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Pure Salvetti complexes and Euclidean Artin groups

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Pure Salvetti complexes and Euclidean Artin groups

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by

Gordon Rojas Kirby

Committee in charge:

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Gordon Rojas Kirby

Dedicado con mucho amor y respeto a mi abuelo, Ernest Rojas.

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Matthias Beck, Alyssa Cuyjet, Gordon Rojas Kirby, Molly Stubblefield, and Michael Young. *Nowhere-zero* **k**-*flows on graphs*, Annals of Combinatorics **18** no. 4 (2014) 579-583.

Fields of Study

Geometric group theory, combinatorics, Coxeter groups, and Artin groups.

Abstract

Pure Salvetti complexes and Euclidean Artin groups

by

Gordon Rojas Kirby

Coxeter groups are a general family of groups that contain the isometry groups of the Platonic solids and the symmetry groups of regular Euclidean tilings. These groups are ubiquitous and well-understood. They are also closely linked to the lesser-known braided versions known as Artin groups. In this dissertation, I investigate the word problem for Artin groups corresponding to Coxeter groups that act naturally on Euclidean space. The corresponding Artin group is the fundamental group of the quotient of the complexified Euclidean space after removing the fixed hyperplanes of the reflections in the Coxeter group.

To understand a Euclidean Artin group, I focus on the structure of the infinite sheeted cover corresponding to the kernel of the homomorphism from the Artin group to the Coxeter group. This space deformation retracts onto a complex constructed as an oriented version of the complex dual to the tessellation preserved by the Coxeter group. This is a multivertex complex with infinitely many vertices. The fundamental groups of its subcomplexes have not been previously studied. The subcomplexes where the inclusion map induces an injection on fundamental groups are of particular interest. This condition is known as π_1-injectivity.

Given a sufficiently rich family of compact π_1-injective subcomplexes, the word problem for the full Artin group can be reduced to the word problem for the fundamental groups of the subcomplexes in this family. The goal is to produce such families of subcomplexes, and then to reduce the word problem of their fundamental groups to the word

problem of a finite list of “atomic” subcomplexes.

In this dissertation I present a solution to the word problem, using this approach, for the Artin groups corresponding to the infinite dihedral group, the 333 triangle group, and 244 triangle group. And I describe the difficulties that one encounters when trying to extend these methods to the Artin group corresponding to the 236 triangle group or to other higher dimensional Euclidean Artin groups.

Contents

Chapter 1

Introduction

Euclidean Artin groups are braided versions of the symmetry groups of certain regular tilings of Euclidean space. This dissertation focuses on constructive solutions to the word problem for low-dimensional Euclidean Artin groups with the hope that it might be generalized to other Artin groups in future work. A solution to the word problem is equivalent to constructing arbitrarily large portions of the universal cover of their presentation complex. I reduce this to a problem of solving the word problem for compact subcomplexes of an intermediate infinite-sheeted cover. The fundamental groups of these compact subcomplexes have never been studied before. The main difficulty with this approach is that one has to balance two competing needs. On the one hand, it is necessary to produce families of subcomplexes whose "atomic" elements have easy-to-understand word problems. On the other hand, it is necessary to have fundamental groups that can be coherently reassembled in order to understand the fundamental group of the intermediate cover. Further refinements of this approach are introduced as the dissertation progresses through various examples.

This dissertation is organized into five parts. Chapter 2 recalls the definitions of Coxeter groups, Artin groups, and the cell complexes associated with them. Chapter 3 focuses on relatively simple low dimensional Artin groups, where the associated complexes and subcomplexes are easy to visualize. This chapter includes a discussion about when subcomplex inclusion induces an injection on fundamental groups. Chapter 4 presents a constructive solution to the word problem for the braided 333 triangle group by expressing the fundamental group of arbitrarily large portions of a particular cover of the presentation complex as an iterated amalgamated free product of fundamental groups of subcomplexes whose word problems are completely understood. Although the word problem for the braided 333 triangle group was already known to be solvable, this is a new approach. Chapter 5 presents a modified version of the previous argument to construct a new solution to the word problem for the braided 244 triangle group. Finally, Chapter 6 outlines why it is difficult to adapt this technique to the braided 236 triangle group and to other higher-dimensional examples. It also includes directions for future work.

Chapter 2

Groups and complexes

This chapter reviews the definitions of Coxeter groups, Artin groups and the cell complexes associated with them. There is a special focus on the Euclidean versions of each and on the examples of type $\widetilde{A}_2$ and $\widetilde{C}_2$ in particular. Euclidean Coxeter groups are infinite groups that act naturally on Euclidean space, preserving an arrangement of affine hyperplanes, and this hyperplane arrangement induces a simplicial cell structure on Euclidean space. The corresponding Euclidean Artin group is the fundamental group of the quotient of the complement of a complexified version of this hyperplane arrangement.

Coxeter groups are discrete groups generated by reflections. Each reflection fixes a hyperplane, and conjugates of reflections are reflections that fix additional hyperplanes. The set of all such hyperplanes is preserved by the action of the Coxeter group. For Euclidean Coxeter groups this hyperplane arrangement induces an infinite cell structure on Euclidean space called the Coxeter complex. This hyperplane arrangement can be succinctly described by a crystallographic root system, and the cells can be labeled by the left cosets of finite parabolic subgroups. The top dimensional simplices are in 1-1 correspondence with the group elements. The cellular dual of the Coxeter complex is called the Davis complex. It has vertices in 1-1 correspondence with group elements, and the higher dimensional cells are now metric polytopes. These polytopes are also labeled by left cosets of finite parabolic subgroups, and the finite parabolic subgroup in the label determines the shape of the metric polytope. There is an "oriented" version of the Davis complex called the pure Salvetti complex. It is homotopy equivalent to the complement of the complexified version of the hyperplane arrangement. The Coxeter group naturally acts on the pure Salvetti complex. The quotient of the pure Salvetti complex by this action is a 1-vertex complex called the Salvetti complex, and the fundamental group of the Salvetti complex is the corresponding Artin group. The fundamental group of the pure Salvetti complex is a subgroup of the Artin group called the pure Artin group. The relationship between the Artin group and the pure Artin group is the same as the relationship between the braid group and the pure braid group. It is important to understand the relationships between these various groups and spaces since the Coxeter versions are used to index aspects of the Artin versions.

2.1 Coxeter groups

Coxeter groups appear throughout mathematics. Examples of finite or spherical Coxeter groups include the symmetry groups of the Platonic solids or, more generally, the symmetry group of a regular polytope. Familiar examples of Euclidean Coxeter groups include the Weyl groups of complex simple irreducible Lie algebras. As the name suggests, spherical Coxeter groups act naturally on a sphere, or on Euclidean space fixing the origin, and Euclidean Coxeter groups act naturally on Euclidean space. In fact, for each Euclidean Coxeter group there is an associated spherical Coxeter group obtained by quotienting out the normal subgroup of pure translations. The Weyl groups of the Lie algebras of Cartan-Killing type A_n, B_n, C_n, F_4, and G_2 respectively correspond to the symmetry groups of the regular simplices, cubes and their dual cross polytopes, the

4-dimensional 24-cell, and the regular hexagon. Both of these families of groups are well-studied and completely classified in terms of labeled graphs that encode easy-to-define group presentations, and both families can be defined from a root system.

Definition 2.1.1 (Roots)**.** Let V be a finite dimensional Euclidean vector space with standard Euclidean inner product $(\cdot, \cdot)$ then a finite collection of nonzero vectors Φ is a *root system* in V if the following conditions hold

1. The roots span V.
2. The only scalar multiples of a root α in Φ are α and $-\alpha$.
3. For every α the root system Φ is closed under reflection through the hyperplane perpendicular to α.

The root system Φ is *crystallographic* if for all $\alpha, \beta \in \Phi$ the projection of β onto the line through α is an integer or half-integer multiple of α.

Spherical Coxeter groups are defined from root systems where the length of each root does not play a role. Roots systems and spherical Coxeter groups have been completely classified. See Remark 2.1.9. Euclidean Coxeter groups are defined from crystallographic root systems, and the length of a root $\alpha \in \Phi$ encodes the spacing between the parallel hyperplanes with α as a normal vector. See Remark 2.2.6 and more generally Section 2.2. Crystallographic root systems and Euclidean Coxeter groups are also completely classified, see Figure 2.1.7.

This section briefly introduces several of the tools associated with these classifications. This section starts with spherical Coxeter groups and then considers the Euclidean ones.

Definition 2.1.2 (Spherical Coxeter group)**.** A *spherical Coxeter group* is a group that is generated by orthogonal reflections acting on Euclidean space, and it acts *geometrically* on a sphere in Euclidean space, fixing its center. Recall that a geometric action is one

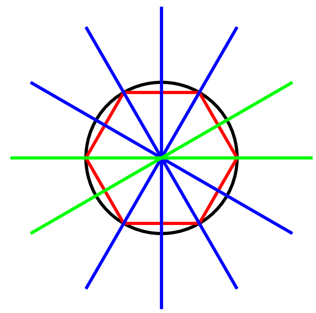


Figure 2.1.1: Symmetries of a regular hexagon.

that acts properly discontinuously, cocompactly by isometries. The group is *irreducible* if there does not exist a nontrivial orthogonal decomposition of the underlying Euclidean space so that the group is a product of subgroups acting on these subspaces.

Remark 2.1.3. Given a root of $\Phi \subset \mathbb{R}^n$, there is a reflection that fixes its orthogonal complement and sends the root to its negative. The group W generated by the set of all reflections in the roots permutes the roots of Φ so that there is a map $W \to \mathrm{Sym}_\Phi$. Since Φ spans $\mathbb{R}^n$ the action of W on Φ determines the action of W on $\mathbb{R}^n$ so that the map $W \to \mathrm{Sym}_\Phi$ is injective. Since a root system is finite the group W is finite and defines a spherical Coxeter group.

Conversely, given a spherical Coxeter group generated by orthogonal Euclidean reflections consider the full set of reflections. The set of unit vectors orthogonal to a hyperplane fixed by a reflection forms a root system.

Example 2.1.4 (Symmetries of a regular hexagon)**.** Consider the spherical Coxeter group W of symmetries of a regular hexagon, generated by the reflections through the blue and green lines in Figure 2.1.1 and rotations by an integer multiple of $\pi/3$. This group W can be generated by orthogonal reflection in two hyperplanes which meet at an angle of $\pi/6$, such as the two green lines in Figure 2.1.1, so that the composition of these two reflections results in a rotation by $\pi/3$, counterclockwise or clockwise depending on the order of composition. Observe that, if the set of all reflections generated by the reflections in two hyperplanes meeting at an angle of $\pi/3$ generates the reflections in the rest of the blue lines in Figure 2.1.1. This group acts geometrically on the circle depicted in black and also acts geometrically on the the red regular hexagon that is homeomorphic to the circle. Letting s and t be the reflections through green lines, the group admits a presentation of the form

$$W = \langle s, t \mid s^2 = t^2 = (st)^6 = 1 \rangle.$$

Example 2.1.5 (Symmetries of a triangular prism)**.** Alternatively, consider the symmetry group W' of a triangular prism, with base an equilateral triangle. The group W' can be expressed as the cross product of the symmetry group of an equilateral triangle and the symmetry group of an interval. However, to realize W' as a group generated by reflections acting geometrically on a sphere consider reflections a and b in two vertical planes in $\mathbb{R}^3$, which intersect at an angle of $\pi/3$, corresponding to any pair of vertical planes depicted in Figure 2.1.2. Since they meet the xy-plane orthogonally, they commute with the reflection c in the xy-plane. Thus, W' admits the presentation

$$W' = \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^3 = (ac)^2 = (bc)^2 = 1 \rangle.$$

Moreover, there is an isomorphism $W \cong W'$ between the group of symmetries of a

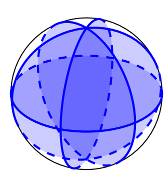


Figure 2.1.2: The symmetry group a triangular prism acting on a sphere.

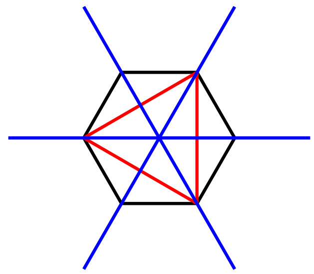


Figure 2.1.3: Symmetries of the red triangle as a subgroup of the symmetries of a regular hexagon.

regular hexagon and a triangular prism by viewing W as the product of the symmetry group of the red equilateral triangle in Figure 2.1.3, generated by reflections in a pair of blue lines, and the group of order 2 generated by rotation by π. Note, that rotation by π commutes with these two reflections that generate the symmetry group of an equilateral triangle. Algebraically, this can be seen as $\varphi : W' \to W$ such that $\varphi(a) = s$, $\varphi(b) = tst$, and $\varphi(c) = (st)^3$.

Remark 2.1.6. Examples 2.1.4 and 2.1.5 illustrate how a spherical Coxeter group is not merely an abstract group, but implicitly it comes equipped with a particular action on Euclidean space. In particular, the symmetries of a regular hexagon and the symmetries of a triangular prism are distinct Coxeter groups even though as groups they are abstractly isomorphic.

One way to encode the action is to fix a special set of generators.

Definition 2.1.7 (Spherical Coxeter system)**.** Given a root, there is a reflection that fixes its orthogonal complement and sends the root to its negative. The set T of all reflections of a root system, Φ, generate a spherical Coxeter group W, but this is a highly redundant generating set. A minimal generating set is obtained by fixing a hyperplane through the origin that does not intersect any of the roots. The two sides of the hyperplane partition the roots into two sets $\Phi = \Phi^+ \sqcup \Phi^-$. The positive linear combinations of the roots in

Φ^+ form a polyhedral cone that has extremal rays. These rays are determined by roots $\Delta \subset \Phi^+$, and the reflections in the roots in Δ form a minimal generating set S for the Coxeter group W. The ordered pair (W, S) is called a spherical Coxeter system. The cardinality $|S| = n$ is the *rank* of the Coxeter system.

When pairs of generating reflections in hyperplanes meet at an angle of π/m they generate a subgroup isomorphic to the dihedral group of order $2m$. In fact, to succinctly store the information of a spherical Coxeter group one need only keep track of the dihedral angles between the hyperplanes that generate the group.

Definition 2.1.8 (Coxeter Matrix)**.** Suppose that (W, S) is a spherical Coxeter system with $S = \{s_1, \ldots, s_n\}$, where s_i is an orthogonal reflection in the hyperplane H_i. The *Coxeter matrix* is an $n \times n$ matrix M with entries m_{ij}, such that $m_{ii} = 1$ for all i and for $i \neq j$ π/m_{ij} is the dihedral angle between H_i and H_j. Thus, $s_i s_j$ has order m_{ij} for all i, j. The integer m_{ij} is called the *braid relation length* since for each $i \leq j$ the corresponding Coxeter group admits a presentation with a relation of the form $(s_i s_j)^{m_{ij}} = 1$, which for $i \neq j$ can be rewritten as a *braid relation*

$$\underbrace{s_i s_j s_i \ldots s_j}_{m_{ij}} = \underbrace{s_j s_i s_j \ldots s_i}_{m_{ij}} \quad \text{or} \quad \underbrace{s_i s_j s_i \ldots s_i}_{m_{ij}} = \underbrace{s_j s_i s_j \ldots s_j}_{m_{ij}}$$

if m_{ij} is even or odd respectively since each s_i is an involution.

Remark 2.1.9. Through a further simplication one can encode this Coxeter matrix in terms of a labeled graph in the following way. There are n vertices, one vertex i for each generator s_i, and each braid relation $m_{ij} \geq 3$ is recorded as an edge between vertex i and j. The edge between vertex i and j is labeled with m_{ij} if $m_{ij} \geq 4$. The classification of spherical Coxeter group in terms of these graphs is a standard fact and can be found in [1, 2] for example.

In order to construct Coxeter groups in full generality from these Coxeter graphs edge labels and the corresponding entries of the Coxeter matrix must be allowed to take the value ∞.

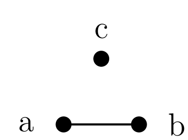
Definition 2.1.10 (Coxeter Graph)**.** A finite simple graph Γ with some subset of the edges labeled with an integer greater than 3 or ∞ is called a *Coxeter graph*. For two vertices i, j of Γ, m_{ij} is the *braid relation length*, where m_{ij} is determined as follows:

1. If $\{i, j\}$ is not an edge of Γ then $m_{ij} = 2$.
2. If $\{i, j\}$ is an unlabeled edge of Γ then $m_{ij} = 3$.
3. If $\{i, j\}$ is a labeled edge of Γ with label $k \in \mathbb{Z}_{\geq 4} \cup \infty$ then $m_{ij} = k$.

Example 2.1.11 (Coxeter graph)**.** Recall Example 2.1.5 of the Coxeter group W' with presentation $W' = \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^3 = (ac)^2 = (bc)^2 \rangle$. The corresponding Coxeter matrix is

$$\begin{bmatrix} 1 & 3 & 2 \\ 3 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix},$$

and it is encoded in the following Coxeter graph.



It is straightfoward to see that the symmetry group of a triangular prism W' acts on a sphere reducibly, as it splits as the product of the symmetry group of an equilateral triangle and an interval. That is,

$$W' = \langle a, b \mid a^2 = b^2 = (ab)^3 = 1 \rangle \oplus \langle c \mid c^2 = 1 \rangle$$

In Figure 2.1.2 this is easy to recognize because the the fixed planes H_a, H_b of the reflections a and b are both orthogonal to the fixed plane H_c of the reflection c and thus the subgroup generated by a and b commutes with the subgroup generated by c. In terms of the braid relations, $m_{ac} = m_{bc} = 2$, which is reflected in the Coxeter graph having two disjoint components.

Next, return to the definition of Coxeter groups in full generality.

Definition 2.1.12 (Coxeter system)**.** Given a Coxeter graph Γ one can define the *Coxeter group of type* Γ, denoted $\textsc{Cox}(\Gamma)$, to be the group generated by $V(\Gamma)$ with one relation of the form

$$\underbrace{s_i s_j s_i \ldots s_j}_{m_{ij}} = \underbrace{s_j s_i s_j \ldots s_i}_{m_{ij}} \quad \text{or} \quad \underbrace{s_i s_j s_i \ldots s_i}_{m_{ij}} = \underbrace{s_j s_i s_j \ldots s_j}_{m_{ij}}$$

if m_{ij} is even or odd respectively for each pair of vertices i, j such that $m_{ij} \neq \infty$ and one relation $s_i^2 = 1$ for each vertex $i \in V(\Gamma)$. Any group, which admits a presentation of this form is called a *Coxeter group*.

A pair (W, S), where W is a Coxeter group so that it admits a presentation of the above form with generating set $S \subset W$ is called a *Coxeter system*. S is referred to as the *simple system*, and the elements of S as the *simple reflections*. The *rank* of a Coxeter group is $|S|$. Moreover, given a Coxeter graph the Coxeter system determined by Γ is defined to be (W, S), where W is the Coxeter system of type Γ and $S = V(\Gamma)$. A Coxeter group is *irreducible* if Γ is a connected graph.

In Example 2.1.11 the Coxeter group W' had a Coxeter graph Γ consisting of two connected subgraphs Γ_1 and Γ_2, where $W' = \textsc{Cox}(\Gamma) = \textsc{Cox}(\Gamma_1) \oplus \textsc{Cox}(\Gamma_2)$ could be written as the direct sum of the Coxeter groups determined by Γ_1 and Γ_2. These are examples of what are called parabolic subgroups.

Definition 2.1.13 (Parabolic subgroups)**.** Naturally one can look at subgraphs and the Coxeter systems corresponding to them. Focusing on an *induced subgraph* of some graph Γ, that is a subgraph $\Gamma[I]$ formed by a subset $I \subset V(\Gamma)$ with all of the edges in Γ that connect pairs of vertices in the subset, the corresponding Coxeter group is a subgroup. Given a Coxeter system (W, S) of type Γ and $I \subset S = V(\Gamma)$, denote $W_I = \text{Cox}(\Gamma[I])$ the subgroup of W generated by I so that $(W_I, \Gamma[I])$ is a Coxeter system. All subgroups of W that are obtainable in this way are called *parabolic subgroups* of W.

This section has so far detailed how to encode a spherical Coxeter group in a Coxeter graph and, conversely, how to define a general Coxeter group given a Coxeter graph. One might hope that given a Coxeter graph of a spherical Coxeter group it is possible to reconstruct the geometry of the group acting geometrically on a sphere. To do so there is the following construction of a faithful linear representation due to Jacques Tits. Accordingly, it is often referred to as the Tits representation or the standard geometric representation. As an additional benefit this representation can be used to not only reconstruct the geometry of spherical Coxeter groups but to better understand the geometry of a Coxeter group coming from an arbitrary Coxeter graph.

Theorem 2.1.14. *Let (W, S) be a Coxeter system of type Γ, then there is a faithful representation $\rho : W \to GL(n, \mathbb{R})$, such that the image of each element of $S = \{s_1, ..., s_n\}$ is a linear involution fixing a hyperplane and for $s_i, s_j \in S$, $i \neq j$, $\rho(s_i)\rho(s_j)$ has order m_{ij}.*

Suppose that Γ is a Coxeter graph with vertices $V(\Gamma) = \{1, \ldots, n\}$. The construction of this representation is outlined starting with a symmetric matrix $n \times n$ matrix B, referred to as the *Schläfli matrix*, with entries $B_{ij} = -2\cos\left(\frac{\pi}{m_{ij}}\right)$ if m_{ij} finite and $B_{ij} = -2$ if $m_{ij} = \infty$. Furthermore, B defines a bilinear form on $\mathbb{R}^n$. Denote $(x, y)_B = \frac{1}{2}x^t By$. Let $\{e_1, ..., e_n\}$ be the standard basis for $\mathbb{R}^n$ and consider

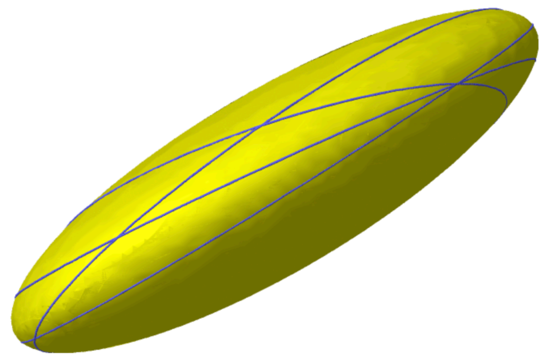
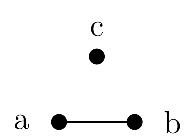


Figure 2.1.4: Tits representation with the unit sphere and the spherical hyperplanes

the hyperplanes $H_i = \{v \in \mathbb{R}^n \mid (e_i, v)_B = 0\}$. Then the map $\sigma_i : \mathbb{R}^n \to \mathbb{R}^n$ defined $\sigma_i(v) = v - (e_i, v)_B e_i$ is a linear map that preserves the bilinear form B, fixes the hyperplane H_i, satisfies $\sigma_i^2 = \mathrm{id}$ and sends e_i to $-e_i$. Thus, σ_i is a reflection with respect to B in the hyperplane H_i. Moreover, it is straightforward to check that $\sigma_i\sigma_j$ has order m_{ij}, and, in the case when $m_{ij} < \infty$, it is a rotation by $\frac{2\pi}{m_{ij}}$, see [1, 3, 2] for details. Therefore, the desired faithful representation is defined by $\rho(s_i) = \sigma_i$.

Example 2.1.15 (Tits representation)**.** Continue the example of the Coxeter group $W' = \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^3 = (ac)^2 = (bc)^2 \rangle$, with Coxeter graph below.



Following the outline above the symmetric Schläfli matrix B with columns indexed by a, b, and c is

$$B = \begin{bmatrix} 2 & \sqrt{3} & \sqrt{2} \\ \sqrt{3} & 2 & \sqrt{2} \\ \sqrt{2} & \sqrt{2} & 2 \end{bmatrix},$$

determining a positive semidefinite bilinear form $(\cdot,\cdot)$ on $\mathbb{R}^3$ defined $(x,y) = \frac{1}{2}x^tBy$. This group is generated by σ_i, defined $\sigma_i(v) = v - 2(e_i, v)e_i$, where e_i is the i-th standard basis vector of $\mathbb{R}^3$ and $i = 1, 2, 3$. This defines a representation $\rho : W' \to \mathrm{GL}(3,\mathbb{R})$. Moreover, this bilinear form B preserves a sphere, e.g. the unit sphere of all vectors v satisfying $(v,v)_B = 1$. Acting on it are the reflections σ_i, which each fix a hyperplane H_i and satisfy $\sigma(e_i) = -e_i$, where $\pm e_i$ are the roots corresponding to H_i. Figure 2.1.4 depicts the unit sphere with respect to B and the intersection of the hyperplanes fixed by the full set of reflections

$$\{wxw^{-1} \mid w \in W', x = a, b, c\} = \{a, b, c, aba\}.$$

Notice that since since B is not the standard Euclidean inner product on $\mathbb{R}^3$ the unit sphere looks elongated and the hyperplanes acting on it meeting do not meet at dihedral angles π/m_{ij} as depicted in Figure 2.1.2. Instead, in the Tits representation the roots are chosen to be the standard basis vectors of $\mathbb{R}^n$ and the bilinear form is adapted to construct the desired dihedral angles between our reflecting hyperplanes. However, this bilinear form is just a conjugate of the standard Euclidean inner product, where the standard basis vectors are mapped to the corresponding crystallographic simple roots.

For general Coxeter systems, the Schläfli matrix is used to characterize the geometry of our Coxeter group. A Coxeter group is Euclidean if B is positive semidefinite with one zero eigenvalue and thus preserves some unit sphere, which is some $\mathbb{S}^{n-1} \times \mathbb{R}$. Next, is the simplest example of a Euclidean Coxeter group, which is commonly referred to as type $\widetilde{A}_1$.

Example 2.1.16 (Geometry when $m_{ij} = \infty$)**.** Suppose W is the Coxeter group with

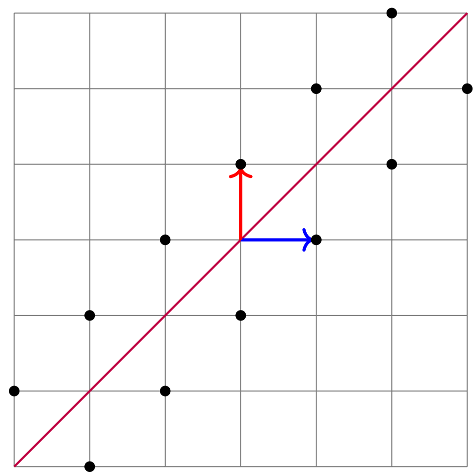


Figure 2.1.5: A portion of the roots and hyperplanes in the Tits representation of the Coxeter group of type $\widetilde{A}_1$.

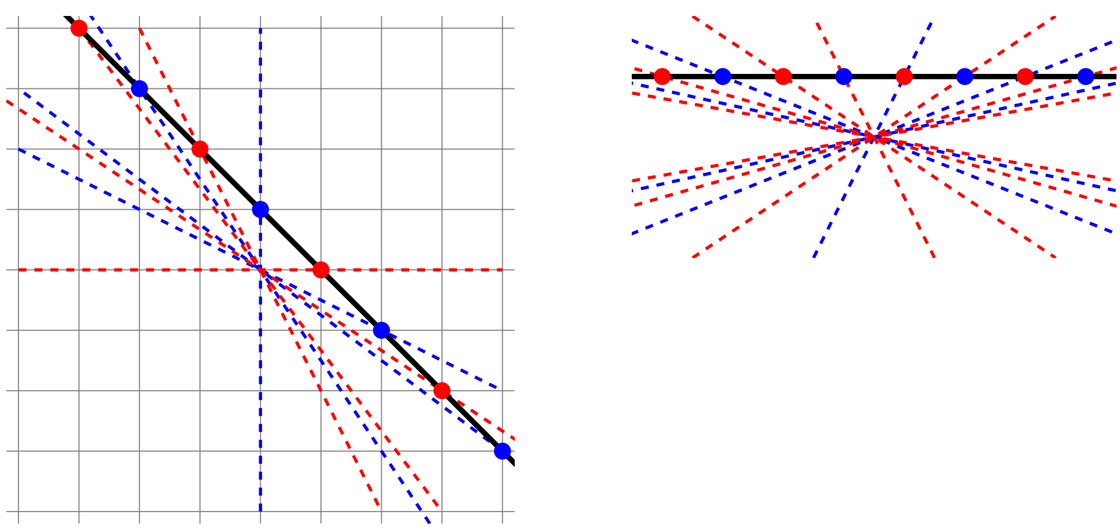


Figure 2.1.6: The contragredient representation of Coxeter group of type $\widetilde{A}_1$ and a conjugate of it.

presentation $W = \langle s_1, s_2 \mid s_i^2 = 1 \rangle$, i.e. $m_{12} = \infty$, and the Schläfli matrix is

$$B = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}.$$

Note that B is postive semidefinite, with a nullspace span$(1, 1)$ so that vectors v in null(B) have $(v, v)_B = 0$. Also the form preserves an $\mathbb{S}^0 \times \mathbb{R}$. Figure 2.1.5 depicts a portion of the orbit of the standard basis vectors e_1 and e_2. Each of the we_i in the orbit $W\{e_1, e_2\}$ defines a reflection with respect to $(\cdot, \cdot)_B$, $w\sigma_i w^{-1}$, which fixes null(B) and sends we_i to $-we_i$. In this setting there is a nice picture of each of the roots, but each of the reflections $w\sigma_i w^{-1}$ fixes the same line $y = x$.

Furthermore, one can look at the quotient $\mathbb{R}^2/\text{null}(B)$, with the induced positive definite form, which can be identified with $\mathbb{R}$, so that the fixed space of each reflection is just the origin. In general, for a Euclidean Coxeter group of rank n with bilinear form $1/2B$, $1/2B$ induces a positive definite form on the quotient $\mathbb{R}^n/\text{null}(B)$, which can be identified with $(n-1)$-dimensional Euclidean space so that W acts as a Euclidean reflection group. However, in this representation, hyperplanes fixed by distinct reflections coincide.

Instead, look at the dual or contragredient representation. In terms of the matrices this is the representation that sends each matrix to its inverse transpose. Under this representation, the roots coincide, but there is a distinct hyperplane for each distinct reflection. Moreover, the contragredient representation preserves the line $y = 1 - x$, i.e. some affine patch of $\mathbb{R}$ on which W acts depicted on the left in Figure 2.1.6. By conjugating by a suitable matrix a "nicer" picture can be obtained, where one can identify the affine patch at $y = 1$ depicted in black with the real line and the hyperplane arrangement restricted to the real line as $\mathbb{Z} \subset \mathbb{R}$ of alternating blues and reds. This defines an action of

W on the real line generated by the reflection s_1 in $x = 0$ and s_2 in $x = 1$. In particular, $s_1(x) = -x$ and $s_2(x) = 2 - x$, which generate the full set of reflections consisting of reflection in n for each $n \in \mathbb{Z}$.

Note this is a Euclidean Artin group so that it can be obtained from a crystallographic root system as well. The root system of type A_1 consists of the vectors $\{1, -1\} \subset \mathbb{R}$. The corresponding hyperplane arrangement consists all vectors in $\mathbb{R}$ that have integer inner product with a root. This again leads to the hyperplane arrangement of $\mathbb{Z} \subset \mathbb{R}$ as above.

This in fact is the general setting for Euclidean Artin groups. Given a Coxeter graph, the Tits representation realizes the corresponding rank $n + 1$ Coxeter group as a group generated by reflections that preserves some $\mathbb{S}^{n-1} \times \mathbb{R}$, which under the contragredient representation preserves some $\mathbb{R}^n \times \mathbb{S}^0$. In the quotient space there is an action on $\mathbb{R}^n$ generated by reflections in codimension-1 hyperplanes. More directly from the crystallographic root system, these hyperplanes may be obtained as the sets of vectors in $\mathbb{R}^n$ that have integer inner product with a fixed root. The irreducible Euclidean Coxeter groups, i.e. the ones that act on $\mathbb{R}^n$ irreducibly, are fully classified.

Theorem 2.1.17 (Euclidean Coxeter graphs)**.** *The irreducible Euclidean Coxeter groups are classified by the table in Figure 2.1.7. Specifically there are 4 infinite families of type A, B, C, D and finitely many that are not part of an infinite family.*

2.2 Coxeter complex

Euclidean Coxeter groups, which are infinite groups, act naturally on Euclidean space. They preserve some hyperplane arrangement that can be obtained from a crystallographic root system. Each hyperplane is the affine subspace of vectors that have a fixed integral inner product with a fixed positive root. The collection of all such hyperplanes induces a

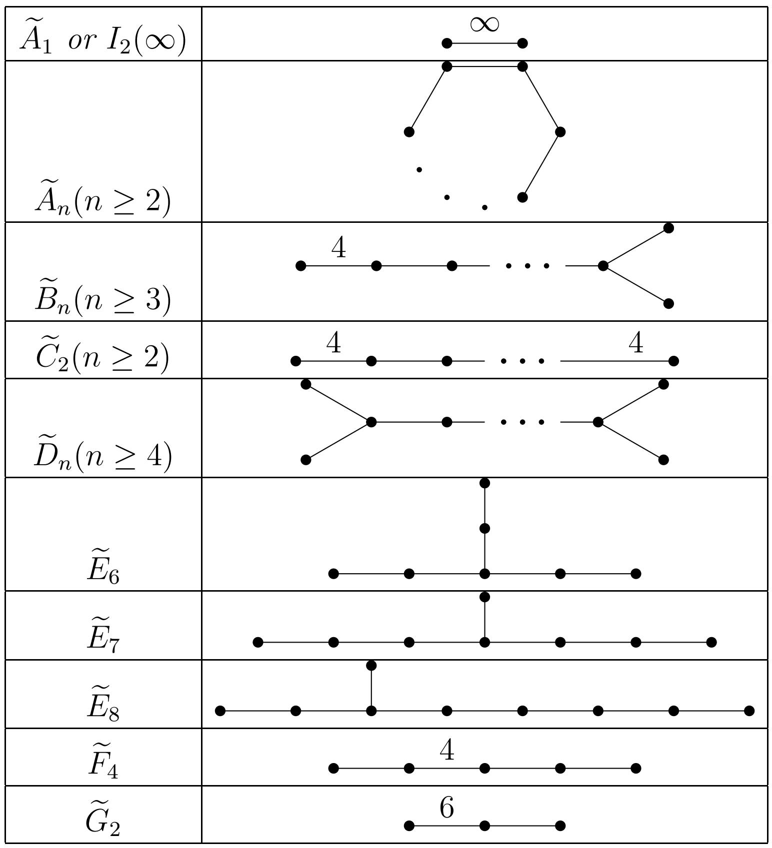


Figure 2.1.7: The Coxeter graphs of Euclidean type

simplicial cellular structure on the space and the Coxeter group acts simply transitively on the interiors of the simplices of this cell structure. This section focuses on the induced cell structure which can be described in terms parabolic subgroups.

Each of the examples presented thus far demonstrates how a Coxeter group admits a representation as a group generated by reflections. And the set of all fixed hyperplanes of the reflections decomposes the space on which the Coxeter group acts into a nice cellular structure. Furthermore, the Coxeter group action permutes these cells. For example, the fixed hyperplanes of all of the reflections of the dihedral group of order 12 acting on the circle in Figure 2.1.1 divide the space into twelve arcs. These arcs are permuted by the action of the Coxeter group. The symmetry group of a triangular prism acts on a sphere in Figure 2.1.2, and the fixed hyperplanes of the four reflections divide it into twelve spherical triangles with angles $\pi/2, \pi/2$, and $\pi/3$. The infinite dihedral group of Example 2.1.16 acts on the real line in Figure 2.1.6 with reflections in $x = n$ for each $n \in \mathbb{Z}$. These affine hyperplanes $x = n$, $n \in \mathbb{Z}$ decompose $\mathbb{R}$ into the intervals between consecutive integers. To formalize this notion, the following definition recalls the definition of a Coxeter complex following [1].

Definition 2.2.1 (Coxeter Complex)**.** Every Coxeter system, (W, S), admits a representation $W \to \mathrm{GL}(V)$ for some V with basis $\{\alpha_s\}_{s\in S}$. However, for general Coxeter groups, including Euclidean Coxeter groups, in order to recover the group as one generated by reflections in distinct hyperplanes consider the contragredient representation $W \to \mathrm{GL}(V^*)$. Denote elements of V^* as f, g, h, etc. and the natural pairing with V, $\langle f, v\rangle$. Then the natural action of W is characterized by:

$$\langle w(f), w(v)\rangle = \langle f, v\rangle \text{ for } w \in W, f \in V^* \text{ and } v \in V.$$

For each $s \in S$ there is a hyperplane $H_{\alpha_s} = \{f \in V^* \mid f(\alpha_s) = 0\}$, together with the associated halfspaces $H_{\alpha_s}^+ = \{f \in V^* \mid f(\alpha_s) > 0\}$ and $H_{\alpha_s}^- = \{f \in V^* \mid f(\alpha_s) > 0\}$, with their respective closures denoted $\overline{H}_{\alpha_s}^+ = H_{\alpha_s}^+ \cup H_{\alpha_s}$ and $\overline{H}_{\alpha_s}^- = H_{\alpha_s}^- \cup H_{\alpha_s}$. Define $D = \bigcap_{s \in S} \overline{H}_{\alpha_s}^+$. Then this is a fundamental domain for the action of W on V^*. To describe the faces of the cone, partition D into subsets corresponding to its parabolic subgroups, defined

$$C_T = \left(\bigcap_{s \in T} H_{\alpha_s} \right) \cap \left(\bigcap_{s \notin T} H_{\alpha_s}^+ \right).$$

Note that $C_T \subseteq C_T'$ if and only if $T \supseteq T'$. The *Tits cone* is the interior of the union of the orbit of D under the action by W. The *Coxeter complex* is the quotient of the Tits cone, with the origin removed, by the positive reals. In the case of Euclidean Coxeter groups this is equivalent to taking a the intersection of its hyperplane arrangement with an affine plane parallel to the roots.

More specifically, in the case of irreducible Coxeter groups of Euclidean type, one can follow the description of the hyperplane arrangement and chambers for affine reflection groups also found in [1]. Suppose that W_0 is a rank n irreducible spherical Coxeter group of type $A - G$ so that it corresponds to the Weyl group of some crystallographic root system Φ, with simple system Δ and set of positive roots Π. For each $\alpha \in \Pi$ define an affine hyperplane $H_{\alpha,k} := \{x \in \mathbb{R}^n \mid (x, \alpha) = k\}$. Then, the collection of all hyperplanes for each $k \in \mathbb{Z}$, i.e $\mathcal{H} = \{H_{\alpha,k} \mid \alpha \in \Phi, k \in \mathbb{Z}\}$ is the set of reflecting hyperplanes corresponding to the Euclidean Coxeter group W. Moreover, the connected components of $\mathbb{R}^n \setminus \mathcal{H}$ are open Euclidean simplices acted on simply transitively by W. Let $\tilde{\Delta} = \Delta \cap \{-\gamma\}$ where γ is the unique highest root of Φ with respect to Δ. Then the chamber bounded by $H_{\alpha,0} \equiv H_\alpha$ for $\alpha \in \Delta$ and $H_{\gamma,1}$ is called the fundamental chamber C. The action $s_{\alpha,k}$ of $H_{\alpha,k}$ on a vector $v \in \mathbb{R}^n$, where $n = |\Delta|$ is defined by

$$s_{\alpha,k}(v) = v - ((v, \alpha) - k)\frac{2\alpha}{(\alpha, \alpha)},$$

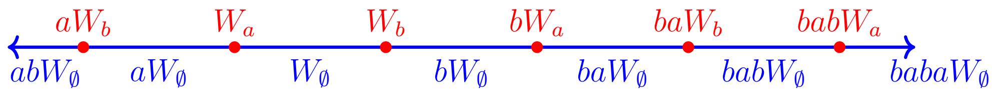


Figure 2.2.1: The Coxeter complex of type $\widetilde{A}_1$

and the orbit $W(C)$ is $\mathbb{R} \setminus \mathcal{H}$. The action of the underlying spherical Coxeter group W_0 on this hyperplane arrangement is defined by $wH_{\alpha,k} = H_{w\alpha,k}$ and $ws_{\alpha,k}w^{-1} = s_{w\alpha,k}$.

In summary, the Coxeter complex of a rank $n+1$ Euclidean Coxeter group is a complex tiled by n-simplices, whose faces can be described in terms of parabolic subgroups of spherical type. Considering the orbit of these parabolic subgroups leads to the following definition.

Definition 2.2.2. (Poset of special cosets) Given a Coxeter system (W, S), the set $\mathcal{S} = \{xW_T \mid x \in W, T \subseteq S, |W_T| < \infty\}$ of left cosets of the spherical parabolic subgroups of W is referred to as the *special cosets*. The set of special cosets comes with a natural partial order $\leq$, ordered by inclusion. The poset $(\mathcal{S}, \leq)$ is precisely the face poset of the Davis complex, which is defined in the subsequent section. Moreover, the dual poset $\mathcal{S}^{\text{op}}$ is the face poset of the Coxeter complex.

Example 2.2.3 (Coxeter complex of type $\widetilde{A}_1$)**.** The underlying spherical Coxeter group of type A_1 has crystallographic root system $\Phi = \{-1, 1\} \subset \mathbb{R}$ so that the set of positive roots is the singleton set $\Pi = \{1\}$. Thus, the set of affine hyperplanes is $\{k \in \mathbb{Z}\}$ can be seen as the red dots in Figure 2.2.1. The chambers are the open blue edges $\{(i, i+1) | i \in \mathbb{Z}\}$.

This coincides, with the above definition where the parabolic subgroups of spherical type are precisely, $W, W_a := W_{\{a\}}, W_b := W_{\{b\}}$. There are 0-cells, which correspond to left cosets of the maximal parabolic subgroups, W_a or W_b, and the 1-cells/chambers correspond to the left cosets of the trivial parabolic subgroup $W_\emptyset$, which are in 1-1 correspondence with the group elements. Notice that the Coxeter complex reflects the geometry of the contragredient representation of Example 2.1.16.

Remark 2.2.4. In general, a Coxeter group W acts on its Coxeter complex by left multiplication. The cells of the Coxeter complex are labeled by special cosets of the form xW_T for $x \in W$ and W_T a special coset of spherical type. For $g \in W$, $g \cdot (xW_T) = (gx)W_T$. Observe that W acts simply transitively on the *chambers*, the cells of maximal dimension, labeled by $xW_\emptyset$.

Example 2.2.5 (Coxeter complex of type $\widetilde{A}_2$)**.** Consider the Coxeter group with presentation

$$W = \langle a, b, c | a^2 = b^2 = c^2 = 1, aba = bab, aca = cac, bcb = cbc \rangle.$$

It is the irreducible Euclidean Coxeter group of type $\widetilde{A}_2$ as classified in Theorem 2.1.17. The underlying spherical Coxeter group of type A_2 corresponds to the Weyl group of the root system of the Lie algebra $\mathfrak{sl}(3, \mathbb{C})$, $\Phi = \pm(e_1 - e_2), \pm(e_1 - e_3), \pm(e_2 - e_e)$, where e_1, e_2, e_3 are the standard basis vectors of $\mathbb{R}^3$. However, these roots all lie in a 2-dimensional plane. One may instead view these roots as the six vectors of length $\sqrt{2}$, $\pm\alpha, \pm\beta, \pm\gamma \in \mathbb{R}^2$ depicted in Figure 2.2.2.

Thus, the set of affine hyperplanes is $\mathcal{H} = \{H_{x,k} | x = \alpha, \beta, \gamma, k \in \mathbb{Z}\}$ so that the hyperplane arrangement corresponds to the tiling of the plane by equilateral triangles. Figure 2.2.3 depicts the Coxeter complex of type $\widetilde{A}_2$, with each of the cells labeled by special cosets.

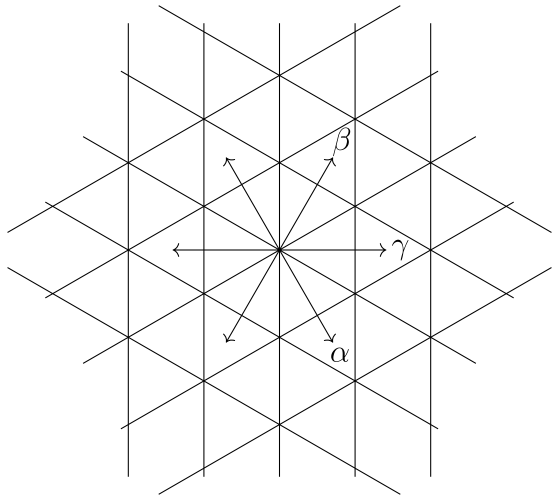
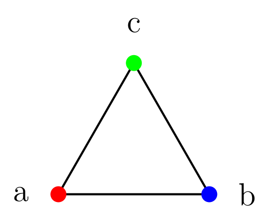


Figure 2.2.2: The hyperplane arrangement of type $\widetilde{A}_2$.

Specifically, the Coxeter group of type $\widetilde{A}_2$ has Coxeter graph

$\Gamma =$



and each of the 8 subsets of the vertices $V(\Gamma) = \{a, b, c\}$ define induced subgraphs and parabolic subgroups of W. The eight parabolic subgroups are ordered by inclusion in the lattice of parabolic subgroups depicted in Figure 2.2.4 along with the corresponding induced subgraphs.

The three parabolic subgroups of rank 2, $W_{a,b}$, $W_{a,c}$, and $W_{b,c}$ are all spherical Coxeter groups isomorphic to the symmetric group on 3 letters which admit a presentation of the form $\text{Sym}_3 = \langle x, y \mid x^2 = y^2 = 1, xyx = yxy \rangle$. The three rank 1 parabolic subgroups W_a, W_b, W_c are each isomorphic to $\mathbb{Z}/2\mathbb{Z}$, admitting presentations of the form $\langle x \mid x^2 = 1 \rangle$. Lastly, $W_\emptyset$ is the trivial subgroup.

In terms of cells of the Coxeter complex in Figure 2.2.3, the cosets of the rank 0

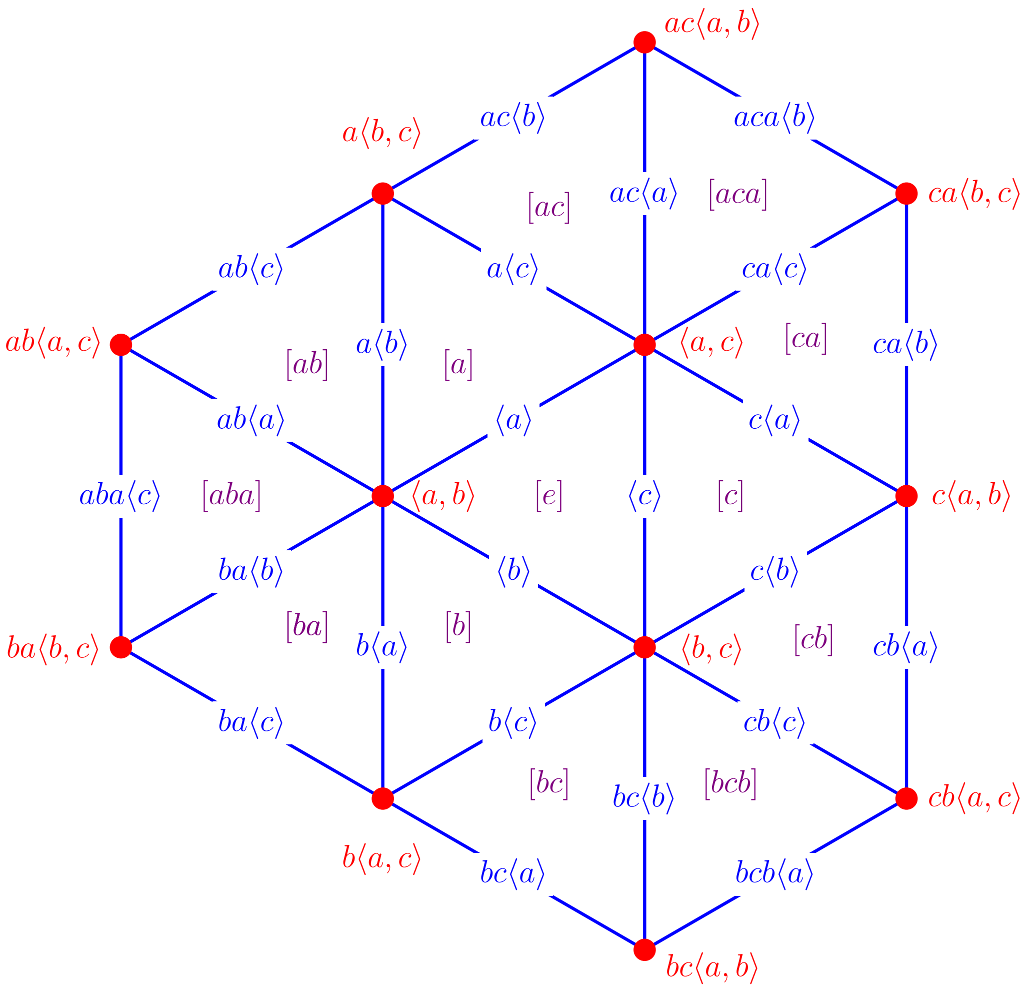


Figure 2.2.3: A portion of the Coxeter complex of type $\widetilde{A}_2$

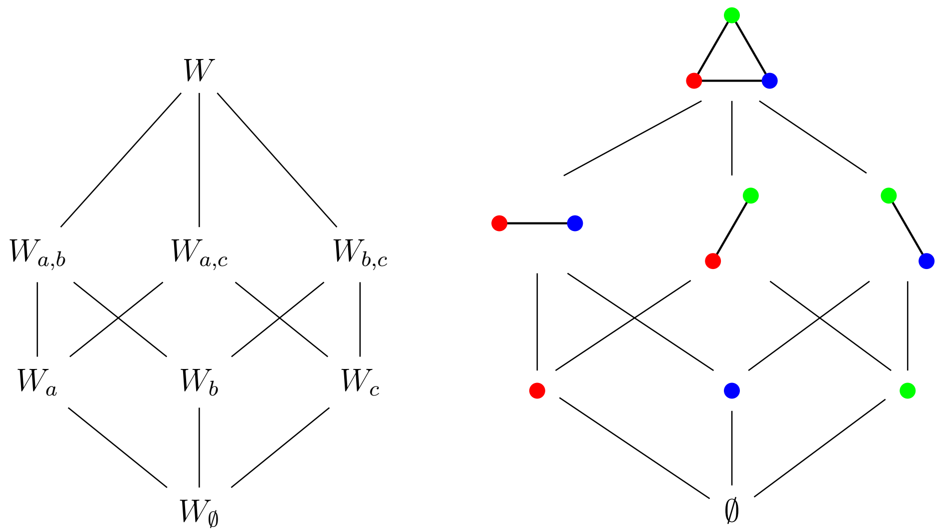


Figure 2.2.4: The poset of parabolic subgroups of $W = \textsc{Cox}(A_2)$

trivial subgroup $W_\emptyset$ are the special cosets that correspond to the chambers or purple 2-cells below. These chambers are in 1-1 correspondence with group elements, which are labeled with a purple representative for the equivalence class of words that represent the corresponding group element. Note that each 2-cell has triangle boundary since it is contained in three rank 1 special cosets, corresponding to the blue edges below. The three 1-cells bounding a 2-cell correspond to three rank 1 special cosets which pairwise are contained in a unique rank 2 special coset. That is, the face poset of Coxeter complex is precisely to dual poset to the poset of special cosets.

Remark 2.2.6 (Type $\widetilde{C}_2$ and $\widetilde{B}_2$ Coxeter complex)**.** The hyperplane arrangement that defines the Coxeter complex is defined in terms of affine hyperplanes with integral inner product with the crystallographic root system. Thus, the choice of roots affects the spacing of the hyperplanes. Although the Weyl groups of the type B_2 and type C_2 root systems are isomorphic as groups to the 244 triangle group, the root systems are slightly different so that the corresponding hyperplane arrangements are slightly different. For example, the C_2 root system is defined $\Phi_{C_2} = \{(\pm 1, 0), (0, \pm 1), (\pm 1, \pm 1)\}$ as depicted in

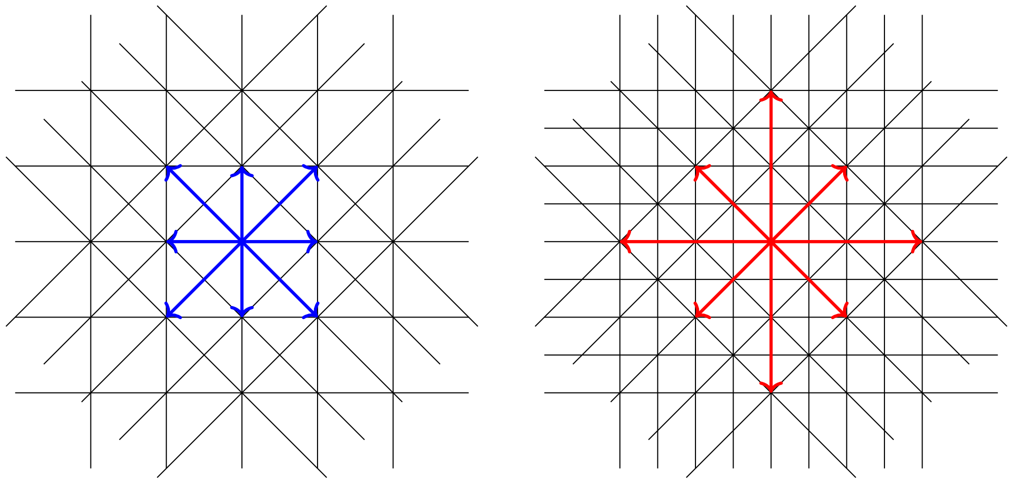


Figure 2.2.5: Type C_2 and B_2 root systems in blue and red respectively along with their corresponding hyperplane arrangements for the Coxeter groups of type $\widetilde{C}_2$ and $\widetilde{B}_2$.

blue in Figure 2.2.5. Then the coroots of short roots become the long roots of length 2 in the root system Φ_{B_2} depicted in red in Figure 2.2.5. Thus, the horizontal and vertical hyperplanes in the hyperplane arrangement of type $\widetilde{B}_2$ are spaced distance half as far apart as in the hyperplane arrangement of type $\widetilde{C}_2$.

2.3 Davis complex

This section recalls the definition of the Davis complex, starting with the Cayley graph, which constitutes its 1-skeleton. To describe the higher dimensional cells, this section defines W-polytopes as metric zonotopes, i.e. linear images of unit cubes defined by the root system of W. These are glued into the "holes" in the Cayley graph, one for each special coset. Moreover, the Davis complex is the cellular dual to the Coxeter complex with face poset the poset of special cosets. Thus, the action of the Coxeter group on the Davis Coxeter group is analogous.

Definition 2.3.1 (Unoriented right Cayley graph)**.** The *unoriented right Cayley graph*

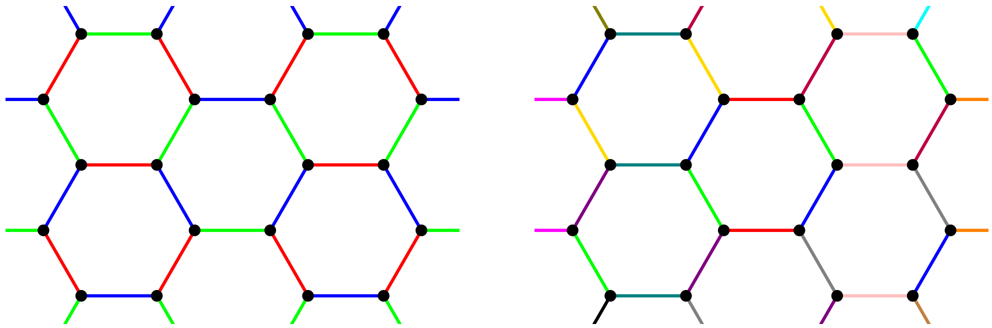


Figure 2.3.1: A right-colored and left-colored portion of the unoriented right Cayley graph for $\textsc{Cox}(\widetilde{A}_2)$ respectively.

for a Coxeter system (W, S) is a graph with vertices indexed by W and an edge between vertices v_g and v_h if there exists $s \in S$ such that $g \cdot s = h$. Notice that since $s^2 = 1$ that this implies that $g = h \cdot s$ as well.

The edges of the right Cayley graph can be labeled by the elements of S. That is, if there exist $g, h \in W$, $s \in S$ such that $g \cdot s = h$ label the edge between v_g and v_h with s or represent each $s \in S$ with a color and color the edges accordingly. For example, consider the type $\widetilde{A}_2$ Coxeter group with presentation

$$\langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^3 = (bc)^3 = (ca)^3 = 1 \rangle.$$

Letting blue correspond to a, red correspond to b, and green to c, Figure 2.3.1 depicts the unoriented right Cayley graph. This labeling or coloring of the edges is referred to as the *right labeling* of the right Cayley graph.

Similarly, in the *left labeling* of the right Cayley graph the edges are labeled by elements of the set of reflections, $T = \{wsw^{-1} \mid w \in W, s \in S\}$. If there exist $g, h \in W$, $s \in S$ such that $g \cdot s = h$, then $(gsg^{-1}) \cdot g = h$, where $gsg^{-1} \in T$. Accordingly, label the edge between v_g and v_h by the unique $t \in T$ such that $t \cdot g = h$. Notice that $t^2 = 1$ so that $g = t \cdot h$ as well.

Remark 2.3.2. Note that the edges of right or left labeling are invariant under action on the opposite side. That is, for a Coxeter group W and an edge labeled s in the right labeling with endpoints x and xs the vertex wx is adjacent to wxs by an edge labeled s for every $w \in W$. Conversely, for an edge labeled t in the left-labeling incident to vertices x and tx the vertex xw is adjacent to txw by an edge labeled t for every $w \in W$. Note that the collection of all edges labeled by t corresponds to a parallel family of all edges bisected by the hyperplane fixed by the reflection t. Since this dissertation focuses on the fundamental group of an oriented version of the Davis complex it will often use a labeling that is invariant under the right action of concatenation.

Definition 2.3.3 (Zonotope)**.** The *Minkowski sum* of polytopes $P_1, \ldots, P_n \subset \mathbb{R}^d$ is defined to be

$$P_1 + \cdots + P_n = \{\mathbf{x}_1 + \cdots + \mathbf{x}_n \mid \mathbf{x}_i \in P_i\}.$$

The structure of the sum is independent of the location of the origin. If a different location is chosen then the resulting Minkowski sum is just a translate of the original. When each of the P_i are line segments then the Minkowski sum is a *zonotope*. Note that a zonotope can be equivalently defined as a translate of the image of the unit cube $[-1, 1]^n$ under a linear transformation.

Definition 2.3.4 (W-permutohedron)**.** Suppose that (W, S) is a Coxeter system with a root system Φ, and let $\Pi = \{\alpha_1, \ldots, \alpha_N\}$ is the set of positive roots. Each of the positive roots α_i defines a line segment ℓ_i from $-\alpha_i$ to α_i. Then the zonotope $\ell_1 + \cdots + \ell_N$ is called a *W-permutohedron.* For more flexibility one can rescale the line segments before taking the Minkowski sum. In this disseration the root system of a Coxeter group W is scaled so that each of the vectors have length $1/2$ and the W-permutohedron has unit length edges.

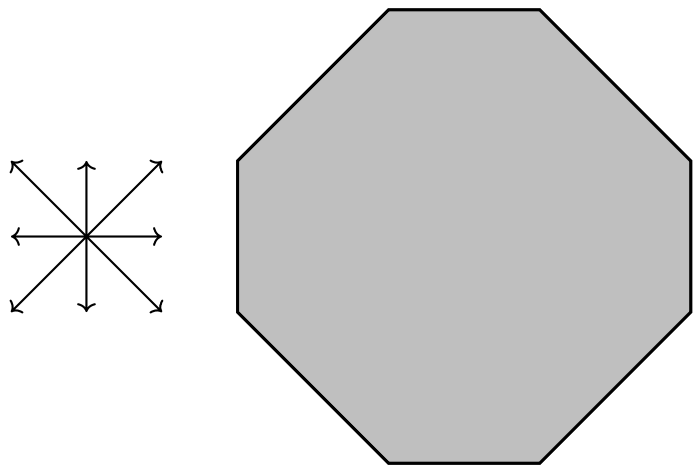


Figure 2.3.2: A W-permutohedron as a Minkowski sum of the crystallographic root system of type C_2.

Example 2.3.5 (Type C_2 permutohedron)**.** For type C_2 the Coxeter group $W = \text{Cox}(C_2)$ is symmetry group of a square, the dihedral group of order 8. The standard crystallographic root system is $\Phi_{C_2} = \{(\pm 1, 0), (\pm 0, 1), (\pm 1, \pm 1)\}$. Thus, Figure 2.3.2 is a W-permutohedron with vertex set $\{(\pm 3, \pm 1), (\pm 1, \pm 3)\}$.

Note that this is not regular octagon. However, as mentioned in Definition 2.3.4 the 1-skeleton of the Davis complex is typically taken to have unit edge length. Scaling each of the roots of Φ_{C_2} to length 1/2 yields the W-permutohedron depicted in Figure 2.3.3 that is a regular octagon with unit length edges.

Remark 2.3.6. Definition 2.3.4 and 2.3.7 are based on the definition found in [4]. In the literature the Davis complex is often the barycentric subdivision of the Davis complex as defined in this dissertation so that it is the order complex of the poset of special cosets. For example the barycentric subdivision of the type C_2 permutohedron is depicted in and is the order complex of the poset of special cosets. See Example 2.3.11 for more details of the special cosets of type $\widetilde{C}_2$.

Definition 2.3.7 (Davis complex)**.** Let W be a Coxeter group and let Γ be its oriented

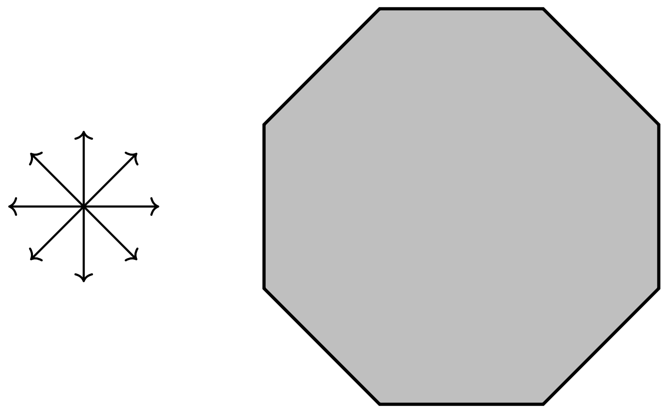


Figure 2.3.3: The permutohedron of type $\widetilde{C}_2$ as a regular octagon and the corresponding root system.

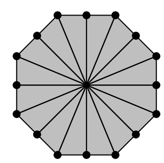


Figure 2.3.4: The barycentric subdivision of the type C_2 permutohedron.

right Cayley graph. For each special coset xW_T attach a metric W_T-permutohedron to the vertices of Γ labeled by the elements of xW_T. When $x'W_{T'} \subset xW_T$ are two such special cosets identify the $W_{T'}$-permutohedron attached to the vertices labeled by the elements in $x'W_{T'}$ as a face of the W_T-permutohedron attached to the vertices labeled by the elements in xW_T. Then the *Davis complex* is the resultant metric zonotopal complex.

Remark 2.3.8. As defined, the Davis complex is the cellular dual to the Coxeter complex, whose face poset is the poset of special cosets. Thus, the cells of the Davis complex are labeled by special cosets of the form xW_T for $x \in W$ and W_T a special coset of spherical type. For $g \in W$ the action is defined $g \cdot (xW_T) = (gx)W_T$. Observe that W acts simply transitively on the vertices, labeled by $xW_\emptyset$.

Example 2.3.9 (Davis complex of type $\widetilde{A}_2$)**.** Figure 2.3.5 displays a portion of the Davis complex of type $\widetilde{A}_2$ with each of its cells labeled by the special cosets, along with the

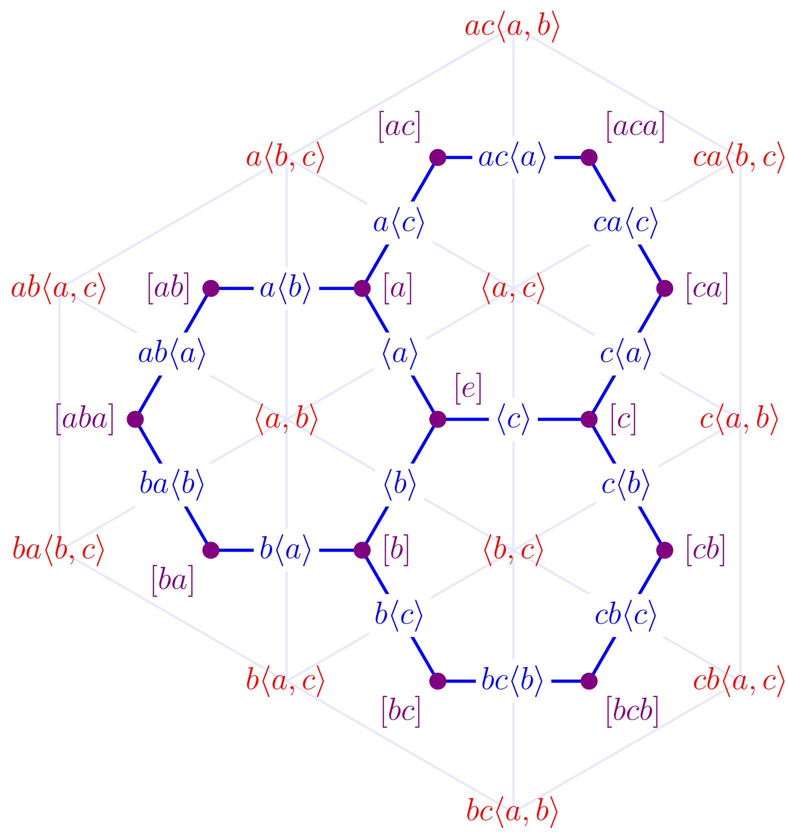


Figure 2.3.5: The Davis complex of type $\widetilde{A}_2$

Coxeter complex dual. Note that each of the 2-cells are hexagons. This follows from the fact that each of the spherical subgroups W_T of rank 2 are isomporphic to the Coxeter group of type A_2 whose roots as depicted in Figure 2.2.2 determine that each of these W_T-zonotopes are regular hexagons.

Remark 2.3.10. In Figure 2.3.5 the Davis complex and Coxeter complex are drawn together in the plane. In order to draw these two complexes as depicted it is necessary to scale the Coxeter complex or Davis complex accordingly. For example, if the edges of the Davis complex are unit length edges then the Coxeter complex would have edges of length $\sqrt{3}$. In general, this dissertation assumes that the Davis complex is a metric zonotopal cell-complex, with each of the edges unit length. This can conveniently be constructed in $\mathbb{R}^n$. Given a Euclidean Coxeter group W there exists some representation conjugate to the contragredient representation where W acts on $\mathbb{R}^{n+1}$. The complement of the hyperplanes fixed by reflections is a disjoint union of simplicial cones, and there is a copy of $\mathbb{R}^n$ spanned by the roots. The intersection of a translate of this copy of $\mathbb{R}^n$ in a direction orthogonal to the roots with the hyperplane arrangement yields a tiling of $\mathbb{R}^n$ by simplices. Each of these simplices has incenter the same distance from its facets. Note, however, that adjusting the distance of the affine patch from the origin adjusts the distance of the incenters from their facets so that there is an affine patch where each incenter is distance $1/2$ from its facets. Fix a fundamental chamber C for this action and the incenter x in the simplex at the intersection of C with an affine patch so that x is distance $1/2$ from each of its facets. Then the W orbit of x, forms the vertices of the Cayley graph. If two vertices $y, z \in W(x)$ are separated by a single facet, so that there is some reflection taking y to z, then they are connected by an edge. Moreover, a W_T-permutohedron glued in for a special coset is simply the convex hull of corresponding vertices in Cayley graph so that they each W_T-permutohedron has unit edge length.

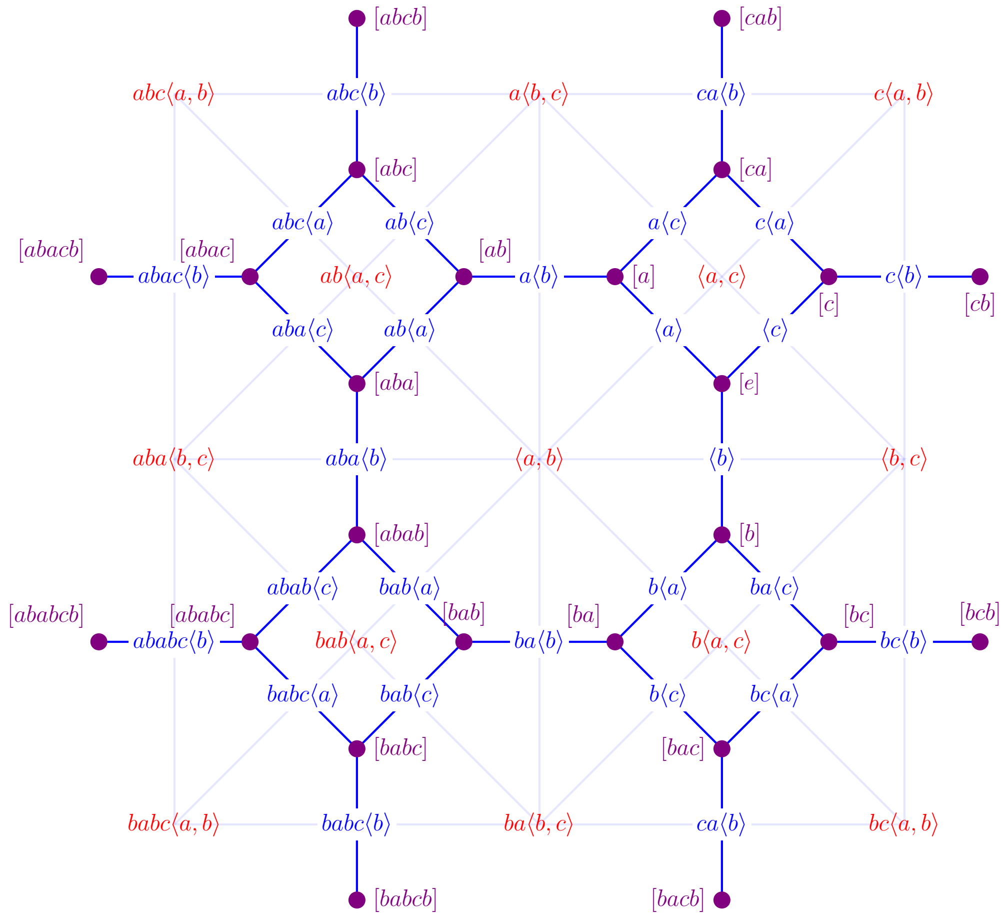


Figure 2.3.6: A portion of the Davis complex of type $\widetilde{C}_2$

Under this construction, the Davis complex lies in a translate of $\mathbb{R}^n$ in a direction orthogonal to the roots. The intersection of this affine subspace with the hyperplane arrangement yields the Coxeter complex. Thus, one can think about scaling the Coxeter complex and Davis complex as choosing some height orthogonal to the root system.

Example 2.3.11 (Davis complex of type $\widetilde{C}_2$)**.** Let W be the Coxeter group of type $\widetilde{C}_2$, i.e. the symmetry group of the 244 triangular tiling of the plane. A portion of the Davis complex of type $\widetilde{C}_2$ is depicted in Figure 2.3.6. The Coxeter graph of type $\widetilde{C}_2$ is

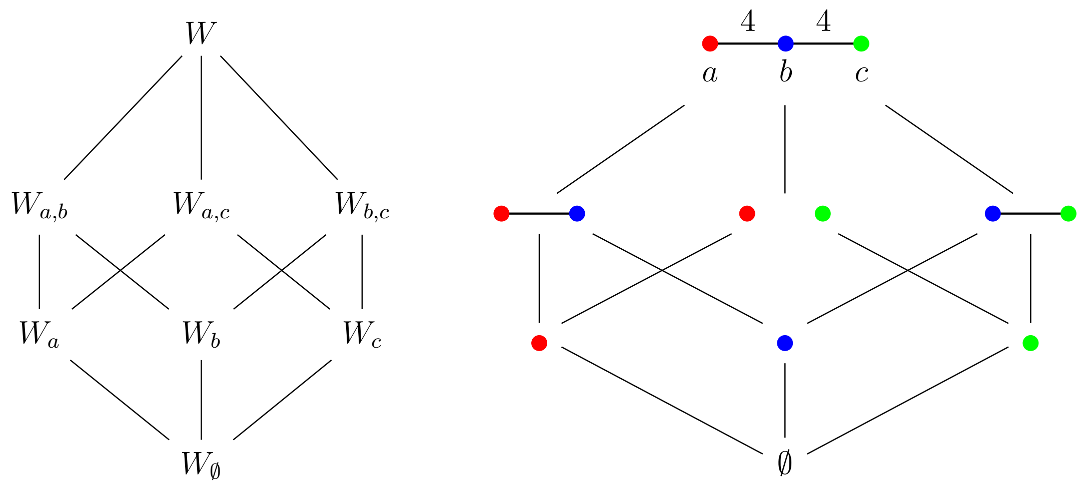
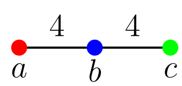


Figure 2.3.7: The poset of parabolic subgroups of $W = \text{Cox}(\widetilde{C}_2)$



and has poset of induced subgraphs that is isomorphic to the poset of parabolic subgroups of depicted in Figure 2.3.7. The two rank 2 spherical subgroups $W_{a,b}, W_{b,c}$ are isomorphic to the dihedral group of order 8 or $\text{Cox}(C_2)$. Thus, as in Example 2.3.5 $W_{a,b}$ and $W_{b,c}$-permutohedra are octagons. The remaining rank 2 spherical subgroup $W_{a,c}$ is of type $A_1 \times A_1$ and isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$ so that the $W_{a,c}$-permutohedron is a square, the product of two intervals. Accordingly, note that the Davis complex corresponds to the semiregular tesselation of the plane by octagons and squares.

2.4 Salvetti complex

There is a deep connection between Artin groups and Coxeter groups. One can think of an Artin group as a braided version of the corresponding Coxeter group. This section recalls the theorem from Harm van der Lek's thesis [5] which, in the Euclidean case, states that a Euclidean Artin group is the fundamental group of the quotient of the complexified hyperplane complement. Instead of working directly with the complexified hyperplane complement, this section reviews a homotopy equivalent space constructed as an oriented version of the Davis complex. The Coxeter group acts freely and properly discontinously so that the quotient of the oriented version of the Davis complex has fundamental group the Artin group. Then the section reviews a presentation for the Artin group analogous to the presentation for its associated Coxeter group. Importantly, this section reframes the question of understanding the word problem for Euclidean Artin groups in terms of the topology of this infinite vertex cell-complex.

Theorem 2.4.1. *Given a Euclidean Coxeter group W, the corresponding Artin group A is the fundamental group of the quotient of the complexified hyperplane complement under the action of the Coxeter group,*

$$A = \pi_1((\mathbb{C}^n \setminus \mathcal{H}^{\mathbb{C}})/W).$$

In the non-Euclidean case, instead take the complexified hyperplane complement of the complexified interior of the Tits cone, which in the Euclidean case is identified with $\mathbb{C}^n$ as some complexified affine patch of $\mathbb{R}^n$.

Example 2.4.2 (Type $A_1 \times A_1$ Artin group)**.** Consider the type $A_1 \times A_1$ case, with defining graph Γ consisting of two isolated vertices. The corresponding Coxeter group has presentation $W = \textsc{Cox}(\Gamma) = \langle a, b \mid a^2 = b^2 = 1, ab = ba\rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ generated by reflections in the hyperplanes $x = 0$ and $y = 0$ in $\mathbb{R}^2$. Note these reflections commute since the fix orthogonal hyperplanes. Thus, a and b are the full set of reflections and the two hyperplanes $x = 0$ and $y = 0$ are the complete hyperplane arrangement $\mathcal{H}$ of type $A_1 \times A_1$.

To complexify the hyperplane arrangement treat the equations $x = 0$ and $y = 0$ as equations with complex variables, using z, w to denote this. Then $\mathcal{H}^{\mathbb{C}}$ consist of the hyperplane $z = 0$ and $w = 0$ in $\mathbb{C}^2$, where W acts on $\mathbb{C}^2$ by $a \cdot (z, w) = (e^{i\pi} z, w)$ and $b \cdot (z, w) = (z, e^{i\pi} w)$. The complexified hyperplane complement in $\mathbb{C}^2$ consists of removing two orthogonal copies of $\mathbb{C}^1$, and it deformation retracts onto the product $\mathbb{S}^1 \times \mathbb{S}^1$, a torus. Quotienting by the action of W yields another torus that has a 4-fold cover by the torus that is a deformation retraction of $\mathbb{C}^2 \setminus \mathcal{H}^{\mathbb{C}}$. Specifically, the covering map restricted to each factor $\mathbb{S}^1$ is the quotient by the antipodal map. The Artin group A of type $A_1 \times A_1$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$ and admits the presentation $A = \langle a, b \mid ab = ba \rangle$.

Remark 2.4.3. The complexified hyperplane arrangement quickly becomes difficult to visualize. Instead of working with the complexified hyperplane complement, this dissertation focuses on a homotopy equivalent infinite vertex cell-complex called the pure Salvetti complex. This section follows the methods introduced by Salvetti in his thesis [6] and by Delucchi in [7].

Example 2.4.4 (Salvetti complex of type $A_1 \times A_1$)**.** As a continuation of Example 2.4.2, let $A = \mathbb{Z}^2$, $W = (\mathbb{Z}/2\mathbb{Z})^2$ be the Artin group and Coxeter group of type $A_1 \times A_1$, with hyperplane arrangement $\mathcal{H}$ consisting of $x = 0$ and $y = 0$ depicted in Figure 2.4.1 in the dashed red and blue lines respectively along with the Davis complex consisting of the square with red and blue edges.

To complexify a hyperplane arrangement, view $\mathbb{C}^n$ as a the tangent bundle of $\mathbb{R}^n$, i.e.

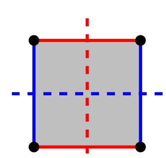


Figure 2.4.1: The Davis Complex and hyperplane arrangement of type $A_1 \times A_1$

express a point $\mathbf{z} = \mathbf{x} + i\mathbf{y} \in \mathbb{C}^n$ as a point in $\mathbf{x} \in \mathbb{R}^n$ and tangent vector $\mathbf{y} \in \mathbb{R}^n$. If this point *avoids* a complexified hyperplane of the form $a_1 z_1 + \ldots a_n z_n = b$, where $a_i, b \in \mathbb{R}$, then either $\mathbf{x}$ does not lie on the hyperplane $a_1 x_1 + \ldots a_n x_n = b$ or $\mathbf{y}$ does not lie on the hyperplane $a_1 y_1 + \ldots a_n y_n = 0$. Equivalently, $\mathbf{z}$ is in the complexified hyperplane complement if either (1) $\mathbf{x}$ is in the real complexified hyperplane complement so that the imaginary part of $\mathbf{z}$ is a single contractible component, (2) $\mathbf{x}$ lies on a single real hyperplane, so that $\mathbf{y}$ must avoid pointing in the direction of the hyperplane, splitting the imaginary part of $\mathbf{z}$ into two contractible halfspaces into which the hyperplane separates $\mathbb{C}^n$, or (3) $\mathbf{x}$ lies on an intersections of real hyperplanes, so that $\mathbf{y}$ must avoid pointing in the direction of any of these hyperplanes, splitting the imaginary part of $\mathbf{z}$ into multiple contractible cones.

In this example, identify $\mathbb{C}^2 = \mathbb{R}^2 + i\mathbb{R}^2$ with the tangent bundle of $\mathbb{R}^2$ so that every $(z, w) \in \mathbb{C}^2$ can be expressed as $(z, w) = (x_1, x_2) + i(y_1, y_2)$, and think of (z, w) as a point $(x_1, x_2) \in \mathbb{R}^2$ and a tangent vector (y_1, y_2). The complex depicted in Figure 2.4.2 is homotopy equivalent to the complexified hyperplane complement. It depicts the 1-skeleton along with labels detailing the correspondence of each cell to a portion of the complexified hyperplane complement, and the four squares below are glued accordingly. Although each of the edges are drawn curved, each of the oriented squares are metric Euclidean zonotopes. To describe the homotopy equivalence between the complexified hyperplane complement and this complex split into the cases (1)-(3) as outlined above.

A point $(z, w) \in \mathbb{C}^2$ can avoid the complexified hyperplanes $z = 0$ and $w = 0$ as summarized in (1) above if $x_1, x_2 \neq 0$. The real part of points (z, w) avoiding the real hyperplane arrangement consists of four contractible cones $x_1, x_2 > 0$, $x_2 > 0 > x_1$, $0 > x_1, x_2$, and $x_1 > 0 > x_2$ and the imaginary part is all of $\mathbb{R}^2$. Thus, the points in the complexified hyperplane complement (z, w) such that $x_1, x_2 \neq 0$ are replaced with the four vertices of Figure 2.4.2.

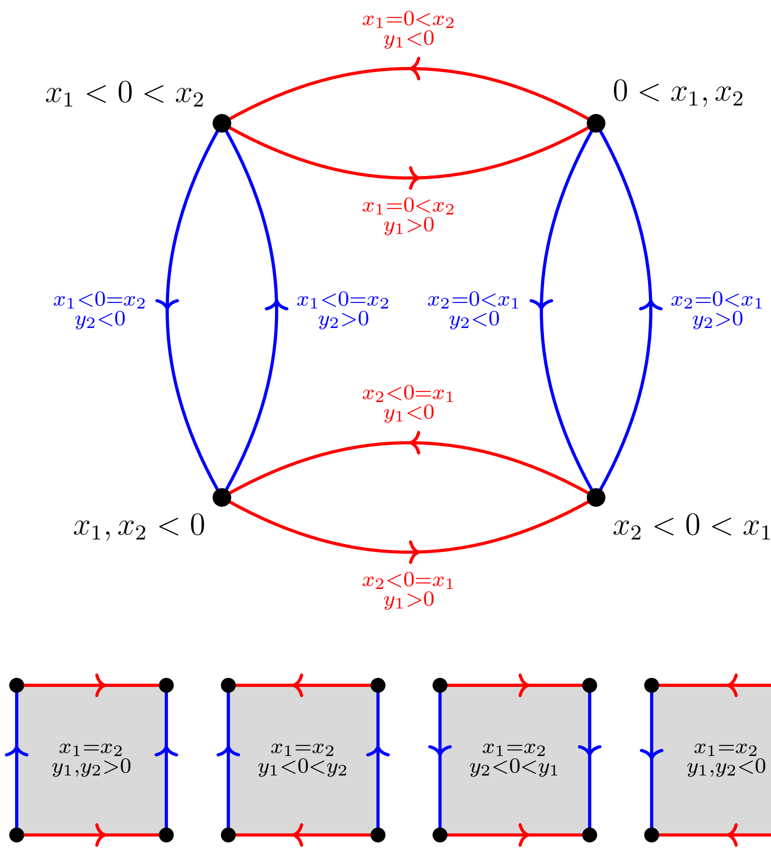


Figure 2.4.2: The pure Salvetti complex of type $A_1 \times A_1$.

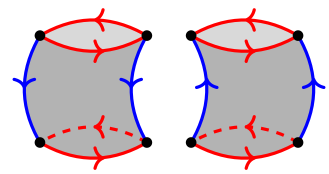


Figure 2.4.3: The pure Salvetti complex of type $A_1 \times A_1$ is a torus constructed out of four squares. These squares are shown as two cylinders glued top to top and bottom to bottom according to orientation.

As in case (2) a point (z, w) can avoid the complexified hyperplanes $z = 0$ and $w = 0$ if it lies on the real part of a single hyperplane and has imaginary part that avoids the direction of the hyperplane. Consider first the points (z, w) in the complexified hyperplane complement with real part only on the hyperplane $x_1 = 0$. If $x_1 = 0$ and $x_2 \neq 0$ then $y_1 \neq 0$, so that the imaginary part consists of two contractible regions $y_1 > 0$ and $y_1 < 0$. Thus, points (z, w) satisfying $x_1 = 0$, $x_2 > 0$ are replaced with the two top red edges in Figure 2.4.2 one edge for imaginary direction $y_1 > 0$ and one for imaginary direction $y_1 < 0$ glued to the two . The points satisfying $x_1 = 0$, $x_2 < 0$ are replaced with the two bottom red edges. Analogously, the set of points (z, w) with real part (x_1, x_2) on $x_2 = 0$, but not on $x_1 = 0$ are replaced by the four blue edges. Cases (1) and (2) form the 1-skeleton of the complex.

As described in case (3) if a point (z, w) has real part that lies at the intersection of the hyperplanes, i.e. $x_1 = x_2 = 0$, then the imaginary part must avoid pointing in the direction of these hyperplanes. That is, the imaginary part consists of four contractible cones $y_1, y_2 > 0$, $y_2 > 0 > y_1$, $0 > y_1, y_2$, and $y_1 > 0 > y_2$. Thus, the points (z, w) satisfying $x_1 = x_2 = 0$ are replaced by the four squares in Figure 2.4.3, glued to the edges according to their orientation.

The complex constructed and depicted in Figure 2.4.3 is a torus. It depicts two cylinders, each comprised of a pair of the two squares that share there orientation on the blue edges. The two cylinders glue top to top and bottom to bottom to form a

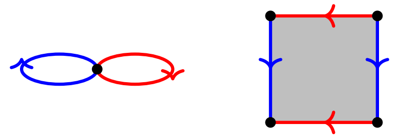


Figure 2.4.4: The Salvetti complex of type $A_1 \times A_1$ is the quotient of the pure Salvetti complex by W and equal to the presentation complex of A.

torus. The action of W as a complex reflection group on the complexified hyperplane arrangement is also translated into an action on pure Salvetti complex X and is precisely the same action as described in Example 2.4.2. Thus, the quotient by this free properly discontinuous action, is the single vertex complex consisting of a single vertex, bouquet of two circles, with a single oriented square glued in as depicted in Figure 2.4.4. This complex is the presentation 2-complex, which is a torus.

Remark 2.4.5. The details of Example 2.4.2 hold for general Artin groups as well. The interior of the Tits cone viewed as a subset of $\mathbb{R}^n$ is complexified as above and the complexified hyperplanes are removed. The complexified Coxeter group action is a π rotation about each complexified hyperplane, and it acts freely on this space. Moreover, the complexified interior of the Tits cone with the complexified hyperplanes removed is homotopy equivalent to an oriented version of the Davis complex referred to as the pure Salvetti complex in this dissertation. This section reviews the construction of the pure Salvetti complex as outlined in [4], where the 1-skeleton is the oriented right Cayley graph and the higher-dimensional cells are oriented W-permutohedra.

Definition 2.4.6 (The oriented right Cayley graph)**.** The *oriented right Cayley graph* for a Coxeter system (W, S) is a graph with vertices indexed by W and an oriented edge from v_g to v_h if there exists $s \in S$ such that $g \cdot s = h$. Notice that since $s^2 = 1$ this implies that $g = h \cdot s$ so there is also an oriented edge from v_h to v_g as well. As in Definition 2.3.1 refer to this as the *right labeling*. The *left labeling* of the oriented right Cayley graph has edges labeled by elements of the set of reflections, $T = \{wsw^{-1} \mid w \in W, s \in S\}$. That is, the edge from v_g to v_h is labeled by the unique $t \in T$ such that $t \cdot g = h$.

Definition 2.4.7 (Oriented W-permutohedron)**.** Let P be a W-permutohedron. Given a vertex v of P there is a unique vertex v' that is opposite v so that the vector from v to v' passes through the center of P. Orient each edge in the 1-skeleton of P so that its dot product with the vector from v to v' is positive. Note that P has as many orientations as it has vertices, which is equal to $|W|$.

Definition 2.4.8 (Salvetti Complex)**.** Let W be a Coxeter group and Y its Davis complex. The *pure Salvetti complex* has the same vertex set as Y, and each edge of the unoriented right Cayley graph 1-skeleton of Y is replaced with two oriented edges. Thus, the pure Salvetti complex has 1-skeleton the oriented right Cayley graph. More generally, each W_T-permutohedron of Y is replaced with $|W_T|$ many oriented permutohedra, one for each possible orientation. When a permutahedron has a smaller permutahedron as a face, the oriented version of the larger one is attached to the oriented version of the smaller one where the orientations are compatible. Note that there is a natural map form the pure Salvetti complex sending each oriented W_T-permutohedron to the corresponding unoriented W_T-permutohedron in the Davis complex.

The Coxeter group W acts freely on the pure Salvetti complex and its quotient is a 1-vertex complex called the *Salvetti complex* that has one oriented W_T-permutahedron for each spherical parabolic subgroup W_T. The 2-skeleton of the Salvetti complex is the presentation 2-complex.

The fundamental group of any Salvetti complex admits a straightforward presentation reminiscent of the presentation of a Coxeter group, which is formalized in the following definition.

Definition 2.4.9 (Artin Group)**.** Given a Coxeter system (W, S) with Coxeter graph Γ the *Artin group of type* Γ, denoted $\text{Art}(\Gamma)$, is defined to be the group generated by S

with one relation of the form

$$\underbrace{iji\ldots j}_{m_{ij}} = \underbrace{jij\ldots i}_{m_{ij}} \quad \text{or} \quad \underbrace{iji\ldots i}_{m_{ij}} = \underbrace{jij\ldots j}_{m_{ij}}$$

if m_{ij} is even or m_{ij} is odd respectively for each pair of vertices i, j such that $m_{ij} \neq \infty$. Any group, which admits a presentation of this form is referred to as an *Artin group*. The *pure Artin group* of type Γ is the kernel of the map of the map from $\text{ART}(\Gamma)$ onto $\text{COX}(\Gamma)$ sending S to S, i.e. sending each generator of the Artin group to the corresponding involutory generator in the Coxeter group.

The following theorem follows from the work of Salvetti [6] and Delucchi [7]

Theorem 2.4.10. *Given a Coxeter group W, the pure Salvetti complex is homotopy equivalent to the complexified hyperplane complement, and it's fundamental group is the pure Artin group. The quotient of the pure Salvetti complex by the action of W is the Salvetti complex which has fundamental group the Artin group.*

Definition 2.4.11. The pure Salvetti complex is homotopy equivalent to the complexified hyperplane complement. Consider first the complement of a single complex codimension 1 hyperplane H in $\mathbb{C}^n$. The space $\mathbb{C}^n \setminus H$ deformation retracts onto an infinite cylinder around H, which has fundamental group generated by a meridional loop around H. It is well-known that the fundamental group of the complexified hyperplane complement in the interior of the Tits cone is generated by based versions of meridional loops around the missing hyperplanes. The non-based version of a meridional loop around a missing hyperplane corresponds to a parallel family of bigons in the pure Salvetti complex.

Given a Coxeter group W with hyperplane arrangement $\mathcal{H}$, let $H \in \mathcal{H}$ be a hyperplane and let t be the reflection that fixes H. Define a *meridional bigon* dual to H in the pure

Salvetti complex X to be any bigon in the left labeled right Cayley graph with both edges labeled by t. This bigon consisting of two vertices in X, v_g and v_h, such that $t \cdot g = h$ and $t = gsg^{-1}$ for some s in the simple system. In the Euclidean case, there is an infinite family of parallel meridional bigons. It is well-known that one can choose meridional bigons, one for each hyperplane $H \in \mathcal{H}$, and that these are sufficient to generate the fundamental group of the pure Salvetti complex, i.e. the pure Artin group.

Chapter 3

Injectivity and subcomplexes

This section focuses on the fundamental groups of subcomplexes of the pure Salvetti complex in relation to the fundamental group of pure Salvetti complex as a whole. The subcomplexes where the inclusion map induces an injection on fundamental groups are of particular interest. This chapter focuses on the types of subcomplexes that meet this injectivity criterion for various types of Artin groups. As the complexity of the Artin groups is increased the requirements that subcomplexes must meet to satisfy the injectivity criterion become increasingly more refined. This chapter progresses through examples starting with 1-dimensional Artin groups, then one-relator Artin groups, and lastly spherical type exemplified by the four strand braid group. Each set of examples yields a more sophisticated strategy for understanding the word problem for Euclidean Artin groups. The chapter concludes with a rephrasing of the word problem in terms of a general strategy for giving a constructive solution. It involves finding an exhaustive family of subcomplexes that is coarse enough to satisfy the injectivity criterion, but fine enough so that the finite list of irreducible elements all have fundamental groups with easy to solve word problems.

3.1 Group elements and subcomplexes

This section introduces the language that will be used throughout the remainder of this dissertation to study the word problem for Euclidean Artin groups in terms of fundamental groups of compact subcomplexes of the pure Salvetti complex. This section formalizes the perspective of this dissertation, viewing an element of the Artin group as a loop in the 1-skeleton of Salvetti complex. However, since Coxeter groups are well-known to have solvable word problem it suffices to study elements in the kernel of the map from an Artin group to its Coxeter group, i.e. loops in the pure Salvetti complex.

Remark 3.1.1. A standard fact of algebraic topology textbooks such as Hatcher [8] is that *for a path-connected CW complex* X *the inclusion* $X^{(1)} \hookrightarrow X$ *of the 1-skeleton induces a surjection on* π_1. Thus, to study the fundamental group of the pure Salvetti complex, which is a path connected CW complex, it suffices to study loops in the 1-skeleton. Furthermore, one need only consider loops that locally don't have backtracks, which is made precise in the following definition.

Definition 3.1.2 (Combinatorial Loop)**.** A map $Y \to X$ between cell complexes is *combinatorial* if its restriction to each open cell of Y is a homeomorphism onto an open cell of X. A cell complex is *combinatorial* if the attaching map of each open cell of X is combinatorial for a suitable subdivision. Note that all of the complexes introduced in Chapter 2 are combinatorial. Given a combinatorial cell complex X a loop $\gamma : \mathbb{S}^1 \to X$ is *combinatorial* if γ is combinatorial for some suitable subdivision of $\mathbb{S}^1$.

Definition 3.1.3. Oftentimes, it will be convenient to describe subcomplexes of the pure Salvetti complex in terms of portions of the Coxeter complex or some subcomplex of the Davis complex. To avoid confusion, a subcomplex of the Coxeter complex is called a *Coxeter subcomplex*, a subcomplex of the Davis complex a *Davis subcomplex*, and a subcomplex of the pure Salvetti complex a *pure subcomplex*.

Note that pure subcomplexes need not contain all of the cells of the full preimage of a Davis subcomplex.

Definition 3.1.4 (Full subcomplex)**.** A subcomplex of the pure Salvetti complex is *full* if it is the preimage of a subcomplex of the Davis complex. That is, if a full pure subcomplex contains one oriented W-permutohedron then it contains all of the oriented W-permutohedra that have the same projection to the Davis complex.

Remark 3.1.5. Given a presentation for a group $G = \langle S|R \rangle$ there is an associated presentation 2-complex so that there is a bijection between words in $S \cup S^{-1}$ and combinatorial loops in the 1-skeleton of based at the single vertex. For an Artin group $A = \langle S|R \rangle$ the 2-skeleton $Z^{(2)}$ of the Salvetti complex Z is the presentation 2-complex so that the combinatorial loops in the $Z^{(1)}$ are in bijection with words in $S \cup S^{-1}$. Since the word problem is well-known for Coxeter groups it suffices to study loops that lift to loops in the pure Salvetti complex X. However, the pure Salvetti complex is a multivertex infinite cell-complex with no standard generating set. However, fixing a basepoint vertex in X a combinatorial loop in $X^{(1)}$ can be described by a word in denoting the edges traversed in the oriented right Cayley graph. Note the right labeling corresponds to words in $S \cup S^{-1}$ and the left labeling corresponding to words in $T \cup T^{-1}$, where T is the set of reflections that is in bijection with the hyperplane arrangement $\mathcal{H}$. One can think of left labeling by elements of T as naturally arising from the perspective of the pure Artin group the group generated by the meridional bigons as in Definition 2.4.11. Unless otherwise specified this dissertation will refer to the word determined by a combinatorial loop in $X^{(1)}$ to be such a description of the edges traversed in the left labeling of the oriented right Cayley graph. Also, this dissertation adopts the convention that edges traversed in the direction opposite their orientation correspond to capital letters.

Definition 3.1.6 (Word subcomplex)**.** Given a word determined by a combinatorial loop

γ in the 1-skeleton of the pure Salvetti complex X, consider the image of $p(\gamma)$, where $p : X \to Y$ is the usual map from the pure Salvetti complex to the Davis complex. The *Davis word subcomplex* is the 1-dimensional Davis subcomplex consisting of all of the edges traversed by γ. The preimage of the Davis word subcomplex in the pure Salvetti complex is the *pure word subcomplex*. Note that for any loop γ in the pure Salvetti complex the Davis and pure word subcomplexes are compact, connected, and full.

3.2 Graphs and subgraphs

This section focuses on the two examples of 1-dimensional Artin groups. That is, the pure Salvetti complex is a graph. In both examples the subcomplexes are subgraphs and the inclusion map induces an injection on fundamental groups. This section proves the fact that if all of the compact full subcomplexes of the pure Salvetti complex have inclusion maps that induce an injection on their fundamental groups then the word problem is solvable.

Example 3.2.1 (Type A_1 subcomplexes)**.** Consider the Artin group of type A_1. Algebraically, the Coxeter group is $\textsc{Cox}(A_1) \cong \mathbb{Z}/2\mathbb{Z}$, the Artin group is $\textsc{Art}(A_1) \cong \mathbb{Z}$, and the pure Artin group is $\text{P}\textsc{Art} \cong \mathbb{Z}$ with homomorphisms

$$\text{P}\textsc{Art}(A_1) \xhookrightarrow{f} \textsc{Art}(A_1) \xrightarrow{g} \textsc{Cox}(A_1)$$

where f maps the generator of the pure Artin group to its square in the Artin group and g is the quotient map sending the square of each generator to the identity. It is straightforward that the word problem is solvable for $\mathbb{Z}$. However, this dissertation focuses on solving the word problem in terms of the topology of subcomplexes of the pure Salvetti complex.

Figure 3.2.1: The hyperplane arrangement and Davis complex of type A_1.

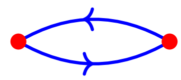


Figure 3.2.2: The pure Salvetti complex of type A_1.

For type A_1 the hyperplane arrangement is a subset of $\mathbb{R}$ and consists of the single hyperplane $x = 0$. One can think of the cell structure induced by the hyperplane arrangement as the two red rays and the blue vertex for the hyperplane $x = 0$ depicted in Figure 3.2.1. Recall, that the cellular dual is the Davis complex, depicted on the right of the figure, with two vertices and an edge connecting them. To construct the pure Salvetti complex, which is homotopy equivalent to the complexified hyperplane complement, orient the Davis complex. As depicted in Figure 3.2.2, it consists of two points connected by two oriented edges and is homeomorphic to a circle, with fundamental group $\mathbb{Z}$. Note that the complexified hyperplane arrangement consists of one hyperplane $z = 0$. Thus, the complexified hyperplane complement is $\mathbb{C}$ with the origin removed, where it is straightforward to see that it is homotopy equivalent to the pure Salvetti complex. Note that every proper subcomplex of the pure Salvetti complex of type A_1 is contractible and has trivial fundamental group. Thus, the word problem for type A_1 can be understood entirely in terms of the single bigon of the pure Salvetti complex.

Furthermore, the action of the Coxeter group on the complexified hyperplane complement $\mathbb{C} \setminus \{0\}$ is generated by a rotation by π. As an action on the pure Salvetti complex, this can be thought of as the reflection about the hyperplane $x = 0 \subset \mathbb{R}$, which exchanges the two vertices and opposite points on the edges. The quotient by this action is the Salvetti complex consisting of a single vertex and single edge that is a loop depicted in

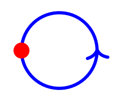


Figure 3.2.3: The Salvetti complex of type A_1.

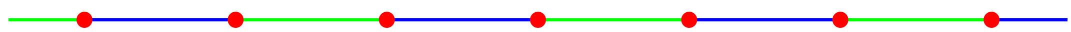


Figure 3.2.4: The Davis complex of type $\widetilde{A}_1$

Figure 3.2.3.

Example 3.2.2 (Type $\widetilde{A}_1$ subcomplexes)**.** In the case of type $\widetilde{A}_1$, the Coxeter group $\text{Cox}(\widetilde{A}_1) = \langle a, b \mid a^2 = b^2 = 1\rangle \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ is the infinite dihedral group, the Artin group $\text{Art}(\widetilde{A}_1) = \langle a, b\rangle \cong \mathbb{Z} * \mathbb{Z} = \mathbb{F}_2$ is a rank 2 free group, and the pure Artin group $\text{PArt}(\widetilde{A}_1) \cong \mathbb{F}_\infty$ is a countably infinite rank free group.

To describe the associated cell complexes recall that the hyperplane arrangement can be identified with $\mathbb{Z} \subset \mathbb{R}$ so that the Coxeter complex is an infinite path graph as described in 2.2.3. The dual Davis complex is just the unoriented right Cayley graph of the infinite dihedral group and is also an infinite path graph depicted in Figure 3.2.5. The green edges correspond to a, and the blue edges correspond to b in the right labeling of the unoriented right Cayley graph. The pure Salvetti complex depicted in 3.2.6 is the oriented right Cayley graph obtained from the Davis complex by replacing each edge with two oriented edges. It is homeomorphic to a countably infinite wedge of circles so that the fundamental group is a countably infinite rank free group. The Salvetti complex is the quotient of the pure Salvetti complex by the action of the Coxeter group. It is homeomorphic to a wedge of two circles with fundamental group $\mathbb{F}_2$.

Algebraically, the action of the Coxeter group induces an injective map from the pure Artin group to the Artin group, whose image is the kernel of the map from the Artin

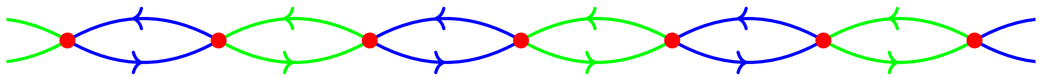


Figure 3.2.5: The pure Salvetti complex of type $\widetilde{A}_1$.

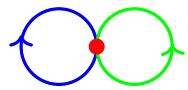


Figure 3.2.6: The Salvetti complex of type $\widetilde{A}_1$

group to the Coxeter group.

$$\mathbb{F}_\infty \hookrightarrow \mathbb{Z} * \mathbb{Z} \twoheadrightarrow \mathbb{Z}_2 * \mathbb{Z}_2$$

Proposition 3.2.3. *Suppose that K is a connected subgraph of a connected graph L and $i : K \hookrightarrow L$ is the inclusion map. Then $i_* : \pi_1(K) \to \pi_1(L)$ is an injection.*

Proof. It suffices to show that no nontrivial loop in K becomes trivial in L under i. Suppose that γ is a nontrivial loop in K. Let T be a spanning tree for L so that $T' = T \cap K$ is a spanning tree for K. Then, K/T' and L/T are both roses with their edges labeling the generators of their respective fundamental groups. In particular, the map $K/T' \to L/T$ induced by i is an injection on π_1 because there exists a retraction from L/T to K/T'. □

Definition 3.2.4 (π_1-injectivity)**.** Given a space L and a subspace $K \subset L$, K is *π_1-injective* (with respect to L) if the inclusion map $i : K \hookrightarrow L$ induces an injection on fundamental groups, $i_* : \pi_1(K) \to \pi_1(L)$.

Lemma 3.2.5. *All subcomplexes of the pure Salvetti complexes of type A_1 and all subcomplexes the pure Salvetti complexes of type $\widetilde{A}_1$ π_1-inject.*

Proof. In both type A_1 and type $\widetilde{A}_1$ the pure Salvetti complex is a graph so all of the subcomplexes are graphs. Thus, Proposition 3.2.3 implies subcomplexes of the pure Salvetti complexes of type A_1 and all subcomplexes the pure Salvetti complexes of type $\widetilde{A}_1$ π_1-inject. □

Example 3.2.6 (Structure of the pure Artin group of type $\widetilde{A}_1$)**.** Continuing Example 3.2.2 consider the word problem for the pure Artin group in terms of the topology of the pure Salvetti complex X and its compact connected full subcomplexes, which are finite subgraphs. Moreover, the edges of the bigons can be labeled by an integer corresponding the hyperplane $x = k \in \mathbb{Z}$ that they traverse. The image of any combinatorial loop is contained in the set of connected compact full subcomplexes corresponding to a finite subset of consecutive integers. Note that it suffices to only study these compact connected subcomplexes of consecutive bigons.

Specifically, suppose γ is a nullhomotopic combinatorial loop in the pure Salvetti complex X. Next, let K be the word subcomplex determined by γ. Then this a connected full subcomplex consisting of the bigons labeled by the finite subset of consecutive integers, one for each hyperplane that is traversed by γ. Consider the image of a nuhomotopy $H : \mathbb{S}^1 \times I \to X$. This, may include cells outside of K. However, consider the cover of the pure Salvetti complex consisting of the subcomplex K with both the leftmost and rightmost vertices connected to a countably infinite valence tree. For example if the loop γ just traverses the edges of two bigons then a portion of this cover is depicted in Figure 3.2.7. Note that the nullhomotopy lifts to this cover and since the union of K and the two trees deformation retracts to K there is a homotopy of the nullhomotopy H so that its image is entirely contained in K. If the loop was nullhomotopic in the full pure Salvetti complex then it is nullhomotopic in K. Thus, every subcomplex of this form is π_1-injective, and it suffices to study the fundamental group of a subcomplex K of this

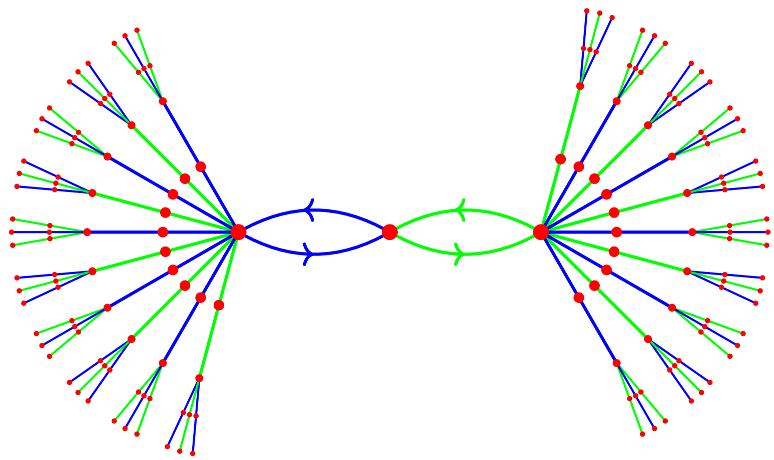


Figure 3.2.7: A portion of the cover of the pure Salvetti complex obtained by unraveling each of the bigons not traversed by a loop whose image is contained in two bigons

form, which is labeled by a finite consecutive string of integers in $[a, b] \subset \mathbb{R}$.

Suppose that K is a compact full subcomplex with edges labeled by integers in $[a, b]$ for $a \leq b \in \mathbb{Z}$. Then it is clear that K consists of the wedge of $r = b - a + 1$ circles and has fundamental group equal to rank r free group $\mathbb{F}_r$ with solvable word problem. However, consider decomposing $\pi_1(K)$ in terms of the family of all connected compact full subcomplexes consisting of finitely many consecutive bigons or the singleton vertices, thought of as zero consecutive bigons. Let c be some integer $a < c < b$ and express K as the union of two such proper connected subcomplexes L and M corresponding to consecutive bigons with edges labeled by integers in $[a, c]$ and $[c, b]$ respectively. Then $A \cap B$ is a single vertex and $\pi_1(K) = \pi_1(L) * \pi_1(M)$. Continuing in this way, each of the pieces $\pi_1(L), \pi_1(M)$ can be iteratively decomposed in an identical fashion to express $\pi_1(K)$ as an iterated free product. At some point this process terminates when each of the pieces of this iterated amalgamated free product is $\mathbb{Z}$, corresponding to the fundamental group of a single bigon. Since any loop in X is contained in a subcomplex K that can be written as a free product of the fundamental groups of the bigons understanding the

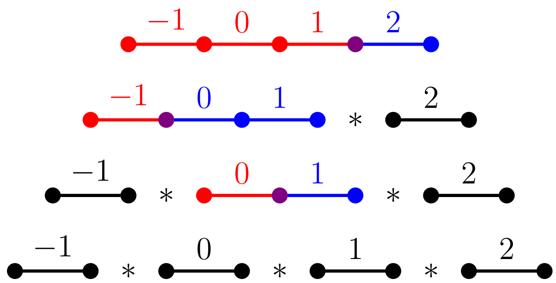


Figure 3.2.8: A decomposition of a word subcomplex.

word problem for the pure Artin group, $\pi_1(X)$ reduces to the word problem for a single bigon.

Remark 3.2.7. In Example 3.2.6 the decomposition of the pure word subcomplex K determined by a loop γ was done implicitly in terms of a decomposition of the corresponding Davis subcomplex. For example suppose that γ was a loop that traversed the hyperplanes labeled by the integers in $[-1, 2]$ so that the Davis word subcomplex determined by γ is depicted at the top of Figure 3.2.8. In each line of the figure any subcomplex that is not a single closed edge is written as a union of two proper subcomplexes with intersection a vertex. Since K_Y is compact this process terminates with K_Y written as the union of edges corresponding to closed intervals $[i, i+1]$, for $i \in \mathbb{Z}$. As a result, this decomposition of K_Y can be applied to K so that the fundamental group of K can be written as the free product of finitely many bigons. Specifically, for the K_Y depicted in Figure 3.2.8 $\pi_1(K) = (\mathbb{Z} * (\mathbb{Z} * \mathbb{Z})) * \mathbb{Z}$.

The remainder of the dissertation will follow in an analogy to the above example, where hyperplanes in the Davis complex will be used to decompose compact subcomplexes satisfying some injectively criterion into some union of “irreducible” pieces.

Definition 3.2.8 (Reducible subcomplex)**.** Suppose that X is the Davis complex and $K \subset X$ is a nonempty connected Davis subcomplex. Then K is *reducible* if there exist connected subcomplexes L, M with neither contained in the other such that $K = L \cup M$ and $L \cap M$ is connected. If K is not reducible then it is *irreducible.* A nonempty connected full pure subcomplex is reducible if it is the preimage of a reducible Davis subcomplex under the projection of the pure Salvetti complex to the Davis complex and is irreducible otherwise.

Lemma 3.2.9. *A compact connected Davis subcomplex is irreducible if and only if it consists of a single closed W-permutohedron.*

Proof. If K consists of a single closed W-permutohedron of dimension d then every proper subcomplex of K is of dimension strictly smaller than d so that K must be irreducible.

Conversely, suppose that K is a compact irreducible subcomplex of the Davis complex dimension d. Then there is a subcomplex L consisting of a single closed d-dimensional W-permutohedron P. If $L = K$ then K consists of a single closed W-permutohedron as desired. If not let M be the closure of $K \setminus L$. If M is not connected then partition the closure of $K \setminus L$ into two sets. Let K_1 be a connected component of the closure of $K \setminus L$, and let K_2 be the union of the remaining connected components. Then adjust L and M so that L is P union K_2 and M is the closure of K_1. Then $K = L \cup M$, L, and M are connected subcomplexes, neither containing the other, so that K is reducible, which is a contradiction. Thus, K must be a closed W-permutohedron of dimension d. □

Remark 3.2.10. Note that a Davis subcomplex consisting of a single closed W-permutohedron corresponds to a full pure subcomplex, which has fundamental group equal to the spherical Artin group corresponding to W. The word problem for all spherical Artin groups has been known to be solvable since the 1970's [9, 10].

Theorem 3.2.11. *Let X be the pure Salvetti complex of some fixed type Γ. If every full subcomplex of X π_1-injects then the fundamental group of every reducible full compact*

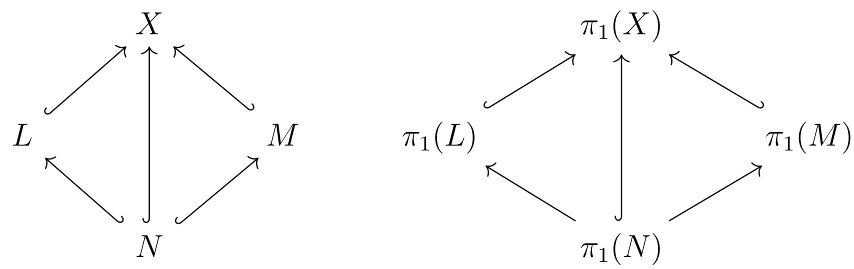


Figure 3.2.9: Inclusion maps of subcomplexes induce maps on fundamental groups.

subcomplex $K \subset X$ can be written as an iterated amalgamated free product of fundamental group of spherical Artin groups. Furthermore, if every amalgamated subgroup has decidable membership problem in each factor then the Artin group of type Γ has solvable word problem.

Proof. If K is reducible suppose that L, M are full subcomplexes of the pure Salvetti complex, X, such that $K = L \cup M$ and $N = L \cap M$ is a connected full subcomplex. Then there are inclusion maps for each subcomplex into the each of the complexes that contain it, depicted in the Figure 3.2.9 on the left.

Note for sets Y, Z, and W if $f : Y \to W$ and $g : Z \to W$ are injections then a map $h : Y \to Z$ such that $g \circ h = f$ is injective. By assumption every subcomplex π_1-injects so that the induced maps on the level of fundamental groups are all injections. This implies that the induced maps $\pi_1(N) \to \pi_1(L)$ and $\pi_1(N) \to \pi_1(M)$ are injections so that

$$\pi_1(K) = \pi_1(L) *_{\pi_1(N)} \pi_1(M).$$

As in Example 3.2.6 one may continue to decompose the pieces of the amalgamated free product, until $\pi_1(K)$ is expressed as an iterated amalgamated free product of the fundamental groups of its irreducible subcomplexes. By Lemma 3.2.9 these irreducible subcomplexes are of spherical type as desired.

To prove that the Artin group of type Γ has solvable word problem it suffices to show that the fundamental group of any compact full subcomplex K has solvable word problem since every combinatorial loop is contained in the pure word subcomplex which is a full compact subcomplex of X.

If K is a full irreducible pure subcomplex then it is the preimage of a closed d-dimensional W_T-permutohedron by Lemma 3.2.9. Since W_T is a spherical Coxeter group, K has fundamental group equal to the corresponding spherical Artin group. By Remark 3.2.10 K has solvable word problem.

If K is a full reducible pure subcomplex then it can be expressed as an iterated amalgamated free product of spherical Artin group, each of which has solvable word problem by Remark 3.2.10. Furthermore, if each of the amalgamated subgroups has decidable membership problem then the word problem for K is solvable, see Corollary 3.6.6. □

Remark 3.2.12. Note the membership problem for the fundamental group of subgraph K into the fundamental group of a graph L, where $\pi_1(K)$ is viewed as a subgroup of $\pi_1(L)$ under the map induced by inclusion, is decidable. To see this, note that since L is a graph every homotopy class of a path in K has a unique representative without backtracks. Thus, a path γ in L with both endpoints in K is an element of $\pi_1(K)$ if and only if the unique representative for γ without backtracks is entirely contained in K.

Corollary 3.2.13. *The word problem is solvable for both 1-dimensional Artin groups.*

3.3 One-relator Artin groups

This section focuses on the π_1-injectivity of subcomplexes of one-relator Artin groups. It is established that every one-relator Artin group can be understood in terms of an Artin group that is just the braided version of a dihedral group, i.e. type $I_2(m)$. Then this section presents the example of type $A_1 \times A_1$, where it is straightforward to see that not all subcomplexes π_1-inject. The section concludes by classifying which connected full pure subcomplexes π_1-inject.

Remark 3.3.1. If A is a one-relator Artin group, then it contains a unique finite rank 2 parabolic subgroup A' of type $I_2(m)$ corresponding to a unique pair $\{a, b\}$ of the standard generators satisfying a nontrivial braid relation. All other rank 2 parabolic subgroups are infinite and thus all other pairs of vertices do not satisfy a nontrivial braid relation, i.e $m_{ij} = \infty$ for all other pairs of vertices. Thus, any 1-relator Artin group A can be expressed as the free product $\mathbb{F}_r * A'$ of a finite rank r free group and an Artin group A' of type $I_2(m)$. Since the word problem is well understood for free products in terms of its pieces, see Proposition 3.6.3, it suffices to study the pure Salvetti complex for groups of type $I_2(m)$

Example 3.3.2 (Type $A_1 \times A_1$)**.** Recall the cell complexes of type $A_1 \times A_1$ described in Example 2.4.4. Topologically, the pure Salvetti complex is a torus with cell structure depicted in Figure 2.4.3. Next consider the full connected pure subcomplexes and their injectivity. Proper connected Davis subcomplexes are vertices or connected 1-dimensional Davis subcomplexes. The corresponding full pure subcomplexes are vertices which trivially π_1-inject, and the preimage of the connected 1-dimensional Davis subcomplexes correspond to the wedge of r bigons, $r = 1, 2, 3, 4$, which have fundamental group a free group of rank r.

Now consider the π_1-injectivity of each of these subcomplexes. It is straightforward that the fundamental group of a single bigon π_1 injects. However, if there are two or more bigons this corresponds to a free group that does not π_1-inject. Consider subcomplex K consisting of two adjacent bigons with fundamental group $\pi_1(K) \cong \mathbb{F}_2$. Note that these loops commute in the full pure Salvetti complex homeomorphic to a torus. For example,

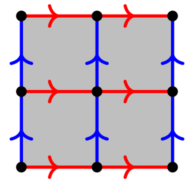


Figure 3.3.1: A disk diagram for the word $a^2b^2A^2B^2$ up to some conjugation depending on choice of basepoint.

the loop γ defined by the word $a^2b^2A^2B^2$ is non trivial in $\pi_1(K)$, which is generated by the red meridional bigon with edges labeled a and blue meridional bigon with edges labeled b as depicted in Figure 2.4.3. However, this loop is nullhomotopic in the pure Salvetti complex X. In particular, there is a map of a disk into X whose boundary is the loop γ as illustrated in Figure 3.3.1.

Remark 3.3.3. In general, full pure subcomplexes do not π_1-inject. Anytime the pure Salvetti complex contains a 2-dimensional W-permutohedron, i.e. a polygon, any full 1-dimensional pure subcomplex K containing at least half of the 1-skeleton does not π_1-inject. In particular, K is a graph and has a fundamental group a free group, but loops pure Salvetti complex satisfy nontrivial relations in the 2-cells. Since the pure Salvetti complexes of the Euclidean Artin groups of type $\widetilde{A}_2$ and $\widetilde{C}_2$ considered in the subsequent chapters are built out of oriented polygons from one-relator Artin groups it will be necessary to understand which connected full pure subcomplexes of type $I_2(m)$ π_1-inject.

Observe, that the only connected 0-dimensional subcomplexes of the pure Salvetti complex of type $I_2(m)$ consist of a single vertex. Also, the only 2-dimensional full connected pure subcomplex is the entire pure Salvetti complex. These 0 and 2-dimensional subcomplexes π_1-inject trivially. Thus, it suffices to study the injectivity of the 1-dimensional full connected pure subcomplexes corresponding to a path in the edges of the Davis complex that traverses less than half of a polygon. The next example considers

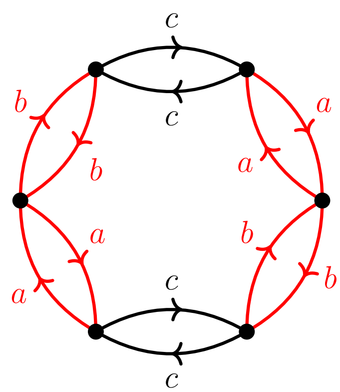


Figure 3.3.2: The oriented right Cayley graph of type A_2 with the left labeling.

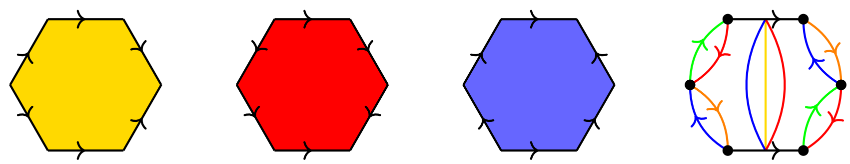


Figure 3.3.3: The subcomplex of the pure Salvetti complex of type consisting of all cells that agree with the choice of orientation on the top and bottom black edges.

the case where the Davis complex is a hexagon, i.e. type $A_2 = I_2(3)$.

Example 3.3.4. Suppose that A is the three strand braid group, i.e. the Artin group of type A_2. This example highlights how the fundamental group of two adjacent bigons injects into the fundamental group of pure Salvetti complex. Let X be the pure Salvetti complex of type A_2 so that X consists of six oriented hexagons glued into the oriented Cayley graph of the symmetric group on three letters depicted in Figure 3.3.2. In particular, the two pairs of consecutive bigons, highlighted in red in the figure, with edges labeled a and b are homotopy equivalent. Any loop in the left pair can be homotoped to a loop in the right pair.

To see this, consider the subcomplex K of X obtained by choosing an orientation for the edges labeled c and gluing in the three oriented hexagons that include the chosen orientation of c. This is depicted in Figure 3.3.3 with 1-skeleton on the left and the

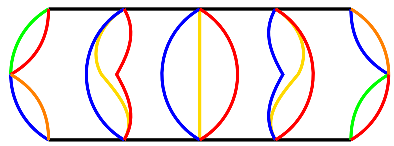


Figure 3.3.4: A depiction of the homotopy equivalence between the the bigons on either side of the hexagon

blue, red, and yellow oriented hexagons glued according to the orientations on the edges. Note that the edges are colored according to the oriented hexagon to which they belong. For example, the green edge is included in both the yellow and blue hexagon. Also, a midsection is depicted, which is the preimage of the Davis hyperplane that bisects the hexagon and the parallel edges labeled c. It consists of a blue, red and yellow edge connecting the midpoints of the two edges labeled c with the chosen orientation. Note that K is homotopy equivalent to either pair of the adjacent bigons with edges a and b as well as the midsection theta graph with red, yellow, and blue edges as depicted in Figure 3.3.4.

Furthermore, to see how a loop in the left pair of adjacent bigons can be rewritten as a loop in the right pair of adjacent bigons consider the following. Fix a basepoint to be the bottom left vertex incident to the edges labeled a and c. Then the word $abbaaBBa$ corresponds to a loop entirely contained in consecutive bigons on the left. Following this loop along the homotopy equivalence as depicted in Figure 3.3.4, one obtains the loop $cbaabbaaBBAabC$, which is just a conjugate of a loop based entirely on the right side. One can view this process of rewriting in terms of the subdivided disk mapped into K as depicited in Figure 3.3.5.

Lemma 3.3.5. *Let X be the pure Salvetti complex of type $I_2(m)$. Then any 1-dimensional subcomplex consisting of r consecutive bigons for $1 \leq r \leq m-1$ π_1-injects.*

Proof. Suppose that A is the Artin group of type $I_2(m)$ and let X denote the pure Salvetti complex. It consists of $2m$ oriented $2m$-gons glued into the oriented right Cayley

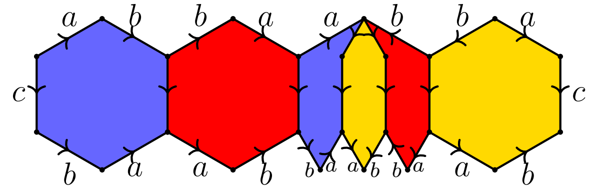


Figure 3.3.5: A representation of the homotopy between a loop in the left two bigons and the right bigons of the pure Salvetti complex of type $\widetilde{A}_2$.

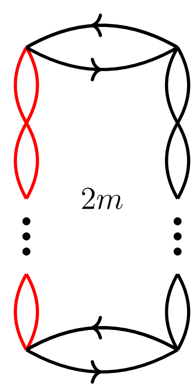


Figure 3.3.6: The 1-skeleton of the pure Salvetti complex of type $I_2(m)$ with a π_1-injective subcomplex highlighted in red. The $2m$ in the middle of the figure indicates the number 2-cells glued in. Specifically, there are $2m$ oriented $2m$-gons that are glued into the 1-skeleton according to orientation.

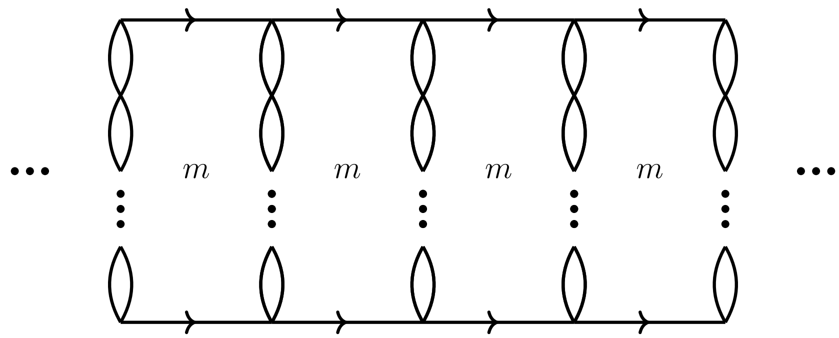


Figure 3.3.7: The cover of the pure Salvetti complex corresponding to the kernel of the map to the winding number around the hyperplane dual to the top and bottom bigons of Figure 3.3.6. The $m's$ in the figure indicate that there are m oriented $2m$-gons glued to the surrounding portion of the 1-skeleton.

graph as in Figure 3.3.6. It suffices to show that the subcomplex K highlighted in red consisting of $m-1$ adjacent bigons π_1-injects since Proposition 3.2.3 implies each of the proper subgraphs of adjacent bigons π_1-injects into the $m-1$ adjacent bigons. To show this, suppose that γ is a nullhomotopic combinatorial loop in the K and argue that the there exists a nullhomotopy of γ that is entirely contained in K. Thus, if a loop γ in K is nullhomotopic in X then it is nullhomotopic in K.

First consider the map from $\pi_1(X) \to \mathbb{Z}$, sending a loop in X to its winding number around the hyperplane dual to the top and bottom oriented edges. The cover $\widetilde{X}$ corresponding to the kernel of that map is depicted in Figure 3.3.7. It consists of $\mathbb{R}$ divided into $\mathbb{Z}$ many intervals across the top and bottom and vertically there are copies of the wedge of $m-1$ consecutive bigons. The 2-cells are glued in groups of m oriented $2m$-gons with orientation matching the top and bottom edges. Note that this entire cover $\widetilde{X}$ deformation retracts onto any single copy of $m-1$ consecutive bigons. Thus, any nullhomotopy of γ in X lifts to $\widetilde{X}$, which is homotopy equivalent to a chosen lift if K. Thus, there exists a nullhomotopy of γ that is entirely contained in K.

□

3.4 Convex subcomplexes

This section introduces the notion of convex subcomplexes in terms of the hyperplane arrangement for a given Artin group. Convexity determines a coarser notion of a subcomplex, and for type $I_2(m)$ the convex full pure subcomplexes do π_1-inject. This section then goes on to characterize the irreducible convex Davis subcomplexes. They are all closed W-permutohedra so that they correspond to full pure subcomplexes with fundamental group a spherical Artin group. As a result, it is possible to conclude that if all of the compact convex full subcomplexes of the pure Salvetti complex π_1-inject then the word problem is solvable.

Definition 3.4.1 (Convexity)**.** Define a *convex Coxeter subspace* to be the subcomplex consisting of the union of all open cells contained in an intersection of open halfspaces determined by the hyperplane arrangement. Note that each of these cells correspond to a special coset. However, this does not correspond to a subcomplex since it is open. A *convex Davis subcomplex* is defined to be the cellular dual of the convex Coxeter subspace, so that the special cosets labeling the cells of the convex Davis subcomplex correspond to a convex Coxeter subspace. A *convex pure subcomplex* is the preimage of a convex Davis subcomplex under the usual map from the pure Salvetti complex to the Davis complex.

Example 3.4.2. Consider the $\widetilde{A}_2$ hyperplane arrangement, and consider the six hyperplanes which bound the dotted regular hexagonal region as depicted in Figure 3.4.1. These hyperplanes determine six open halfspaces, which intersect to form the convex subset of $\mathbb{R}^2$ shaded in red.

Figure 3.4.2 depicts the convex Coxeter subspace, as well as the corresponding Davis and pure subcomplexes determined by this convex subset of $\mathbb{R}^2$. The convex Coxeter subspace, show on the left of the figure, consists of six open two cells, six open 1-cells, and one 0-cell. Its dual is a convex Davis subcomplex depicted in the center of the figure.

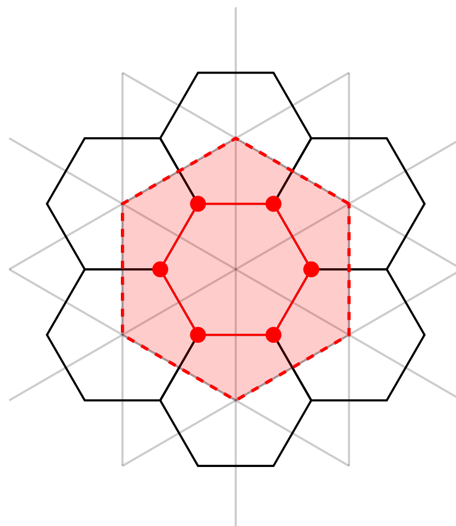


Figure 3.4.1: A convex region defined by the intersection of open halfspaces of the $\widetilde{A}_2$ hyperplane arrangement.

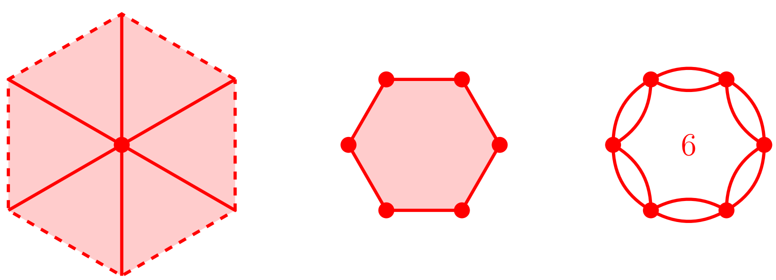


Figure 3.4.2: A convex Coxeter subspace, convex Davis subcomple, and convex pure subcomplex corresponding to the same intersection of halfspaces in the hyperplane arrangement.

The preimage of the convex Davis subcomplex under the map from the the pure Salvetti complex to the Davis complex has 1-skeleton depicted on the right. The edges in the 1-skeleton also have orientation and there are also six oriented hexagons glued into the 1-skeleton according to orientation.

Remark 3.4.3. If K and L are two convex Coxeter subspaces then $K \cap L$ is just the intersection of each of the halfspaces that defines either K or L. Thus $K \cap L$ is convex, which implies that the dual subcomplex and its preimage in the pure Salvetti complex are convex. Thus, the intersection of convex subcomplexes is convex.

Note also that every convex subcomplex is connected. To see this, suppose that Y is the Davis complex of some fixed type Γ and K is a convex Davis subcomplex. If K consists of a single vertex then it is connected. Suppose that v and w are vertices, then they correspond to adjacent open chambers in the Coxeter complex. Since it is convex it must contain all of the boundary points between them since they lie on some line segment between points in each of the chambers. Thus, there is an edge between v and w in the Davis complex, and two oriented edges between v and w in the pure Salvetti complex.

Definition 3.4.4 (Convex hull)**.** The *convex hull* of a Davis subcomplex K, denoted $\text{conv}(K)$ is the intersection of all the halfspaces containing all of K. Given a pure subcomplex K_X, let $K = p(K_X)$ be the image of K_X under the usual map $p : X \to Y$ from the pure Salvetti complex to the Davis complex. The convex hull of K_X, denoted $\text{conv}(K_X)$, is the preimage under p of the convex hull of K.

Lemma 3.4.5. *The convex hull of a compact Davis subcomplex K is compact.*

Proof. The Davis complex Y has a natural metric defined on the set of vertices coming from the word metric on the unoriented right Cayley graph. Define the length of a combinatorial path in $Y^{(1)}$ to be the number of edges that it traverses, and for two vertices

$u, v \in Y^{(0)}$ define the distance $d(u, v)$ to be the minimum length among all combinatorial paths in $Y^{(1)}$ from u to v. Thus, if K is a compact then the $K^{(1)}$ has bounded diameter, i.e. the maximum distance between any two points of K is a bounded.

Next suppose that w is a vertex of $\text{conv}(K) \setminus K$. If H is a hyperplane where K lies entirely on one side then w is contained on the same side of H as K. Since K is compact there are only finitely many hyperplanes which intersect K. Also, each edge of path from w to a vertex of K crosses exactly one of these finitely many hyperplanes that intersect K. Moreover, the deletion condition as in [11] implies that if the path traverses a edge more than once, the path may be shortened to one that crosses an edge in the Davis complex at most one time. Then every vertex w in $\text{conv}(K)$ is a bounded distance away from every vertex of K. Therefore the 1-skeleton of $\text{conv}(K)$ has bounded diameter. Since the Cayley graph $Y^{(1)}$ is a locally finite graph $\text{conv}(K)$ contains finitely many vertices. Note, that there are finitely many closed cells of Y that contain the vertices of $\text{conv}(K)$. Therefore, $\text{conv}(K)$ is compact. □

Recall, Remark 3.3.3, which stated that pure subcomplexes corresponding to Davis subcomplexes that contain at least half of the edges of a 2-dimensional W-permutohedron, but don't contain the W-permutohedron itself do not π_1-inject. Convexity guarantees that this situation does not arise. Thus, Lemma 3.3.5 implies the following corollary.

Corollary 3.4.6. *All full convex subcomplexes of the pure Salvetti complex of type $I_2(m)$ π_1-inject.*

Proof. Consider the Davis complex Y of type $I_2(m)$ and its subcomplexes. Note that Y consists of a single closed $2m$-gon and has dual hyperplane arrangement $\mathcal{H}$ consisting of the perpendicular bisectors of each pair of parallel edges. Thus, any open halfspace does not contain any vertices of the Coxeter complex, which implies that the only 2-dimension convex Davis subcomplex is all of Y. The corresponding full pure subcomplex is all of the pure Salvetti complex X and trivially π_1-injects.

Next, any hyperplane of $\mathcal{H}$ intersects a pair of opposite parallel edges of the Davis complex so that any proper convex Davis subcomplex can contain at most $m-1$ edges of the Davis complex. Moreover, a convex Davis subcomplex is connected so that any 1-dimensional convex Davis subcomplex must be the union of between 1 and $m-1$ consecutive edges. The corresponding pure subcomplex is the union of between 1 and $m-1$ consecutive bigons, which by Lemma 3.3.5 π_1-injects.

The only 0 dimensional convex subcomplexes consist of a single vertex so that they π_1-inject. The desired result follows. □

The importance of π_1-injectivity for subcomplexes of a certain type is that it guarantees that the fundamental group of compact subcomplexes of the same type can be expressed in terms of amalgamated free products of "atomic" subcomplexes.

Definition 3.4.7 (Convex reducible)**.** Suppose that Y is the Davis complex and $K \subset Y$ is a nonempty convex Davis subcomplex. Then K is *convex reducible* if there exist convex subcomplexes L, M with neither contained in the other such that $K = L \cup M$. If K is not convex reducible then it is *convex irreducible*. A nonempty connected full pure subcomplex is convex reducible if it is the preimage of a convex reducible Davis subcomplex under the projection to the Davis complex and is irreducible otherwise.

Remark 3.4.8. Note that if a convex Davis subcomplex K has nonempty intersection with a pair of parallel hyperplanes H_1, H_2 then it is reducible. This follows from the fact that the union of the halfspace determined by H_1 containing H_2 and the halfspace determined by H_2 containing H_1 is the entire Davis complex. Therefore $K = L \cup M$, where L is the union of all of the cells of K entirely contained in the halfspace of H_1 containing H_2 and M is the union of all of the cells of K entirely contained in the halfspace of H_2 containing H_1 so that L, M are convex and neither contains the other, and their union is K.

Remark 3.4.9. If a convex Davis subcomplex is 0-dimensional, then it must be a single vertex since any convex Davis subcomplex is connected. If a convex Davis subcomplex is 1-dimensional and consists of more than one edge then it is reducible just as the subcomplexes of type $\widetilde{A}_1$. See Example 3.2.6. Thus, the convex irreducible subcomplexes of the Davis complex of some fixed type Γ of dimension less than 2 are either a single vertex or a single edge, which can be thought of as $W_\emptyset$-permutohedron or W_s-permutohedron for some $s \in S$, where (W, S) is the Coxeter system of type Γ.

Lemma 3.4.10. *If Y is the Davis complex of some fixed type Γ then the compact convex irreducible subcomplexes of Y are precisely the subcomplexes consisting of a single closed W-permutohedron.*

Proof. It suffices to consider the case for convex Davis subcomplexes of dimension at least 2 by Remark 3.4.9.

If K is a single closed W-permutohedron then every hyperplane that intersects K bisects K. In particular, every proper convex subcomplex of K must be of strictly smaller dimension. So there exist no convex subcomplexes L, M neither containing the other such that $L \cup M = K$. In other words K convex irreducible.

Conversely suppose that K is a compact irreducible convex Davis subcomplex. Denote the set of all hyperplanes that bisect an edge of K as $\mathcal{H}'$. For $H \in \mathcal{H}'$ denote the intersection of H with K as H^K. Each hyperplane $H \in \mathcal{H}'$ determines an orthogonal reflection r_H that fixes H. Consider the group W' generated by all such r_H. Suppose that the every element of W' does not preserve K setwise to obtain a contradiction. This implies that there are two hyperplanes $H_1, H_2 \in \mathcal{H}'$ such that $r_{H_1}(H_2) = H_3 \notin \mathcal{H}'$. Since K is irreducible Remark 3.4.8 implies that H_1 and H_2 intersect. Moreover, since r_{H_1} is a reflection, $H_1 \cap H_2 = H_1 \cap H_3$. But since $H_3 \notin \mathcal{H}'$ the intersection $H_1 \cap H_2$ is not in K. Hence, K lies entirely in a halfspace determined by H_3. This halfspace of H_3 is contained in the union of the halfspace of H_1 that contains H_2^K and the halfspace of H_2 that contains H_1^K. Therefore, K is reducible as $K = L \cup M$, where L is the union of all cells of K contained in halfspace of H_1 containing H_2^K and M is the union of all cells of K contained in halfspace of H_2 containing H_1^K. This contradiction implies W' must fix K setwise.

Suppose that elements of W' fix K setwise. Recall that the Davis complex is a CAT(0) space, see [11], and note that K is bounded. Because K is a bounded convex subset of a complete CAT(0) space it has a unique (circum)center, and this center is contained in K. In particular, each of the isometries in W' must fix this center, and it lies in all of the hyperplanes in $\mathcal{H}'$. The center of K is in some cell of K, i.e. some permutohedron P.

Since every hyperplane in $\mathcal{H}'$ intersects P, every hyperplane that doesn't intersect P is not in $\mathcal{H}'$ and therefore doesn't intersect K. This means that all of K is in one halfspace determined by such a hyperplane and K is entirely contained in the intersection of all such halfspaces. Finally, since the intersection of all of the halfspaces containing P is P itself, K can be no larger than P. This shows that K and P are equal. Therefore, the convex irreducible subcomplexes of Y are precisely the subcomplexes consisting of a single closed W-permutohedron.

□

Lemma 3.4.11. *If every full compact convex subcomplexes of the pure Salvetti complex X of fixed type Γ π_1-inject then the fundamental group of a reducible full compact convex pure subcomplex K can be written as an amalgamated free product of fundamental groups of proper subcomplexes of the same type as K.*

Proof. Let $p : X \to Y$ be the usual map from the pure Salvetti complex to the Davis complex of type Γ, and denote the image of K as $p(K) = K_Y$. By assumption K_Y is not a closed W-permutohedron. Thus, Lemma 3.4.10 implies that $K_Y = L_Y \cup M_Y$ for convex Davis subcomplexes L_Y, M_Y with neither containing the other. Hence, $L = p^{-1}(L_Y)$ and $M = p^{-1}(M_Y)$ are proper full compact convex subcomplexes of K such that $L \cup M = K$. Moreover, $N = L \cap M = p^{-1}(L_Y \cap M_Y)$ so that N is a full convex pure subcomplex. By assumption the L, M, and N π_1-inject in X. Since the composition of the maps induced by inclusion $\pi_1(N) \to \pi_1(L) \hookrightarrow \pi_1(X) = \pi_1(N) \hookrightarrow \pi_1(X)$ is an injection, the map $\pi_1(N) \to \pi_1(L)$ is injective. By the same argument N π_1-injects into M so that $\pi_1(K) = \pi_1(L) *_{\pi_1(N)} \pi_1(M)$ as desired. □

Corollary 3.4.12. *If the subcomplexes of the pure Salvetti complex X of fixed type Γ π_1-inject then the fundamental group of every full compact convex subcomplex K of X can be written as an iterated amalgamated free product of spherical Artin groups. If the iterated amalgamated free product has amalgamated subgroups with solvable membership problem, then $\pi_1(K)$ has a solvable word problem.*

Proof. If K is convex irreducible, by Lemma 3.4.10 it is the preimage of a closed W-permutohedron. By remark 3.2.10 this implies $\pi_1(K)$ is spherical and has solvable word problem.

Otherwise, if K is reducible, then $\pi_1(K)$ can be expressed $\pi_1(K) = \pi_1(L) *_{\pi_1(N)} \pi_1(M)$ as the amalgamated free product of fundamental groups of proper full compact convex subcomplexes L, M, and N of the pure Salvetti complex. Similarly if a subcomplex L, M, or N is reducible then it can be expressed similarly as an amalgamated free product. Since K is compact this process of expressing each of the pieces of an amalgamated product as another amalgamated free product of proper subcomplexes of the same type eventually terminates. In this fashion $\pi_1(K)$ can be expressed as an iterated amalgamated free product of irreducible full compact convex subcomplexes of X. Each irreducible full compact convex subcomplex of X is the preimage of a closed W-permutohedron and has solvable word problem, see Remark 3.2.10. Thus, $\pi_1(K)$ can be expressed as an iterated amalgamated free product of groups with solvable word problem. If the iterated amalgamated free product has amalgamated subgroups with solvable membership problem, then K has solvable word problem, see Corollary 3.6.6. □

Theorem 3.4.13. *Let X be the pure Salvetti complex of some fixed type Γ. If every full compact convex subcomplex of X π_1-injects and every full, compact, convex, subcomplex can expressed as an iterated amalgamated free product as outlined in Corollary 3.4.12 then the word problem is solvable for Artin group of type Γ.*

Proof. Suppose that γ is a combinatorial loop in the 1-skeleton of X, and let $p : X \to Y$ be the usual map to the Davis complex. Then the image $\gamma_Y = p(\gamma)$ is contained in the Davis word subcomplex, which is compact. Let K be the preimage of the convex hull of the Davis word complex under p. By Lemma 3.4.5 the convex hull of the Davis word subcomplex is compact. Thus, K is full compact convex subcomplex. By Corollary 3.4.12, K has solvable word problem. Thus, every combinatorial loop in X is contained in a full compact convex subcomplex with decidable word problem. Therefore, the desired result follows. □

Corollary 3.4.14. *The word problem is solvable for one-relator Artin groups.*

3.5 Spherical type

This section focuses on the π_1-injectivity of subcomplexes of a spherical Artin group. Although their word problem has been known to be solvable since the 1970's they provide a good testing ground for techniques and conjectures about how to approach the word problem for Euclidean Artin groups. In particular, the pure Salvetti complex of a Euclidean type is built out of spherical pieces and one is guaranteed to have a finite family of irreducible elements in any family of subcomplexes since the pure Salvetti complex is finite. This section focuses specifically on pure Salvetti complex of type A_3 corresponding to the four strand braid group. I show that convex subcomplexes do not necessarily π_1-inject and use it to exemplify some of the difficulties that arise in higher-dimensional examples.

Definition 3.5.1 (Diagram)**.** A *diagram* over a combinatorial CW complex X is a nonempty connected finite 2-complex D with a combinatorial map $D \to X$ and a specific embedding in $\mathbb{R}^2$, i.e.

$$\mathbb{R}^2 \longleftarrow D \longrightarrow X$$

A contractible diagram is a *disk diagram.* The boundary of a diagram D is the topological boundary of D viewed as a subset of $\mathbb{R}^2$, denoted ∂D. When D is a disk diagram homeomorphic to a disk, ∂D is homeomorphic to a circle. A *boundary* 0*-cell* or a *boundary* 1*-cell* is a 0-cell or a 1-cell which lies in ∂D. A boundary 2-cell is a closed 2-cell which has a nonempty intersection with ∂D. The diagram is *reduced* if the neighborhoods of points on interiors of edges map over to X locally injectively.

Example 3.5.2. For example, consider the Salvetti complex X in Figure 3.5.1 for the Artin group of type $\widetilde{A}_2$ given by

$$\textsc{Art}(\widetilde{A}_2) = \langle a, b, c \mid aba = bab, aca = cac, bcb = cbc \rangle$$

. Since there are three generators, the 1-skeleton, $X^{(1)}$ has as single vertex and three loops, one for each generator. There are three 2-cells, one for each relation. Note each relation has length six so that it can be visualized as a regular hexagon, whose boundary is glued into the $X^{(1)}$ according the the relation to which it corresponds.

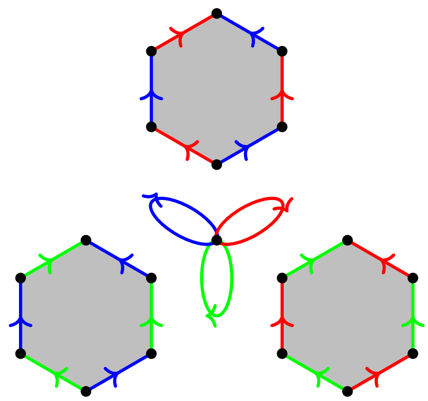


Figure 3.5.1: The Salvetti complex of type $\widetilde{A}_2$.

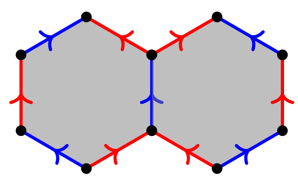


Figure 3.5.2: An example of an unreduced disk diagram over the Salvetti complex of type $\widetilde{A}_2$.

Figure 3.5.2 depicts an unreduced disk diagram over X, whose boundary is given by the word $abAaBABaAb$, or some cyclic permutation of it by changing the basepoint. In particular, no neighborhood of a point on the interior of the edge in the interior of the diagram is mapped to X injectively. Next, Figure 3.5.3 illustrates an example of a reduced disk diagram over X with boundary given by the word a^2BAB^2ab. It can be readily checked that the neighborhoods of points on interiors of edges map over locally injectively. The disk diagram can be viewed as providing a discretized version of the nullhomotopy of the loop in X as depicted in Figure 3.5.3.

Remark 3.5.3. A standard fact from an algebraic topology or combinatorial group theory textbook such as in [8, 12] is Van Kampen's Lemma. Following the setup and definitions of [13] Van Kampen's Lemma is restated in the following lemma.

Lemma 3.5.4. *Suppose that X is a combinatorial CW complex. A combinatorial loop γ*

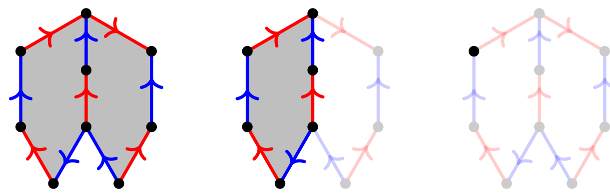


Figure 3.5.3: A reduced diagram of the pure Salvetti complex X depicting a discretized nullhomotopy of a loop in X.

in $X^{(1)}$ *is nullhomotopic if and only if there exists a reduced disk diagram over* X *with* γ *as its boundary.*

Definition 3.5.5. Let $\mathcal{H}$ be the hyperplane arrangement corresponding to an Artin group. A hyperplane in $\mathcal{H}$ with the cellular structure inherited from the Coxeter complex in the Coxeter complex is referred to as a *Coxeter hyperplane*. When viewing a Coxeter hyperplane as a subset of the Davis complex, it is called a *Davis hyperplane*. Note that a Coxeter hyperplane is a subcomplex of the Coxeter complex, but a Davis hyperplane is not a subcomplex of the Davis complex. Davis hyperplanes are subsets of Davis complex that are transverse to the cell-structure of the Davis complex.

A *pure hyperplane* is the preimage of a Davis hyperplane under the map $p : X \to Y$ from the pure Salvetti complex to the Davis complex. Given a disk diagram D over X with map $f : D \to X$ *a diagram hyperplane* is the preimage of a Davis hyperplane under the map $p \circ f$.

Remark 3.5.6. Note that pure hyperplanes are not connected. For example in dimension 2, pure hyperplanes are graphs with two components as illustrated in Example 3.5.7. In general, a pure hyperplane is a codimension 1 cell complex with cell structure inherited from the Coxeter complex that consists of 2 components corresponding to the two orientations on the edges to that the components of the pure hyperplane bisect. The remainder of the dissertation refers to a component of a pure hyperplane as an *oriented pure hyperplane*.

Also, the diagram hyperplanes are not necessarily connected and may have many components. However, each of the connected components of a diagram hyperplane is a 1-manifold. Thus, it is either a circle with no endpoints on the boundary of the disk diagram or it is 1-manifold with boundary and is a line segment with both of its endpoints on the boundary. The blue dotted hyperplane in Figure 3.5.7 of Example 3.5.8 depicts a diagram hyperplane with no points on the boundary.

Example 3.5.7. For type $\widetilde{A}_2$ the Davis complex Y corresponds to the tiling $\mathbb{R}^2$ by regular hexagons. Figure 3.5.4 depicts a portion of a Davis hyperplane in purple running through a parallel family of hexagons as the perpendicular bisector to the horizontal family of parallel edges. The preimage of each of these hexagons, is six oriented hexagons so that there is a corresponding parallel family of oriented hexagons in the pure Salvetti complex. The parallel family of edges that was bisected by the purple Davis hyperplane has preimage a parallel family of bigons. Each oriented edge of the bigon is contained in three oriented hexagons, and each of these three hexagons contains a portion of the preimage of the purple Davis hyperplane. To depict this in Figure 3.5.4 the 1-skeleton of the parallel family of oriented hexagons is drawn in two components. One blue component for the "front" parallel family of oriented edges and one red component for the "back" parallel family of oriented edges. Note that the corresponding vertices of these two components are identified in the pure Salvetti complex. Thus, the corresponding pure hyperplane has two disjoint components, each consisting of consecutive theta graphs, where a theta graph is the union of three disjoint edges meeting that each have the same distinct endpoints so that it resembles the greek letter θ.

Example 3.5.8. Let Y denote the Davis complex for A_3 so that Y is a closed 3-dimensional permutohedron, with 1-skeleton depicted on the left of Figure 3.5.5. Let X be the pure Salvetti complex. Note it consists of the 24 oriented 3-dimensional per-

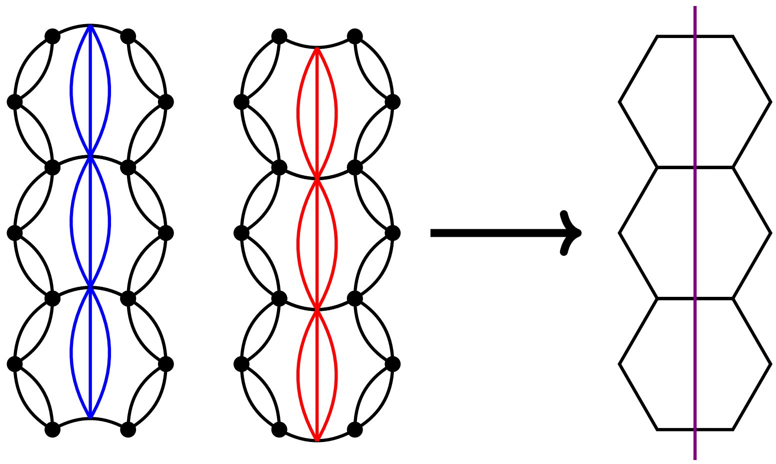


Figure 3.5.4: A portion of the pure hyperplane in red and blue that is the preimage of the Davis hyperplane depicted in purple.

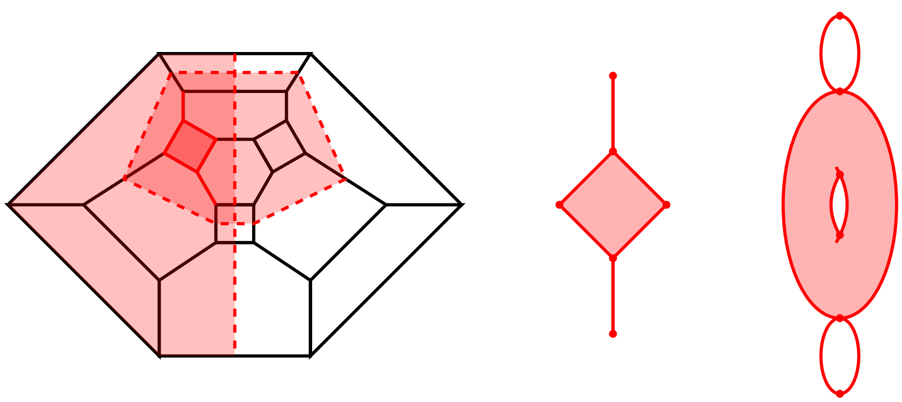


Figure 3.5.5: A convex Davis subcomplex and corresponding pure subcomplex of type A_3.

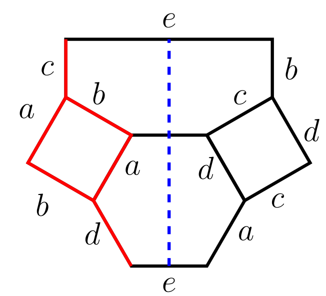


Figure 3.5.6: A portion of the A_3 Davis complex depicted with a convex subcomplex highlighted in red along with a hyperplane that is not crossed by any loop in the red subcomplex.

mutohedra. Also depicted in the figure is a convex Coxeter subspace as the shaded red portion. The corresponding edges of the convex Davis subcomplex K are highlighted in red. In the middle and right are the corresponding Davis and pure subcomplexes, K and K' respectively. This example illustrates that the preimage K' of K in the pure Salvetti complex does not π_1-inject. Note that K consists of a square with two edges attached to opposite vertices of the square. Thus, the preimage is K', which is a full convex subcomplex, consisting of a torus with two bigons attached to opposite vertices.

Figure 3.5.7 depicts a portion of a left labeling of the edges of the Davis and pure Salvetti complex so that the portion of $Y^{(1)}$ in the figure has edges labeled a through e. In particular, this implies that the fundamental group $\pi_1(K')$ is generated by loops a^2, b^2, c^2, d^2, where a^2 and b^2 commute, but there are no relations between the other loops.

Next let γ be a combinatorial loop in $X^{(1)}$ defined by the word $Ac^2aBD^2bAC^2aBd^2b$ and based at the leftmost vertex K. To see that γ is non trivial in K consider the map that quotients the torus to a single point and induces the map f on fundamental groups $f : \mathbb{Z}^2 * \mathbb{Z} * \mathbb{Z} \to \mathbb{Z} * \mathbb{Z}$. If the labels on bigons are maintained under the quotient then γ is sent to the loop defined by the word $c^2D^2C^2d^2$, i.e. a nontrivial commutator in $\mathbb{Z} * \mathbb{Z}$, the fundamental group of the two bigons. To see that γ is nullhomotopic in X see the following disk diagram in Figure 3.5.7, with boundary γ, when read clockwise with

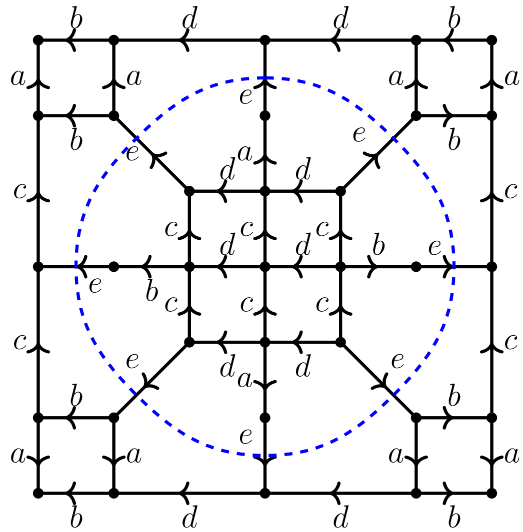


Figure 3.5.7: A disk diagram over the Salvetti complex of type A_3 with diagram hyperplane in blue.

basepoint the bottom left vertex.

One key feature to note is that this is a reduced diagram with a component of a diagram hyperplane that has no boundary. Consider the hyperplane in the Davis complex Y that is a perpendicular bisector to each of the edges labeled e. This intersects the portion of Y depicted in Figure 3.5.6 as the dashed blue line. The corresponding diagram hyperplane is depicted in Figure 3.5.7 in blue as well.

Remark 3.5.9 (Convex subcomplexes do not π_1-inject)**.** Example 3.5.8 illustrates an obstruction to π_1-injectivity of convex subcomplexes that can occur in Artin groups of dimension greater than 1. Notice that loops around the orthogonal c and d hyperplanes meet in a square of the Davis complex. One can show that any convex subcomplex that does not contain this square, but contains the edges dual to the hyperplanes meeting in this square has a corresponding full pure subcomplex that does not π_1-inject by constructing a similar example as above in Example 3.5.8.

In type $\widetilde{G}_2$ (see Figure 6.0.2) or in 3 dimensions and above this happens frequently.

However, type $\widetilde{A}_2$, which is the focus of Chapter 4 is a special situation where this does not happen since it is the unique Euclidean Artin group of dimension greater than 2, where there is no pair of standard generators that commute. That is, no pair of hyperplanes are orthogonal to each other so that there can be no convex Davis subcomplex containing bigons dual to orthogonal hyperplanes without the corresponding square in which the hyperplanes intersect. For type $\widetilde{C}_2$ in Chapter 5 a coarser notion of convexity is introduced to avoid this obstruction.

3.6 A Scott and Wall type approach

This section synthesizes the various results about injectivity and subcomplexes into a general approach for proving the solvability of the word problem for Euclidean Artin groups. This section first recalls the concepts of a graph of groups and a graph of spaces introduced by Scott and Wall [14]. This approach is then adapted to Euclidean Artin group. This section concludes with sufficient conditions for the word problem to be solvable in this context.

Remark 3.6.1 (Graph of groups and graph of spaces)**.** The notion of a graph of groups is a generalization of an amalgamated free product. An amalgamated free product such as $G = A *_C B$ can be expressed as a graph of groups, with a single edge between two distinct vertices u, v with vertex groups A, B respectively, edge group C, and boundary morphisms $C \hookrightarrow A$ and $C \hookrightarrow B$.

To construct a corresponding graph of spaces, start with CW complexes K_A, K_B, and K_C such that $\pi_1(K_X) = X$ for each $X = A, B, C$. Note that, $\pi_1(K_C \times I) = \pi_1(K_C) = C$. Then glue $K_C \times \{0\}$ to K_A so that the induced map on fundamental groups realizes the injection $C \hookrightarrow A$. Similarly glue $K_C \times \{1\}$ to K_B so that the induced map realizes the injection $C \hookrightarrow B$. There are various conditions one can impose on K_A, K_B, and K_C to guarantee that such maps exist, see [15].

Moreover, the universal cover of this graph of spaces is a "tree-like" graph of spaces built out of the universal covers $\widetilde{K_A}, \widetilde{K_B}, \widetilde{K_C} \times I$, with underlying tree, T, on which the group acts without edge inversions.

More specifically, the tree T has vertex set the set of cosets of A and B, that is $\{gA : g \in G\} \sqcup \{gB : g \in G\}$, with an edge between gA and hB if they have nonempty intersection. Moreover, the degree of the vertex gA and hB is the index $[A : C]$ and $[B : C]$ respectively. Thus, there is a tree of spaces by constructing the universal cover of the graph of spaces, where each vertex space is the universal cover of the corresponding vertex space or edge space. Then one can collapse each copy of these universal covers to a point and obtain the underlying tree, T.

Example 3.6.2. Consider $A *_C B$, where $A, B, C = \mathbb{Z}$ and $C \hookrightarrow A$ is the multiplication by 11 map and $C \hookrightarrow B$ is the multiplication by 7 map. This corresponds to a graph of groups depicted in Figure 3.6.1 consisting of a single edge, with edge group $\mathbb{Z}$, between two distinct vertices the left vertex with vertex group $\mathbb{Z}$ and the right with vertex group $\mathbb{Z}$. The boundary morphisms are the multiplication by 11 map and the multiplication by 7 map. To construct the graph of spaces, replace each vertex with a circle, and replace the edge with a cylinder, i.e. a circle cross an interval. The boundary circle on the left of the cylinder is attached to the blue circle by the map which winds it around the blue circle 11 times. Similarly the boundary circle on the right of the cylinder is attached to the green circle by the map which winds it around the green circle 7 times.

To construct the universal cover, note that each circle is covered by the real line $\mathbb{R}$, the cylinder is covered by the real line cross an interval $\mathbb{R} \times I$, the blue vertex space has degree 11, with 11 intervals connecting it to copies of $\mathbb{R}$ of the opposite color, and the green vertex space has degree 7, with 7 intervals connecting it to copies of $\mathbb{R}$ of the

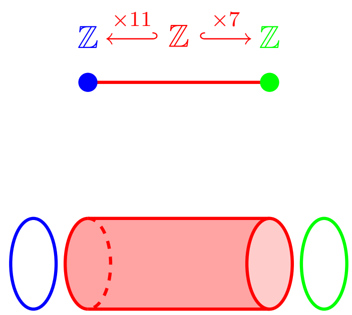


Figure 3.6.1: A graph of groups and the corresponding graph of spaces.

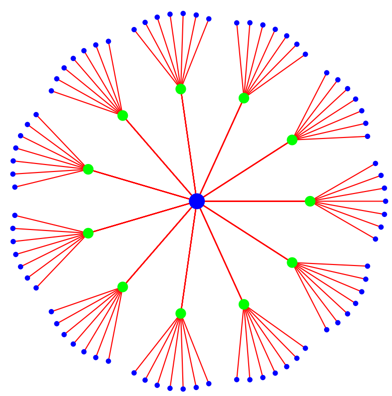


Figure 3.6.2: A portion of the universal cover of the graph of spaces.

opposite color. Moreover, collapsing each of the copies of $\mathbb{R}$ to a point yields a bipartite tree with where each blue vertex has degree 11, and each green vertex has degree 7, a portion of which is depicted in Figure 3.6.2.

A consequence viewing an amalgamated free product as graph of groups with corresponding graph of spaces is the following proposition.

Proposition 3.6.3. *If A, B, and C have solvable word problem and C has solvable membership problem in both A and B, then $A *_C B$ has solvable word problem.*

Proof. By assumption A, B and C have solvable word problem so that the universal covers $\widetilde{K_A}, \widetilde{K_B}$, and $\widetilde{K_C} \times I$ are constructible. The universal cover of the corresponding graph of spaces associated to $A *_C B$ can constructed out of $\widetilde{K_A}, \widetilde{K_B}$, and $\widetilde{K_C} \times I$ in a tree like fashion. By assumption the membership problem for C in both A and B is decidable so that the desired result holds. □

Definition 3.6.4 (Iterated amalgamated free product)**.** If a group G is written as an amalgamated free product $A *_C B$, then the groups A, B, and C are referred to as the *inputs* of the amalgamated free product and G is called the *output*. Note that an input of an amalgamated free products may be written as the output of another amalgamated free product whose inputs are also written as outputs of other amalgamated free products, etc. A group that is written in such a way in terms of finitely many amalgamated free products so that there are finitely many inputs and finitely many outputs is called an *iterated amalgamated free product*. Note that there is a single *global output* that is not written as the input of any amalgamated free in the iterated amalgamated free product, and there are a set of *initial inputs* that are not written as the output of any amalgamated free product.

Example 3.6.5. Suppose that A, B, and C are groups such that $A = D *_F E$, $B = G *_K H$, and $C = L *_M N$ then a group $\mathcal{G}$ written as

$$\mathcal{G} = A *_C B = (D *_F E) *_{(L *_N M)} (G *_K H).$$

is an iterated amalgamated free product, with global output $\mathcal{G}$ and initial inputs D, E, F,G, H, K, L, M, and N.

Corollary 3.6.6. *If G is an iterated amalgamated free product such that the initial inputs have solvable word problem and each of the amalgamated subgroups have decidable membership problem in the corresponding factors, then G has solvable word problem.*

Proof. If G is an iterated amalgamated free product such that the initial inputs have solvable word problem then repeated applications of Proposition 3.6.3 imply that G has solvable word problem. □

Remark 3.6.7. The normal form theorem implies the solvability of the word problem for an amalgamated free product whose inputs have solvable word problem, when the membership problem for C is decidable. From the perspective of a graph of spaces this has a topological interpretation for a group $G = A *_C B$. Specifically, the image of a closed loop can be lifted to the universal cover of the corresponding graph of spaces, and then it can be projected to T. For each loop there is an element of the homotopy class, which has image in T that is a path without backtracks. Moreover, for a nullhomotopic loop, one can construct a nullhomotopy so that its image in T does not grow past the original image.

The approach of Section 3.4 can be seen as a direct analogue of this topological interpretation. The analogue of the graph of spaces is the Salvetti complex, and the analogue of the universal cover is the pure Salvetti complex, which is an intermediate cover of a Salvetti corresponding to the pure Artin group. Then, the π_1-injectivity of convex subcomplexes guarantees that a nullhomotopic loop has a nullhomotopy that does not grow past the convex hull of the loop. Specifically, the method of proof in Chapters 4 and 5 uses the topology of the pure Salvetti complex between adjacent parallel hyperplanes to ensure that every nullhomotopic loop γ in the 1-skeleton of the pure Salvetti complex has a nullhomotopy that is entirely contained in a canonical convex subcomplex containing γ. This method can be readily adapted to ensure that there exists a way of expressing the fundamental group of subcomplexes as an iterated amalgamated free product so that the amalgamated subgroups at each step have decidable membership problem.

However, there was one additional requirement that there was an exhaustion of the Davis complex by compact convex subcomplexes.

Definition 3.6.8 (Exhaustive family)**.** An *exhaustive family* of Davis subcomplexes is a set $\mathcal{F}$ of compact Davis subcomplexes of some fixed type Γ such that every compact Davis subcomplex is contained in a subcomplex of $\mathcal{F}$. Analogously, an *exhaustive family* of full pure subcomplexes is a set $\mathcal{F}$ of full compact pure subcomplexes of some fixed type Γ such that every compact pure subcomplex is contained in a subcomplex of $\mathcal{F}$.

Definition 3.6.9 (Reducible with respect to a family)**.** Let $\mathcal{F}$ be a family of Davis subcomplexes. Then a subcomplex $K \in \mathcal{F}$ is *reducible* with respect to $\mathcal{F}$ or $\mathcal{F}$*-reducible* if there exists subcomplexes $L, M \in \mathcal{F}$ with $L \cap M \in \mathcal{F}$ such that neither contains the other and $K = L \cup M$. Otherwise K is $\mathcal{F}$*-irreducible*. Analogously for a family of pure subcomplexes $\mathcal{F}$, a pure subcomplex $K \in \mathcal{F}$ is $\mathcal{F}$-reducible if there exists subcomplexes $L, M \in \mathcal{F}$ with $L \cap M \in \mathcal{F}$ such that neither contains the other and $K = L \cup M$.

Remark 3.6.10. Note that decomposition of compact convex subcomplexes in terms of iterated amalgamated free products extends more generally to exhaustive families of π_1-injective subcomplexes. In particular, a compact subcomplex K of an exhaustive family $\mathcal{F}$ of π_1-injective subcomplexes has fundamental group $\pi_1(K)$ that can be expressed as an iterated amalgamated free product $\mathcal{F}$-irreducible subcomplexes. The proof follows identically as the one in Section 3.4. Specifically if K is reducible as $K = L \cup M$, with L, M, and $N = L \cap M \in \mathcal{F}$ it is assumed that the subcomplexes L, M, and N π_1-inject in X. Since the maps $\pi_1(L) \hookrightarrow \pi_1(X)$, $\pi_1(M) \hookrightarrow \pi_1(X)$, and $\pi_1(N) \hookrightarrow \pi_1(X)$ are all injections the maps $\pi_1(N) \to \pi_1(L)$ and $\pi_1(N) \to \pi_1(M)$ must also be injective so that $\pi_1(K) = \pi_1(L) *_{\pi_1(N)} \pi_1(M)$. Similarly, if a subcomplex L, M, or N is reducible then it can be similarly expressed as an amalgamated free product. Since K is compact it has finitely many cells and this process of expressing each of the pieces of the amalgamated product as an amalgamated free product of proper subcomplexes in $\mathcal{F}$ eventually terminates. In this fashion a compact subcomplex of $\mathcal{F}$ has fundamental group $\pi_1(K)$ that can be expressed as an iterated amalgamated free product $\mathcal{F}$- irreducible subcomplexes.

Theorem 3.6.11. *Let X be the pure Salvetti complex of some fixed type Γ. If there exists an exhaustive family $\mathcal{F}$ of π_1-injective subcomplexes, then the fundamental group of a subcomplex of $\mathcal{F}$ can be written as an iterated amalgamated free product of $\mathcal{F}$-irreducible subcomplexes. If the iterated amalgamated free product has amalgamated subgroups with solvable membership problem and the $\mathcal{F}$-irreducible subcomplexes have solvable word problem then the Artin group of type Γ has solvable word problem.*

Proof. Suppose that γ is a combinatorial loop in the 1-skeleton of X, and let $p : X \to Y$ be the usual map to the Davis complex. Then the image $\gamma_Y = p(\gamma)$ is contained in the Davis word subcomplex, which is compact. Let K be the preimage of the convex hull of the Davis word complex under p. Since $\mathcal{F}$ is exhaustive there exists a pure subcomplex $K \in \mathcal{F}$ containing γ. By Remark 3.6.10 $\pi_1(K)$ can be expressed as an iterated amalgamated free product of $\mathcal{F}$-irreducible pure subcomplexes. Suppose that the iterated amalgamated free product has amalgamated subgroups with solvable membership problem and the $\mathcal{F}$-irreducible subcomplexes have solvable word problem. Then, Corollary 3.6.6 implies that $\pi_1(K)$ has solvable word problem. Thus, every combinatorial loop in $X^{(1)}$ is contained in a full compact convex subcomplex with solvable word problem. Therefore, the desired result follows. □

Chapter 4

Type $\widetilde{A}_2$

This chapter describes a new way to solve the word problem for the Euclidean Artin group of type $\widetilde{A}_2$ by considering the full convex subcomplexes of the pure Salvetti complex. The family of full compact convex subcomplexes is an exhaustive family of subcomplexes that is both coarse enough so that its elements π_1-inject and fine enough so that the irreducible elements have fundamental groups with solvable word problem. This chapter first focuses on the injectivity of convex subcomplexes of the pure Salvetti complex of type $\widetilde{A}_2$. To do so I show that any nullhomotopic loop that is contained in a convex subcomplex admits a nullhomotopy that is entirely contained in that same convex subcomplex. Specifically, this reduces to showing that a reduced disk diagram must have each component of its diagram hyperplanes with endpoints on the boundary of the disk. This result paired with the fact that the convex irreducible Davis subcomplexes are all closed W_T permutoheadra implies that the word problem is solvable in type $\widetilde{A}_2$.

There are two initial observations that are crucial to showing that convex subcomplexes π_1-inject.

Remark 4.0.1. It is straightforward to observe that there do not exist any two cells between consecutive parallel hyperplanes of the $\widetilde{A}_2$ arrangment $\mathcal{H}$. See Figure 4.0.1.

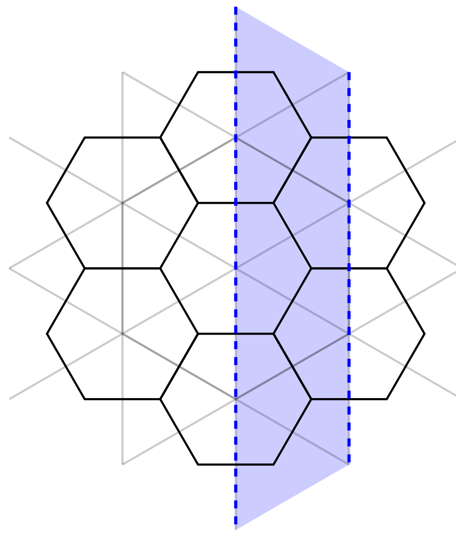


Figure 4.0.1: Two consecutive parallel hyperplanes determine a convex Davis subcomplex that does not have any two cells.

Remark 4.0.2. Given a parallel family of oriented hexagons in the pure Salvetti complex of type $\widetilde{A}_2$ consider the smallest subcomplex K that contains a component of a pure oriented hyperplane. More specifically, choosing a given side of the hyperplane dual to a parallel family of edges in the Davis complex determines a choice of orientation in their preimage in the Davis complex as well as an oriented pure hyperplane. The closure of all of the 2-cells containing that oriented pure hyperplane is K. In the proof of Lemma 3.3.5 and depicted in Figure 3.3.4 the subcomplex consisting of three oriented hexagons that share the same orientation on a pair of opposite edges is homotopy equivalent to portion to the oriented pure hyperplane that contains the midpoint of the chosen oriented edge. In fact, this homotopy equivalence can be extended to all of K, which is homotopy equivalent to the union of the purple theta graphs depicted in Figure 4.0.2 as well as either the blue or red families of bigons.

Theorem 4.0.3. *Suppose that D is a reduced disk diagram over the pure Salvetti complex of type $\widetilde{A}_2$. Then there are no components of any diagram hyperplane that are closed loops. In particular each component of every diagram hyperplane is a 1-manifold with boundary*

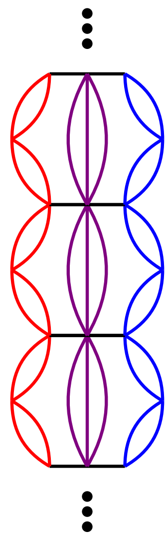


Figure 4.0.2: The subcomplex K of the pure Salvetti complex of type $\widetilde{A}_2$ that contains the image of an innermost disk.

with both of its endpoints on the boundary of D.

Proof. Let X be the pure Salvetti complex of type $\widetilde{A}_2$, let Y denote the Davis complex of type $\widetilde{A}_2$, and let $p : X \to Y$ be the usual projection. If $f : D \to X$ is a reduced disk diagram over X such that there exists some component of a diagram hyperplane that is a circle, then look at corresponding parallel family of hyperplanes in the Davis complex. The entire preimage of this family inside D is still a 1-manifold that contains at least one circular component inside D, and then there exists an innermost circular component H of this 1-manifold. In particular, this innermost circle bounds a disk, E, which contains no other components of that 1-manifold. Note that the assumption that H is an innermost circle along with Remark 4.0.1 implies that E does not contain any complete 2-cells. Thus, H is a nullhomotopic loop in D by a homotopy that avoids the other components of the preimage of the parallel family.

Since the edges bisected by H all must have the same orientation the image $f(H)$ is contained in a single oriented pure hyperplane H_X, as depicted in Figure 4.0.2 in purple. Let K be the smallest subcomplex containing H_X consisting of the parallel family of closed oriented hexagons that are bisected by an edge of H_X. The 1-skeleton is depicted in black, red and blue in Figure 4.0.2. Recall that the purple graph is not part of the 1-skeleton of the pure Salvetti complex but its edges are the bisectors of the hexagons with vertices at the midpoints of the edges. Note that $f(H)$ is nullhomotopic in K using $f(E)$. By Remark 4.0.2 K is homotopy equivalent to the H_X which is a graph.

Therefore the loop $f(H)$ is nullhomotopic in H_X, and, as a consequence, there exists a backtrack in this graph. That is, there is some edge that the loop traverses and then immediately traverses again in the opposite direction. The portion of H in D that creates this backtrack in H_X occurs at the midpoint of an oriented edge in D, and consequently has a neighborhood that is not mapped to X injectively. Therefore, D is not a reduced disk diagram, and this contradiction implies the desired result. □

Corollary 4.0.4. *Every full convex subcomplex K of the pure Salvetti complex of type $\widetilde{A}_2$ π_1-injects.*

Proof. Let X be the pure Salvetti complex of type $\widetilde{A}_2$, let Y denote the Davis complex of type $\widetilde{A}_2$, and let $p : X \to Y$ be the usual projection. Suppose that $H \subset Y$ is a Davis hyperplane that does not intersect $K_Y = p(K)$. Let γ be nullhomotopic loop in K so that its image $\gamma_Y = p(\gamma)$ is entirely contained in one of the halfspaces determined by H, say H^+. Then there exists a reduced disk diagram D with γ as its boundary so that it must have the diagram hyperplane H_D corresponding to H entirely contained in its interior. If H_D is nonempty then H_D must be a union of circles. However, Theorem 4.0.3 implies that since D is reduced H_D must be empty. Thus, for any Davis hyperplane H such that one of its halfspaces contains K_Y the halfspace also contains the image of D in Y. Thus, there is a null homotopy of γ whose image in Y is entirely contained in K_Y. Therefore, any loop that is nullhomotopic in X is nullhomotopic in K so that the desired result follows. □

Lemma 4.0.5. *If K is a convex, reducible, full, compact pure subcomplex of type $\widetilde{A}_2$, then $\pi_1(K)$ can be written as an iterated amalgamated free product such that each of the initial inputs are spherical Artin groups and each of the amalgamated subgroups are finite rank free groups. Moreover, each of the amalgamated subgroups are fundamental groups of subgraphs of the pure Salvetti complex entirely contained between two consecutive parallel hyperplanes.*

Proof. Suppose that K is convex reducible, full, compact, pure subcomplex of type $\widetilde{A}_2$ and contains at least one oriented hexagon P, and let K_Y and P_Y be the corresponding Davis subcomplex and corresponding unoriented hexagon. Then the set of all separating hyperplanes of K_Y has at least one hyperplane from each of the three distinct families of parallel hyperplanes of type $\widetilde{A}_2$ determined by a positive root of the crystallographic root system of type A_2. Since K is reducible Lemma 3.4.10 implies that K_Y must consist of some other edge not contained in P_Y. Thus, there are two adjacent parallel hyperplanes H_1 and H_2 that intersect K_Y. Thus, K_Y can be expressed as $K_Y = L_Y \cup M_Y$, where L_Y is the largest subcomplex of K_Y entirely contained in the halfspace of H_1 containing H_2 and M_Y is the largest subcomplex of K_Y entirely contained in the halfspace of H_2 containing H_1. Denote $N_Y = L_Y \cap M_Y$, and let L, M, and N be the full pure subcomplexes corresponding to L_Y, M_Y, and N_Y. Remark 4.0.1 implies that N is some finite subgraph of X so that $\pi_1(N)$ is a finite rank free group. Corollary 4.0.4 implies that $\pi_1(K) = \pi_1(L) *_{\pi_1(N)} \pi_1(M)$. Every finite rank free group arising as the fundamental group of some N as a above, i.e. the corresponding Davis subcomplex N_Y is contained between two consecutive parallel hyperplanes, can be written as a free product of finitely many copies of $\mathbb{Z}$ and thought of as an iterated free product with initial inputs the fundamental groups of bigons. See Example 3.2.6. Continuing in this way it follows that any convex, reducible, compact, full, pure subcomplex of type $\widetilde{A}_2$ can be systematically written as an iterated amalgamated free group such that each of the initial inputs are spherical Artin groups and each of the amalgamated subgroups are finite rank free groups. □

Remark 4.0.6. Let X be the pure Salvetti complex of type $\widetilde{A}_2$. Fixing a parallel family of hyperplanes $\mathcal{H}'$ determines a natural projection from X to X' where X' is the pure Salvetti complex of type $\widetilde{A}_1$ consisting of an infinite path of bigons by identifying $\mathcal{H}'$ with the hyperplane arrangement of type $\widetilde{A}_1$. Note that this map induces a map on fundamental groups. Specifically, a loop γ in X can be decomposed into a finite list of consecutive subpaths that only traverse a hyperplane in $\mathcal{H}'$ a single time with a specific orientation. Then γ in X is mapped to the loop in X' that traverses the hyperplanes of type $\widetilde{A}_1$ in the same order and with the same orientation as the subpaths of γ.

Next, suppose that γ is a combinatorial path in $X^{(1)}$ and that the image of γ in X' is a path in X' with a backtrack in an edge traversing some hyperplane H. Then there is a combinatorial subpath of γ in X that starts by traversing some edge bisected by H, then proceeds to traverse some edges between H and the next consecutive parallel hyperplane, and ends by traversing H along an edge with the same orientation as at the start of the subpath. The two edges sharing the same orientation that are crossed at the start and end of this subpath determine an oriented pure hyperplane H_X. Let K be the smallest subcomplex containing H_X, as depicted in Figure 4.0.2. By Remark 4.0.2, this subpath is homotopic to a path that is entirely contained on one side of H and has image a trivial path in X'.

If a path in X is systematically reduced in this way, then its projection to X' is a path whose length continues shortens. This process will stop when the projection to X' has no backtracks. Remark 3.2.12 implies that a path in X with endpoints on one side of a hyperplane is homotopic to a path that stays on that side of the hyperplane if and only if there is a finite sequence of reductions that produce a homotopic path whose projection to X' is a path without backtracks staying on one side of the hyperplane. Therefore, a path in X with endpoints between two consecutive parallel hyperplanes in X is homotopic to a path that stays between two consecutive parallel hyperplanes if and only if there is a finite sequence of reductions that produce a homotopic path whose projection to X' is a trivial path.

Lemma 4.0.7. *Every convex reducible, compact, full, pure subcomplex K of the pure Salvetti complex X of type $\widetilde{A}_2$ has fundamental group that can be written as an amalgamated free product of spherical Artin groups, such that the amalgamated subgroups are free groups with decidable membership problem in each of the factors of the amalgamated free product.*

Proof. By Lemma 4.0.5 $\pi_1(K)$ can be written as an iterated amalgamated free product such that each of the initial inputs are spherical Artin groups and each of the amalgamated subgroups are finite rank free groups, where each of the amalgamated subgroups are fundamental groups of subgraphs of X entirely contained between two consecutive parallel hyperplanes. Remarks 4.0.6 and 3.2.12 imply that each of the amalgamated subgroups have decidable membership problem in each of the factors of the amalgamated free product. □

The following corollary is a direct consequence of Theorem 3.4.13.

Corollary 4.0.8. *The word problem for the Artin group of type $\widetilde{A}_2$ is solvable.*

Chapter 5

Relative convexity and type $\widetilde{C}_2$

This chapter focuses on the Artin group of type $\widetilde{C}_2$. The chapter begins with an example to demonstrate that there are convex subcomplexes which do not π_1-inject. Thus, the approach used for type $\widetilde{A}_2$ in Chapter 4 does not apply directly. Instead this chapter defines a new proper subfamily of convex subcomplexes that do π_1-inject, and the family remains exhaustive. Then it is shown that the finite list of irreducible subcomplexes with respect to this family are free products of spherical Artin groups. Together these two results imply that the word problem is solvable in type $\widetilde{C}_2$.

5.1 Counterexample to convex π_1-injectivity

This section begins with an example to illustrate why convexity is not a sufficient condition to ensure that a full pure subcomplex π_1-injects.

Example 5.1.1. Let X denote the pure Salvetti complex of type $\widetilde{C}_2$, and let Y denote the Davis complex of the same type. Let $K \subset Y$ be the Davis subcomplex depicted in red in Figure 5.1.1 along with a portion of the left labeling. The subcomplex K consists of a square and two path of length two with each path attached opposite vertices of the

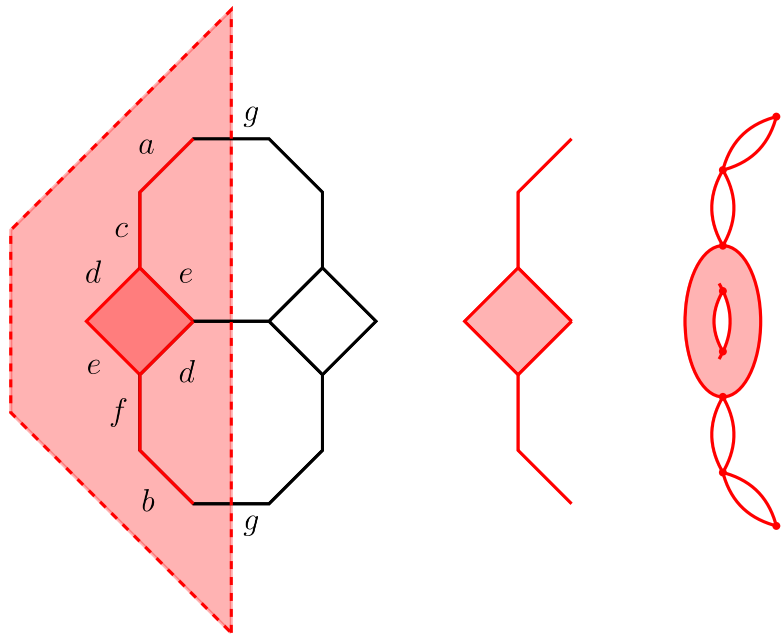


Figure 5.1.1: A convex Coxeter subspace of type $\widetilde{C}_2$ whose corresponding pure subcomplex does not π_1-inject.

square. The corresponding pure subcomplex K_X is depicted on the right of the figure and has fundamental group $\pi_1(K_X) = \mathbb{Z} * \mathbb{Z} * \mathbb{Z}^2 * \mathbb{Z} * \mathbb{Z}$ generated by the loops $a^2, \ldots, g^2$, where only the loops d^2 and e^2 commute in $\pi_1(K_X)$.

Next, let γ be the combinatorial loop in $X^{(1)}$ defined by the word

$$eca^2CEdfb^2FDecA^2CEDFb^2fd$$

based at the right vertex of the square in K. To see that γ is nontrivial in K consider the map quotienting the union of the torus and the two adjacent bigons to a single point that induces a map on fundamental groups $\mathbb{Z} * \mathbb{Z} * \mathbb{Z}^2 * \mathbb{Z} * \mathbb{Z} \to \mathbb{Z} * \mathbb{Z}$. If the labels on the bigons are maintained in the quotient then γ is sent to the loop defined by word $a^2b^2A^2B^2$, which is nontrivial in $\mathbb{Z} * \mathbb{Z}$, i.e. the fundamental group of quotient consisting of the two bigons labeled by a and b.

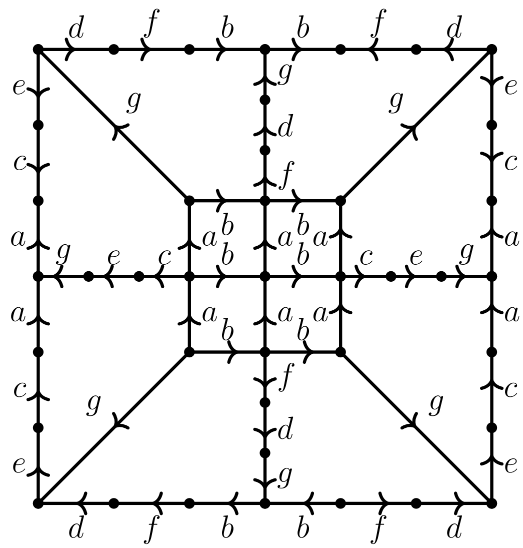


Figure 5.1.2: A disk diagram illustrating a nullhomotopy of its boundary loop.

However, note that γ is nullhomotopic in X via the disk diagram in Figure 5.1.2. Thus, K_X does not π_1-inject.

Remark 5.1.2. Note that the proof of π_1-injectivity of convex subcomplexes of type $\widetilde{A}_2$ from Chapter 4 cannot be immediately adapted to type $\widetilde{C}_2$ because there are 2-cells between consecutive parallel hyperplanes in the type $\widetilde{C}_2$ hyperplane arrangement. The vertical dashed hyperplanes in Figure 5.1.1 are two such hyperplanes. Roughly speaking, when there is a 2-cell between consecutive parallel hyperplanes one can construct a reduced disk diagram with an inner most disk with respect to a parallel family of hyperplanes that contains an entire 2-cell. In particular, a disk diagram with an innermost circle need not be reduced. As a result, it is possible to construct a loop in a convex subcomplex so that a reduced disk diagram has image outside that convex subcomplex.

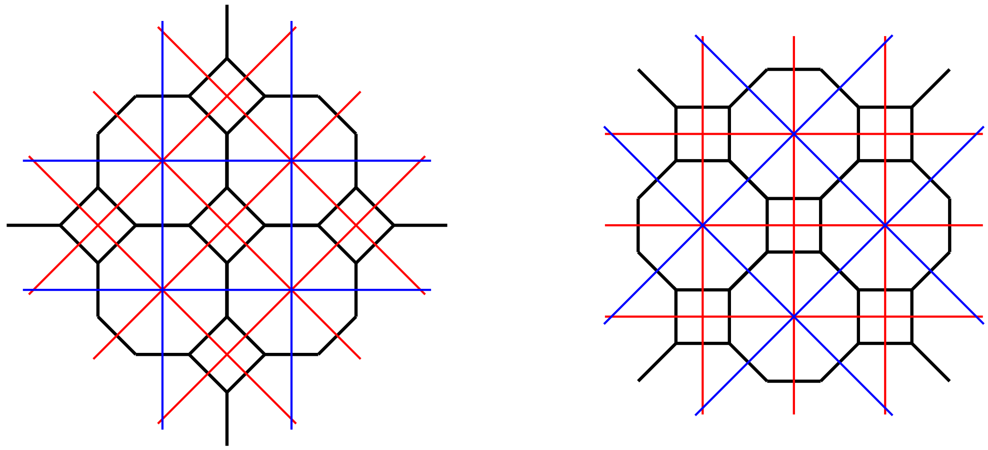


Figure 5.2.1: The type $\widetilde{C}_2$ hyperplane arrangement partitioned into blue and red families of hyperplanes and the dual Davis complex.

5.2 Relative Convexity

This section introduces a coarser family of subcomplexes by considering convexity with respect to a subarrangement of the hyperplane arrangement consisting of two orthogonal parallel families of hyperplanes. This section concludes by proving that the subcomplexes that are convex relative to this subarrangement do π_1-inject.

Definition 5.2.1 (Dense and sparse families of hyperplanes)**.** Suppose Φ is a crystallographic root system with two different length roots that defines a Euclidean Artin group with hyperplane arrangement $\mathcal{H}$. The family of hyperplanes corresponding to the long roots of Φ is said to be *dense*, and the family corresponding to the short roots is said to be *sparse*. An element of the dense family of hyperplanes is said to be dense, and an element of a sparse family of hyperplanes is said to be sparse. A diagram hyperplane or a pure hyperplane is said to be dense or sparse if it has image a dense or sparse hyperplane respectively.

Remark 5.2.2. Figure 5.2.1 depicts on the left the hyperplane arrangement of type $\widetilde{C}_2$ partitioned into two sets, blue and red. Note that the blue family corresponds to the shorter roots and the red family to the longer roots. Thus, the red hyperplanes are the dense family and the blue hyperplanes are the sparse family of hyperplanes. In fact, observe that there are no 2-cells contained between consecutive parallel hyperplanes in the dense family of type $\widetilde{C}_2$.

Note that the right of Figure 5.2.1 is just a $\pi/4$ rotation of the image on the left. The remainder of this chapter will rotate the hyperplane arrangement and Davis complex so that the dense family of red hyperplanes are the vertical and horizontal hyperplanes are the dense family in the type $\widetilde{C}_2$ hyperplane arrangement.

Definition 5.2.3 (Relative convexity)**.** Define a *relatively convex Coxeter subspace* of the Coxeter complex of type $\widetilde{C}_2$ to be the subcomplex consisting of the union of all open cells contained in an intersection of open halfspaces determined by hyperplanes in the dense family. A *relatively convex Davis subcomplex* is defined to be the cellular dual of the convex Coxeter subspace, and a *relatively convex pure subcomplex* is the preimage of a convex Davis subcomplex under the usual map from the pure Salvetti complex to the Davis complex.

Remark 5.2.4. Note that the collection of all compact relatively convex subcomplexes is still an exhaustive family of subcomplexes for type $\widetilde{C}_2$. To see this it suffices to observe that every compact convex Davis subcomplex is contained in a compact relatively convex Davis subcomplex. Let K be a compact subcomplex and let K' be the relatively convex Davis subcomplex defined by the intersection of all dense halfspaces containing all of K. Note that there are only finitely many hyperplanes of the full hyperplane arrangement of type $\widetilde{C}_2$ that intersect K'. Therefore, K' must be compact as well.

Remark 5.2.5. Given a parallel family of oriented octagons in the pure Salvetti complex of type $\widetilde{C}_2$ consider the smallest subcomplex K that contains a component of a pure oriented hyperplane. More specifically, choosing a given side of the hyperplane dual to a

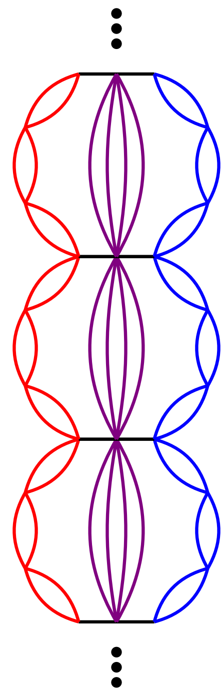


Figure 5.2.2: The subcomplex of the pure Salvetti complex of type $\widetilde{C}_2$ that contains the image of the innermost disk

parallel family of edges in the Davis complex determines a choice of orientation in their preimage in the Davis complex as well as an oriented pure hyperplane. The closure of all of the 2-cells containing that oriented pure hyperplane is K. Just as in Remark 4.0.2 the subcomplex K is homotopy equivalent the purple graph depicted in Figure 5.2.2 as well as either the blue or red families of bigons.

Theorem 5.2.6. *Suppose that D is a reduced disk diagram over the pure Salvetti complex of type $\widetilde{C}_2$. Then there are no components of any dense diagram hyperplane that are closed loops. In particular each component of every dense diagram hyperplane is a 1-manifold with boundary with both of its endpoints on the boundary of D.*

Proof. Let X be the pure Salvetti complex of type $\widetilde{C}_2$, let Y denote the Davis complex of type $\widetilde{C}_2$, and let $p : X \to Y$ be the usual projection. If $f : D \to X$ is a reduced disk diagram over X such that there exists some component of a dense diagram hyperplane that is a circle, then there exists an innermost circular component H of the same diagram hyperplane. In particular, this innermost circle bounds a disk, E, which contains no other components of that diagram hyperplane. Note that the assumption that H is an innermost circle along with Remark 5.2.2 implies that E does not contain any complete 2-cells. Thus, H is a nullhomotopic loop in D by a homotopy that avoids the other components of the preimage of the parallel family.

Since the edges bisected by H all must have the same orientation the image $f(H)$ is contained in a single oriented pure hyperplane H_X, as depicted in Figure 5.2.2 in purple. Let K be the smallest subcomplex containing H_X consisting of the parallel family of closed oriented octagons that are bisected by an edge of H_X. The 1-skeleton is depicted in black, red and blue in Figure 5.2.2. Recall that the purple graph is not part of the 1-skeleton of the pure Salvetti complex but its edges are the bisectors of the octagons with vertices at the midpoints of the edges. Note that $f(H)$ is nullhomotopic in K using $f(E)$. By Remark 5.2.5 K is homotopy equivalent to the H_X which is a graph.

Therefore the loop $f(H)$ is nullhomotopic in H_X, and, as a consequence, there exists a backtrack in this graph. That is, there is some edge that the loop traverses and then immediately traverses again in the opposite direction. The portion of H in D that creates this backtrack in H_X occurs at the midpoint of an oriented edge in D, and consequently has a neighborhood that is not mapped to X injectively. Therefore, D is not a reduced disk diagram, and this contradiction implies the desired result. □

Corollary 5.2.7. *Every full relatively convex subcomplex K of the pure Salvetti complex of type $\widetilde{C}_2$ π_1-injects.*

Proof. Let X be the pure Salvetti complex of type $\widetilde{C}_2$, let Y denote the Davis complex of type $\widetilde{C}_2$, and let $p : X \to Y$ be the usual projection. Suppose that $H \subset Y$ is a dense Davis hyperplane that does not intersect $K_Y = p(K)$. Let γ be nullhomotopic loop in

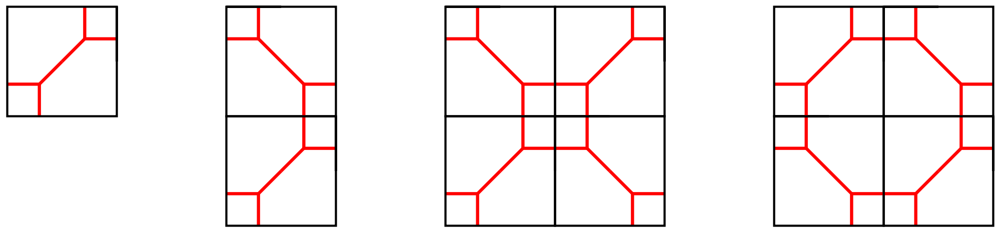


Figure 5.3.1: The relatively convex irreducible Davis subcomplexes.

K so that its image $\gamma_Y = p(\gamma)$ is entirely contained in one of the halfspaces determined by H, say H^+. Then there exists a reduced disk diagram D with γ as its boundary so that it must have the diagram hyperplane H_D corresponding to H entirely contained in its interior. If H_D is nonempty then H_D must be a union of circles. However, Theorem 5.2.6 implies that since D is reduced H_D must be empty. Thus, for any Davis hyperplane H such that one of its halfspaces contains K_Y the halfspace also contains the image of D in Y. Thus, there is nullhomotopy of γ whose image in Y is entirely contained in K_Y. Therefore, any loop that is nullhomotopic in X is nullhomotopic in K so that the desired result follows. □

5.3 The word problem for type $\widetilde{C}_2$

This section classifies the irreducible relatively convex subcomplexes and proves that they are free products of spherical pure Artin groups. As a result, this section concludes that the word problem in type $\widetilde{C}_2$ is solvable.

Lemma 5.3.1. *The irreducible relatively convex subcomplexes of the Davis complex of type $\widetilde{C}_2$ with respect to $\mathcal{H}'$ are either an edge, a path of three consecutive edges of a single octagon, a square union the four edges adjacent to a single vertex of that square, and a single octagon, as depcited in Figure 5.3.1*

Proof. Let K be an irreducible relatively convex subcomplex of the Davis complex of type $\widetilde{C}_2$. Observe that the dense family of hyperplanes in type $\widetilde{C}_2$ is the the union of two parallel families of hyperplanes, with each hyperplane in the first parallel family orthogonal to each of the hyperplanes in the second family. By Remark 3.4.8 there can be at most 2 dense hyperplanes with nonempty intersection with K, otherwise there will be two parallel hyperplanes. It is straightforward to observe that Figure 5.3.1 depicts the only four possibilities. When there is no dense hyperplane with nonempty intersection with K then K must consist of a single edge. When there is one dense hyperplane with nonempty intersection with K then K consists of three consecutive edges of a single octagon. When there are two mutually orthogonal dense hyperplanes with nonempty intersection with K then K either consists of a square union four edges each adjacent to a different vertex of that square, or K is a single octagon. □

Corollary 5.3.2. *The fundamental groups of the corresponding relatively convex, irreducible, full, pure subcomplexes are free products of spherical pure Artin groups. Thus, the fundamental group of each irreducible relatively convex pure subcomplexes has solvable word problem.*

Proof. Consider the four pure subcomplexes corresponding to the four irreducible subcomplexes of Corollary 5.3.1. The single edge corresponds to a bigon with fundamental group $\mathbb{Z}$ corresponding to a rank one spherical parabolic subgroup of type A_1. The path of three edges corresponds to 3 consecutive bigons with fundamental group free group of rank 3, $\mathbb{F}_3 = \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$, which is a free product of 3 spherical pure Artin groups of type A_1. The square union four additional edges corresponds to a torus wedge four bigons. It has fundamental group $\mathbb{F}_4 * \mathbb{Z}^2$, i.e. the free product of four spherical pure Artin groups of type A_1 with $\mathbb{Z}^2$, the spherical pure Artin group of type $A_1 \times A_1$. Lastly the octagon corresponds to the pure Salvetti complex of type C_2, which is spherical. Therefore, the fundamental group of each irreducible relatively convex pure subcomplex is a free product of spherical pure Artin groups, so that Corollary 3.6.6 implies it has solvable word problem. □

Lemma 5.3.3. *The set of compact relatively convex subcomplexes is an exhaustive family.*

Proof. Let K be a compact Davis subcomplex. Recall, that the dense family of hyperplanes in type $\widetilde{C}_2$ is the union of two parallel families of hyperplanes, with each hyperplane in the first parallel family orthogonal to each of the hyperplanes in the second family. Since K is compact, there exists two hyperplanes in the first family as well as two hyperplanes in the second family so that K is contained between them. The intersection of the halfspaces containing K determined by these four hyperplanes is compact and relatively convex. □

Lemma 5.3.4. *If K is a relatively convex, reducible, full, compact pure subcomplex of type $\widetilde{C}_2$, then $\pi_1(K)$ can be written as an iterated amalgamated free product such that each of the initial inputs have solvable word problem and each of the amalgamated subgroups are finite rank free groups. Moreover, each of the amalgamated subgroups are fundamental groups of subgraphs of the pure Salvetti complex entirely contained between two consecutive parallel hyperplanes.*

Proof. Suppose that K is relatively convex reducible, full, compact, pure subcomplex of type $\widetilde{C}_2$, and let K_Y be the corresponding Davis subcomplex. Lemma 5.3.1 implies that there are two dense adjacent parallel hyperplanes H_1 and H_2 that intersect K_Y. Thus, K_Y can be expressed as $K_Y = L_Y \cup M_Y$, where L_Y is the largest subcomplex of K_Y entirely contained in the halfspace of H_1 containing H_2 and M_Y is the largest subcomplex of K_Y entirely contained in the halfspace of H_2 containing H_1. Denote $N_Y = L_Y \cap M_Y$, and let L, M, and N be the full pure subcomplexes corresponding to L_Y, M_Y, and N_Y. Remark

5.2.2 implies that N is some finite subgraph of X so that $\pi_1(N)$ is a finite rank free group. Corollary 5.2.7 implies that $\pi_1(K) = \pi_1(L) *_{\pi_1(N)} \pi_1(M)$. Every finite rank free group arising as the fundamental group of some N as a above, i.e. the corresponding Davis subcomplex N_Y is contained between two consecutive dense parallel hyperplanes, can be written as a free product of finitely many copies of $\mathbb{Z}$ and thought of as an iterated free product with initial inputs the fundamental groups of bigons. See Example 3.2.6. Continuing in this way it follows that any relatively convex reducible, compact, full, pure subcomplex of type $\widetilde{C}_2$ can be systematically written as an iterated amalgamated free group such that each of the initial inputs are relatively convex irreducible Artin groups and each of the amalgamated subgroups are finite rank free groups. □

Remark 5.3.5. Let X be the pure Salvetti complex of type $\widetilde{C}_2$. Fixing a parallel family of dense hyperplanes $\mathcal{H}'$ determines a natural projection from X to X' where X' is the pure Salvetti complex of type $\widetilde{A}_1$ consisting of an infinite path of bigons by identifying $\mathcal{H}'$ with hyperplane arrangement of type $\widetilde{A}_1$. Note that this map induces a map on fundamental groups. Specifically, a loop γ in X can be decomposed into a finite list of consecutive subpaths that only traverse a hyperplane in $\mathcal{H}'$ a single time with a specific orientation. Then γ in X is mapped to the loop in X' that traverses the hyperplanes of type $\widetilde{A}_1$ in the same order and with the same orientation as the subpaths of γ.

Next, suppose that γ is a combinatorial path in $X^{(1)}$ and that the image of γ in X' is a path in X' with a backtrack in an edge traversing some hyperplane H. Then there is a combinatorial subpath of γ in X that starts by traversing some edge bisected by H, then proceeds to traverse some edges between H and the next consecutive parallel hyperplane, and ends by traversing H along an edge with the same orientation as at the start of the subpath. The two edges sharing the same orientation that are crossed at the start and end of this subpath determine an oriented pure hyperplane H_X. Let K be the smallest subcomplex containing H_X, as depicted in Figure 5.2.2. By Remark 5.2.5, this subpath is homotopic to a path that is entirely contained on one side of H and has image a trivial path in X'.

If a path in X is systematically reduced in this way, then its projection to X' is a path whose length continues to shorten. This process will stop when the projection to X' has no backtracks. Remark 3.2.12 implies that a path in X with endpoints on one side of a hyperplane is homotopic to a path that stays on that side of the hyperplane if and only if there is a finite sequence of reductions that produce a homotopic path whose projection to X' is a path without backtracks staying on one side of the hyperplane. Therefore, a path in X with endpoints between two consecutive parallel hyperplanes in X is homotopic to a path that stays between two consecutive parallel hyperplanes if and only if there is a finite sequence of reductions that produce a homotopic path whose projection to X' is a trivial path.

Lemma 5.3.6. *Every relatively convex reducible, compact, full pure subcomplex K of the pure Salvetti complex X of type $\widetilde{A}_2$ has fundamental group that can be written as an amalgamated free product of spherical Artin groups, such that the amalgamated subgroups are free groups with decidable membership problem in each of the factors of the amalgamated free product.*

Proof. By Lemma 5.3.4 $\pi_1(K)$ can be written as an iterated amalgamated free product such that each of the initial inputs are spherical Artin groups and each of the amalgamated subgroups are finite rank free groups, where each of the amalgamated subgroups are fundamental groups of subgraphs of X entirely contained between two consecutive parallel hyperplanes. Remarks 5.3.5 and 3.2.12 imply that each of the amalgamated subgroups have decidable membership problem in each of the factors of the amalgamated free product. □

The following corollary is a direct consequence of Theorem 3.6.11.

Corollary 5.3.7. *The word problem for the Artin group of type $\widetilde{C}_2$ is solvable.*

Chapter 6

Future directions

This chapter focuses on directions for future research highlighting where the tools established in the previous chapters fall short. This leads to directions for further work.

Remark 6.0.1. Consider first the crystallographic root system of type $\widetilde{G}_2$ depicted in Figure 6.0.1 with the long roots depicted in red and the short roots depicted in blue. The figure also depicts the corresponding hyperplane arrangement for type $\widetilde{G}_2$. Since there are long and short roots there are sparse and dense parallel families of hyperplanes. Thus, there are two types of parallel families of hyperplanes of type $\widetilde{G}_2$ that are depicted in Figure 6.0.2, with the color indicating the spacing between the hyperplanes determined by the length of the corresponding root. Observe that between any two consecutive parallel hyperplanes in any parallel family there exists a 2-cell, specifically a square. In fact, by a similar argument as Example 5.1.1, it can be shown that there does not exist a subfamily of hyperplanes so that the convex subcomplexes with respect to that family π_1-inject. As a result, the methods for type $\widetilde{C}_2$ in Chapter 5 do not directly apply.

Recall that the approach of this dissertation has been to argue that any nullhomotopy of a loop in the pure Salvetti complex can be entirely contained in a canonical convex subcomplex K. However, solvability of the word problem for the fundamental group

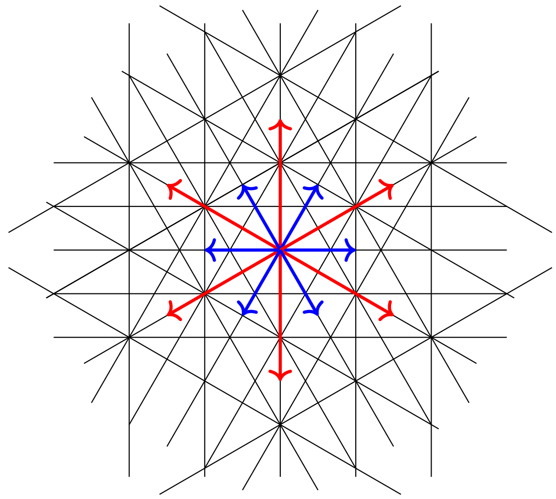


Figure 6.0.1: The $\widetilde{G}_2$ root system with a portion of the corresponding hyperplane arrangement

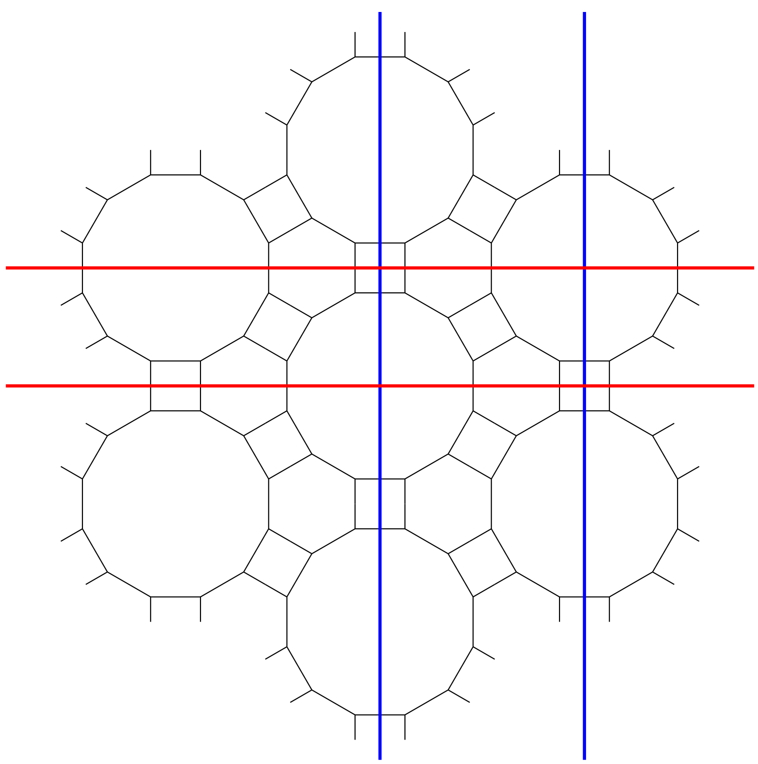


Figure 6.0.2: The two types of parallel hyperplanes in type $\widetilde{G}_2$.

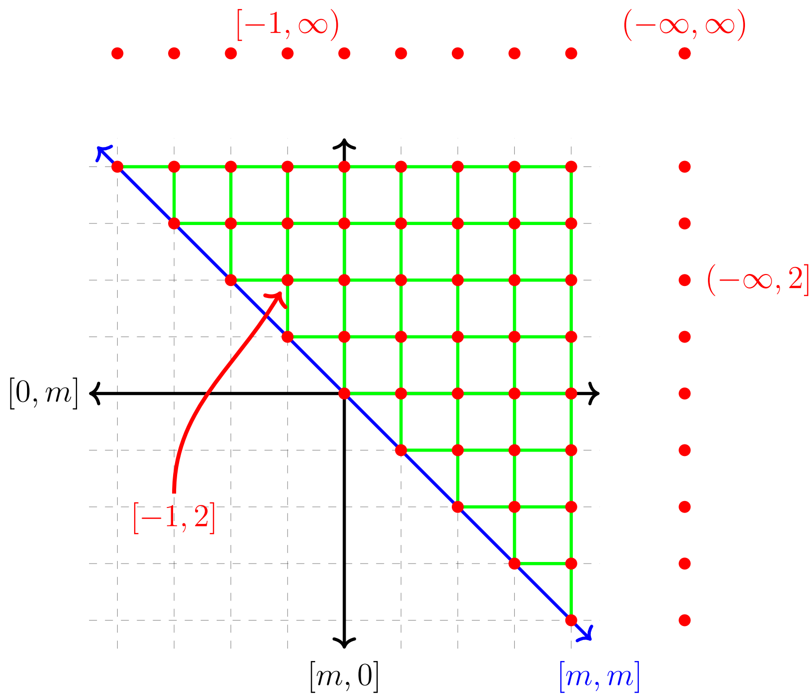


Figure 6.0.3: The poset of convex subcomplexes of the Davis complex ordered by inclusion.

of a subcomplex only relies on guaranteeing that the nullhomotopy has image that is a bounded distance from K. There are, however, additional difficulties associated with having fewer π_1-injective subcomplexes. It implies that not only is it more difficult to find a large enough compact subcomplexes L that contain a nullhomotopy of a nullhomotpic loop, but also decomposing $\pi_1(L)$ as an amalgamted free product of smaller subcomplexes with nice fundamental groups is more difficult as well.

Remark 6.0.2. Consider the poset $\mathcal{P}$ of all full subcomplexes of the pure Salvetti complex X ordered by inclusion. Note that $\mathcal{P}$ is also a directed set since inclusion is both reflexive and transitive, and for every $K, L \in \mathcal{P}$ there is always some $M \in \mathcal{P}$ such that $K \subset M$ and $L \subset M$. Thus, $(\mathcal{P}, \{\pi_1(X)\}_{X\in\mathcal{P}}, i_{KL})$ is a directed system in **Grp**, where i_{KL} are the maps on fundamental groups $i_{KL} : \pi_1(K) \to \pi_1(L)$ induced by inclusion $K \hookrightarrow L$. The case when these maps have no kernel is precisely the case when all subcomplexes π_1-inject, such as for the 1-dimensional Artin groups. However, the perspective of this dissertation was to look at the directed system corresponding to the directed set of full convex pure subcomplexes. For example, consider the poset of full convex pure subcomplexes of type $\widetilde{A}_1$, where each subcomplex could be described in terms of a subset of consecutive integers. This is depicted in Figure 6.0.3, where the red vertices represent a subcomplex, and the green edges represent covering relations. In general, however, the maps i_{KL} may have very large kernel.

Next, recall the observation of Remark 6.0.1 stating that for a nullhomotopic loop it suffices to guarantee that there is a sufficiently large compact convex subcomplex containing a nullhomotopy of the loop. In terms of the language of directed systems, it suffices to show that the kernels of the maps i_{KM} eventually stabilize as M grows. That is, if K is the convex hull of the pure word subcomplex of a loop, then the kernel eventually stabilizes if there is some full compact convex pure subcomplex L such that the kernel $\ker i_{KL}$ is equal to $\ker i_{KM}$ for any full compact convex pure subcomplex $M \supset L$.

Remark 6.0.3. The Artin group of type $\widetilde{G}_2$ is a great initial testing ground for future work because of the aforementioned difficulties, but also because Craig Squier in [16] proved that the Artin group of type $\widetilde{G}_2$ admits a decomposition as an amalgamated free product to prove that its word problem is solvable. It would be interesting to try to adapt this into a statement about the topology of the pure Salvetti complex.

Additionally, in the case of 2-dimensional *hyperbolic* Artin groups, hyperplanes are geodesics which diverge so that there are many examples of consecutive parallel hyperplanes that contain entire 2-cells. This is also true in many Euclidean Artin groups of dimension 3 or greater. In both cases, the pure Salvetti complexes and their convex subcomplexes are straightforward to describe. Thus, these are good candidates for future directions of research.

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