Title
Certifying optimality in hybrid control systems via Lyapunov-like conditions

Journal
IFAC-PapersOnLine, 52(16)

ISSN
2405-8963

Authors
Ferrante, F
Sanfelice, RG

Publication Date
2019-09-01

DOI
10.1016/j.ifacol.2019.11.786

Peer reviewed

Certifying Optimality in Hybrid Control Systems via Lyapunov-like Conditions ⋆

Francesco Ferrante * **Ricardo G. Sanfelice** **

* *Univ. Grenoble Alpes, CNRS, GIPSA-lab, F-38000 Grenoble, France (e-mail: francesco.ferrante@gipsa-lab.fr).*
** *Electrical and Computer Engineering Department, University of California, Santa Cruz, CA 95064, USA (e-mail: ricardo@ucsc.edu).*

Abstract: We formulate an optimal control problem for hybrid systems with inputs and propose conditions for the design of state-feedback laws solving the optimal control problem. The optimal control problem has the flavor of an infinite horizon problem, but it also allows solutions to have a bounded domain of definition, which is possible in hybrid systems with deadlocks. Design conditions for optimal feedback laws are obtained by relating a quite general hybrid cost functional to a Lyapunov like function. These conditions guarantee closed-loop optimality and are given by constrained steady-state-like Hamilton-Jacobi-Bellman-type equations. Applications and examples of the proposed results are presented.

Keywords: Hybrid systems, Optimal Control, Lyapunov theory

1. INTRODUCTION

Over the last decade, new contributions to the problem of asymptotically stabilizing a hybrid dynamical system to a set, with robustness to general perturbations have emerged within a comprehensive theory of hybrid systems, see Goebel et al. (2009, 2012). In these references, hybrid dynamical systems are given by the combination of differential and difference inclusions with constraints, called *hybrid inclusions*. The theoretical results therein have been found useful to derive solutions to several outstanding control problems, such as estimation and control with intermittent information Ferrante et al. (2016); Carnevale et al. (2007); Phillips and Sanfelice (2019); Li et al. (2018), even-triggered control Tolic et al. (2015); Chai et al. (2017) and control of systems with impacts Short and Sanfelice (2018), to just list a few.

Optimal control aspects in hybrid systems have seen a growing interest in the community. First results on optimal control of hybrid systems can be traced back to the 90s in the work of Sussmann (1999), later followed by Caines et al. (2006); Shaikh and Caines (2007), where maximum principles for some class of hybrid systems are formulated. Results that assure optimality for general hybrid inclusions are just now becoming available, with initial results on the existence of optimal control reported in Goebel (2019), on model predictive control in Altin et al. (2018), on linear-quadratic control for specific classes of hybrid systems Cristofaro et al. (2018); Possieri and Teel (2016), and in Ferrante and Sanfelice (2018) for the evaluation of cost of solutions.

In this paper, we present sufficient conditions in terms of Lyapunov-like conditions that assure that a static state-feedback law is optimal. For this purpose, we exploit ideas from the literature of classical optimal control, in particular, those in Bernstein (1993). We formulate an infinite horizon-like optimal control problem for hybrid dynamical systems that extends the one in Goebel (2019) and Altin et al. (2018). Similar to those references, the problem allows the use of different stage cost functions during flows and at jumps. Our main result shows that when a Lyapunov-like function, the state-feedback laws, and the stage cost functions satisfy certain infinitesimal inequalities, then the optimal control problem is solved. These conditions are checkable as they do not require computing solutions to the system. On the other hand, due to the inverse optimality nature of the result, the stage cost functions need to satisfy conditions that depend on the other functions, as well as the data that models the hybrid system. The applicability of the conditions proposed in the paper is showcased in two examples. The first example is reminiscent of the bouncing ball system in Goebel et al. (2012) and illustrates how an optimal stabilizing feedback control can be designed when the stage cost is suitably selected. The second example pertains to the case of hybrid systems with linear maps and periodic jumps. This very special case has been analyzed in Possieri and Teel (2016); Carnevale et al. (2014). Specifically, we show that our main result, when specialized to this class of systems, covers the results in Carnevale et al. (2014). Due to space constraints, the proofs of Corollary 1 and Proposition 1 are here omitted.

Notation: The set $\mathbb{N}_{>0}$ is the set of strictly positive integers, $\mathbb{N}_{\geq 0} = \mathbb{N}_{>0} \cup \{0\}$, $\mathbb{R}_{\geq 0}$ represents the set of non-negative real scalars, $\mathbb{S}_+^n$ denotes the set of real symmetric

⋆ Research by R. G. Sanfelice has been partially supported by the National Science Foundation under Grant no. ECS-1710621 and Grant no. CNS-1544396, by the Air Force Office of Scientific Research under Grant no. FA9550-16-1-0015, Grant no. FA9550-19-1-0053, and Grant no. FA9550-19-1-0169, and by CITRIS and the Banatao Institute at the University of California. Research by Francesco Ferrante is funded in part by ANR via project HANDY, number ANR-18-CE40-0010.

positive semidefinite matrices of dimension n and $\mathbb{S}^n_{++}$ denotes the set of real symmetric positive definite matrices of dimension n. The symbol $\mathbb{R}^{n\times m}$ represents the set of $n\times m$ real matrices. Let $A \in \mathbb{R}^{m\times n}$, we denote its transpose by A^T, and, when $n = m$, we define $\mathrm{He}(A) = A + A^\mathsf{T}$. For a vector $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean norm. Given two vectors x and y, we use the equivalent notation $(x, y) = [x^\mathsf{T}\ y^\mathsf{T}]^\mathsf{T}$. Given a vector x and a closed set $\mathcal{A}$, the distance of x to $\mathcal{A}$ is defined as $|x|_\mathcal{A} = \inf_{y\in\mathcal{A}} |x - y|$. Given a set S, we denote by $\overline{S}$ the closure of S.

1.1 Hybrid Systems with Inputs

We consider controlled hybrid systems with state $x \in \mathbb{R}^n$ and input $u = (u_C, u_D) \in \mathbb{R}^{m_C} \times \mathbb{R}^{m_D}$ of the form

$$\mathcal{H}_u \begin{cases} \dot{x} &= f(x, u_C) \qquad (x, u_C) \in C \\ x^+ &= g(x, u_D) \qquad (x, u_D) \in D \end{cases} \tag{1}$$

where $f\colon \mathbb{R}^n \times \mathbb{R}^{m_C} \to \mathbb{R}^n$ is the *flow map*, $C \subset \mathbb{R}^n \times \mathbb{R}^{m_C}$ is the *flow set*, $g\colon \mathbb{R}^n \times \mathbb{R}^{m_D} \to \mathbb{R}^n$ is *jump map*, and $D \subset \mathbb{R}^n \times \mathbb{R}^{m_D}$ is the *jump set*.

Solutions to the hybrid system $\mathcal{H}_u$ are defined on hybrid time domains. A hybrid time domain E is a subset of $\mathbb{R}_{\geq 0} \times \mathbb{N}_{\geq 0}$ such that for each $(T, J) \in E$ one has $E \cap ([0, T] \times \{0, 1, \ldots, J) = \bigcup_{j=0}^{J-1}([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \cdots \leq t_J$. A hybrid signal u is a function defined over a hybrid time domain. Given a hybrid signal u, we denote by $\mathrm{dom}_t\, u := \{t \in \mathbb{R}_{\geq 0} \colon \exists j \in \mathbb{N}_{\geq 0}$ s.t. $(t, j) \in \mathrm{dom}\, u\}$ and by $\mathrm{dom}_j\, u := \{j \in \mathbb{N}_{\geq 0} \colon \exists t \in \mathbb{R}_{\geq 0}$ s.t. $(t, j) \in \mathrm{dom}\, u\}$. A hybrid signal u is called a hybrid input if $u(\cdot, j)$ is measurable and locally essentially bounded for each j. In particular, we denote by $\mathcal{U}$ the set of hybrid inputs with values in $\mathbb{R}^{m_C} \times \mathbb{R}^{m_D}$. A hybrid signal ϕ is called a hybrid arc if for each $j \in \mathbb{N}_{\geq 0}$, the function $t \mapsto \phi(t, j)$ is locally absolutely continuous on the interval $I^j := \{t \colon (t, j) \in \mathrm{dom}\, \phi\}$. In particular, we denote by $\mathcal{M}$ the set of hybrid arc with values in $\mathbb{R}^n$. Given a hybrid time domain E and $(s, i) \in E$, we denote by $t(i)$ the smallest time t such that $(t, i) \in E$ and by $j(s)$ the smallest index j such that $(s, j) \in E$. A pair $(\phi, u) \in \mathcal{M} \times \mathcal{U}$, with $\mathrm{dom}\, \phi = \mathrm{dom}\, u$, and $u := (u_C, u_D)$ defines a solution pair (ϕ, u) to $\mathcal{H}_u$ if $(\phi(0, 0), u_C(0, 0)) \in \overline{C}$ or $(\phi(0, 0), u_D(0, 0)) \in D$ and (ϕ, u) satisfies the dynamics of $\mathcal{H}_u$; see, e.g., Cai and Teel (2009) for more details on solution pairs to hybrid systems. We say that a solution pair (ϕ, u) to $\mathcal{H}_u$ is maximal if it cannot be extended and is complete if $\mathrm{dom}\, \phi$ is unbounded.

Definition 1. (Set of maximal solution pairs). Given $\xi \in \mathbb{R}^n$, we denote by $\mathcal{S}_u(\xi)$ the set of maximal solution pairs $(\phi, u) \in \mathcal{M} \times \mathcal{U}$ to (1) such that $\phi(0, 0) = \xi$. ◇

Definition 2. (Set of maximal responses). Given $\xi \in \mathbb{R}^n$ and $u := (u_C, u_D) \in \mathcal{U}$, we denote the set of maximal responses to (1) by

$$\mathcal{R}(\xi, u) = \{\phi \in \mathcal{M} \colon (\phi, u) \in \mathcal{S}_u(\xi)\}$$

◇

In particular, given $\xi \in \mathbb{R}^n$ and $u \in \mathcal{U}$, $\mathcal{R}(\xi, u)$ is the "state component" of solution pairs in $\mathcal{S}_u(\xi)$ with input u.

We define the projections of C and D onto $\mathbb{R}^n$, respectively, as

$$\Pi_C = \{\xi \in \mathbb{R}^n \colon \exists u_C \in \mathbb{R}^{m_C} \text{ s.t. } (\xi, u_C) \in C\}$$
$$\Pi_D = \{\xi \in \mathbb{R}^n \colon \exists u_D \in \mathbb{R}^{m_D} \text{ s.t. } (\xi, u_D) \in D\}$$

1.2 Autonomous Hybrid Systems

In this paper, we also consider autonomous hybrid systems with state $x \in \mathbb{R}^n$ of the form

$$\mathcal{H} \begin{cases} \dot{x} &= f(x) \qquad x \in C \\ x^+ &= g(x) \qquad x \in D \end{cases}$$

where $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is the *flow map*, $C \subset \mathbb{R}^n$ is the *flow set*, $g\colon \mathbb{R}^n \to \mathbb{R}^n$ is *jump map*, and $D \subset \mathbb{R}^n$ is the *jump set*. Such systems result from the interconnection of system (1) with feedback controllers. Such systems are introduced next, along with the corresponding definition of maximal solutions.

Definition 3. (Closed-loop maximal solutions). Given $\xi \in \mathbb{R}^n$ and a function $\kappa := (\kappa_D, \kappa_D) \colon \mathbb{R}^n \to \mathbb{R}^{m_C + m_D}$, we denote by $\mathcal{S}_\kappa(\xi)$ the set of maximal solutions ϕ_κ to the autonomous hybrid system

$$\mathcal{H}_\kappa \begin{cases} \dot{x} &= f(x, \kappa_C(x)) \qquad x \in C_\kappa \\ x^+ &= g(x, \kappa_D(x)) \qquad x \in D_\kappa \end{cases}$$

where $C_\kappa := \{x \in \mathbb{R}^n \colon (x, \kappa_C(x)) \in C\}$ and $D_\kappa := \{x \in \mathbb{R}^n \colon (x, \kappa_D(x)) \in D\}$. ◇

2. LYAPUNOV-LIKE FUNCTIONS AND OPTIMAL CONTROL

2.1 Problem Statement

By retracing the same approach as in Bernstein (1993) and (Sontag, 1998, Chapter 8), in this section we define an optimal control problem that has the flavor of an infinite horizon problem, but it also allows solutions to have a bounded domain of definition, which is possible in hybrid systems as in (1) due to with deadlocks, constraints, and jumps outside $C \cup D$.

To begin, let $\mathcal{A} \subset \mathbb{R}^n$ be closed and consider the following definitions:

$$\mathcal{M}_\mathcal{A} := \left\{ \phi \in \mathcal{M} \colon \lim_{\substack{(t,j)\in \mathrm{dom}\, \phi \\ (t,j)\to \sup \mathrm{dom}\, \phi}} |\phi(t, j)|_\mathcal{A} = 0 \right\}$$

and for each $\xi \in \mathbb{R}^n$

$$\mathcal{U}_\mathcal{A}(\xi) := \{u \in \mathcal{U} \colon \exists \phi \in \mathcal{R}(\xi, u) \cap \mathcal{M}_\mathcal{A}\}$$

Essentially, $\mathcal{M}_\mathcal{A}$ is the set of hybrid arcs in $\mathcal{M}$ converging to $\mathcal{A}$, while $\mathcal{U}_\mathcal{A}(\xi)$, for each $\xi \in \mathbb{R}^n$, is the set of hybrid inputs such that there exists a response to (1) from ξ that converges to $\mathcal{A}$. For each initial condition $\xi \in \mathbb{R}^n$ and hybrid input $u \in \mathcal{U}_\mathcal{A}(\xi)$, consider the following cost

$$\mathcal{J}(\xi, u) := \sup_{\phi\in\mathcal{R}(\xi,u)\cap\mathcal{M}_\mathcal{A}} \left(\int_{\mathrm{dom}_t\, \phi} L_C(\phi(s, j(s)), u_C(s, j(s)))ds + \sum_{j=1}^{\sup \mathrm{dom}_j\, \phi} L_D(\phi(t(j), j-1), u_D(t(j), j-1)) \right) \tag{2}$$

where $L_C \colon \mathbb{R}^n \times \mathbb{R}^{m_C} \to \mathbb{R}_{\geq 0}$ is continuous and $L_D \colon \mathbb{R}^n \times \mathbb{R}^{m_D} \to \mathbb{R}_{\geq 0}$. The problem we solve in this paper is formalized as follows.

Problem 1. Let $\mathcal{H}_u$ and $\mathcal{J}$ be defined, respectively, as in (1) and (2), $\mathcal{A} \subset \mathbb{R}^n$ be a closed set, and $\xi \in \mathbb{R}^n$ be such that $\mathcal{U}_\mathcal{A}(\xi)$ is nonempty. Determine

$$\kappa := (\kappa_C, \kappa_D)\colon \mathbb{R}^n \to \mathbb{R}^{m_C} \times \mathbb{R}^{m_D}$$

such that

$$\min_{u \in \mathcal{U}_\mathcal{A}(\xi)} \mathcal{J}(\xi, u) = \int_{\mathrm{dom}_t \phi^\star_\kappa} L_C(\phi^\star_\kappa(s, j(s)), \kappa_C(\phi^\star_\kappa(s, j(s))))ds + \sum_{j=1}^{\sup \mathrm{dom}_j \phi^\star_\kappa} L_D(\phi^\star_\kappa(t(j), j-1), \kappa_D(\phi^\star_\kappa(t(j), j-1))) \tag{3}$$

for each $\phi^\star_\kappa \in \mathcal{S}_\kappa(\xi)$. ◇

2.2 Lyapunov-like Sufficient Conditions

Next, we provide sufficient conditions for the solution of Problem 1.

Theorem 1. Let $\mathcal{H}_u$ and $\mathcal{J}$ be defined, respectively, as in (1) and (2), $\mathcal{A} \subset \mathbb{R}^n$ be closed, and $\xi \in \Pi_D \cup \Pi_{\overline{C}}$. Assume that there exist $\kappa_C\colon \mathbb{R}^n \to \mathbb{R}^{m_C}$, $\kappa_D\colon \mathbb{R}^n \to \mathbb{R}^{m_D}$, and a function $V\colon \mathrm{dom}\, V \to \mathbb{R}$, with $\mathrm{dom}\, V \supset \Pi_D \cup \Pi_{\overline{C}} \cup g(D)$, that is continuously differentiable on an open set containing $\Pi_{\overline{C}}$ and uniformly continuous on a neighborhood of $\mathcal{A}$. Furthermore, assume that $V(\mathcal{A} \cap \mathrm{dom}\, V) = \{0\}$, and

$$\langle \nabla V(x), f(x, \kappa_C(x))\rangle + L_C(x, \kappa_C(x)) = 0 \qquad \forall x \in C_\kappa \tag{4a}$$

$$\langle \nabla V(x), f(x, u_C)\rangle + L_C(x, u_C) \geq 0 \qquad \forall (x, u_C) \in C \tag{4b}$$

$$V(g(x, \kappa_D(x))) - V(x) + L_D(x, \kappa_D(x)) = 0 \qquad \forall x \in D_\kappa \tag{4c}$$

$$V(g(x, u_D)) - V(x) + L_D(x, u_D) \geq 0 \qquad \forall (x, u_D) \in D \tag{4d}$$

and that for any $\phi^\star_\kappa \in \mathcal{S}_\kappa(\xi)$

$$s \mapsto L_C(\phi^\star_\kappa(s, j(s)), \kappa_C(\phi^\star_\kappa(s, j(s))))$$

is measurable and

$$\lim_{t+j \to \sup \mathrm{dom}\, \phi^\star_\kappa} |\phi^\star_\kappa(t, j)|_\mathcal{A} = 0 \tag{5}$$

Then $\kappa = (\kappa_C, \kappa_D)$ solves Problem 1. In particular

$$\min_{u \in \mathcal{U}_\mathcal{A}(\xi)} \mathcal{J}(\xi, u) = V(\xi) \tag{6}$$

Proof sketch. Pick $(\phi, u) \in \mathcal{S}_u(\xi) \cap (\mathcal{M}_\mathcal{A} \times \mathcal{U}_\mathcal{A}(\xi))$ and observe that for each $(t, j) \in \mathrm{dom}\, \phi$

$$V(\phi(t, j)) - V(\phi(0, 0)) = \int_0^t \frac{d}{ds} V(\phi(s, j(s)))ds + \sum_{i=1}^{j} [V(\phi(t(i), i)) - V(\phi(t(i), i-1))] \tag{7}$$

Using (4b), one gets, for almost all $s \in [0, t]$

$$\frac{d}{ds} V(\phi(s, j(s))) \geq -L_C(\phi(s, j(s)), u_C(s, j(s))) \tag{8a}$$

Similarly, for each $i \in \{1, 2, \ldots, j\}$, (4d) implies

$$V(\phi(t(i), i)) - V(\phi(t(i), i-1)) \geq - L_D(\phi(t(i), i-1), u_D(t(i), i-1)) \tag{8b}$$

Combining (7), (8a), and (8b) one gets

$$V(\phi(t, j)) - V(\phi(0, 0)) \geq -\widetilde{\mathcal{J}}_{(\phi,u)}(t, j) \tag{9}$$

where, for each $(t, j) \in \mathrm{dom}\, \phi$,

$$\widetilde{\mathcal{J}}_{(\phi,u)}(t, j) := \int_0^t L_C(\phi(s, j(s)), u_C(s, j(s)))ds + \sum_{i=1}^{j} L_D(\phi(t(i), i-1), u_D(t(i), i-1))$$

Therefore, (9) implies

$$\widetilde{\mathcal{J}}_{(\phi,u)}(t, j) \geq V(\xi) - V(\phi(t, j)) \qquad \forall (t, j) \in \mathrm{dom}\, \phi \tag{10}$$

Since $(\phi, u) \in \mathcal{S}_u(\xi) \cap (\mathcal{M}_\mathcal{A} \times \mathcal{U}_\mathcal{A}(\xi))$, it follows that $\lim_{t+j \to \sup \mathrm{dom}\, \phi} |\phi(t, j)|_\mathcal{A} = 0$. Using uniform continuity of V on a neighborhood of $\mathcal{A}$ and $V(\mathcal{A}) = \{0\}$, it can be shown that

$$\lim_{\substack{(t,j) \in \mathrm{dom}\, \phi \\ (t,j) \to \sup \mathrm{dom}\, \phi}} V(\phi(t, j)) = 0$$

which, thanks to (10), gives

$$\lim_{\substack{(t,j) \in \mathrm{dom}\, \phi \\ (t,j) \to \sup \mathrm{dom}\, \phi}} \widetilde{\mathcal{J}}_{(\phi,u)}(t, j) \geq V(\xi)$$

Then, from the definition of $\mathcal{J}$, we get

$$\mathcal{J}(\xi, u) \geq V(\xi) \tag{11}$$

Pick $\phi^\star_\kappa \in \mathcal{S}_\kappa(\xi)$. By repeating the same arguments as above, using (4a) and (4c) one obtains

$$-V(\phi^\star_\kappa(t, j)) + V(\phi^\star_\kappa(0, 0)) = \int_0^t L_C(\phi^\star_\kappa(s, j(s)), \kappa_C(\phi^\star_\kappa(s, j(s))))ds \sum_{i=1}^{j} L_D(\phi^\star_\kappa(t(i), i-1), \kappa_D(\phi^\star_\kappa(t(i), i-1))) =: \widetilde{\mathcal{J}}_{\phi^\star_\kappa}(t, j) \tag{12}$$

Taking limits in (12), due to V being uniformly continuous on a neighborhood of $\mathcal{A}$ and $V(\mathcal{A}) = \{0\}$, thanks to (5) gives

$$\lim_{\substack{(t,j) \in \mathrm{dom}\, \phi^\star_\kappa \\ (t,j) \to \sup \mathrm{dom}\, \phi^\star_\kappa}} \widetilde{\mathcal{J}}_{\phi^\star_\kappa(t,j)} = V(\xi)$$

The latter, using (11), gives $\min_{u \in \mathcal{U}_\mathcal{A}(\xi)} \mathcal{J}(\xi, u) = V(\xi)$. ■

Remark 1. In this paper, following a similar philosophy as in Bernstein (1993), we denote the function V in Theorem 1 as "Lyapunov-like function." This is because of the infinitesimal conditions it satisfies, i.e., (4). Nonetheless, observe that, as emphasized by (6), V coincides with the value function of the considered optimal control problem.

The result given next provides sufficient conditions for the solution to Problem 1 that are easier to check compared to those given in Theorem 1. These conditions take the form of Hamilton-Jacobi-Bellman steady-state equations.

Corollary 1. Define the following set-valued maps

$$\Pi_u(x, C) := \{u_C \in \mathbb{R}^{m_C} : (x, u_C) \in C\}$$
$$\Pi_u(x, D) := \{u_D \in \mathbb{R}^{m_D} : (x, u_D) \in D\}$$

and let $V\colon \mathrm{dom}\, V \to \mathbb{R}$, with $\mathrm{dom}\, V \supset \Pi_D \cup \Pi_{\overline{C}} \cup g(D)$, be continuously differentiable on an open set containing $\Pi_{\overline{C}}$. Define:

$$\dot{V}(x, u_C) := \langle \nabla V(x), f(x, u_C)\rangle \qquad \forall (x, u_C) \in C$$
$$\Delta V(x, u_D) := V(g(x, u_D)) - V(x) \quad \forall (x, u_D) \in D$$

and let

$$\kappa_C(x) \in \underset{u_C\in\Pi_u(x,C)}{\arg\min} \; (\dot{V}(x,u_C)+L_C(x,u_C)) \quad \forall x\in\Pi_C \quad (13)$$

$$\kappa_D(x) \in \underset{u_D\in\Pi_u(x,D)}{\arg\min} \; (\Delta V(x,u_D)+L_D(x,u_D)) \; \forall x\in\Pi_D \quad (14)$$

If

$$0 = \dot{V}(x,\kappa_C(x)) + L_C(x,\kappa_C(x)) \qquad \forall x\in\Pi_C \qquad (15)$$

$$0 = \Delta V(x,\kappa_D(x)) + L_D(x,\kappa_D(x)) \qquad \forall x\in\Pi_D \qquad (16)$$

then (4a), (4b), (4c), and (4d) hold. □

Remark 2. Conditions for the design of feedback laws satisfying (13) and (14) are given in Sanfelice (2013).

Remark 3. It is worthwhile to remark that, given $\kappa = (\kappa_C,\kappa_D)$ and V such that (4a)-(4b)-(4c)-(4d) are satisfied, if $C_\kappa = \Pi_C$ and $D_\kappa = \Pi_D$, then V and κ satisfy (13), (14), (15), and (16). This is for example the case when $C = C_x\times C_u$ and $D = D_x\times D_u$, for some sets $C_x, D_x \subset \mathbb{R}^n$, $C_u \subset \mathbb{R}^{m_C}$, and $D_u \subset \mathbb{R}^{m_D}$. In this sense, under some additional assumptions, Corollary 1 provides alternative conditions for the solution to Problem 1 that are equivalent to the conditions in Theorem 1, yet easier to check. ◇

Next we show an example in which Corollary 1 is used to solve Problem 1.

Example 1. Let $\lambda > 0$ and consider the following hybrid system with state $x\in\mathbb{R}^2$ and input u_D

$$\mathcal{H}_u \begin{cases} \dot{x} = \begin{bmatrix} x_2 \\ -1 \end{bmatrix} & x\in\mathbb{R}_{\geq 0}\times\mathbb{R} \\ x^+ = \begin{bmatrix} 0 \\ -\lambda x_2 + u_D \end{bmatrix} & (x,u_D)\in\{0\}\times\mathbb{R}_{\leq 0}\times\mathbb{R} \end{cases}$$

which is somehow reminiscent of the bouncing ball system analyzed in (Goebel et al., 2012, Chapter 1). For this example, we select $\mathcal{A} = \{0\}$, $L_C \equiv 0$, and $L_D(x,u_D) := x_2^2 Q_D + u_D^2 R_D$, with $Q_D, R_D > 0$. Pick $V(x) = \frac{1}{2}x_2^2 + x_1$ and observe that for all $x\in\Pi_C$, $\dot{V}(x) = 0$, thereby implying that (15) holds. In addition, it can be easily shown that for any $x\in\Pi_D$

$$\underset{u_D\in\Pi_u(x,D)}{\arg\min} \; (\Delta V(x,u_D)+L_D(x,u_D)) = \left\{\frac{\lambda}{2R_D+1}x_2\right\}$$

Following Corollary 1, we select $\kappa_D(x) = \frac{\lambda}{2R_D+1}x_2$. Straightforward manipulations show that, for this selection of κ_D, if

$$Q_D = \frac{-2R_D\lambda^2+2R_D+1}{4R_D+2} \qquad (17)$$

then, (16) holds. In particular, one has that

$$V(g(x,\kappa_D(x))) - V(x) = -\left(Q_D + \frac{\lambda^2 R_D}{4R_D^2+4R_D+1}\right)x_2^2 \quad \forall x\in\Pi_D$$

where Q_D is selected as in (17). Because of Q_D being strictly positive, similar arguments as in (Goebel et al., 2012, Example 8.5, page 173) enables one to conclude that maximal solutions to the closed-loop system obtained by selecting $u_D = \kappa_D(x)$ converge to $\mathcal{A}$. As such, Theorem 1 and Corollary 1 can be invoked to conclude on the optimality of the feedback law κ_D.

One challenging aspect in the solution to this problem is that one needs to guarantee that Q_D selected as in (17) is nonnegative. This depends on the value of R_D and λ. It is worthwhile to observe that when $\lambda\in(0,1)$ Q_D is nonnegative for any value of R_D. As a matter of fact, it can be easily shown that for $u_D\equiv 0$ maximal solutions pairs to $\mathcal{H}_u$ approach $\mathcal{A}$, i.e., for any $\xi\in\mathbb{R}^2$, $0\in\mathcal{U}_\mathcal{A}(\xi)$. On the other hand, as λ gets larger, R_D needs to be suitably selected to ensure that $Q_D \geq 0$. This is made clear in Fig. 1, which shows the value of Q_D in (17) as a function of R_D for different values of λ. In Fig. 2 we report the evolution of the closed-loop system from $x(0,0) = (1,1)$ with $\lambda = 1.5$ for different values of R_D. Fig. 2 emphasizes that as R_D gets smaller, the control action gets more aggressive and is able to bring the state to zero faster.

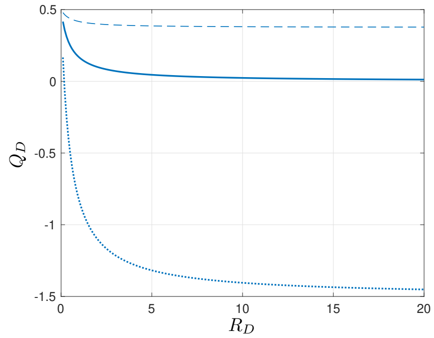

Fig. 1. Plot of Q_D in (17) vs. R_D for different values of λ: $\lambda = 2$ (dotted line), $\lambda = 1$ (solid line), and $\lambda = 0.5$ (dashed line).

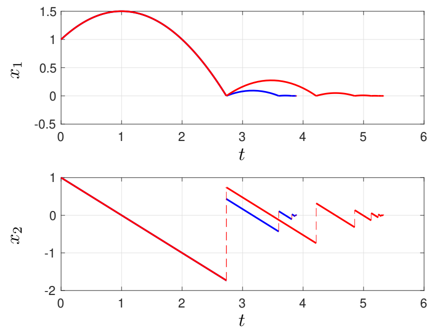

Fig. 2. Plot of the closed-loop trajectory in Example 1 from $x(0,0) = (1,1)$ with $\lambda = 1.5$ for different values of R_D: $R_D = 0.2$ (red) and $R_D = 0.1$ (blue).

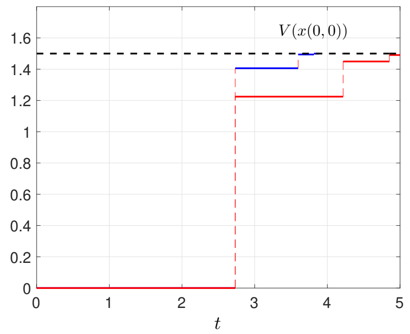

Fig. 3. Evolution of the incurred cost in Example 1 for different values of R_D: $R_D = 0.2$ (red) and $R_D = 0.1$ (blue).

In Fig. 3, we report the evolution of the cost (12) along the solution of the closed-loop system from $x(0,0) = (1,1)$ and for different values of R_D. The picture clearly shows that the cost incurred by the two solutions correspond and, as foreseen by Theorem 1, is equal to $V(x(0,0))$.

2.3 Linear-Quadratic Problems with Periodic Jumps

In this section, we specialize our results to the case of hybrid systems with linear maps, periodic jumps, and quadratic cost. Such type of systems can be found in numerous applications. In particular, in Possieri and Teel (2016) and in Carnevale et al. (2014), specific tools have been provided for the solution to quadratic optimal control problems for hybrid systems linear flow and jump maps and periodic jumps. In this paper, we show how our general tools, when specialized to this class of hybrid systems, give rise to similar conditions as in Possieri and Teel (2016); Carnevale et al. (2014).

Consider the following hybrid system $\mathcal{H}_u^P$ with state $x = (x_p, \tau) \in \mathbb{R}^n \times [0,T]$ and input $(u_C, u_D) \in \mathbb{R}^{m_C+m_D}$ given by

$$\mathcal{H}_u^P \begin{cases} \left.\begin{aligned} \dot{x}_p &= A_c x_p + B_c u_C \\ \dot{\tau} &= 1 \end{aligned}\right\} & (x, u_C) \in C_P \\ \left.\begin{aligned} x_p^+ &= A_d x_p + B_d u_D \\ \tau^+ &= 0 \end{aligned}\right\} & (x, u_D) \in D_P \end{cases} \tag{18}$$

where

$$C_P := \mathbb{R}^n \times [0,T] \times \mathbb{R}^{m_C}, D_P := \mathbb{R}^n \times \{T\} \times \mathbb{R}^{m_D}$$

and $A_c, A_d \in \mathbb{R}^{n\times n}$, $B_c \in \mathbb{R}^{n\times m_C}$, $B_d \in \mathbb{R}^{n\times m_D}$, and $T > 0$ are given. Observe that, in this case, $\Pi_C = \mathbb{R}^n \times [0,T]$ and $\Pi_D = \mathbb{R}^n \times \{T\}$. In addition, one has

$$\begin{aligned} \Pi_u(x, C) &= \mathbb{R}^{m_C} \qquad \forall x \in \Pi_C \\ \Pi_u(x, D) &= \mathbb{R}^{m_D} \qquad \forall x \in \Pi_D \end{aligned}$$

The following result can be established:

Proposition 1. Let $\mathcal{A} = \{0\} \times [0,T]$, $\xi = (\xi_p, \xi_\tau) \in \mathbb{R}^n \times [0,T]$, $(x, u_C) \mapsto L_C(x, u_C) := x_p^\mathsf{T} Q_C x_p + u_C^\mathsf{T} R_C u_C$, and $(x, u_D) \mapsto L_D(x, u_D) := x_p^\mathsf{T} Q_D x_p + u_D^\mathsf{T} R_D u_D$, where $Q_C, Q_D \in \mathbb{S}_+^n$, $R_C \in \mathbb{S}_{++}^{m_C}$, and $R_D \in \mathbb{S}_{++}^{m_D}$. Assume that there exists $P\colon [0,T] \to \mathbb{S}_{++}^n$ continuously differentiable such that

$$\begin{aligned} \frac{d}{d\tau}P(\tau) + \operatorname{He}\left(A_c^\mathsf{T} P(\tau) - P(\tau) B_c R_C^{-1} B_c^\mathsf{T} P(\tau)\right) \\ + Q_C = 0 \quad \forall \tau \in (0,T) \end{aligned}$$

$$\begin{aligned} P(T) = Q_D &+ A_d^\mathsf{T} P(0) A_d \\ &- A_d^\mathsf{T} P(0) B_d (R_D + B_d^\mathsf{T} P(0) B_d)^{-1} B_d^\mathsf{T} P(0) A_d \end{aligned}$$

For all $x \in \mathbb{R}^n \times [0,T]$, let

$$\begin{aligned} \kappa_C(x) &:= -R_C^{-1} B_c^\mathsf{T} P(\tau) x_p \\ \kappa_D(x) &:= -(B_d^\mathsf{T} P(0) B_d + R_D)^{-1} B_d^\mathsf{T} P(0) A_d x_p \end{aligned}$$

and consider the corresponding closed-loop system $\mathcal{H}_\kappa^P$ obtained by setting $(u_C, u_D) = (\kappa_C(x), \kappa_D(x))$ in (18). Assume that any maximal solution to $\mathcal{H}_\kappa^P$ converges to $\mathcal{A}$. Then, $\kappa = (\kappa_C, \kappa_D)$ solves Problem 1. In particular

$$\min_{u \in \mathcal{U}_\mathcal{A}(\xi)} \mathcal{J}(\xi, u) = \xi_p^\mathsf{T} P(\xi_\tau) \xi_p$$

□

Remark 4. It is worthwhile to observe that when Q_C or Q_D are positive definite, by using (15) and (16), one can easily show that any maximal solution to (18) approaches the set [1] $\mathcal{A}$. This renders the application of Proposition 1 to the solution of Problem 1 easier. ◇

Remark 5. Conditions (15) and (16) are strongly related to the results in Carnevale et al. (2014), especially with Theorem 1 therein. On the other hand, Proposition 1, as opposed to (Carnevale et al., 2014, Theorem 1), does not require Q_D and Q_C to be positive definite and gives rise to checkable conditions for the design of optimal feedback laws. ◇

Example 2. Consider the following data for (18):

$$A_c = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, A_d = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}, B_d = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, T = 1$$

In this paper we want to illustrate an approach that is reminiscent of the *inverse optimal control design* paradigm considered in Doyle et al. (1996); Krstic and Li (1998). In particular, our approach is a follows: First we select $Q_C \in \mathbb{S}_+^n$, $R_C \in \mathbb{S}_{++}^{m_C}, R_D \in \mathbb{S}_{++}^{m_D}$, and $X \in \mathbb{S}_{++}^n$. Second, by using (Ntogramatzidis and Ferrante, 2011, Theorem 2.1), we determine the unique solution P to (15) such that $P(T) = X$. By using standard arguments from LQ control theory, it can be easily shown that if $X \in \mathbb{S}_{++}^n$, then for all $\tau \in [0,T]$, $P(\tau) \in \mathbb{S}_{++}^n$. Third, using the expression of $\tau \mapsto P(\tau)$ available from the previous step, we select $Q_D \in \mathbb{S}_+^n$ (if any) such that (16) holds. Let us showcase the application of this approach for this example. We pick $Q_C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, R_C = R_D = 1, X = 9I$. By applying (Ntogramatzidis and Ferrante, 2011, Theorem 2.1), one gets $P(0) \approx \begin{bmatrix} 5.1 & 2.6 \\ 2.6 & 2.3 \end{bmatrix} \in \mathbb{S}_{++}^n$. Since Q_C is positive semidefinite, the results in Carnevale et al. (2014) do not apply. Solving for $Q_D \in \mathbb{S}_+^n$ in (16), one gets $Q_D \approx \begin{bmatrix} 1.7 & -1 \\ -1 & 7.8 \end{bmatrix} \in \mathbb{S}_{++}^n$. It is worthwhile to observe that since Q_D is positive definite, from Remark 1, one has that the optimal feedback law resulting from Proposition 1 renders the set $\mathcal{A}$ globally attractive for the closed-loop system (18). As such, Proposition 1 can be applied directly to conclude. In Fig. 4 and Fig. 5, we report, respectively, the evolution of the state x_p and of the cost functional (12) along the solution to the closed-loop system from $x(0,0) = (10,10,0)$. Fig. 4 shows that the closed-loop system state approaches the set $\mathcal{A}$ and Fig. 5 points out that the cost incurred by the two solutions, as established by Proposition 1, is equal to $V(x(0,0))$.

3. CONCLUSION

In this paper we addressed the design of optimal static state-feedback laws for hybrid systems in the framework in Goebel et al. (2012). The results are obtained by relating a quite general hybrid cost functional to a Lyapunov like function. Sufficient conditions for optimality of feedback laws are given in terms of Hamilton-Jacobi-Bellman steady state equations. The proposed results are illustrated in some numerical examples. Future research directions in-

[1] When Q_C or Q_D are positive definite, convergence of maximal solutions to (18) towards $\mathcal{A}$ can be shown by observing that for any maximal solution ϕ to (18), $\operatorname{dom}\phi$ is unbounded in both the t and j-directions.

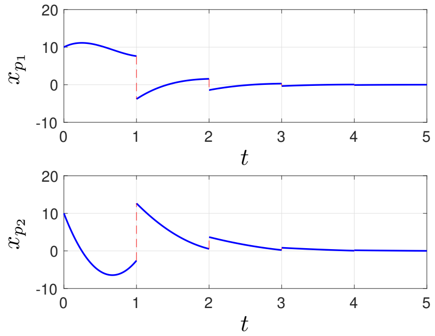

Fig. 4. Evolution of the closed-loop system from $x(0,0) = (10, 10, 0)$ in Example 2.

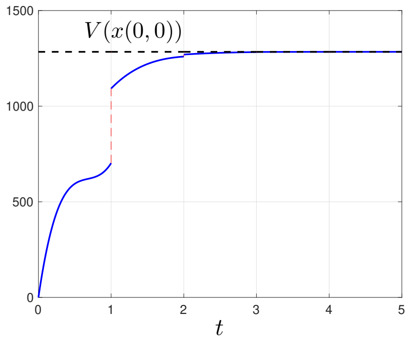

Fig. 5. Evolution of the incurred cost in Example 2.

clude the extension of the proposed approach to hybrid dynamical games in the spirit of L'Afflitto (2017).

REFERENCES

Altin, B., Ojaghi, P., and Sanfelice, R.G. (2018). A model predictive control framework for hybrid systems. In *Proceedings of the 6th IFAC Conference on Nonlinear Model Predictive Control*, 128–133.

Bernstein, D.S. (1993). Nonquadratic cost and nonlinear feedback control. *International Journal of Robust and Nonlinear Control*, 3(3), 211–229.

Cai, C. and Teel, A.R. (2009). Characterizations of input-to-state stability for hybrid systems. *Systems & Control Letters*, 58(1), 47–53.

Caines, P.E., Clarke, F.H., Liu, X., and Vinter, R.B. (2006). A maximum principle for hybrid optimal control problems with pathwise state constraints. In *Proceedings of the 45th IEEE Conference on Decision and Control*, 4821–4825.

Carnevale, D., Galeani, S., and Sassano, M. (2014). A linear quadratic approach to linear time invariant stabilization for a class of hybrid systems. In *22nd Mediterranean Conference of Control and Automation*, 545–550.

Carnevale, D., Teel, A.R., and Nešić, D. (2007). A Lyapunov proof of an improved maximum allowable transfer interval for networked control systems. *IEEE Transactions on Automatic Control*, 52(5), 892–897.

Chai, J., Casau, P., and Sanfelice, R.G (2017). Analysis and design of event-triggered control algorithms using hybrid systems tools. In *Proceedings of the 56th IEEE Conference on Decision and Control*, 6057–6062.

Cristofaro, A., Possieri, C., and Sassano, M. (2018). Linear-quadratic optimal control for hybrid systems with state-driven jumps. In *Proceedings of the 2018 European Control Conference*, 2499–2504.

Doyle, J., Primbs, J.A., Shapiro, B., and Nevistic, V. (1996). Nonlinear games: examples and counterexamples. In *Proceedings of the 35th IEEE Conference on Decision and Control*, 3915–3920.

Ferrante, F., Gouaisbaut, F., Sanfelice, R.G., and Tarbouriech, S. (2016). State estimation of linear systems in the presence of sporadic measurements. *Automatica*, 73, 101–109.

Ferrante, F. and Sanfelice, R., G. (2018). Cost evaluation for hybrid inclusions: A Lyapunov approach. In *Proceedings of the 57th IEEE Conference on Decision and Control*, 855–860.

Goebel, R., Sanfelice, R.G., and Teel, A.R. (2012). *Hybrid Dynamical Systems: Modeling, Stability, and Robustness*. Princeton University Press.

Goebel, R., Sanfelice, R.G., and Teel, A. (2009). Hybrid dynamical systems. *IEEE Control Systems Magazine*, 29(2), 28–93.

Goebel, R. (2019). Existence of optimal controls on hybrid time domains. *Nonlinear Analysis: Hybrid Systems*, 31, 153–165.

Krstic, M. and Li, Z.H. (1998). Inverse optimal design of input-to-state stabilizing nonlinear controllers. *IEEE Transactions on Automatic Control*, 43(3), 336–350.

L'Afflitto, A. (2017). Differential games, continuous Lyapunov functions, and stabilisation of non-linear dynamical systems. *IET Control Theory & Applications*.

Li, Y., Phillips, S., and Sanfelice, R.G. (2018). Robust distributed estimation for linear systems under intermittent information. *IEEE Transactions on Automatic Control*, 63(4), 973–988.

Ntogramatzidis, L. and Ferrante, A. (2011). On the exact solution of the matrix Riccati differential equation. In *Proceedings of the 18th IFAC World Congress*, 14556–14561.

Phillips, S. and Sanfelice, R.G. (2019). Robust distributed synchronization of networked linear systems with intermittent information. *Automatica*, 105, 323–333.

Possieri, C. and Teel, A.R. (2016). LQ optimal control for a class of hybrid systems. In *Proceedings of the 55th IEEE Conference on Decision and Control*, 604–609. IEEE.

Sanfelice, R.G. (2013). Pointwise minimum-norm control laws for hybrid systems. In *Proceedings of the 52nd IEEE Conference on Decision and Control*, 2665–2670.

Shaikh, M.S. and Caines, P.E. (2007). On the hybrid optimal control problem: theory and algorithms. *IEEE Transactions on Automatic Control*, 52(9), 1587–1603.

Short, B. and Sanfelice, R.G. (2018). A hybrid predictive control approach to trajectory tracking for a fully actuated biped. In *Proceedings of the American Control Conference 2018*, 3526–3531.

Sontag, E.D. (1998). *Mathematical control theory: deterministic finite dimensional systems*, volume 6. Springer Science & Business Media, 2nd edition.

Sussmann, H.J. (1999). A maximum principle for hybrid optimal control problems. In *Proceedings of the 38th IEEE Conference on Decision and Control*, volume 1, 425–430.

Tolic, D., Sanfelice, R.G., and Fierro, R. (2015). Input-output triggered control using Lp stability over finite horizons. *International Journal of Robust and Nonlinear Control*, 2299–2327. doi:10.1002/rnc.3203.