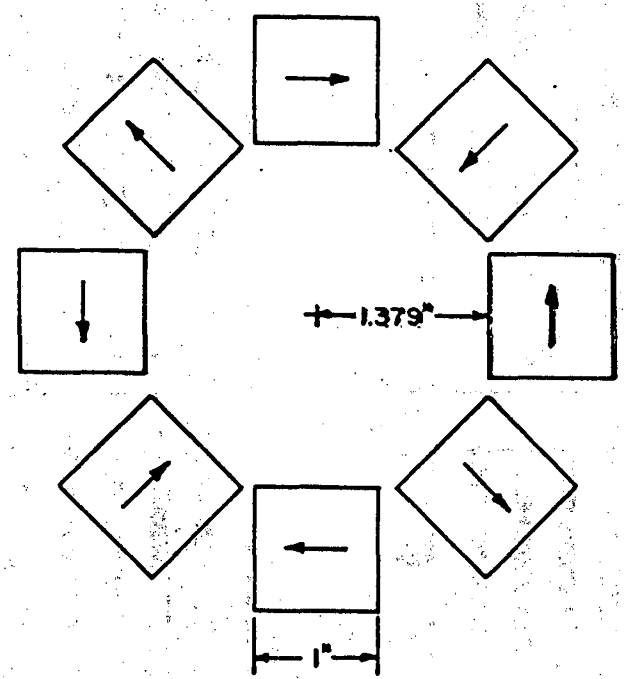
$$\Delta C_{n0} = -i n \Delta \alpha \cdot C_{n0}$$

} Extremely useful for correction of field errors

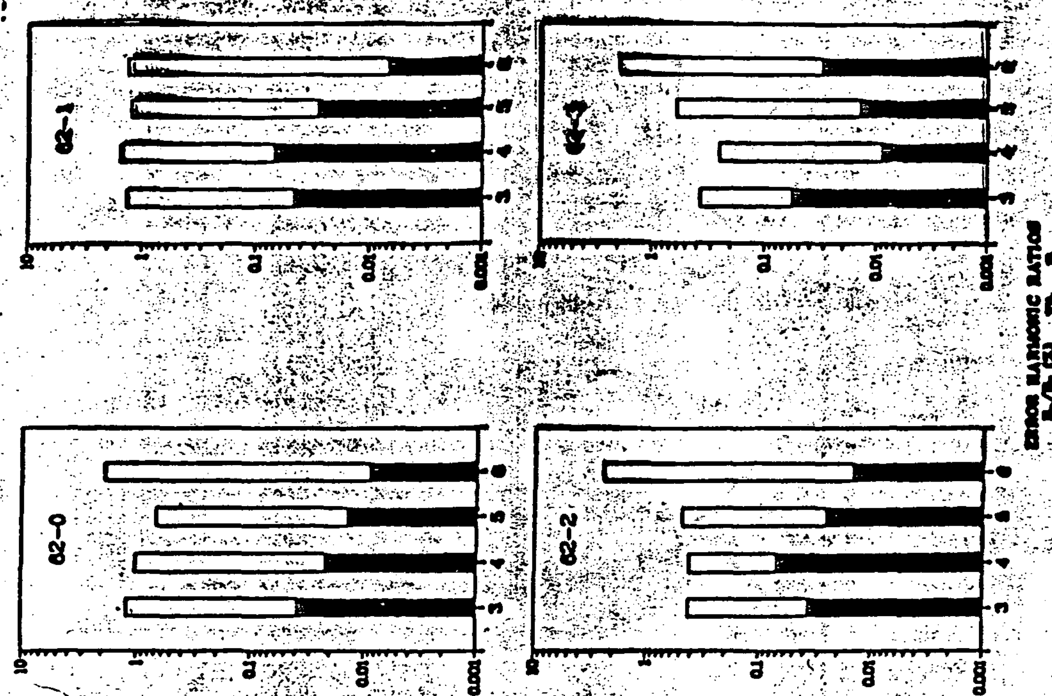
(NIM 198, 213 (82))

nd of ummary

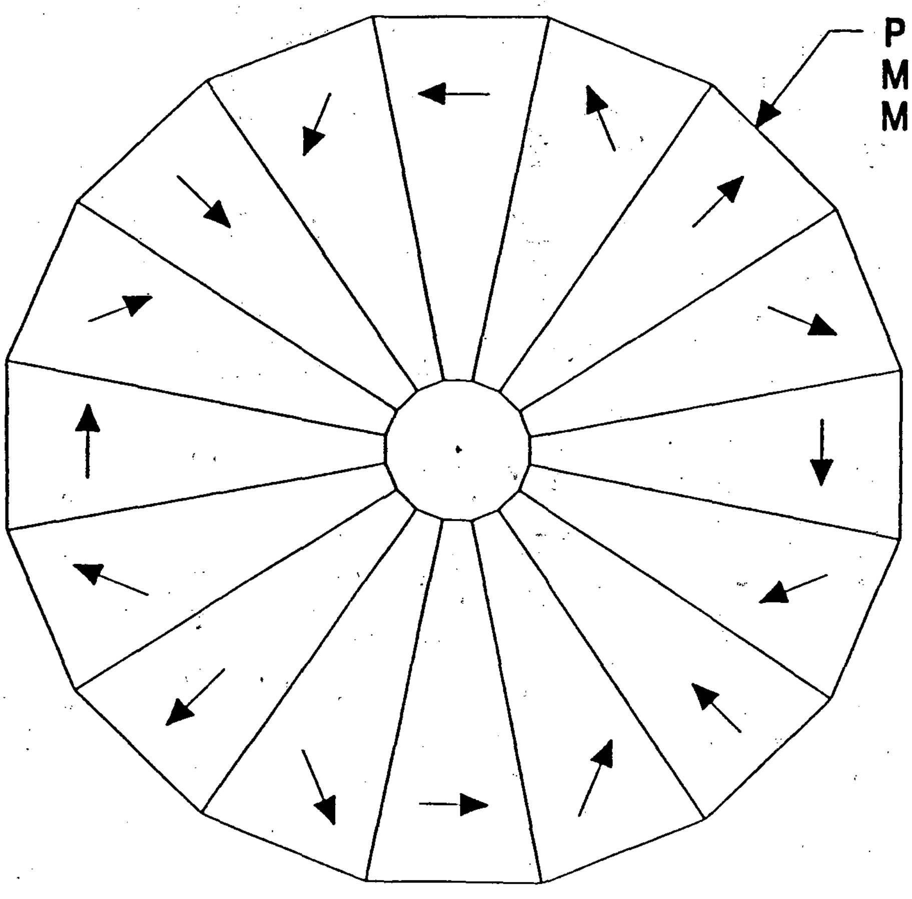
(4.6)



1.26 kG/cm REC QUADRUPOLE



(47)



PERMANENT
MAGNET
MATERIAL

4.7a $\left(\frac{}{n-1}[1-(r_1/r_2)\ldots\right)_{n=1} = \ln(r_2/r_1)$

For the geometry indicated by dashed lines in fig. 4, i.e., for circular arcs of radii r_1, r_2 (the inner and outer boundaries) C_n is most easily calculated with eqs. (15) and (18a), and K_n in eq. (24a) has to be replaced by

$$K_n = \frac{\sin[(n+1)\varepsilon\pi/M]}{(n+1)\pi/M}. \tag{24b}$$

It follows from eq. (24) that for a given B_r, and

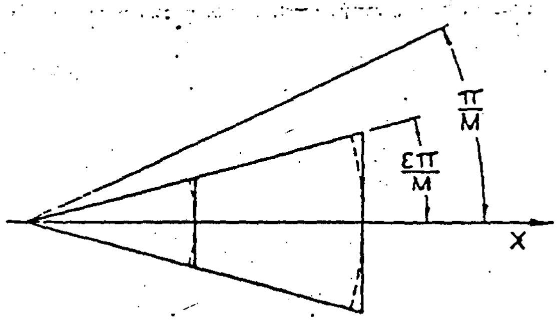


Fig. 4. One piece of a segmented REC multipole.

(4.8)

$b_n = C_{n0}\cdot\sum_{m=0}^{M-1}\exp(i\cdot 2\pi\cdot m(N-n)/M)$ $\qquad \sum_0^{M-1} q^n = \frac{1-q^M}{1-q}$

$b_n \neq 0$ only for $n = N+\nu\cdot M$; $\nu = 0, 1, \ldots$

$B^*(z_0) = \sum_{\nu=0} b_n z_0^{n-1} \qquad n = N+\nu\cdot M$

$b_n = M\cdot\frac{B_{r0}}{4\pi i}\cdot\oint\frac{dz^*}{z^n}$

Refer. block geometry: CSEM with $r_1 < r < r_2$, within $\varphi = \pm\varepsilon\cdot\frac{\pi}{M}$

$B^*(z_0) = B_r\sum_0\left(\frac{z}{r_1}\right)^{n-1}\cdot\frac{n}{n-1}\left(1-\left(\frac{r_1}{r_2}\right)^{n-1}\right)\cdot K_n$

$K_n = \frac{\sin(\varepsilon(n+1)\pi/M)}{(n+1)\pi/M} \qquad n = N+\nu\cdot M, \; \nu = 0, 1, \ldots$

Linear array of CSEM:

$z = r_1 + W$ (change of coordinate origin)

$r_2 = r_1 + D$ $\quad$ D = radial thickness of block; fixed.

$2\pi r_1/N = \lambda$ = period length; fixed

$2\pi/\lambda = k$; → $N = k r_1$

(4.10)

PURE CSEM CONFIGURATION PERFORMANCE

$$B^* = i \cdot 2 \cdot B_r \sum_{\mu=0} \cos(n\,k\,z) \cdot e^{-nkh} \cdot \frac{\sin(n\,\epsilon\,\pi/M')}{(n\pi/M')} \cdot (1 - e^{-nkL})$$

$n = 1 + \mu M'$
$k = 2\pi/\lambda$
$z = x + iy$
$B^* = B_x - i B_y$

Example:
for: $L = \lambda/2$
$M' = 4$
$B_r = 0.9$ Teslas (REC)

$$B^*_{\mu=0}(\text{Teslas}) = i \cdot 1.55 e^{-kh} \cdot \cos(kz)$$

(4.9)

$M' = M/N$ = # of blocks / period; fixed

$n = N(1+\mu M') = k r_1 \cdot (1+\mu M')$

let $r_1 \to \infty$:

$$\left(\frac{z}{r_1}\right)^{n-1} \longrightarrow \left(1 + \frac{kW}{kr_1}\right)^{kr_1 \cdot (1+\mu M')} = e^{kW(1+\mu M')}$$

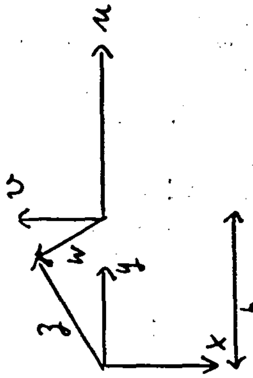
$$\left(\frac{r_1}{r_2}\right)^{n-1} \longrightarrow \frac{1}{\left(1+\frac{kD}{kr_1}\right)^{kr_1(1+\mu M')}} = e^{-kD(1+\mu M')}$$

$(n+1)/M = N(1+\mu M')/M'N = (1+\mu M')/M'$

Re-introduce n with new meaning $n = 1+\mu M'$

$$B^*(W) = B_r \cdot \sum_\mu e^{nkW} (1 - e^{-nkD}) \cdot \frac{\sin(\epsilon n \pi/M')}{n\pi/M'}$$

New coordinate system:

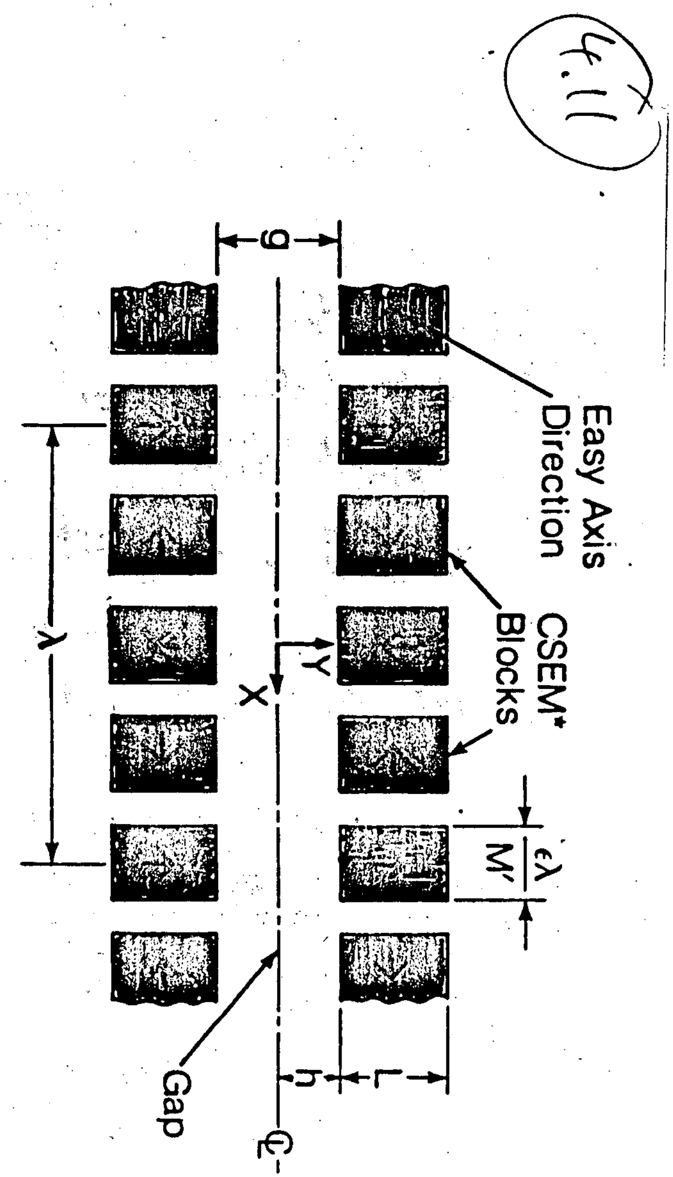


$y = u + h$; $u = y - h$
$x = -v$; $v = -x$
$W = -h + y - ix = -iz - h$

$$B^*(z) = B_r \cdot \sum_{\mu,\; n=1+\mu M'} e^{-inkz} \cdot e^{-nkh} (1 - e^{-nkD}) \frac{\sin(\epsilon n\pi/M')}{n\pi/M'}$$

Lower 1/2 gives same, except $z \to -z$

$e^{-inkz} + e^{inkz} = 2\cos nkz$



PURE CSEM* W/U CROSS SECTION

*Current Sheet Equivalent Material - e.g. REC

(4.12)

Hybrid Theory

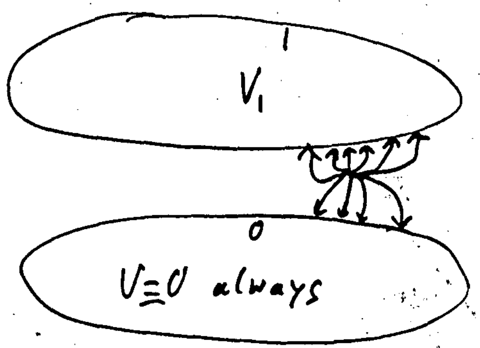
$\mu = \infty$. Reason: Nearly always, when μ is small enough to make a significant difference device will be too sensitive to μ to be usable. $\mu = \infty$ does not prevent calculation of flux density in iron to sufficent accuracy.

$\mu_{\parallel}, \mu_{\perp} \neq 1$ for general theory, but usually $\mu_{\parallel} = \mu_{\perp} = 1$ in some part of applications

General 3D theory.

Represent CSEM by $\mu_{\parallel}, \mu_{\perp}$, charges. Start with • charge and 2 iron surfaces, then proceed to dipole, + finally distribution of dipoles $\Leftrightarrow \vec{B}_r$. Later any number of iron surfaces.

4.13



"Construct" solution that satisfies M-equ's in space outside iron and has total flux entering surface 1 equal 0. Solution = linear superposition of 2 solutions that satisfy M-eqn's outside iron:

1) $q\neq 0$; $V_1 = V_q(\vec r_1) = 0$; $V_q(\vec r) \to \vec H_q \to \Phi_q = \int_1 \mu_0 \vec H_q\cdot d\vec a = q\cdot c_1$

↑ direct fields ← indirect fields

2) $q = 0$; $V_1 = V_s(\vec r_1) = V_{s0}$; $V_s(\vec r) \to \vec H_s \to \Phi_s = \int_1 \mu_0 \vec H_s\cdot d\vec a = V_{s0}\, c_2$

3) $V = V_q - V_s \to \vec H = \vec H_q - \vec H_s$; $\Phi = \Phi_q - \Phi_s = q\cdot c_1 - V_{s0} c_2 = 0$

$V_{s0} = q\cdot c_1/c_2$

4.14

Calculation of c_1

Result : $c_1 = V_s(\vec r_q)/V_{s0}$

Proof : Consider $I = \int (V_s \vec B_q - V_q \vec B_s)\cdot d\vec a$ over all surfaces, enclosing total volume ≠ iron

On surface 0 : $V_q = V_s = 0$

On surface 1 : $V_q = 0$; $V_s = V_{s0}$

"At ∞", V·B goes stronger to 0 than a goes to ∞

$I = V_{s0}\cdot \Phi_q$

$$\mathrm{div}(V_s\vec B_q - V_q \vec B_s) = V_s \overbrace{\mathrm{div}\,\vec B_q}^{q} - V_q \overbrace{\mathrm{div}\,\vec B_s}^{0} + \underbrace{\vec H_q\cdot\vec B_s - \vec H_s\cdot\vec B_q}_{=0}$$

$$\tfrac{1}{\mu_0}\,\vec H_q\cdot\vec B_s = (\vec H_{q\parallel} + \vec H_{q\perp})\cdot(\mu_\parallel \vec H_{s\parallel} + \mu_\perp \vec H_{s\perp}) = \mu_\parallel H_{q\parallel}H_{s\parallel} + \mu_\perp H_{q\perp}H_{s\perp}$$

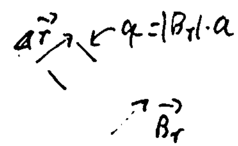
$I = V_{s0}\,\Phi_q = V_s(\vec r_q)\cdot q$ q.e.d.

$\Phi_q = q\cdot V_s(\vec r_q)/V_{s0}$

Dipole $\Delta\vec r$: $-q \to +q$

$$\Phi_D = q\,(V_s(\vec r + \Delta\vec r) - V_s(\vec r))/V_{s0} = -\underbrace{q\Delta\vec r}_{\text{dipole moment}}\cdot\vec H_s/V_0$$

(4.15)



$\vec{B}_r$: $q \cdot \Delta\vec{r} = |B_r| \cdot a \cdot \Delta\vec{r} = \vec{B}_r \, dv$

$$\phi_{B_r} = -\int \vec{B}_r \cdot \vec{H}_s \, dv \, / \, V_{so}$$

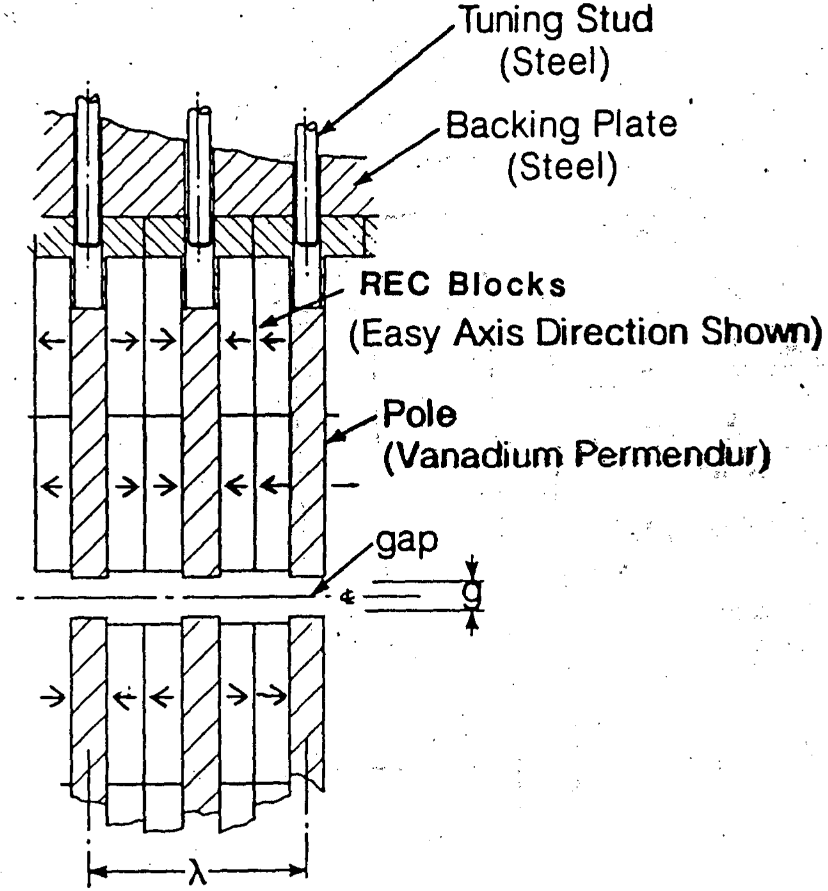
Optimum $\vec{B}_r \parallel \vec{H}_s$, <u>but</u>: $\cos 20° \approx .94$, i.e. exact easy axis orientation not worth great effort + expense.

In most systems, all, or most CSEM surfaces "with surface charges" are in direct contact with iron. This is practically always true in vicinity of field region used → fields there = <u>indirect fields</u>

Most computational effort spent on calculating $c_2 = \phi_s / V_{so}$.

(4.16)

Hybrid Insertion Device configuration with field tuning capability.



(4.17)

"Broad-brush" design procedure

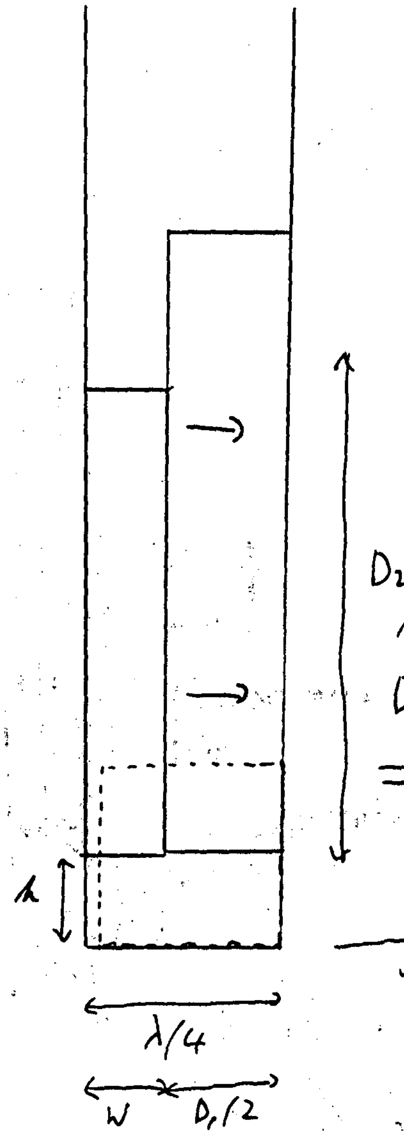
1) Design surfaces to which $\vec{B}$ is $\perp$ (either because of symmetry or because they are $\mu = \infty$ iron surfaces) to get desired field distribution. Preferably "business region" has only indirect fields

2) Determine the scalar potential(s) necessary on pole(s) to get desired field distribution and strength

3) Design rest of iron, and placement of CSEM, to produce these potentials.

Step 3 involves usually the most work, since 3D effects have to be taken into account.

(4.18)

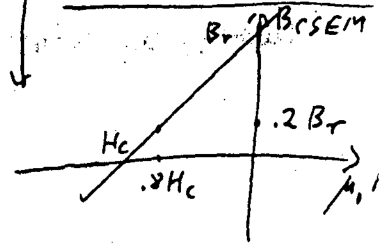


$$\lambda \cdot \bar{H} = \frac{D_1}{2} H_{CSE}$$

$$\bar{H} = H_{CSEM} \cdot \frac{D_1}{2\lambda}$$

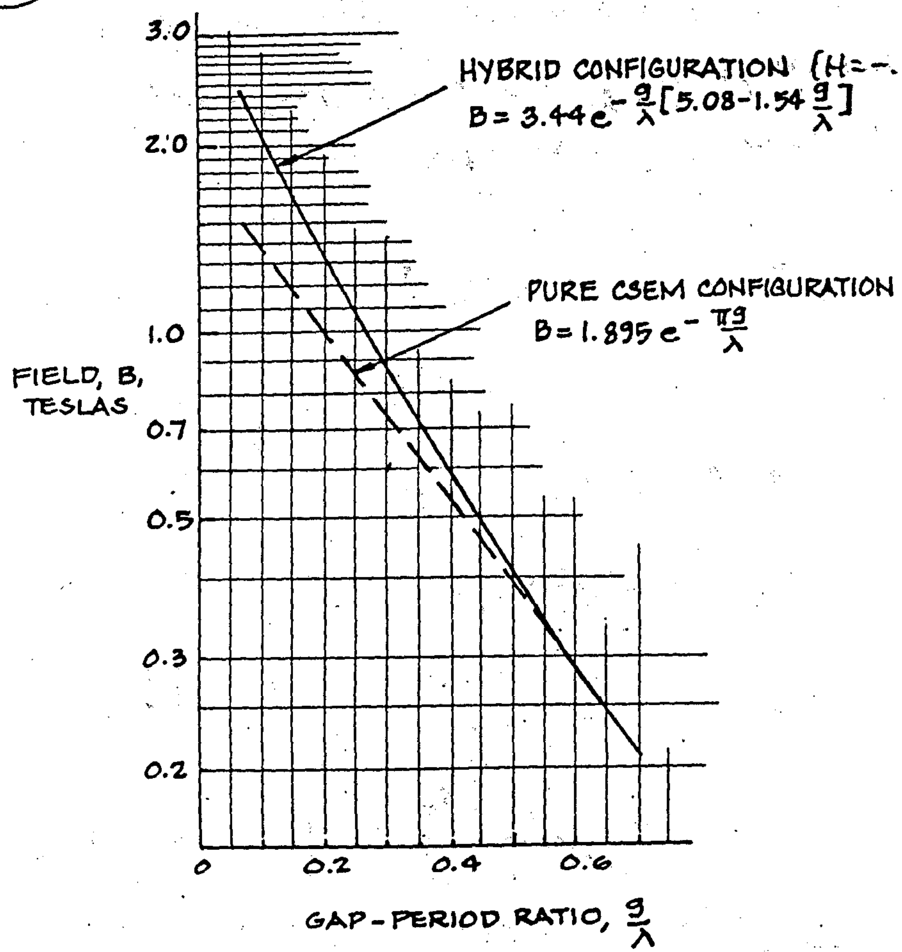
$$\mu_0 \bar{H} \cdot W_{eff} = B_{CSEM} \cdot D$$

$$D_{2\,eff.} = \frac{\mu_0 \bar{H} \cdot W_{eff}}{B_{CSEM}}$$



(4.19)

PURE CSEM AND HYBRID UNDULATOR / WIGGLER PERFORMANCE FOR NdFe (B_r = 1.1 TESLAS)



(4.20)

2D Hybrid U/W Design.

2D design <u>not</u> adequate; do it to develop idea.

Assume CSEM does not overhang:

$$\phi'_{CSEM} = B_r \cdot \text{height of CSEM.}$$

Necessary $\tilde{V}$ ($\sim B_0$) with POISSON (or analytical equ's. to be developed).

Units $\tilde{V} = B_0 \times$ length.

My notation: $\tilde{V} = B_0 \cdot D_4$

For pole length in x-dir $\gtrsim \lambda/4$, $D_4 = 1/2$ gap.

First approxim. beyond that:

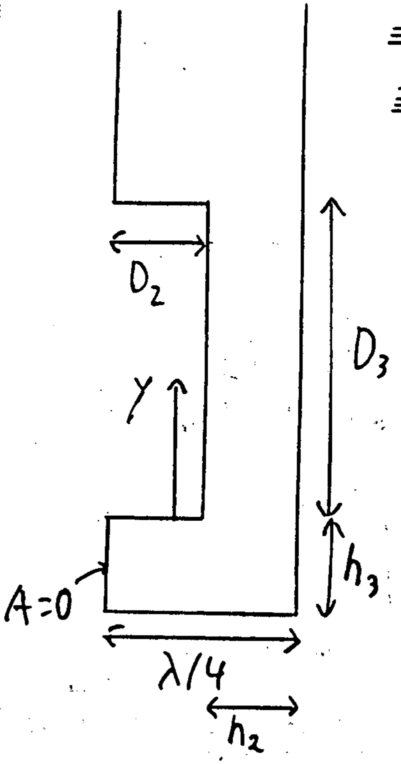
$$B^x = B_0 \cosh kz \; ; \; k = 2\pi/\lambda$$

$$\tilde{V} = B_0 \cdot \int_0^h \cosh ky\, dy = B_0 \cdot \frac{\sinh kh}{k}$$

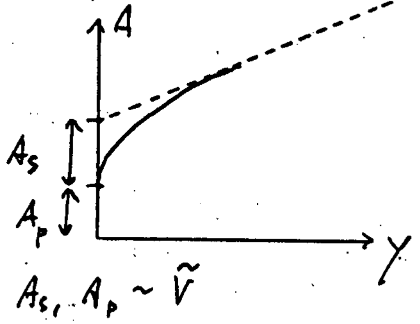
$$D_4 \approx h \cdot \frac{\sinh(kh)}{kh}$$

(4.21)

ϕ_s-calculation.



Central concept: excess flux → excess flux coefficients.



$A_s, A_p \sim \tilde{V}$

$$A(Y) = \tilde{V}(E_p + E_s + Y/h_2)$$

From POISSON (or analytically)

$$E_p = A(0)/\tilde{V}$$

$$E_s = \left(A(Y)/\tilde{V} - E_p - Y/h_2\right)_{Y=\text{large enough}}$$

"large enough" not "large" because of exponential decay ($\sim e^{-\pi y/h_2}$) of deviation of field from homogeneous field.

(4.22)

"Complete" Design of this simple model

Assume CSEM touches pole over length D_3:

$$B_r \cdot D_3 = \overbrace{B_0}^{\tilde{V}_{pole} = \tilde{V}_0} \cdot D_4 \left(E_p + \mu_{11} \cdot E_s + E_T + \mu_{11} \cdot D_3/h_2\right)$$

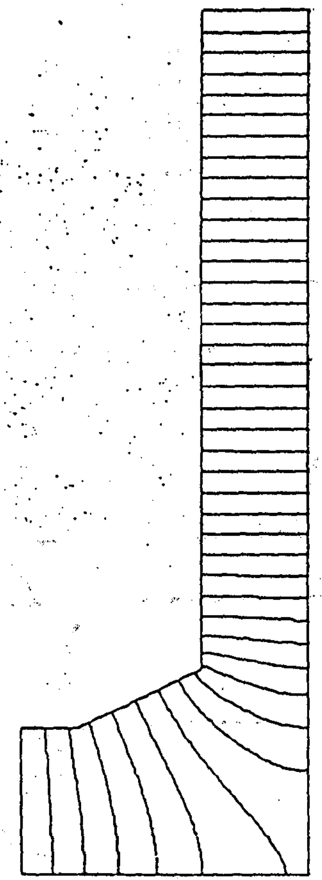
$$D_3 = \frac{B_0/B_r \cdot D_4 (E_p + \mu_{11} E_s + E_T)}{1 - \frac{B_0}{B_r} \cdot \frac{D_4}{h_2} \cdot \mu_{11}}$$

Yes, it is that simple!!!

CSEM overhang, 3D adds more terms, but structure of design equation remains essentially unchanged!!!

Additional terms require development of a number of additional formulae (not all of which are very simple), but structure of design equation does <u>not</u> change.

(4.22a)



PROB. -InDev Pol V: 1.22 11/09/88/ CYCLE - 800

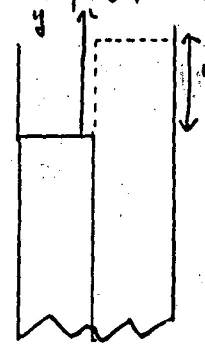
$A(x,0) = A(0,0)\left(1 + a_3 + a_5 + a_7 + a_9\right)$ $\qquad A(x, y=0.44)$

	$A(x,0)$	$A(x, y=0.44)$
$a_3 =$	$4.03\cdot10^{-6}$	$-7.30\cdot10^{-6}$
$a_5 =$	$1.16\cdot10^{-6}$	$9.48\cdot10^{-6}$
[illegible]	[illegible]	$9.12\cdot10^{-6}$

(4.23)

This design equation is characteristic for most hybrid devices!!!

Why overhang on top?



More flux on iron from CSEM.

New design equ:

$$B_r D_3 + B_r \int_0^{y_1} V(y)\,dy/V_0 = \tilde{V}_0\left(E_{tot} + \mu_{\parallel} D_3/h_2\right)$$

$$D_3 = \frac{\tilde{V}_0 E_{tot}/B_r - \int_0^{y_1} V(y)\,dy/V_0}{1 - \frac{\tilde{V}_0 \mu_{\parallel}}{B_r}\cdot\frac{1}{h_2}}$$

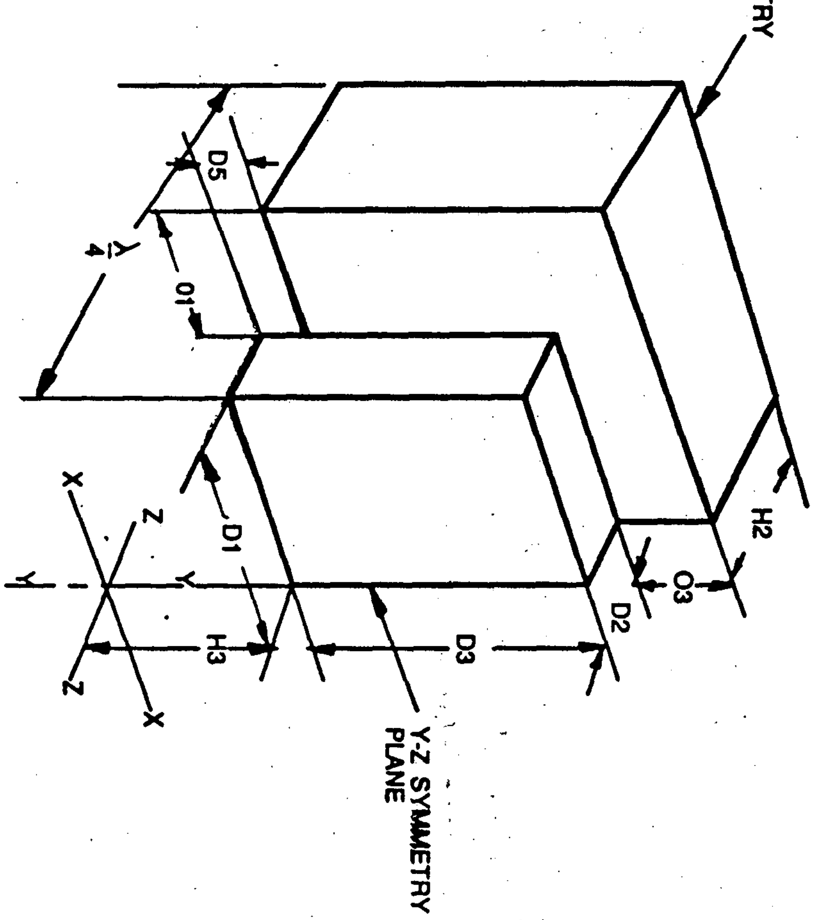
$$L_{CSEM} = D_3 + y_1 \; ; \; L'_{CSEM} = 1 - \frac{V(y_1)/V_0}{\frac{V_0/h_2}{H_c}} = 0$$

$$V(y_1)/V_0 = 1 - H_{CSEM}/H_c$$

For $H_{CSEM}/H_c \approx .8$, $V(y_1)/V_0 = .2$

Overhanging CSEM on top reduces amount of CSEM; overhang on side increases achievable B_0

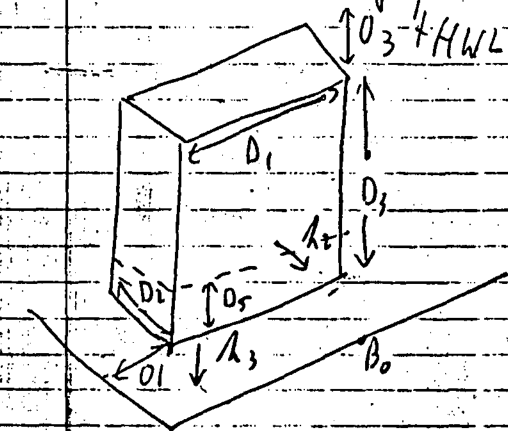
HYBRID CONFIGURATION GEOMETRY



4-25 (1)

because of modest modifications now called HW2

Formula summary for HW1, for Bob L. 1/23/86



D3 + HW2 = HW1, but FNGI(O1/H2) only in
O1, O3 = CSEM overhang

h_2 = dist. to symm. plane

h_3 = 1/2 gap

$D_1 = \frac{1}{2}$ width of pole } 1/4 of iron pole

$D_2 = \frac{1}{2}$ length of pole

D_3 = height of pole

D_5 = start of CSEM above pole

$D_2 + h_2 = \lambda/4$

V_0 = sc. pot. of pole

E2 = Excess flux coeff.

E1 = Excess flux

B3 = B_r

$V_0 = B_0 \cdot D_4$

↑ from POLE2, or POISSON

$\frac{V_0}{B_r} = D_0$; Calculation below determines D_3 = only unknown

(1) $\phi = V_0 \left(D_3 \left(\frac{\mu D_1}{h_2} + G_0 \right) + D_1 (E_1 + G_0) + D_2 E_2 \right)$ — terms numbered ① ② ③ ④ ⑤

$a = \frac{D_2 + h_2}{h_2} = 1 + \frac{D_2}{h_2}$; $G_0 = \left[(a+1) \ln(a+1) - (a-1) \ln(a-1) \right] / \pi$

In (1), ① = "homogeneous field flux"

② = excess flux at lateral end

66 4-26

③ flux into pole face, + excess flux into side at bottom ($\to A_p + dA_s$)

from POLE2 or POISSON

④ Excess flux into top.

⑤ Excess flux into D_7-edge., $E_2 \simeq .5$

ϕ = flux out of pole, = ϕ into pole from CSEM, from (2

(2) $\phi = B_r\left((D_3 - D_5)\left(D_1 + h_2 \cdot G_1(O_1/h_2)\right) + D_1\, h_2\, G_1(O_3/h_2)\right)$

① ② ③

In (2), ① = flux from "touching" charge sheet.

② = flux from "overhanging" CSEM on lateral side

③ = " " " " on top.

Set right side of (1) equal right side of (2) + solve for D_3

$$D_3\left(D_1 + h_2 G_1(O_1/h_2) - D_0\left(\frac{M D_1}{h_2} + G_0\right)\right)$$

$$= D_0\left(D_1(E_1 + G_0) + D_2 E_2\right) + D_5\left(D_1 + h_2 G_1(O_1/h_2)\right)$$

$$- D_1 h_2 G_1(O_3/h_2)$$

4-27

```
CLS
PRINT DATE$;"  ";TIME$;"  HW4"
PI=4*ATN(1)
A1$="D1=##.### D2=##.### H2=##.### H3=##.### D4=##.### D5=#.### "
A2$="33=##.### E2=#.### E1=#.### E3=#.### M1=#.###"
A3$="D7=##.### D8=##.###"
A4$="D3=##.###^^^^ V3=##.###^^^^ CO=##.###^^^^ C1=##.###^^^^"
GOSUB DAT
PRINT
PRINT USING A1$;D1;D2;H2;H3;D4;D5
PRINT USING A2$;B3;E2;E1;E3;M1
INPUT "BO=";DO
DO=D4*DO/B3:A9=1+D2/H2
GO=((A9+1)*LOG(A9+1)-(A9-1)*LOG(A9-1))/PI:B9=SQR(A9*A9-1)
G1=GO+2*((A9-1)*LOG(A9-1)-A9*LOG(A9))/PI:G1=G1*(H2+D2)*2/PI
D7=D1*(E1+GO)+D2*E2:D8=M1*D1/H2+GO
PRINT USING A3$;D7;D8
PRINT
K9=(B9/(A9+1))^(1/A9):K8=8*A9*A9/B9/PI/PI
S1=SIN(.5*PI/A9):S3=SIN(1.5*PI/A9)*(1-2/B9/B9)/9
100 INPUT "O1,O3=";O1,O3
D3=H2*FNG1(O1/H2)
D6=DO*D7-D1*H2*FNG1(O3/H2)+D5*(D1+D3)
D3=D1+D3-DO*D8:D3=D6/D3
V3=(D1+O1)*(D3+O3-D5)*H2
C1=D7+D3*D8:CO=D1*E3+(H2+D2)*2/PI*LOG(1+(D3+.5*D1)/(H3+G1))
C1=C1-CO:CO=4*CO
PRINT USING A4$;D3;V3;CO;C1
PRINT:GOTO 100

DEF FNG1(X)
E9=K9*EXP(-.5*PI*X/A9):FNG1=GO-K8*E9*(S1+S3*E9*E9)
IF X=0 THEN FNG1=0
END DEF

DAT:
READ D1,D2,H2,H3,D4,D5,B3,E2,E1,E3,M1
DATA 2.5,.25,.45,.5,.62519,.1,10.6,.5,1.002676,1.147476,1.03
RETURN
```

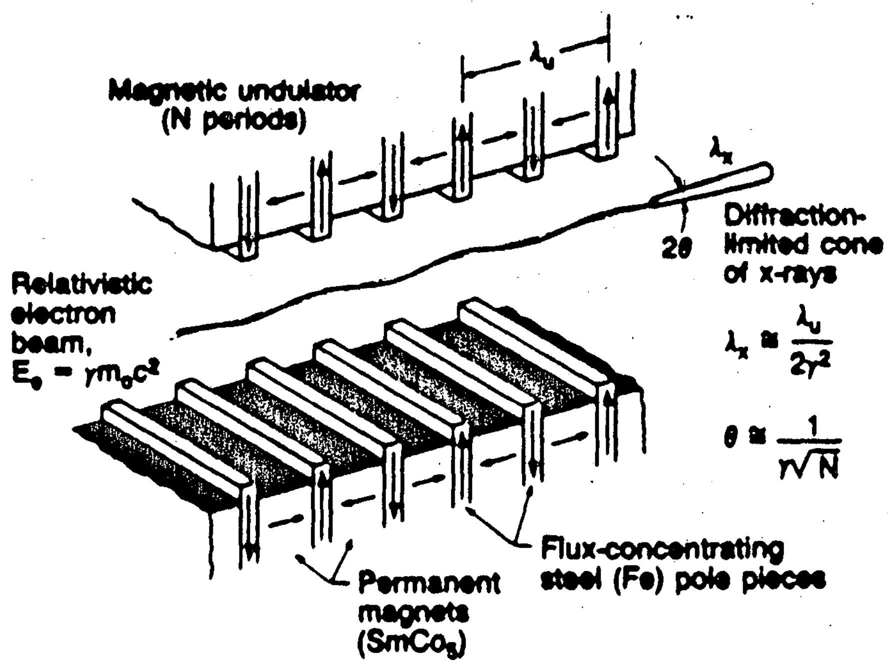
Insertion Device Design

Klaus Halbach

Lecture 5.

November 18, 1988

Lecture 6 – Dec. 2
Lecture 7 – Dec 13



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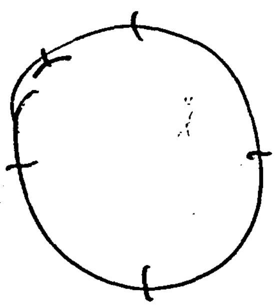
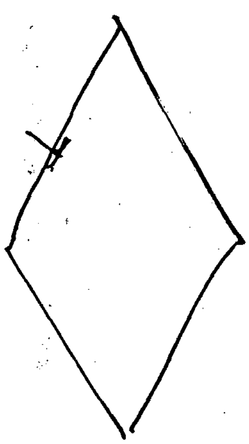
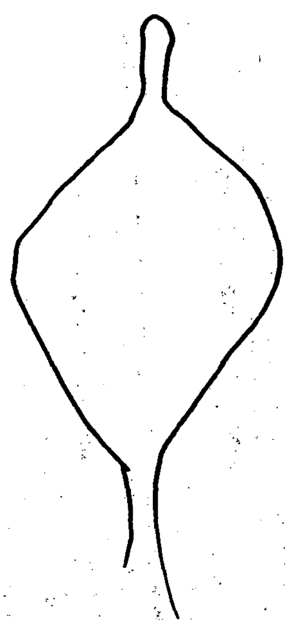






(10.24)

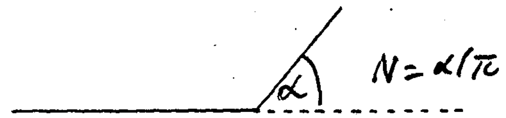
```
DEFDBL A-I,K-Z
DEFINT J
PI=4*ATN(1)
CLS
PRINT DATE$;"  ";TIME$;"  FEXP2"
REM--------Expansion of F' of semi 1/0-dipole with slanted (angle=N1*PI) side.

A$="+#.####^^^^ "
START:
PRINT
INPUT "N,J9=",N1,J9
DIM A1(0:J9),B1(0:J9),C1(0:J9)
A1(0)=1:A1(1)=1:CALL POWER2(J9,N1,A1()):REM-------------(1+T)^N1
B1(1)=1
FOR J1=2 TO J9
  S1=0
  FOR J2=1 TO J1-1
    S1=S1+B1(J2)*A1(J1-J2)
  NEXT J2
  B1(J1)=S1/(J1-1):REM--------------------------------W(T) with B1(1)=1
NEXT J1

T1=.5:REM--S1 leads to W(-T1), S2 leads to D1. C2 for renormalization of W(T).
F1=-T1:F2=1-T1
S1=B1(J9):S2=1/(N1+1+J9)
FOR J1=J9-1 TO 0 STEP -1
  S1=S1*F1+B1(J1)
  S2=S2*F2+1/(N1+1+J1)
NEXT J1
D1=S2*F2^(N1+1):C2=-EXP(-D1)/S1:PRINT USING A$;D1/PI

FOR J1=1 TO J9:B1(J1)=C2*B1(J1):NEXT J1:REM----------Renormalization of W(T).
A1(0)=1:A1(1)=1:CALL POWER2(J9,-N1,A1()):REM----------F' coefficients.
CALL INVERT(J9,B1(),C1()):REM--------------------------C1=T(W)
CALL INSERT(J9,C1(),A1(),B1()):B1(0)=A1(0):REM------------Insert T(W) in F'(T)
REM------------------------------------------------------- and get F'(W).
FOR J1=0 TO J9:PRINT USING A$;B1(J1);:NEXT J1
ERASE A1,B1,C1
GOTO START

SUB POWER2(J9,E,A1(1)):REM-------------------Raises A(0)+A(1)*X to power E.
K=A1(1)/A1(0):A1(0)=A1(0)^E
FOR J1=1 TO J9
  A1(J1)=A1(J1-1)*K*(E+1-J1)/J1
NEXT J1
END SUB

SUB INSERT(J9,A1(1),B1(1),C1(1)):REM------Insert one series into another.
DIM A2(0:J9,0:J9)
CALL MATR(J9,A1(),A2())
C1(1)=A2(1,1)*B1(1)
FOR J1=2 TO J9
  S=0
  FOR J2=1 TO J1
    S=S+A2(J1,J2)*B1(J2)
  NEXT J2
  C1(J1)=S
NEXT J1
ERASE A2
END SUB
```

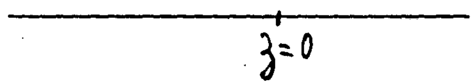
Other procedures same as in FEXP1

(10.25)

01-14-1989 18:47:49 FEXP2
Coefficients of expansion of F' in exponentials in dipole with sloping side (N=angle/PI).

N	A1	A2	A3	A4	A5
0.1	-8.5773E-02	+4.7821E-02	-3.3603E-02	+2.6078E-02	-2.1393E-02
0.2	-0.1499E+00	+8.9911E-02	-6.5715E-02	+5.2378E-02	-4.3841E-02
0.3	-0.1995E+00	+0.1260E+00	-9.4898E-02	+7.7232E-02	-6.5687E-02
0.4	-0.2338E+00	+0.1568E+00	-0.1209E+00	+9.9999E-02	-8.6148E-02
0.5	-0.2707E+00	+0.1832E+00	-0.1438E+00	+0.1205E+00	-0.1049E+00
0.6	-0.2970E+00	+0.2058E+00	-0.1639E+00	+0.1390E+00	-0.1220E+00
0.7	-0.3190E+00	+0.2254E+00	-0.1818E+00	+0.1554E+00	-0.1375E+00
0.8	-0.3378E+00	+0.2425E+00	-0.1975E+00	+0.1702E+00	-0.1514E+00
0.9	-0.3539E+00	+0.2574E+00	-0.2115E+00	+0.1834E+00	-0.1640E+00
1.0	-0.3679E+00	+0.2707E+00	-0.2240E+00	+0.1954E+00	-0.1755E+00





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(10.22)

$$\pi z = \ln W \; ; \; \pi \dot z = \dot W / W = \frac{(1+t)^N}{t}$$

Shows clearly that W is determined uniquely, except for freely choosable multiplication factor that is obviously related to where one wants $z=0$ to be.

Ansatz: $W = C\cdot \sum_1 b_n t^n = C\cdot g(t)\,;\; b_1 = 1$

C to be determined later.

$$(1+t)^N = \sum_0 a_m t^m \;;\; a_m = \text{known};\; a_1 = 1$$

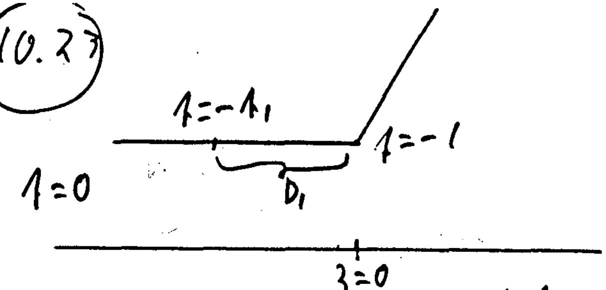
$$t\dot W = W(1+t)^N$$

$$\sum_1 n b_n t^n = \sum_{m=1} b_m a_\mu t^{m+\mu} \overset{m+\mu=n}{=} \sum_{m=1} b_m a_{n-m} t^n$$

$$b_n(n-a_0) = b_n(n-1) = \sum_{m=1}^{n-1} b_m a_{n-m} \qquad n\geq 2,\; b_1 = 1$$

Determination of C. Use $z=0$ in midplane under corner.

(10.23)



$$\pi D_1 = \int_{t_1}^{1} \frac{(1-t)^N}{t}\,dt = \int_0^{1-t_1} \frac{t^N}{1-t}\,dt$$

Do this (integrator) either with Romberg or Taylor series.

T-s: $\pi D_1 = (1-t_1)^{N+1} \sum_0 \frac{(1-t_1)^n}{N+n+1}$

Use also $W(-t_1) = e^{\pi(-D_1+i)} = -e^{-\pi D_1} = C\,g(-t_1)$

$$C = -e^{-\pi D_1} / g(-t_1)$$

I now have Taylor series for expansion of $W = e^{\pi z}$ in t. Invert that series, i.e. get series for $t = t(W)$, and use that in series for $F' = (1+t)^{-N}$

10.20

01-07-1989 15:30:38 FEXP1 ← coeff. for expansion of F' (first row of each block); coeff. for expansion of F (second row of each block)

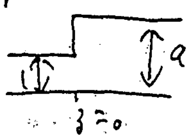
1.2						
	1.1102E+00	-0.5336E+00	0.2818E+00	-8.3557E-02	-6.9646E-02	0.1680E+00
	-0.8481E+00	0.1359E+00	-4.3054E-02	9.1190E-03	5.9118E-03	-1.1670E-02
1.4 ← a						
	1.0764E+00	-0.3243E+00	-6.8501E-02	0.2547E+00	-0.2247E+00	5.8169E-02
	-0.9594E+00	9.6341E-02	1.2211E-02	-3.2433E-02	2.2257E-02	-4.7131E-03
1.6						
	1.0127E+00	-0.1142E+00	-0.2693E+00	0.2360E+00	2.4519E-02	-0.1902E+00
	-1.0315E+00	3.8784E-02	5.4863E-02	-3.4342E-02	-2.7750E-03	1.7617E-02
1.8						
	0.9436E+00	5.0406E-02	-0.3227E+00	7.2309E-02	0.1926E+00	-0.1140E+00
	-1.0813E+00	-1.9254E-02	7.3951E-02	-1.1837E-02	-2.4524E-02	1.1875E-02
2.0						
	0.8774E+00	0.1689E+00	-0.2925E+00	-8.1299E-02	0.1962E+00	5.6283E-02
	-1.1171E+00	-7.1663E-02	7.4475E-02	1.4788E-02	-2.7753E-02	-6.5147E-03
2.2						
	0.8167E+00	0.2506E+00	-0.2265E+00	-0.1781E+00	0.1120E+00	0.1557E+00
	-1.1438E+00	-0.1170E+00	6.3440E-02	3.5632E-02	-1.7426E-02	-1.9823E-02
2.4						
	0.7620E+00	0.3053E+00	-0.1513E+00	-0.2206E+00	1.0164E-02	0.1651E+00
	-1.1642E+00	-0.1555E+00	4.6226E-02	4.8160E-02	-1.7255E-03	-2.2926E-02
2.6						
	0.7130E+00	0.3407E+00	-7.9494E-02	-0.2246E+00	-7.4608E-02	0.1188E+00
	-1.1802E+00	-0.1880E+00	2.6316E-02	5.3112E-02	1.3721E-02	-1.7872E-02
2.8						
	0.6692E+00	0.3626E+00	-1.6225E-02	-0.2051E+00	-0.1321E+00	5.1965E-02
	-1.1929E+00	-0.2155E+00	5.7944E-03	5.2218E-02	2.6163E-02	-8.4208E-03
3.0						
	0.6300E+00	0.3756E+00	3.7205E-02	-0.1731E+00	-0.1637E+00	-1.3249E-02
	-1.2031E+00	-0.2387E+00	-1.4211E-02	4.7221E-02	3.4738E-02	2.3003E-03
	1	3	5	7	9	11

01-07-1989 15:33:02 FEXP1

0.9670E+00	-4.9790E-08	-0.3170E+00	0.1317E+00	0.1559E+00	-0.1641E+00
-1.0662E+00	1.8300E-08	6.9910E-02	-2.0751E-02	-1.9100E-02	1.6451E-02

↑ $a = \sqrt{3}$

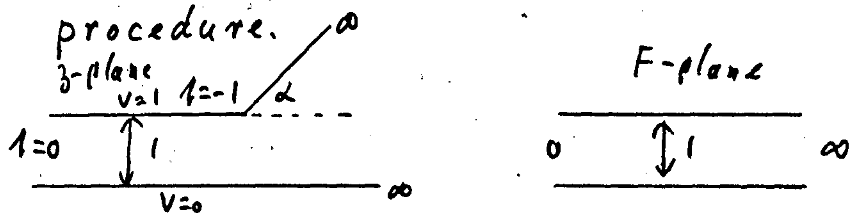
Coefficient for expansion of F', F in $e^{-\pi z/(2a)}$ in



10.21

Expansion of F' in exponentials when $\dot{z}$ can not be integrated in closed form.

Use specific example to explain general procedure.



$$\pi \dot{z} = \frac{(1+t)^N}{t} \; ; \; N = \alpha/\pi \; ; \; \pi \dot{F} = 1/t$$

$$F' = (1+t)^{-N}$$

Physics / Math: $|t| \ll 1$: $\pi z \sim \ln t$; $t \sim e^{\pi z}$

Problem: where is $z=0$? Or: how do I put $z=0$ where I want it to be? Will show up as an indeterminate constant that has to be chosen to locate $z=0$ as wanted

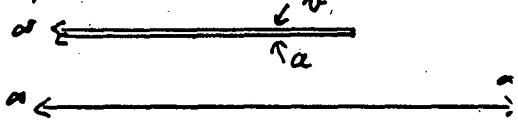
Procedure: get from equ. for $\dot{z}$ Taylor series of $W = e^{\pi z}$ in t; invert + use in F!

(10.14)

#3

$F(z) = \int_0^z \sqrt{z}\cdot \exp(z + a z^3)\, dz$. Express $F(z)$ with the <u>help</u> of a Taylor series, and give the recursion formula for the coefficients of that series.

#4



For capacitor with zero-thickness electrode (Rogowski-capacitor; viewgraph 8.10) and halfgap = 1, calculate the excess flux coefficient for the flux entering the lower surface (a) of the electrode

#5

Calculate the excess flux coefficient for the upper surface (b) of the electrode of the Rogowski capacitor.

(10.15)

<u>Hint</u> for #4 and #5: While "ideal" flux in #4 is obvious, for #5 one has to "invent" an appropriate model for the "ideal" flux formula. This formula is not unique, but it has to have the correct asymptotic behaviour. Use $z(1)$, $F(1)$,

#6

For Rogowski capacitor, expand the error fields between the electrodes in exponentials to 3. order by hand, i.e. give closed expressions.

<u>Hint</u>: Use $z(1)$, $F(1)$, $F'(1)$

(10.12)

A_{nm} : $w = \sum_1 a_n z^n = z \sum_1 a_n z^{n-1}$

Obviously: $A_{n1} = a_n$; $A_{nm} = \begin{cases} 0 & n<m \\ a_1^m & n=m \end{cases}$

$$w^{m-1} = \sum z^\mu A_{\mu\, m-1}$$

$$w^m = \sum z^{\mu+\kappa} A_{\mu\, m-1}\, a_\kappa = \sum z^n A_{n m}$$

$\mu + \kappa = n$; $\kappa = n - \mu$

$$A_{n m} = \sum_{\mu=m-1}^{n-1} A_{\mu\, m-1}\, a_{n-\mu} \qquad w^m = \sum z^n A_{nm}$$

Simple recursion formula to calculate new columns in A_{nm}.

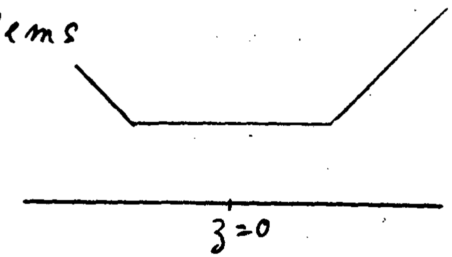
blem # 4: Using $w = \sum_1 a_n z^n$ in $F(z) = \sum_1 w^m b_m$

$$F(z) = \sum z^n A_{nm}\, b_m$$

New coefficient array = product of A-matrix x old coefficient array b, = algorithm for problem 4).

(10.13)

Homework Problems



#1

Assume that a symmetric dipole is wide enough so that for analysis of error fields, error fields at each end can be obtained from semi-∞ dipole model. Using these coefficients for exponential decay of error fields, write formula for error fields for the finite width dipole

#2

Develop recursion formula for coefficients of a Taylor series if one known Taylor series is divided by another Taylor series with known coefficients.

$A(x) = \sum a_n x^n$; $B(x) = \sum b_n x^n$

$C(x) = A(x)/B(x) = \sum c_n x^n$; a_n, b_n = known

c_n = wanted.

54

(10.11)

Problem #1: $F = \sum_{\mu=0}^{M} a_\mu x^\mu$; $G = \sum_{n=0}^{\infty} b_n x^n = F^\varepsilon$;

(a_μ known; b_n to be computed, $b_0 = a_0^\varepsilon$)

Method, applicable to many problems:
Transform original problem into a differential equation that can be solved by Taylor series expansion:

$\ln G = \varepsilon \ln F$; $G'/G = \varepsilon F'/F$

$F G' = \varepsilon G F'$

$$\sum_{n,\mu} n b_n a_\mu x^{n+\mu-1} = \sum_{n,\mu} \varepsilon \mu b_n a_\mu x^{n+\mu-1}$$

$n+\mu = m$; $\mu = m-n$

$$\sum_{m=1} x^{m-1} \cdot \sum_{n=n_0}^{m} b_n a_{m-n} \left(\varepsilon(m-n) - n \right) = 0$$

n_0 = larger of 0 ; $m-M$

$$b_m a_0 m = \sum_{n=n_0}^{m-1} b_n a_{m-n} \left(\varepsilon(m-n) - n \right)$$

$$b_m = \left(\sum_{n=n_0}^{m-1} b_n a_{m-n} \left(\varepsilon - n \cdot \frac{\varepsilon+1}{m} \right) \right) / a_0$$

$b_0 = a_0^\varepsilon$

(10.12.1)

Problem #3 + #4: Inversion of Taylor series.

Given: $W = \sum_1 a_n z^n$ a_n = given

Wanted: $z = \sum_1 b_m W^m$ b_m = wanted.

Important: $a_0 = 0$ for our problem, and $a_0 = 0$ is necessary for simple solution: if $a_0 \neq 0$, there are as many different solutions as the order of the original series, since b_0 gives z for $W=0$, i.e. b_0 can be any one of the solutions to the equation $\sum_0 a_n z^n = 0$

Procedure: From $W = \sum_1 a_n z^n$, develop recursion formula for A_{nm} in

$$W^m = \sum z^n A_{nm}, \quad m = \text{integer} \geq 1.$$

Using that in $\sum b_m W^m$ solves problem 4) when the b_m are considered known; and yields expressions for b_m if $\sum b_m W^m$ is set equal z

(10.9)

Execution by hand to order $g^{3/2}$ (= first non-trivial term)

$$K = \left(\frac{a-1}{a+1}\right)^{1/a}$$

$$F' = \sqrt{Kg}\cdot\frac{1}{b}\left(2+\frac{u}{a}\right)\left(1-u\left(\frac{1}{a-1}-\frac{1}{a+1}\right)\right)^{1/2a}$$

$$F' = \sqrt{Kg}\cdot\frac{2}{b}\left(1+u\cdot\left(\frac{1}{2a}-\frac{1}{ab^2}\right)\right)$$

$$F' = \sqrt{Kg}\cdot\frac{2}{b}\left(1+\frac{u}{2a}\left(1-2/b^2\right)\right)$$

$$\frac{u}{2a} = Kg = \left(\frac{a-1}{a+1}\right)^{1/a}\cdot e^{-i\,3/a}$$

$$F' = \frac{2}{b}\cdot\sqrt{Kg}\left(1+Kg\left(1-2/b^2\right)\right)$$

no 3. harmonic for $a=\sqrt{3}$ $\to 1-1/a = .4226$

$$F = -\frac{4a}{\pi b}\cdot\sqrt{Kg}\left(1+Kg\left(1-2/b^2\right)/3+\cdots\right)$$

For calculation of flux from overhanging CSEM, need to integrate $V = \Im m F$ from $x+i$ to $\infty+i$. To calculate flux from CSEM attached to surface $z=i$ to $z=ia$, have to integrate V from $x+i$ to $x+ia$. Both integrals trivial

(10.10)

To get "all" expansion coefficients, need following algorithms:

1) Expansion coefficients for $(1+u\cdot x)^{\varepsilon}$.
2) Expansion coefficients for product of T.-series
3) Inversion of Taylor series
4) Use Taylor series as variable in a T.-series.

Because of the importance of the result, and because of the wide applicability of the methodology used to derive result, do, instead of 1), $\left(\sum_{\mu=0}^{M} a_\mu x^\mu\right)^{\varepsilon}$.

Problem 2: But first, because it is trivial, 2):

$$\sum_n a_n x^n\cdot\sum_\mu b_\mu x^\mu = \sum_{n,\mu} a_n b_\mu x^{n+\mu} = \sum_m c_m x^m$$

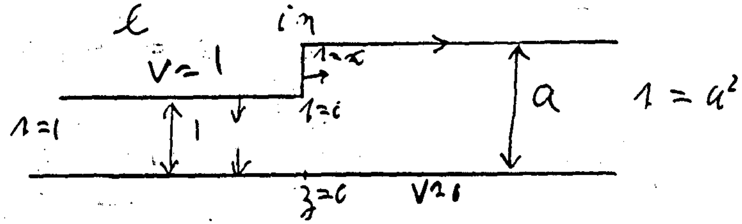
$n+\mu = m$; $\mu = m-n$

$$\sum_m c_m x^m = \sum_m x^m \sum_n a_n b_{m-n}$$

$$c_m = \sum_{n=0}^{m} a_n b_{m-n}$$

(10.7)

Expansion of F in Taylor series of $e^{-\pi z/2a}$ in



From excess flux calculation, with z=c moved from corner to lower boundary below corner.

$F' = \frac{1}{b}\cdot\frac{\sqrt{a^2-W^2}}{W}$; $\pi z = \ln\frac{W-1}{W+1} + a\ln\frac{a+W}{a-W}$

$b=\sqrt{a^2-1}$; $W=\sqrt{t}$

"Program": know from expansion of fields in exponentials that F', F must be expandable in Taylor series that has only odd powers of $\exp(-\pi z/2a)$. Will do first 2 terms explicitely, and give then Taylor series coefficient manupilation algorithms that allow

(10.8)

very simple and fast! calculation of expansion coefficients with computer.

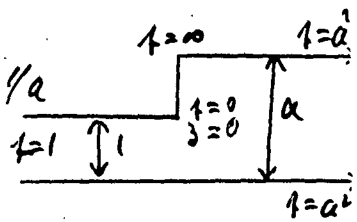
$W=\sqrt{t}$

$b^2=a^2-1$

$$\pi z/a = \ln\frac{a+W}{a-W} + \ln\left(\frac{W-1}{W+1}\right)^{1/a}$$

$$g = e^{-\pi z/a} = \frac{a-W}{a+W}\cdot\left(\frac{W+1}{W-1}\right)^{1/a}$$

$$F'/\sqrt{g} = \frac{1}{b}\cdot\frac{a+W}{W}\left(\frac{W-1}{W+1}\right)^{\frac{1}{2a}}$$



(u = new complex variable, not Real part of W)

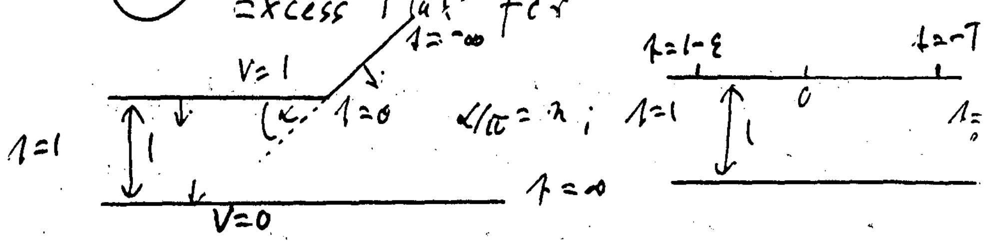
$a-W=u$; $W=a-u$

$$g = \frac{u}{2a-u}\cdot\left(\frac{a+1-u}{a-1-u}\right)^{1/a}$$

$$F'/\sqrt{g} = \frac{1}{b}\left(1+\frac{a}{a-u}\right)\left(\frac{a-1-u}{a+1-u}\right)^{1/2a}$$

↑ = Starting point for "hand" and computer calculation. Basic thought/procedure: Can expand $F'/\sqrt{g}$ in Taylor series in u. Can expand g in Taylor series in u, get from that Taylor series of u in g, and use that in Taylor series for $F'/\sqrt{g}$

(10.5) Excess Flux for



$\pi \dot{z} = \frac{1^n}{1-1}$; $\pi \dot{F} = \frac{1}{1-1}$; $F' = 1/1^n$

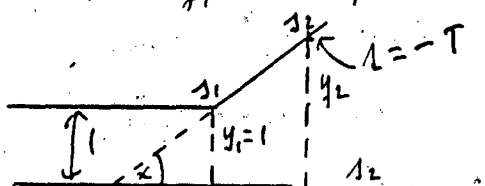
$F(0) - F(1-\varepsilon) = z(0) - z(1-\varepsilon) + \Delta A_{10}$

$\pi \Delta A_{10} = \int_{1-\varepsilon}^{0} (\pi \dot{F} - \pi \dot{z}) d1 = \int_0^1 \frac{1-1^n}{1-1} d1 = \pi \Delta A_{10}$

$\alpha = \pi/2$; $n = 1/2$;

$\pi \Delta A_{10} = \int_0^1 \frac{1-\sqrt{x}}{1-x} dx = \int_0^1 \frac{dx}{1+\sqrt{x}} = 2 \cdot \int_1^2 \left(1 - \frac{1}{y}\right) dy = 2(1 - \ln 2)$

$1+\sqrt{x} = y$; $x = (y-1)^2$; $dx = 2(y-1) dy$



$F(1_2) - F(1_1) = \int_{1_1}^{1_2} \frac{d1}{\alpha \cdot 1} + \Delta A_{0\infty} = \Delta A_{0\infty} + \frac{1}{\alpha} \ln \frac{1_2}{1_1}$

$\pi \Delta A_{0\infty} = \ln(1+T) - \frac{1}{n} \ln \frac{y_2}{y_1}$

$T = -1$

(10.6)

$y_2 = y_1 + \frac{\sin\alpha}{\pi} \cdot \int_0^T \frac{1^n}{1+1} d1$

$y_1 = 1$

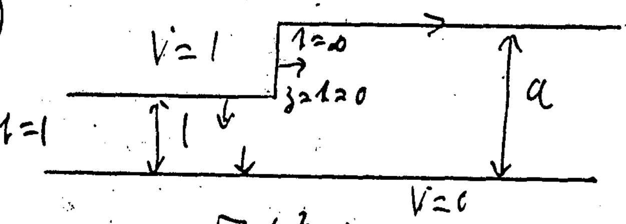
$n \pi \Delta A_{0\infty} = \left(\ln \left(\frac{(1+T)^n}{1 + \frac{\sin\alpha}{\pi} \cdot \int_0^T \frac{1^n}{1+1} d1} \right) \right)_{T \to \infty}$

$= \left(\ln \frac{n(1+T)^{n-1}}{\frac{\sin\alpha}{\pi} \cdot T^{n-1}} \right)_{T \to \infty} = \ln \frac{\alpha}{\sin\alpha}$

$\pi \Delta A_{0\infty} = \frac{1}{n} \ln \frac{\alpha}{\sin\alpha}$

$\pi \Delta A_{tot} = \int_0^1 \frac{1-1^n}{1-1} d1 + \frac{1}{n} \ln \frac{\alpha}{\sin\alpha} \approx \pi n = \alpha$

10.3 Excess Flux in



$t = a^2$

$$\pi \dot{z} = -\frac{\sqrt{t}\,(a^2-1)}{(t-1)(t-a^2)}$$

check: $-i\pi a = -i\pi \cdot \frac{a(a^2-1)}{a^2-1}$ = o.k.

$z = w^2$; $dz = 2w\,dw$

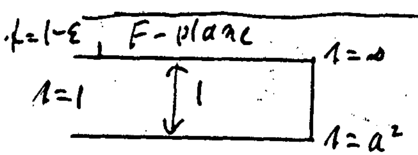
$$\pi\, dz/dw = -\frac{2(a^2-1)t}{(t-1)(t-a^2)} = 2\left(\frac{1}{t-1} - \frac{a^2}{t-a^2}\right)$$

$$\frac{1}{t-a^2} = \frac{1}{w^2-a^2} = \frac{1}{2a}\left(\frac{1}{w-a} - \frac{1}{w+a}\right)$$

$$\pi\, dz/dw = \frac{1}{w-1} - \frac{1}{w+1} - a\left(\frac{1}{w-a} - \frac{1}{w+a}\right)$$

$$\pi z = \ln\frac{1-\sqrt{t}}{1+\sqrt{t}} + a\ln\frac{a+\sqrt{t}}{a-\sqrt{t}}$$

$b^2 = a^2 - 1$



$$\pi \dot{F} = i\frac{b}{(t-1)(t-a^2)^{1/2}}$$

$t = a^2 + q^2$; $dt = 2q\,dq$

$$\pi\, dF/dq = \frac{2ib}{q^2+b^2} = \frac{1}{q-ib} - \frac{1}{q+ib}$$

$$\pi F = \ln\frac{ib - \sqrt{t-a^2}}{ib + \sqrt{t-a^2}} = \ln\frac{b - \sqrt{a^2-t}}{b+\sqrt{a^2-t}}$$

10.4

$dF/dz = -i\sqrt{1-a^2/t}\,/\,b$ (For completeness only)

For $\varepsilon > 0$; $\varepsilon \downarrow 0$:

$$F(\infty) - F(1-\varepsilon) = z(0) - z(1-\varepsilon) + \Delta A$$

$$\pi(F(\infty) - F(1-\varepsilon)) = \ln\frac{b^2-(a^2-t)}{(b+\sqrt{a^2-t})^2}\Bigg|_{1-\varepsilon}^{\infty} = \ln\frac{t-1}{(b+\sqrt{a^2-t})^2}\Bigg|_{1-\varepsilon}^{\infty}$$

$$\pi(F(\infty) - F(1-\varepsilon)) = \ln\frac{4b^2}{\varepsilon}$$

$$\pi(z(0) - z(1-\varepsilon)) = \ln\frac{1-t}{(1+\sqrt{t})^2}\Bigg|_{1-\varepsilon}^{0} + a\ln\frac{a+\sqrt{t}}{a-\sqrt{t}}\Bigg|_{1-\varepsilon}^{0}$$

$$\pi(z(0) - z(1-\varepsilon)) = \ln\frac{4}{\varepsilon} + a\ln\frac{a-1}{a+1}$$

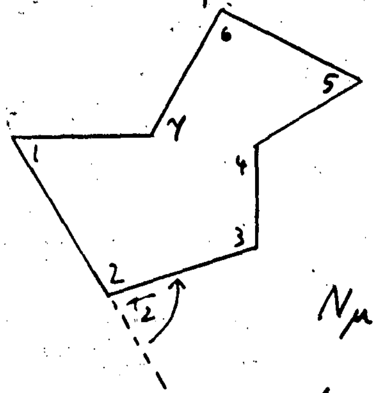
$$\pi\Delta A = \ln\frac{4b^2}{\varepsilon}\cdot\frac{\varepsilon}{4} + a\ln\frac{a+1}{a-1}$$

$$\pi\Delta A = \ln(a^2-1) + a\ln\frac{a+1}{a-1}$$

$$\Delta A = \big((a+1)\ln(a+1) - (a-1)\ln(a-1)\big)/\pi$$

(10.1)

S-C-Transformation Memory Jogger



$N_\mu = \alpha_\mu / \pi$

$$dz/dt = \frac{A}{\pi (t - t_\mu)^{N_\mu}}$$

$N_\mu \gtrless 0$ for $\alpha_\mu \lessgtr 0$

i.e. when $\alpha_\mu < 0$, factor appears in numerator (above ——)

$t_{\mu-1} < t_\mu < t_{\mu+1}$

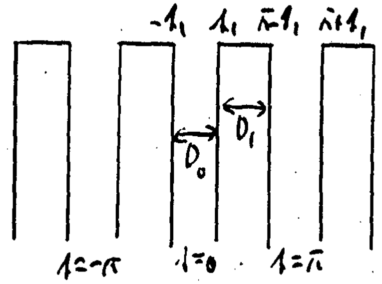
all t_μ = real

Can always choose one $t_\mu = \infty \rightarrow$ it disappears from equ. Two other t_μ can be located arbitrarily, usually $t_\mu = 0;\ t = \pm 1$

(10.2)

S-C Map of ∞ Array of ID Poles

z-plane



$$\pi z' = a \frac{\sqrt{\sin(t - t_1)\sin(t + t_1)}}{\sin t}$$

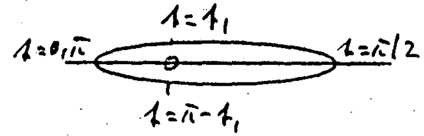
$$\sin(t - t_1)\sin(t + t_1) = \sin^2 t \cos^2 t_1 - \cos^2 t \sin^2 t_1$$
$$= \sin^2 t - \sin^2 t_1 = \cos^2 t_1 - \cos^2 t$$
$$= \tfrac{1}{2}(\cos 2t_1 - \cos 2t) = W/2$$

To check correctness of phase at corners of degenerate polygon, use $t \Rightarrow t + i\varepsilon$, $0 < \varepsilon \ll 1$

$$W = \cos 2t_1 - \cos 2t \cosh 2\varepsilon + i \sinh 2\varepsilon \cdot \sin 2t$$

$$W = u + iv: \quad \left(\frac{u - \cos 2t_1}{\cosh 2\varepsilon}\right)^2 + \left(\frac{v}{\sinh 2\varepsilon}\right)^2 = 1$$

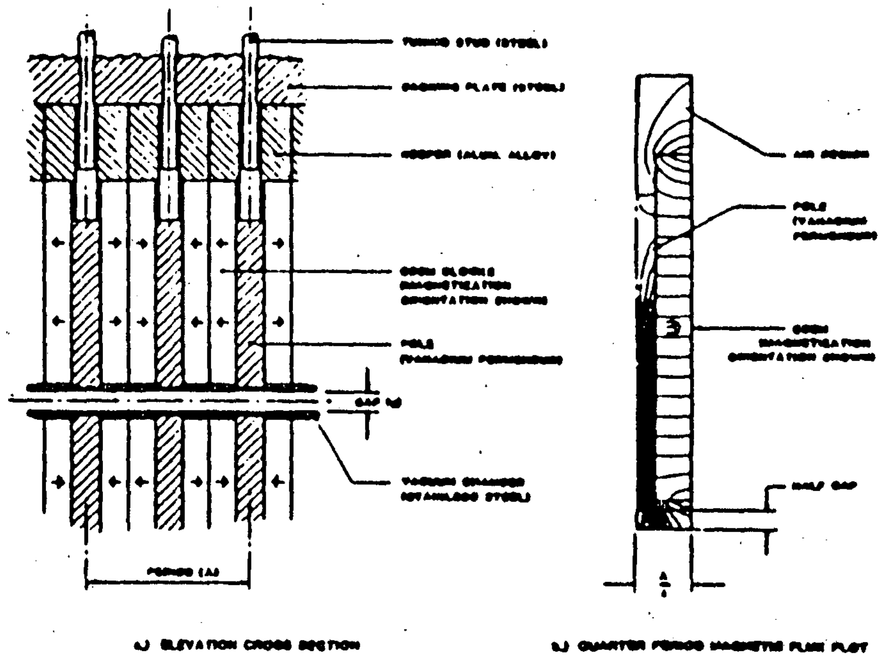
As t increases, W describes an ellipse in clockwise (i.e. mathematically negative) direction.



Conclusion: $\sqrt{W}$ behaves as "needed"

Insertion Device Design

Klaus Halbach



CSEM-STEEL HYBRID INSERTION DEVICE

Lecture 10.

January 19, 1989

Next Lecture:
Febr. 3, $8^{30} - 10^{30}$

134

(9.15)

very simple (and fast!) calculation of expansion coefficients with computer.

$$\pi z/a = \ln\frac{a+W}{a-W} + \ln\left(\frac{W-1}{W+1}\right)^{1/a}$$

$$g = e^{-\pi z/a} = \frac{a-W}{a+W}\cdot\left(\frac{W+1}{W-1}\right)^{1/a}$$

$$F'/\sqrt{g} = \frac{1}{b}\cdot\frac{a+W}{W}\left(\frac{W-1}{W+1}\right)^{\frac{1}{2a}}$$

$a-W=u$; $W=a-u$ (u = new complex variable, not Real part of W)

$$g = \frac{u}{2a-u}\cdot\left(\frac{a+1-u}{a-1-u}\right)^{1/a}$$

$$F'/\sqrt{g} = \frac{1}{b}\left(1+\frac{a}{a-u}\right)\left(\frac{a-1-u}{a+1-u}\right)^{1/2a}$$

↑ = Starting point for "hand" and computer calculation. Basic thought/procedure: Can expand $F'/\sqrt{g}$ in Taylor series in u. Can expand g in Taylor series in u, get from that Taylor series of u in g, and use that in Taylor series for $F'/\sqrt{g}$

(9.16)

Execution by hand to order $g^{3/2}$ (= first non-trivial term)

$$K = \left(\frac{a-1}{a+1}\right)^{1/a}$$

$$F' = \sqrt{Kg}\cdot\frac{1}{b}\left(2+\frac{u}{a}\right)\left(1-u\left(\frac{1}{a-1}-\frac{1}{a+1}\right)\right)^{1/2a}$$

$$F' = \sqrt{Kg}\cdot\frac{2}{b}\left(1+u\cdot\left(\frac{1}{2a}-\frac{1}{ab^2}\right)\right)$$

$$F' = \sqrt{Kg}\cdot\frac{2}{b}\left(1+\frac{u}{2a}\left(1-2/b^2\right)\right)$$

$$\frac{u}{2a} = Kg = \left(\frac{a-1}{a+1}\right)^{1/a}\cdot e^{-\pi z/a}$$

$$F' = \frac{2}{b}\cdot\sqrt{Kg}\left(1+Kg\left(1-2/b^2\right)\right)$$

no 3. harmonic for $a=\sqrt{3}$ → $1-1/a = .4226$

$$F = -\frac{4a}{\pi b}\cdot\sqrt{Kg}\left(1+Kg\left(1-2/b^2\right)/3+\cdots\right)$$

For calculation of flux from overhanging CSEM, need to integrate $V = \mathrm{Im}\,F$ from $x+i$ to $\infty+i$. To calculate flux from CSEM attached to surface $z=i$ to $z=ia$, have to integrate V from $x+i$ to $x+ia$. Both integrals trivial

(9.13)

$$y_2 = y_1 + \frac{\sin\alpha}{\pi} \cdot \int_0^T \frac{t^n}{1+t}\,dt$$

$$y_1 = 1$$

$$n\pi\Delta A_{0\infty} = \left(\ln\left(\frac{(1+T)^n}{1+\frac{\sin\alpha}{\pi}\cdot\int_0^T \frac{t^n}{t+1}\,dt}\right)\right)_{T\to\infty}$$

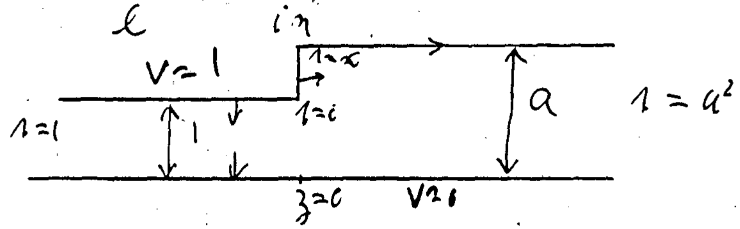
$$= \left(\ln \frac{n(1+T)^{n-1}}{\frac{\sin\alpha}{\pi}\cdot T^{n-1}}\right)_{T\to\infty} = \ln\frac{\alpha}{\sin\alpha}$$

$$\pi\Delta A_{0\infty} = \frac{1}{n}\ln\frac{\alpha}{\sin\alpha}$$

$$\pi\Delta A_{\infty 1} = \int_0^1 \frac{1-t^n}{1-t}\,dt + \frac{1}{n}\ln\frac{\alpha}{\sin\alpha} \approx \pi n = \alpha$$

(9.14)

Expansion of F in Taylor series of $e^{-\pi z/2a}$ in



From excess flux calculation, with $z=0$ moved from corner to lower boundary below corner.

$$F' = \frac{1}{b}\cdot\frac{\sqrt{a^2-W^2}}{W}\;;\quad \pi z = \ln\frac{W-1}{W+1} + a\ln\frac{a+W}{a-W}$$

$$b = \sqrt{a^2-1}\;;\quad W = \sqrt{\lambda}$$

"Program": Know from expansion of fields in exponentials that F', F must be expandable in Taylor series that has only odd powers of $\exp(-\pi z/2a)$. Will do first 2 terms explicitely, and give then Taylor series coefficient manipulation algorithms that allow

(9.11)

$$\bar{\pi}\left(V(\infty) - V(a^2+\varepsilon)\right) = \ln\frac{4b^2}{\varepsilon} = \bar{\pi}\left(\mathfrak{z}(a^2+\varepsilon) - \mathfrak{z}(\infty)\right)\cdot B_{\infty} + \bar{\pi}\Delta l$$

$$\bar{\pi}\Delta V = \ln\frac{4b^2}{\varepsilon} - \left(\frac{1}{a}\ln\frac{a-1}{a+1} + \ln\frac{4a^2}{\varepsilon}\right)$$

$$\bar{\pi}\Delta V = \ln\frac{a^2-1}{a^2} + \frac{1}{a}\ln\frac{a+1}{a-1}$$

$$\Delta V = \left((a+1)\ln(a+1) + (a-1)\ln(a-1) - 2a\ln a\right)/(a\bar{\pi})$$

Special case: pole thickness $=0 \to a=1$: $\Delta V = \ln(4)/\bar{\pi}$

"Translation" into movement of pole

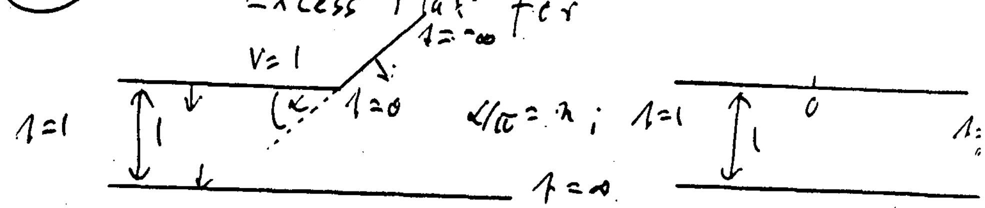
$\Delta V = B_{\infty}\cdot\Delta x = \Delta x/a \longrightarrow \Delta x = a\Delta V$

$$\Delta x = \left((a+1)\ln(a+1) + (a-1)\ln(a-1) - 2a\ln a\right)/\bar{\pi}$$

Normalization: Δx is measured in length of dimension that was set $=1$ in original geometry. ΔV is given for flux $=1$ between the 2 extreme fieldlines. (In 2D, flux and potential have same dimensions)

(9.12)

Excess Flux for



$$\bar{\pi}\dot{\mathfrak{z}} = \frac{1^n}{1-1} \quad ; \bar{\pi}\dot{F} = \frac{1}{1-1} \quad ; \quad F' = 1/1^n$$

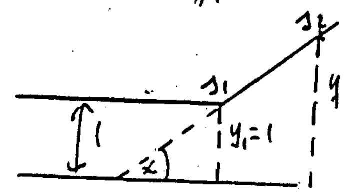
$$F(0) - F(1-\varepsilon) = \mathfrak{z}(0) - \mathfrak{z}(1-\varepsilon) + \Delta A_{10}$$

$$\bar{\pi}\Delta A_{10} = \int_{1-\varepsilon}^{0}(\bar{\pi}\dot{F} - \bar{\pi}\dot{\mathfrak{z}})d1 = \int_0^1 \frac{1-1^n}{1-1}d1 = \bar{\pi}\Delta A_{10}$$

$\alpha = \bar{\pi}/2$; $n = 1/2$;

$$\bar{\pi}\Delta A_{10} = \int_0^1 \frac{1-\sqrt{x}}{1-x}dx = \int_1^1 \frac{dx}{1+\sqrt{x}} = 2\cdot\int_1^2\left(1-\frac{1}{y}\right)dy = 2(1-\ln 2)$$

$1+\sqrt{x} = y$; $x = (y-1)^2$; $dx = 2(y-1)dy$



$$F(1_2) - F(1_1) = \int_{1_1}^{1_2}\frac{d1}{\alpha\cdot 1} + \Delta A_{1\infty} = \Delta A_{0\infty} + \frac{1}{\alpha}\ln\frac{1_2}{1_1}$$

$$\bar{\pi}\Delta A_{0\infty} = \ln(1+T) - \frac{1}{n}\ln\frac{y_2}{y_1}$$

$T = -1$

9.9

$dF/dz = -i\sqrt{1-a^2/t}\,/\,b$ (For completeness only)

For $\varepsilon > 0$; $\varepsilon \downarrow 0$:

$$F(\infty) - F(1-\varepsilon) = z(0) - z(1-\varepsilon) + \Delta A$$

$$\pi\left(F(\infty) - F(1-\varepsilon)\right) = \ln \frac{b^2-(a^2-t)}{(b+\sqrt{a^2-t})^2}\Bigg|_{1-\varepsilon}^{\infty} = \ln \frac{t-1}{(b+\sqrt{a^2-t})^2}\Bigg|_{1-\varepsilon}^{\infty}$$

$$\pi\left(F(\infty) - F(1-\varepsilon)\right) = \ln \frac{4b^2}{\varepsilon}$$

$$\pi\left(z(0) - z(1-\varepsilon)\right) = \ln \frac{1-t}{(1+\sqrt{t})^2}\Bigg|_{1-\varepsilon}^{0} + a \ln \frac{a+\sqrt{t}}{a-\sqrt{t}}\Bigg|_{1-\varepsilon}^{0}$$

$$\pi\left(z(0) - z(1-\varepsilon)\right) = \ln \frac{4}{\varepsilon} + a \ln \frac{a-1}{a+1}$$

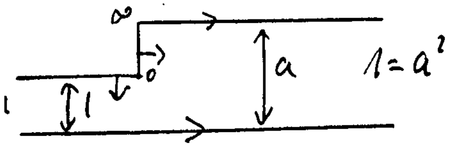
$$\pi \Delta A = \ln \frac{4b^2}{\varepsilon}\cdot\frac{\varepsilon}{4} + a \ln \frac{a+1}{a-1}$$

$$\pi \Delta A = \ln(a^2-1) + a \ln \frac{a+1}{a-1}$$

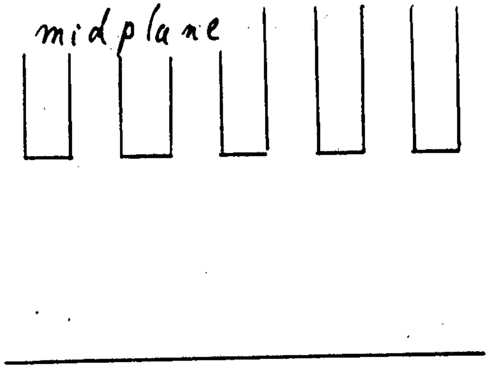
$$\Delta A = \left((a+1)\ln(a+1) - (a-1)\ln(a-1)\right)/\pi$$

9.10

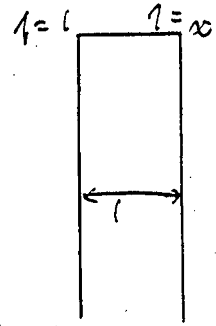
Excess V-drop in



Motivation: "laminated" magnet → Flux between sides of poles of hybrid ID and midplane



F-plane



Normalization: Flux going to pole = 1. $t = a^2$

→ far enough to the right, $B_\infty = 1/a$

Map $z \to t$ as before, but $\dot{F}(t)$, $F(t)$ from

$$\pi \dot{F} = \frac{i\,b}{\sqrt{t-1}\,(t-a^2)}\ ; \quad b = \sqrt{a^2-1}\ \text{(as before)}$$

$$t = 1 + W^2;\quad \pi\frac{dF}{dW} = \frac{2ib}{W^2-b^2} = i\left(\frac{1}{W-b} - \frac{1}{W+b}\right)$$

$$\pi F = i \ln \frac{W-b}{W+b} = i \ln \frac{\sqrt{t-1}-b}{\sqrt{t-1}+b} = i \ln \frac{t-a^2}{(\sqrt{t-1}+b)^2}$$

9.7

$$\pi z/a = \int \frac{\sqrt{\cos^2 t_1 - \cos^2 t}}{\sin t}\, dt \quad ; \quad \cos t_1 = c_1 ; \sin t_1 = s_1$$

$$\cos t = \cos t_1 \cdot \sin\varphi \ ; \ dt = -\cos t_1 \cos\varphi \, d\varphi / \sin t$$

$$\pi z/a = -\int \frac{\cos^2 t_1 \cos^2\varphi \, d\varphi}{1-\cos^2 t_1 \sin^2\varphi} = \int \frac{c_1^2(\sin^2\varphi - 1)\, d\varphi}{1 - c_1^2 \sin^2\varphi}$$

$$\pi z/a = \int \frac{c_1^2 \sin^2\varphi - 1 + 1 - c_1^2}{1 - c_1^2\sin^2\varphi} d\varphi = -\varphi + \int \frac{s_1^2}{1 - c_1^2 \sin^2\varphi} d\varphi$$

$$\pi z/a + \varphi = \int \frac{s_1^2}{\frac{1}{\sin^2\varphi} - c_1^2} \cdot \frac{d\varphi}{\sin^2\varphi}$$

$$\cot g\,\varphi = \frac{\cos\varphi}{\sin\varphi} = g \ ; \ dg = -d\varphi/\sin^2\varphi$$

$$\pi z/a + \varphi = -\int \frac{s_1^2\, dg}{g^2 + s_1^2} = -s_1 \tan^{-1}(g/s_1)$$

$$g = \frac{\cos\varphi}{\sin\varphi} = \frac{c_1}{\cos t} \cdot \sqrt{1 - \cos^2 t / c_1^2} = \sqrt{\cos^2 t_1 / \cos^2 t - 1}$$

$$\pi z/a = -\sin^{-1}(\cos t/\cos t_1) - \sin t_1 \tan^{-1}\left(\frac{\sqrt{\cos^2 t_1/\cos^2 t - 1}}{\sin t_1}\right)$$

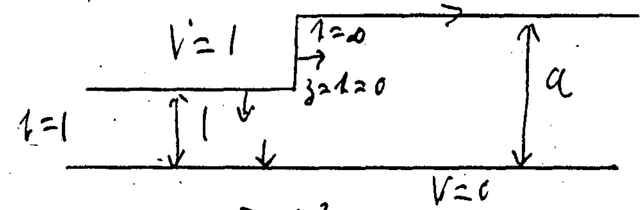
For complex t, use

$$\sin^{-1}(u) = \frac{1}{i}\ln\left(iu + \sqrt{1-u^2}\right)$$

$$\tan^{-1}(u) = \frac{1}{2i}\ln\frac{1+iu}{1-iu}$$

9.8

Excess Flux in



$$1 = a^2$$

$$\pi \dot{z} = -\frac{\sqrt{t}\,(a^2-1)}{(t-1)(t-a^2)}$$

check: $-i\pi a = -i\pi \cdot \frac{a(a^2-1)}{a^2-1} = $ o.k.

$$z = w^2 \ ; \ dz = 2w\,dw$$

$$\pi\, dz/dw = -\frac{2(a^2-1)t}{(t-1)(t-a^2)} = 2\left(\frac{1}{t-1} - \frac{a^2}{t-a^2}\right)$$

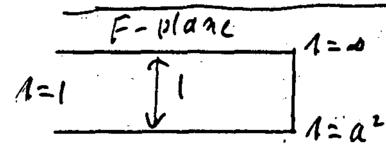
$$\frac{1}{t-a^2} = \frac{1}{w^2-a^2} = \frac{1}{2a}\left(\frac{1}{w-a} - \frac{1}{w+a}\right)$$

$$\pi\, dz/dw = \frac{1}{w-1} - \frac{1}{w+1} - a\left(\frac{1}{w-a} - \frac{1}{w+a}\right)$$

$$\pi z = \ln\frac{1-\sqrt{t}}{1+\sqrt{t}} + a\ln\frac{a+\sqrt{t}}{a-\sqrt{t}}$$

$$b^2 = a^2 - 1$$

F-plane



$$\pi \dot{F} = i\frac{b}{(t-1)(t-a^2)^{1/2}}$$

$$t = a^2 + q^2 \ ; \ dt = 2q\,dq$$

$$\pi\, dF/dq = \frac{2ib}{q^2+b^2} = \frac{1}{q-ib} - \frac{1}{q+ib}$$

$$\pi F = \ln\frac{ib - \sqrt{t-a^2}}{ib + \sqrt{t-a^2}} = \ln\frac{b - \sqrt{a^2-t}}{b+\sqrt{a^2-t}}$$

9.5

Behaviour of $\frac{g(t)}{\sin t}$ at $t = n\pi$

$$\left(\frac{g(t)}{\sin t}\right)_{t\to n\pi} = \frac{1}{t-n\pi}\,\frac{(t-n\pi)g(t)}{\sin t} = \frac{1}{t-n\pi}\left(\frac{g(t)}{\cos t}\right)_{t=n\pi}$$

$$\left(\frac{g(t)}{\sin t}\right) = \frac{1}{t-n\pi}\cdot\frac{g(n\pi)}{(-1)^n}$$

9.6

Integrate around $t=0$:

$-D_0\pi = a\cdot i\pi\cdot i\sin t_1 \longrightarrow \underline{D_0 = a\cdot\sin t_1}$

Integrate (on separate sheet) $\dot{z}$ to get D_1:

$$\pi z = -a\left(\sin^{-1}\left(\frac{\cos t}{\cos t_1}\right) + \sin t_1 \tan^{-1}\left(\frac{\sqrt{\cos^2 t_1/\cos^2 t - 1}}{\sin t_1}\right)\right)$$

$$\pi D_1/2 = \pi z\Big|_{t_1}^{\pi/2} = -a\left(\frac{\pi}{2}\cdot\sin t_1 - \frac{\pi}{2}\right) = a\cdot\frac{\pi}{2}(1-\sin t_1)$$

$$\underline{D_1 = a(1-\sin t_1)}$$

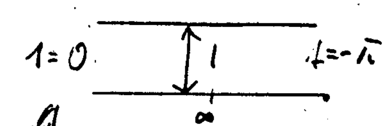
For complex argument z_1 use

$$\sin^{-1}(z) = \frac{1}{i}\ln\left(\sqrt{1-z^2} + iz\right)$$

$$\tan^{-1}(z) = \frac{1}{2i}\ln\left(\frac{1+iz}{1-iz}\right)$$

Pole between $t=-\pi$ and $t=0$ on $V=1$:

$$\dot{F} = \frac{1}{t(t+\pi)} = \frac{1}{\pi}\left(\frac{1}{t} - \frac{1}{t+\pi}\right)$$

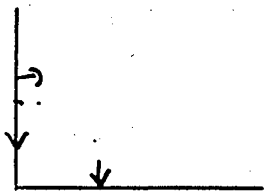


$$\pi F = \ln\frac{t}{t+\pi} = -\ln(1+\pi/t)$$

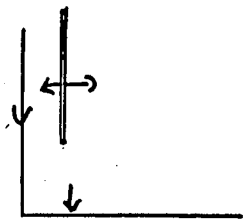
$$\underline{t = \frac{\pi}{e^{-\pi F}-1}} \longrightarrow \text{Field line / } V=\text{const plots.}$$

(9.3)

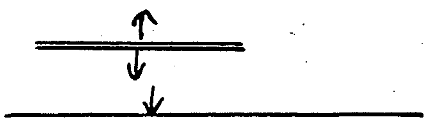
Problems solved so far, together with functions used.



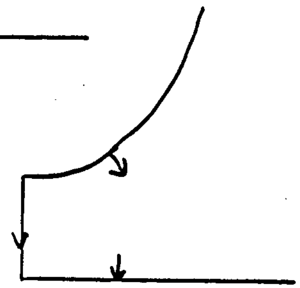
$z\,;\,F$



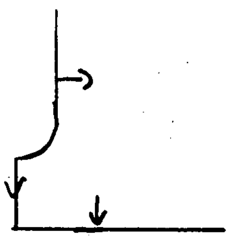
$z\,;\,F$



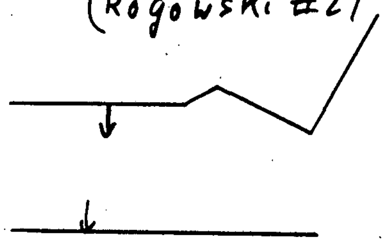
$z\,,\,F$ (Rogowski #1)



$iB_0/B^*\,;\,F$ (Rogowski #2)



$\ln i\,\Xi/E^*\,;\,z$

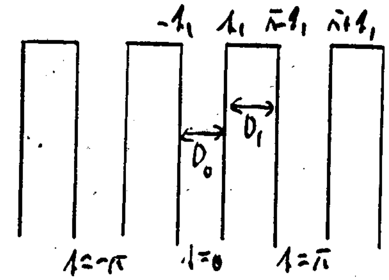


$z\,,\,F$

Notation $\dot{\ } = \frac{d}{dt}\;;\;{}' = \frac{d}{dz}$

(9.4)

S-C Map of ∞ Array of ID Poles



$$i\pi\dot{z} = a\,\frac{\sqrt{\sin(t-t_1)\sin(t+t_1)}}{\sin t}$$

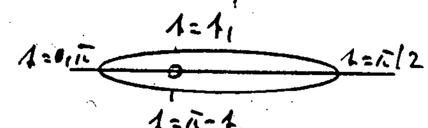
$$\sin(t-t_1)\sin(t+t_1) = \sin^2 t\cos^2 t_1 - \cos^2 t\sin^2 t_1$$
$$= \sin^2 t - \sin^2 t_1 = \cos^2 t_1 - \cos^2 t$$
$$= \tfrac{1}{2}(\cos 2t_1 - \cos 2t) = W/2$$

To check correctness of phase at corners of degenerate polygon, use $t \Rightarrow t + i\varepsilon$, $0<\varepsilon\ll 1$

$$W = \cos 2t_1 - \cos 2t\cosh 2\varepsilon + i\sinh 2\varepsilon\cdot\sin 2t$$

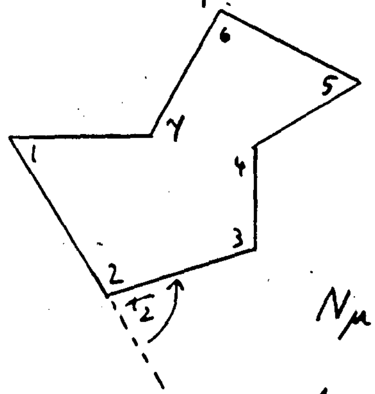
$$W = u + iv\;:\;\left(\frac{u-\cos 2t_1}{\cosh 2\varepsilon}\right)^2 + \left(\frac{v}{\sinh 2\varepsilon}\right)^2 = 1$$

As t increases, W describes an ellipse in clockwise (i.e. mathematically negative) direction.



Conclusion: $\sqrt{W}$ behaves as "needed"

(9.1) S-C-Transformation Memory Jogger



$N_\mu = \alpha_\mu / \pi$

$$dz/dt = \frac{A}{\prod (t - t_\mu)^{N_\mu}}$$

$N_\mu \gtrless 0$ for $\alpha_\mu \gtrless 0$

i.e. when $\alpha_\mu < 0$, factor appears in numerator (above ——)

$t_{\mu-1} < t_\mu < t_{\mu+1}$

all t_μ = real

Can always choose one $t_\mu = \infty \rightarrow$ it disappears from equ. Two other t_μ can be located arbitrarily, usually $t = 0$; $t = \pm 1$

(9.2) Summary + Extension of Lecture #8.

More applications of S-C-transformation

<u>General Procedure</u>

Establish, from Physics / Geometry, relationship between 2 relevant complex quantities that are analytical functions of each other (e.g. z, F, B^*, ...) on boundary of problem. Choose functions such that when complex value of functions are plotted as function of a parameter that identifies (conceptually only in many cases) points on problem boundary, a, usually degenerate, polygon is formed. Map the interior of both polygons on the upper ½ of t-plane, with identical points on polygons mapped onto the same points on real axis of t-plane.

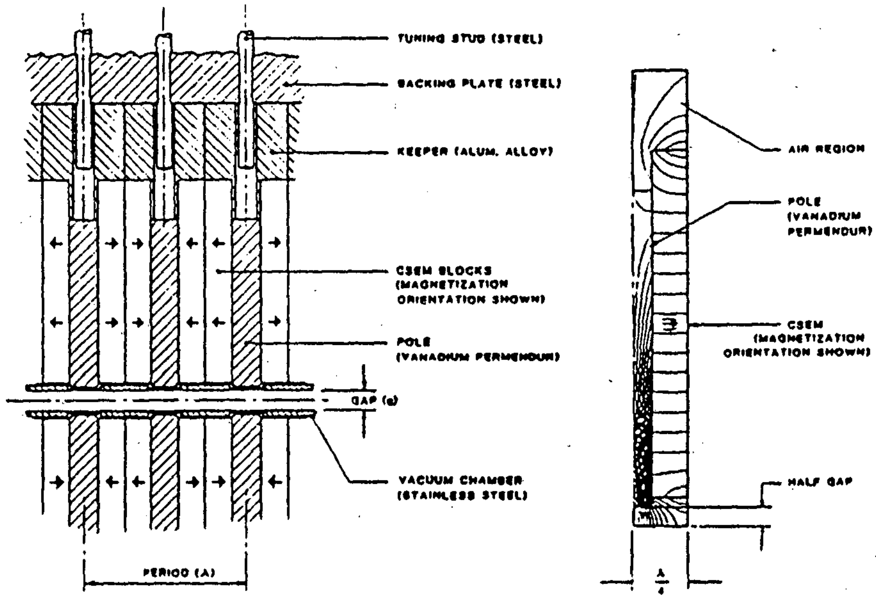
This then establishe functional relationship between 2 functions

Insertion Device Design

Klaus Halbach

Lecture 9.

January 13, 1989



a.) ELEVATION CROSS SECTION

b.) QUARTER PERIOD MAGNETIC FLUX PLOT

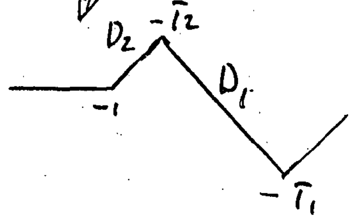
CSEM-STEEL HYBRID INSERTION DEVICE

8.17

Arbitrary specific case: $\alpha_1 = \alpha_2 = \pi/2$

$n_1 = 1/2$; $n_2 = -1/2$; $n_3 = -\alpha_0/\pi = -n_0$

$$\left.\begin{aligned} k_2 - k_1 &= 2n_0 \\ k_2^2 - k_1^2 &= 2n_0 \end{aligned}\right| \to k_2 + k_1 = 1 \to k_2 = \tfrac{1}{2} + n_0, \quad k_1 = \tfrac{1}{2} - n_0$$



$$\pi \dot{z} = \frac{(t+1)^{n_0}(1+k_1 t)^{1/2}}{t\,(1+k_2 t)^{1/2}}$$

$$\pi D_2 = \frac{\sqrt{T_2}}{\sqrt{T_1}} \cdot \int_1^{T_2} \frac{(t-1)^{n_0}(T_1-t)^{1/2}}{t\,(T_2-t)^{1/2}}\, dt$$

$\sqrt{T_2 - t} = u$; $t = T_2 - u^2$; $dt = -2u\,du$

$$\pi D_2 = \frac{\sqrt{T_2}}{\sqrt{T_1}} \cdot 2 \cdot \int_0^{\sqrt{T_2-1}} \frac{(T_2-1-u^2)^{n_0}(T_1-T_2+u^2)^{1/2}}{T_2-u^2}\, du$$

$$\pi D_1 = \frac{\sqrt{T_2}}{\sqrt{T_1}} \cdot \int_{T_2}^{T_1} \frac{(t-1)^{n_0}(T_1-t)^{1/2}}{t\,(t-T_2)^{1/2}}\, dt$$

$\sqrt{t - T_2} = u$; $t = T_2 + u^2$; $dt = 2u\,du$

$$\pi D_1 = \frac{\sqrt{T_2}}{\sqrt{T_1}} \cdot 2 \cdot \int_0^{\sqrt{T_1-T_2}} \frac{(T_2-1+u^2)^{n_0}(T_1-T_2-u^2)^{1/2}}{T_2+u^2}\, du$$

8.18

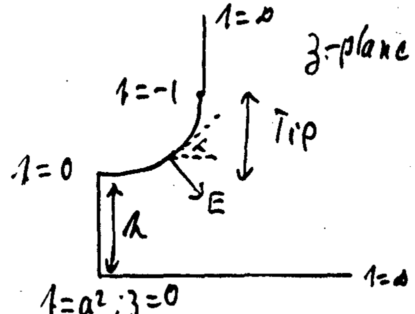
$\alpha_0 = \pi/4$; $n_0 = 1/4 \to D_2 = .189$; $D_1 = .387$.

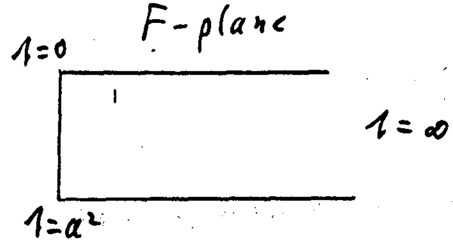
$\sqrt{.5} \cdot D_2 = .134$; $\sqrt{.5}\, D_1 = .274$

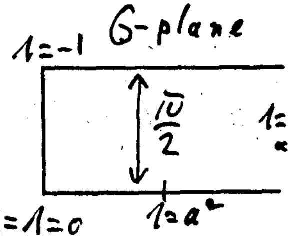
$\sqrt{.5}\,(D_1 - D_2) = .14$; $\sqrt{.5} \cdot (D_1 + D_2) = .408$

8.15

2D Needle with $|E|$ = constant on "Tip"







$$G = \ln\frac{iE_0}{E^*} = \ln\frac{E_0}{|E|} + i\left(\frac{\pi}{2}+\beta\right)$$

$$G = \ln\frac{E_0}{|E|} + i\alpha = -\ln F'/E_0$$

$$\dot G = \frac{1}{2\sqrt{t}\sqrt{t+1}}\,;\quad G = \ln\left(\sqrt{t}+\sqrt{t+1}\right) = -\ln F'/E_0$$

$$F(0)/F'(a^2) = a + \sqrt{1+a^2}$$

$$F' = \frac{\dot F}{\dot z} = E_0\, e^{-G}\,;\quad \dot z = \dot F e^{G}/E_0$$

$$\dot F = \frac{E_0 \cdot b}{\sqrt{t}\sqrt{t-a^2}} \longrightarrow \dot z = b\cdot\left(\frac{1}{\sqrt{t-a^2}} + \frac{\sqrt{t+1}}{\sqrt{t}\sqrt{t-a^2}}\right)$$

$$h = b\cdot\int_0^{a^2} \frac{\sqrt{t}+\sqrt{t+1}}{\sqrt{t}\sqrt{a^2-t}}\,dt \Longrightarrow b$$

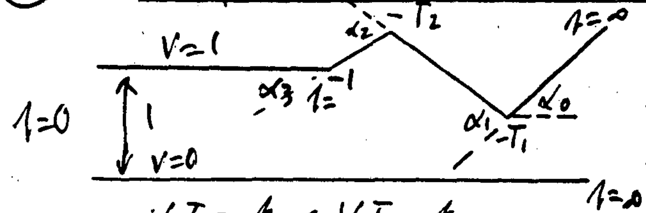
$$z - ih = \int_0^{-T} \dot z(t)\,dt = \int_0^{T} -\dot z(-t)\,dt = b\int_0^T\left(\frac{i}{\sqrt{t+a^2}} + \frac{\sqrt{1-t}}{\sqrt{t}\sqrt{t+a^2}}\right)dt$$

Tip: $0 \le T \le 1$; $x = b\cdot\int_0^T \frac{\sqrt{1-t}}{\sqrt{t}\sqrt{t+a^2}}\,dt$; $y = h + b\cdot\int_0^T \frac{dt}{\sqrt{t+a^2}}$

Integrals → Elliptic integrals

8.16

Analytical 2. Order Shim for semi-∞ Dipole



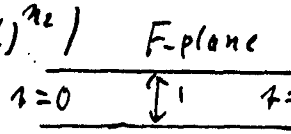
$\alpha_1 - \alpha_2 + \alpha_3 = \alpha_0$

$n_1 = -\alpha_1/\pi$; $n_2 = \alpha_2/\pi$

$n_3 = -\alpha_3/\pi$

$1/T_1 = k_1 < 1/T_2 = k_2$

$$\pi \dot z = 1/\left(t(1+t)^{n_3}(1+k_1 t)^{n_1}(1+k_2 t)^{n_2}\right)$$



For $|t| \ll 1$: $t \sim e^{\pi z}$

$\pi \dot F = 1/t$; $F' = (1+t)^{n_3}(1+k_1 t)^{n_1}(1+k_2 t)^{n_2}$

F' can be expanded in form $\sum_0 a_n\, e^{n\pi z}$.

To make $a_1 = a_2 = 0$, expand F' in t to 2. order and make coefficients of t, t^2 zero:

$$F' = (1+t)^{n_3}\left(1 + n_1 k_1 t + \frac{n_1(n_1-1)}{2}k_1^2 t^2\right)\left(1 + n_2 k_2 t + \frac{n_2(n_2-1)}{2}k_2^2 t^2\right)$$

$$F' = \left(1 + t(n_1 k_1 + n_2 k_2) + \frac{t^2}{2}\left(n_2(n_2-1)k_2^2 + 2n_1 n_2 k_1 k_2 + n_1(n_1-1)k_1^2\right)\right)$$

$$\times\left(1 + n_3 t + \frac{n_3(n_3-1)}{2}t^2\right)$$

$$n_3 + n_1 k_1 + n_2 k_2 = 0$$

$$n_3(n_3-1) + 2n_3\underbrace{(n_1 k_1 + n_2 k_2)}_{-n_3} + \underbrace{(n_2 k_2 + n_1 k_1)^2}_{n_3^2} - n_1 k_1^2 - n_2 k_2^2 = 0$$

$$n_3 + n_1 k_1^2 + n_2 k_2^2 = 0$$

(8.13)

$kz = \ln\left(\sqrt{t} + \sqrt{t-1}\right) + b\sqrt{t-1} \quad (-0)$

Contour of surface for $-t = T^2 > 0$:

$kz = \ln\left(i\left(T + \sqrt{T^2+1}\right)\right) + ib\sqrt{T^2+1}$

$T = \sinh\alpha$

$kz = kx + iky = i\frac{\pi}{2} + \alpha + ib\cosh\alpha$

$ky = \frac{\pi}{2} + b\cosh(kx)$

Re-write this to see what it means

$\frac{\pi}{2k} = x_1 \; ; \; y = x_1 + b/k \cdot \cosh\left(\frac{\pi}{2} x/x_1\right)$

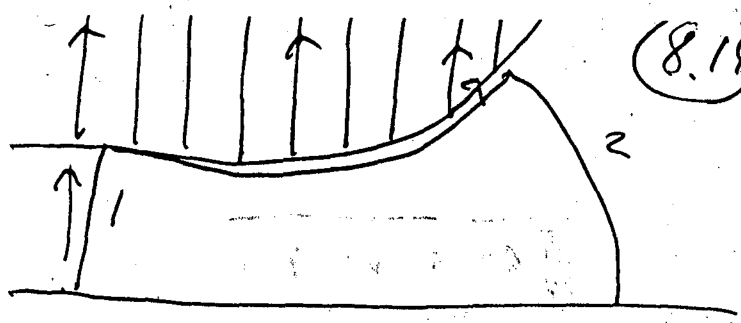
$b/k = y_0 - x_1 \; : \; y = (y_0 - x_1)\cosh\left(\frac{\pi}{2}\cdot\frac{x}{x_1}\right) + x_1$

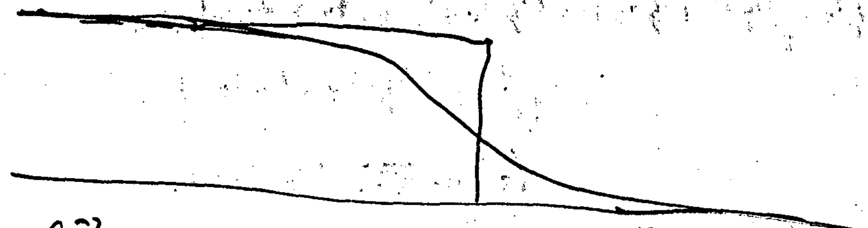
Clearly: $x_1 < y_0$ = necessary.

Limiting case: $x_1 \to y_0 \Longrightarrow y = y_0 + c \cdot \exp\left(\frac{\pi}{2}\cdot\frac{x}{y_0}\right)$

↑ original Rogowski formula.

(8.14)





$\int_2 \vec{H}\cdot d\vec{s} = \int_1 \vec{H}\cdot d\vec{s} + \int_{inner} H_{\parallel}\cdot ds$

8.11

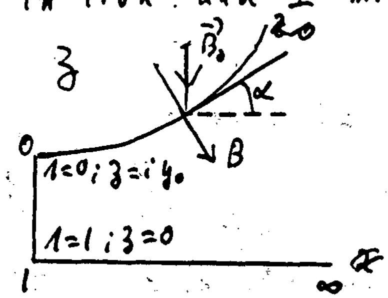
With general procedure formulated before ($\mathfrak{z} = f_1(t)$; $\dot{F} = f_2(t)$), can not "get" curved contours, like Rogowski surface systematically "from first principles".

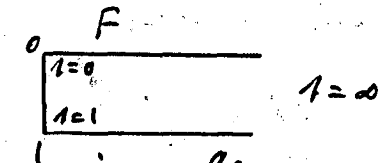
To do that, use other analytical functions that have to come from formulation of Physics of problem, and will therefore be different for different problems. But one will, of course, practically alway utilize general relationships, like $B^* = iF' = i\dot{F}/\dot{\mathfrak{z}}$.

8.12

Equation for Rogowski surface in 2D for $\mu = \infty$ (Finite pole width)

Definition: Field in iron $= B_0 =$ homogeneous in iron and $\perp$ midplane.

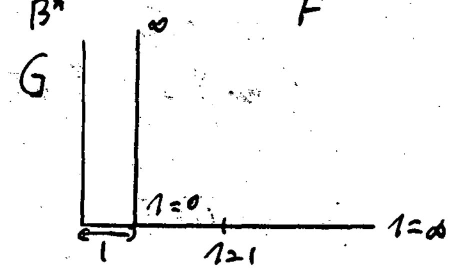




$$\dot{F} = \frac{a_0}{2\sqrt{t}\sqrt{t-1}}$$

$B_0 = |\vec{B}_0|$; $B = -iB_0 e^{i\alpha} \cdot \cos\alpha$; $B^* = iB_0 e^{-i\alpha} \cos\alpha$

$$\frac{iB_0}{B^*} = G(\mathfrak{z}) = B_0 \cdot \frac{\dot{\mathfrak{z}}}{\dot{F}} \Rightarrow 1 + i\,tg\,\alpha \text{ on pole surface}$$



$$\dot{G} = \frac{b}{2\sqrt{t}};$$

$$G = 1 + b\sqrt{t} =$$

$$\dot{\mathfrak{z}} = \frac{\dot{F}}{B_0} \cdot G = (1 + b\sqrt{t}) \frac{a}{2\sqrt{t}\sqrt{t-1}} \quad ; \ a = a_0 / B_0 = 1/h$$

$$\frac{1}{2}\int \frac{dt}{\sqrt{t}\sqrt{t-1}} = \int \frac{dW}{\sqrt{W^2-1}} = \ln\left(W + \sqrt{W^2-1}\right)$$

$t = W^2$; $dt = 2W\,dW$.

(8.9)

$s=-2: \pi z = 2i\sqrt{2} + \frac{1}{i}\ln\frac{1-\sqrt{2}}{1+\sqrt{2}} = \pi + i\cdot 2(\sqrt{2}+\ln(1+\sqrt{2}))$

$z(-2) \cong 1 + i\cdot 1.46$

$F'(0) = 0.5$ ("ideally" $\frac{1}{1.46} = .68$)

$s = w^2 \rightarrow \frac{\pi}{2}\frac{dF}{dw} = \frac{1}{\sqrt{w^2+1}}$; $\frac{\pi}{2}F = \ln(w+\sqrt{w^2+1})$

$w = \sinh(\frac{\pi}{2}F)$

$\pi z = 2w + \frac{1}{i}\ln\frac{1+iw}{1-iw}$

Very convenient for "production" of field line patterns: Field line from

$F = A + iV$

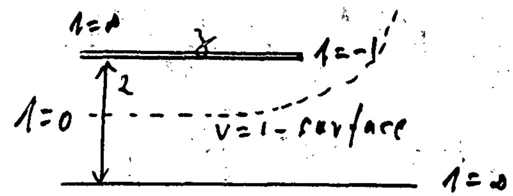
A ↑ const. for field line

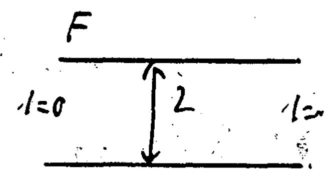
V ← vary to get field line

Similarly: V = const. surfaces

(8.10)

Rogowski surface from semi ∞ "capacitor"
(Rogowski's derivation)





$\pi \dot{z} = \frac{2(s+1)}{s}$

$\pi \dot{F} = \frac{2}{s}$

$z = \frac{2}{\pi}\cdot s + \frac{2}{\pi}\ln s$

$\pi F = 2\ln s$

$s = e^{\frac{\pi}{2}F}$

$z = \frac{2}{\pi}e^{\frac{\pi}{2}(A+iV)} + A + iV$

$V=1: \; x+iy = \frac{2}{\pi}\cdot i\cdot e^{\frac{\pi}{2}A} + A + i$

$x = A \rightarrow$ homogeneous field in "material"

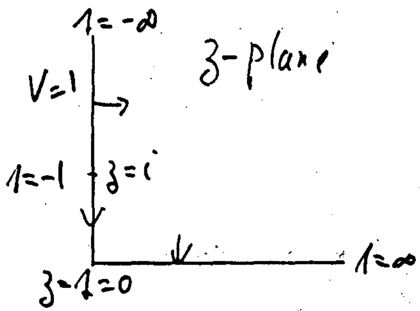
$y = 1 + \frac{2}{\pi}\cdot e^{\frac{\pi}{2}x}$

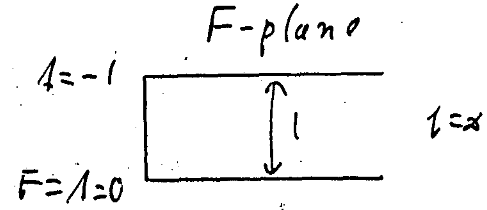
De-normalize: $y = y_0 + \frac{2}{\pi}\cdot y_0\cdot e^{\frac{\pi}{2}\cdot\frac{x}{y_0}}$

↳ origin-dependant!

(8.7)

Dipole with 0-thickness pole





$\dot{z} = \frac{1}{2\sqrt{t}} \Rightarrow z = \sqrt{t}$

$F = \frac{A}{\sqrt{t}\sqrt{t+1}}$

$\operatorname{Im}(\Delta F) = 1 = A \int \frac{dt}{t\sqrt{1+1/t}} = A \cdot i\pi \quad A = 1/\pi$

$F' = \frac{2}{\pi} \cdot \frac{1}{\sqrt{t+1}} = \frac{2/\pi}{\sqrt{1+z^2}}$

$F'(0) = 2/\pi$

Special case of "general" procedure:

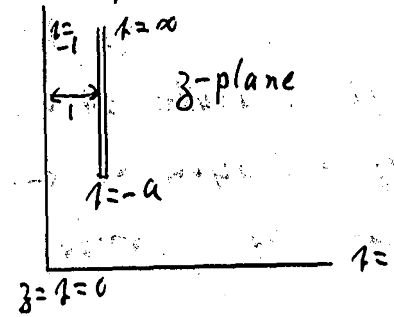
Identify straight lines (= sides of polygon) in z and F-planes that are mapped on each other through $F(z)$ = Physics. Map corresponding polygon sides in z, F onto <u>same</u> intervall on real axis of t-plane.

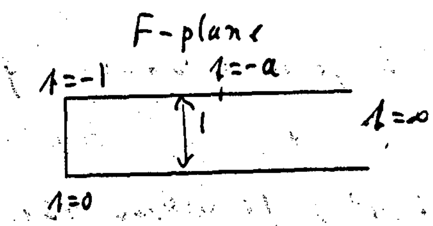
$\Rightarrow \dot{z} = f_1(t)\ ;\ \dot{F} = f_2(t) \Rightarrow F' = f_2(t)/f_1(t)$

$\hookrightarrow z(t) \qquad \hookrightarrow F(t) \rightarrow z(F)$ or $F(z)$

(8.8)

Dipole with two 0-thickness poles





$\dot{z} = \frac{A(t+a)}{\sqrt{t}(t+1)}$

$t = -1$

$\pi \dot{F} = \frac{1}{\sqrt{t}\sqrt{t+1}}$

$F' = (a-1)\frac{\sqrt{1+t}}{a+t}$

$\operatorname{Re}\Delta z = 1 = A \cdot \operatorname{Re} \int \frac{t+a}{\sqrt{t}(1+t)}\, dt$

$1 = A \cdot \pi i \cdot \frac{a-1}{i}\ ;\ A = \frac{1}{\pi(a-1)}$

$|t| \ll 1:\ \pi\dot{z} = \frac{a}{a-1}\frac{1}{\sqrt{t}}\ ;\ \pi z = \frac{2a}{a-1}\sqrt{t}$

$F' = \left(1-\frac{1}{a}\right)\cdot\left(1 + t\left(\frac{1}{2}-\frac{1}{a}\right) + \cdots\right)$

$a = 2 \rightarrow$ no term $\sim z^2$

$\sqrt{t} = w\ ;\ t = w^2\ ;\ dt = 2w\,dw$

$\pi \cdot \frac{dz}{dw} = \frac{2(t+2)}{t+1} = 2 + \frac{2}{(w-i)(w+i)} = 2 + \frac{1}{i}\left(\frac{1}{w-i} - \frac{1}{w+i}\right)$

$\pi z = 2w + \frac{1}{i}\ln\frac{i-w}{i+w} = 2w + \frac{1}{i}\ln\frac{1+iw}{1-iw} = z\pi$

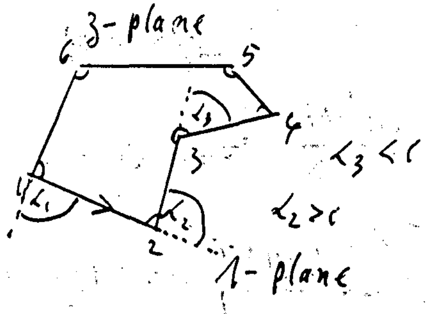
(8.5)

Schwarz-Christoffel Transformation.

What is it? Procedure to get transformation that maps interior of polygon to ½ plane or interior (usually) of circular disk.

Polygons very often degenerate, i.e. one or more corners at ∞.

What good is it? Best seen with specific applications.

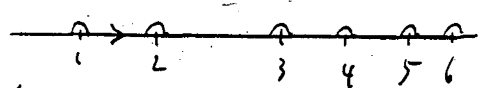


Recipe

Number corners

- sequentially, and

map them on sequentially numbered points



on real axis of λ-plane with

$$dz/d\lambda = A \cdot \prod_{\mu=1} (\lambda - \lambda_\mu)^{-n_\mu}$$

(7.26) (8.6)

All λ_μ = real ; $n_\mu = \alpha_\mu / \pi$; $\sum n_\mu = 2$

$(\lambda - \lambda_\mu)^{n_\mu}$ = real for $\lambda - \lambda_\mu$ = real, > 0.

$\lambda_{\mu-1} < \lambda_\mu$

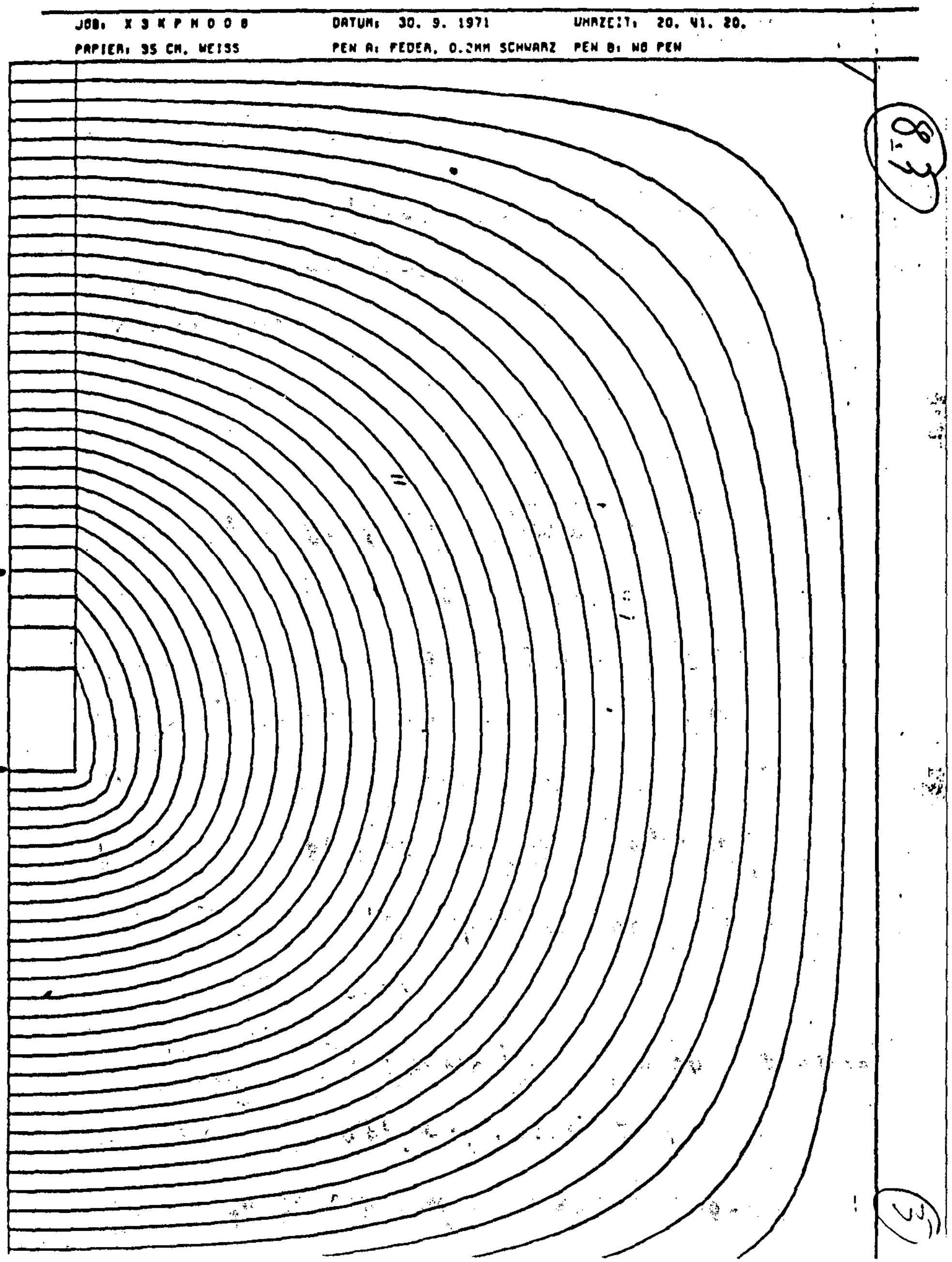
For λ = real ; $\lambda_{\mu-1} < \lambda < \lambda_\mu$, $dz/d\lambda$ does not change phasefactor $\rightarrow z(\lambda)$ = straight.

$\lambda - \lambda_\mu > 0$: phase of $(\lambda - \lambda_\mu)^{n_\mu}$ is zero

$\lambda - \lambda_\mu < 0$: phase of $(\lambda - \lambda_\mu)^{n_\mu}$ is $e^{i\pi \cdot \alpha_\mu / \pi} = e^{i\alpha_\mu}$

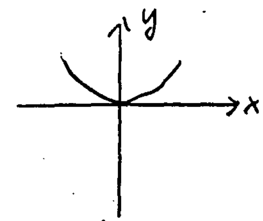
Conclusion: going from "a little" to the left of λ_μ to "a little" to the right of λ_μ, the phase of $dz/d\lambda$ increases by $\alpha_\mu \rightarrow$ interior of polygon is mapped unto upper ½ of λ-plane.

Choice of λ_μ : can change origin of λ-plane $\rightarrow$ can make one $\lambda_\mu = 0$. Now: $T = -1/\lambda$ maps upper ½ plane of λ to upper ½ plane of T.



(8.4)

Curvature of $A(x,y)=\text{const}_1$ = field line.



Rotate x-y-system so that $x=y=0$ at point of interest, and tangent to field line ∥ x-axis. For that field line:

$$\frac{dy}{dx} = -\frac{A'_x}{A'_y} = \frac{B_y}{B_x} = \frac{H_y}{H_x} \quad (\text{isotropic } \mu)$$

$$\frac{1}{R} = \frac{d^2y}{dx^2} = \frac{\partial y'}{\partial x} + \frac{\partial y'}{\partial y}\cdot \underbrace{y'}_{0} = \frac{H_x \cdot \partial H_y/\partial x - \underbrace{H_y}_{\to 0} \cdot \partial H_x/\partial x}{H_x^2}$$

$$\frac{1}{R} = \frac{\partial H_y/\partial x}{H_x} = \frac{\partial H_x/\partial y}{H_x}$$

Change of field in direction ⊥ field line

Curvature of $V=\text{const.}$ Rotate x-y as above, with tangent to $V=\text{const}$ ∥ x-axis. Curvature of $V=\text{const.}$ from

$$\frac{dy}{dx} = -\frac{V'_x}{V'_y} = -\frac{H_x}{H_y} = -\frac{B_x}{B_y}$$

$$\frac{1}{R} = \frac{\partial y'}{\partial x} + \frac{\partial y'}{\partial y}\cdot \underbrace{y'}_{=0} = -\frac{B_y \cdot \partial B_x/\partial x - \overbrace{B_x}^{=0} \cdot \partial B_y/\partial x}{B_y^2}$$

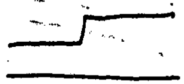
$$\frac{1}{R} = -\frac{\partial B_x/\partial x}{B_y} = \frac{\partial B_y/\partial y}{B_y}$$

Change of field in direction ∥ fieldline

(8.1)

Topics to be covered on + after 1-6-89

4 non-ID-applications of S-C

In  : Excess flux / V-drop; expansion of F

S-C polygon → O

In  : V_0/B_0, pole flux, excess flux, e.t.c.

Many $\mu=\infty$ bodies in 3D; capacities

Non ID-applications of C's

Error propagation in hybrid ID

Entry/Exit for hybrid ID → tapered ID

Field from □ charge sheet

Table of excess flux formulae

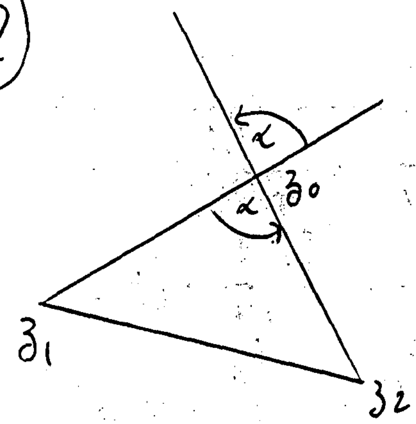
?

OA-model

Eddy current effects

Perturbation effects in symmetric multipoles

(8.2)



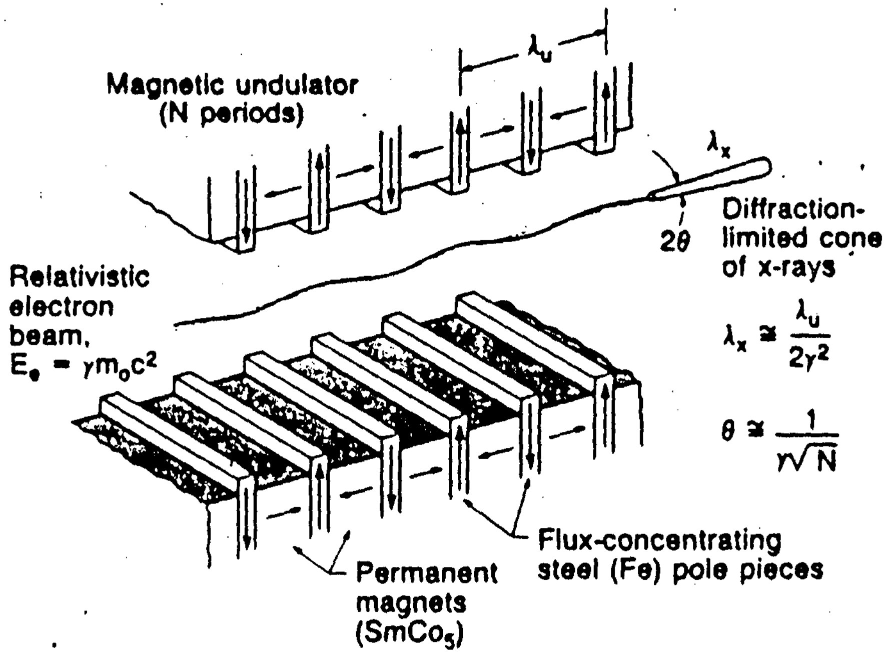
$$\ln \frac{z_0 - z_2}{z_0 - z_1} = \ln \left| \frac{z_0 - z_2}{z_0 - z_1} \right| + i\alpha$$

(8.0)

Future Lectures: 1-13 8-10
1-19 8^{30}-10^{30}
2-3 8^{30}-10^{30}
2-10 8-10

Insertion Device Design

Klaus Halbach

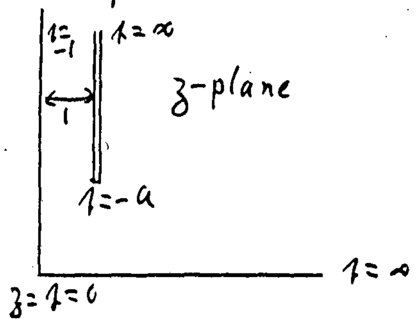


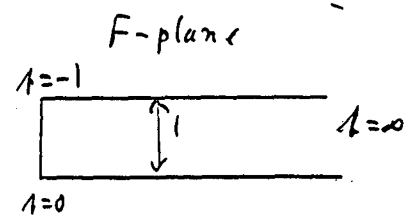
Lecture 8.

January 6, 1989

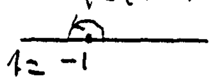
(7.29)

Dipole with two 0-thickness poles





$$\dot{z} = \frac{A(t+a)}{\sqrt{t}(t+1)}$$



$$\pi \dot{F} = \frac{1}{\sqrt{t}\sqrt{t+1}}$$

$$F' = (a-1)\frac{\sqrt{1+t}}{a+t}$$

$$\mathrm{Re}\,\Delta z = 1 = A \cdot \mathrm{Re} \oint \frac{t+a}{\sqrt{t}(1+t)}\,dt$$

$$1 = A \cdot \pi i \cdot \frac{a-1}{i} \;;\; A = \frac{1}{\pi(a-1)}$$

$$|t| \ll 1 : \pi\dot{z} = \frac{a}{a-1}\frac{1}{\sqrt{t}} \;;\; \pi z = \frac{2a}{a-1}\sqrt{t}$$

$$F' = \left(1 - \frac{1}{a}\right)\cdot\left(1 + t\left(\frac{1}{2} - \frac{1}{a}\right) + \cdots\right)$$

$a = 2 \rightarrow$ no term $\sim z^2$

$\sqrt{t} = W \;;\; t = W^2 \;;\; dt = 2W\,dW$

$$\pi \cdot \frac{dz}{dW} = \frac{2(t+2)}{t+1} = 2 + \frac{2}{(W-i)(W+i)} = 2 + \frac{1}{i}\left(\frac{1}{W-i} - \frac{1}{W+i}\right)$$

$$\pi z = 2W + \frac{1}{i}\ln\frac{i-W}{i+W} = 2W + \frac{1}{i}\ln\frac{1+iW}{1-iW} = z$$

(7.30)

$$t = -2 : \pi z = 2i\sqrt{2} + \frac{1}{i}\ln\frac{1-\sqrt{2}}{1+\sqrt{2}} = \pi + i \cdot 2\left(\sqrt{2} + \ln(1+\sqrt{2})\right)$$

$$z(-2) \approx 1 + i \cdot 1.46$$

$$F'(0) = 0.5 \quad \left(\text{"ideally" } \frac{1}{1.46} = .68\right)$$

7.27

Number points so that $t_1 = 0$

$$\frac{dz}{dT} = \frac{dz}{dt}\cdot\frac{1}{T^2} = \frac{A}{T^2}\cdot\frac{1}{\frac{1}{T^{n_1}}\prod\left(\frac{1}{T}-\frac{1}{T_\mu}\right)^{n_\mu}}$$

$$\frac{dz}{dT} = \frac{A}{T^2}\,\frac{T^{\sum n_\mu}}{\prod_2 (1-T/T_\mu)^{n_\mu}}$$

$\sum n_\mu = 2$; with new A_n

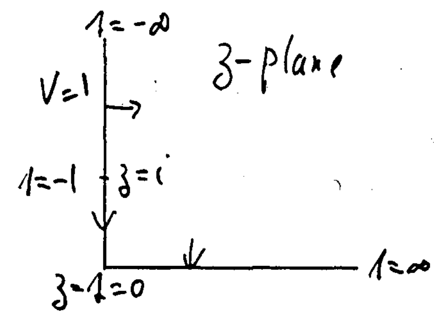
$$\frac{dz}{dT} = \frac{A_n}{\prod_2 (T-T_\mu)^{n_\mu}}$$

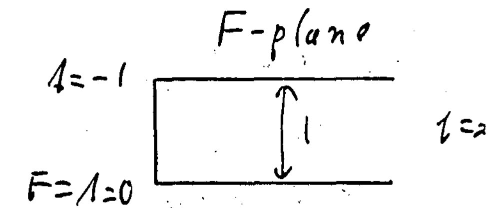
Same as before, but $\cdot 1$ in T plane is now at ∞ → it has disappeared from formula!!!

By shifting origin again, and scaling T plane, can move two points on T-axis to arbitrary locations (usually $T=0$, and $T=1$ or $T=-1$) $\neq \infty$ without changing polygon.

7.28

Dipole with 0-thickness pole





$\dot z = \frac{1}{2\sqrt{t}} \Rightarrow z = \sqrt{t}$

$$\dot F = \frac{A}{\sqrt{t}\,\sqrt{t+1}}$$

$$\operatorname{Im}(\Delta F) = 1 = A\cdot \operatorname{Im}\int \frac{dt}{t\sqrt{1+1/t}} = A\cdot i\pi \qquad A = 1/\pi$$

$$F' = \frac{2}{\pi}\cdot\frac{1}{\sqrt{t+1}} = \frac{2/\pi}{\sqrt{1+z^2}}$$

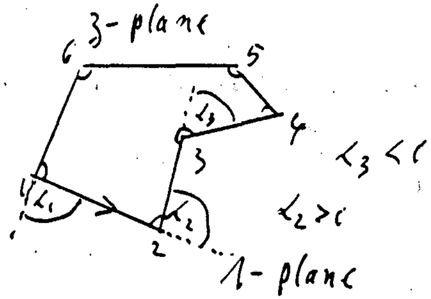
$F'(0) = 2/\pi$

(7.25)

Schwarz-Christoffel Transformation.

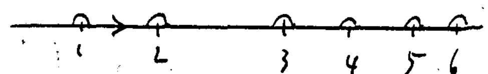
What is it? Procedure to get transformation that maps interior of polygon to ½ plane or interior (usually) of circular disk.
Polygons very often degenerate, i.e. one or more corners at ∞.

What good is it? Best seen with specific applications.



Recipe

Number corners sequentially, and map them on sequentially numbered points



on real axis of t-plane with

$$dz/dt = A \cdot \prod_{\mu=1} (t - t_\mu)^{-n_\mu}$$

(7.26)

All t_μ = real ; $n_\mu = \alpha_\mu / \pi$; $\sum n_\mu = 2$

$(t - t_\mu)^{n_\mu}$ = real for $t - t_\mu$ = real, > 0.

$t_{\mu-1} < t_\mu$

For t = real ; $t_{\mu-1} < t < t_\mu$, dz/dt does not change phasefactor $\rightarrow z(t)$ = straight

$t - t_\mu > 0$: phase of $(t - t_\mu)^{n_\mu}$ is zero

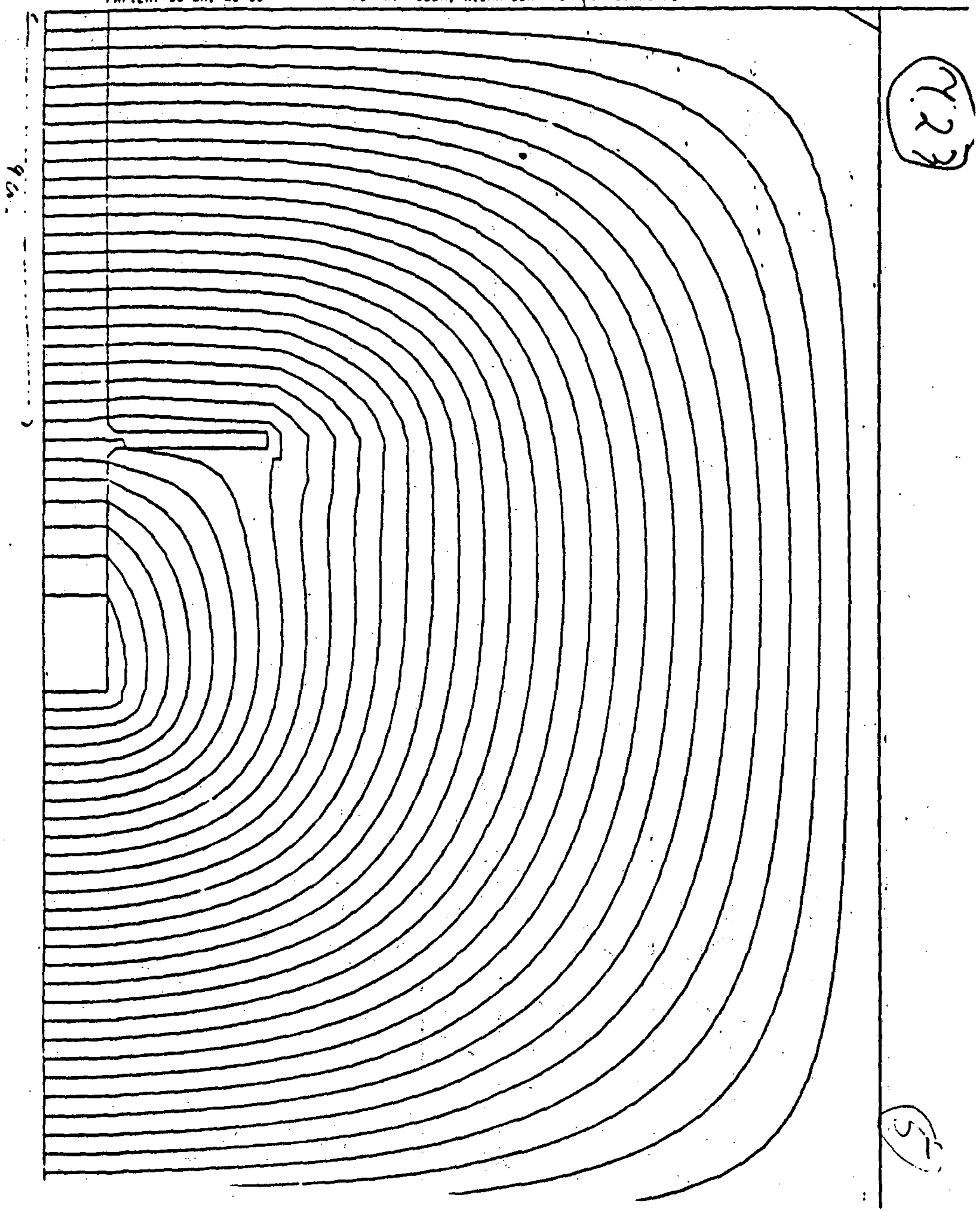
$t - t_\mu < 0$: phase of $(t - t_\mu)^{n_\mu}$ is $e^{i\pi \alpha_\mu / \pi} = e^{i\alpha_\mu}$

Conclusion: going from "a little" to the left of t_μ to "a little" to the right of t_μ, the phase of dz/dt increases by $\alpha_\mu \rightarrow$ interior of polygon is mapped unto upper ½ of t-plane.

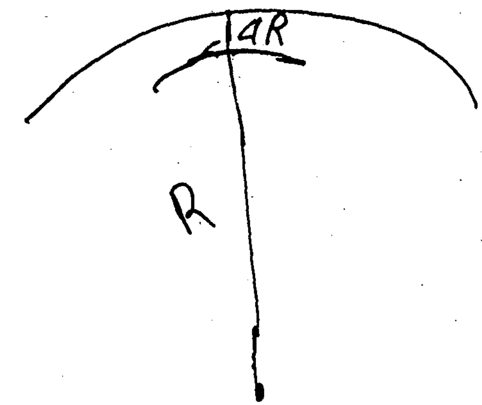
Choice of t_μ : can change origin of t-plane $\rightarrow$ can make one $t_\mu = 0$. Now: $T = -1/t$ maps upper ½ plane of t to upper ½ plane of T.

JOB: T 3 K P H 0 0 8   DATUM: 30. 9. 197.
PAPIER: 35 CM, WEISS   PEN A: FEDER, 0.2MM SCHWARZ PEN D: NO PEN

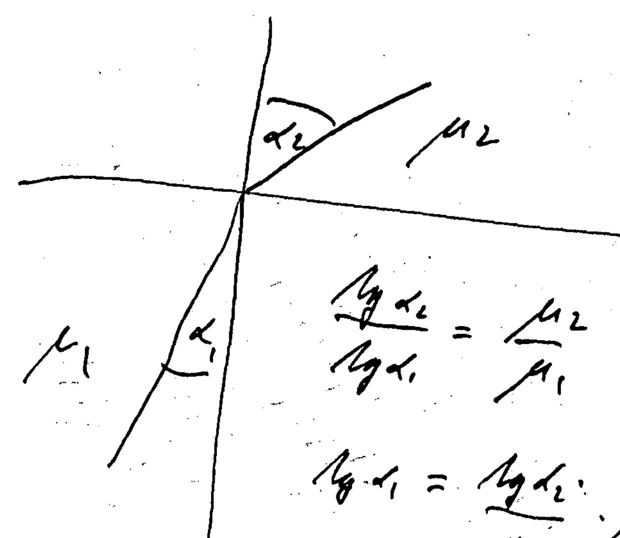
(7.23)



(7.24)



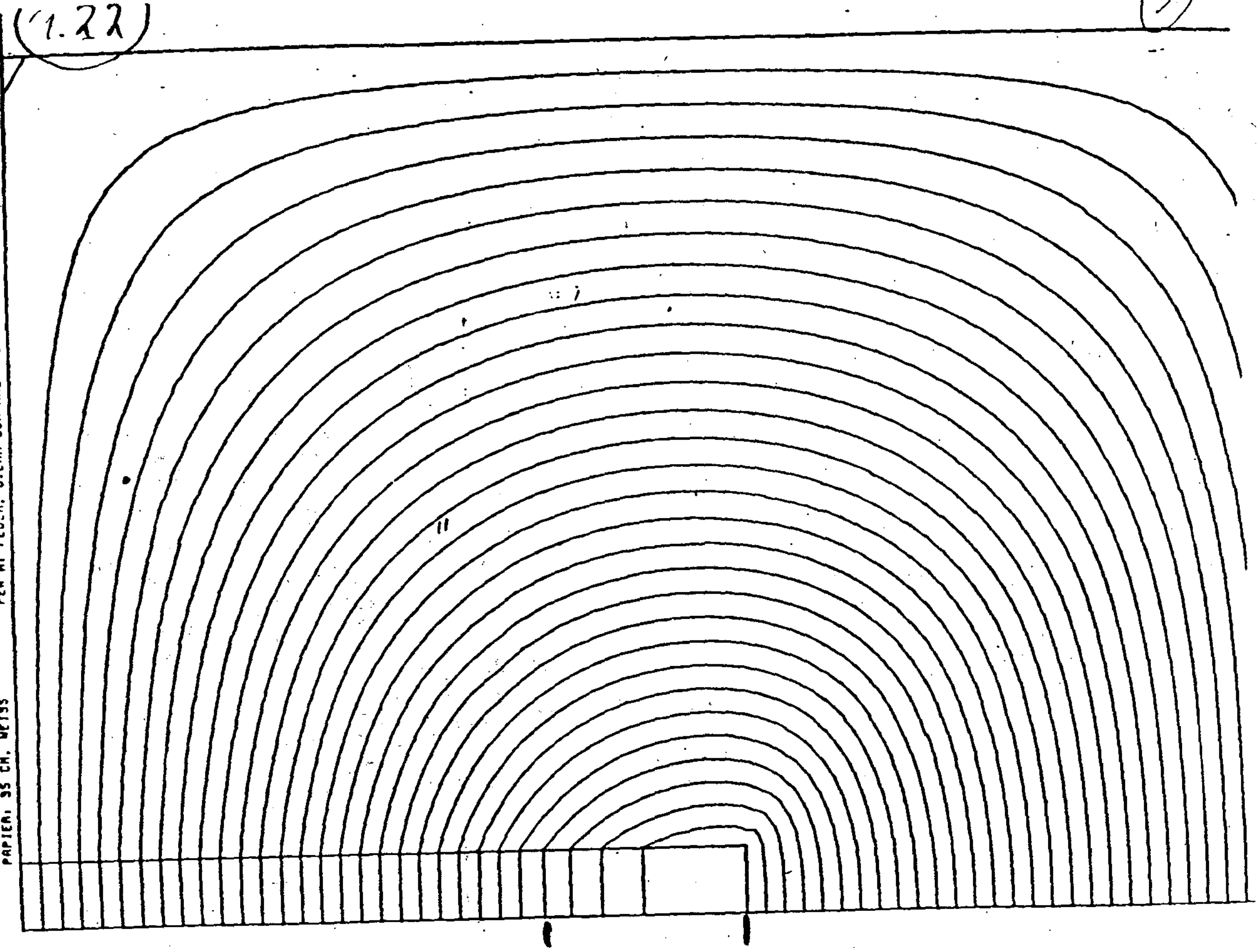
$$\frac{\Delta H}{H} = \frac{\Delta R}{R}$$



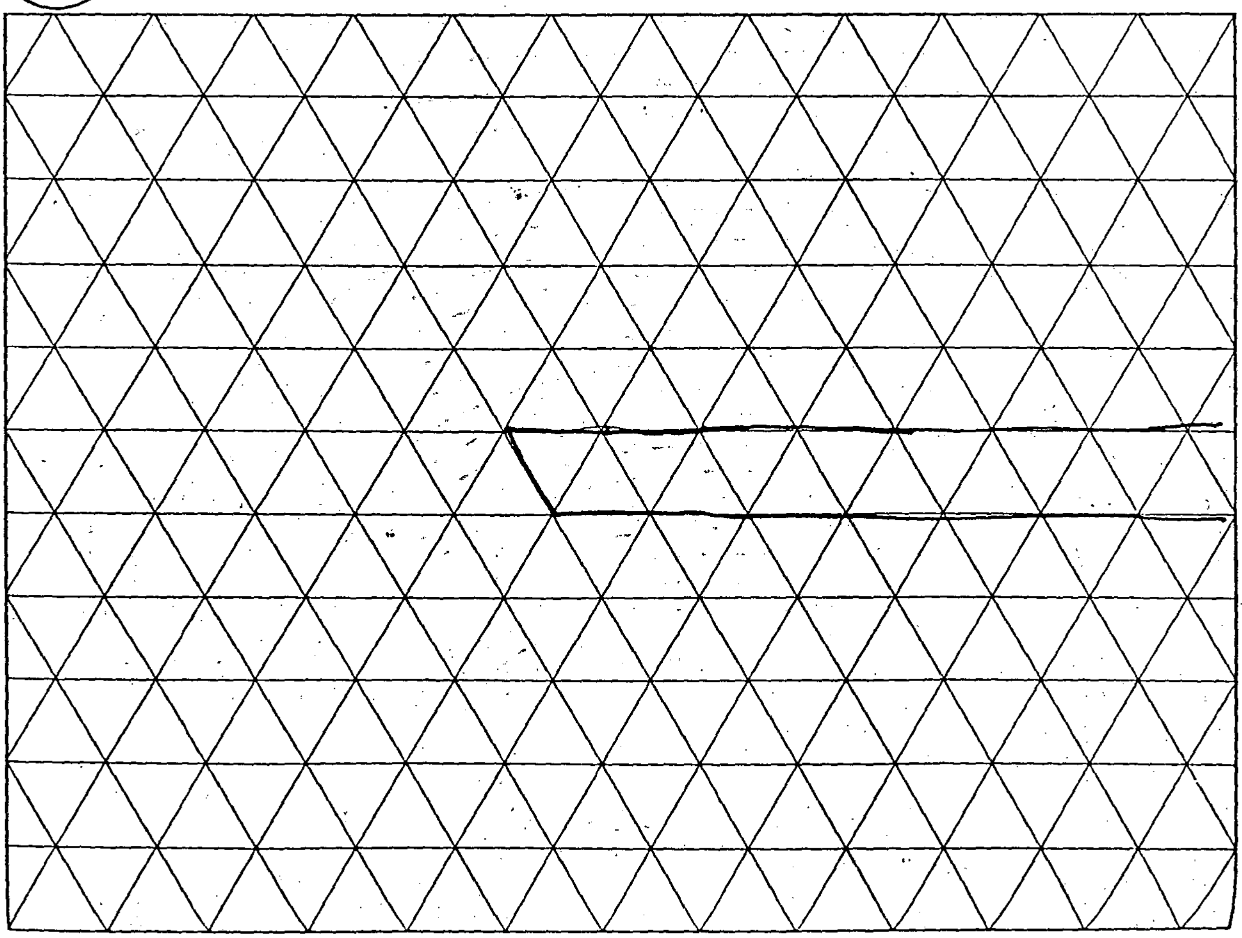
$$\frac{\operatorname{tg}\alpha_2}{\operatorname{tg}\alpha_1} = \frac{\mu_2}{\mu_1}$$

$$\operatorname{tg}\alpha_1 = \frac{\operatorname{tg}\alpha_2}{\mu_2} \cdot \mu_1$$

(1.22)

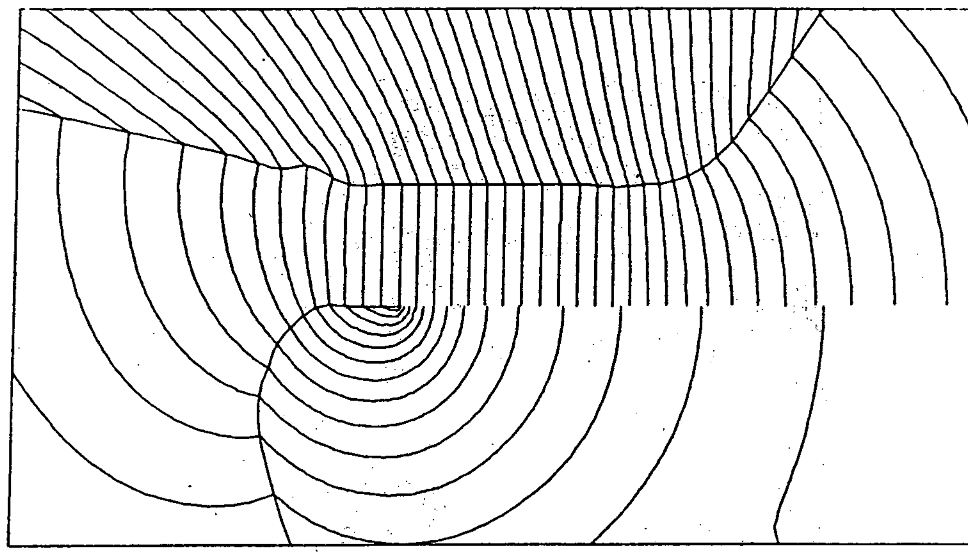

JOB: X 3 K P M O O 8 DATUM: 30. 9. 1971 UHRZEIT: 20. 41. 20.
PAPIER: 35 CM. WEISS PEN A: FEDER, 0.2MM SCHWARZ PEN B: NO PEN

(1.21)

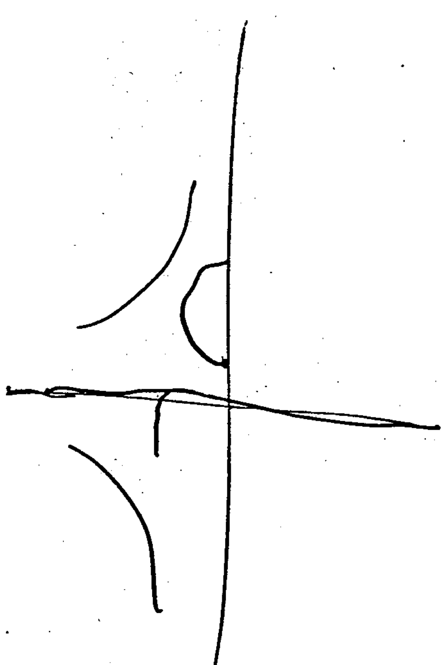

Isometric 20 millimeters/Division

7.20

TYPE INPUT DATA- NUM, ITRI, NPHI, INAP, NSUXV,

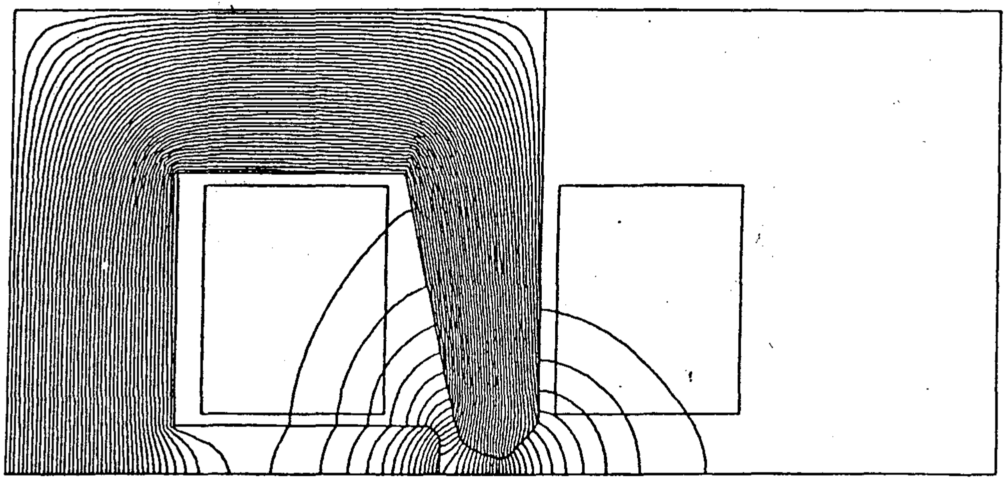


PROB. NAME = SLC L31 : M=1, OPT. POLE FROM SA   CYCLE = 70

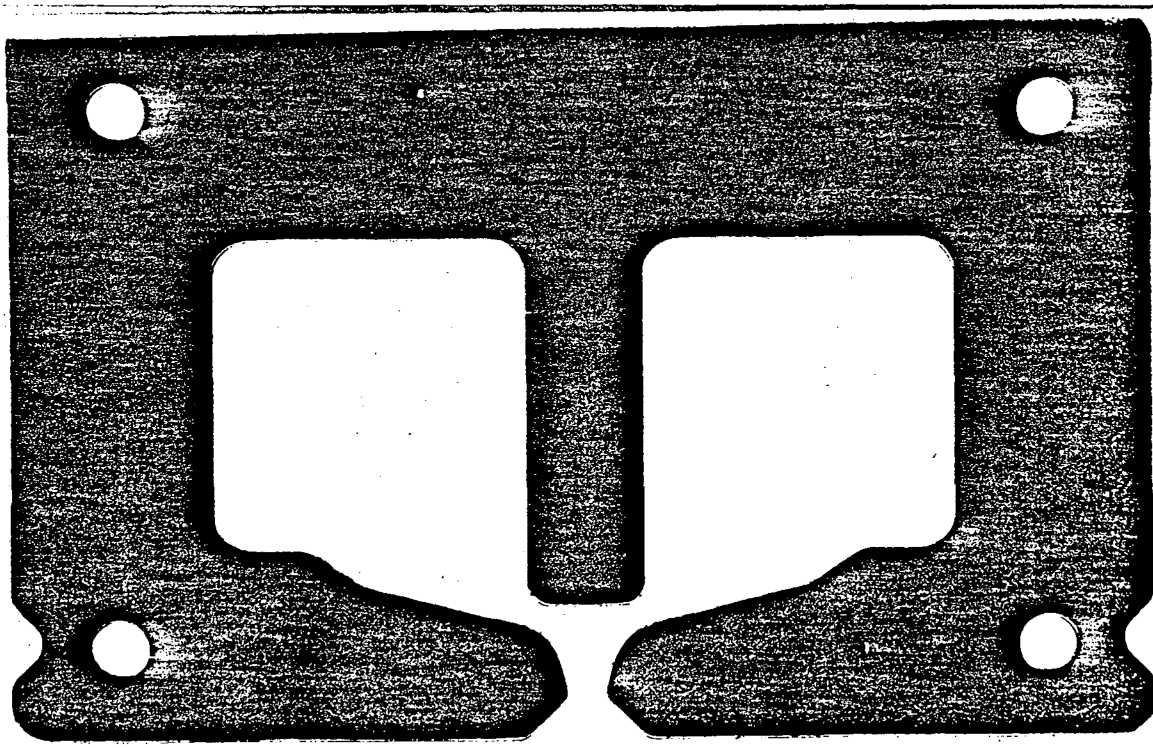


7.19

7.18

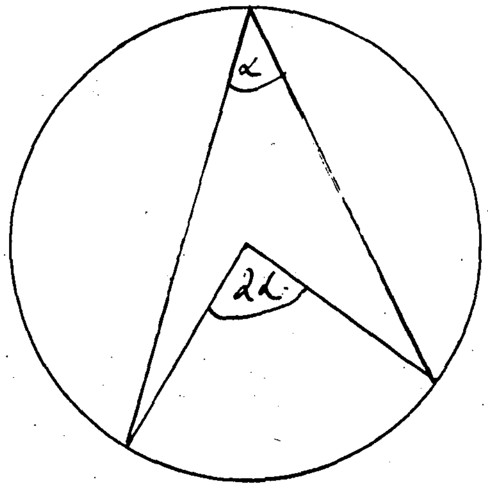


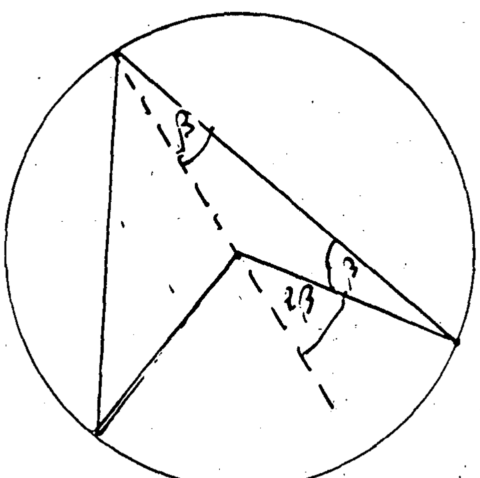
PROB. NAME = SAM30 : N=1,1ST CORRECTED FIN. M   CYCLE = 2350



7.19

(7.15)





(7.16)

Result : $\underline{|H_{12y}| \le H_c/2}$

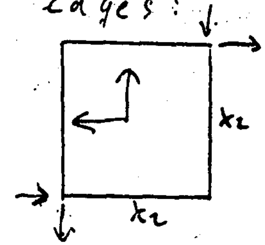
$\underline{\underline{H_{12x}}}$ $\quad z_1 = 0$; $z_2 = x_2$; $|z_0| = r_0 \ll x_2$

$\left|1 - \frac{x_2}{z_0}\right| \approx \frac{x_2}{r_0}$

$H_{12x}(z_0) = \frac{H_c}{2\pi} \ln(x_2/r_0)$

$|H_{12x}| \geqq H_c$: $r_0 \leqq x_2 e^{-2\pi} = x_2/535$

No problem if block magnetized ∥, ⊥ edges. <u>But</u>: if magnetized at 45° to edges:



Problem at 2 corners; but area affected very small:

$a = 2 \cdot \frac{\pi}{4} \cdot r_0^2 = x_2^2 \cdot \frac{\pi}{2(535)^2} = x_2^2 \cdot 5.5 \cdot 10^{-6}$

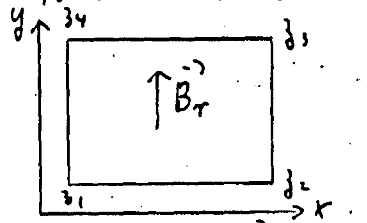
(7.13)

Field at edge of block of CSEM.

2 Reasons: 1) Useful to understand effects high field at edge may have on material.

2) While not major concept, methodology used to deal with ln can be extremely useful.

Use charge sheet.



$$\tilde{H}^*(z_0) = \frac{q'}{2\pi(z_0 - z)}$$

$$\mu_{\parallel} = \mu_{\perp} = 1$$

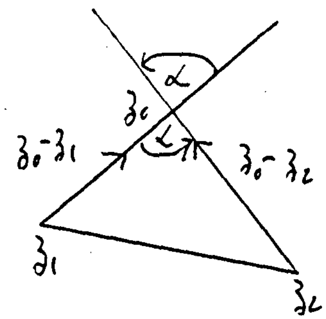
$$\tilde{H}^*(z_0) = \frac{1}{2\pi}\int_{z_1}^{z_2} \frac{-B_r\,dx}{z_0 - z} + \frac{1}{2\pi}\int_{z_4}^{z_3} \frac{B_r\,dx}{z_0 - z}$$

$$B_r/\mu_0 = H_c$$

$$H^*(z_0) = \underbrace{\frac{H_c}{2\pi}\ln\frac{z_0 - z_2}{z_0 - z_1}}_{H^*_{12}} + \underbrace{\frac{H_c}{2\pi}\ln\frac{z_0 - z_4}{z_0 - z_3}}_{H^*_{34}}$$

(7.14)

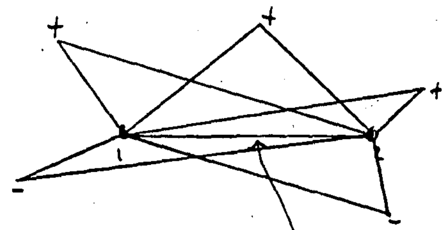
$H^*_{12}(z_0)$

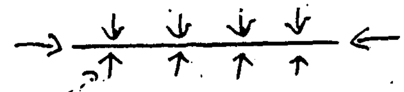


$$\ln\frac{z_0 - z_2}{z_0 - z_1} = \ln\left|\frac{z_0 - z_2}{z_0 - z_1}\right| + i\alpha$$

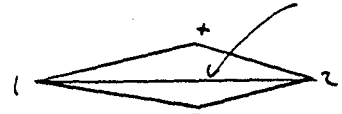
$$= \ln\left|1 - \frac{z_2 - z_1}{z_0 - z_1}\right| + i\alpha$$

α





charge sheet

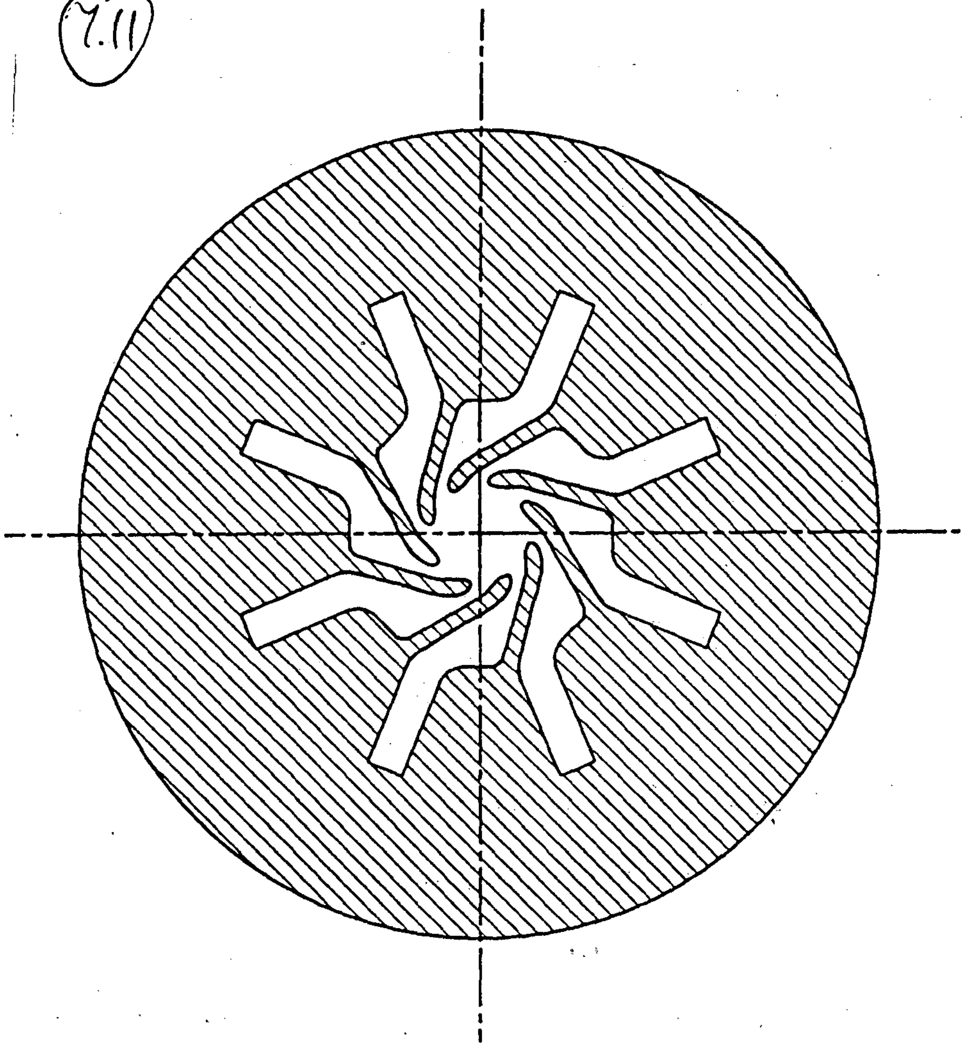


"Regular" rule

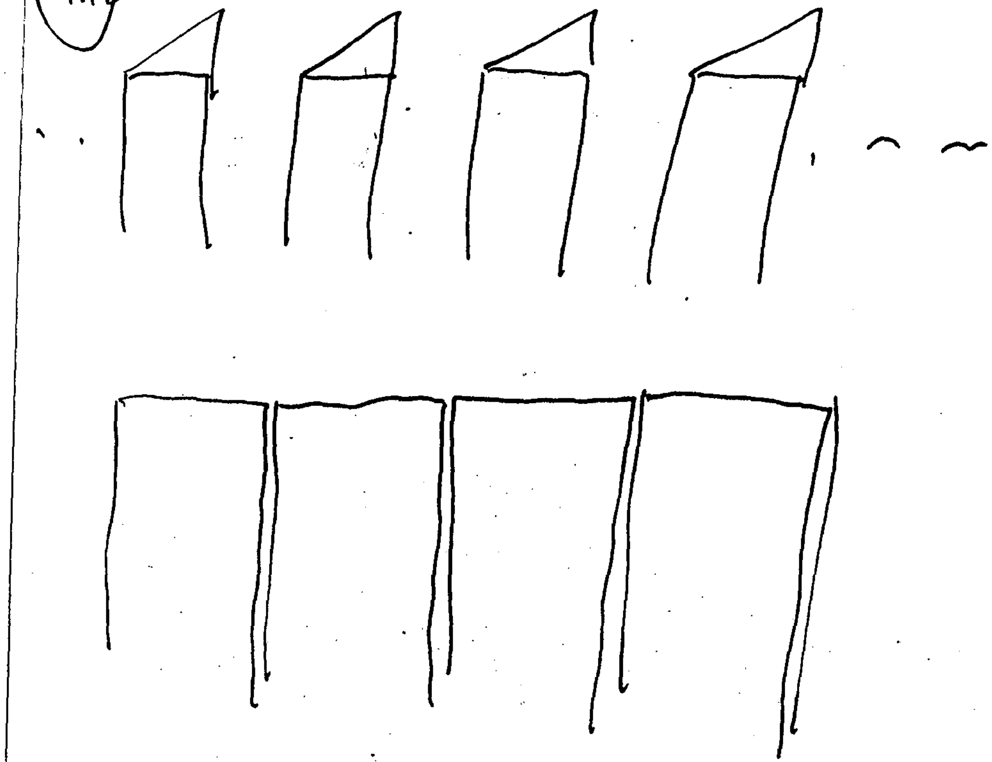
$$-\pi \le \mathrm{Im}(\ln(z)) \le \pi$$

describes/reflects physics correctly in this case, but this needs to be checked in every application, and sometimes "regular" rule has to be changed to describe physics correctly!!!

7.11



7.12



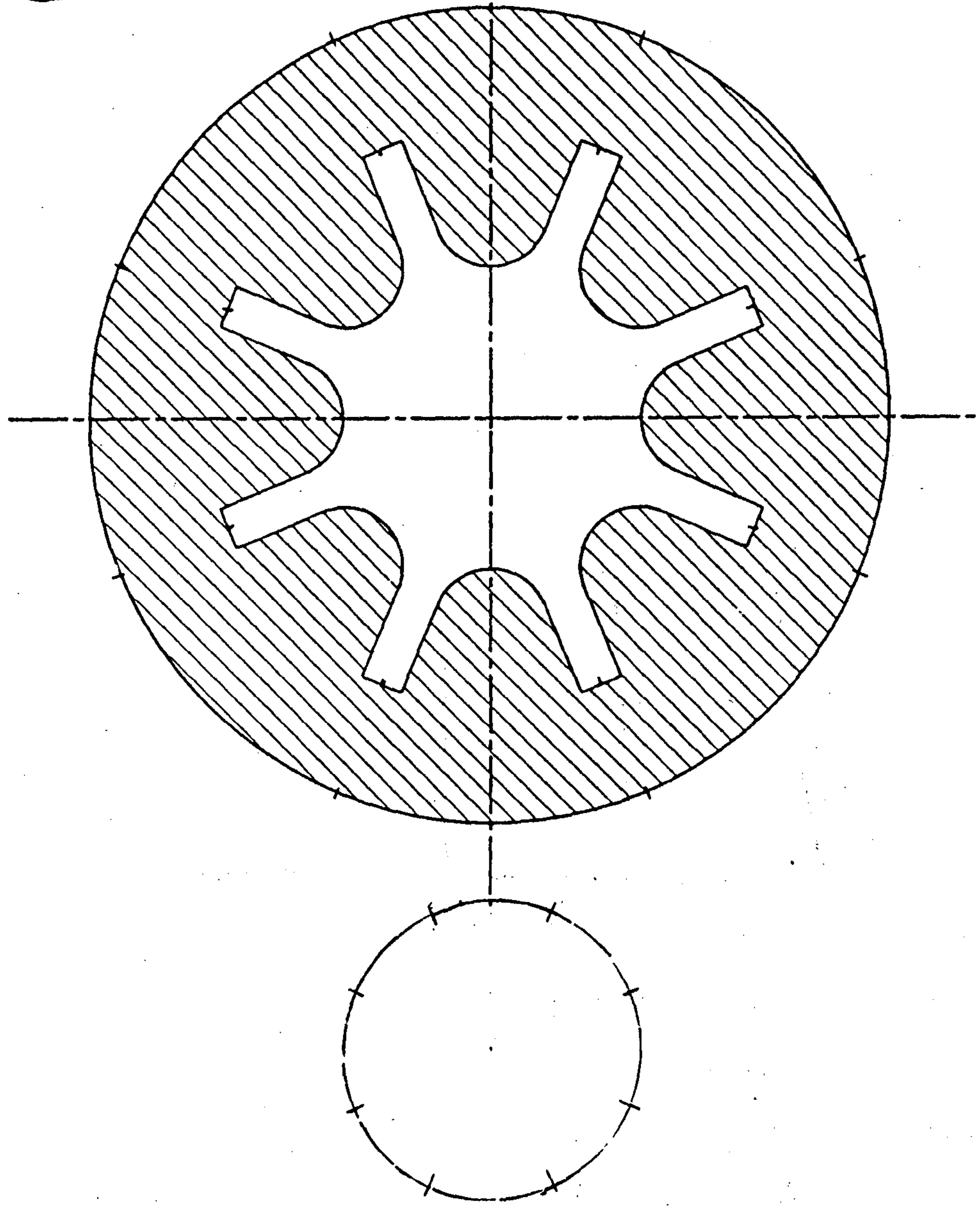
(7.9)

Mapping of inside of perfect multipole onto circular disk = example to get mapping function from Physics.

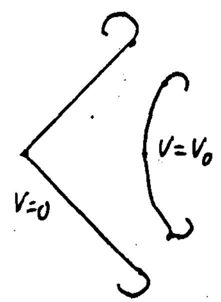
Flux from a pole in symmetric multipole to (non-immediate-neighbor) another pole depends only on multipolarity and "distance" between poles, not on pole shape/geometry.

→ same to ∞ linear array = poles of I.D.

(7.10)

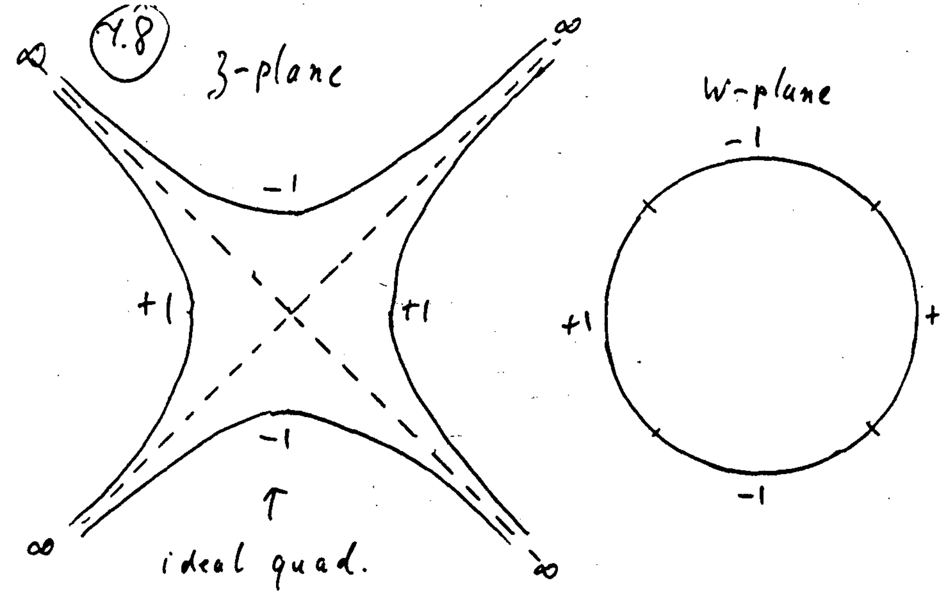


(7.7)



Better design

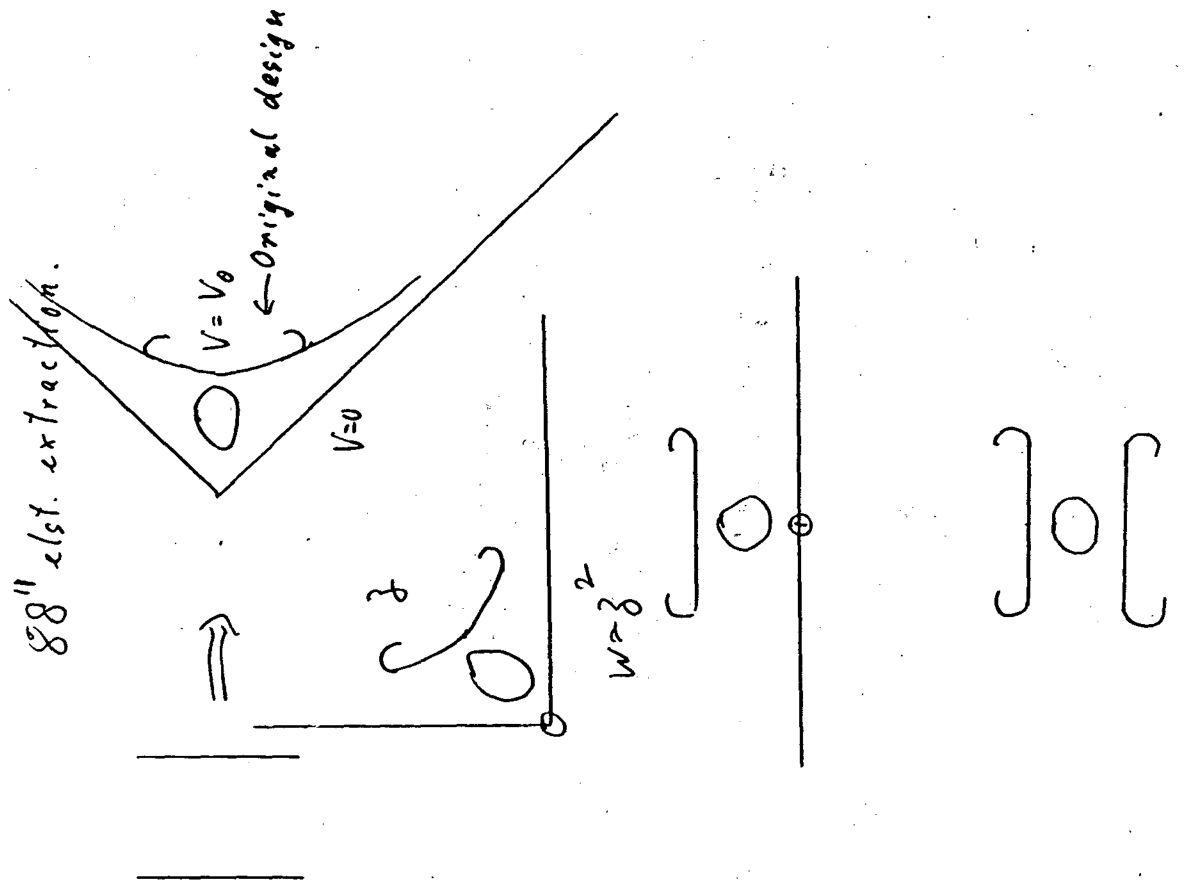
(7.8)



For ±1 excitation: $F = i\zeta^2 = \frac{2}{\pi} \ln\left(\frac{1+iW^2}{1-iW^2}\right)$

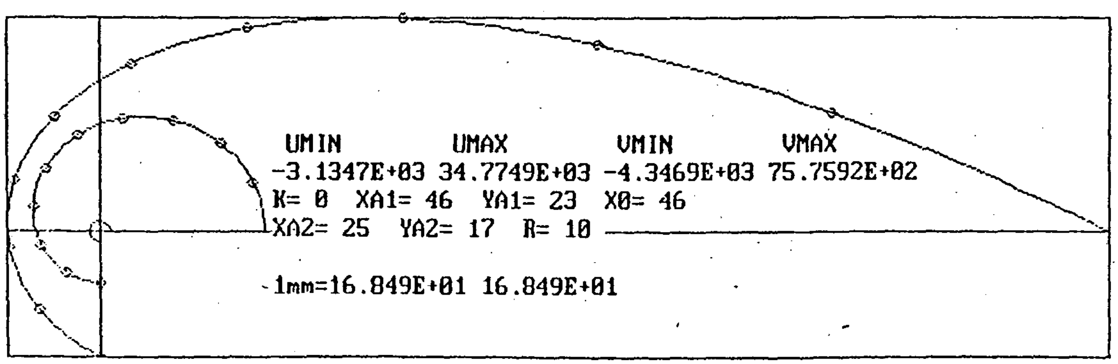
2N-pole $\quad W = \left(tg(\pi \zeta^N/4)\right)^{1/N}$

$$\zeta = \left(\frac{2}{i\pi} \ln\left(\frac{1+iW^N}{1-iW^N}\right)\right)^{1/N}$$

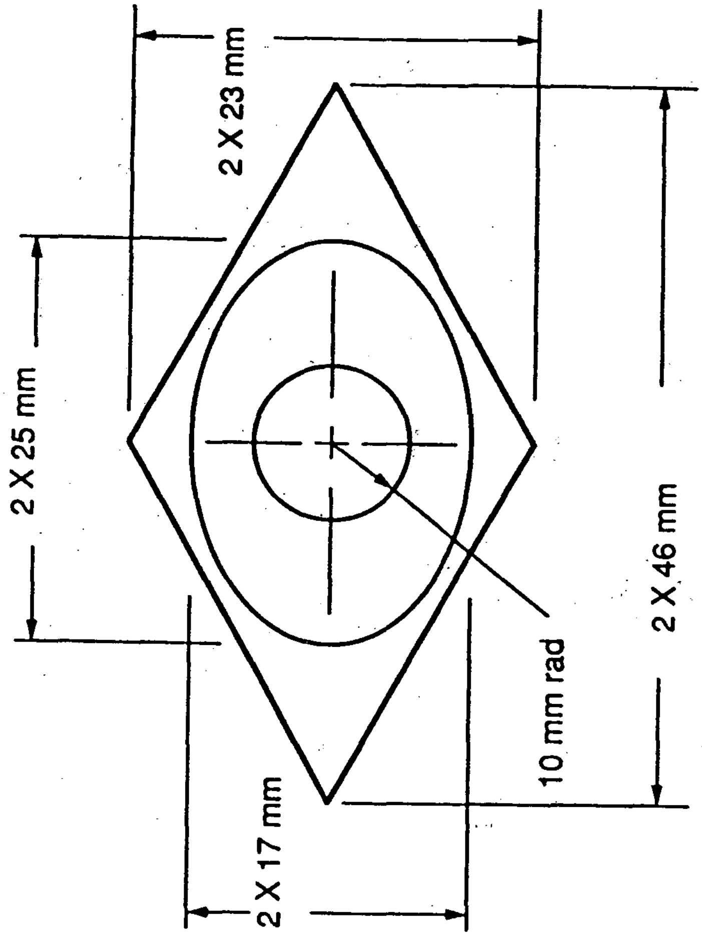

88" elst. extraction.
V = V0
← Original design
V=0

7.6

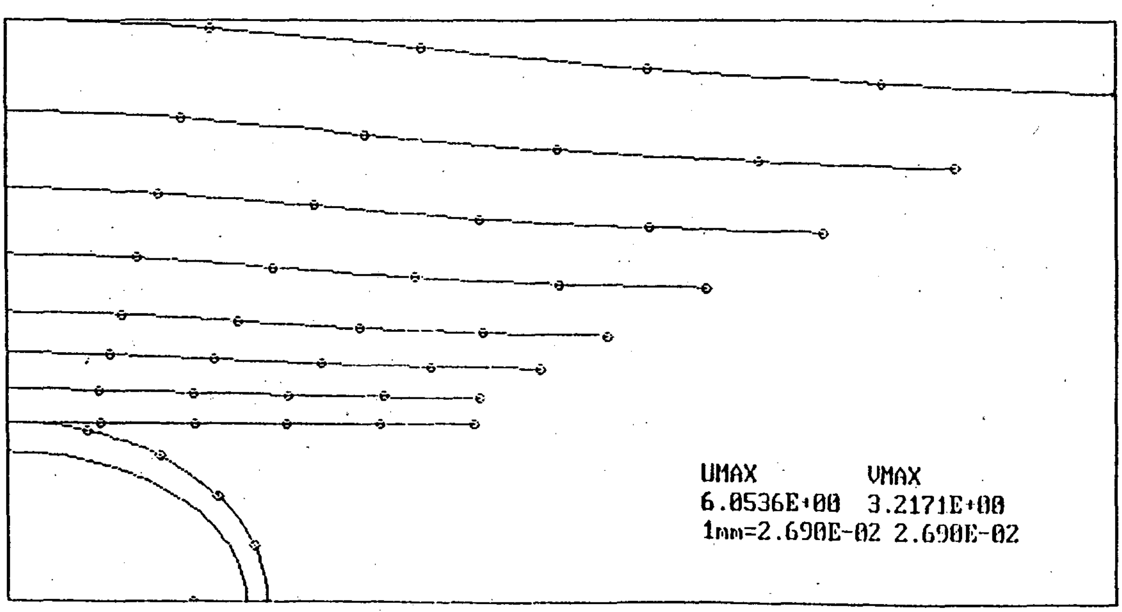
7.5


UMIN UMAX VMIN VMAX
-3.1347E+03 34.7749E+03 -4.3469E+03 75.7592E+02
K= 0 XA1= 46 YA1= 23 X0= 46
XA2= 25 YA2= 17 R= 10
1mm=16.849E+01 16.849E+01

(7.4)


2 X 23 mm
2 X 25 mm
2 X 46 mm
10 mm rad
2 X 17 mm

(7.3)


UMAX VMAX
6.0536E+00 3.2171E+00
1mm=2.690E-02 2.690E-02
G/L in Z-plane for flat pole/cap of good field/shifted poles in W-plane
0.250 0.240 0.220 0.260 0.400 0.450 0.500 0.550 0.600

(7.1)

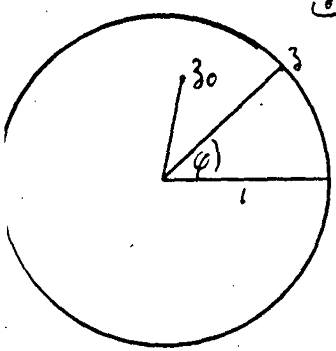
Lecture # 7

Summary of # 6:

- Mapping non-dipole into dipole geometry for design: General; Specifically: hybrid I.D., multipole.
- $B_z^* = B_w^* \cdot \frac{dw}{dz} \approx B_w^* \, \Delta w / \Delta z$
- Useful to mark maps of points equidistant in $z \rightarrow$ information about B_z^*.
- Conformal mapping as thinking tool (: 88" extr.

Dirichlet problem in circle:

$$\pi F(z_0) - \underbrace{\int_0^{2\pi} F(e^{i\varphi})\, d\varphi / 2}_{\pi F(0)} = z_0 \cdot \int_0^{2\pi} \frac{A(\varphi)\cdot\alpha + i(1-\alpha)V(\varphi)}{e^{i\varphi} - z_0}\, d\varphi$$



$$\int_0^{2\pi} F(e^{i\varphi})\, d\varphi = \frac{1}{i} \oint \frac{F(z)\,dz}{z} = 2\pi F(0)$$

$$e^{i\varphi} = z; \; d\varphi = \frac{1}{i}\frac{dz}{z}$$

(7.2)

Complete Design Procedure

1) Establish mapping function from desired field.
2) Map good field region from z into w
3) Map outside of vacuum chamber from z to w
4) In w, draw pole of sufficient width to produce dipole field of sufficient quality in w ($\leftrightarrow z$).
5) Map that pole from w into z.
6) Design rest of pole, coils, e.t.c. in z.

For some details, one may need to go back and forth between z and w. Make sure nothing "dangerous" comes too close to good field region in w.
Narrow pole more important for non-dipoles than dipoles, because of saturation.

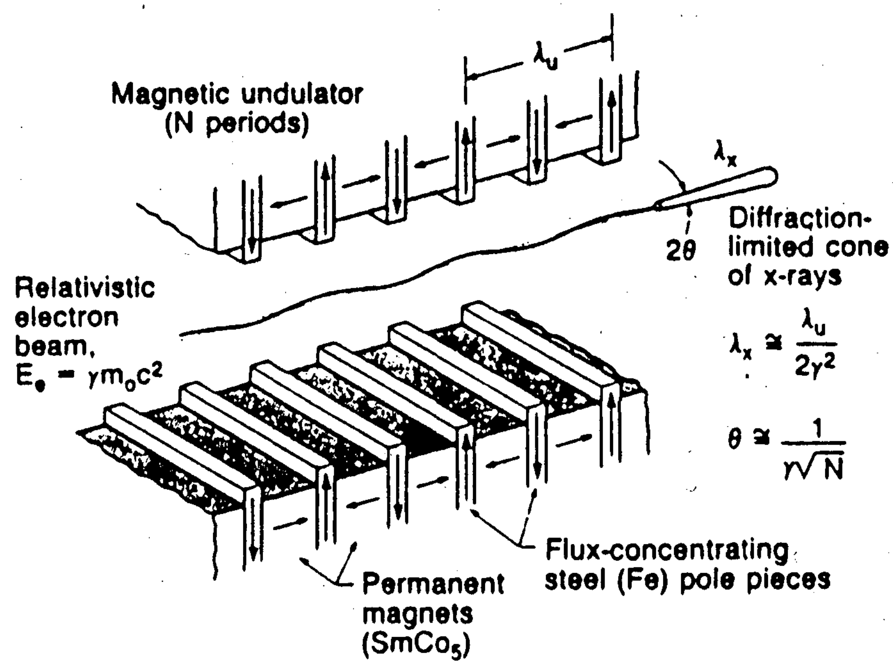
POISSON can do "everything" in w plane, even for non-linear iron.

Insertion Device Design

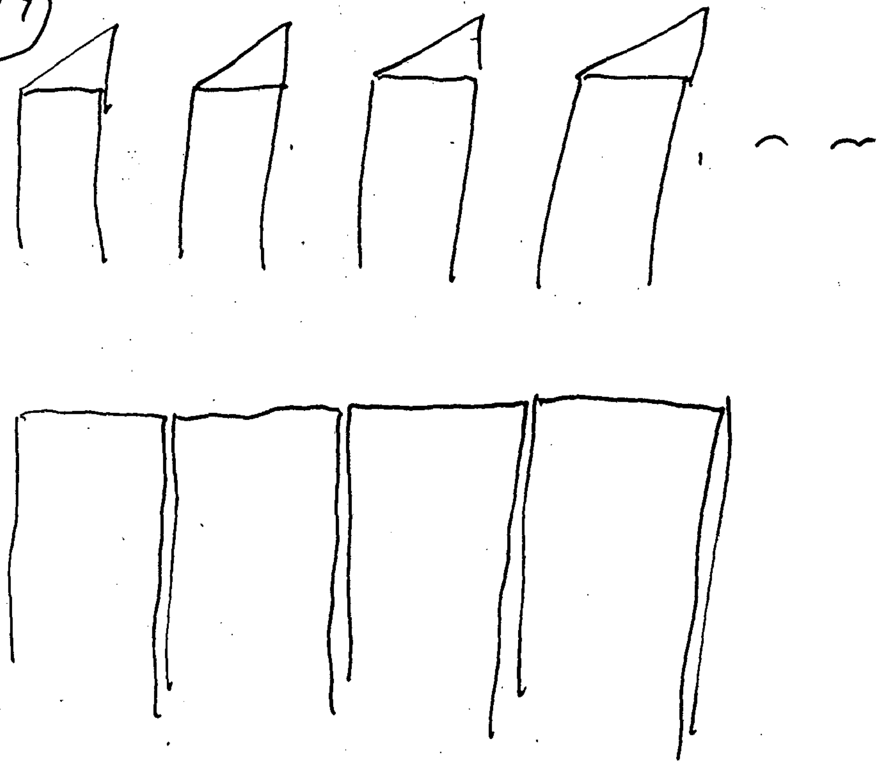
Klaus Halbach

Lecture 7.

December 21, 1988

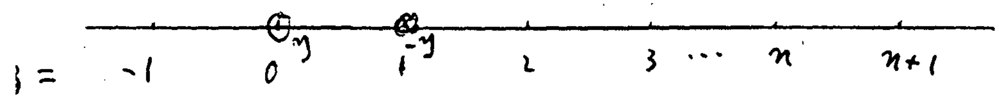


6.27



6.28

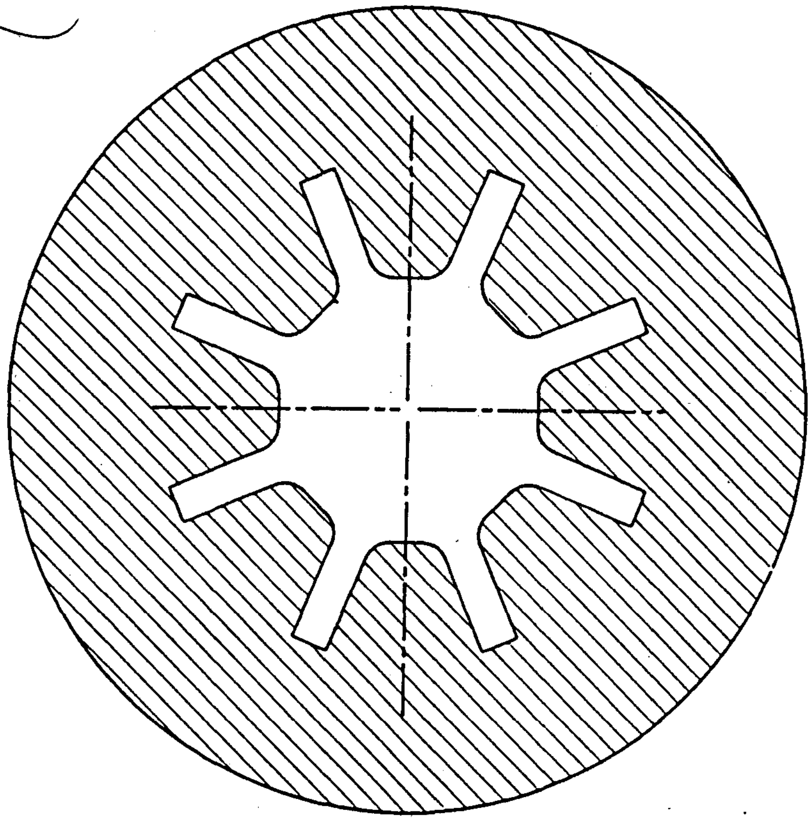
Calculation of flux from pole 0 on V to pole 1



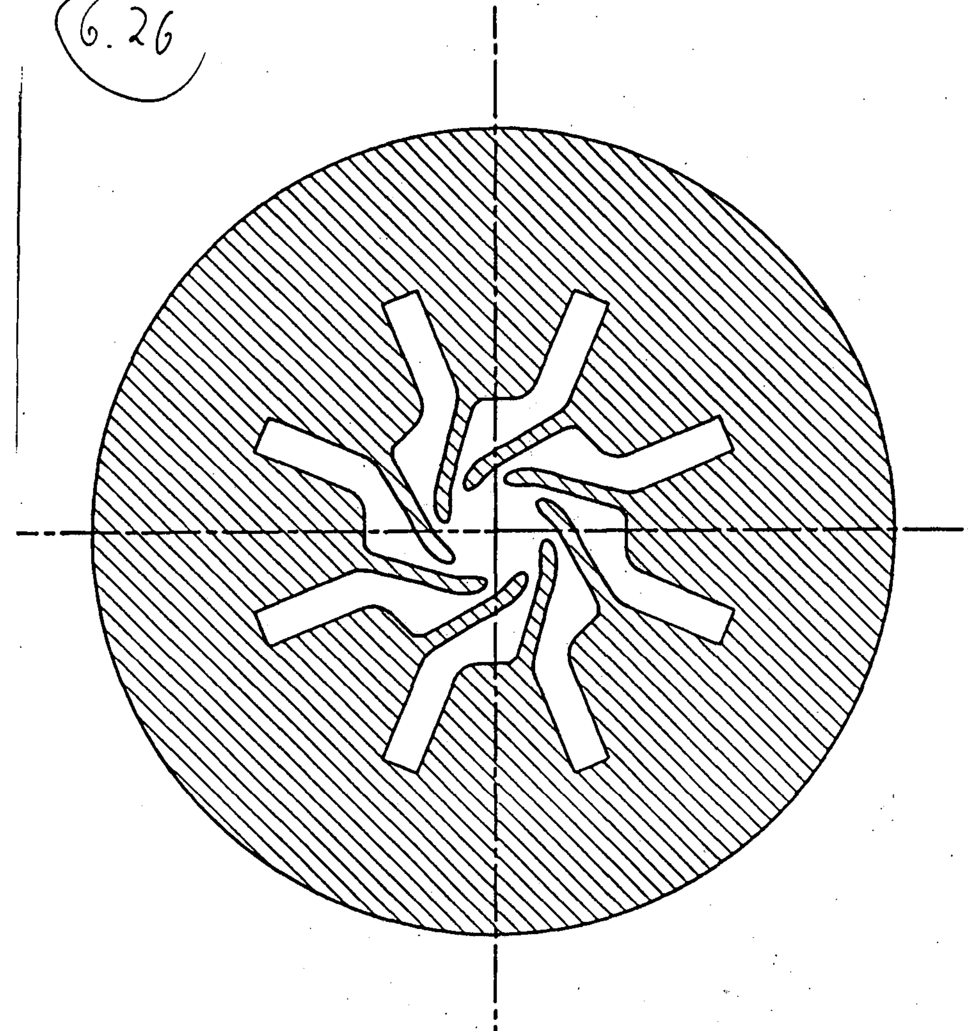
$$F(\zeta) = \frac{\eta}{2\pi} \cdot \ln \frac{\zeta - 1}{\zeta} \quad ; \quad V = \eta/2$$

$$\Delta A_n = F(n+1) - F(n) = \frac{V}{\pi} \ln \frac{n}{n+1} \cdot \frac{n}{(n-1)} = \frac{V}{\pi} \ln \frac{1}{1 - 1/n^2}$$

6.25

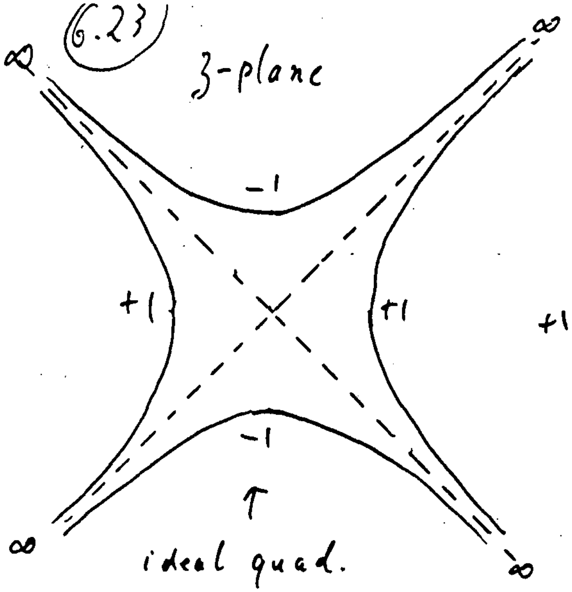


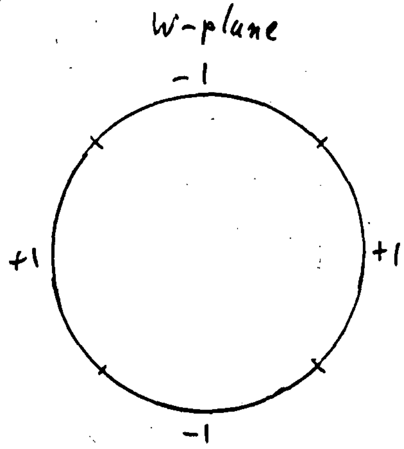
6.26



87

(6.23)



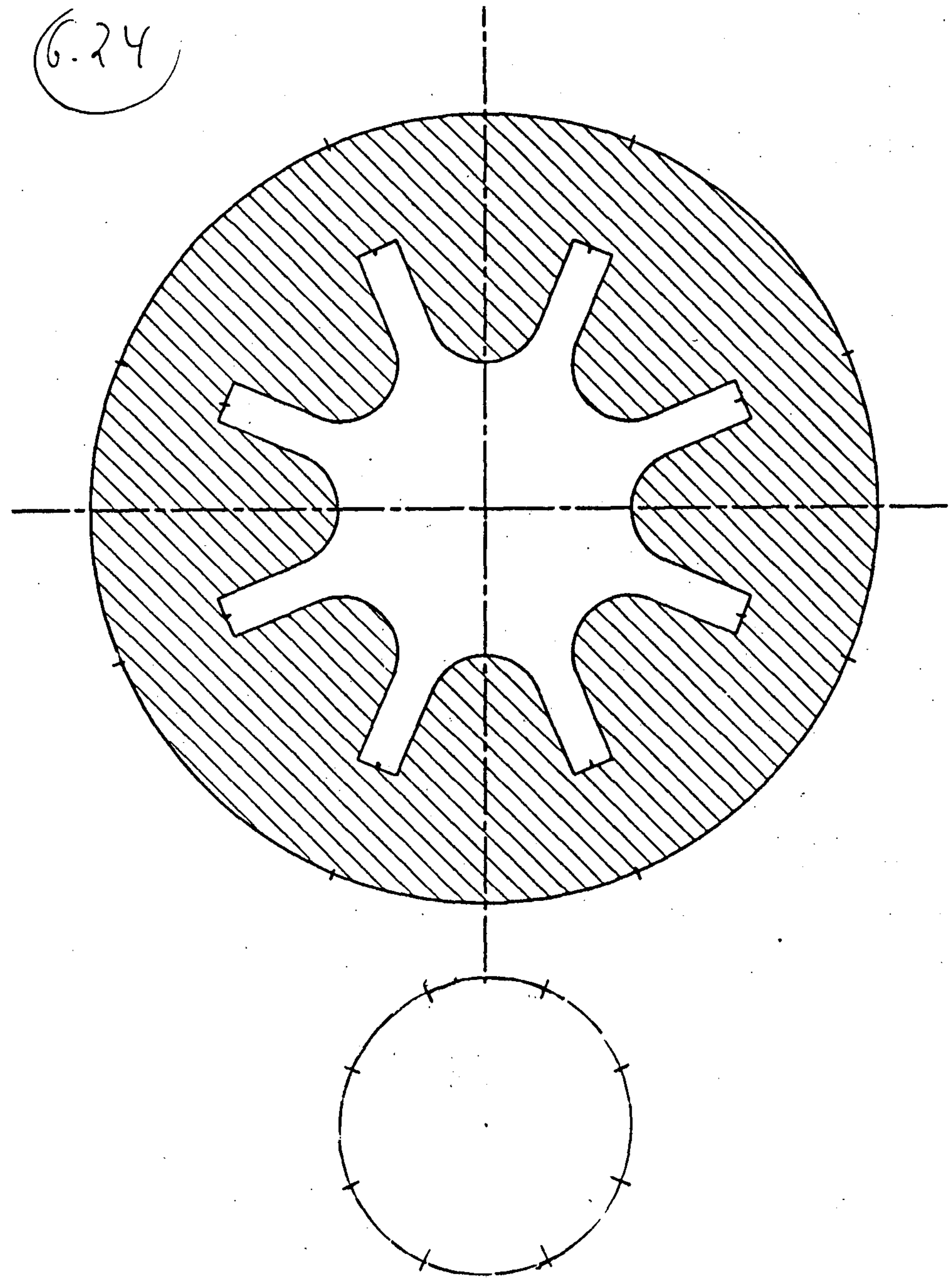


For ±1 excitation: $F = i\zeta^2 = \frac{2}{\pi} \ln\left(\frac{1+iw^2}{1-iw^2}\right)$

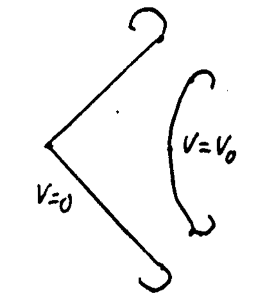
2N-pole $\quad w = \left(tg\left(\pi \zeta^N/4\right)\right)^{1/N}$

$$\zeta = \left(\frac{2}{i\pi} \ln\left(\frac{1+iw^N}{1-iw^N}\right)\right)^{1/N}$$

(6.24)



6.21



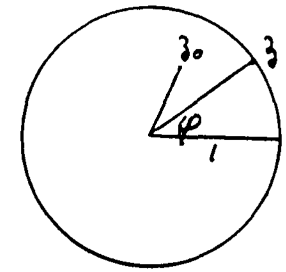
Better design

6.22

Dirichlet-problem in unit circle.

Problem: from V, or A, on circumference → $F(z_0)$.

$|z_0| < 1$



$$F(z_0) = \frac{1}{2\pi i} \oint \frac{A + iV}{z - z_0}\, dz = \frac{1}{2\pi} \int \frac{A + iV}{e^{i\varphi} - z_0} e^{i\varphi} d\varphi$$

$$\int \frac{A + iV}{e^{i\varphi} - 1/z_0^*} e^{i\varphi} d\varphi = \int \frac{(A + iV)\, z_0^*}{z_0^* - e^{-i\varphi}}\, d\varphi \to 0$$

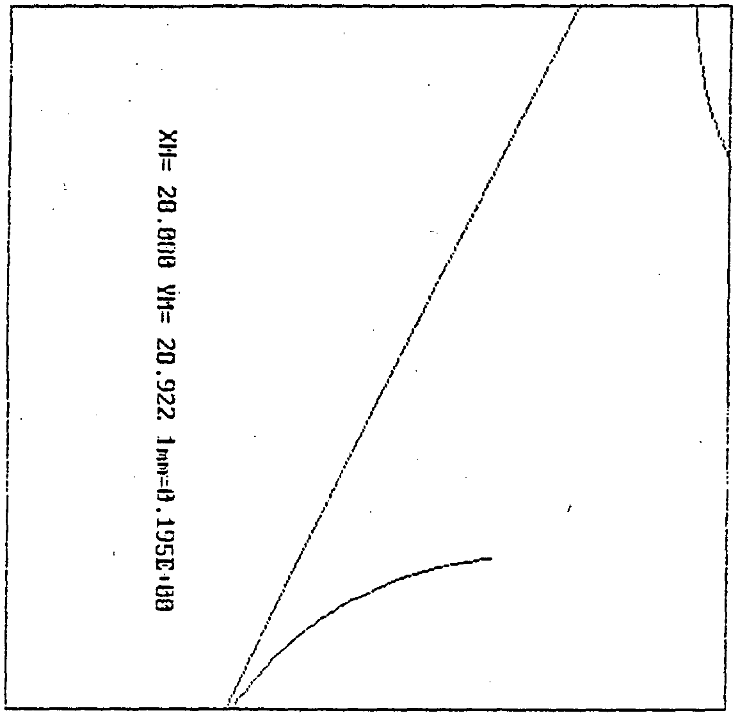
$$\frac{1}{2\pi} \cdot \int \frac{A - iV}{e^{i\varphi} - z_0}\, z_0\, d\varphi = 0$$

$$e^{i\varphi} = e^{i\varphi} - z_0 + z_0$$

$$F(z_0) = \frac{1}{2\pi} \cdot \int F(e^{i\varphi})\, d\varphi + \frac{z_0}{2\pi} \cdot \int \frac{A + iV}{e^{i\varphi} - z_0}\, d\varphi$$

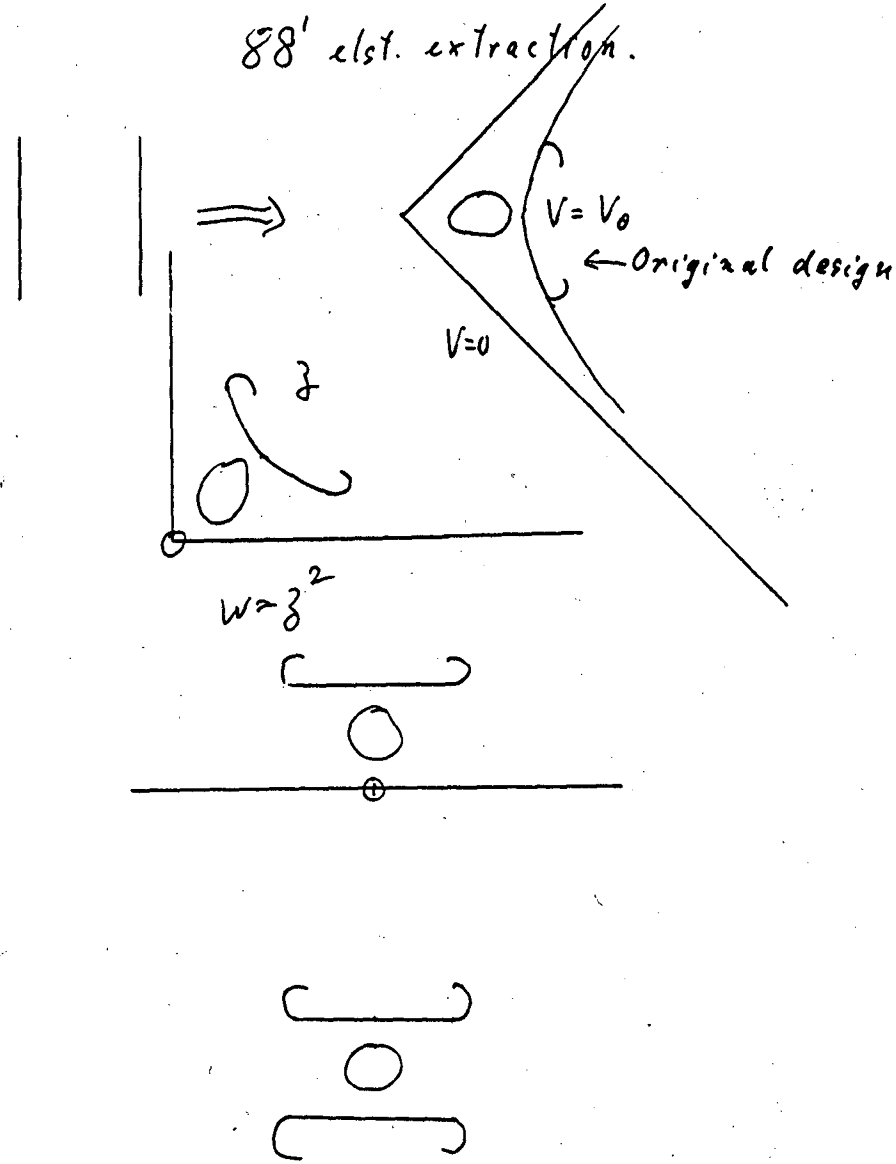
$$\pi F(z_0) - \underbrace{\int F(e^{i\varphi})\, d\varphi / 2}_{\pi F(0)} = z_0 \cdot \int \frac{A(\varphi)}{e^{i\varphi} - z_0}\, d\varphi = i z_0 \cdot \int \frac{V(\varphi)}{e^{i\varphi} - z_0}\, d\varphi$$

(6.19)



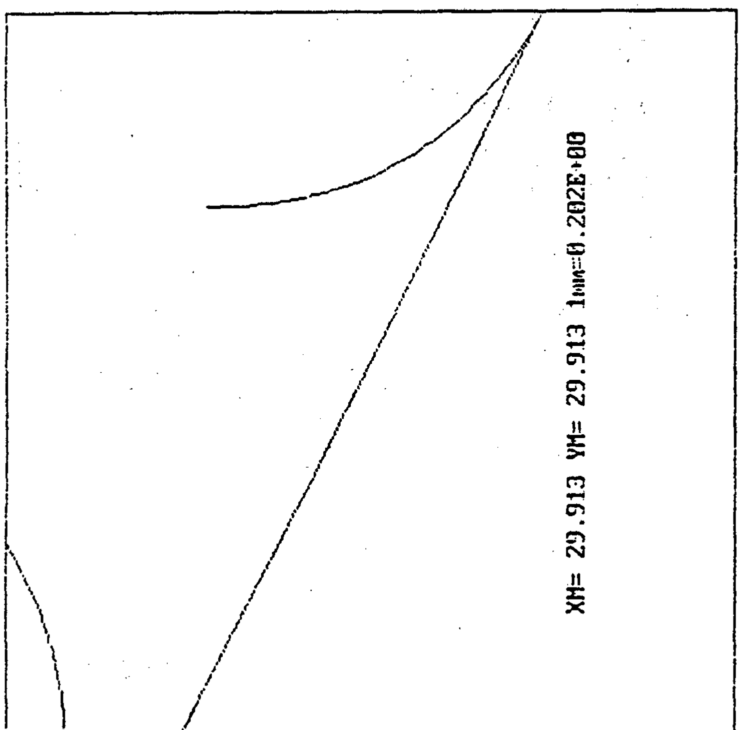
K= 0.0000E+00 UL1=-5.0000E-03 UR1=5.0000E-03 V1=7.0000E+03
UL2=-5.0000E+03 V2=-7.0000E+03 B= 1.000 BM1=8.733E-02 BM2=8.733E+02

(6.20)



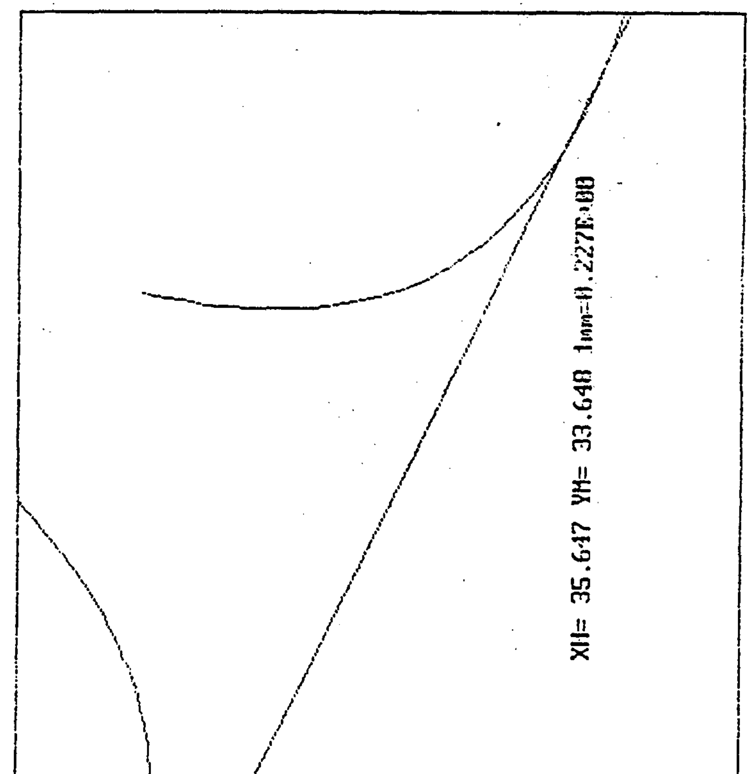
63

K= 0.0000E+00 UL1=-7.0000E+03 UR1=7.0000E+03 V1=7.0000E+03
UL2=-7.0000E+03 V2=-7.0000E+03 B= 1.000 BM1=9.590E+02 BM2=9.590E+02



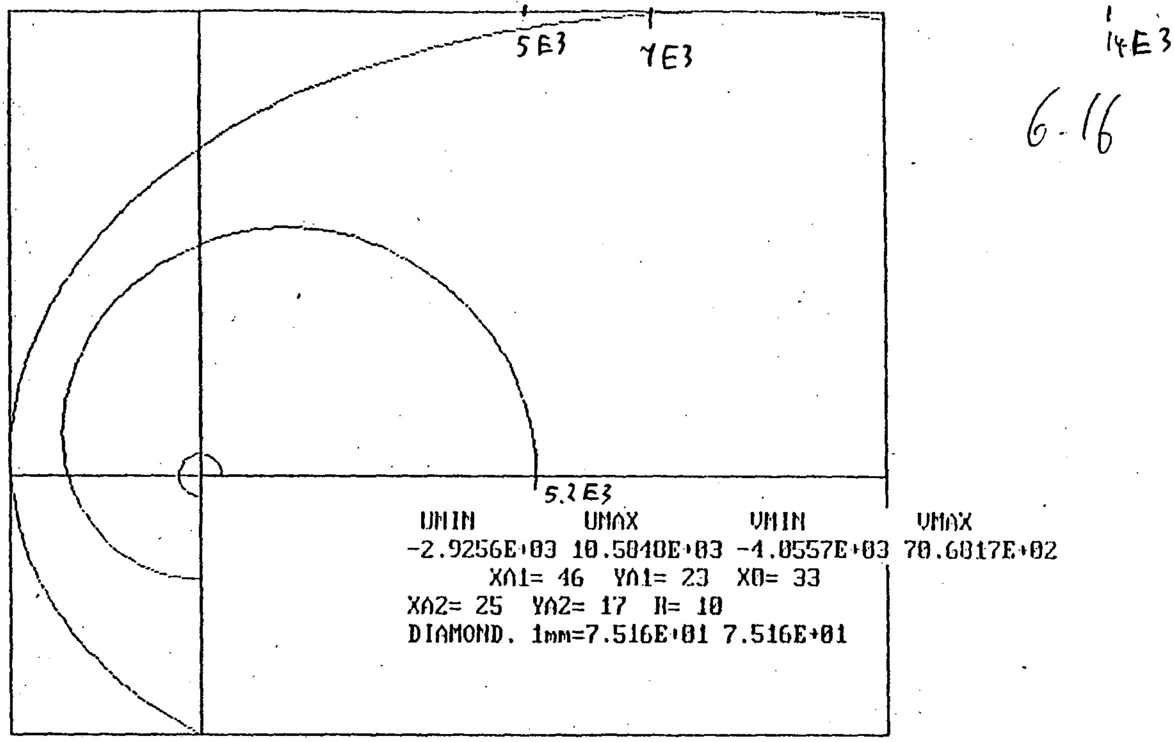
6.18

K= 0.0000E+00 UL1=-1.4000E+04 UR1=1.4000E+04 V1=7.0000E+03
UL2=-1.4000E+04 V2=-7.0000E+03 B= 1.000 BM1=1.302E+03 BM2=1.302E+03

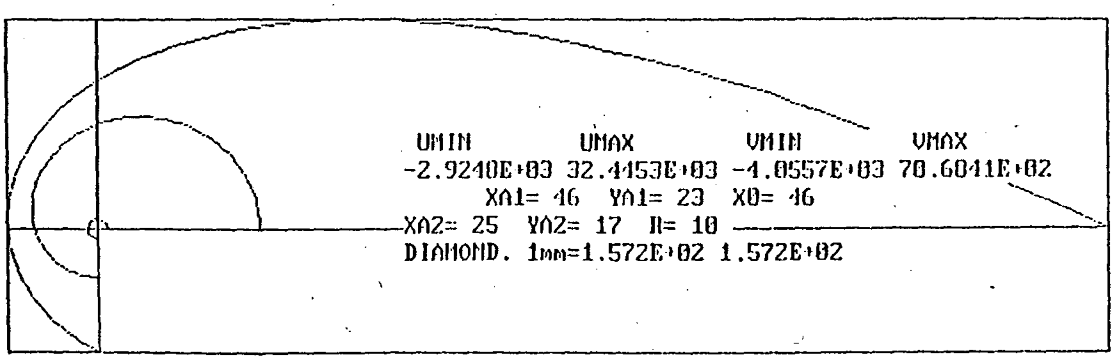


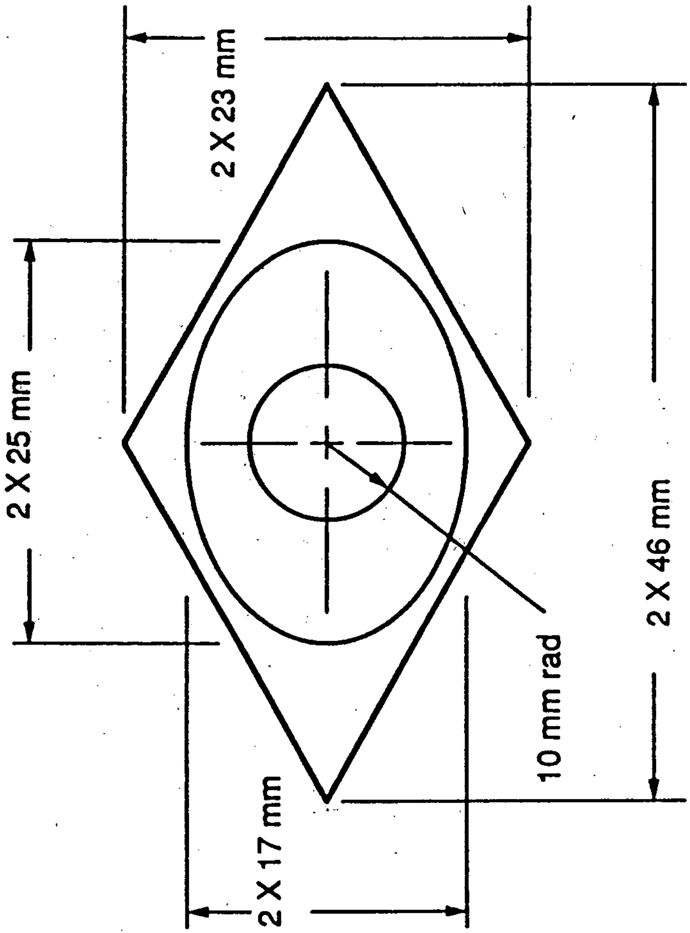
6.17

6.16


5E3
7E3
14E3
5.2E3
UMIN UMAX VMIN VMAX
-2.9256E+03 10.5040E+03 -4.0557E+03 70.6017E+02
XA1= 46 YA1= 23 X0= 33
XA2= 25 YA2= 17 N= 10
DIAMOND. 1mm=7.516E+01 7.516E+01

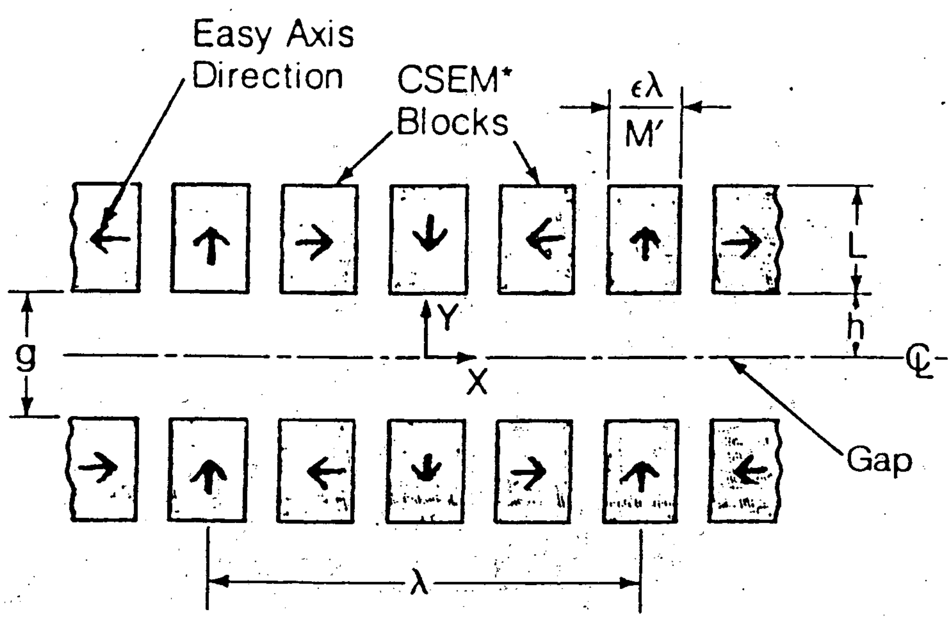
6.15


UMIN UMAX VMIN VMAX
-2.9240E+03 32.4453E+03 -4.0557E+03 70.6041E+02
XA1= 46 YA1= 23 X0= 46
XA2= 25 YA2= 17 N= 10
DIAMOND. 1mm=1.572E+02 1.572E+02



6.14

6.13

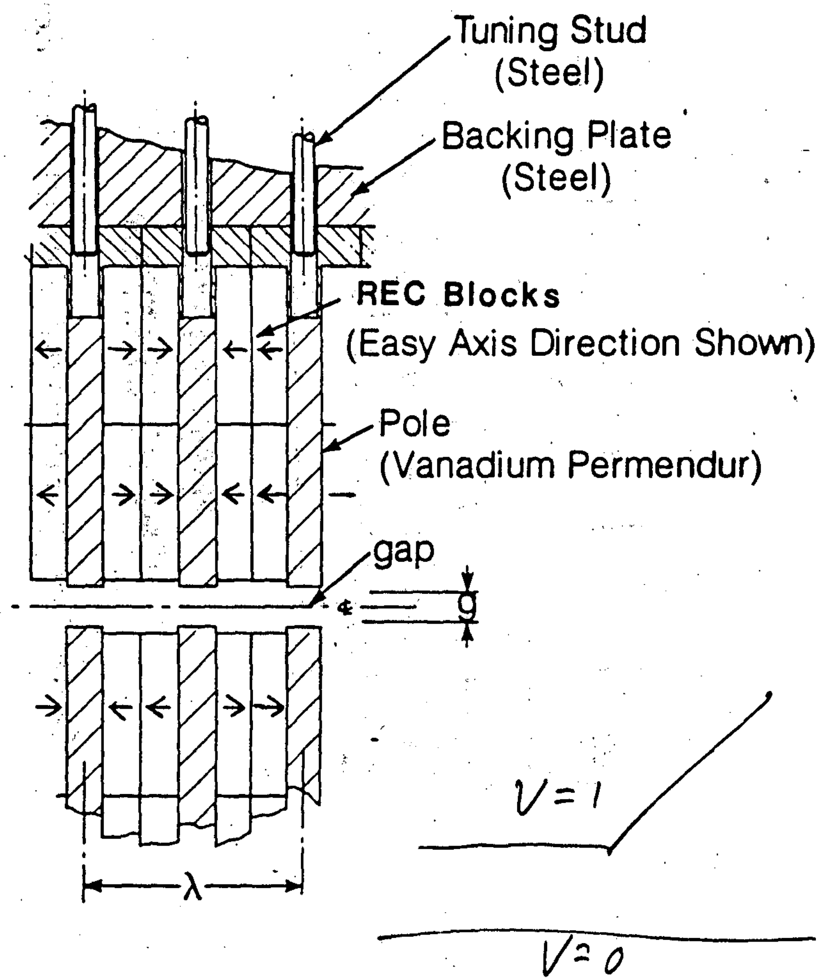


PURE CSEM* W / U CROSS SECTION

*Current Sheet Equivalent Material - e.g. REC

(6.11)

Hybrid Insertion Device configuration with field tuning capability.

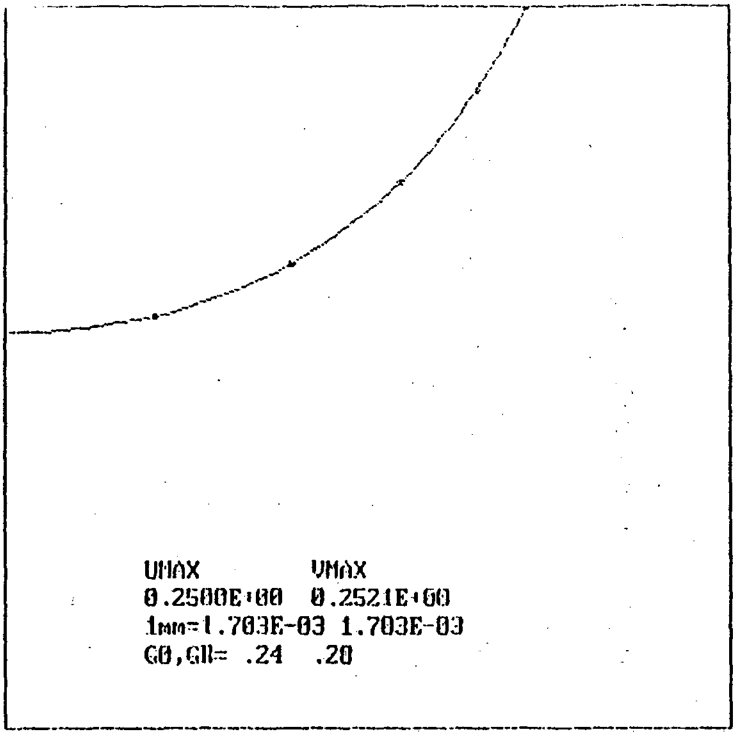


(6.12)

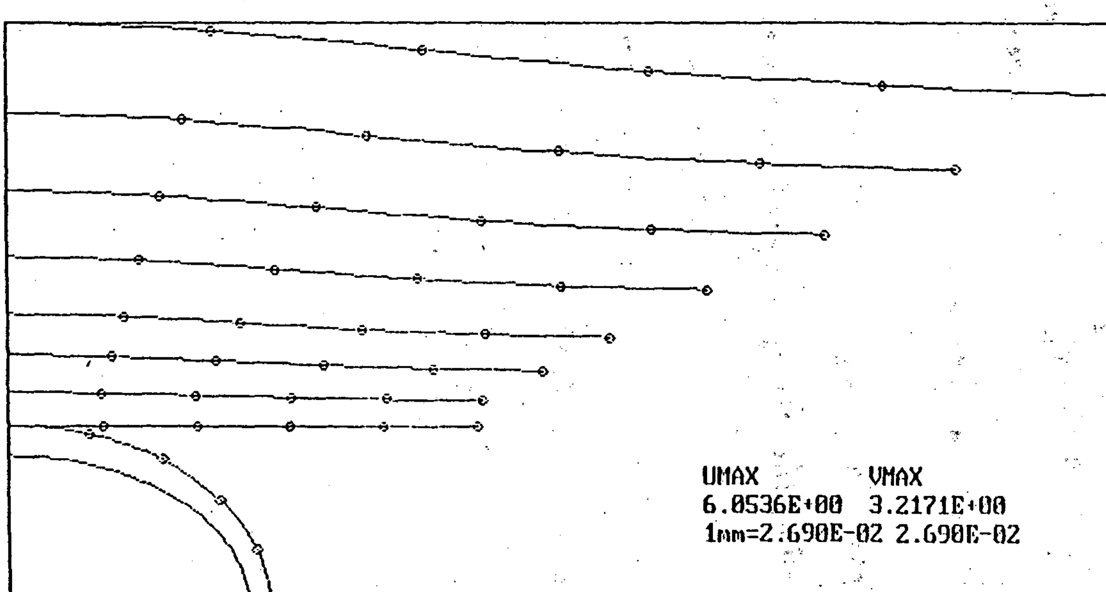
Looking at, and designing, good field quality ID in dipole geometry has many advantages:

- Better field distribution
- Better understanding, particularly of effects caused by changing gap, and of improvement of field by going from flat pole in z to flat pole in w
- Design actually becomes <u>easier</u>, because D_4, flux into pole face, and excess flux associated with corner, are all very simple in w-geometry
- It is also clear that field with shaped iron pole will be much better than with iron-free ID, quite aside from tolerance problems.

5.19


UMAX VMAX
0.2500E+00 0.2521E+00
1mm=1.703E-03 1.703E-03
G0,GR= .24 .20

6.9


UMAX VMAX
6.0536E+00 3.2171E+00
1mm=2.690E-02 2.690E-02
G/L in Z-plane for flat pole/top of good field/shifted poles in W-plane
0.290 0.240 0.320 0.360 0.400 0.450 0.500 0.550 0.600

6.7

Relation between relative field errors:

$\Delta B_z^{*} / B_z^{*} = \Delta B_w^{*} / B_w^{*}$: <u>exact</u> equation, <u>not</u> approximation!!

Complete Design procedure.

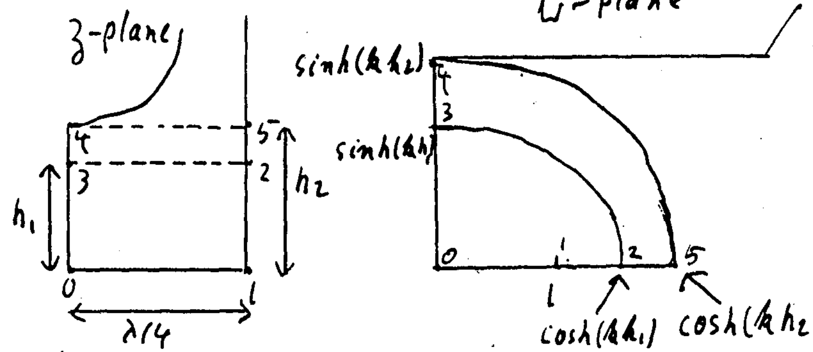
Reason for designing "rest of pole, coils, e.t.c" in z: Most of the time, magnet becomes "too large" in w. Example: sextupole, with outside dimensions ÷ good field aperture $= 10$ in z. In w, that ratio becomes ($w = z^3 \cdot$ const.) 10^3; ratio of corresponding areas goes from 10^2 to 10^6.

$(B_z^{*})_{des} = (B_w^{*})_{des} \cdot w'$

$(B_z^{*})_{real} = (B_w^{*})_{real} \cdot w'$

6.8

Design of "perfect" ID



$B_z^{*} = i B_0 \cos kz = i F'$; $F = B_0/k \sin kz = \frac{B_0}{k} \cdot w$; $k = \frac{2\pi}{\lambda}$

$W = \sin kz$

$W_1 = 1 + i0$; $W_2 = \sin(\pi/2 + i k h_1) = \cosh k h_1 + i \cdot 0$

$W_3 = i \sinh k h_1$

Between 3 and 2 : $W = u + iv = \sin(kx + ikh_1)$

$u = \sin kx \cdot \cosh kh_1$; $v = \cos kx \cdot \sinh kh_1$

$\left(\frac{u}{\cosh kh_1}\right)^2 + \left(\frac{v}{\sinh kh_1}\right)^2 = 1$: ellipse.

$W \to z$: $e^{ikz} - e^{-ikz} = 2iW$; $e^{2ikz} - 2ie^{ikz}W - 1 = 0$

$e^{ikz} = iW + \sqrt{1 - W^2}$

$kz = \ln(iW + \sqrt{1 - W^2}) / i$

(6.5)

Need only 4 maps / procedures

1) Non-dipole ⟷ dipole (P)

2) S-C (P)

3) ○ ⟷ ½ plane (M)

4) ½ plane with bump ⟷ ½ plane without bump.

Also: Dirichlet problem in ½ plane; circular disk.

<u>Non-Dipole ⟷ Dipole</u>; Design of non-Dipole

Now: "continous" transition from review to new material.

Need: desired field specified (uniquely!).

Usually: $B_x - iB_y$ in midplane, but occasionally other specs are used, e.g. $B_x(x,y)$; or $B_x(x,y) \cdot B_y(x,y)$; (see Nehari)

$W(z)$ maps field producing / modifying entities:

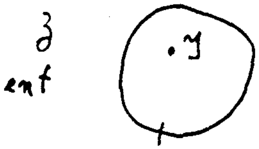
V = const. surfaces (e.g. $\mu = \infty$ surfaces)

A = const. surfaces (e.g. Cu-surfaces for AF)

J, q distributions

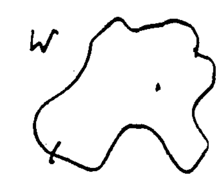
(6.6)

$F(z) = F(z(w))$: complex pot., describing arrangement of all field producing / modifying entities in space.



Going once around J-filament in $z \to F$ changes by iJ

Same in W.



Relationship between B_z^* ; B_w^*

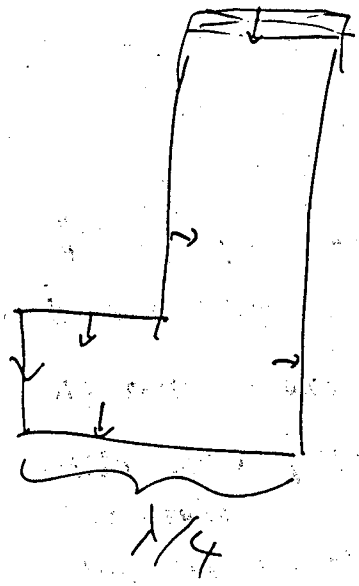
$B_z^* = i\,dF/dz$; $B_w^* = i\,dF/dW = i\,dF/dz \cdot dz/dW$

$$\boxed{B_W^* = B_z^* / W'}$$

↑ True no matter what map is.

Transformation that maps perfect desired non-dipole into perfect dipole: $W'(z) = \text{const.}\left(B_z^*(z)\right)_{ideal}$

$B_W^* = B_z^* / W'$ applies whether or not magnets are actually perfect.



6.3.1

$$a = 1 + D_2/h_2 \,;\quad b = \sqrt{a^2 - 1}\,;\quad K = \left(b/(a+1)\right)^{1/a}$$

$$T = \exp(-2\pi z/\lambda)$$

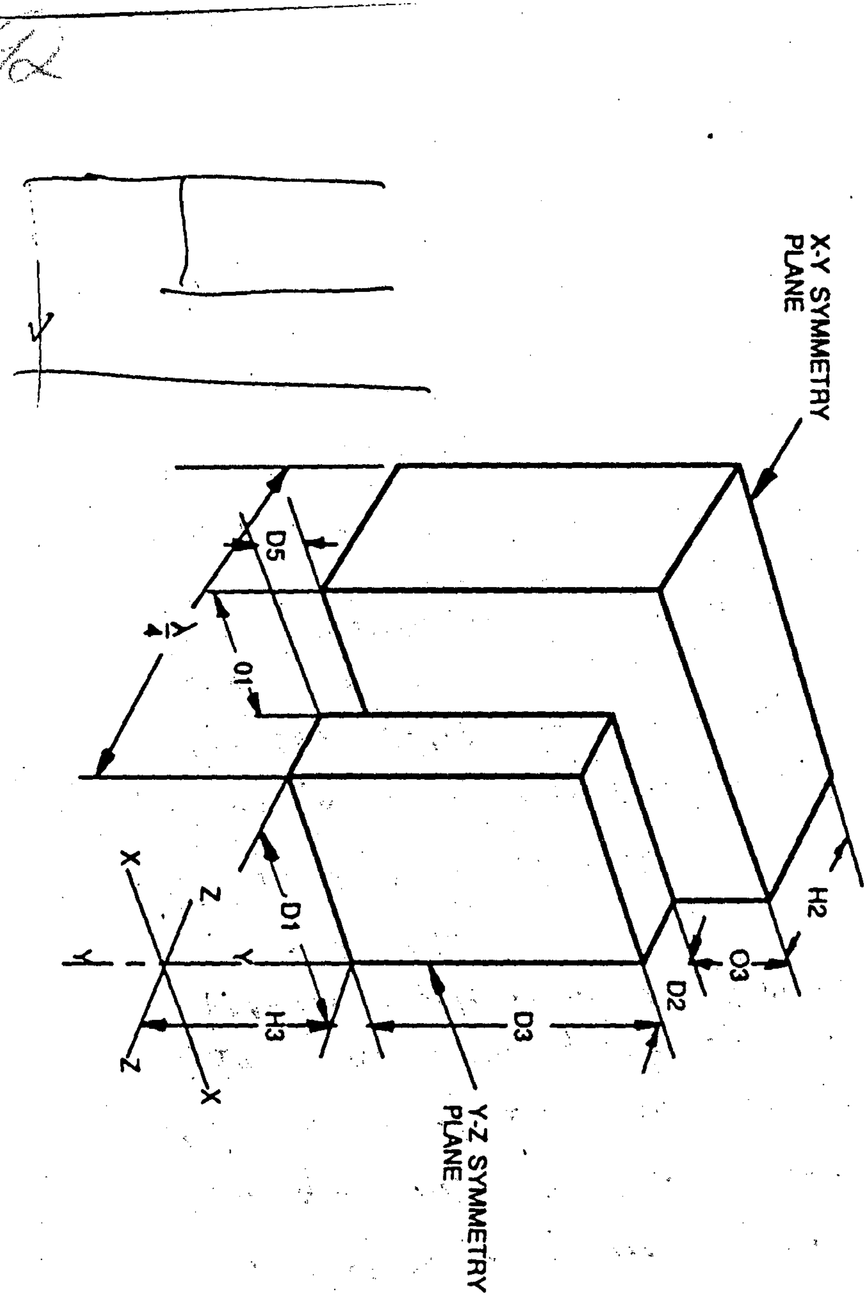
$$F(z) = -4aKT/(\pi b)\cdot\left(1 + T^2K^2(1 - 2/b^2)/3 + \cdots\right)V_0$$

$$\int_{x_1}^{\infty} V\,dx/V_0 = \mathrm{Im}\int_{x_1}^{\infty} F(X + i h_2)\,dX/V_0$$

$$E_0 = E_T - \int_{x_1}^{\infty} V\,dx/V_0 \,/\, h_2$$

$$E_T = \left((a+1)\ln(a+1) - (a-1)\ln(a-1)\right)/\pi$$

(6.2)



HYBRID CONFIGURATION GEOMETRY

(6.3)

3D ID Design

$$\phi_s = \tilde{V}_p \left(D_3\left(\frac{\mu_{11} D_1}{\lambda_2} + E_T\right) + D_1\left(E_p + E_s + E_T\right) + D_2 E_c \right)$$

$\tilde{V}_p = B_0 \cdot D_4$; from POISSON, or analytically

$\tilde{V}_p \cdot E_p$ = 2D flux into pole face; POISSON or analyt.

$\tilde{V}_p \cdot E_s$ = 2D excess flux into side of pole; POISSON or a

$\tilde{V}_p \, E_T$ = 2D excess flux into top/side of pole; analyt.

$\tilde{V}_p \cdot E_c$ = 2D excess flux into corner; analytical

$$\phi_{Br} = B_r \left((D_3 - D_5)(D_1 + \lambda_2 \cdot E_{03}) + D_1 \lambda_2 E_{01} \right)$$

$B_r \lambda_2 E_{03, 01}$ = 2D flux from overhang; analytical.

Solve $\phi_s = \phi_{Br}$ for D_3

$$D_3 = \frac{\frac{B_0 D_4}{B_r}\left(D_1(E_p + E_s + E_T) + D_2 E_c \right) + D_5 (D_1 + \lambda_2 E_{03}) - D_1 \lambda_2 E_{01}}{D_1 + \lambda_2 E_{03} - \underbrace{\frac{B_0 D_4}{B_r}\left(\frac{\mu_{11} D_1}{\lambda_2} + E_T \right)}_{\text{Performance limitation!!}}}$$

If CSEM is also attached to top, side, effect can be included in E_{01}, E_{03}

Denominator in equ. for D_3 looks dangerous. It isn't for B_e!

(6.0)

Complete Design Procedure

1) Establish mapping function from desired field
2) Map good field region from z into w
3) Map outside of vacuum chamber from z to w
4) In w, draw pole of sufficient width to produce dipole field of sufficient quality in w ($\leftrightarrow z$).
5) Map that pole from w into z.
6) Design rest of pole, coils, e.t.c. in z.

For some details, one may need to go back and forth between z and w. Make sure nothing "dangerous" comes too close to good field region in w

Narrow pole more important for non-dipoles than dipoles, because of saturation.

POISSON can do "everything" in w plane, even for non-linear iron.

(6.1)

Summary of lecture #5

Finished reason for overhanging CSEM.

3D ID design

Relationship between $\tilde{V}_p$, B_0 : $\tilde{V}_p = B_0 D_4$

ϕ_S ; ϕ_{Br}

Achievable performance decreased by excess flux along edge of length D_3, increased by flux induced by B_r along edge of length D_3.

Attaching CSEM on surface of side would also increase performance limit.

Conformal mapping.

For: thinking, design, computations

$w(z)$; $\Delta w = w' \Delta z$ conformality

$k_w = (k_z + \mathrm{Im}(e^{i\alpha_z} \cdot w''/w'))/|w'|$

$k_w = |z'| \cdot k_z - \mathrm{Im}(e^{i\alpha_w} \cdot z''/z')$

$k > 0$: curve turns left when moving in direction $e^{i\alpha_z}$

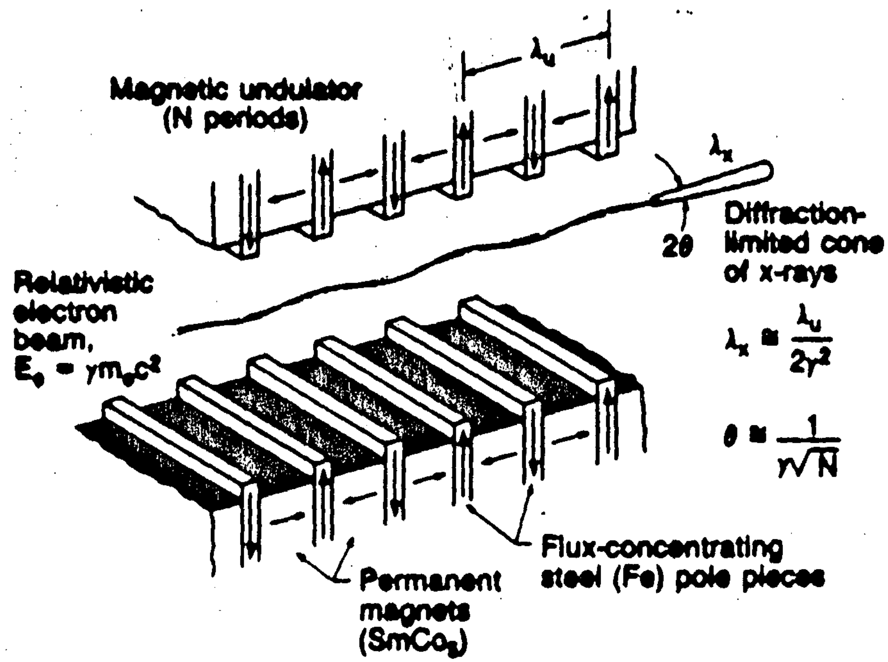
$e^{i\alpha_w} = e^{i\alpha_z} \cdot w'/|w'| = e^{i\alpha_z} \cdot |z'|/z'$

Insertion Device Design

Klaus Halbach

Lecture 6.

December 2, 1988



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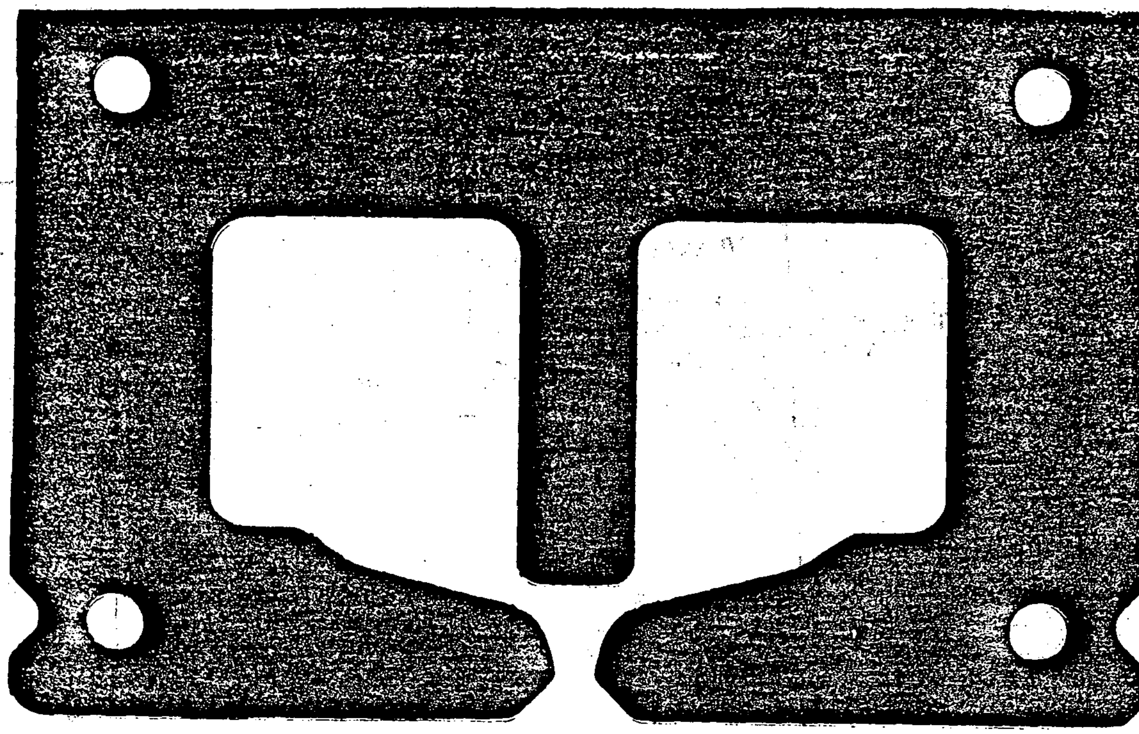
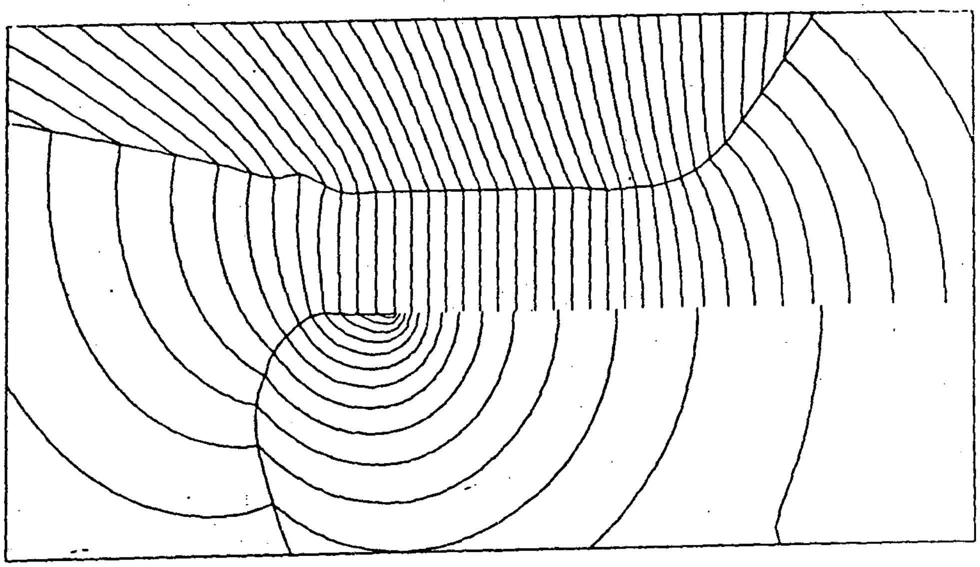
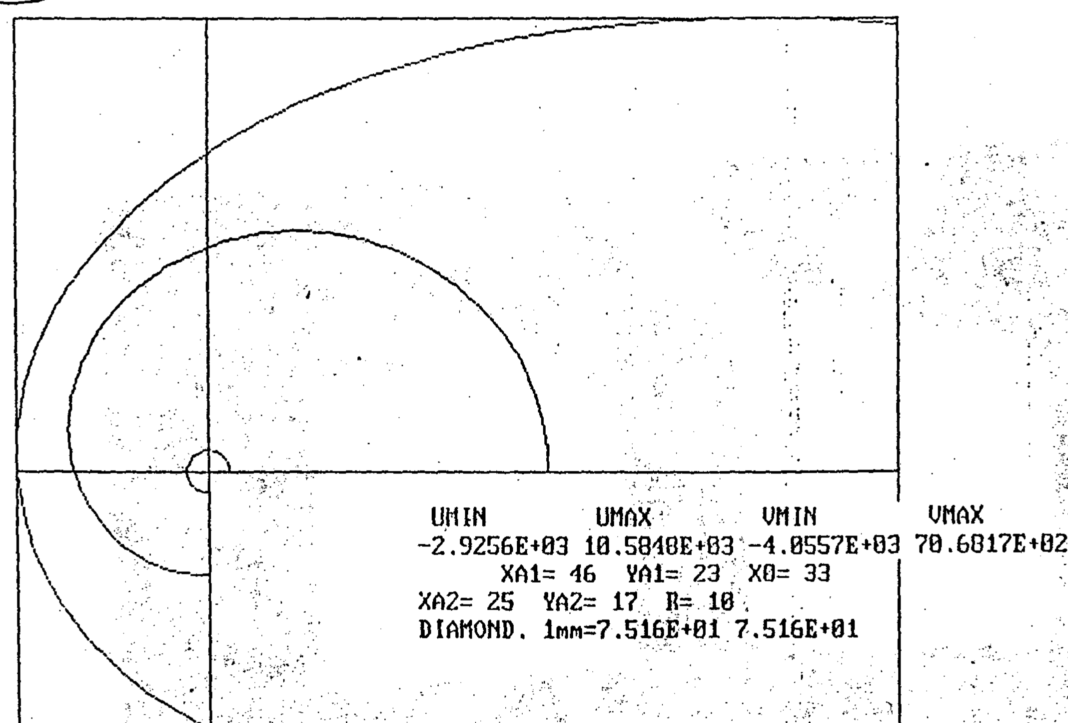


TYPE INPUT DATA- NUM, ITRI, NPHI, INAP, NSUXY.

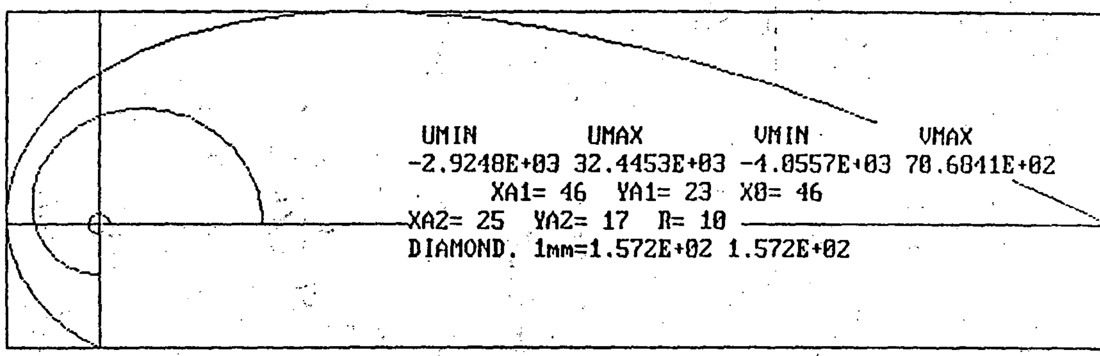


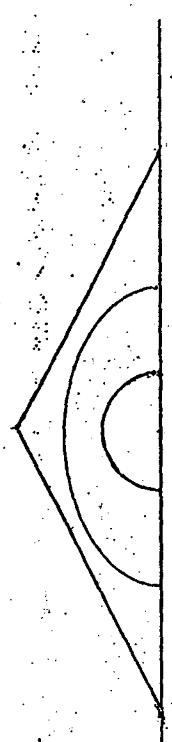
PROB. NAME = SLC L31 ; M=1, OPT. POLE FROM SA CYCLE = 70

5.21



5.20


UMIN UMAX VMIN VMAX
-2.9248E+03 32.4453E+03 -4.0557E+03 70.6841E+02
XA1= 46 YA1= 23 X0= 46
XA2= 25 YA2= 17 R= 10
DIAMOND. 1mm=1.572E+02 1.572E+02



5.19

(5.17)

Complete Design Procedure

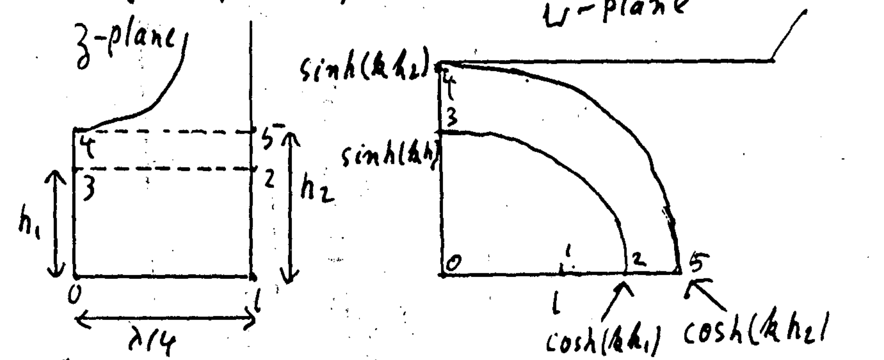
1) Establish mapping function from desired field.
2) Map good field region from z into w
3) Map outside of vacuum chamber from z to w
4) In w, draw pole of sufficient width to produce dipole field of sufficient quality in w ($\leftrightarrow z$).
5) Map that pole from w into z.
6) Design rest of pole, coils, e.t.c. in z. For some details, one may need to go back and forth between z and w

Narrow pole more important for non-dipoles than dipoles, because of saturation.

POISSON can do "everything" in w plane, even for non-linear iron.

(5.18)

Design of "perfect" ID



$B^* = iB_0 \cos kz = iF'$; $F = B_0/k \sin kz = \frac{B_0}{k} \cdot w$; $k = \frac{2\pi}{\lambda}$

$W = \sin kz$

$W_1 = 1 + i0$; $W_2 = \sin(\pi/2 + ikh_1) = \cosh kh_1 + i \cdot 0$

$W_3 = i \sinh kh_1$.

Between $\cdot 3$ and $\cdot 2$: $W = u + iv = \sin(kx + ikh_1)$

$u = \sin kx \cdot \cosh kh_1$; $v = \cos kx \cdot \sinh kh_1$

$\left(\frac{u}{\cosh kh_1}\right)^2 + \left(\frac{v}{\sinh kh_1}\right)^2 = 1$: ellipse.

$W \to z$: $e^{ikz} - e^{-ikz} = 2iW$; $e^{2ikz} - 2ie^{ikz} W - 1 = 0$

$e^{ikz} = iW + \sqrt{1 - W^2}$

$kz = \ln(iW + \sqrt{1 - W^2}) / i$

(5.15)

But also: Want potential, fields to satisfy "standard" equations.

Use $W(z) \leftrightarrow z(W)$ = analytical functions → conformal map. $F(z) = F(z(W))$.

$\nabla_W^2 F = 0$ is obvious.

Other condition: which $W(z)$ maps non-dipole into dipole?

From B_z^*, know $F(z) = \int B_z^*(z)\,dz/i$

In mapped geometry, want complex potential proportional to w:

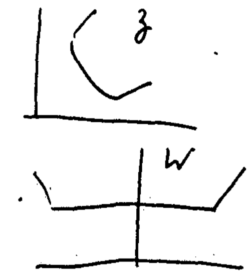
Map: $\underline{F(z(W)) = W \cdot \text{constant}}$

With more detail:

$$W = a \cdot \int B_z^*(z)\,dz / ib$$

↑ arbitrary scaling of lengths (under a); ↑ take out field strength (under b)

(5.16)



Example:

$B_z^* = i B_0 \cdot z / r_1$ = quad

$W = a\, z^2 / 2 r_1$ = map

$B_W^* = i\,dF/dW = i\,dF/dz \cdot dz/dW = B_z^* \cdot z' = B_z^* / W'$

$\Delta B_z^* / B_z^* = \Delta B_W^* / B_W^*$: relative field errors are same in w as in z.

Other example:

Optics man wants:

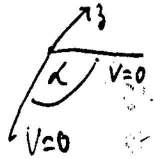
$B_y(x,0) = a_1 + a_2 x + a_3 x^2$; $B_x(x,0) = 0$

↳ $B_z^*(z) = -i(a_1 + a_2 z + a_3 z^2) = i F'$

$F(z) = -(a_1 z + a_2 z^2/2 + a_3 z^3/3) = -g \cdot W(z)$ = Map.

(5.13)

2) Scalar pot. surfaces in vicinity of corner.



$F = a z^n$; a = real ; $n = 1/(2-\alpha/\pi)$

$$k_z = |F'| \cdot k_F - \mathrm{Im}\left(\frac{F''}{F'} e^{i\alpha_z}\right) \longrightarrow -\mathrm{Im}\left(\frac{F''}{F'^2}\right)\cdot |F'|$$

$$e^{i\alpha_z} = e^{i\alpha_F}\frac{|F'|}{F'} \; ; \; F' = n a z^{n-1} \; ; \; F'' = n(n-1) a z^{n-2}$$

$$k_z = -(n-1)\, r^{n-1}\, \mathrm{Im}\, z^{n-2+2-2n} = (n-1)\sin(n\varphi)/r$$

Kober: Dictionary of Conformal Representations (Dover)

↑ Beautiful, but we really need only 4 procedures / maps:

1) Map non-dipole with prescribed field into dipole (Proc.)
2) Schwarz-Christoffel transform. (Proc.)
3) Circular disc ⟷ ½ plane (Map)
4) ½ plane with "elliptical bump" onto ½ plane with straight boundary (Map)

(5.14)

Often very useful: Solution of "Dirichlet Problem" in circular disk.

Often use CM to understand magnetic fields. Reverse also true: get map from Physics.

Mapping of non-Dipole into Dipole

Field distribution given/controlled by geometry of field producing/modifying entities: V = const. surfaces, A = const. surfaces in case of "superconducting" surfaces (RT Copper qualifies at sufficiently large frequencies, e.g. kicker magnets), current and charge distributions.

"Re-locate" with $u(x,y)$, $v(x,y)$, such that desired non-dipole field becomes dipole field.

(5.11)

$z(t) = z_0 + \dot{z}\cdot t + \ddot{z}\, t^2/2 + \cdots \quad ; \; \dot{} = \frac{\partial}{\partial t}$

$\dot{z} = |\dot{z}|\cdot e^{i\alpha_z} \quad ; \; \alpha_z$ = direction of tangent

$\ddot{z}\cdot e^{-i\alpha_z} = a + ib$

$z(t) = z_0 + e^{i\alpha_z}\left(|\dot{z}|\cdot t + t^2\cdot(a+ib)/2\right)$

$\xi = |\dot{z}|\cdot t + a\,t^2/2 + \cdots \; ; \; \eta = b\,t^2/2 + \cdots$

$d\eta/d\xi = \dot{\eta}/\dot{\xi} = (b\,t + \cdots)/(|\dot{z}| + a\,t + \cdots)$

$(d\eta/d\xi)_{t=0} = 0$

$k_z = \left(d^2\eta/d\xi^2\right)_{t=0} = \left(\frac{d(\dot{\eta}/\dot{\xi})}{dt}\cdot\frac{1}{\dot{\xi}}\right)_{t=0} = b/|\dot{z}|^2$

$b/|\dot{z}| = \mathrm{Im}\left(\ddot{z}/(|\dot{z}|e^{i\alpha_z})\right) = \mathrm{Im}\left(\ddot{z}/\dot{z}\right)$

$k_z = \mathrm{Im}\left(\ddot{z}/\dot{z}\right)/|\dot{z}|$

$w(z) = w(z(t))$

$k_w = \mathrm{Im}\left(\ddot{w}/\dot{w}\right)/|\dot{w}|$

$\dot{w} = w'\dot{z} \; ; \; \ddot{w} = w''\dot{z}^2 + w'\ddot{z}$

(5.12)

$\ddot{w}/(\dot{w}|\dot{w}|) = \frac{w''\dot{z}^2 + w'\ddot{z}}{w'\cdot\dot{z}\cdot|\dot{z}|\cdot|w'|}$

$k_w = \mathrm{Im}\left(\frac{\ddot{z}}{\dot{z}\,|\dot{z}|} + \frac{w''}{w'}\cdot\frac{\dot{z}}{|\dot{z}|}\right)/|w'|$

$k_w = \left(k_z + \mathrm{Im}\left(\frac{w''}{w'}\cdot e^{i\alpha_z}\right)\right)/|w'|$

$k > 0$: curve turns left when looking in direction of tangent.

$k_w = |z'|\cdot k_z - \mathrm{Im}\left(\frac{z''}{z'}\,e^{i\alpha_w}\right)$

$k_w > 0$ → tangent

$e^{i(\alpha_w - \alpha_z)} = w'/|w'| = |z'|/z'$

Most of the time, $k_z = 0$

2 Applications:

1.) curve in polarcoordinates: $r = r(\varphi)$

$w(\varphi) = r(\varphi)\cdot e^{i\varphi}$. Consider φ = real part of z

$w' = e^{i\varphi}(r' + ir)$;

$\ln w' = i\varphi + \ln(r' + ir)$

$\frac{w''}{w'} = i + \frac{r'' + ir'}{r' + ir} = \frac{(r'' - r + 2ir')(r' - ir)}{r'^2 + r^2}$

$k = \frac{r^2 + 2r'^2 - rr''}{\sqrt{r'^2 + r^2}^{\,3}}$

(5.9)

$a = 1 + D_2/h_2$; $b = \sqrt{a^2-1}$; $K = (b/(a+1))^{1/a}$

$T = \exp(-2\pi z/\lambda)$

$$F(z) = -4aKT/(\pi b)\cdot\left((1+T^2K^2(1-2/b^2)/3+\cdots\right)V_0$$

$$\int_{x_1}^{\infty} V\,dx/V_0 = \operatorname{Im}\int_{x_1}^{\infty} F(x+ih_2)\,dx/V_0$$

$$E_0 = E_T - \int_{x_1}^{\infty} V\,dx/V_0 \cdot /h_2$$

(5.10)

Conformal Mapping

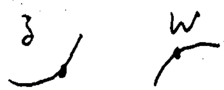
CM extremely useful for:

1) Understanding, and thinking about, magnetic fields, magnets
2) Designing magnets
3) Use as a computational tool (e.g. derivation of excess flux formulae)

$w = W(z)$ = conformal map (except where $|W'| = \infty$ or 0)

Conformality: $\Delta W = W'\cdot\Delta z = |W'|\cdot e^{i\alpha}\cdot\Delta z$

How does curvature of a curve in z-plane transform?



Parameter representation of curve in z-plane: $x = x(t)$; $y = y(t) \to z = z(t)$. t = real, dimensionless quantity. Look in vicinity of $t=0$, and calculate curvature k_z

(5.7)

3D ID Design

$$\phi_S = \tilde{V}_P \left(D_3\left(\frac{\mu_{11} D_1}{h_2} + E_T\right) + D_1\left(E_P + E_S + E_T\right) + D_2 E_C \right)$$

$\tilde{V}_P = B_0 \cdot D_4$; from POISSON, or analytically

$\tilde{V}_P \cdot E_P$ = 2D flux into pole face; POISSON or analyt.

$\tilde{V}_P \cdot E_S$ = 2D excess flux into side of pole; POISSON or a

$\tilde{V}_P \cdot E_T$ = 2D excess flux into top/side of pole; analyt.

$\tilde{V}_P \cdot E_C$ = 2D excess flux into corner; analytical

$$\phi_{Br} = B_r \left((D_3 - D_5)\left(D_1 + h_2 \cdot E_{03}\right) + D_1 h_2 E_{01} \right)$$

$B_r \cdot h_2 E_{03,01}$ = 2D flux from overhang; analytical.

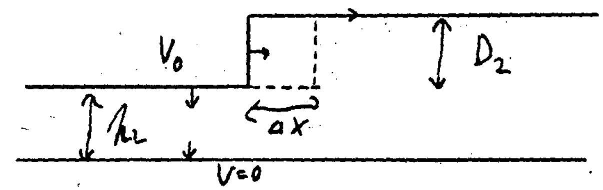
Solve $\phi_s = \phi_{Br}$ for D_3

$$D_3 = \frac{\frac{B_0 D_4}{B_r}\left(D_1(E_P + E_S + E_T) + D_2 E_C\right) + D_5(D_1 + h_2 E_{03}) - D_1 h_2 E_0}{D_1 + h_2 E_{03} - \underbrace{\frac{B_0 D_4}{B_r}\left(\frac{\mu_{11} D_1}{h_2} + E_T\right)}_{\text{Performance limitation!!}}}$$

If CSEM is also attached to top, side, effect can be included in E_{01}, E_{03}

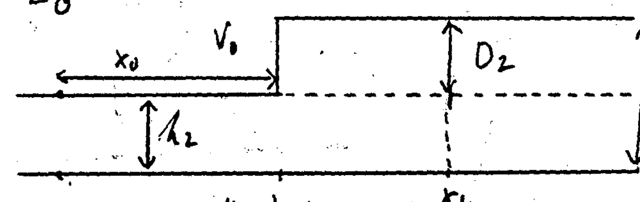
(5.8)

$$E_T = \left((a+1)\ln(a+1) - (a-1)\ln(a-1) \right)/\pi \; ; \; a = \frac{h_2 + D_2}{h_2}$$



$$a = 2 : E_T = 1.05 = \frac{\tilde{V}_0}{h_2} \Delta x / \tilde{V}_0 = \Delta x / h_2$$

E_0



$$\int_0^{x_1} V\,dx = \int_0^{\infty} V\,dx - \int_{x_1}^{\infty} V\,dx$$

↖ with far field expansion in exponential

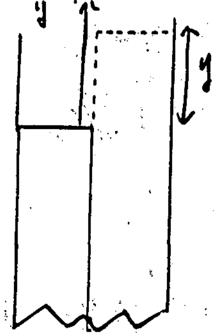
$$\operatorname{Im} \oint F\,dz = x_0 V_0 + \int_0^{\infty} V\,dx - h_2 A(\infty) + h_2 A(-x_0) =$$

$$\int_0^{\infty} V\,dx = h_2 \underbrace{\left(A(\infty) - A(-x_0)\right)}_{\frac{V_0}{h_2} \cdot x_0 + V_0 E_T} - x_0 V_0 = V_0 h_2 E_T$$

$$\int_0^{\infty} V\,dx / V_0 = h_2 \cdot E_T$$

(5.5) This design equation is characteristic for most hybrid devices!!!

Why overhang on top?



More flux on iron from CSEM.

New design equ:

$$B_r D_3 + B_r \int_0^{y_1} V(y)\,dy / V_0 = \tilde{V}_0 \left(E_{tot} + \mu_{11} D_3 / h_2 \right)$$

$$D_3 = \frac{\tilde{V}_0 E_{tot} / B_r - \int_0^{y_1} V(y)\,dy / V_0}{1 - \frac{\tilde{V}_0 \mu_{11}}{B_r} \cdot \frac{1}{h_2}}$$

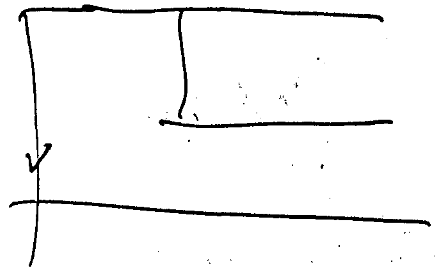
$$L_{CSEM} = D_3 + y_1 \; ; \; L'_{CSEM} = 1 - \frac{V(y_1)/V_0}{\frac{V_0/h_2}{H_c}} = 0$$

$$V(y_1)/V_0 = 1 - H_{CSEM}/H_c$$

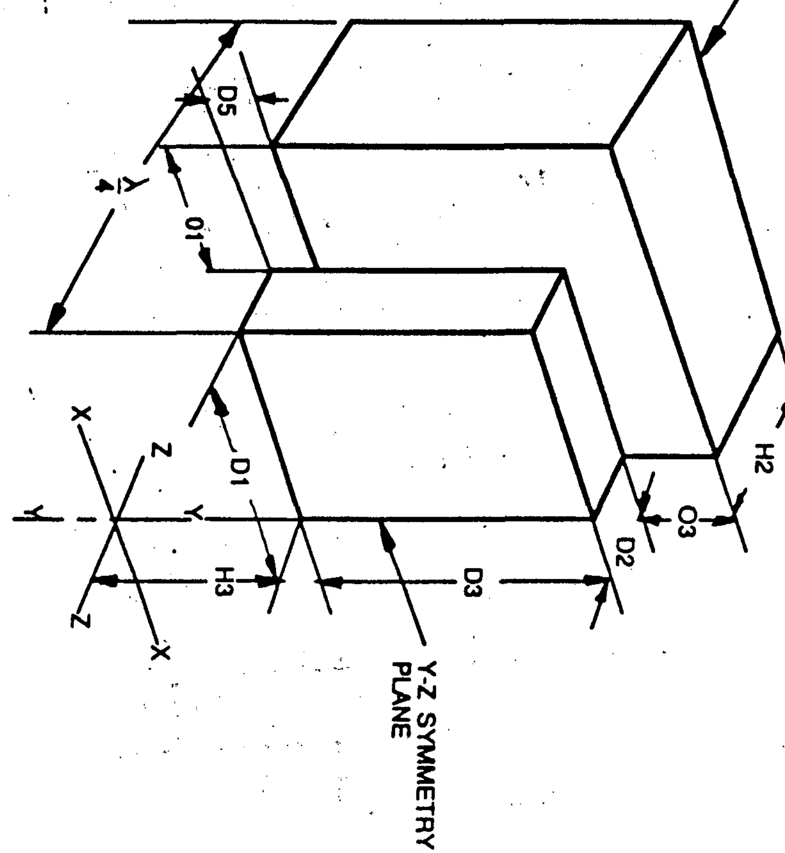
For $H_{CSEM}/H_c \approx .8$, $V(y_1)/V_0 = .2$

Overhanging CSEM on top reduces amount of CSEM; overhang on side increases achievable B_0

(5.6)



HYBRID CONFIGURATION GEOMETRY



(5.3) Error fields must have "disappeared" over distance $\approx D_3$ (not $D_3/2$).

Benefit of CSEM overhang.

$(V_s(y_1)/V_{s0})_{opt} = 1 - H_{CSEM}/H_c$, quite small for strong ID → large overhang.

End of summary of lect. #4

(5.4) Qualitative reason for benefit:

Keep height of CSEM ($= D_{PM}$) fixed. Vary height of iron, and look at $\tilde{V}_0$ ($\sim B_0$)

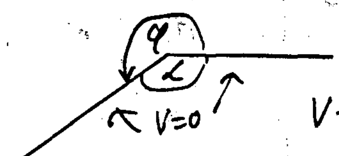
1) $D_3 = D_{PM} + \Delta D$; $\Delta D \gtrsim 0$

$$\tilde{V}_0 = \frac{B_r D_{PM}}{E_{tot} + (D_{PM}+\Delta D)\mu_\parallel/\mu_\perp} \; ; \; \Delta D \uparrow : \tilde{V}_0 \downarrow$$

2) $D_3 = D_{PM} - y_1$; $y_1 \gtrsim 0$

$$\tilde{V}_0 = B_r \cdot \frac{D_{PM} - y_1 + \int_0^{y_1} V(y)\,dy/V_0}{E_{tot} + (D_{PM}-y_1)\mu_\parallel/\mu_\perp}$$

Behaviour of V in vicinity of corner:



Ansatz: $F = a z^n$; a = real

$V = a r^n \sin(n\varphi)$

$n\varphi = n(2\pi - \alpha) = \pi \to n = 1/(2 - \alpha/\pi)$

$\alpha = \pi/2 \to n = 2/3$ ($|B| \sim 1/r^{1/3}$)

$V(y) = V_0(1 - b\, y^{2/3})$; $\int_0^{y_1} V\,dy/V_0 = y_1 - \frac{3}{5} b\, y_1^{5/3}$

$$\tilde{V}_0 = B_r \frac{D_{PM} - .6 \cdot b\, y_1^{5/3}}{E_{tot} + (D_{PM}-y_1)\mu_\parallel/\mu_\perp} \; ; \; y_1 \uparrow : \tilde{V}_0 \uparrow$$

(5.1)

Summary of lecture #4

Iron free system analysis / design

Because of material imperfections, performance usually not as field-error-free as theory indicates; but theory works perfectly for correction of error fields.

Multipole order, inside dimension $\to \infty \Rightarrow$ linear array

2. Linear arrays $\to$ 1D.

Hybrid Theory

Complete solution: linear superposition of 2 solutions.

1) $V_s = 0$; $q \neq 0 \to$ dipole $\to \vec{B}_r \to$ direct fields

ϕ_q into surface $\sim q, \vec{m}, \vec{B}_r$. Dominant part = easy

2) $q, \vec{m}, \vec{B}_r = 0$; $V_s = V_{s0} \to \phi_s$ into surface $\sim V_{s0}$.

ϕ_s more difficult to calculate than ϕ_q.

V_{s0} for system from $\phi_s = \phi_q$. $V_{s0} \to \vec{H}_s$ = indirect fields.

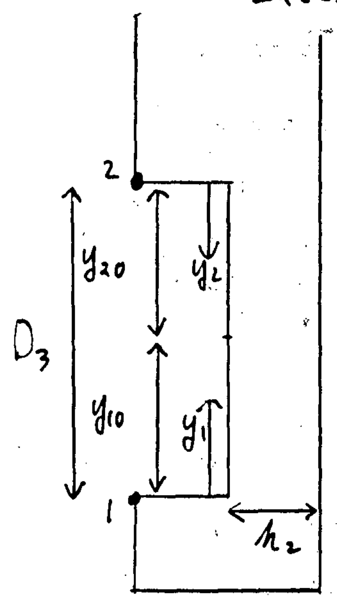
(5.2)

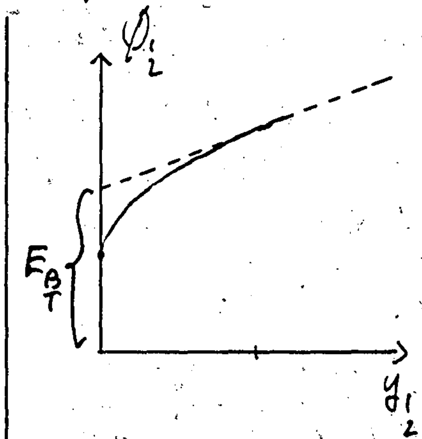
Calculation of ϕ_q:

$\phi_q = q \cdot V_s(\vec{r}_q)/V_{s0}$ or $\phi_q = -\int \vec{B}_r \vec{H}_s \, dv / V_{s0}$

Central for ϕ_s-calculation:

Excess flux concept / coefficient.





E accurate to 1% for $D_3 \geq h_2/2$!!

ϕ between •1 and y_1: $\phi_1 = \tilde{V}(E_B + y_1/h_2)$

ϕ between •1 and y_{10}: $\phi_{10} = \tilde{V}(E_B + y_{10}/h_2)$

ϕ between •2 and y_{20}: $\phi_{20} = \tilde{V}(E_T + y_{20}/h_2)$

ϕ between •1 and •2: $\phi_{12} = \tilde{V}(E_B + E_T + D_3/h_2)$

(10.18)

```
DEFINT J
CLS
PRINT DATE$;"  ";TIME$;"  FEXP1"
J9=5:PI=4*ATN(1)
DIM A1(0:J9),B1(0:J9),C1(0:J9),F1(0:J9)
A$="##.####^^^^ "
REM--Procedures:POWERT(J8,J9,E,A1(),B1());POWER2(J9,E,A1())
REM--INSERT(J9,A1(),B1(),C1());INVERT(J9,A1(),B1());PROD(J9,A1(),B1(),C1())

START:
'FOR A0=1.2 TO 3.05 STEP .2:PRINT USING "#.#";A0
A0=SQR(3)
B0=SQR(A0*A0-1):BD=1/B0:AD=1/A0
A1(0)=A0-1:A1(1)=-1
CALL POWER2(J9,.5*AD,A1()):REM-----------------A1=(A-1-U)^(.5/A)
B1(0)=A0+1:B1(1)=-1
CALL POWER2(J9,-.5*AD,B1()):REM----------------B1=(A+1-U)^(-.5/A)
CALL PROD(J9,A1(),B1(),C1()):REM---------------C1=A1*B1
A1(0)=2*BD:A1(1)=BD*AD
FOR J1=2 TO J9:A1(J1)=A1(J1-1)*AD:NEXT J1:REM--A1=(1+A/(A-U))/B
CALL PROD(J9,A1(),C1(),F1()):REM---------------F1=C1*A1
AD2=.5*AD:A1(1)=AD2:A1(0)=0
FOR J1=2 TO J9:A1(J1)=A1(J1-1)*AD2:NEXT J1:REM--A1=1/(2*A-U)
CALL POWERT(J9,J9,-2,C1(),B1()):REM------------B1=C1^(-2)
CALL PROD(J9,A1(),B1(),C1()):REM-------------------C1=A1*B1
CALL INVERT(J9,C1(),B1())
CALL INSERT(J9,B1(),F1(),A1())
A1(0)=F1(0)
CONV=-2*A0/PI
FOR J1=0 TO J9:PRINT USING A$;A1(J1);:NEXT J1
FOR J1=0 TO J9:A1(J1)=A1(J1)*CONV/(2*J1+1):NEXT J1
FOR J1=0 TO J9:PRINT USING A$;A1(J1);:NEXT J1:PRINT
'NEXT A0
END

SUB POWERT(J8,J9,E,A1(1),B1(1)):REM-------------Raise series to power E.
REM--B1(0:J9)=Coeff. of series defined by A1(0:J8), raised to power E.
B1(0)=A1(0)^E
FOR J1=1 TO J9
  B=0
  E1=(E+1)/J1:J3=J1-J8
  IF J3<0 THEN J3=0
  FOR J2=J3 TO J1-1
    B=B+B1(J2)*A1(J1-J2)*(E-J2*E1)
  NEXT J2
  B1(J1)=B/A1(0)
NEXT J1
END SUB

SUB POWER2(J9,E,A1(1)):REM------------------Raises A(0)+A(1)*X to power E.
K=A1(1)/A1(0):A1(0)=A1(0)^E
FOR J1=1 TO J9
  A1(J1)=A1(J1-1)*K*(E+1-J1)/J1
NEXT J1
END SUB

SUB INSERT(J9,A1(1),B1(1),C1(1)):REM------Insert one series into another.
DIM A2(0:J9,0:J9):REM--Y=S A1(N)*X^N. Z=S B1(M)*Y^M=C1(N)*X^N.
CALL MATR(J9,A1(),A2())
C1(1)=A2(1,1)*B1(1)
FOR J1=2 TO J9
```

(10.19)

```
    S=S+A2(J1,J2)*B1(J2)
  NEXT J2
  C1(J1)=S
NEXT J1
ERASE A2
END SUB

SUB INVERT(J9,A1(1),B1(1)):REM--------------Coeff. of inverted series.
DIM A2(1:J9,1:J9)
CALL MATR(J9,A1(),A2())
B1(1)=1/A1(1)
FOR J1=2 TO J9
  B=0
  FOR J2=1 TO J1-1
    B=B+A2(J1,J2)*B1(J2)
  NEXT J2
  B1(J1)=-B/A2(J1,J1)
NEXT J1
ERASE A2
END SUB

SUB MATR(J9,A1(1),A2(2)):REM-------Matrix for series raised to integer powers.
FOR J1=1 TO J9:A2(J1,1)=A1(J1):NEXT J1
FOR J1=2 TO J9
  FOR J2=J1 TO J9
    A=0
    FOR J3=J1-1 TO J2-1
      A=A+A2(J3,J1-1)*A1(J2-J3)
    NEXT J3
    A2(J2,J1)=A
NEXT J2:NEXT J1
END SUB

SUB PROD(J9,A1(1),B1(1),C1(1)):REM--------------------Product of 2 series.
FOR J1=0 TO J9
  C=0
  FOR J2=0 TO J1
    C=C+A1(J2)*B1(J1-J2)
  NEXT J2
  C1(J1)=C
NEXT J1
END SUB
```

(10.16)

Problem #3: Problem / Algorithm 3): $F(z) = z \to$ solve for b_n ← Inversion of Taylor series.

$n=1$: $A_{11} b_1 = a_1 b_1 = 1$; $b_1 = 1/a_1$ = obvious

$n>1$: $\sum_{m=1}^{n} A_{nm} b_m = \sum_{m=1}^{n-1} A_{nm} b_m + A_{nn} b_n = 0$

$$b_n = -\sum_{m=1}^{n-1} A_{nm} b_m / A_{nn} \qquad A_{nn} = a_1^n$$

Comments to Taylor series inversion.

$W = \sum_1 a_n z^n$; $z = \sum_1 b_n W^n$

1) Algorithm = procedure to get

$b_n = \frac{1}{n!} \frac{d^n}{dW^n} z$ from $a_n = \frac{1}{n!} \frac{d^n}{dz^n} W$

2) More reasons to require $a_0 = 0$:

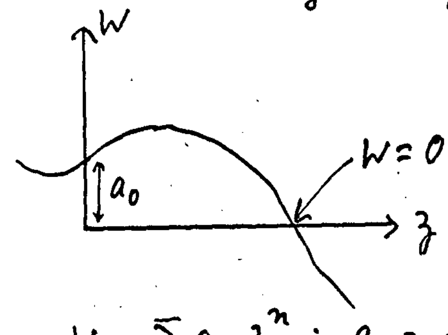
2.1) Assume $a_0 \neq 0$: $W - a_0 = \sum_1 a_n z^n$

$\to z = \sum_1 b_n (W-a_0)^n$, obtained with algorithm given. To get from that Taylor series $z = \sum_1 b_n (W-a_0)^n = \sum_0 c_m W^m$ means that to get c_m, one needs to

(10.17)

know all b_n , $n \geq m$, meaning also that one needs to use all a_n.

2.2) Qualitative reason. Assume $a_0 \neq 0$, and assume z = real, all a_n = real



$W = \sum_0 a_n z^n$: $a_n = \frac{1}{n!} \left(\frac{d^n}{dz^n} W \right)_{z=0}$

$z = \sum_0 c_m W^m$ requires $c_m = \frac{1}{m!} \left(\frac{d^m}{dz^m} z \right)_{W=0}$

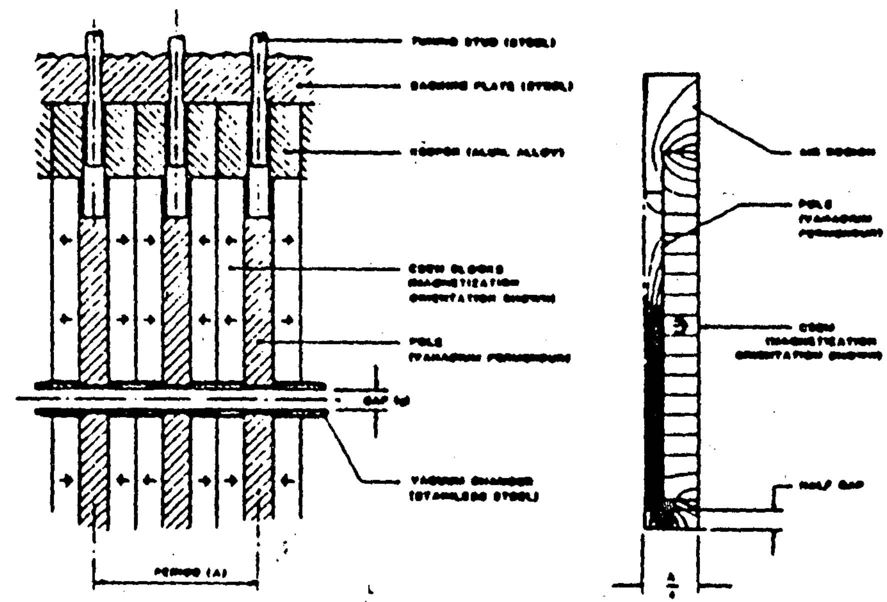
It is fairly safe to assume that it is "impossible" to get c_m from a_n.

Insertion Device Design

Klaus Halbach

Lecture 11.

February 3, 1989

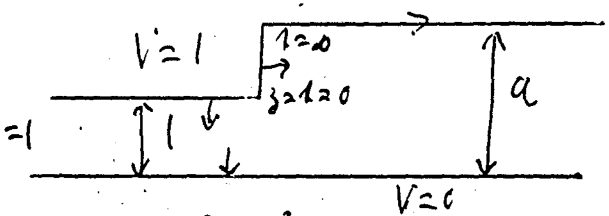


a) ELEVATION CROSS SECTION

b) QUARTER PERIOD MAGNETIC FLUX PLOT

CSEM-STEEL HYBRID INSERTION DEVICE

11.1 Excess Flux in



$t = a^2$

$$\pi \dot{z} = -\frac{\sqrt{t}\,(a^2-1)}{(t-1)(t-a^2)}$$

check: $-i\pi a = -i\pi \cdot \frac{a(a^2-1)}{a^2-1} = \text{o.k.}$

$t = w^2;\ dt = 2w\,dw$

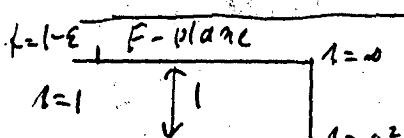
$$\pi\, dz/dw = -\frac{2(a^2-1)t}{(t-1)(t-a^2)} = 2\left(\frac{1}{t-1} - \frac{a^2}{t-a^2}\right)$$

$$\frac{1}{t-a^2} = \frac{1}{w^2-a^2} = \frac{1}{2a}\left(\frac{1}{w-a} - \frac{1}{w+a}\right)$$

$$\pi\, dz/dw = \frac{1}{w-1} - \frac{1}{w+1} - a\left(\frac{1}{w-a} - \frac{1}{w+a}\right)$$

$$\pi z = \ln\frac{1-\sqrt{t}}{1+\sqrt{t}} + a\ln\frac{a+\sqrt{t}}{a-\sqrt{t}}$$

$b^2 = a^2 - 1$



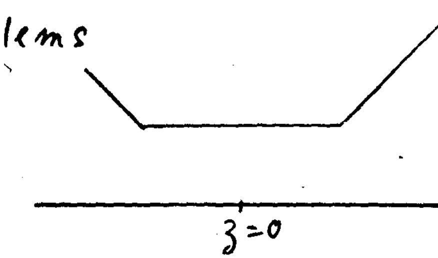
$$\pi \dot{F} = i\frac{b}{(t-1)(t-a^2)^{1/2}}$$

$t = a^2 + q^2;\ dt = 2q\,dq$

$$\pi\, dF/dq = \frac{2ib}{q^2+b^2} = \frac{1}{q-ib} - \frac{1}{q+ib}$$

$$i\pi F = \ln\frac{ib - \sqrt{t-a^2}}{ib + \sqrt{t-a^2}} = \ln\frac{b-\sqrt{a^2-t}}{b+\sqrt{a^2-t}}$$

11.2 Homework Problems



#1

Assume that a symmetric dipole is wide enough so that for analysis of error fields, error fields at each end can be obtained from semi-∞ dipole model. Using these coefficients for exponential decay of error fields, write formula for error fields for the finite width dipole

#2

Develop recursion formula for coefficients of a Taylor series if one known Taylor series is divided by another Taylor series with known coefficients.

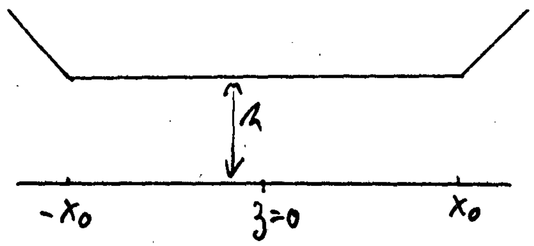
$A(x) = \sum a_n x^n;\ B(x) = \sum b_n x^n$

$C(x) = A(x)/B(x) = \sum c_n x^n;\quad a_n, b_n = \text{known}$

$c_n = \text{wanted}.$

(11.3)

Homework #1



$B^* = i \sum a_n e^{-\bar{n} z/h}$ if origin under left corner

$B^* = i \sum a_n e^{\bar{n} z/h}$ if origin under right corner

$B^* = i \sum a_n \left(e^{\bar{n}(z-x_0)/h} + e^{-\bar{n}(z+x_0)/h} \right)$

$B^* = i \sum a_n \cdot 2 e^{-\bar{n} x_0/h} \cdot \cosh(\bar{n} z/h)$

$B^* \approx i \sum a_n \dfrac{\cosh(\bar{n} z/h)}{\cosh(\bar{n} x_0/h)}$

(11.4)

Homework #2

$A(x) = \sum a_n x^n \ ; \ B(x) = \sum b_n x^n$

$C(x) = A(x)/B(x) = \sum c_n x^n$

$a_n, b_n = \text{known} \ ; \ c_n = ?$

$A(x) = C(x) \cdot B(x) = \sum c_m b_\mu \cdot x^{m+\mu} = \sum a_n x^n$

$m + \mu = n \ ; \ \mu = n - m$

$\sum_{m=0}^{n} c_m b_{n-m} = a_n = c_n \cdot b_0 + \sum_{m=0}^{n-1} c_m b_{n-m}$

$c_n = \left(a_n - \sum_{m=0}^{n-1} c_m b_{n-m} \right) / b_0$

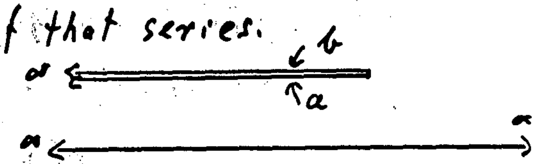
$c_0 = a_0 / b_0$

(11.5)

#3

$F(z) = \int_0^z \sqrt{z} \cdot \exp(z + a z^3)\, dz$. Express $F(z)$ with the help of a Taylor series, and give the recursion formula for the coefficients of that series.

#4



For capacitor with zero-thickness electrode (Rogowski-capacitor; viewgraph 8.10) and halfgap = 1, calculate the excess flux coefficient for the flux entering the lower surface (a) of the electrode

#5

Calculate the excess flux coefficient for the upper surface (b) of the electrode of the Rogowski capacitor.

(11.6) Homework #3

Expansion of $\int_0^z \sqrt{z} \exp(z + \frac{a}{3} z^3)\, dz$ in Taylor s

$$\int_0^z \sqrt{z} \exp(z + \tfrac{a}{3} z^3)\, dz = z^{3/2} G(z) = z^{3/2} \sum_0 b_n z^n$$

$$\exp(z + \tfrac{a}{3} z^3) = \tfrac{3}{2} G + z G'$$

$$(1 + a z^2) \exp(z + \tfrac{a}{3} z^3) = (\tfrac{3}{2} G + z G') \cdot (1 + a z^2)$$

$$= \tfrac{5}{2} G' + z G''$$

$$\sum (n(n-1) + \tfrac{5}{2} n) b_n z^{n-1}$$

$$= \sum (\tfrac{3}{2} + n) b_n z^n + a \sum (\tfrac{3}{2} + n) b_n z^{n+2}$$

$$n(n + 3/2) b_n = b_{n-1}(n + 1/2) + b_{n-3}(n - 3/2) \cdot a$$

$$b_n = \frac{b_{n-1}(n + 1/2) + b_{n-3} \cdot a(n - 3/2)}{n(n + 3/2)} \qquad n \geq 3$$

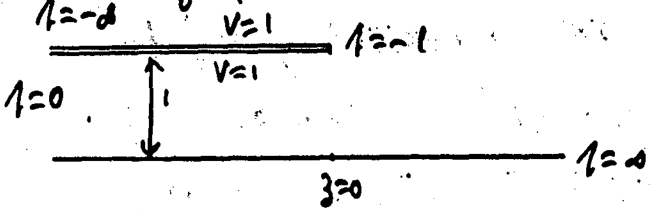
b_0, b_1, b_2:

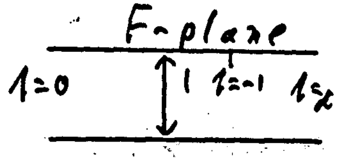
$$\frac{1}{z^{3/2}} \int_0^z (z^{1/2} + z^{3/2} + \tfrac{1}{2} z^{5/2} + \cdots)\, dz = b_0 + b_1 z + b_2 z^2$$

$$b_0 = \tfrac{2}{3}; \quad b_1 = \tfrac{2}{5}; \quad b_2 = 1/7$$

(11.7)

For Homework #4, #5, #6





$\pi \dot z = \frac{t+1}{t}$; $F' = \frac{1}{1+t}$ $\qquad \pi \dot F = \frac{1}{t}$

$\pi z = t + 1 + \ln(+t)$ $\qquad \pi F = \ln t$

Homework #4

"Ideal" flux model for lower pole surface:

Field 1 into surface.

$F(-1) - F(-\varepsilon) = z(-1) - z(-\varepsilon) + \Delta A$

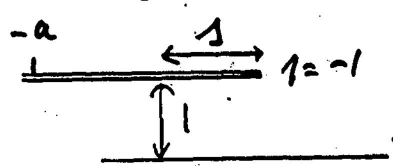
$\pi \Delta A = \int_{-\varepsilon}^{-1} (\pi \dot F(t) - \pi \dot z) dt = \int_{-\varepsilon}^{-1} -1 \cdot dt = 1$; $\Delta A = 1/\pi$ ∎

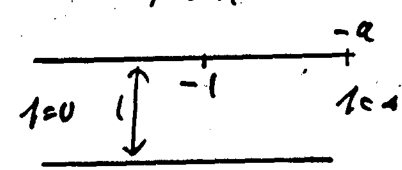
(11.8)

Homework #5

"Ideal" flux model for upper pole surface:

Field = $1/\pi(1 +$ distance from edge) into surface





$F(-a) - F(-1) = \frac{1}{\pi} \cdot \int_0^{s(-a)} \frac{ds}{1+s} + \Delta A$ $\qquad \int_0^{s} \frac{ds}{1+s} = \ln(1+s)$

$\pi \Delta A = \ln a - \ln\left(1 + z(t)\Big|_{-a}^{-1}\right)$

$\pi z(t)\Big|_{-a}^{-1} = (t + \ln(+t))\Big|_{-a}^{-1} = -1 + a + \ln 1 - \ln a$

$z(t)\Big|_{-a}^{-1} = (a - 1 - \ln a)/\pi$

$\pi \Delta A = \left(-\ln\left(\frac{1}{a} + \frac{1}{\pi}\left(1 - \frac{1+\ln a}{a}\right)\right)\right)\Big|_{a\to\infty}$

$\Delta A = \frac{\ln(\pi)}{\pi}$ ∎

10/15

11.9 <u>Hint</u> for #4 and #5: While "ideal" flux in #4 is obvious, for #5 one has to "invent" an appropriate model for the "ideal" flux formula. This formula is not unique, but it has to have the correct asymptotic behaviour. Use $z(\lambda)$, $F(\lambda$

#6

For Rogowski capacitor, expand the error fields between the electrodes in exponentials to 3. order by hand, i.e. give closed expressions.

<u>Hint</u>: Use $z(\lambda)$, $F(\lambda)$, $F'(\lambda)$

11.10

Homework #6

$$F' = \frac{1}{1+\lambda} = 1 - \lambda + \lambda^2 - \lambda^3 + \cdots$$

$$\pi z = 1 + \lambda + \ln\lambda \,;\quad e^{\pi z - 1} = W = e^{\lambda + \ln\lambda}$$

$$W = \lambda \cdot e^{\lambda} = \lambda + \lambda^2 + \lambda^3/2 + \cdots$$

$$\lambda = W + a_2 W^2 + a_3 W^3 + \cdots$$

$$0 = a_2 W^2 + a_3 W^3 + W^2(1 + 2a_2 W) + W^3/2 + \cdots$$

$$a_2 = -1 \,;\quad a_3 = 3/2$$

$$F' = 1 - W - a_2 W^2 - a_3 W^3 + W^2(1 + 2a_2 W) - W^3 + \cdots$$

$$F' = 1 - W + W^2(1 - a_2) - W^3(1 + a_3 - 2a_2)$$

$$F' = 1 - e^{\pi z} \cdot e^{-1} + e^{2\pi z} \cdot 2e^{-2} - e^{3\pi z} \cdot \frac{a}{2} \cdot e^{-3} + \cdots$$

170

(11.11)

Summary of Algorithms for Taylor Series Manipulation.

$A(x)=\sum_0 a_n x^n$; $B(x)=\sum_0 b_n x^n$; $C(x)=\sum_0 c_n x^n$

1) $C(x)=A(x)\cdot B(x)$; $c_n=\sum_{m=0}^{n} a_m b_{n-m}$

2) $C(x)=A(x)/B(x)$; $c_n=\left(a_n-\sum_{m=0}^{n-1} c_m b_{n-m}\right)/b_0$

$y=\sum_1 a_n x^n$; $z=\sum b_m y^m=\sum c_n x^n$

3)
- $m=$ integer : $y^m=\sum A_{nm} x^n$
- $A_{n1}=a_n$
- $A_{nm}= 0 \quad n<m$; $\quad A_{nm}= a_1^m \quad n=m$
- $A_{nm}=\sum_{\mu=m-1}^{n-1} A_{\mu\, m-1}\, a_{n-\mu} \qquad n>m>1$

4) $c_n=\sum_{m=1}^{n} A_{nm} b_m$; $(c_0=b_0)$

(11.12)

Inversion of Taylor series:

$y=\sum_1 a_n x^n$; $x=\sum_1 d_n y^n$

5) $d_n=-\sum_{m=1}^{n-1} A_{nm} b_m / A_{nn}$

Very often, describing a closed expressions by a differential equation that is easyly solved with a Taylor series is the most convenient way to expand the original closed expression into a Taylor series.

11.13

SC-transformation of polygon to ○

$W = \frac{1+i\zeta}{1-i\zeta}$ maps upper ½ plane to unit circle.

$$\zeta = \frac{1}{i}\frac{W-1}{W+1} = \frac{1}{i}\left(1 - \frac{2}{W+1}\right)$$

$$\zeta - \zeta_1 = \frac{2}{i}\left(\frac{1}{W_1+1} - \frac{1}{W+1}\right) = \frac{2}{i}\cdot\frac{W-W_1}{(W+1)(W_1+1)}$$

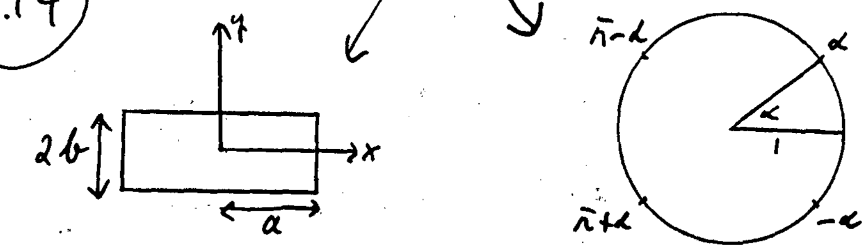
$$\frac{dz}{dW} = \frac{dz}{d\zeta}\cdot\frac{d\zeta}{dW} = \frac{2}{i}\cdot\frac{dz}{d\zeta}\cdot\frac{1}{(W+1)^2}$$

$$\prod_\mu (\zeta-\zeta_\mu)^{-n_\mu} = \text{const}\cdot(W+1)^{\Sigma n_\mu}\prod_\mu (W-W_\mu)^{-n_\mu}$$

$$\Sigma n_\mu = 2$$

Conclusion: same formula as before, but W_μ are now points on unit circle, and no point can be removed from formula.

11.14



$$(W-e^{i\alpha})(W+e^{i\alpha}) = W^2 - e^{2i\alpha}$$

$$R = (W^2-e^{2i\alpha})(W^2-e^{-2i\alpha}) = W^4 - 2W^2\cos 2\alpha + 1$$

$$\frac{dz}{dW} = \frac{A}{\sqrt{R}}\ ;\quad a = A\underbrace{\int_0^1 \frac{du}{\sqrt{A(\alpha)}}}_{y}\ ;\quad b = A\cdot\int_0^1\frac{du}{\sqrt{A(\alpha+\frac{\pi}{2})}}$$

$$u = \tan\varphi\ ;\ du = \frac{d\varphi}{\cos^2\varphi}\ ;\quad y = \int_0^{\pi/4}\frac{d\varphi}{\sqrt{\cos^4\varphi R}}$$

$$\cos^4\varphi R = \sin^4\varphi + \cos^4\varphi - 2\sin^2\varphi\cos^2\varphi\cos 2\alpha$$

$$= \underbrace{\sin^4\varphi + \cos^4\varphi + 2\sin^2\varphi\cos^2\varphi}_{(\sin^2\varphi+\cos^2\varphi)^2 = 1} - \underbrace{4\sin^2\varphi\cos^2\varphi}_{\sin^2 2\varphi}\cos^2\alpha$$

$$2y = \int_0^{\pi/2}\frac{d\varphi}{\sqrt{1-\sin^2\varphi\cos^2\alpha}} = K(\cos^2\alpha) = \text{complete elliptic integral}$$

$$a/b = K(\cos^2\alpha)/K(\sin^2\alpha)\ ▨$$

(11.16) (There is no sheet # 11.15)

General 3D Hybrid Theory with many $\mu=\infty$ Blocks/Poles.

Same basic procedure as before:

1) Direct fields, and flux induced onto poles, from charges/dipole moment distributions/CSEM when all ($\mu=\infty$) blocks on $V=0$

2) Indirect fields from each ($\mu=\infty$) block on V_{n0} (with block 0 always on $V_0=0$), with $\mu_\parallel, \mu_\perp$ from CSEM present, but active part (charges, dipole moments) "off".

Superimpose linearly all fields and get all V_{n0} from condition that total flux (from CSEM and all other $\mu=\infty$ blocks) into each $\mu=\infty$ block $= C$.

(11.17)

Flux induced from charge the same way as before: $\mu=\infty$ block/pole under consideration on V_{n0}, with all other blocks on $V=0$: $\phi_{mq} = q \cdot V_n(\vec{r}_q)/V_{n0}$.

Indirect fields/flux: Put each ($\mu=\infty$) block in turn on V_{n0}, with all others on $V=0$, and calculate fields, and flux into pole m: $\phi_{nm} = \left(\int \vec{B}_n \, d\vec{a}\right) = C_{nm} \cdot V_{n0}$

($\vec{B}_n$: field from V_{n0}; integral over surface of block m)

Proof that $C_{nm} = C_{mn}$

Without loss of generality $n=1$, $m=2$

$$\int \left(\vec{B}_2(\vec{r}) V_1(\vec{r}) - \vec{B}_1(\vec{r}) \cdot V_2(\vec{r})\right) d\vec{a} = I$$

Integral to be taken over all surfaces.

$$I = V_{10} \cdot V_{20} \cdot C_{21} - V_{20} \cdot V_{10} \cdot C_{12}$$

(11.18)

Also: $I = \int \text{div}(\vec{B}_2 V_1 - \vec{B}_1 V_2)\,dv$

$I = \int (\vec{B}_1 \cdot \vec{H}_2 - \vec{B}_2 \cdot \vec{H}_1)\,dv$

At all locations: $\vec{B}_1 = \mu_{\parallel} \cdot \vec{H}_{1,\parallel} + \mu_{\perp} \cdot \vec{H}_{1,\perp}$

$\vec{B}_1 \cdot \vec{H}_2 = \mu_{\parallel} \cdot \vec{H}_{1,\parallel} \cdot \vec{H}_{2,\parallel} + \mu_{\perp} \cdot \vec{H}_{1\perp} \cdot \vec{H}_{2\perp}$.

Therefore $I = 0 \Rightarrow C_{12} = C_{21}$

Flux balance for pole #1 (V_n designates now V of pole)

Flux from CSEM to pole 1

$$\underbrace{V_1(C_{10} + C_{12} + C_{13} + \dots)}_{\text{Flux going from pole 1 to pole 0, 2, 3, \dots}} = Q_1 + \underbrace{V_2 C_{12} + V_3 C_{13} + V_4 C_{14} + \dots}_{\text{Flux going from pole "0", 2, 3, \dots to pole 1}}$$

Equivalent equ. for pole # 2, 3 ... , → as many equ's as unknown V's → enough equ's to solve for V's if Q's are known.

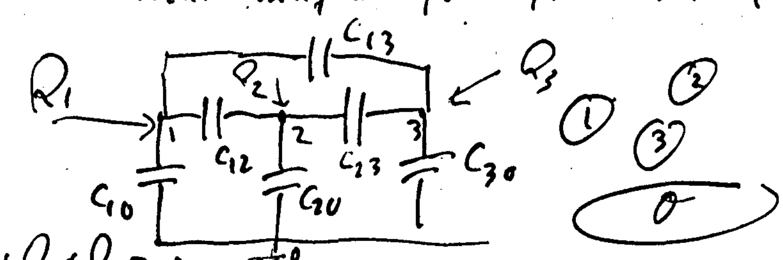
Equ's identical to electrostatic equ's.

C = capacities.

Can us same graphical representation, + methods to write + solve equ's.

(11.19)

Circuit diagram for system with 4 poles/surfaces



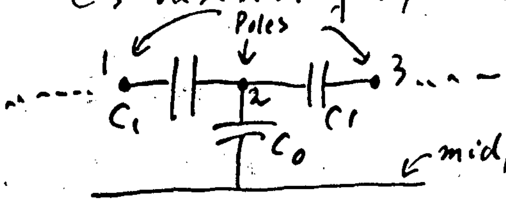
$Q_1 + Q_2 + Q_3 = 0$

Important: C_{nm}'s can be calculated in the manner described in development of general Theory, but they don't have to be calculated that way. Very often, systems have symmetries that allow simple C-calculations by calculating fluxes for specific excitation patterns. Conversely, one does not need C's if one needs fluxes only for a specific excitation pattern. That seems to be true for hybrid ID, and it is true if one wants to know only fieldstrength in device. To get answers to other questions (e.g. propagation of errors) one needs to know capacities.

JS

(11.20)

C's describing hybrid ID.



Ignore $C_{13} = C_{24} = \cdots$

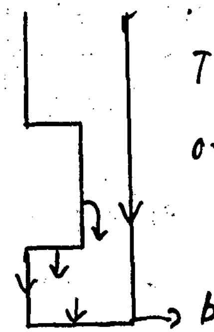
1.) $V_1 = V_3 = -V_2$; $Q_{\underset{+-}{2}} = V_2(C_0 + 4C_1)$

Know from design equ. $C_0 + 4C_1 = C_2$

2) $V_1 = V_3 = V_2$; $Q_{\underset{++}{2}} = V_2 \cdot C_0$

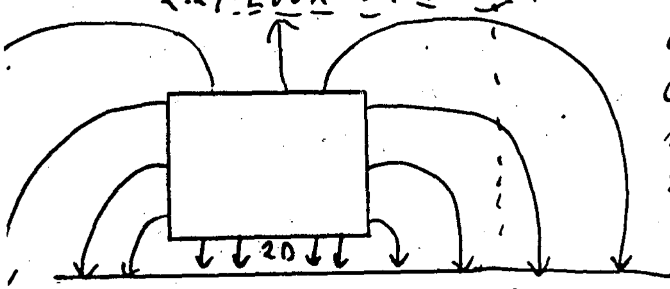
2 parts contribute to $Q_{\underset{++}{2}}$:

2.1)



This 2D flux with POISSON, or analytically.

2.2) Look in dir. ∥ beam: 3D flux obtainable by considering very many poles ⟹ 2D flux with excess V-drop correction/# of poles



↑analytically or with POISSON.

(11.21)

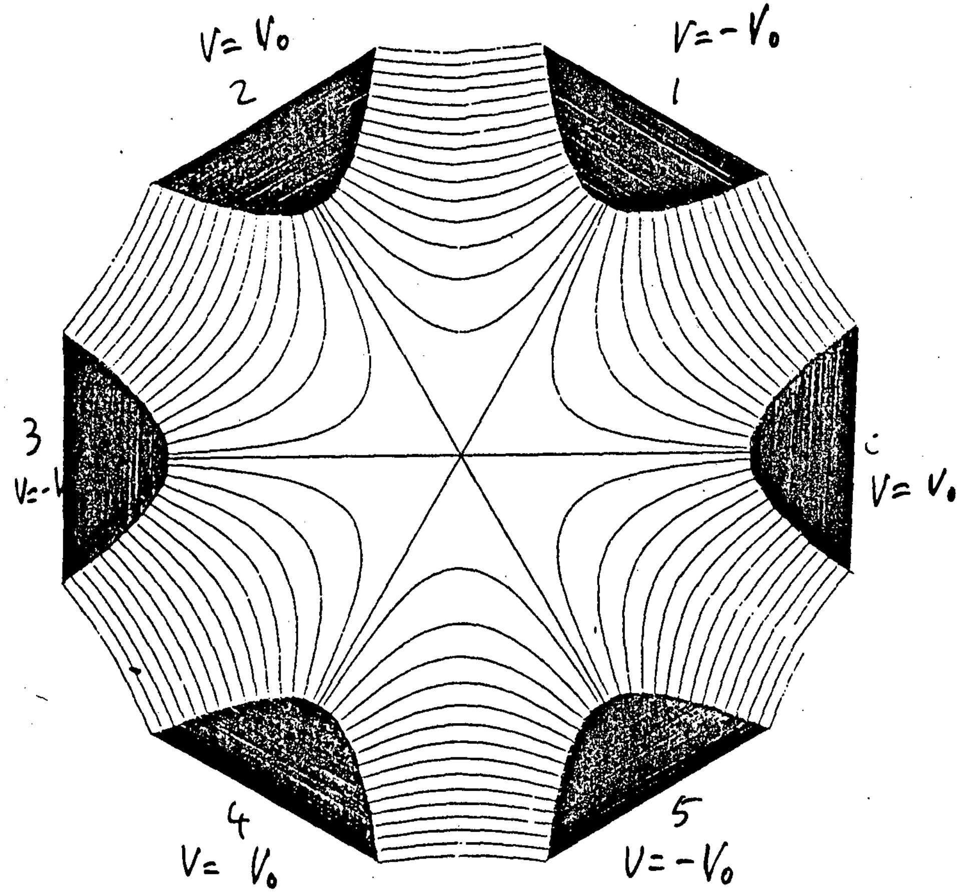
Why does this C_0-related 3D flux not show up in "normal" design equation. When one looks at flux for <u>specific</u> excitation pattern (like +-+- pattern), at least some of flux associated with specific capacity may be "invisible" in field line pattern. Transparent example: sextupole with poles excited in regular sextupole +- pattern:

Flux into pole 0 for regular excitation:

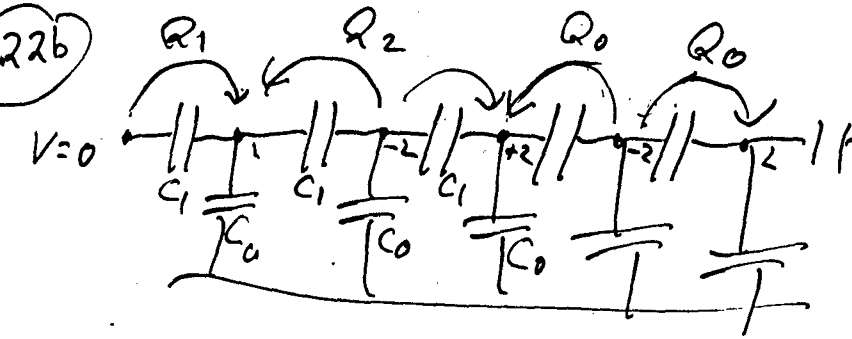
$\Phi = 2V_0(2C_{01} + C_{03})$

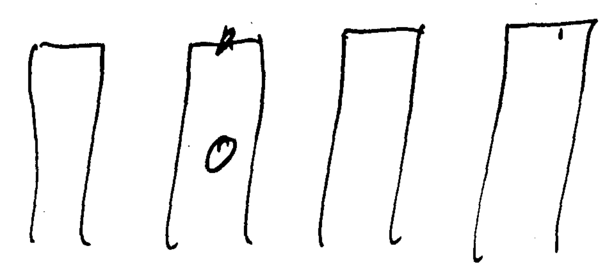
↑into pole #0

11.22a



11.22b

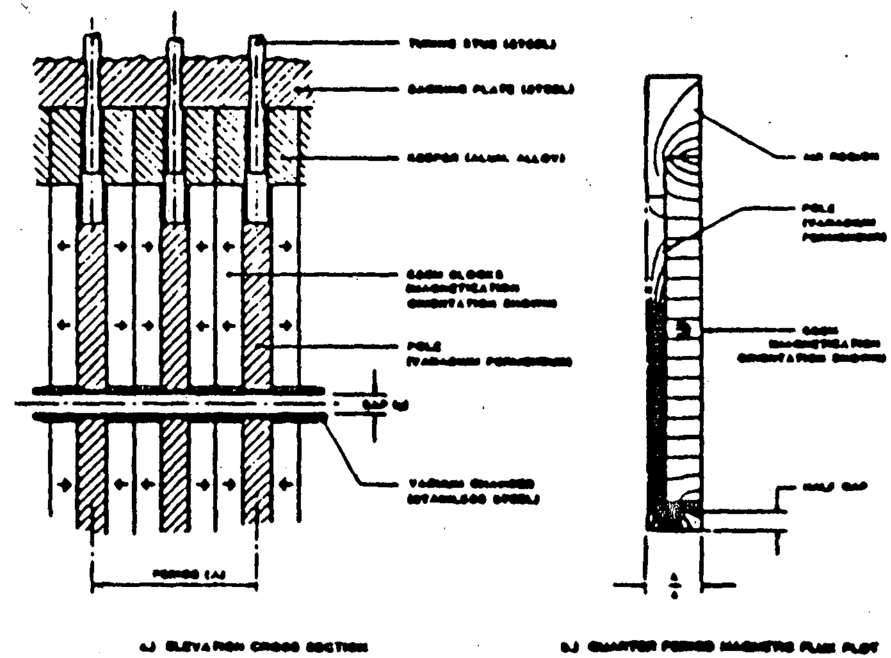




$C_0 n, \; n \geq 2$

Insertion Device Design

Klaus Halbach



Lecture 12.

February 10, 1989

Next Lectures

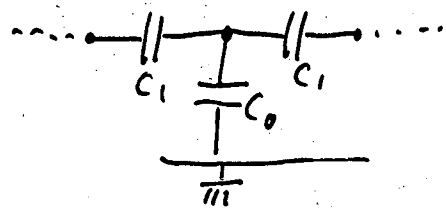
02/17/ 8^{30}
03/ 3/ 8^{30}
03/ 10/ 8^{00}

(12.1)

Lecture # 12

Summary of # 11:

- Solution to homework problems.
- Summary of algorithms for Taylor series manipulation.
- S-C polygon → circle.
- 3D hybrid theory for many $\mu=\infty$ blocks.
  - $C_{nm} = C_{mn}$
  - Flux-balance equ. $Q_n = \sum_{m=0} (V_n - V_m) C_{nm}$
  - Hybrid 1D equivalent circuit diagram



- C-calculation / flux calculation for specific excitation pattern
- "invisible" flux in sextupole.
- C_0 calculation for 1D.

(There is no sheet # 11.15)

(12.2)

General 3D Hybrid Theory with many $\mu=\infty$ Blocks/Poles.

Same basic procedure as before:

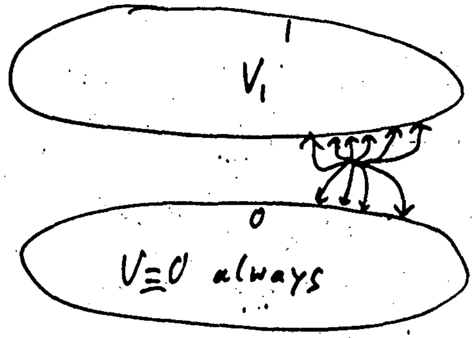
1) Direct fields, and flux induced onto poles, from charges/dipole moment distributions/CSEM when all ($\mu=\infty$) blocks on $V=0$

2) Indirect fields from each ($\mu=\infty$) block on V_{n0} (with block 0 always on $V_0=0$), with $\mu_{\parallel}, \mu_{\perp}$ from CSEM present, but active part (charges, dipole moments) "off".

Superimpose linearly all fields and get all V_{n0} from condition that total flux (from CSEM and all other $\mu=\infty$ blocks) into each $\mu=\infty$ block $= 0$.

From Lecture # 4

(12.3)



"Construct" solution that satisfies M-equ's in space outside iron and has total flux entering surface 1 equal 0. Solution = linear superposition of 2 solutions that satisfy M-equ's outside iron:

1) $q \neq 0$; $V_1 = V_q(\vec{r}_1) = 0$; $V_q(\vec{r}) \rightarrow \vec{H}_q \rightarrow \Phi_q = \int_1 \mu_0 \vec{H}_q \cdot d\vec{a} = q \cdot c_1$

↑ direct fields

← indirect fields

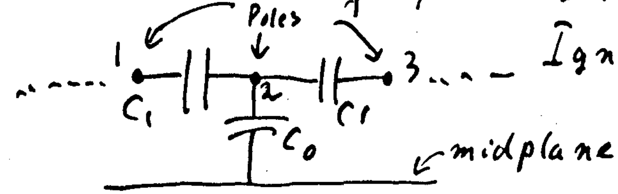
2) $q = 0$; $V_1 = V_s(\vec{r}_1) = V_{s0}$; $V_s(\vec{r}) \rightarrow \vec{H}_s \rightarrow \Phi_s = \int_1 \mu_0 \vec{H}_s \cdot d\vec{a} = V_{s0} \cdot c_2$

3) $V = V_q - V_s \rightarrow \vec{H} = \vec{H}_q - \vec{H}_s$; $\Phi = \Phi_q - \Phi_s = q \cdot c_1 - V_{s0} c_2 = 0$

$V_{s0} = q \cdot c_1 / C_2$

(12.4)

C's describing hybrid ID.



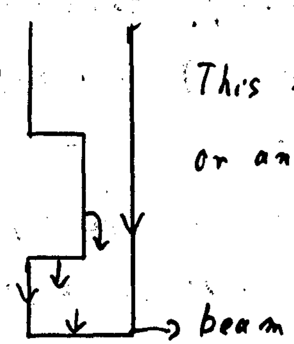
Ignore $C_{13} = C_{24} = \cdots$

1.) $V_1 = V_3 = -V_2$; $Q_{\underset{+-}{2}} = V_2 (C_0 + 4 C_1)$

Know from design equ. $C_0 + 4 C_1 = C_2$

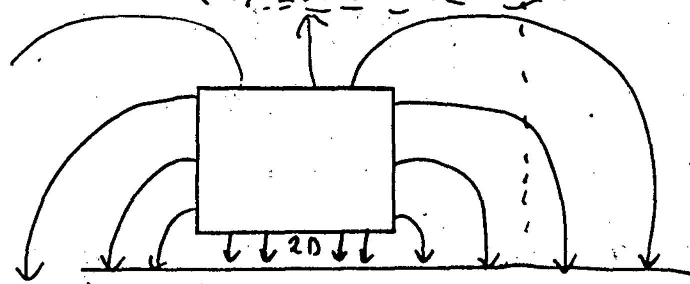
2) $V_1 = V_3 = V_2$; $Q_{\underset{++}{2}} = V_2 \cdot C_0$

2 parts contribute to $Q_{\underset{++}{2}}$:

2.1) 

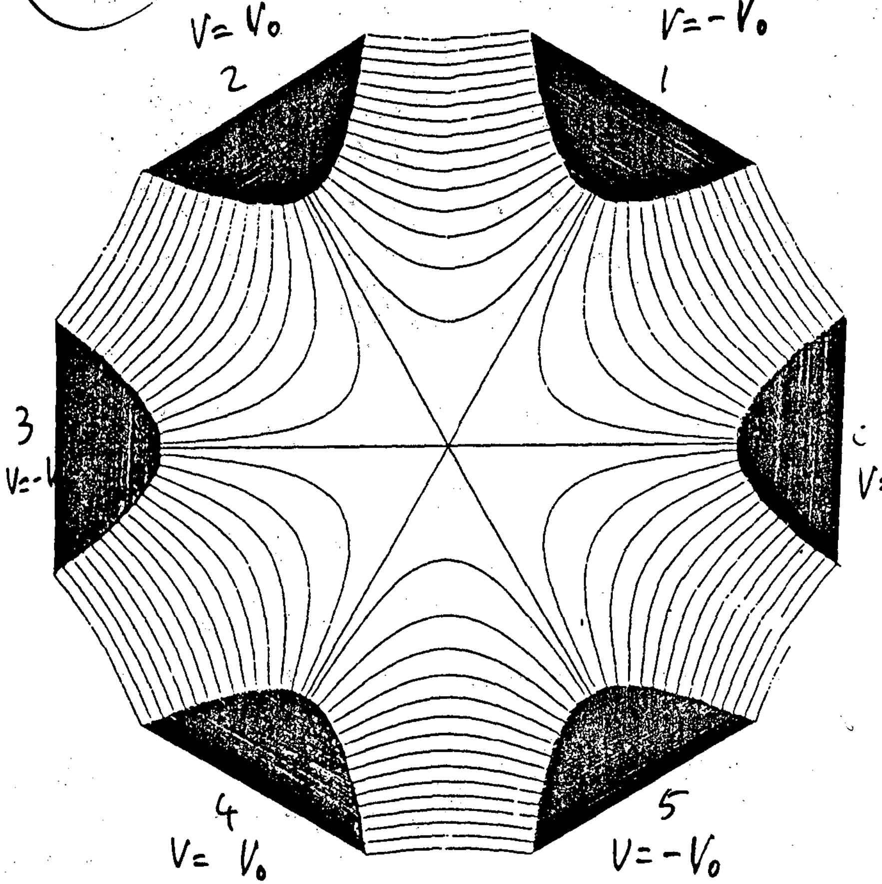
This 2D flux with POISSON, or analytically.

2.2) Look in dir. ∥ beam: 3D flux obtainable by considering very many poles ⟹ 2D flux with excess V-drop correction/# of poles



↑ analytically or with POISSON.

12.5a

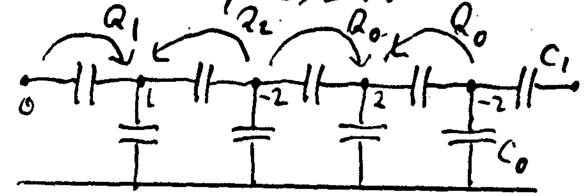


12.5b

Entrance into Hybrid I D.

Want poles at entrance/exit end to be on potentials 0, 1, -2, +2, -2, + - + ···

<trajectory> = straight, but slightly displaced. Achieve that pattern with flux Q_n induced by CSEM.



$2C_0 + 8C_1 = 2Q_0$

$C_0 + 4C_1 = Q_0$ ▨ (1)

$2C_0 + 7C_1 = Q_0 + Q_2$ ▨ (2)

$C_0 + 4C_1 = Q_1 + Q_2$ ▨ (3)

(1), (3): $Q_1 + Q_2 = Q_0$ ▨

(2), (3): $Q_0 - Q_1 = C_0 + 3C_1$ (4)

(1), (4): $Q_1 = C_1 = Q_0/(4 + C_0/C_1)$ ▨

1 -2 1
 1 -2 1
  1 -2 1

1 -3 3 -1
 1 -3 3 -1
  1 -3 3 ·

1 -3 4 -4 x ·

(12.6)

C between non-adjacent poles on "open" side of ID.

C = independant of pole geometry.

Choice: poles fill "all" available space.

pole #		0	1	2	...	n
$x/x_0 =$	-1 ($\odot\, \psi$)	0 ($\otimes\, -\psi$)	1	2	$n-1$	n

Pot. difference between pole 0 and other poles $= V = \tilde{\psi}/2$; $\tilde{\psi} = 2V$

$$F(z) = \frac{V}{\pi} \ln \frac{z-0}{z+x_0}$$

$$A(nx_0) - A((n-1)x_0) = \frac{V}{\pi} \ln \frac{n}{n+1} \cdot \frac{n}{n-1} = \frac{V}{\pi} \ln \frac{1}{1-1/n}$$

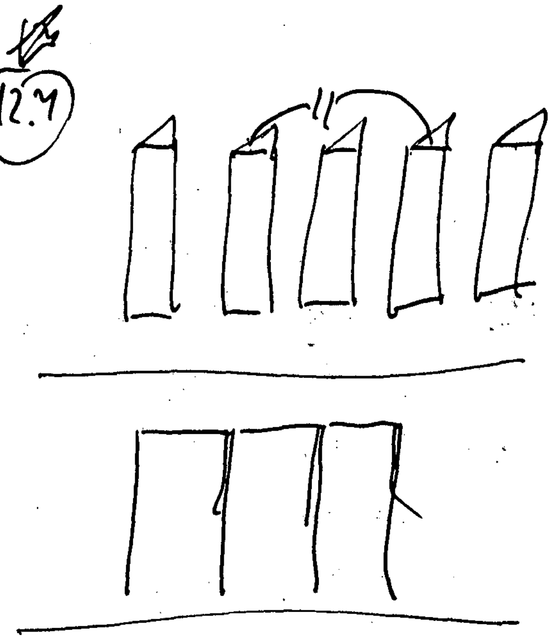
$$C_{0n} = \frac{1}{\pi} \ln\left(1/(1-1/n^2)\right)$$

$C_{02} = .092$; $C_{03} = .037$; $C_{04} = .021$

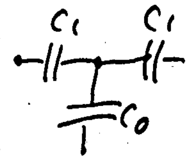
$$\prod_2^N \frac{n^2}{n^2-1} = \frac{\left(\prod_2^N n\right)^2}{\prod_3^{N+1} n \cdot \prod^{N-1} n} = \frac{1}{\frac{N+1}{2} \cdot \frac{1}{N}} = 2/(1+1/N)$$

$$\sum_2^N C_{0n} = \frac{1}{\pi} \ln\left(\frac{2}{1+1/N}\right) = .22 - \frac{1}{\pi} \ln(1+1/N)$$

(12.7)



(12.8)

"Real" calculation of CC, CI : 

Calculation #1 : Flux for + - + -
excitation of ID. → C_2 → $C_0 + 4C_1$
Use "recipe" of Lecture #6

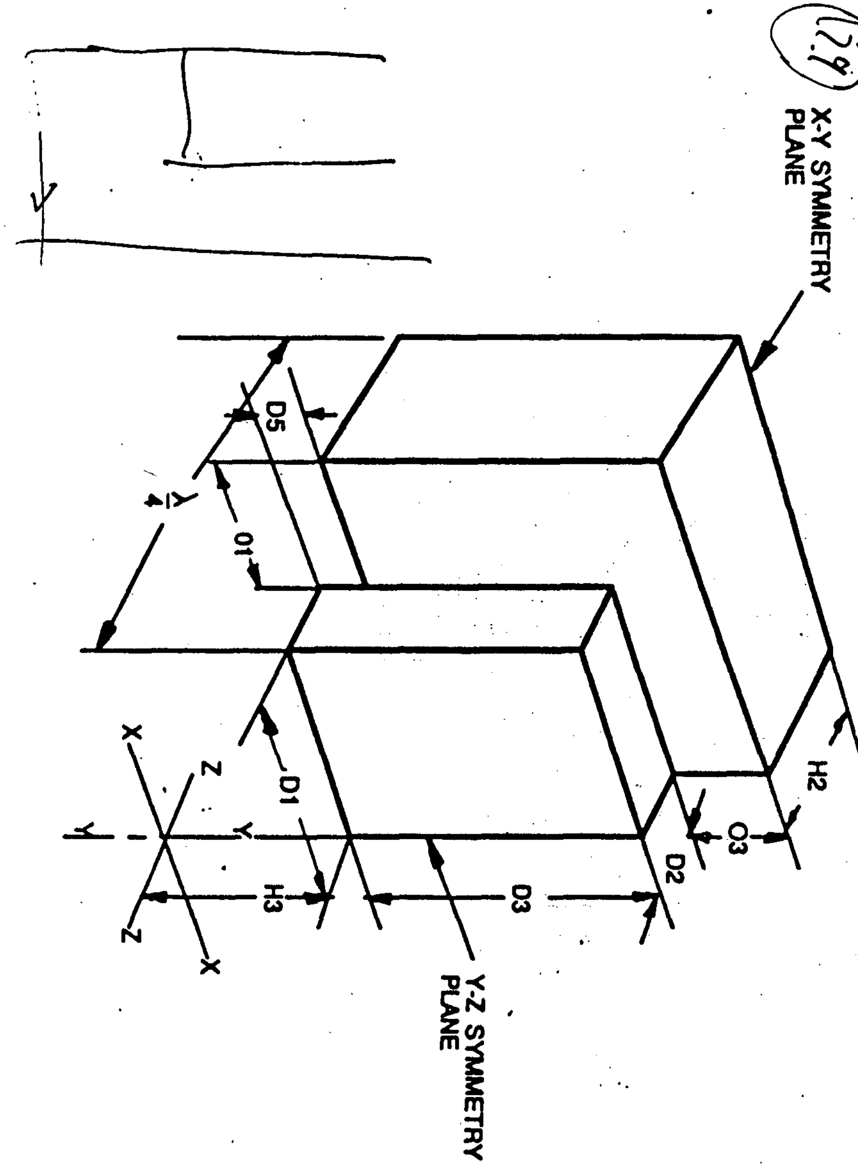
Calculation # 2.1 : Flux for +++ excitation
of ID in 2D crossection → EM → C_{0M}

Calculation #2.2 : Flux for +++ excitation
into "beams" and midplane "outside" ID
→ EB → C_{0B}

Program IDCAP1 to "digest" that
information and extract C_{0M}, C_{0B}, C_0, C_1

(12.9)

HYBRID CONFIGURATION GEOMETRY



(12.10)

3D ID Design

$$\phi_s = \tilde{V}_p \left(D_3\left(\frac{\mu_{11} D_1}{\lambda_2} + E_T\right) + D_1\left(E_p + E_s + E_T\right) + D_2 E_c \right)$$

$\tilde{V}_p = B_0 \cdot D_4$; from POISSON, or analytically

$\tilde{V}_p \cdot E_p$ = 2D flux into pole face; POISSON or analyt.

$\tilde{V}_p \cdot E_s$ = 2D excess flux into side of pole; POISSON or a

$\tilde{V}_p\, E_T$ = 2D excess flux into top/side of pole; analyt.

$\tilde{V}_p \cdot E_c$ = 2D excess flux into corner; analytical

$$\phi_{Br} = B_r \left((D_3 - D_5)\left(D_1 + \lambda_2 \cdot E_{03}\right) + D_1 \lambda_2 E_{01} \right)$$

$B_r\, \lambda_2 E_{03}$, $B_r\, \lambda_2 E_{01}$ = 2D flux from overhang; analytical.

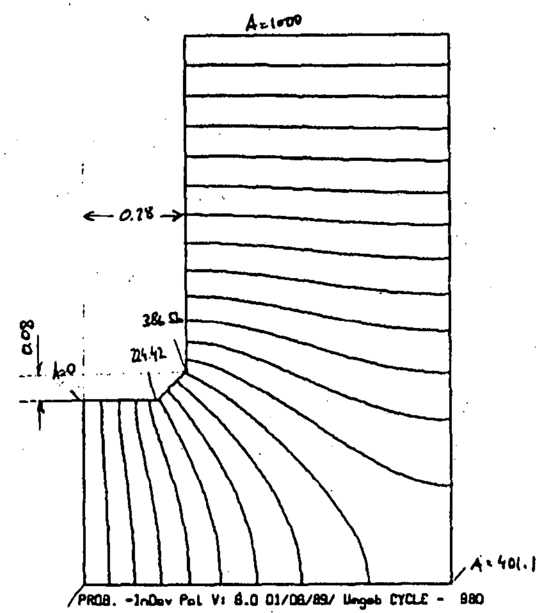
Solve $\phi_s = \phi_{Br}$ for D_3

$$D_3 = \frac{\frac{B_0 D_4}{B_r}\left(D_1(E_p + E_s + E_T) + D_2 E_c\right) + D_5\left(D_1 + \lambda_2 E_{03}\right) - D_1 \lambda_2 E_0}{D_1 + \lambda_2 E_{03} - \underbrace{\frac{B_0 D_4}{B_r}\left(\frac{\mu_{11} D_1}{\lambda_2} + E_T\right)}_{\text{Performance limitation!!}}}$$

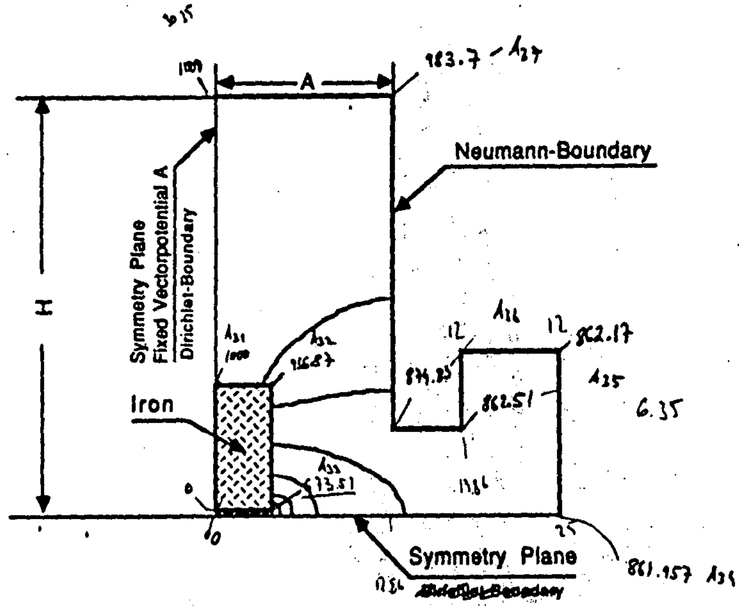
If CSEM is also attached to top, side, effect can be included in E_{01}, E_{03}

Denominator in equ. for D_3 looks dangerous. It isn't for B_r!

(12.11)

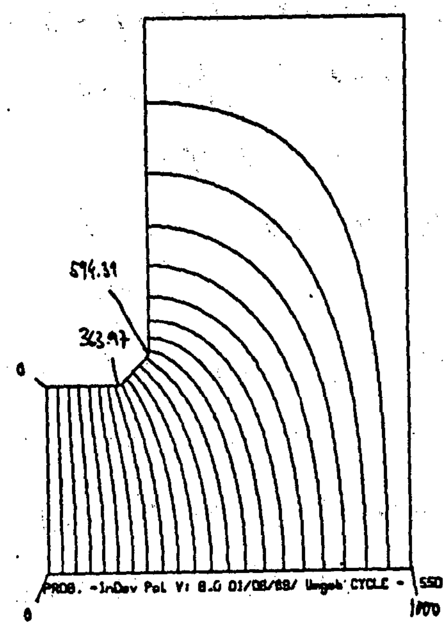


(12.12)



H
A
Symmetry Plane
Fixed Vectorpotential A
Dirichlet-Boundary
Neumann-Boundary
Iron
Symmetry Plane
983.7
862.17
862.51
6.35
861.957

(12.13)

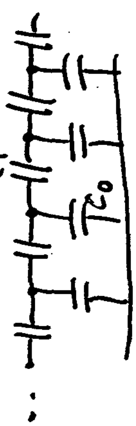


594.39
363.97
PROB. -InDav Pol V: 8.0 01/08/89/ Umgeb CYCLE - 550

(12.15)

A subtle point about C_0; C_1 model of ID.

For ∞ ID, calculation of C_0, C_1 by calculation of flux



$\Phi_2 = C_2 V_{20} = V_{20}(C_0 + 4C_1)$

on pole when poles excited to potentials $\pm V_{20}$, and then determining C_0 from flux on pole when all poles are on V_{10}:

$\Phi_1 = C_0 V_{10}$

must give correct answers.

$C_1 = (C_2 - C_0)/4$.

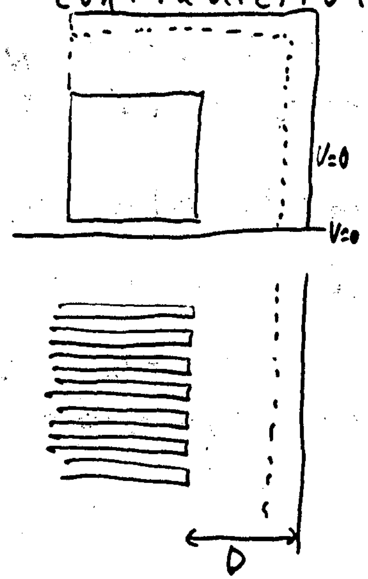
When part of C_0 comes from flux to V=0 - beams outside ID, and one changes the distance to the beam, Φ_2, and with it, C_2, does not change; since C_0 changes,

(12.14)

```
02-10-1989  07:14:47   IDCAP1

L1= 4.000   D1= 3.2000   H2=0.7200  D3= 6.2000
MU= 1.05   EPT= 1.1200   EM=1.4250  EB= 3.0800
COM=1.824E+01  COB=1.2320E+01  CO=3.0560E+01  C1=3.234E+01

DEFDBL A-Z
CLS
PRINT DATE$;"  ";TIME$;"  IDCAP1":PRINT
A1$="COM=#.###^^^^  COB=#.####^^^^  CO=#.####^^^^  C1=#.###^^^^"
A2$="L1=##.###   D1=##.####   H2=#.####  D3=##.####
A3$="MU=##.##   EPT=##.####   EM=#.####  EB=##.####"
READ L1,D1,H2,D3,MU,EPT,EM,EB
REM--L1=Lambda; D1=1/2 length of pole in direction perpendicular to beam;
REM--H2=distance from pole to symmetry plane between poles; D3=height of pole.
REM--MU=permeability of CSEM; EPT=excess fluxcoeff. for pole face and side;
REM--EM=fluxcoeff. for flux from pole to midplane for +++ excitation;
REM--EB=fluxcoeff. for flux from (one) side and top of 2D ID.
PI=4*ATN(1):D2=L1/4-H2
ET=L1/4/H2:ET=((ET+1)*LOG(ET+1)-(ET-1)*LOG(ET-1))/PI
C2=4*(D3*(MU*D1/H2+ET)+D1*(EPT+ET)+D2*.5)
COM=4*EM*D1:COB=2*EB*L1/2
C1=(C2-COM-COB)/4
PRINT USING A2$;L1;D1;H2;D3
PRINT USING A3$;MU;EPT;EM;EB
PRINT USING A1$;COM;COB;COM+COB;C1

DATA 4,3.2,.72,6.2,1.05,1.12,1.425,3.08
```

(12.16)

C_1 changes: Why? Is there a contradiction somewhere?



No, because:

1) Effect of beam on $\phi_2 \to C_2$ is $\sim e^{-2\pi D/\lambda}$

→ negligible as long as D not too small.

2) Contribution of presence of beam to C_0 is $\sim 1/D$.

$C_1 = (C_2 - C_0)/4$ expresses the consequence of that quantitatively.

Direct view: to get C_1, put a pole on V_{10}, and everything else on V=0. The closer beam to poles, the more flux goes from pole face on V_{10} to beam → less flux from face to next pole.

(12.17)

Topics that have not yet been covered. (random order)

C's for ID between pole that are not next to each other; far/close to symmetry plane.

quantitative treatment of gap errors, and easy axis orientation errors.

propagation of perturbation along length of ID.

- CSEM placement at entrance/exit of ID.

Analytical treatment of ⌐┘-geometry

Model for 3D ID-fields

Complete magnetic design procedure for ID

Excess flux formulae table

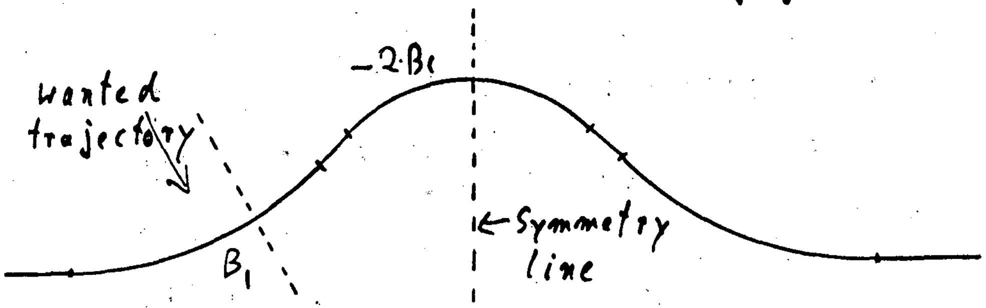
Equation of motion in S-C-mapped geometry.

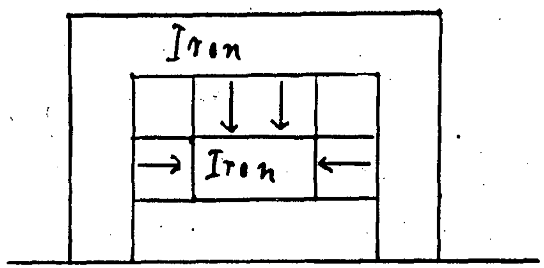
Re-visit design of "exotic" 2D magnets in dipole geometry.

OAM.

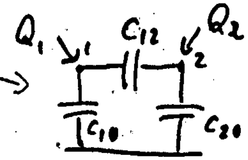
Please make suggestions for additions/omissions.

(12.18) Application of capacitor concept to non-ID permanent magnet: "jog-magnet"





For symmetrical operation → Q from CSEM

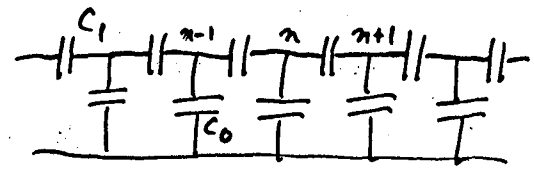


$V_1(C_{10}+C_{12}) - V_2 C_{12} = Q_1$ ‖ Need $V_2 = -2V_1 \rightarrow$

$-V_1 C_{12} + V_2(C_{20}+C_{12}) = Q_2$ ‖ $Q_2/Q_1 = -\dfrac{2C_{20}+3C_{12}}{C_{10}+3C_{12}}$

Design system. Build iron structure. Insert CSEM block in gap1; gap2 → measure fields → V → precise experimental values for $C_{nm} \rightarrow Q_1; Q_2$.

(12.19) Propagation of perturbation in a array of poles



Put perturbing charge Q_0 on pole 0. → V_0 (to be calculated later). Error-V will appear in symmetrical pattern to the right ($n>0$) and left ($n<0$) of pole 0. Can use matrix methods developed for such problems in electrical engineering. Expect exponential decay of error V's → Use that Ansatz, and if we find a solution satisfying "everything" it must be the solution

Ansatz: $V_n = V_0 \cdot \varepsilon^n$.

No net flux on pole n:

$$V_n(C_0 + 2C_1) - (V_{n-1} + V_{n+1})C_1 = 0$$

$1 + C_0/2C_1 = \alpha$; $\varepsilon + 1/\varepsilon = 2\alpha$

(12.20)

$\varepsilon^2 - 2\varepsilon\alpha + 1 = 0$ $\qquad \alpha = 1 + \frac{C_0}{2C_1}$

$\varepsilon_{1,2} = \alpha \pm \sqrt{\alpha^2 - 1}$ $\qquad \varepsilon_1 \cdot \varepsilon_2 = 1$

physically meaningful solutions

$V_n = V_0 \cdot \varepsilon^n \; ; \; \varepsilon = \begin{cases} \varepsilon_1 & n > 0 \\ \varepsilon_2 & n < 0 \end{cases}$.

(V_0 at source location.)

$V_0 \cdot (C_0 + 2C_1(1-\varepsilon_1)) = Q_0$

$$V_0 = \frac{Q_0}{\sqrt{C_0^2 + 4C_0C_1}}$$

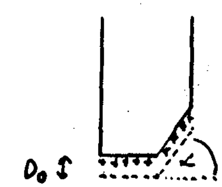
C0/C1	e1
0.2	0.6417
0.4	0.5367
0.6	0.4693
0.8	0.4202
1.0	0.3820
1.2	0.3510
1.4	0.3252
1.6	0.3033
1.8	0.2845
2.0	0.2679
2.2	0.2534
2.4	0.2404
2.6	0.2288
2.8	0.2183
3.0	0.2087
3.2	0.2000
3.4	0.1920
3.6	0.1847
3.8	0.1779
4.0	0.1716
4.2	0.1657
4.4	0.1603

(12.21)

Line integral error due to gap error / CSEM easy axis orientation error, or pole thickness error.

1) Gap error.

Error fields by: $\sigma = B_{1\perp}$ on surface of iron to be removed. ($B_{1\perp}$ = field from normal (+ - + -) excitation)



Remove (now field-free) iron → no effect. Remove charges = add charges of opposite polarity → error fields.

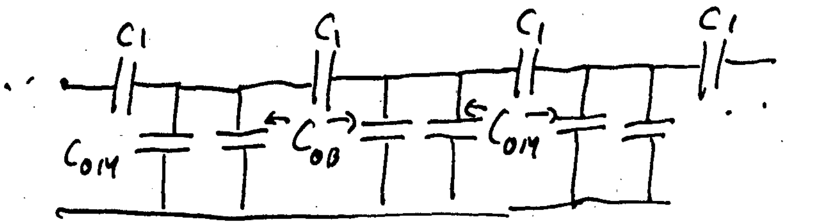
Calculate (later) direct fields → flux Q_0 going to midplane.

Equal flux, but opposite polarity, must go to pole(s). Indirect fields from poles must deposit that charge on V=0 surface, but only fraction $\frac{C_{0M}}{C_{0M} + C_{0B}}$ goes to midplane between

(12.22)

poles, the rest goes to midplane "outside"

I D



Net flux to midplane between poles

$$Q_N = Q_0\left(1 - \frac{C_{OM}}{C_{OM}+C_{OB}}\right) = Q_0 \cdot \frac{C_{OB}}{C_{OM}+C_{OB}}$$

Calculation of Q_0:

Need to calculate flux induced into midplane by charge very close to pole:
Use standard recipe: put all poles on $V=0$ and midplane on $V_{20} \to B_{2\perp}$ on pole surface, obtained from analysis of potentials / fields for ++ excitation.
Charge finds itself on $V = B_{2\perp} \cdot D_0 \cdot \cos\alpha$

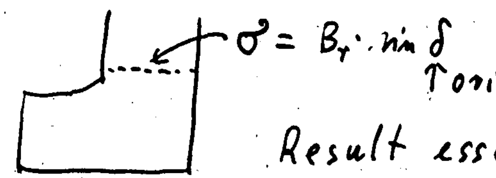
In 2D: $Q_0 = D_0 \cdot \int B_{1\perp} B_{2\perp} \cos\alpha \, ds / V_{20}$

$\perp \int B_y dz = Q_N$

(12.23)

Under most circumstances, average values for $B_{1\perp}$, $B_{2\perp}$ on surfaces will be good enough. For flat pole faces, will later see that the integral can be expressed by complete elliptical integral.

2) CSEM easy axis orientation error.
Represent CSEM by charge sheet →

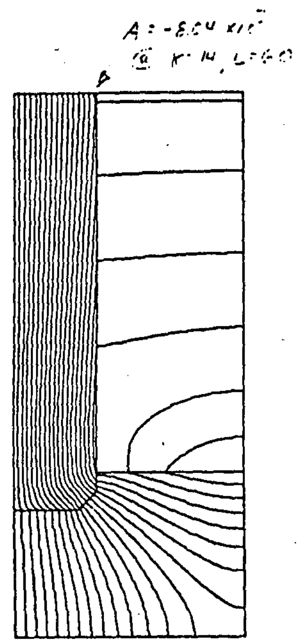


Result essentially the same, except $Q_0 = B_r \sin\delta \int V_2(x,y)\,dx / V_{20}$,
with $V_2(y,x)$ for all poles on $V=0$, and midplane on $V=V_{20}$.

3) Pole thickness error.
Same treatment as gap change, except this time field-error-causing charge

(12.24)

is "sitting" on side of pole. Notice: $B_{1\perp}$ very small where CSEM is located.



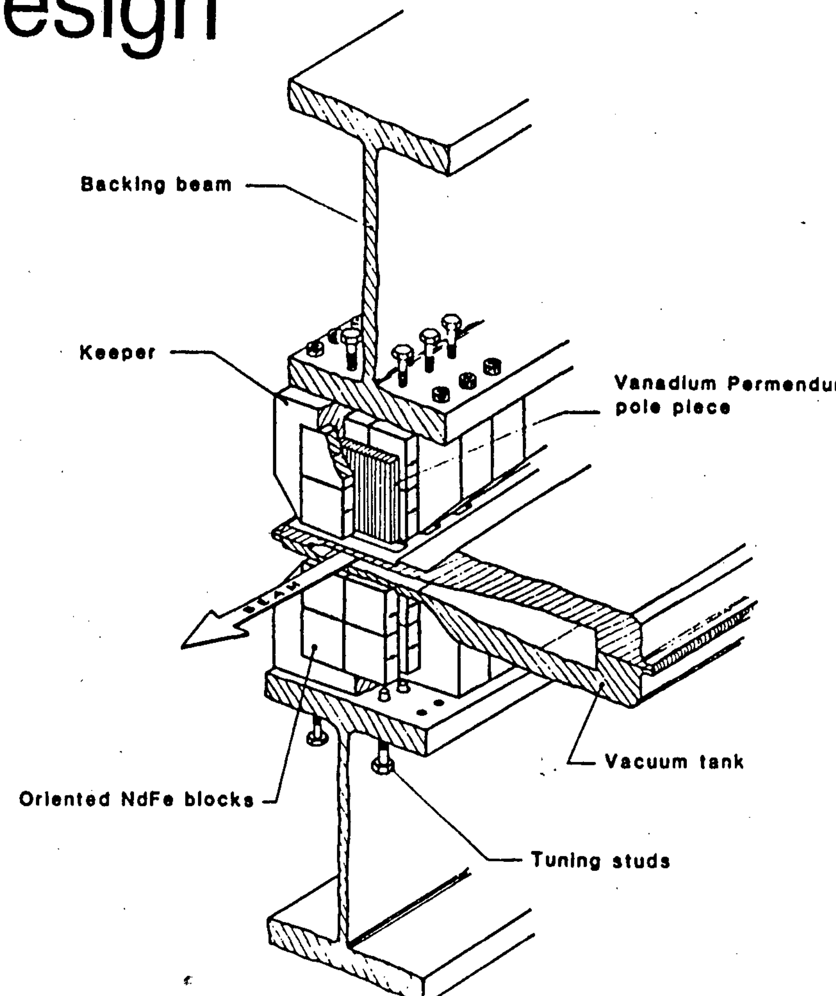
PROB. -IDMS100 : Tp/2=.44,Gp=1.4,Pop=.9 CYCLE = 10

Insertion Device Design

Klaus Halbach

Lecture 13.

February 17, 1989



LIGHT SOURCE INSERTION DEVICE

XBL 867-2534

(13.1)

Lecture # 13, 2-17-89

Summary of # 12

Placement of CSEM in entrance/exit region of I D. Very good example of powerful concepts + trivial math.

C between distant poles

Actual calculation of C's (comp. program; and a subtle property of the value of these C's

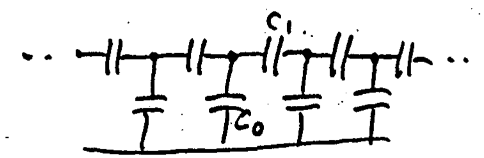
Propagation of perturbation along ID.

Line integral error due to various tolerances, with partial cancellation due to ΔV of poles.

(12.15) (13.2)

A subtle point about C_0; C_1 model of I D.

For ∞ I D, calculation of C_0, C_1



by calculation of flux

$\Phi_2 = C_2 V_{20} = V_{20}(C_0 + 4C_1)$

on pole when poles excited to potentials $\pm V_{20}$, and then determining C_0 from flux on pole when all poles are on V_{10}:

$\Phi_1 = C_0 V_{10}$

must give correct answers.

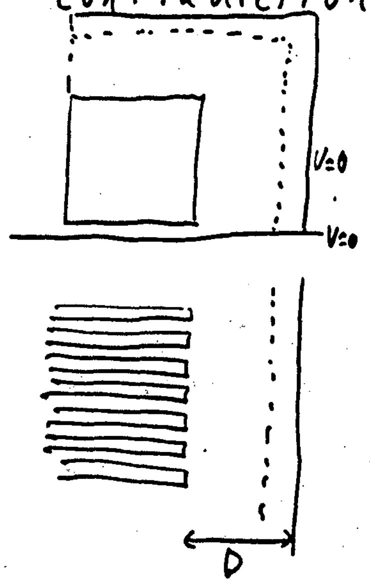
$C_1 = (C_2 - C_0)/4$.

When part of C_0 comes from flux to $V=0$ - beams outside I D, and one changes the distance to the beam, Φ_2, and with it, C_2, does not change; since C_0 changes,

→

(12.16) (13.3)

C_1 changes. Why? Is there a contradiction somewhere?



· No, because:

1) Effect of beam on $\phi_2 \to C_2$ is $\sim e^{-2\pi D/\lambda}$

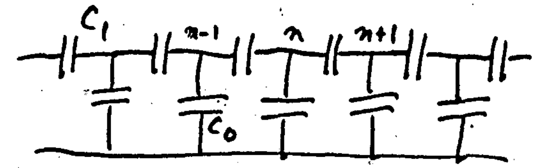
→ negligible as long as D not too small.

2) Contribution of presence of beam to C_0 is $\sim 1/D$. $C_1 = (C_2 - C_0)/4$ expresses the consequence of that quantitatively.

Direct view: to get C_1, put a pole on V_{10}, and every thing else on $V=0$. The closer beam to poles, the more flux goes from pole face on V_{10} to beam → less flux from face to next pole.

(12.19) (13.4)

Propagation of perturbation in a array of poles



Put perturbing charge Q_0 on pole 0. → V_0 (to be calculated later). Error - V will appear in symmetrical pattern to the right ($n>0$) and left ($n<0$) of pole 0. Can use matrix methods developed for such problems in electrical engineering. Expect exponential decay of error V's → Use that Ansatz, and if we find a solution satisfying "everything" it must be the solution

Ansatz: $V_n = V_0 \cdot \varepsilon^n$.

No net flux on pole n:

$$V_n (C_0 + 2C_1) - (V_{n-1} + V_{n+1}) C_1 = 0$$

$1 + C_0/2C_1 = \alpha$; $\varepsilon + 1/\varepsilon = 2\alpha$

(12.20) (13.5)

$\varepsilon^2 - 2\varepsilon\alpha + 1 = 0$ $\qquad \alpha = 1 + \frac{C_0}{2C_1}$

$\varepsilon_2 = \alpha \pm \sqrt{\alpha^2 - 1}$ $\qquad \varepsilon_1 \cdot \varepsilon_2 = 1$

physically meaningful solutions

$V_n = V_0 \cdot \varepsilon^n$; $\varepsilon = \begin{matrix} \varepsilon_1 & n > 0 \\ \varepsilon_2 & n < 0 \end{matrix}$.

(V_0 at source location.)

$V_0 \cdot (C_0 + 2C_1(1-\varepsilon_1)) = Q_0$

$$V_0 = \frac{Q_0}{\sqrt{C_0^2 + 4C_0C_1}}$$

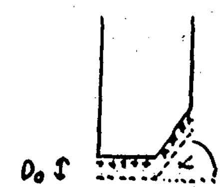
CO/C1	e1
0.2	0.6417
0.4	0.5367
0.6	0.4693
0.8	0.4202
1.0	0.3820
1.2	0.3510
1.4	0.3252
1.6	0.3033
1.8	0.2845
2.0	0.2679
2.2	0.2534
2.4	0.2404
2.6	0.2288
2.8	0.2183
3.0	0.2087
3.2	0.2000
3.4	0.1920
3.6	0.1847
3.8	0.1779
4.0	0.1716
4.2	0.1657
4.4	0.1603

(12.21) (13.6)

Line integral error due to gap error / CSEM easy axis orientation error, or pole thickness error.

1) Gap error.

Error fields by: $\sigma = B_{1\perp}$ on surface of iron to be removed. ($B_{1\perp}$ = field from normal (+ - + -) excitation)

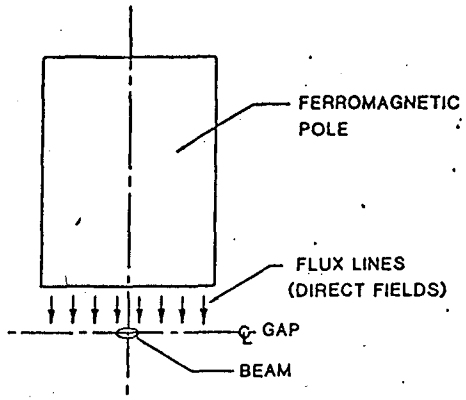


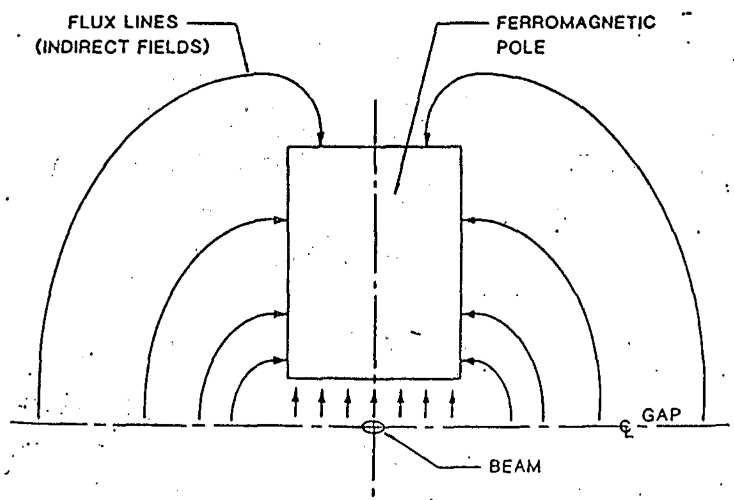
Remove (now field-free) iron → no effect. Remove charges = add charges of opposite polarity → error fields.

Calculate (later) direct fields → flux Q_0 going to midplane.

Equal flux, but opposite polarity, must go to pole(s). Indirect fields from poles must deposit that charge on V=0 surface, but only fraction $\frac{C_{0M}}{C_{0M} + C_{0B}}$ goes to midplane between

(13.7)

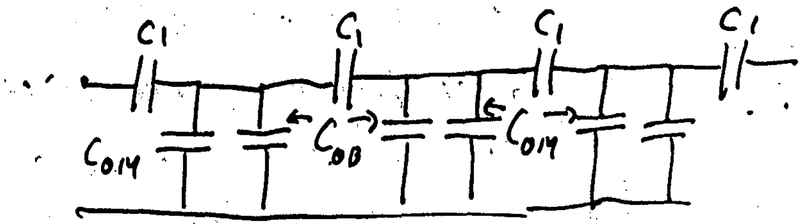




~~[2.22]~~ (13.8)

poles, the rest goes to midplane "outside"

I D



Net flux to midplane between poles

$$Q_N = Q_0\left(1 - \frac{C_{0M}}{C_{0M}+C_{0B}}\right) = Q_0 \cdot \frac{C_{0B}}{C_{0M}+C_{0B}}$$

Calculation of Q_0:

Need to calculate flux induced into midplane by charge very close to pole:
Use standard recepe: put all poles on $V=0$ and midplane on $V_{20} \rightarrow B_{2\perp}$ on pole surface, obtained from analysis of potentials / fields for +++ excitation.
Charge finds itself on $V = B_{2\perp} \cdot D_0 \cdot \cos\alpha$

In 2D: $Q_0 = D_0 \cdot \int B_{1\perp} B_{2\perp} \underbrace{\cos\alpha \, ds}_{dx} / V_{20}$

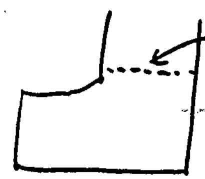
$\perp \int B_y \, dZ = Q_N$

~~2.23~~

(13.9) Under most circumstances, average values for $B_{1\perp}$, $B_{2\perp}$ on surfaces will be good enough. For flat polefaces, will later see that the integral can be expressed by complete elliptical integral.

2) CSEM easy axis orientation error.

Represent CSEM by charge sheet →



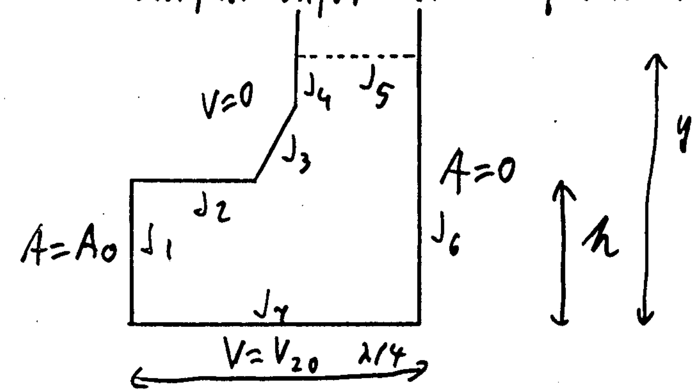
↑ orientation angle error

Result essentially the same, except $Q_0 = B_r \sin\delta \int V_2(x,y)\,dx / V_{20}$, with $V_2(y,x)$ for all poles on $V=0$, and midplane on $V=V_{20}$.

3) Pole thickness error.

Same treatment as gap change, except this time field-error-causing charge

(13.10) Calculation of $\int V_2(x,y)\,dx/V_{20}$ with output information from POISSON



$$\mathrm{Im}\oint F(z)\,dz = \oint (A\,dy + V\,dx) = \sum J_n = 0$$

$J_1 = A_0 \cdot h$; $J_2 = 0$; $J_3 = \int A_3\,dy$; $J_4 = \int A_4\,dy$

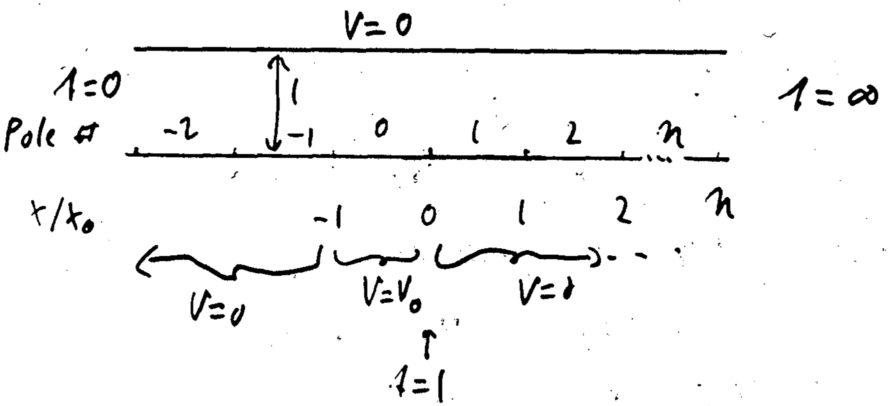
$J_6 = 0$; $J_7 = -V_{20} \cdot \lambda/4$

$$\int V(x, y_5)\,dx/V_{20} = \lambda/4 - (A_0 h + \int A_3\,dy + \int A_4\,dy)/V_{20}$$

13.11

Reduction of C between "distant" $\mu = \infty$ – blocks by presence of $V=0$ surface ("beam").

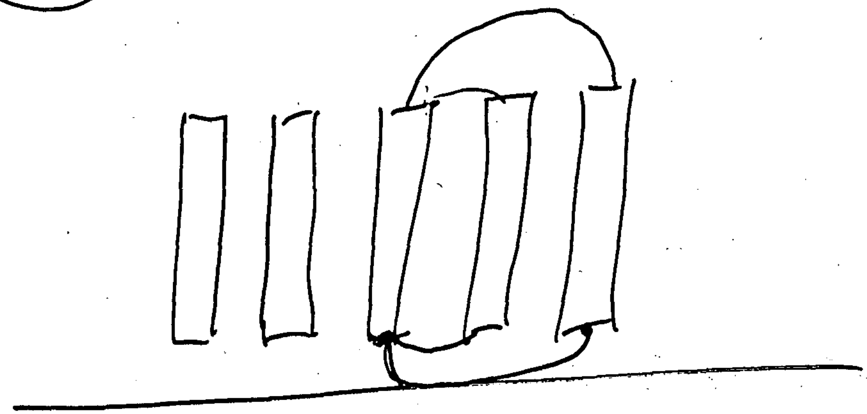
Because this is the only thing that is easy enough to execute, and for C between distant (i.e. not directly neighbor-blocks) this is probably quite adequate, assume again that poles fill "all" available space.



Map interior to upper ½-t-plane:

$\hat{z} = 1/t$; $\hat{z} = \ln t$; $t = e^{\hat{z}}$

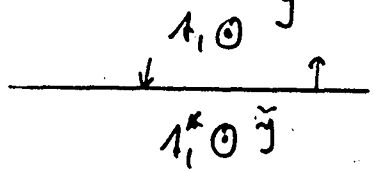
13.12



(3.13)

"Produce" V with filament-$\tilde{\gamma}$-pair.

F(t) in general case for γ not on real t-axis, with field ⊥ real t-axis:



$$F = -\frac{\tilde{\gamma}}{2\pi} \ln(t-t_1)(t-t_1^*)$$

When moving γ to axis → $t_1^* = t_1$; γ becomes V_0-jump on real axis:

$$F(t) = -\frac{V_0}{\pi} \ln(t-t_1) = -\frac{V_0}{\pi} \ln\left(e^{\pi z} - e^{\pi x_1}\right) = F(z)$$

$-\gamma$ at $z=0$; γ at $z=-x_0$

$$F(z) = \frac{V_0}{\pi} \ln \frac{e^{\pi z} - 1}{e^{\pi z} - e^{-\pi x_0}}$$

$$F(z) = \frac{V_0}{\pi} \ln\left(e^{\pi x_0/2} \cdot \frac{\sinh \frac{\pi}{2} z}{\sinh \frac{\pi}{2}(z+x_0)}\right)$$

(3.14)

$$A(n x_0) - A((n-1)x_0) = V_0 C_{on} \quad ; \quad \frac{\pi}{2} \cdot x_0 = \alpha$$

$$\pi C_{on} = \ln \frac{\sinh(n\alpha) \cdot \sinh(\alpha n)}{\sinh(\alpha(n+1)) \cdot \sinh(\alpha(n-1))}$$

$$\sinh(\beta-\gamma)\sinh(\beta+\gamma) = \sinh^2\beta(1+\sinh^2\alpha) - (1+\sinh^2\beta)\sinh^2\alpha = \sinh^2\beta - \sinh^2\alpha$$

$$\pi C_{on} = \ln\left(1 / \left(1 - (\sinh\alpha / \sinh n\alpha)^2\right)\right)$$

$$\pi \sum_{n=2}^{N} C_{on} = \ln \frac{\left(\prod_2^N \sinh n\alpha\right)^2}{\prod_3^{N+1} \sinh n\alpha \cdot \prod_1^{N-1} \sinh n\alpha}$$

$$= \ln \frac{1}{\frac{\sinh \alpha(N+1)}{\sinh \alpha \cdot 2} \cdot \frac{\sinh \alpha}{\sinh N\alpha}}$$

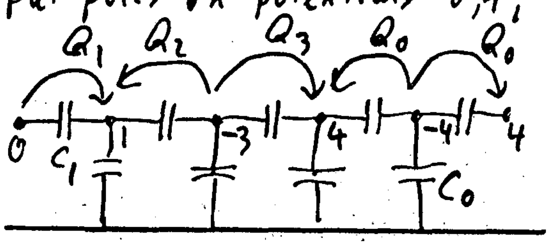
$$e^{\pi \sum_2^\infty C_{on}} = 2\cosh\alpha \cdot e^{-\alpha} = 1 + e^{-2\alpha}$$

$$\sum_2^\infty C_{on} = \frac{1}{\pi} \ln\left(1 + e^{-2\alpha}\right)$$

De-normalization: h = distance between pole and V=0 plane; $x_0 = \lambda/2 \to \alpha = \frac{\pi}{4} \cdot \frac{\lambda}{h}$

(13.15)

To avoid displacement of <trajectory>,
put poles on potentials 0, +1, −3, +4, −4, +4, −+ ⋯



$2C_0 + 8C_1 = Q_0$ ▮ (1)

$4C_0 + 15C_1 = Q_0 + Q_3$ ▮ (2)

$3C_0 + 11C_1 = Q_2 + Q_3$ ▮ (3)

$C_0 + 5C_1 = Q_2 + Q_1$ ▮ (4)

(1),(2): $Q_3 = 2C_0 + 7C_1$

(3),(4): $Q_3 - Q_1 = 2C_0 + 6C_1$

(5) $Q_1 = C_1 = Q_0 / (8 + 2C_0/C_1)$

(5),(4): $Q_2 = C_0 + 4C_1 = Q_0/2 = Q_2$ ▮

(1),(2),(5) $Q_1 + Q_3 = Q_0$ ▮

(13.16)

Antisymmetric Fields in ID.

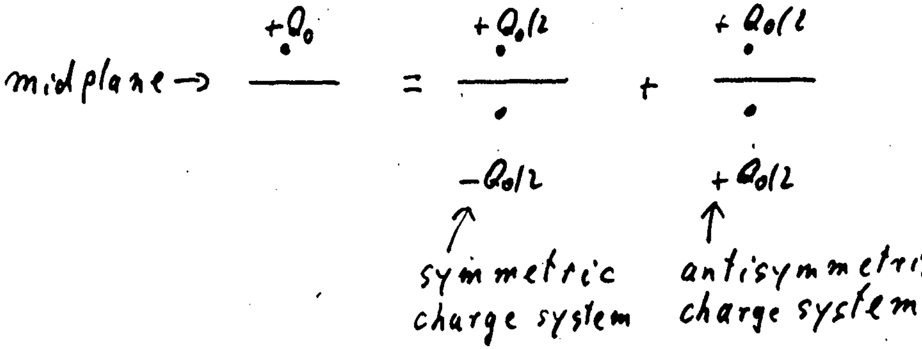
Always break up fields / field errors into fields that are symmetric (i.e. ⊥ midplane in midplane) and fields that are antisymmetric (i.e. ∥ midplane in midplane) relative to midplane.

Antisymmetric fields are usually nearly perfectly ∥ e-trajectory, but still need to discuss them because of possibility of "exotic" ID, and to point out major differences between treatment of symmetric and antisymmetric fields.

General methodology: 1) Describe basic perturbation by equivalent magnetic charges.

(13.17)

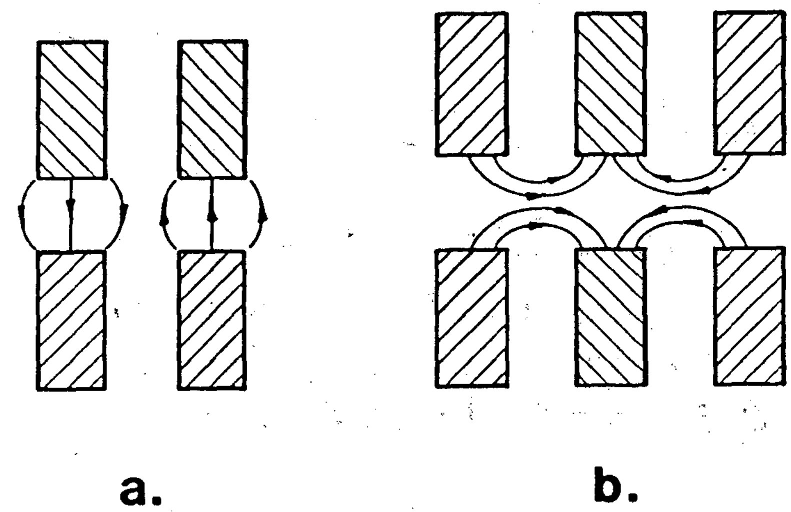
2) Decompose these charges into symmetric and antisymmetric charge systems, i.e.



3) Handle consequences of symmetric/anti-symmetric charges (separately).

Notice: for antisymmetric charges, midplane behaves like a superconducting surface, i.e. $B_\perp = 0$ in midplane → capacities quite different from "normal" capacities! In particular: If there is no shielding-beam: $C_0 = 0$!! → C_2 becomes essential for propagation of perturbations!

(13.17a)



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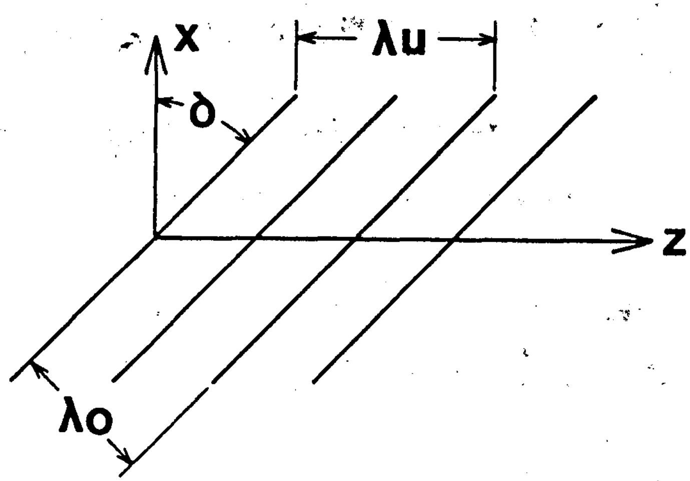
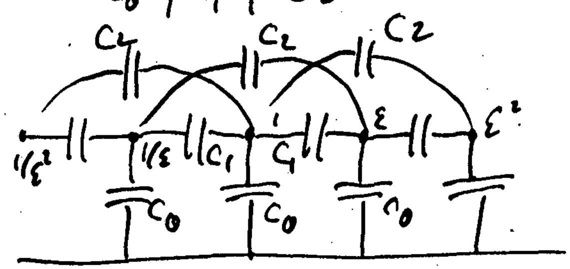
13.18



$\cos\delta \cdot \cosh(2\pi Y_0/\lambda_0) = 1 \rightarrow$ helical fields

13.18

Propagation of perturbations in 1D with C_0, C_1, C_2



2 Problems: A) Propagation constants.
B) Amplitude of "wave" caused by perturbing charge(s).

A) Propagation constants.

$$C_0 + C_1(2-\varepsilon-1/\varepsilon) + C_2\underbrace{(2-\varepsilon^2-1/\varepsilon^2)}_{-(\varepsilon-1/\varepsilon)^2} = 0$$

$$\varepsilon + 1/\varepsilon = 2u$$

$$\varepsilon^2 - 2\varepsilon u + 1 = 0;\ \varepsilon = u \pm \sqrt{u^2-1}$$

$$(\varepsilon-1/\varepsilon)^2 = 4(u^2-1)\ ;\ \frac{C_1}{4C_2} = a_1\ ;\ \frac{C_0}{4C_2} = a_0$$

$$u^2 - 1 + a_1(2u-2) - a_0 = 0$$

$$u^2 + 2ua_1 - 1 - 2a_1 - a_0 = 0$$

13.19

$$(\varepsilon-\varepsilon_1)(\varepsilon-1/\varepsilon_1)(\varepsilon-\varepsilon_2)(\varepsilon-1/\varepsilon_2)$$

13.20

$$u = -a_1 \pm \sqrt{a_1^2+2a_1+1+a_0} = -a_1 \pm \sqrt{(a_1+1)^2+a_0}$$

2 Solution "Families":

1) $u_2 = -a_1 + \sqrt{(a_1+1)^2+a_0} > 0 \implies \varepsilon > 0$

check case $c_2 \to 0$:

$$u_2 = a_1\left(\sqrt{1+\frac{2}{a_1}+\frac{a_0}{a_1^2}+\frac{1}{a_1^2}} - 1\right)$$

$$u_2 \Rightarrow a_1\left(\frac{1}{a_1}+\frac{a_0}{2a_1^2}\right) = 1+\frac{a_0}{2a_1} = 1+\frac{c_0}{2c_1} = \text{o.k.}$$

2) $u_1 = -a_1 - \sqrt{(a_1+1)^2+a_0} = -v < 0 \implies \varepsilon < 0.$

$$\varepsilon = -v+\sqrt{v^2-1} \xrightarrow[c_2\to 0]{} v\left(1-\frac{1}{2v^2}-1\right) = -\frac{1}{2v} \Rightarrow 0$$

---

3) $a_0 \to 0$ (antisymm. perturbation!) $\Rightarrow$ only differences in v of adjacent poles is of significance.

$$u = -a_1 \pm (a_1+1) \begin{array}{l} = 1 \\ = -(2a_1+1) \end{array}$$

$u_2 = 1 \to \varepsilon = 1$

$u_1 = -(2a_1+1) \to \varepsilon = -(2a_1+1) \pm \sqrt{(2a_1+1)^2-1}$

186

(13.21)

4) No approximations: Designate with $\varepsilon_1, \varepsilon_2$ the two ε from u_1, u_2 with absolute value ≤ 1.

$$u_2 = -a_1 + \sqrt{(a_1+1)^2 + a_0} = -a_1 + W$$

$$u_1 = -a_1 - W$$

$$\varepsilon_2 = u_2 - \sqrt{u_2^2 - 1} = \frac{1}{u_2 + \sqrt{u_2^2-1}}$$

$$\varepsilon_1 = u_1 + \sqrt{u_1^2 - 1} = \frac{1}{u_1 - \sqrt{u_1^2-1}}$$

$$1/\varepsilon_1 + 1/\varepsilon_2 = u_1 + u_2 - \sqrt{u_1^2-1} + \sqrt{u_2^2-1}$$

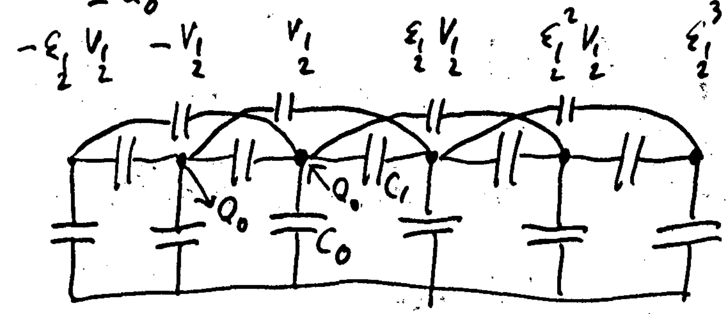
$$1/\varepsilon_1 + 1/\varepsilon_2 = -2a_1 - \sqrt{(a_1+W)^2-1} + \sqrt{(a_1-W)^2-1} < 0$$

$$1/\varepsilon_2 < -1/\varepsilon_1 \; ; \quad \underline{\varepsilon_2 > -\varepsilon_1} \quad \underline{|\varepsilon_2| > |\varepsilon_1|}$$

(13.22)

B.) Amplitude of evanescent waves.

Excitation: 2 adjacent poles receive $\pm Q_0$



$$V_1 b_1 + V_2 b_2 = Q_0 \qquad |\varepsilon| \leq 1$$

$$V_1 g_1 + V_2 g_2 = 0$$

$$b = C_0 + C_1(2+1-\varepsilon) + C_2(2+\varepsilon-\varepsilon^2)$$

$$g = C_0\varepsilon + C_1(2\varepsilon-1-\varepsilon^2) + C_2(2\varepsilon+1-\varepsilon^3)$$

$$d = C_0 + C_1(2-\varepsilon-1/\varepsilon) + C_2(2-\varepsilon^2-1/\varepsilon^2) = 0$$

$$g - \varepsilon d = C_2(2\varepsilon+1-\varepsilon^3-2\varepsilon+\varepsilon^3+1/\varepsilon) = g$$

$$g = C_2(1+1/\varepsilon)$$

$$b - d = C_1(1+1/\varepsilon) + C_2(\varepsilon+1/\varepsilon^2)$$

$$= (1+1/\varepsilon)(C_1 + C_2\varepsilon(1-1/\varepsilon+1/\varepsilon^2))$$

13.23

$V_{1,2}(1+1/\varepsilon_{1,2}) = v_{1,2}$

$V_1 g_1 + V_2 g_2 = 0 \longrightarrow v_1 + v_2 = 0 \;;\; v_2 = -v_1$

$v_1\left(C_1 + C_2(\varepsilon_1 + 1/\varepsilon_1 - 1)\right) + v_2\left(C_1 + C_2(\varepsilon_2 + 1/\varepsilon_2 - 1)\right) = Q_0$

$v_1 C_2\left(\varepsilon_1 + 1/\varepsilon_1 - (\varepsilon_2 + 1/\varepsilon_2)\right) = v_1 \cdot 2C_2(u_1 - u_2) = Q_0$

$u_{1,2} = -a_1 \mp \sqrt{(a_1+1)^2 + a_0} \;;\quad \varepsilon = u \pm \sqrt{u^2 - 1}$

$Q_0 = -v_1 \cdot 4C_2\sqrt{(a_1+1)^2 + a_0} = -V_1 \cdot 4C_2\sqrt{(a_1+1)^2 + a_0}\,(1 + 1/\varepsilon_1)$

$V_2 = -V_1 \cdot \dfrac{1 + 1/\varepsilon_1}{1 + 1/\varepsilon_2}$

$$V_2 = Q_0 \Big/ \left(4C_2(1 + 1/\varepsilon_2)\sqrt{(a_1+1)^2 + a_0}\right)$$

If V_{1s}, V_{2s} are amplitudes for single pole excitation,

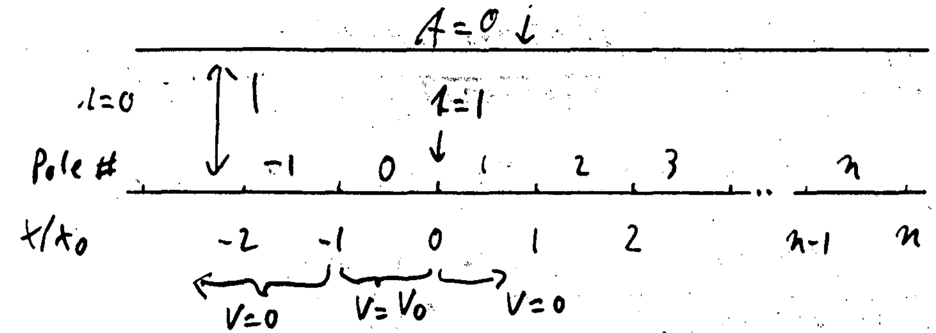
$V_1 = V_{1s}(1 - \varepsilon_1) \;;\; V_2 = V_{2s}(1 - \varepsilon_2)$

$$V_{1s} = V_1/(1 - \varepsilon_1) \;;\; V_{2s} = V_2/(1 - \varepsilon_2)$$

13.24

C between distant $\mu = \infty$ blocks "next" to superconducting plane (for antisymmetric system)

(For completeness, and to introduce one more important procedure about "handling" current filaments)



$i z = 1/A \;;\; i z = \ln A \;;\; A = e^{iz}$

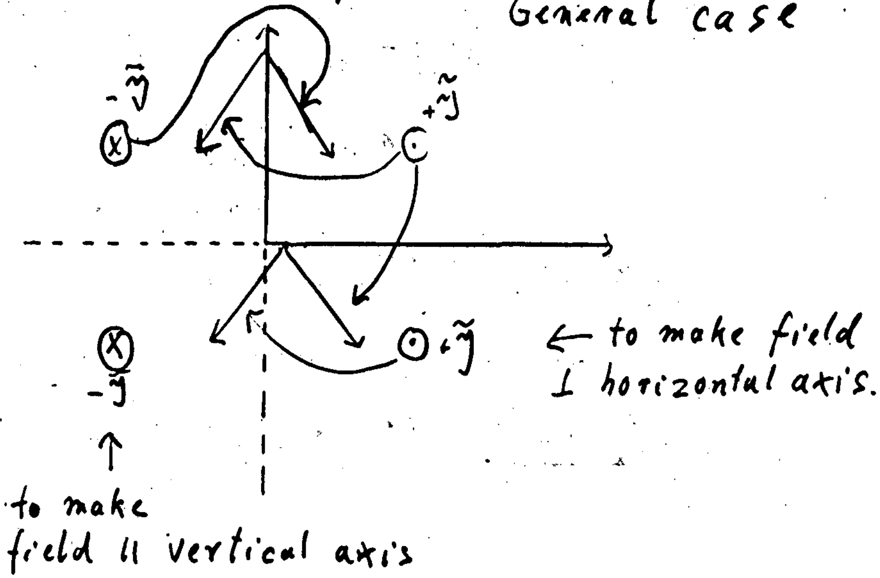
"Produce" V again with pair of y-filaments such that field $\perp$ to $0 < A < \infty$, and field $\parallel$ to $-\infty < A < 0$ } real A-axis.

↑ new twist.

(13.25)

$W=\sqrt{z}$; $z=W^2$

W-plane. General case



In our case, all currents are on real W-axis, with $\tilde{y}=V_0$

$$F\cdot\pi=\tilde{y}\ln\frac{W-1}{W+1}\cdot\frac{W+W_{-1}}{W-W_{-1}}$$

↑ from filament at $z=0$, + image current

↑ from filament at $z=-x_0$ + image current.

(13.26)

$W=\sqrt{z}=e^{\frac{\pi}{2}z}$; $\frac{W-1}{W+1}=\mathrm{tgh}\frac{\pi}{4}z$

$\tilde{y}=V_0=1$ $\qquad \frac{W-W_{-1}}{W+W_{-1}}=\mathrm{tgh}\frac{\pi}{4}(z+x_0)$

$\pi(A_n-A_{n-1})=\pi C_n=\ln H_n$; $\frac{\pi}{4}x_0=\gamma$

$$H_n=\frac{\mathrm{tgh}\,n\gamma}{\mathrm{tgh}(n+1)\gamma}\cdot\frac{\mathrm{tgh}\,n\gamma}{\mathrm{tgh}(n-1)\gamma}$$

$$H_n=\frac{\sinh^2 n\gamma}{\sinh(n-1)\gamma\cdot\sinh(n+1)\gamma}\cdot\frac{\cosh(n-1)\gamma\cdot\cosh(n+1)\gamma}{\cosh^2 n\gamma}$$

$$\cosh(\beta-\alpha)\cdot\cosh(\beta+\alpha)=\cosh^2\beta+\sinh^2\alpha$$

$$H_n=\frac{1+(\sinh\gamma/\cosh n\gamma)^2}{1-(\sinh\gamma/\sinh n\gamma)^2}\ ;\ C_n=\frac{1}{\pi}\ln H_n$$

$$\sum_2^{\infty}C_n=\frac{1}{\pi}\ln(1+1/\cosh 2\gamma)\qquad \gamma=\frac{\alpha}{2}=\frac{\pi}{4}x_0$$

De-normalization: distance pole − $A=0$ plane $=h$; $x_0=\lambda/2\rightarrow\gamma=\frac{\pi}{8}\cdot\frac{\lambda}{h}$

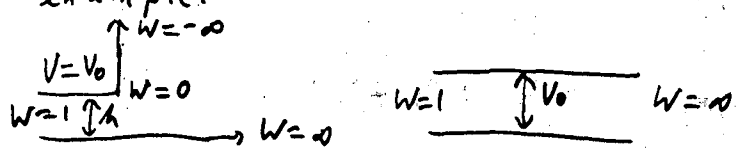
13.27

Equation of motion in "S-C-plane".

Statement of problem: Want to solve (numerically) equ. of motion of particle in 2D electric field that can best be calculated with (non-trivial) S-C-transformations.

Sine time is involved, use W as S-C plane variable. Use $\frac{V=V_0}{V=0}$ as

example.



↑ z-plane ↑ F-plane

$$\pi \frac{dz}{dW} = h \cdot \frac{\sqrt{W}}{W-1} \quad ; \quad \pi \frac{dF}{dW} = \frac{V_0}{W-1}$$

$$\frac{dF}{dz} = \frac{V_0/h}{\sqrt{W}}$$

13.28

General case: I know $\frac{dz}{dW} = z'$, and $\frac{dF}{dz}$ as functions of W.

Equ. of motion: $\left(E^* = i\frac{dF}{dz}\right)$

$$\ddot{z} = \frac{e}{m} E$$

$$\dot{z} = z' \cdot \dot{W} \; ; \; \ddot{z} = z'' \cdot \dot{W}^2 + z' \cdot \ddot{W} = -\frac{e}{m} \cdot i \left(\frac{dF}{dz}\right)^*$$

$$\ddot{W} = -\dot{W}^2 \cdot \frac{z''}{z'} - i\frac{e}{m} \cdot \frac{(dF/dW)^*}{z' z'^*}$$

↑ Easily solved with Runge-Kutta.

R-K is a numeric procedure that solves the following set of first order differential equations

$$\dot{s}_n(t) = G_n(s_1, s_2, \cdots s_N, t) \; ; \; n = 1, 2, \cdots N.$$

(13.29)

Variable assignment and G_n for this case:

$N = 4$. $W = u + iv$

$s_1 = u$; $s_2 = v$; $s_3 = \dot{u}$; $s_4 = \dot{v}$;

$G_1 = s_3$; $G_2 = s_4$

$$G_3 = \mathrm{Re}\left(-(s_3^2 + s_4^2 + 2i\, s_3 s_4) \cdot \frac{z''}{z'} - i\frac{e}{m}\frac{(dF/dW)^*}{z' z'^*}\right)$$

$$G_4 = \mathrm{Im}\left(-(s_3^2 + s_4^2) \cdot \frac{z''}{z'} - i\frac{e}{m}\frac{(dF/dW)^*}{z' z'^*}\right)$$

↑ known functions of s_1, s_2

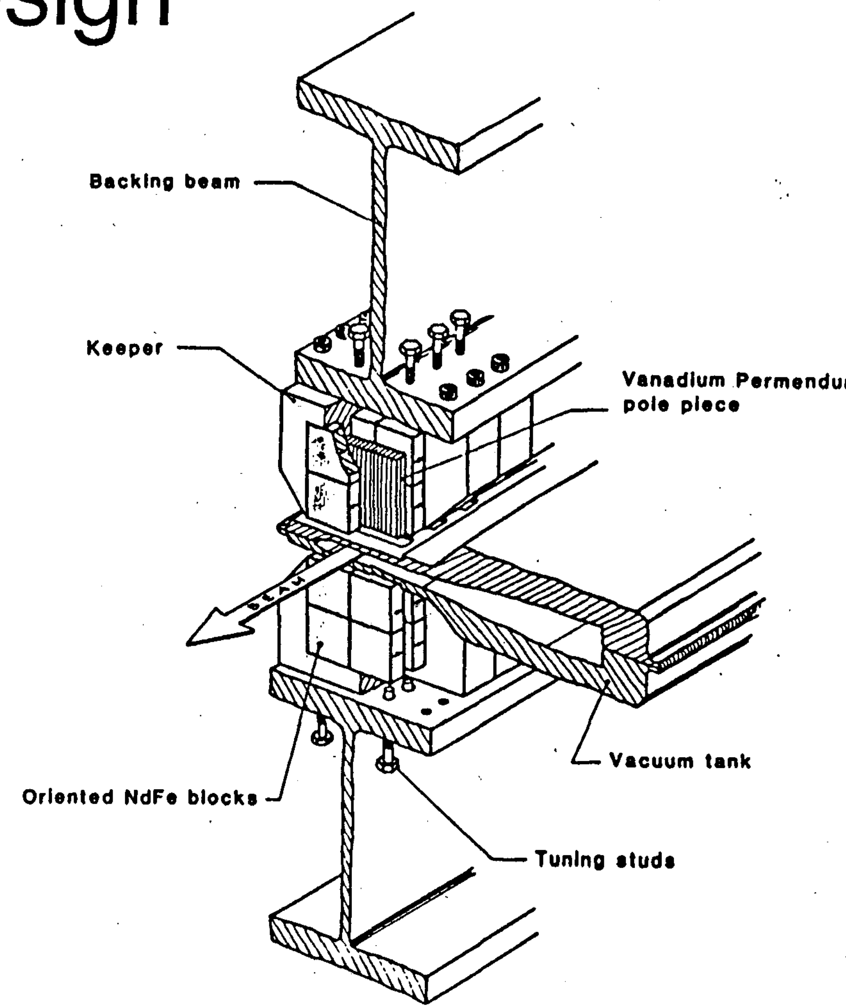
To solve, need to know (obviously) initial conditions, i.e. G_1, G_2, G_3, G_4 for $t=0$.

Insertion Device Design

Klaus Halbach

Lecture 14.

March 3, 1989



LIGHT SOURCE INSERTION DEVICE

XBL 867-2534

(14.1)

Lecture # 14 ; 3-3-89

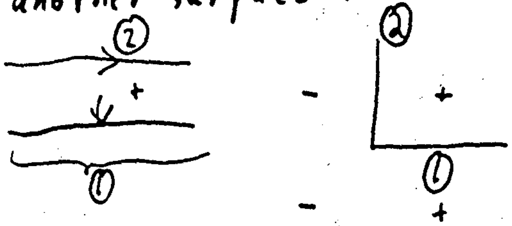
Summary of # 13

- Finished discussion of consequences of major perturbation effects in ID.
- Now: additions to that.

Also in # 13:

- C between "distant" poles that face $V=0$ surfaces / "superconducting" surface.

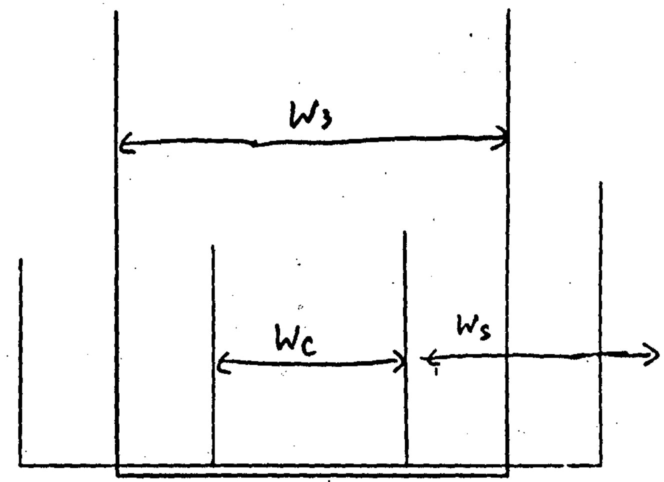
"Trick" to deal in same problem with surface to which $\vec{B}$ must be ⊥, and another surface to which $\vec{B}$ must be ∥,



with $\vec{B}$ produced by current filament

- Placement of CSEM to get entrance/exit V-pattern $V = 0, 1, -3, 4, -4, \cdots$

(14.2) Effect of 3 blocks of CSEM with easy axis orientation error



This time "true" 3D calculation.

Notation: as before, $Q_0 = 2D$ flux = flux/unit length

$[Q_0] = G\,cm$

$\phi = 3D$ flux ; $[\phi] = G\,cm^2$

Direct flux to midplane : $\phi = Q \cdot W_{eff}$.

$W_{eff} = W_c$ for center block

$W_{eff} = W_s$ for side blocks

(14.3)

Indirect flux to midplane:

$\Phi_i = \phi \cdot s \quad ; \quad s = C_{0cu} / C_{0tot}$

↑ direct flux to midplane

Total Q ($= 4\int B_y dz$) seen by beam, caused by center block:

$$Q = Q_0 - Q_0 \cdot W_c \cdot s / W_3 = Q_0 (1 - s \cdot \frac{W_c}{W_3})$$

Still compensation by s, but reduced!

From either side block:

$$Q = - Q_0 \cdot W_s \cdot s / W_3 = - Q_0 \cdot s \cdot \frac{W_s}{W_3}$$

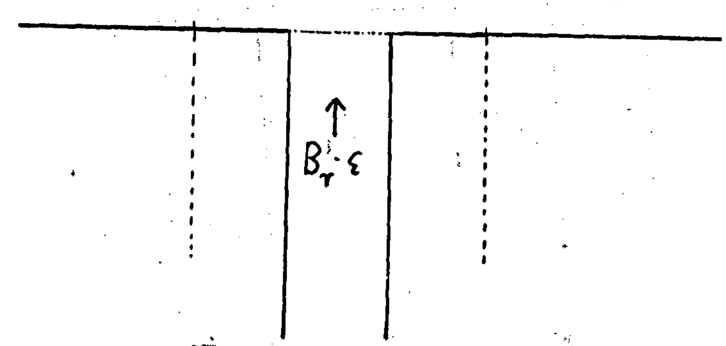
Electrons "see" only indirect flux.

Remedies: grind off material so that easy axis ∥ surface. Or: sort and place 3 CSEM blocks so that steering cancels at smallest gap.

(14.4)

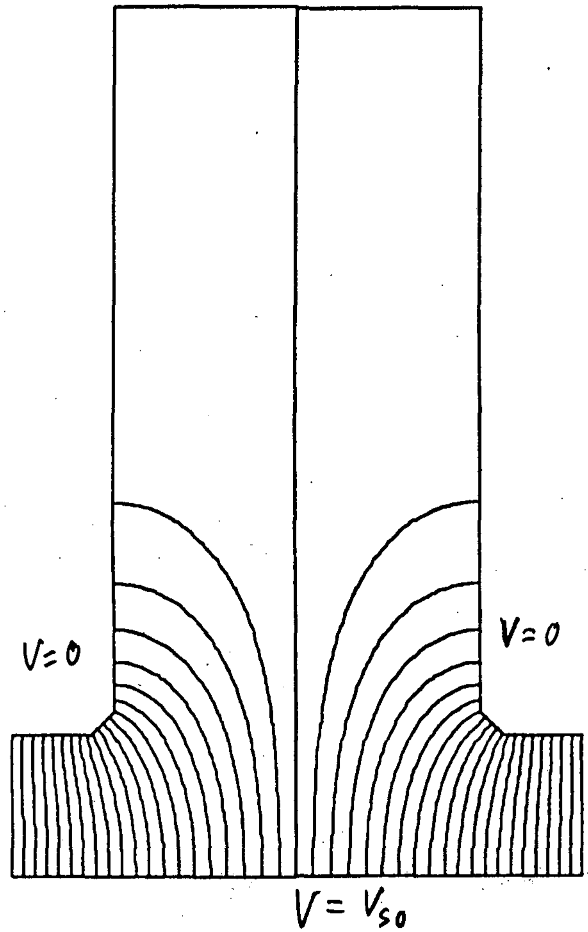
Homework: Q_{Dir} from thin gap between CSEM and pole: on one side; the other side; unequal gaps on both sides thin gap between CSEM and CSEM along vertical center line; Q_{Dir} because of 2 blocks of CSEM of unequal strength to right + left of vertical symmetry line

2D Device to measure easy axis orientation error.



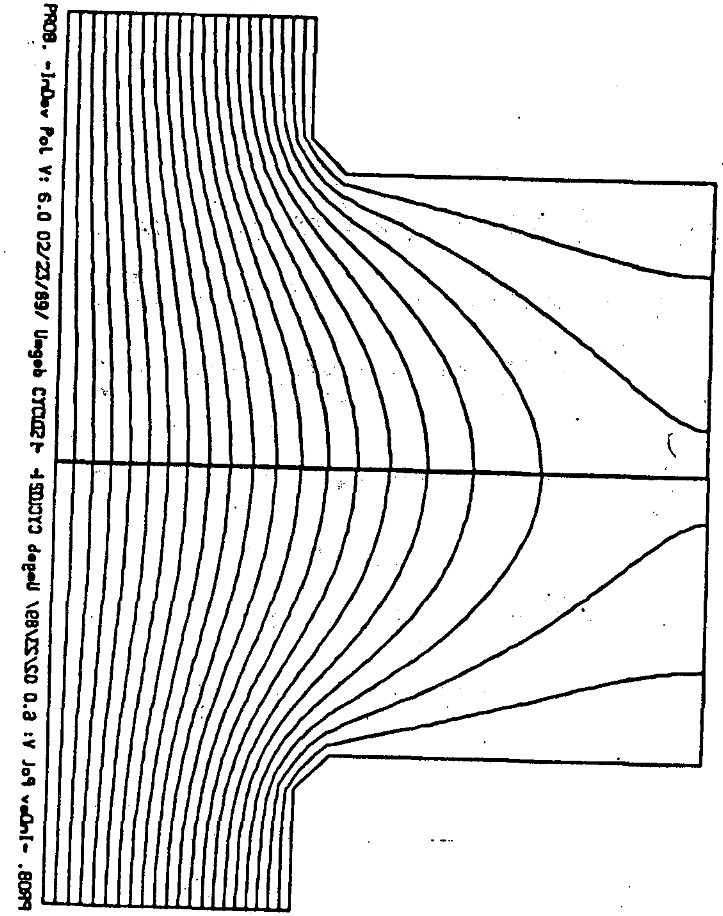
Derive expressin to calculate flux passing between ⅂ Γ corners of μ=∞ iron at upper edge of CSEM.

(14.5)



$$Q_{Dir} = \int \vec{B}_r \cdot \vec{H}_s \, dv / V_{s0} W_3$$

(14.6)



(14.7)

- Discussed systems that are symmetric/antisymmetric relative to midplane.
- Propagation of antisymmetric perturbations.
- Solution of 2D equations of motion in Schwarz-Christoffel-mapped geometry

(14.8) List of tolerance problems discussed

Symmetric/antisymmetric errors

Steering/displacement only - errors.

Excitation strength

Gap error

Pole thickness error

Easy axis orientation error

Gap between CSEM and pole

2 unequal strength blocks of CSEM between poles

Compensation/generation of steering-field errors by indirect flux

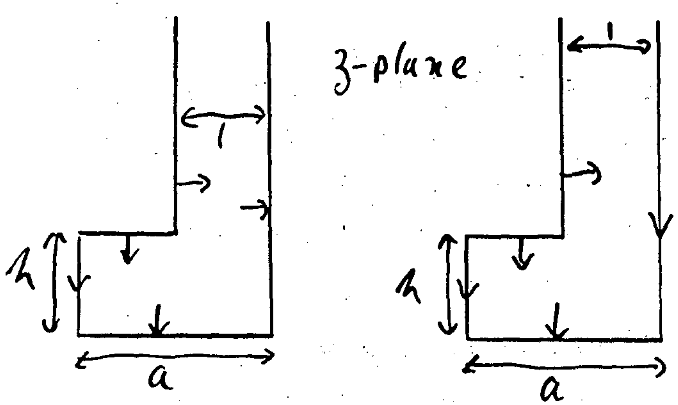
Error propagation (capacities)

(14.9)

Lower part of upper ½ of λ/4 of hybrid I D

Important for "analytical" hybrid I D design, and many of solutions for problems are of great general interest. Explain only those techniques that are not "common knowledge".

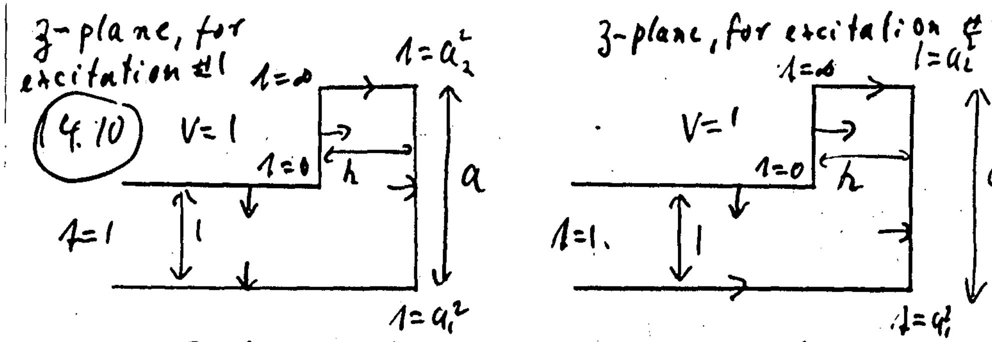
Geometry in common orientation

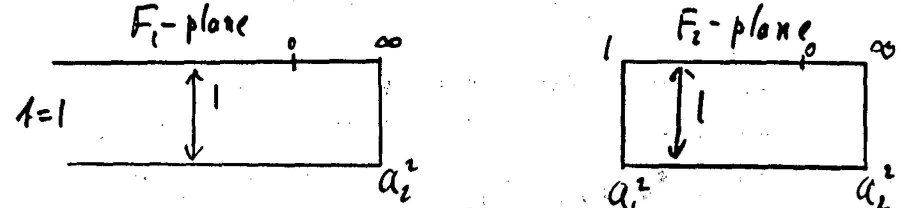


Excitation #1 Excitation #2

Because of previous work, do calculations in differently arranged geometry.

(4.10)





$$\pi \dot{F}_1 = i\,\frac{b_2}{(t-1)\sqrt{t-a_2^2}} \qquad \dot{F}_2 = \frac{i\,C}{\sqrt{t-1}\,\sqrt{t-a_1^2}\,\sqrt{t-a_2^2}}$$

$$b_1^2 = a_1^2-1\,;\; b_2^2 = a_2^2-1\,;\; b^2 = a^2-1$$

$$0 < 1 < a_1 < a_2 < a$$

$$\pi \dot{z} = -\frac{\sqrt{t}\, b_1 b_2}{(t-1)\sqrt{t-a_1^2}\,\sqrt{t-a_2^2}}$$

$$F_1' = -i\sqrt{1-a_1^2/t}\,/b_1 \qquad F_2' = -\frac{i\pi C}{b_1 b_2}\sqrt{1-1/t}$$

$$\pi a = b_1 b_2 \cdot \int_{a_1^2}^{a_2^2} \frac{\sqrt{t}\,dt}{(t-1)\sqrt{t-a_1^2}\,\sqrt{a_2^2-t}}$$

$$\pi h = b_1 b_2 \int_{a_2^2}^{\infty} \frac{\sqrt{t}\,dt}{(t-1)\sqrt{t-a_1^2}\,\sqrt{t-a_2^2}}$$

(4.11)

Have to do the following:

1) Use equations for h, a, to determine a_1, a_2. To do that, have to

   1.1) Describe secant equation solver for >1 dimension

   1.2) Describe method to remove singularities from limit(s) of integrand.

   1.3) Prove $a_1 < a < a_2$

   1.4) Use 1.3) to introduce hard, smooth range restrictions on a_1, a_2, to insure convergence of equ. solver

2) Integrate $\dot{F}_1$ to get flux into pole

3) Derive formula to get excess flux into side of pole for excitation #1

4) Derive flux into midplane for excitation #2

5) Develop procedure to get harmonics for excitation #1

6) $D_4 = V_0/B_0 = |F_1'|_{q=a_2^2}$ = done. (Don't forget to de-normalize!!)

(4.12)

Secant equation solver in N dimensions

$N=1$. x_0 ; $y_0 = y(x_0)$

Assume: $y - y_0 = C \cdot (x - x_0)$

Determine C: $y_1 - y_0 = C \cdot (x_1 - x_0)$

Solve for $y=0$: $x = x_0 - C^{-1} \cdot y_0$

$C^{-1} = (x_1 - x_0) \cdot (y_1 - y_0)^{-1}$

$$x = x_0 - (x_1 - x_0)(y_1 - y_0)^{-1} \cdot y_0$$

$N > 1$: x, y = vectors with N elements.

Assume: $y - y_0 = M(x - x_0)$

Determine M:

$$\underbrace{(y_1 - y_0 \quad y_2 - y_0 \cdots y_N - y_0)}_{Y} = M \cdot \underbrace{(x_1 - x_0 \quad x_2 - x_0 \cdots x_N - x_0)}_{X}$$

$Y \leftarrow$ Square matrices $\rightarrow X$

Solve for $y=0$: $x = x_0 - M^{-1} \cdot y_0$

$M^{-1} = X \cdot Y^{-1}$

$$x = x_0 - X \cdot Y^{-1} y_0$$

(4.13)

Removal of singularities from integrands at limit(s) of integration.

$$\int_{\Lambda_1} \frac{f(\Lambda)}{(\Lambda-\Lambda_1)^{\varepsilon}} d\Lambda = m \cdot \int_0 f(\Lambda) \cdot W^{m(1-\varepsilon)-1} dW$$

↑ well behaved

$\Lambda - \Lambda_1 = W^m$; $d\Lambda = m \cdot W^{m-1} dW$ $m-1-\varepsilon m = m(1-\varepsilon)-1$

Choose m so that $m(1-\varepsilon)-1 = 0$ or

$m(1-\varepsilon)-1 \geq 1$

Procedure works only when $\varepsilon < 1$ (otherwise singularity is not integrable). Use it also for $-\varepsilon < 1$ to avoid infinite first derivative at $t = 0$. That is also reason to choose $m(1-\varepsilon)-1 \geq 1$. $m(1-\varepsilon)-1 = 0$ not practical (see below).

For $\int_{\Lambda_1}^{\Lambda_2} \frac{g(\Lambda)}{(\Lambda-\Lambda_1)^{\varepsilon_1}(\Lambda_2-\Lambda)^{\varepsilon_2}} d\Lambda$, use same thought, but use only integer m_1, m_2:

$$d\Lambda = a \cdot W^{m_1-1} \cdot (1-W)^{m_2-1} dW$$

$$\Lambda = \Lambda_1 + a \cdot \int_0^W W^{m_1-1} (1-W)^{m_2-1} dW$$

(4.14)

a from: $\Lambda_2 = \Lambda_1 + a \cdot \int_0^1 W^{m_1-1} (1-W)^{m_2-1} dW$

Behaviour at limits is now o.k. if one again chooses $m_{1,2}(1-\varepsilon_{1,2})-1 = 0$ or ≥ 1

Need integer values for m_1, m_2 to be able to write simple closed expression for $\Lambda(W)$.

All carried out for most frequent case:

$\varepsilon_1 = \varepsilon_2 = 1/2$: $m_1 = m_2 = 2$.

With a little more symmetrization:

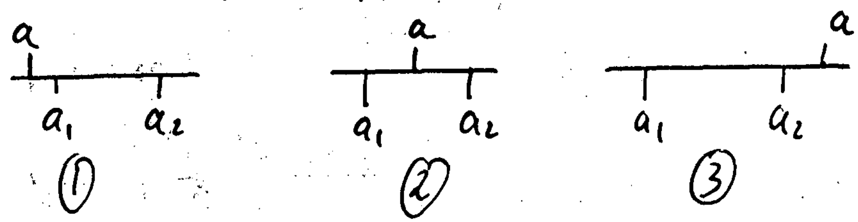
$$\int_{\Lambda_1}^{\Lambda_2} \frac{g(\Lambda) \cdot d\Lambda}{\sqrt{\Lambda-\Lambda_1}\sqrt{\Lambda_2-\Lambda}} = 3 \cdot \int_{-1/2}^{1/2} \frac{g(\Lambda)}{\sqrt{1-W^2}} dW$$

$$\Lambda = \frac{\Lambda_2+\Lambda_1}{2} + \frac{\Lambda_2-\Lambda_1}{2} \cdot W(3-4W^2)$$

14.15

Proof that $a_1 < a < a_2$

Since $a_1 < a_2$, 3 possibilities



Prove that ① and ③ are impossible.

With π ʒ formulae for ⌐ and ⊓ :

$$\pi(a-1) = \int_0^\infty \frac{\sqrt{t}\cdot b_1 b_2\, dt}{(t+1)\sqrt{t+a_1^2}\sqrt{t+a_2^2}} = \int_0^\infty \frac{\sqrt{t}\; b^2\, dt}{(t+1)(t+a^2)}$$

$b_1^2 = a_1^2 - 1$; $b_2^2 = a_2^2 - 1$; $b^2 = a^2 - 1$

$$\int_0^\infty \frac{\sqrt{t}\cdot b^2}{(t+1)(t+a^2)}\cdot\Big(\underbrace{\frac{b_1}{b}\cdot\frac{\sqrt{t+a^2}}{\sqrt{t+a_1^2}}}_{T_1}\cdot\underbrace{\frac{b_2}{b}\cdot\frac{\sqrt{t+a^2}}{\sqrt{t+a_2^2}}}_{T_2} - 1\Big)\, dt = 0$$

$$T_1 = \sqrt{\frac{t+1+b^2}{t+1+b_1^2}}\cdot\frac{b_1}{b} = \sqrt{\frac{1+(t+1)/b^2}{1+(t+1)/b_1^2}}$$

For $a_1, a_2 > a$; $b_1, b_2 > b$; $T_1, T_2 > 1$; $T_1 \cdot T_2 - 1 > 0$

For $a_1, a_2 < a$: $b_1, b_2 < b$; $T_1, T_2 < 1$; $T_1 \cdot T_2 - 1 < 0$

14.16

Hard, smooth range restrictions on a_1, a_2

Force $1 < a_1 < a$ with

$$a_1 = \frac{a+1}{2} + \frac{a-1}{2}\cdot G(x_1)$$

Properties of $G(x)$ for real x, $-\infty < x < \infty$.

$G(-x) = -G(x)$; $dG/dx > 0$; $(G(x))_{x\to\infty} = 1$

Examples: $G(x) = \frac{2}{\pi}\,\mathrm{arctg}(x)$; $\frac{x}{\sqrt{1+x^2}}$; $\mathrm{tgh}(x)$.

Force $a < a_2 < \infty$ with

$$a_2 = \frac{2a}{1 - G(x_2)}$$

(14.17)

Excess flux into side of pole for excitation #1

$$\pi(\zeta(0)-\zeta(\lambda)) = \int_0^{\lambda} \frac{\sqrt{\lambda}\, b_1 b_2 \, d\lambda}{(1-\lambda)\sqrt{a_1^2-\lambda}\sqrt{a_2^2-\lambda}}$$

$$\pi(F(0)-F(\lambda)) = \int_0^{\lambda} \frac{b_2 \, d\lambda}{(1-\lambda)\sqrt{a_2^2-\lambda}}$$

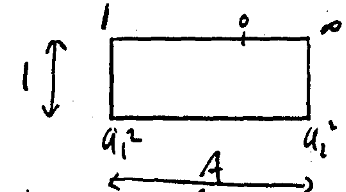
$$\pi \Delta A = \left(\int_0^{\lambda} \frac{b_2 \, d\lambda}{(1-\lambda)\sqrt{a_2^2-\lambda}} \underbrace{\left(1-\frac{\sqrt{\lambda}\, b_1}{\sqrt{a_1^2-\lambda}}\right)}_{T} \right)_{\lambda\to 1}$$

$$T = \frac{\sqrt{a_1^2-\lambda}-\sqrt{\lambda}\, b_1}{\sqrt{a_1^2-\lambda}} = \frac{a_1^2-\lambda-\lambda(a_1^2-1)}{\sqrt{a_1^2-\lambda}\left(\sqrt{a_1^2-\lambda}+\sqrt{\lambda}\, b_1\right)}$$

$$\pi \Delta A = b_2 a_1^2 \cdot \int_0^1 \frac{d\lambda}{\sqrt{a_2^2-\lambda}\sqrt{a_1^2-\lambda}\left(\sqrt{a_1^2-\lambda}+\sqrt{\lambda}\cdot b_1\right)}$$

(14.18)

Flux into midplane for excitation #2



$$\dot{F}_2 = \frac{i\,C}{\sqrt{\lambda-1}\sqrt{\lambda-a_1^2}\sqrt{\lambda-a_2^2}}$$

$$1/C = \int_1^{a_1^2} \frac{d\lambda}{\sqrt{\lambda-1}\sqrt{a_1^2-\lambda}\sqrt{a_2^2-\lambda}} = 2\cdot\int_0^{b_1} \frac{dw}{\sqrt{b_1^2-w^2}\sqrt{b_2^2-w^2}}$$

$\sqrt{\lambda-1}=w$; $\lambda=w^2+1$; $d\lambda=2w\,dw$; $w=b_1\sin\varphi$; $dw=b_1\cos\varphi\, d\varphi$

$$1/C = 2\cdot\int_0^{\pi/2} \frac{d\varphi}{\sqrt{b_2^2-b_1^2\sin^2\varphi}} = \frac{2}{b_2}\cdot\int_0^{\pi/2}\frac{d\varphi}{\sqrt{1-\frac{b_1^2}{b_2^2}\sin^2\varphi}} = \frac{2}{b_2}\cdot K\!\left(\frac{b_1^2}{b_2^2}\right)$$

$$A/C = \int_{a_1^2}^{a_2^2} \frac{d\lambda}{\sqrt{\lambda-1}\sqrt{\lambda-a_1^2}\sqrt{a_2^2-\lambda}} = 2\int_0^{\sqrt{a_2^2-a_1^2}} \frac{dw}{\sqrt{b_1^2+w^2}\sqrt{a_2^2-a_1^2-w^2}}$$

$\sqrt{\lambda-a_1^2}=w$; $\lambda=w^2+a_1^2$; $d\lambda=2w\,dw$

$w=\sqrt{a_2^2-a_1^2}\sin\varphi$; $A/C = 2\cdot\int_0^{\pi/2}\frac{d\varphi}{\sqrt{b_1^2+(b_2^2-b_1^2)(1-\cos^2\varphi)}}$

$$A/C = \frac{2}{b_2}\cdot\int_0^{\pi/2}\frac{d\varphi}{\sqrt{1-\left(1-\frac{b_1^2}{b_2^2}\right)\sin^2\varphi}} = \frac{2}{b_2}\cdot K\!\left(1-\frac{b_1^2}{b_2^2}\right)$$

$$A = \frac{A/C}{1/C} = K\!\left(1-\frac{b_1^2}{b_2^2}\right)\Big/K\!\left(b_1^2/b_2^2\right)$$

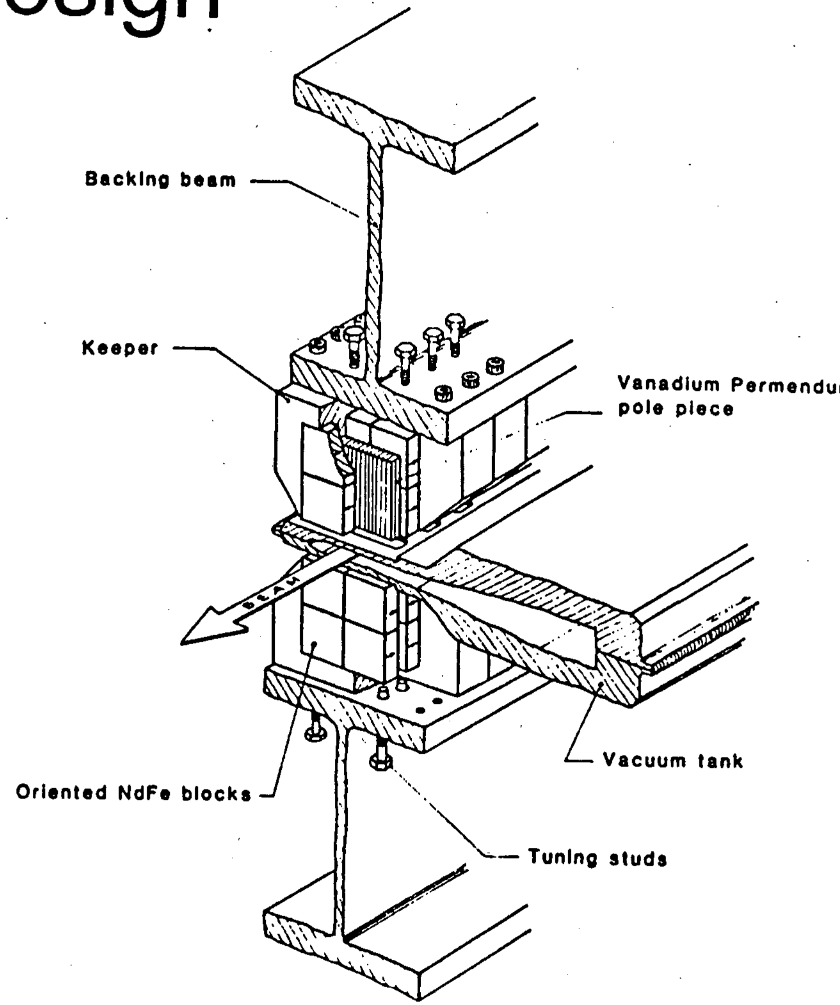
Insertion Device Design

Klaus Halbach

Lecture 15.

March 10, 1989

NOTE:
Final Lecture March 17, 1989
@ 8:00 AM



LIGHT SOURCE INSERTION DEVICE

XBL 867-2534

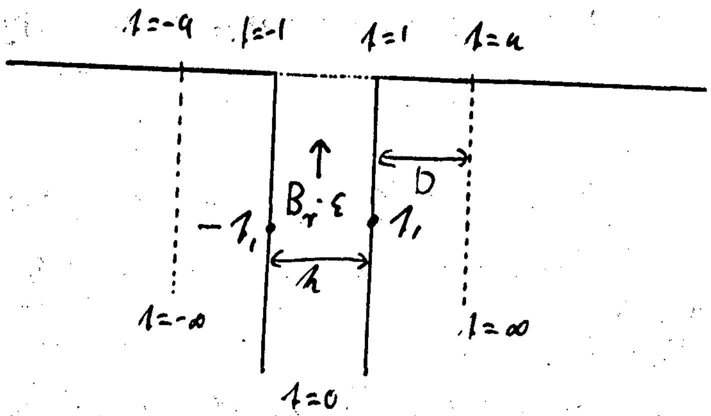
205

(15.1)

Homework: Q_{Dir} from thin gap between CSEM and pole: on one side; the other side; unequal gaps on both sides; thin gap between CSEM and CSEM along vertical center line; Q_{Dir} because of 2 blocks of CSEM of unequal strength to right + left of vertical symmetry line

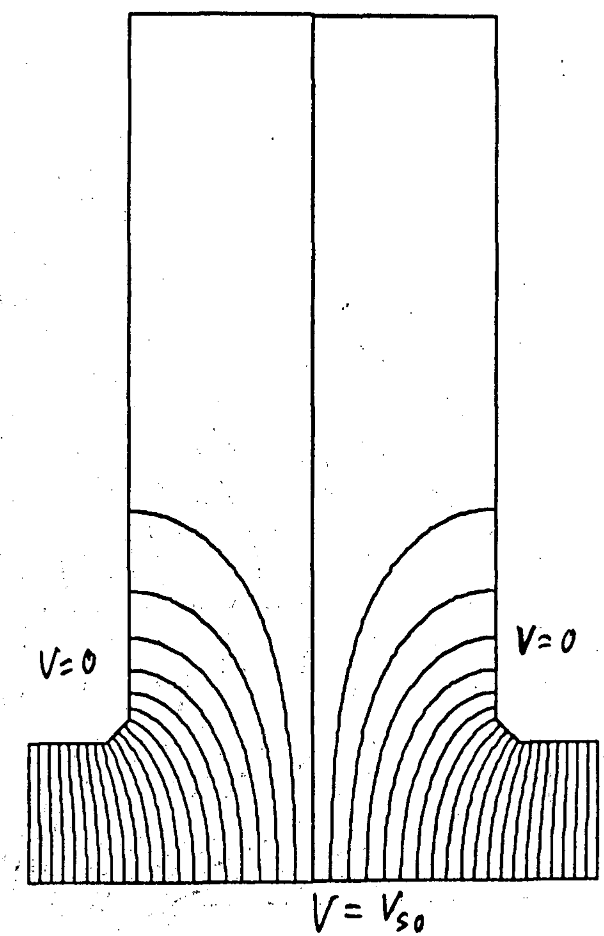
---

2D Device to measure easy axis orientation error.



Derive expressin to calculate flux passing between ⅂ Γ corners of μ=∞ iron at upper edge of CSEM.

(15.2)



$$Q_{Dir} = \int \vec{B}_r \cdot \vec{H}_s \, dv / V_{s0} W_3$$

(15.3)

Q_{Dir} from gap D between CSEM and pole.

$$Q_{Dir} = \int \vec{B_r} \cdot \vec{H_s} \, dv / V_{s0} W_3$$

$$Q_{Dir} = B_r D \int B_r \, dy / V_{s0} \doteq B_r \cdot D \cdot \Delta A / V_{s0}$$

(ΔA: top-bottom difference)

$\vec{H_s}$ - sign on right pole opposite to sign on left pole $\rightarrow Q_{Dir} = B_r \cdot \Delta D \cdot \Delta A / V_{s0}$

(ΔD: right-left gap difference)

Thin gap between 2 CSEM blocks along vertical center line: $\vec{B_r} \cdot \vec{H_s} = 0 \rightarrow Q_{Dir} = 0$

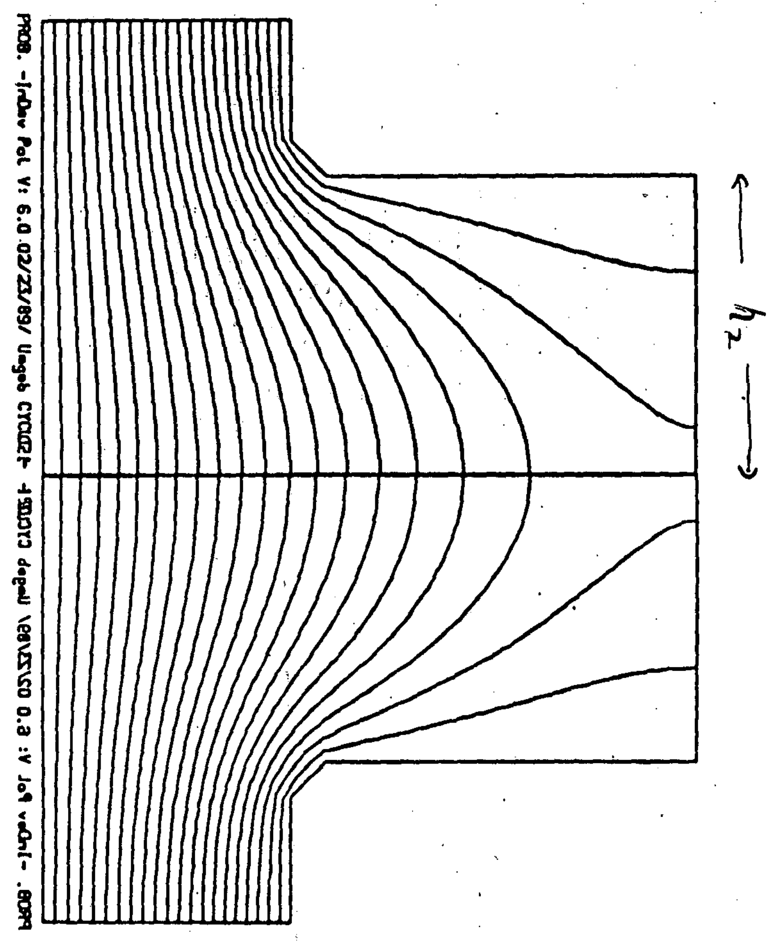
---

2 CSEM of different strength to right and left of vertical center line: charge density "ΔB_r" along vertical center line $\rightarrow Q_{Dir} = \Delta B_r \int_{y_{\text{Bottom of CSEM}}} V(y) \, dy / V_{s0}$

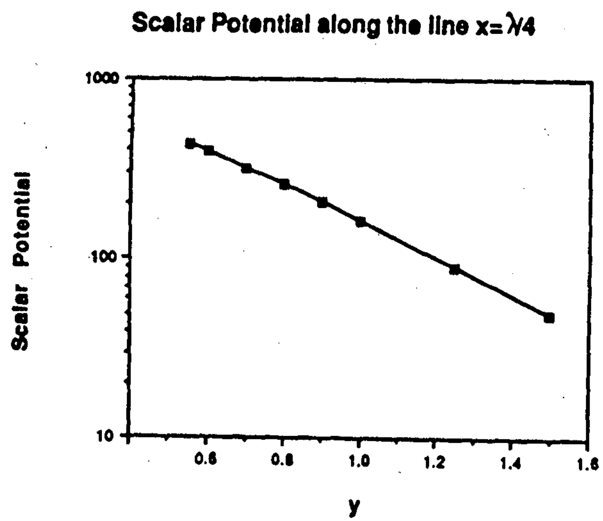
$V(y) \sim \exp(-\pi y / 2h_z)$ + odd harmonics

$$\rightarrow Q_{Dir} \approx \Delta B_r \cdot \frac{2h_z}{\pi} \cdot V_{\text{Bottom of CSEM}} / V_{s0}$$

(15.4)



(15.5)



(15.6)

Flux passing between ⌝ ⌜ corners

Represent CSEM by $\tilde{\eta}' = \varepsilon B_r$

$$\pi \dot{j} = -i \frac{\lambda}{a} \frac{\sqrt{\lambda^2-1}\sqrt{\lambda^2-a^2}}{\lambda}.$$

F from $+\tilde{\eta}$ at $\lambda = \lambda_1$, $-\tilde{\eta}$ at $\lambda = -\lambda_1$:

$$F = -\frac{\tilde{\eta}}{\pi} \ln \frac{\lambda - \lambda_1}{\lambda + \lambda_1}$$

$$\Delta A = A(1) - A(-1) = -\frac{\tilde{\eta}}{\pi} \ln \frac{1-\lambda_1}{1+\lambda_1} \cdot \frac{-1+\lambda_1}{-1-\lambda_1} = \frac{2\tilde{\eta}}{\pi} \ln \frac{1+\lambda_1}{1-\lambda_1}$$

2 current sheets:

$$\Delta A = \varepsilon B_r \cdot \frac{2}{\pi} \cdot \int_0^1 \ln \frac{1+\lambda_1}{1-\lambda_1} \, |\dot{j}(\lambda_1)| \, d\lambda_1$$

Drop subscript 1:

$$\Delta A = \varepsilon B_r \cdot h \cdot \frac{2}{\pi^2} \cdot \int_0^1 \ln \frac{1+\lambda}{1-\lambda} \cdot \frac{\sqrt{1-\lambda^2}\sqrt{1-\lambda^2/a^2}}{\lambda} \, d\lambda$$

Can show, with some effort (+ experience!):

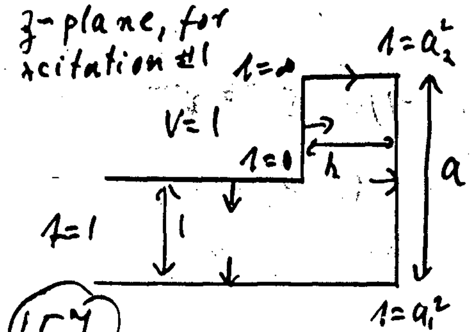
for $a = \infty$, $\int_0^1 = \pi\left(\frac{\pi}{2} - 1\right) = \frac{\pi^2}{2}\left(1 - 2/\pi\right)$

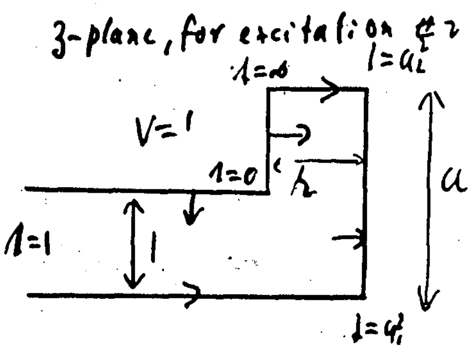
$$\Delta A = \varepsilon B_r h \left(1 - 2/\pi\right).$$

(15.7)

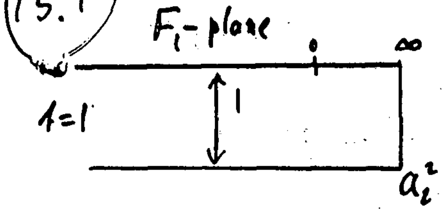
Also, with some effort (+stamina)

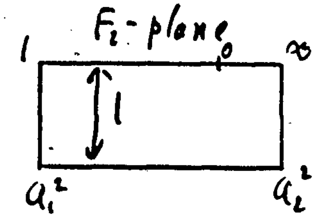
$$a = 1 + 2D/h + \sqrt{(1+2D/h)^2 - 1}$$





(15.7)





$$\pi \dot{F}_1 = i \frac{b_2}{(t-1)\sqrt{t-a_2^2}} \qquad \dot{F}_2 = \frac{i\,c}{\sqrt{t-1}\sqrt{t-a_1^2}\sqrt{t-a_2^2}}$$

$$b_1^2 = a_1^2 - 1\,; \quad b_2^2 = a_2^2 - 1\,; \quad b^2 = a^2 - 1$$

$$\pi \dot{z} = -\frac{\sqrt{t}\, b_1 b_2}{(t-1)\sqrt{t-a_1^2}\sqrt{t-a_2^2}} \qquad 0 < 1 < a_1 < a_2 < \infty$$

$$F_1' = -i\sqrt{1 - a_1^2/t}\,/b_1 \qquad F_2' = -\frac{i\pi c}{b_1 b_2}\sqrt{1 - 1/t}$$

$$\pi a = b_1 b_2 \cdot \int_{a_1^2}^{a_2^2} \frac{\sqrt{t}\, dt}{(t-1)\sqrt{t-a_1^2}\sqrt{a_2^2 - t}}$$

$$\pi h = b_1 b_2 \int_{a_2^2}^{\infty} \frac{\sqrt{t}\, dt}{(t-1)\sqrt{t-a_1^2}\sqrt{t-a_2^2}}$$

15.8

Have to do the following:

1) Use equations for h, a, to determine a_1, a_2. To do that, have to
   1.1) Describe secant equation solver for >1 dimension
   1.2) Describe method to remove singularities from limit(s) of integrand.
   1.3) Prove $a_1 < a < a_2$
   1.4) Use 1.3) to introduce hard, smooth range restrictions on a_1, a_2, to insure convergence of equ. solver
2) Integrate $\dot{F}_1$ to get flux into pole
3) Derive formula to get excess flux into side of pole for excitation #1
4) Derive flux into midplane for excitation #2
5) Develop procedure to get harmonics for excitation #1
6) $D_t = V_0/B_0 = |F_1'|_{t=a_2^2}$ = done. (Don't forget to de-normalize!!)

15.9

Flux into poleface, and harmonic coefficients.

A_{pole}

$$\pi F_1(t) = \int \frac{i b_2\, dt}{(t-1)\sqrt{t-a_2^2}} = \int \frac{2 i b_2\, dW}{W^2 + b_2^2}$$

$\sqrt{t-a_2^2} = W$; $t = W^2 + a_2^2$; $dt = 2W\,dW$

$$\pi F_1(t) = \int \left(\frac{1}{W-ib_2} - \frac{1}{W+ib_2}\right) dW = \ln \frac{\sqrt{t-a_2^2} - ib_2}{\sqrt{t-a_2^2} + ib_2}$$

$$\pi\left(F_1(\infty) - F_1(0)\right) = \pi A_{pole} = \ln \frac{a_2 + \sqrt{a_2^2-1}}{a_2 - \sqrt{a_2^2-1}}$$

$$A_{pole} = \frac{2}{\pi} \cdot \ln\left(a_2 + \sqrt{a_2^2-1}\right)$$

Harmonics

Need aliasing theorem:

$f(\varphi)$ = periodic with period 2π

$f(\varphi) = \sum_{-\infty}^{\infty} a_n e^{in\varphi}$; a_n = exact coeff.

Knowing $f(\mu\varepsilon)$ for $\mu = 0, 1 \cdots M-1, M$; $\varepsilon = 2\pi/M$ (↗ integer)

$$\sum_{\mu=0}^{M-1} f(\mu\varepsilon)\, e^{-im\mu\varepsilon} = \sum_{n;\,\mu=0}^{M-1} a_n\, e^{i\mu\varepsilon(n-m)}$$

(15.10)

With ν = integer

$$\sum_{\mu=0}^{M-1} e^{i\mu\varepsilon(n-m)} = M \quad \text{for } n = m + \nu \cdot M$$
$$= 0 \quad \text{for } n \neq m + \nu \cdot M$$

$$\frac{1}{M}\sum_{\mu=0}^{M-1} f(\mu\varepsilon)\, e^{-im\mu\varepsilon} = (a_m)_{comp} = \sum_\nu a_{m+\nu \cdot M}$$

$$(a_m)_{comp} = a_m + a_{m-M} + a_{m+M} + a_{m-2M} + \dots$$

"contamination"

In case of interest here, know $A(t)$ in midplane as function of t:

$$\pi A(t) = \pi\left(F_1(a_2^2) - F_1(t)\right) = \ln \frac{\sqrt{a_2^2 - t} - b_2}{\sqrt{a_2^2 - t} + b_2}\Bigg|_t^{a_2^2}$$

$$\pi A(t) = \ln \frac{b_2 + \sqrt{a_2^2 - t}}{b_2 - \sqrt{a_2^2 - t}}$$

Simpler
$$F_1' = -i\sqrt{1 - a_1^2/t}\,/\,b_1$$

Need to find t for a sufficiently large number of equidistant locations in midplane.

(15.11)

$$\sum_{\mu=0}^{M-1} x^\mu = \frac{1 - x^M}{1 - x}$$

$$x = e^{i\varepsilon(n-m)}$$

$$x^M = e^{iM\varepsilon(n-m)}$$

$$\varepsilon = 2\pi/M$$

$$x^M = e^{i2\pi(n-m)}$$

(15.12)

If f is known for M_1 equidistant locations in $\lambda/4$-section (requiring M_1-1 determinations by computation), the contamination of lowest order that contaminates harmonic m will be

(count only one end o)

$$m_{cont.} = 4M_1 - m.$$

Requiring (we are dealing with odd m only) $m_{cont} \geq m+2$ requires

$$M_1 \geq (m+1)/2$$

When executing harmonic analysis on less than one complete period, use first and last point only with ½ weight.

(15.13)

$\int B_1 B_2 \, dx$ for flat pole face.

(B_1 for $V_{pole} = 1$)

$$\left| F_1' F_2' \dot{z} \right|_{t \to -1} = \frac{\sqrt{1+a_1^2/\lambda}}{b_1} \cdot \frac{\pi C \sqrt{1+1/\lambda}}{b_1 b_2} \cdot \frac{\sqrt{\lambda} \cdot b_1 b_2}{(\lambda+1)\sqrt{\lambda+a_1^2}\sqrt{\lambda+a_2^2}}$$

$$= \frac{\pi C}{b_1} \cdot \frac{1}{\sqrt{\lambda}\sqrt{1+\lambda}\sqrt{\lambda+a_2^2}}$$

$$\int B_1 B_2 \, dx = \frac{\pi C}{b_1} \int_0^\infty \frac{d\lambda}{\sqrt{\lambda}\sqrt{\lambda+1}\sqrt{\lambda+a_2^2}}$$

$$\lambda = \operatorname{tg}^2\varphi; \quad d\lambda = 2 \operatorname{tg}\varphi \, d\varphi / \cos^2\varphi$$

$$\int B_1 B_2 \, dx = \frac{2\pi C}{b_1} \int_0^{\pi/2} \frac{d\varphi}{\sqrt{\sin^2\varphi + a_2^2 \cos^2\varphi}} = \frac{2\pi C}{b_1 a_2} K\left(1 - \frac{1}{a_2^2}\right)$$

$$a_2^2 c^2 + s^2 = a_2^2 - s^2(a_2^2 - 1)$$

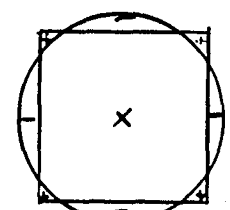
(15.14) Orthogonal Analog Model.

To make understanding easier: state model; use it; prove it; use it some more.

OAM	Magnet
ρ	μ
$-J_3/D$	J_3
V	A
$\vec{l}_3 \times \vec{J}$	$\vec{H}$
$\vec{l}_3 \times \vec{E}$	$\vec{B}$

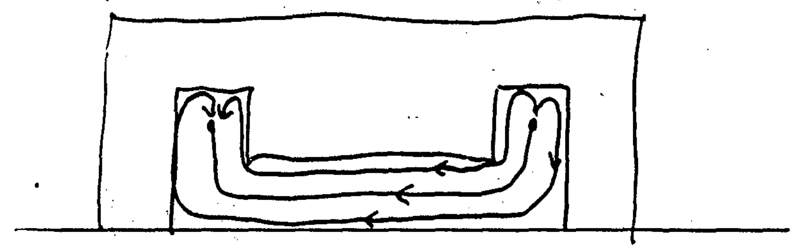
(15.15) Applications of 2D OAM.

1) Square conductor / round conductor.

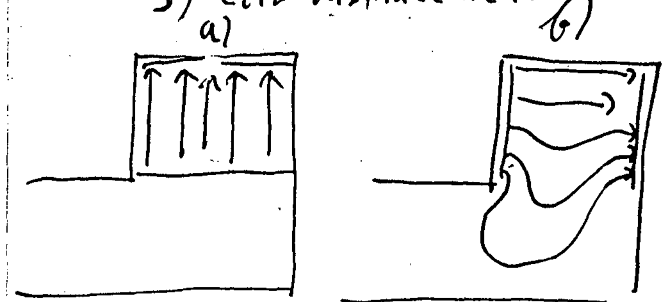


Sq. conductor = filament + currents between sq. conductor and circle of equal area.

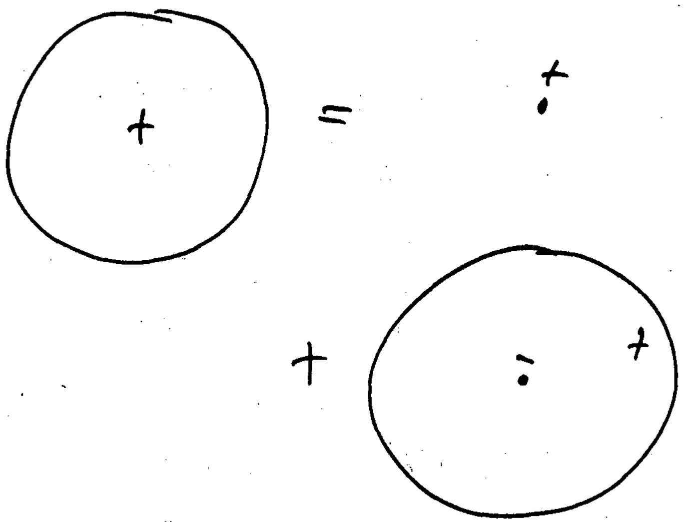
2) H-magnet



3) Coil displacement



15.16



15.17

Proof of OAM equivalences.

1) Magnet

$\mathrm{curl}\,\vec{A} = \vec{B}$; $\vec{A} = \vec{e}_z A$

$B_x = A'_y$; $B_y = -A'_x$

$\gamma = 1/\mu$

$H_x = \gamma A'_y$; $H_y = -\gamma A'_x$

$(\mathrm{curl}\,\vec{H})_z = -\frac{\partial}{\partial x}\gamma A'_x - \frac{\partial}{\partial y}\gamma A'_y = j_z$

2.) Conducting sheet of thickness D

$E_x = -V'_x$; $E_y = -V'_y$

$j_x = -\sigma V'_x$; $j_y = -\sigma V'_y$

$\mathrm{div}\,\vec{j} = \frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y} + \frac{\partial j_z}{\partial z} = 0$;

Integrate ↑ over z-thickness D :

$\frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y} = -j_z/D$ ← "injected" from 3. dim.

$-\frac{\partial}{\partial x}\sigma V'_x - \frac{\partial}{\partial y}\sigma V'_y = -j_z/D$

(15.18)

Comparisons:

$\vec{\sigma} \longleftrightarrow \gamma = 1/\mu$

$\rho \longleftrightarrow \mu$

$V \longleftrightarrow A$

$-j_z/D \longleftrightarrow j_z$

$\vec{e}_z \times \vec{E} = \vec{e}_x V'_y - \vec{e}_y V'_x \qquad \vec{e}_z \times \vec{E} \longleftrightarrow \vec{B}$

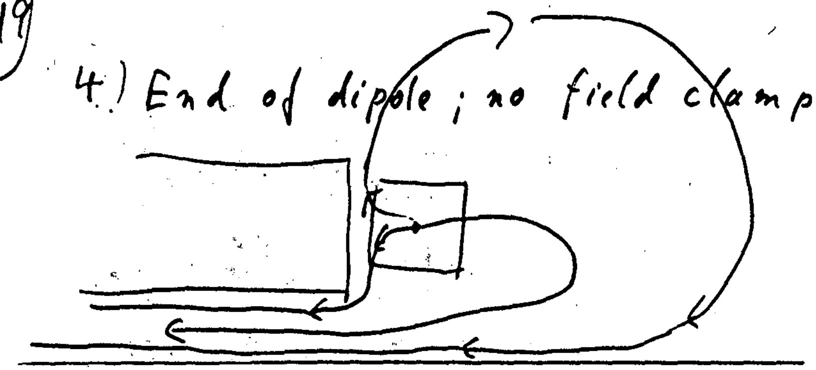
$\vec{e}_z \times \vec{j} \longleftrightarrow \vec{H}$

Generalization for cylindrical magnet.

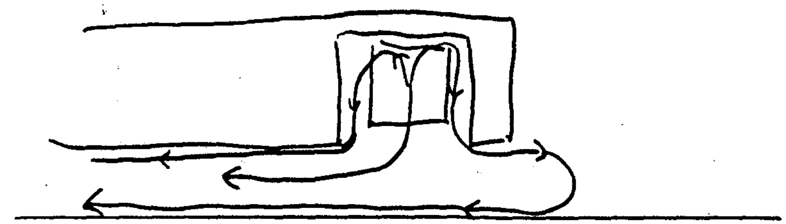
OAM	cyl. Magnet
ρ	$r\mu$
$-j_z/D$	j_z
V	rA
$\vec{e}_z \times \vec{j}$	$\vec{H}$
$\vec{e}_z \times \vec{E}$	$r\vec{B}$

(15.19)

4.) End of dipole; no field clamp.

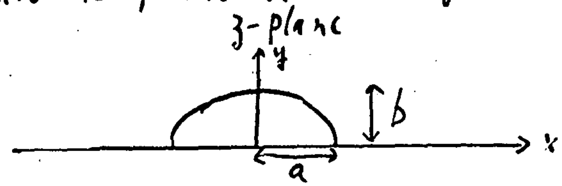


5) End of dipole; with field clamp

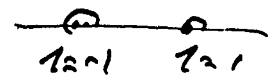


15.20

Map of ½ plane with elliptical "bump" onto ½ plane with straight boundary.



$z = a \cdot \cos\alpha + ib \sin\alpha$



$\rightarrow z = a \cdot w + b\sqrt{w^2-1}$

$w^2(a^2-b^2) - 2waz + z^2 + b^2 = 0$

$w = \frac{az}{a^2-b^2} \pm \frac{1}{a^2-b^2}\sqrt{a^2z^2+(z^2+b^2)(b^2-a^2)}$

$w = \frac{az - b\sqrt{z^2+b^2-a^2}}{a^2-b^2}$

– sign chosen so that for large z in upper ½-plane, $w = z/(a+b)$

Application: B-field ⊥ x-axis + ellipse:

$F(w) = C \cdot w$

$dF/dz = C \cdot \frac{a - zb/\sqrt{z^2+b^2-a^2}}{a^2-b^2}$

5.21

$B_\infty = C/(a+b)$

$F(z) = B_\infty \left(az - b\sqrt{z^2+b^2-a^2}\right)/(a-b)$

$F(a) = B_\infty (a+b)$; $F(a)/a = B_\infty (1+b/a)$

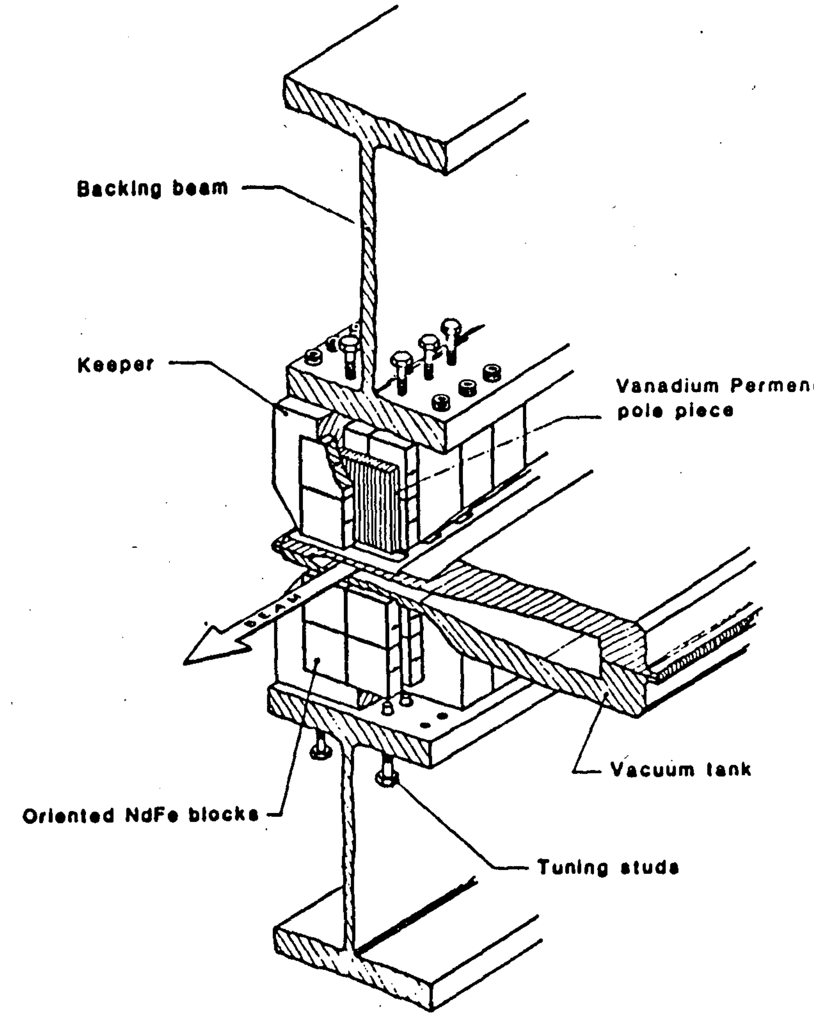
$F'(ib) = \frac{C}{a^2-b^2} \cdot \left(a - \frac{ib^2}{ia}\right) = B_\infty \cdot (1+b/a)$

For w = real ; $-1 \le w \le 1$:

$F(w) = B_\infty (a+b) \cdot x/a = B_\infty (1+b/a) \cdot x$

Insertion Device Design

Klaus Halbach



LIGHT SOURCE INSERTION DEVICE

Lecture 16. (Final)

March 17, 1989

Note: Next Lecture Series To Be Announced for Fall '89.
Klaus is currently soliciting suggestions.
(415) 486 - 5868

16.0

Lecture # 16

March 17, 1989

(Last Lecture)

Q_{Dir} from gap D between CSEM and pole

$$Q_{Dir} = B_r \cdot D \cdot \int B_x \, dy / V_{s0} = B_r \cdot D \cdot \Delta A / V_{s0}$$

↑ ΔA: top-bottom difference

↑ unreadable in xerox copies of lecture #15

16.1

Re-visit design of 2D non-dipole in dipole geometry.

Reason: I<u>n</u> case of really "exotic" magnets, there are pitfalls that one should be aware of.

Explain basic principles by discussing a specific class of magnets that I have worked on.

Memory refresher:

Assume that desired $B_z^*(z)$ = known.

Apply conformal map $z(w) \longleftrightarrow w(z)$ to geometry that shapes and produces fields (V=const., A=const. surfaces; currents, charges)

$$F(z) = F(z(w))$$

$$B_w^* = i \frac{dF}{dw} = B_z^* / w'$$

8/5

(16.2)

Map that gives for perfect desired B_z^* a perfect dipole: $B_w^* = -iB_0 = B_z^*/w'$

$\frac{dw}{dz} = i\,B_z^*(z)/B_0 = -F'(z)/B_0 \longrightarrow w(z) = -F(z)/B_0$

(↑ ideal, wanted B_z^*, $F(z)$)

Because of $B_z^* = w'\,B_w^* \rightarrow \Delta B_z^*/B_z^* = \Delta B_w^*/B_w$

→ relative field errors same in z and w plane.

Design procedure:

Map boundary of region not accessible to field shaping/producing from z to w; map also good field region from z to w.

Design dipole with sufficiently good field in w and map pole into z-plane.

(16.3)

Specific problem:

In region where e-beam is large in x-direction and small in y-direction, want a sextupole with a $B_y(x,0) = b_3 \cdot x^2$ (↑ real) for small x, but with a field that increases less rapidly for large x.

2 general properties of such fields:

Use theorem: if $G(z)$ = analytical within circle $|z| = |re^{i\varphi}| = r$, then

$$\int_0^{2\pi} G(re^{i\varphi})\,d\varphi/2\pi = G(0)$$

Apply that to $G(z) = \ln\left(B_z^*(z)/b_3 z^2\right)$:

$\int \ln(|B_z|)\,d\varphi/2\pi = \ln(b_3 r^2) \longrightarrow$ if $|B| < b_3 r^2$ on one part of circle, $|B| > b_3 r^2$ must be true on other parts of circle.

Also: $G(z) = B^*(z)/b_3 z^2 \rightarrow \int_0^{2\pi} \frac{B^*(z)}{b_3 z^2}\,d\varphi/2\pi = -i$

(16.4)

Some possible functions for $W'(z)$, and potential problems for these functions

$W'(\sim B_z^*)$	Problem
$z\tanh(kz)$	Poles at $kz = \pm i\cdot\frac{\pi}{2}$
$z^2/\cosh(kz)$	Poles at $kz = \pm i\pi/2$
$\sqrt{1+k^2z^2}-1$	Branch points at $kz=\pm i$
$z^2/\sqrt{1+k^2z^2}$	Singularities at $kz=\pm i$
$z^2 e^{-k^2z^2}$	No problems?
$1-\cos(kz)+\varepsilon(1-\cos(3kz))$	No problems?

$z^2 e^{-k^2z^2}$: $x\to\infty$ maps to $W=\sqrt{\pi}/4k^3+i\cdot 0$, giving problem similar to the problem encountered in:

$$W' = 1-\cos(kz)+\varepsilon\,(1-\cos(3kz)); \quad (\varepsilon\approx 0.01;\ k=.1\,mm^{-1})$$

(16.5)

Comment to functions $W'(z)$ that have a singularity on y axis: If the location of the singularity is sufficiently far from the boundary of the inaccessible region, there is nothing wrong with it. But the distance of that singularity from $z=0$ may be smaller than the x-coordinate of the extreme electrons. This means: aside from the fact that a multipole expansion of the fields may not be practical, it may be <u>impossible</u> to do it and use it to describe the fields for <u>all</u> locations where electrons are.

(16.8)

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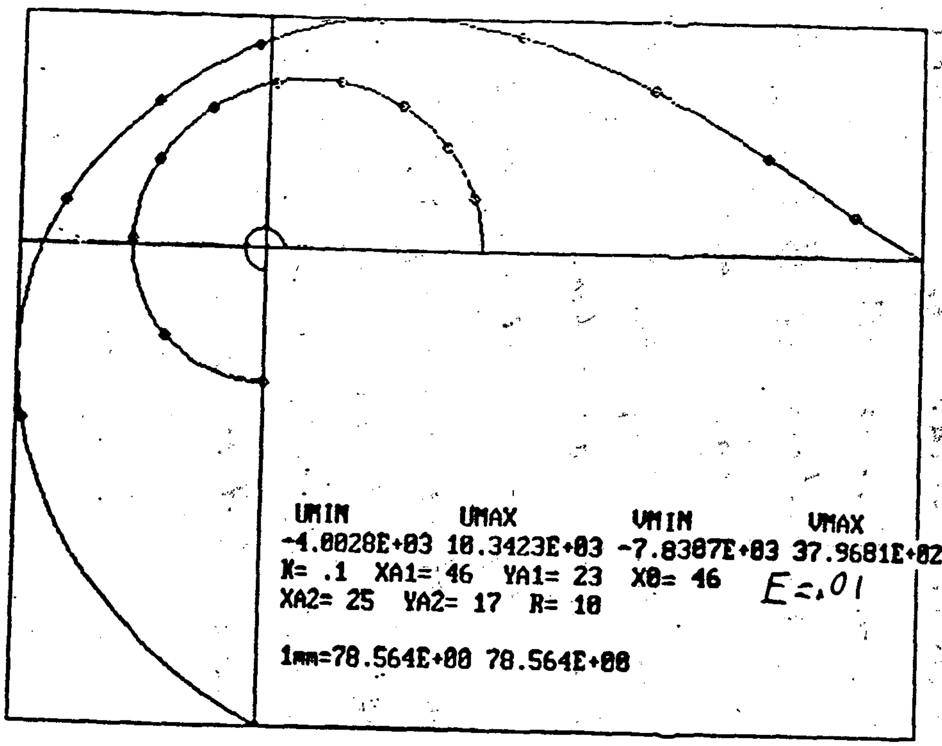
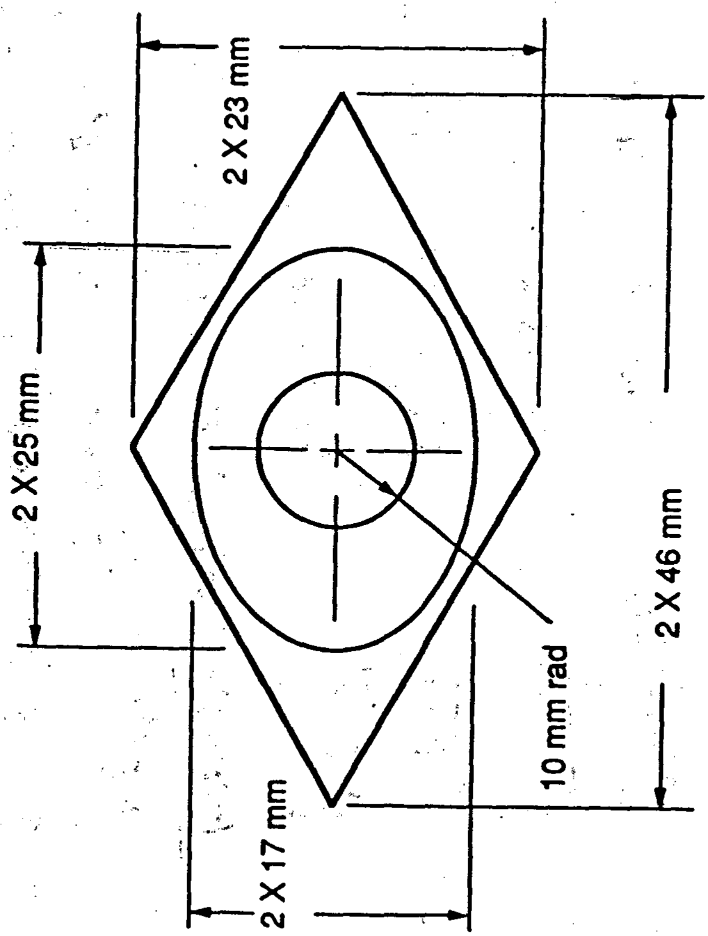


$W' = \text{const.} (1+\varepsilon - \cos 2kz - \varepsilon \cos 3kz)$

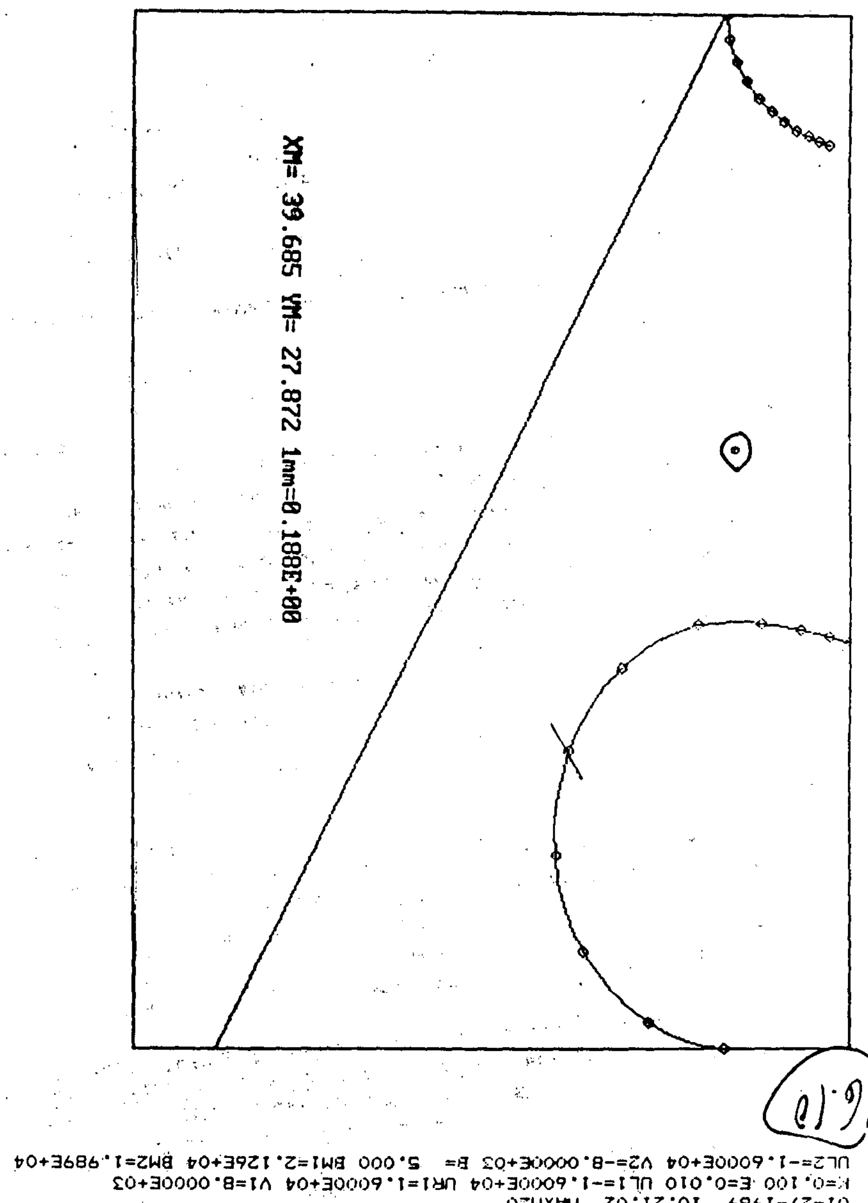
(16.7)



(16.9) Memory refresher: because of symmetry, $\vec{B} \perp$ u-axis for $u>0$ when field line "comes" from $v>0$. No such condition exist for field lines that cross u-axis for $u>0$ from $v<0$-region. To avoid infinitely large fields from this "knife-edge" boundary condition at $w=0$ (leading to small dipole field in z-plane at $z=c$), it is desirable to have 2 perfectly symmetric poles in w-geometry, i.e. poles with equal ½ gaps, and widths.

→ Do it → map wide dipole-poles into z-plane; to be on safe side, evaluate with POISSON
→ fields in error by 10–20% !!!

The "only" conceivable reason: $w'(z)=0$ somewhere in region of interest → map not conformal at that point.



(16.10)

(16.11)

$W'(z) = 1-\cos(kz) + \varepsilon\,(1-\cos(3kz)) = 0$ does not seem reasonable in region of interest.

Argum. principle

Check with "argument principle": "Investigate", with $G(z) = |G| \cdot e^{i\psi}$

$$J = \oint (\ln G)'\, dz = \ln|G| + i\psi \Big|_{z_{beg}}^{z_{end}} = i\Delta\psi$$

Obtain $\Delta\psi$ by actually mapping z-contour into G-plane; or, going along contour in small increments, and adding all incremental changes in ψ to get $\Delta\psi$. Assume that $G(z)$ has some poles and zeroes at locations z_m, with $G(z)$ behaviour in vicinity of z_m describable by

$G(z) = g_m(z) \cdot (z - z_m)^{N_m}$

g_m = analytical;

$N_m \gtrless 0$ = multiplicity of Zero / Pole.

16.12

LAWRENCE BERKELEY LABORATORY – UNIVERSITY OF CALIFORNIA ENGINEERING NOTE	CODE AA0123	SERIAL M6239	PAGE 1 of 2
AUTHOR Klaus Halbach	DEPARTMENT MECHANICAL	LOCATION BERKELEY	DATE 4/23/84

PROGRAM – PROJECT – JOB: MECHANICAL ENGINEERING - GENERAL

MATHEMATICS, DERIVATIONS, COMPUTERS, PROGRAMMING, ETC.

TITLE: CALCULATION OF INVERSE TRIGONOMETRIC FUNCTIONS ON THE IBM PC OR OTHER MICROCOMPUTERS

It has recently come to my attention that the BASIC that comes with the ubiquitous IBM PC (and many other microcomputers) provides the user with only one inverse trigonometric function, namely the arctangent. Furthermore, it is recommended to use the following algorithms ((3), (4)) when only the value S of the sin function, or only the value C of the cos function, is known.

```
(1)  S = SIN(A)
(2)  C = COS(A)
(3)  A = ATN(S/SQR(1-S*S))
(4)  A = 1.570796 - ATN(C/SQR(1-C*C))
```

Both (3) and (4) have the problem that an overflow can occur. It would be folly to assume that this will not happen: depending on what type of calculation one is doing, it is not at all unlikely that the angle A is exactly ± PI/2 or 0 or PI. Since it is a rather poor programming practice to enter a number like PI/2 in digital form, I assume below that the computer has PI, or it has been established by executing early in the program

```
(5)  PI = 4*ATN(1).
```

The following algorithms avoid the overflow problems:

```
(6)  A = 2*ATN(S/(1+SQR(1-S*S)))

(7)  A = PI/2-2*ATN(C/(1+SQR(1-C*C))) .
```

(6) returns - PI/2<A<PI/2, and (7) returns 0<A<PI.

7600-54250

(6.13)

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AUTHOR Klaus Halbach / DEPARTMENT MECHANICAL	LOCATION BERKELEY	DATE 4/23/84	
PROGRAM – PROJECT – JOB			
TITLE CALCULATION OF INVERSE TRIGONOMETRIC FUNCTIONS ON THE IBM PC OR OTHER MICROCOMPUTERS			

Closely related to this is the problem of conversion from Cartesian to polar coordinates.

If X and Y are known,

(8) X = R*COS(A)

(9) Y = R*SIN(A),

R and A are obtained from

(10) R = SQR(X*X+Y*Y)

(11) A = SG(Y)*(PI/2-2*ATN(X/(R+ABS(Y))))

This formula returns A in the correct quadrant, i.e., -PI≤A≤PI.

The signum function in (11) is defined as 1 for Y>0, -1 for Y<0, and 1, or -1, but not 0, for Y = 0. Unfortunately, the signum function SGN(Y) usually supplied has the values ∓1 for Y <> 0, but 0 for Y = 0. However, SG(Y) is easily "constructed" from SGN(Y):

(12) SG(Y) = 1+SGN(Y)*(1-SGN(Y))

If the true/false statment (Y < 0) can be used (as is legal on the IBM PC and many other microcomputers), one can use

(13) SG(Y) = 1-2*(Y<0),

and gets as a slightly better form of (11)

(14) A = (.5-(Y<0))*(PI-4*ATN(X/(R+ABS(Y))))

If one knows S and C, one replaces in (11) or (14) X by C, Y by S, and R by 1.

600-54250

(6.14)

In vicinity of $z = z_m$

$$(\ln G)' = \frac{g_m'}{g_m} + \frac{N_m}{z - z_m}$$

$$\longrightarrow J = 2\pi i (Z - P) = i\,\Delta\varphi$$

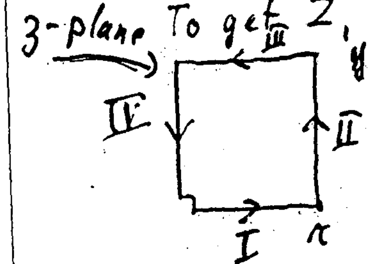
$$Z - P = \Delta\varphi / 2\pi$$

argum. princ.

application to our problem.

For our function, $P = 0$

To get Z, look first at map of "Large" rectangle in z plane, mapped with $g_1 = 1 - \cos z$



Map of line I: $g_1(x) = 1 - \cos x$; $g_1(\pi) = 2$

Map of line II: $g_1(\pi + iy) = 1 + \cosh(y)$

choose y_0 so that $\cosh(y_0) \gg 1$;

$g_1(\pi + iy_0) = 1 + \cosh(y_0)$

Map of line III: use $z = \pi + iy_0 - \Delta x$, Δx increasing from 0 to π

(16.15)

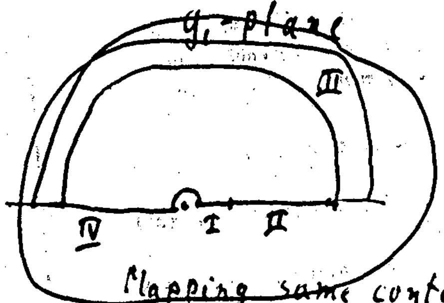
$g_1(\pi - \Delta x + iy_0) = 1 + \cos(\Delta x - iy_0)$

$g_1(\pi - \Delta x + iy_0) = 1 + \cos\Delta x \cdot \cosh y_0 + i \sin\Delta x \cdot \sinh y_0$

$g_1(iy_0) = 1 - \cosh y_0$

Since $\sinh y_0 \gg 1$, when Δx goes from 0 to π, g_1 describes a nearly circular 1/2-ellipse.

Map of line IV: g_1 goes back to very close to origin.



Mapping same contour with $G(z) = g_1(z) + \varepsilon g_3(z)$;

$g_3(z) = 1 - \cos 3z$; $\varepsilon \simeq .01 (\ll 1)$

Behaviour of map of line I dominated by g_1 because $\varepsilon \ll 1$. If y_0 = large enough, at the end of line II, and along line III, εg_3 dominates. Map of line III

(16.16)

therefore is 1+1/2 "circle" of very large radius. Going along line IV gives in G-plane again a straight line toward origin of G; because of 1/4 circle around z-origin, G-origin is outside region when z goes back to starting point.

Conclusion: if y_0 is large enough, $G(z)$ has exactly one (and not more) zero in region of interest.

For $W'(z) = 1 - \cos z + \varepsilon(1 - \cos 3z)$,

$W' = 0$ for $z = 1.6686 + i \cdot 2.3171$,

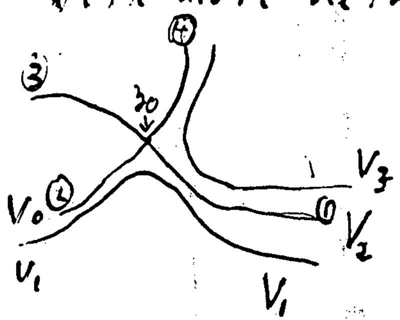
and the map of this point is $W = -3204 + i\,4270$

16.17

To understand problem, go back to meaning of $W'(z)$ as const x $B_z^*(z)$:

If $W'(z)$ has a (single) zero at $z = z_0$, field there will be $= 0$, i.e. field in vicinity of z_0 must be a <u>quadrupole</u> field. If pole on fixed V is "outside" that point, $B_z^*(z_0)$ is obviously not zero → rest of fields can not be correct either.

With more detail: Obviously, if



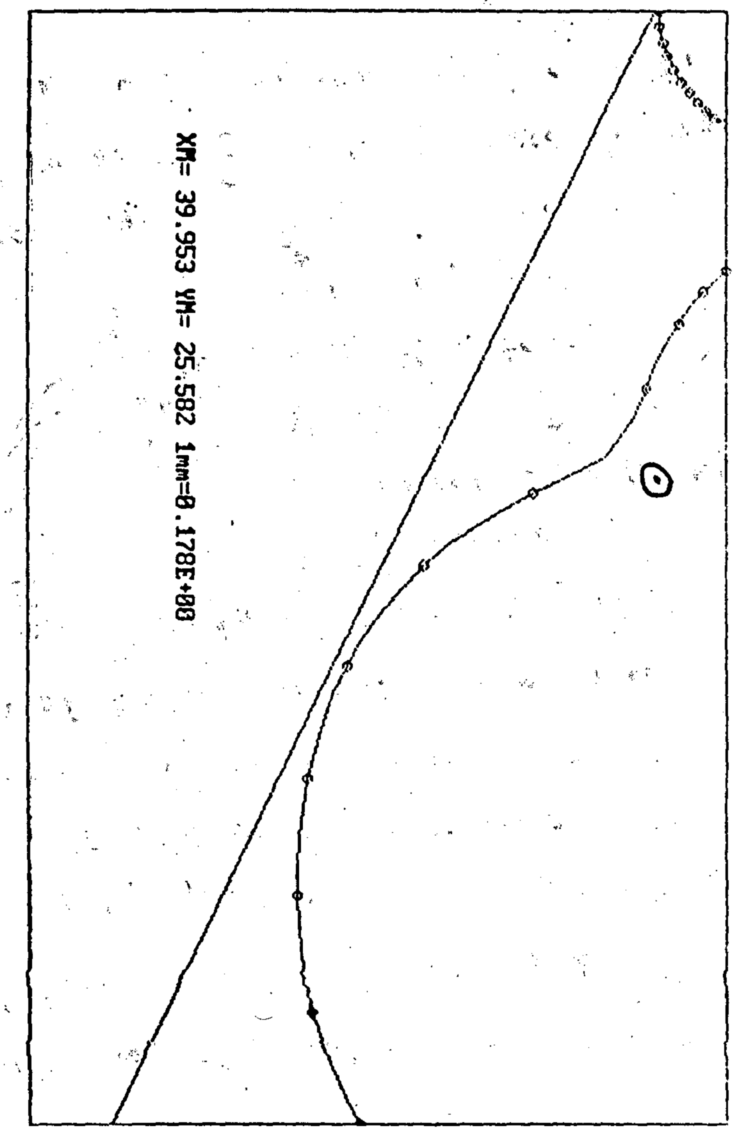
$V = V_3$ - surface is implemented, $B^*(z_0) \neq 0$. $V = V_2$ from ① to $z = z_0$ and then to ③ or ④ will not produce in "business" region $B^*(z_0) = 0$, but $V = V_2$ from ① to z_0 to ② will produce desired field, as will any other surface with $V = V_1 < V_2$

16.18

Remedy: use pole that is on lower potential than the pole that is bisected by y-axis → feasable in this case → works. Not the most desirable outcome, because this solution is less symmetric than solution with poles on $\pm V_4$. There are several ways to "fix" dipole field if it is bothersome.

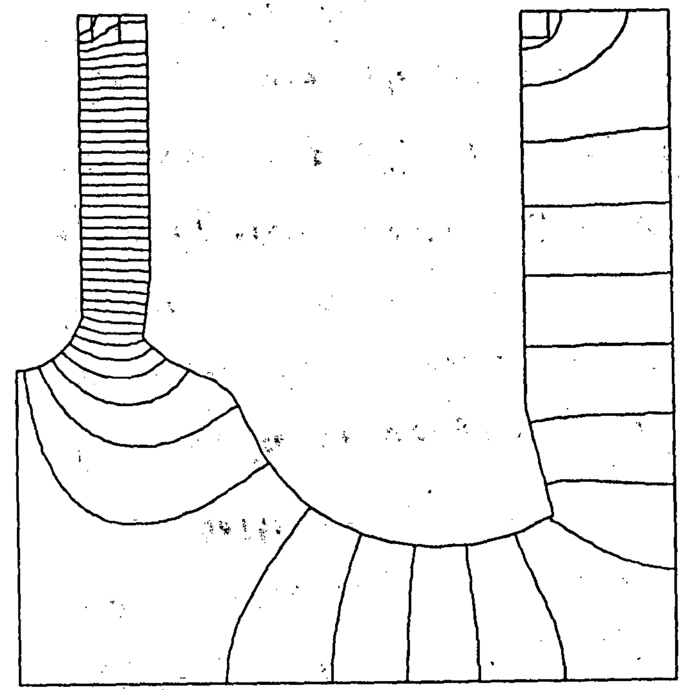
It can also be worse: z_0 still outside boundary of inaccessible region, but too close to mid plane to allow a pole of sufficient width. Remedy: split pole into 2 parts. From OAM: right edge of lower pole should be "sharp". If z_0 inside inaccessible reason, I do not see a solution → different parameters or function.

(16.19)



(16.18)

(16.19)



PROB. -Exotic Sextupole 2-13-89 Singula CYCLE - 940

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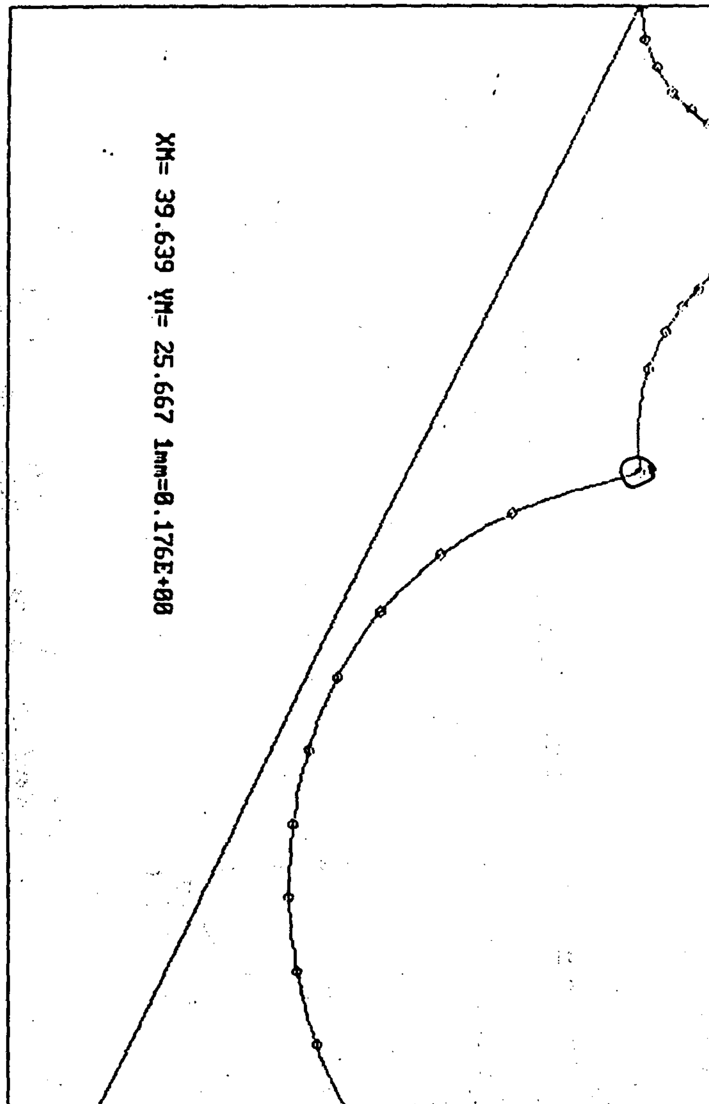
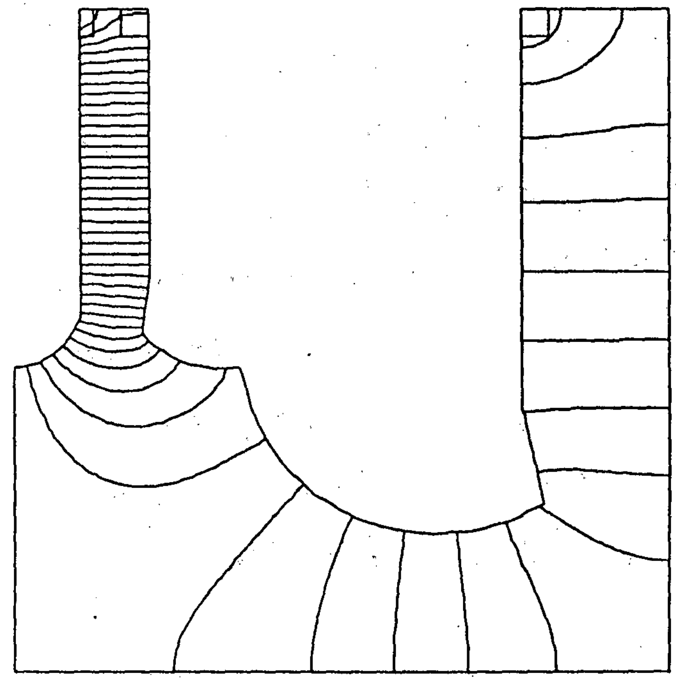
16.20



16.21

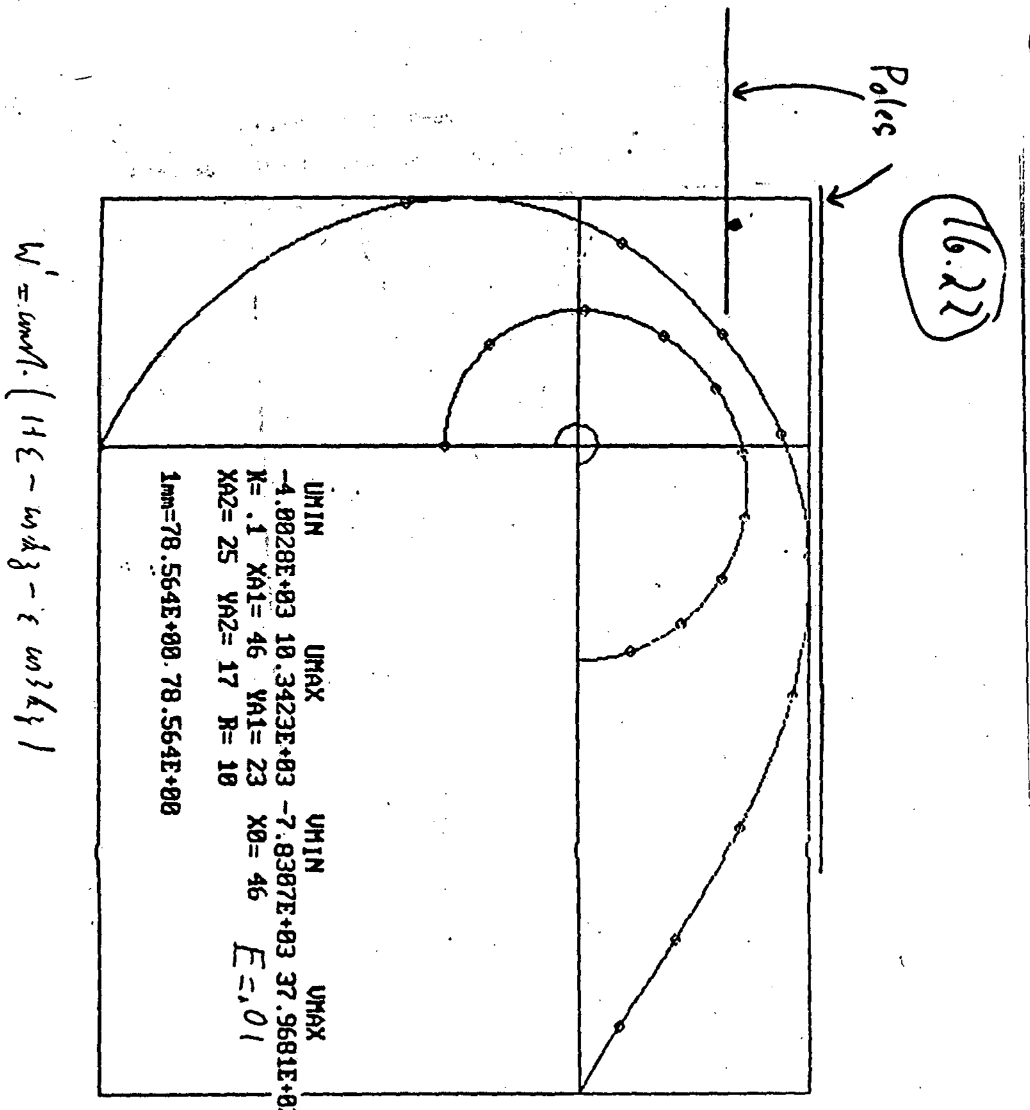


PROB. -Exotic Sextupole 02-14-89 Singul CYCLE - 930

16.23

Evaluation of $J = \int_0^1 \ln\frac{1+\lambda}{1-\lambda}\cdot\frac{\sqrt{1-\lambda^2}}{\lambda}\,d\lambda$

Integral not important enough to justify effort, but many aspects of methodology used to evaluate J are of great general importance.

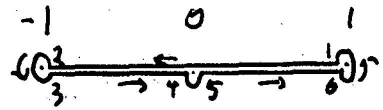


General approach to such problems: use Cauchy's integral- and residue-theorem

→ closed contour integral, with at least part of path making a contributio[n] to contour integral that is proportional to J. Then: deform path and calculate integral in a different way.

Path: no general rule, but usually "obvious" when one studies integrand.

(16.24)

Path: 

$$I = \oint_{1-2-3-4-5-6-1} \ln\frac{1+\lambda}{1-\lambda}\,\frac{\sqrt{1-\lambda^2}}{\lambda}\,d\lambda$$

Since integrand contains functions that are multiple-valued, must make integrand unique by defining precisely sign, or whatever is non-unique, of every function contained in integrand.

Here, define: on 1-2, $\sqrt{1-\lambda^2} > 0$, $\mathrm{Im}\left(\ln\frac{1+\lambda}{1-\lambda}\right) = 0$.

$\int_1^2$: clearly, integrand is even function of λ; so: $\int_1^2 = -2J$.

$\int_2^3$: since $\left(x^n \ln x\right)_{x\to 0} = 0$ for $n > 0$,

$\to \int_2^3 = 0$.

(16.25)

But: on 2-3, $\ln\frac{1+\lambda}{1-\lambda}$, $\sqrt{1-\lambda^2}$ change because of branch points at $\lambda = -1$! ! !

$\lambda = -1 + \rho\cdot e^{i\varphi}$

$\left(\sqrt{1+\lambda}\right)_3 = -\left(\sqrt{1+\lambda}\right)_2$

$\left(\ln(1+\lambda)\right)_3 = \left(\ln(1+\lambda)\right)_2 + 2\pi i$

Consequence of ↑ :

Integrand has singularity at $\lambda = 0$ on path "below" real λ-axis!! Go around that singularity in ½ circle centered at $\lambda = 0$

$$\int_3^6 = -2J - 2\pi i \underbrace{\int \frac{\sqrt{1-\lambda^2}}{\lambda}\,d\lambda}$$

$\sqrt{1-\lambda^2}/\lambda$ = odd function of λ for λ = real → only contribution ≠ from $\underbrace{\int}$ from ½ circle →

(16.26)

$$\int_{\gamma_6} \frac{\sqrt{1-\lambda^2}}{\lambda}\, d\lambda = \pi i \,.$$

$$\int_3^6 = -2J + 2\pi^2 \,.$$

$$I = -4J + 2\pi^2 \quad \blacksquare$$

Now: deform path to circle with radius $\to \infty$.

Have to be careful, again, what $\sqrt{1-\lambda^2}/\lambda$, $\ln\frac{1+\lambda}{1-\lambda}$ mean on that large circle.

$\sqrt{1-\lambda^2}/\lambda$: $\lambda = ia$. Let a (real) grow from small to large values:

$$\sqrt{1-\lambda^2}/\lambda = \sqrt{1+a^2}/ia = -i\sqrt{1+1/a^2} = -i\sqrt{1-1/\lambda^2},$$

with $\sqrt{1-1/\lambda^2} > 0$ for $|\lambda| \gtrsim 1$, $\Im\lambda = 0$.

$$\ln\frac{1+\lambda}{1-\lambda} = \underbrace{\ln\frac{1+1/\lambda}{1-1/\lambda}}_{\text{real for } \lambda \gtrsim 1,\ \Im\lambda = 0} + c$$

(16.27)

$$I = -i\oint\left(\ln\frac{1+1/\lambda}{1-1/\lambda} + c\right)\cdot\sqrt{1-1/\lambda^2}\, d\lambda$$

Expand in $1/\lambda$, apply residue theorem

$$\ln\frac{1+1/\lambda}{1-1/\lambda} = 2\left(\frac{1}{\lambda} + \frac{1/3}{\lambda^3} + \cdots\right)$$

$$I = -i\cdot 4\pi i = 4\pi = -4J + 2\pi^2$$

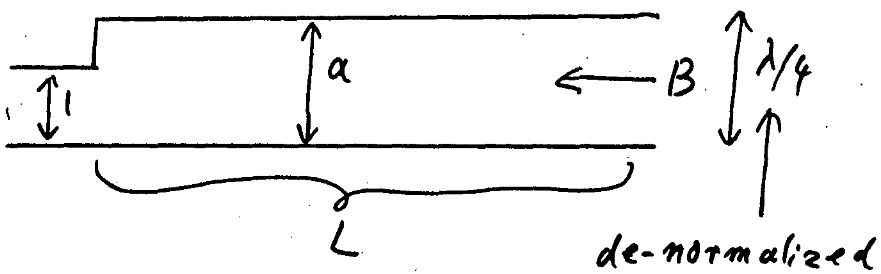
$$J = \int_0^1 \ln\frac{1+\lambda}{1-\lambda}\cdot\frac{\sqrt{1-\lambda^2}}{\lambda}\, d\lambda = \pi\left(\frac{\pi}{2} - 1\right) \quad \blacksquare$$

To do $\int_0^1 \ln\frac{1+\lambda}{1-\lambda}\cdot\frac{\sqrt{1-\lambda^2}}{\lambda}\sqrt{1-\lambda^2/a_1^2}\, d\lambda$:

Closed expression probably not possible, but expansion in $1/a_1^2$ and term by term integration with same technique leads to good results with only first few terms. Notice: $1/a_1^2$- independant term is the only one that leads to pole at $\lambda = 0$ on path 3-6.

(16.28)

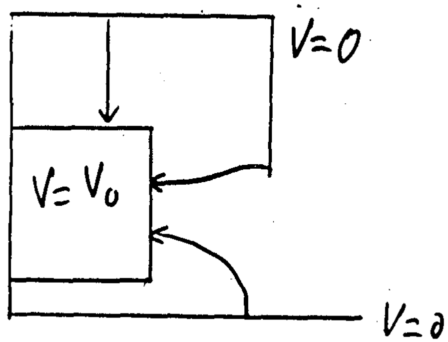
Correction of C_{oB} for excess V- drop.



$$V = BL + \Delta V$$

$$\Delta V = \phi \cdot K(a) = B \cdot \frac{\lambda}{4} K(a) \ ;$$

$$K(a) = \left((a+1)\ln(a+1) + (a-1)\ln(a-1) - 2a \ln a \right) / \pi a$$



$$V_0 = B\left(L + \frac{\lambda}{4} k\right)$$

$$V_0 = B_0 \cdot L$$

(16.29)

Use known V_0, B_0 to get, for general case, "equivalent" L:

$$B = \frac{V_0}{V_0/B_0 + \frac{\lambda}{4} k} = \frac{B_0}{1 + B_0/B_1}$$

$$B_1 = \frac{V_0}{\frac{\lambda}{4} \cdot K(a)}$$

To get "real" flux into side and top, integrate B over surface. Correction small from part of contour where $B_0 \ll B_1$.

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