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AN ARITHMETIC CONSTRUCTION OF THE GAUSSIAN AND EISENSTEIN TOPOGRAPHS

A dissertation submitted in partial satisfaction of the requirements for the degree of

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in

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by

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Abstract

An Arithmetic Construction of the Gaussian and Eisenstein Topographs

by

Christopher Shelley

We demonstrate a purely arithmetic construction of the Eisenstein and Gaussian topographs of Bestvina and Savin. Influenced by Conway's approach, we recover these topographs as incidence geometries over sets of "generalized lax bases". We use the results of Johnson and Weiss to identify our constructions with the Coxeter geometries arising from projective general linear groups.

To my parents.

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Chapter 1

Introduction

In [6], Conway describes a rank three incidence geometry $\mathcal{T}$; a construction he calls a "topograph". The geometry is built from the arithmetic properties of the module $\mathbb{Z}^2$ as follows. A "lax vector" is defined as a primitive element of $\mathbb{Z}^2$ modulo sign, and a pair of lax vectors is called a "lax basis" if a (any) set of representatives form a basis. Conway introduces what he terms a "superbasis": an ordered triple (u, v, w) for which $\{u, v\}$ is a basis and $u + v + w = 0$. Accordingly, he calls a set of lax vectors $\{\pm u, \pm v, \pm w\}$ a "lax superbasis" whenever (u, v, w) is a superbasis. The set of lax vectors, lax bases, and lax superbases determine the vertices, edges, and faces of the geometry $\mathcal{T}$, respectively. An incidence relation is given by symmetric inclusion. Conway argues that $\mathcal{T}$ has a realization in the hyperbolic plane $\mathbb{H}^2$, where the lax superbases and lax bases are identified with the vertices and edges of an infinite trivalent tree $T \subset \mathbb{H}^2$. The lax vectors correspond to the connected components of $\mathbb{H}^2 \backslash T$.

The geometry $\mathcal{T}$ has been studied in numerous guises prior to Conway. Serre [14] constructs the tree T as a $\mathrm{PSL}_2(\mathbb{Z})$-invariant retract of $\mathbb{H}^2$. On the other hand, the presentation

$$\mathrm{PGL}_2(\mathbb{Z}) = \left\langle s_0, s_1, s_2 \mid (s_1 s_2)^3 = (s_0 s_2)^2 = s_0^2 = s_1^2 = s_2^2 = 1 \right\rangle,$$

where

$$s_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad s_1 = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}, \text{ and } s_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

identifies $\mathrm{PGL}_2(\mathbb{Z})$ as a hyperbolic Coxeter group. A general construction, known to Tits [15], then recovers $\mathcal{T}$ as an abstract coset geometry associated to this Coxeter group (its "Coxeter geometry") . See [8], [13] or [1] for details.

The significance of Conway's interpretation of this geometry is its application to the study of integer valued binary quadratic forms. Suppose $f : \mathbb{Z}^2 \to \mathbb{Z}$ is a binary quadratic form. Since $f(u) = f(-u)$ for every $u \in \mathbb{Z}^2$, the form determines a well defined function on the set of lax vectors of $\mathcal{T}$. Conway labels each of these top dimensional faces u by its norm $f(u) \in \mathbb{Z}$ to generate the "range topograph" ${}^f\mathcal{T}$ of f. He shows that the arithmetic structure of $\mathcal{T}$ establishes certain arithmetic relations in the range topograph. Conway exploits these relations to recover much of the classical theory of integral binary quadratic forms. In particular, when f is an anisotropic indefinite form, he demonstrates that the set of lax bases at which f represents both positive and negative integers determines a subgeometry R_f of ${}^f\mathcal{T}$. Conway calls R_f the "river" of the form f and argues that the river of any such form is topologically a line with compact fundamental domain.

Recently, Bestvina and Savin [3] have adapted Conway's ideas to study binary Hermitian forms over the ring of integers in certain imaginary quadratic number fields. Their approach is based on the work of Ash [2] and Mendoza [12] who, in the spirit of Serre, show that if K is an imaginary quadratic number field with fundamental discriminant, and $\mathcal{O}$ is the ring of integers in K, then there exists a two dimensional $\mathrm{GL}_2(\mathcal{O})$-invariant retract $X_{\mathcal{O}}$ of $\mathbb{H}^3$. When K has class number one, Bestvina and Savin interpret $X_{\mathcal{O}}$ as a topograph for $\mathcal{O}^2$ and identify the cusps of $\mathbb{H}^3 \backslash X_{\mathcal{O}}$ with "lax vectors" - primitive elements of $\mathcal{O}^2$ modulo units. Their work focuses on anisotropic indefinite forms, for which they define an analogy of Conway's river they call the "ocean". Their main result states that the ocean of an anisotropic indefinite binary hermitian form is topologically a plane.

In this thesis we derive a purely algebraic formulation of Bestvina and Savin's topographs for the cases $A = \mathbb{Z}[e^{2\pi i/3}]$ and $A = \mathbb{Z}[i]$. Our approach, heavily influenced by Conway, is to define certain "generalized bases" as a means of constructing an incidence geometry from the arithmetic of A^2. Motivation comes from Johnson and Weiss [11] who prove that the projective semilinear groups $\mathrm{P\Gamma L}_2(\mathbb{Z}[i])$ and $\mathrm{P\Gamma L}_2(\mathbb{Z}[e^{2\pi i/3}])$ admit Coxeter group presentations. The coset geometries corresponding to these Coxeter groups are known [7] to have realizations as hyperbolic tessellations of $\mathbb{H}^3$ and the rank three (two dimensional) truncations are topologically equivalent to the respective retract X_A [9].

This thesis is organized as follows. Chapter 2 provides the reader with the essential definitions and results concerning incidence geometry, Coxeter groups and coset geometries. The construction of the Coxeter geometry (a coset geometry arising from an indexed Coxeter group) is discussed.

Chapter 3 focuses on the Eisenstein integers $\mathbb{Z}[e^{2\pi i/3}]$. We begin by introducing lax tribases (equivalent to Conway's "lax superbases") and lax tetrabases. These objects are equipped with an incidence structure and the Eisenstein topograph $\mathcal{E}$ is defined. We then identify $\tilde{\Gamma} = \mathrm{P\Gamma L}_2(\mathbb{Z}[e^{2\pi i/3}])$ as a Coxeter group (reproving a result of [11]) and use this to prove our main result: the Coxeter geometry associated to $\tilde{\Gamma}$ is isomorphic to the topograph $\mathcal{E}$. We end with a description and illustrations of the local structure of $\mathcal{E}$.

Chapter 4 is a direct analogy of Chapter 3 for the Gaussian integers $\mathbb{Z}[i]$. We give the definition of a lax octobasis and use this to define the Gaussian topograph $\mathcal{G}$. We discuss the local properties of $\mathcal{G}$ and prove that it is isomorphic to the Coxeter geometry arising from the Coxeter group $\mathrm{P\Gamma L}_2(\mathbb{Z}[i])$.

Chapter 2

Preliminaries

2.1 Incidence Geometry

This section is intended to provide the reader with a brief introduction to incidence geometry. Buekenhaut and Cohen's forthcoming book [5] serves as an excellent and thorough reference for this subject.

Definition 2.1.1. *An* ***incidence relation*** *on a set X is a reflexive, symmetric relation on X. A non-empty set of pairwise incident elements is called a* ***flag****. A flag is said to be* ***maximal*** *if it is not properly contained in any other flag.*

Definition 2.1.2. *Let X and I be non-empty sets, let $*$ be an incidence relation on X, and let Typ be a function from X to I. The triple $\Pi = (X, *, \mathit{Typ})$ is said to be a* ***pregeometry***[1] *over I if the restriction of Typ to every flag is injective. If the restriction of Typ to every maximal flag is a bijection, then Π is said to be an* ***incidence geometry*** *(or simply a "geometry" if the context is clear).*

If $\Pi = (X, *, \mathrm{Typ})$ is a pregeometry over I, then the elements of X are also referred to as elements of Π. The set I is called the **type set**. Its elements, as well as its subsets, are called **types**. The cardinality of I is called the **rank** of Π. No restrictions are placed on the type set in general, though for our purposes we always choose I as a subset of

$$I_n := \{0, 1, \ldots, n\},$$

[1]Some authors refer to a pregeometry as an "incidence system".

for some $n \in \mathbb{N} \cup \{0\}$. In this case, we refer to elements of type 0, 1, 2, and 3 as **vertices**, **edges**, **faces**, and **cells**, respectively. Elements of type $k \in \mathbb{N} \setminus \{0, 1, 2, 3\}$ are called **k-cells**. For our applications, the reader may (informally) consider the abstract function Typ as an assignment of dimension to the elements of Π.

By Zorn's Lemma, every flag of Π is contained in at least one maximal flag. A flag of type I is called a **chamber**. Chambers are necessarily maximal flags. The converse is also true for geometries. However, pregeometries may contain maximal flags that fail to be chambers. In fact, this gives an alternative characterization of a geometry; a geometry is a pregeometry whose maximal flags are all chambers.

Definition 2.1.3. *Suppose $\Pi = (X, *, Typ)$ is a pregeometry over I and $X' \subset X$. Let $*'$ and Typ' be the restrictions of $*$ and Typ to $X' \times X'$ and X', respectively. The triple $\Pi' = (X', *', Typ')$ is called the **subsystem** of Π induced by X'. If the subsystem is a geometry over $Typ(X')$, then we call Π' the **subgeometry** of Π induced by X'.*

A subsystem need not be a subgeometry; even if Π is a geometry to begin with. However, two important methods of constructing subsystems arise naturally when considering incidence geometries. These are known as "truncations" and "residues". For both of these operations, the resulting subsystems are necessarily subgeometries whenever Π is a geometry. The proof is straightforward; we refer the reader to Chapter 1 of [5] for the details.

Definition 2.1.4. *Let Π be a geometry over I and let $J \subseteq I$. The subgeometry of Π induced by $Typ^{-1}(J)$, written Π_J, is called the **J-truncation** of Π.*

Definition 2.1.5. *Let $\mathcal{F}$ be a flag of a geometry Π over I. Write $\mathcal{F}^*$ for the set of elements belonging to $\Pi \setminus \mathcal{F}$ that are incident to every element of $\mathcal{F}$. The **residue** of $\mathcal{F}$ in Π, written $Res(\mathcal{F}, \Pi)$ (or simply $Res(\mathcal{F})$ if the context is clear), is the subgeometry of Π induced by $\mathcal{F}^*$.*

Proposition 2.1.6 ([5], p. 45)**.** *Let Π be a geometry over I, let $\mathcal{F}$ be a flag of Π, and let $J \subseteq I$. Then*

$$Res(\mathcal{F}, \Pi)_J = Res\left(\mathcal{F}, \Pi_{J \cup Typ(\mathcal{F})}\right).$$

We have occasion to consider both the the truncation of a residue of a flag belonging to a geometry and the residue of a flag belonging to the truncation of a geometry. Proposition 2.1.6 guarantees that no ambiguity arises in writing $\mathrm{Res}_J(\mathcal{F}, \Pi)$ for both.

Definition 2.1.7. *Let $\Pi = (X, *, Typ)$ be a pregeometry over I. We call the undirected graph $(V, E) = (X, *)$ the **incidence graph** of Π. We say that Π is **connected** if its incidence graph is connected.*

Definition 2.1.8. *A geometry is said to be **thin** if every flag of corank one is contained in exactly two chambers.*

Definition 2.1.9. *Let $\Pi = (X, *, Typ)$ and $\Pi' = (X', *', Typ')$ be geometries over I and I', respectively. If $X \cap X' = \emptyset$ and $I \cap I' = \emptyset$, then we define a relation $*_\sqcup$ on $X \sqcup X'$ by*

$$*_\sqcup = * \cup *' \cup \big(X \times X'\big) \cup \big(X' \times X\big)\,,$$

and define the function $Typ_\sqcup : X \sqcup X' \to I \sqcup I'$ by

$$Typ_\sqcup(x) = \begin{cases} Typ(x) & \text{if } x \in X; \\ Typ'(x) & \text{if } x \in X'. \end{cases}$$

*The geometry $\Pi \sqcup \Pi' := (X \sqcup X', *_\sqcup, Typ_\sqcup)$ over $I \sqcup I'$ is called the **coproduct geometry** of Π and Π'.*

Definition 2.1.10. *Let $\Pi = (X, *, Typ)$ and $\Pi' = (X', *', Typ')$ be pregeometries over I. A **geometric homomorphism** from Π to Π' is a function from X to X' which preserves both incidence and type. A **geometric isomorphism** is a bijective geometric homomorphism whose inverse is a geometric homomorphism. We write $Aut(\Pi)$ for the group of automorphisms of Π.*

Definition 2.1.11. *Let Π be a pregeometry over I and let G be a group. A (group) homomorphism $\rho : G \to Aut(\Pi)$ is called a **geometric representation** of G in Π. We say the action of G on Π is **flag transitive** if G acts transitively on the set of flags of type J, for every $J \subseteq I$.*

2.2 Coxeter Groups

We briefly review the theory of Coxeter groups. Notation follows Humphreys [10].

Definition 2.2.1. *A **Coxeter system** is a pair (W, S) consisting of a group W and a set of generators S such that W admits a presentation of the form*

$$W = \left\langle S \mid (ss')^{m(s,s')} = 1 \text{ for all } s, s' \in S \right\rangle;$$

where $m(s, s') = m(s', s)$ is a natural number or ∞, and $m(s, s') = 1$ if and only if $s = s'$. The expression $(ss')^{\infty} = 1$ is interpreted as meaning no relation exists between s and s'.

We shall always assume that S is finite. A Coxeter system is uniquely determined by the set S and the data

$$\left\{m(s, s') \in \mathbb{N} \cup \{\infty\} \ : \ s, s' \in S\right\}.$$

This information is encoded in the **Coxeter graph** of (W, S); the undirected labeled graph having S as its vertex set and an edge joining s and s' whenever $m(s, s') \geq 3$. We label edges by $m(s, s')$ whenever this value is greater than 3. It is understood that for distinct vertices s and s', not joined by an edge, $m(s, s') = 2$.

When the generators and relations of a Coxeter system (W, S) are implied, we often refer to W simply as a **Coxeter group**. The cardinality of S is called the **rank** of W.

Coxeter groups are sometimes referred to as reflection groups. The following discussion justifies this terminology, the details of which may be found in [10], Chapter 5.

Let (W, S) be a Coxeter system and let V be a real vector space with basis $\{\alpha_s \ : \ s \in S\}$ in bijective correspondence with S. Define $B(\ , \ )$ as the symmetric bilinear form on V uniquely determined by

$$B(\alpha_s, \alpha_s') = \begin{cases} -\cos\left(\dfrac{\pi}{m(s, s')}\right) & \text{if } m(s, s') \neq \infty; \\ -1 & \text{if } m(s, s') = \infty. \end{cases}$$

Each generator $s \in S$ determines a linear involution $\sigma_s \in \mathrm{GL}(V)$, given by:

$$\sigma_s(v) = v - 2\, B(\alpha_s, v)\alpha_s.$$

Since σ_s maps α_s to $-\alpha_s$ and fixes the hyperplane

$$H_\alpha = \{v \in V \ : \ (v, \alpha_s) = 0\}$$

pointwise, σ_s is a reflection. We have the following.

Theorem 2.2.2 ([10], p. 110)**.** *There is a unique homomorphism $\sigma\colon W \to GL(V)$ sending s to σ_s. This homomorphism is a faithful representation of W which preserves the bilinear form $B(\ ,\ )$.*

The homomorphism σ is called the **reflection representation** of W. We remark that the reflection representation is not always the most obvious interpretation of W as a reflection group. Often W has a more natural action on an affine or hyperbolic space. See (for example) [10], Chapter 6.

Let (W, S) be a Coxeter system. An expression for $w \in W$ as a product of k elements from S, where k is as small as possible, is called a **reduced expression** of w. The **length** of $w \in W$, written $l(w)$, is the number of elements of S (counting repetitions) that occur in a (any) reduced expression of w. There is a unique homomorphism $\epsilon : W \to \{\pm 1\}$ given by $w \mapsto (-1)^{l(w)}$. See [10], p. 107.

Definition 2.2.3. *The kernel of the map $\epsilon : W \to \{\pm 1\}$, denoted W^+, is called the **even subgroup** of W.*

Theorem 2.2.4 ([4], Chap. IV, Sec. 1)**.** *Let (W, S) be a Coxeter system of rank $n+1$ with generators $S = \{s_0, s_1, \dots, s_n\}$. If $r_i = s_0 s_i$ for each $1 \le i \le n$, then $R = \{r_1, r_2, \dots, r_n\}$ is a set of generators for W^+, with the following presentation:*

$$W^+ = \left\langle R \mid r_i^{m_{0,i}} = \left(r_i^{-1} r_j\right)^{m_{i,j}} = 1 \text{ for } 1 \le i < j \le n \right\rangle .$$

Definition 2.2.5. *Let (W_1, S_1) and (W_2, S_2) be Coxeter systems with respective Coxeter graphs γ_1 and γ_2. The **Coxeter system product** of (W_1, S_1) and (W_2, S_2), written $(W_1 \times W_2, S_1 \sqcup_\times S_2)$, is the Coxeter system with generators*

$$S_1 \sqcup_\times S_2 := \{(s, 1) \ : \ s \in S_1\} \cup \{(1, t) \ : \ t \in S_2\},$$

and Coxeter graph $\gamma_1 \cup \gamma_2$.

Definition 2.2.6. *Let (W, S) be a Coxeter system. A subgroup of W is said to be **special** if it is generated by a subset of S. Given $T \subseteq S$, we define:*

$$W_T := \langle s \mid s \in T \rangle ,$$
$$W^T := \langle s \mid s \in S \backslash T \rangle .$$

*A subgroup that is conjugate (in W) to a special subgroup is called a **parabolic subgroup**.*

If (W,S) is a Coxeter system then, at the two extremes, we have $W_\emptyset = W^S = \{1\}$, and $W_S = W^\emptyset = W$. The following theorems are well known.

Theorem 2.2.7 ([10], p. 113)**.** *For every subset $T \subseteq S$, the pair (W_T, T) is a Coxeter system. The Coxeter graph of (W_T, T) is exactly the restriction of the Coxeter graph of (W,S) to the vertex set T.*

Theorem 2.2.8 ([10], p. 113)**.** *The map sending T to W_T defines an inclusion-preserving bijection from the power set of S onto the set of special subgroups of W.*

Theorem 2.2.9 ([9], p. 45)**.** *Let $\{T_i\}_{i\in I}$ be a family of subsets of S. If $T = \bigcap_{i\in I} T_i$, then*

$$W_T = \bigcap_{i\in I} W_{T_i}.$$

Corollary 2.2.10. *Let $\{T_i\}_{i\in I}$ be a family of subsets of S. If $T = \bigcup_{i\in I} T_i$, then*

$$W^T = \bigcap_{i\in I} W^{T_i}.$$

Proof. The result follows immediately from the Theorem and the observation

$$S\backslash T = \bigcap_{i\in I} S\backslash T_i.$$

□

The next results will be of technical importance in the following section. For ease of notation we write W^s in place of $W^{\{s\}}$.

Lemma 2.2.11 ([4], p. 37)**.** *If (W,S) is a Coxeter system and X, Y, $Z \subseteq S$, then*

$$(W_Y \cap W_Z)\, W_X = (W_Y W_X) \cap (W_Z W_X)\,.$$

Proposition 2.2.12. *Let (W,S) be a Coxeter system of finite rank. For any proper non-empty subset $T \subseteq S$, and for any $s \in S\backslash T$,*

$$W^T W^s = \bigcap_{t\in T} \left(W^t W^s\right).$$

Proof. (By induction). Let n be the (finite) cardinality of T. If $n = 1$, then there is nothing to prove. For the inductive step, let r be any element of T. Then

$$\begin{aligned} W^T W^s &= \left(W^{T\setminus\{r\}} \cap W^r\right) W^s \\ &= W^{T\setminus\{r\}} W^s \cap W^r W^s \\ &= \bigcap_{t\in T} \left(W^t W^s\right); \end{aligned}$$

where the first equality follows by Corollary 2.2.10, the second by Lemma 2.2.11, and the third by the inductive hypothesis. □

We end this section by introducing some non-standard notation. For our purposes it will prove convenient to label the generators of our Coxeter systems by integers.

Definition 2.2.13. *A Coxeter system (W, S) together with a bijection from S to $I \subset \mathbb{N} \cup \{0\}$ (expressed via subscripts), is said to be a **Coxeter system indexed over** I. We refer to W as an **indexed Coxeter group**.*

Suppose (W_1, S_1) and (W_2, S_2) are Coxeter systems indexed via bijections $i_1 : S_1 \to I_1$ and $i_2 : S_2 \to I_2$, respectively. If $I_1 \cap I_2 = \emptyset$, then the product Coxeter system $(W_1 \times W_2, S_1 \sqcup_\times S_2)$ inherits an natural indexing over $I_1 \sqcup I_2$ via the bijection $i : S_1 \sqcup_\times S_2 \to I_1 \sqcup I_2$,

$$i(s) = \begin{cases} i_1(s) & \text{if } \pi_1(s) \in S_1; \\ i_2(s) & \text{if } \pi_2(s) \in S_2, \end{cases}$$

where $\pi_1 : W_1 \times W_2 \to W_1$ and $\pi_2 : W_1 \times W_2 \to W_2$ are the natural projections. We refer to this construction as the **indexed Coxeter system product** of (W_1, S_1) and (W_2, S_2).

Let $n \in \mathbb{N}$ and let (W, S) be a Coxeter system indexed over I_n. Following the notation introduced in Definition 2.2.6, for any $J \subseteq I_n$ we suppress the generators and write W_J for the special subgroup

$$W_J := \langle s_j \mid j \in J \rangle .$$

The analogous convention is followed for superscripts. In the case that J is a singleton $\{j\}$, we suppress the set notation and write

$$W_j := W_{\{j\}} = W_{\{s_j\}}.$$

Similarly for superscripts.

2.3 The Coset Incidence System

In this section we describe the "Coxeter geometry" associated to a Coxeter group. We begin with an overview of the general theory of coset incidence systems (of which the Coxeter geometry is a specific example). The majority of the results may be found in Chapter 11 of [5]. Throughout this section, I denotes a finite, nonempty indexing set.

Definition 2.3.1. *Let G be a group with subgroups $(G_i)_{i\in I}$. For $i \in I$, let $X_i := G/G_i$ be the set of elements of type i, and let X be the union of the X_i. Write $Typ : X \to I$ for the resulting type map. Define $*$ as the incidence relation on X given by*

$$* = \{(a, b) \in X \times X \ : \ a \cap b \neq \emptyset\}.$$

The pregeometry $(X, *, Typ)$, written $\Pi(G, (G_i)_{i\in I})$, is called the ***coset incidence system*** *of G over $(G_i)_{i\in I}$.*

Since $1 \in \cap_{i\in I} G_i$, any subset of $\{G_i \ : \ i \in I\}$ is a flag of $\Pi\left(G, (G_i)_{i\in I}\right)$.

Definition 2.3.2. *Let Π be the coset incidence system of G over $(G_i)_{i\in I}$. For $J \subseteq I$, we define*

$$\mathcal{F}_J := \{G_j \ : \ j \in J\}$$

and call $\mathcal{F}_J$ the ***standard flag*** *of Π of type J.*

Definition 2.3.3. *Let G be a group with subgroups $(G_i)_{i\in I}$. For $J \subseteq I$, we define*

$$G_J := \bigcap_{j\in J} G_j$$

and call G_J the ***standard parabolic subgroup*** *of G of type J.*

The group G naturally acts on the coset incidence system $\Pi\left(G, (G_i)_{i\in I}\right)$ via left multiplication.

Definition 2.3.4. *Let $\Pi\left(G, (G_i)_{i\in I}\right)$ be a coset incidence system. The homomorphism $\rho\colon G \to Aut(\Pi)$ defined by*

$$[\rho(g)]\,(hG_i) = ghG_i$$

is called the ***canonical geometric representation*** *of* G *in* Π*.*

The notation in Definitions 2.3.2 and 2.3.3 reflects that G_J is the stabilizer in G of the standard flag $\mathcal{F}_J$ with respect to the canonical geometric representation.

Let $\Pi\left(G, (G_i)_{i\in I}\right)$ be a coset incidence system. Given $J \subseteq I$, define the map

$$\alpha_J\colon \Pi\left(G_J, \left\{G_{J\cup\{i\}} \ : \ i \in I\backslash J\right\}\right) \to \mathrm{Res}\left(\mathcal{F}_J, \Pi\right),$$

determined by $gG_{J\cup\{i\}} \mapsto gG_i$ for every $i \in I\backslash J$ and $g \in G_J$. The map α_J is well defined since $G_{J\cup\{i\}}$ is a subset of G_i. It preserves incidence because for all $i \in I\backslash J$ and for all $j \in J$, one finds $g \in gG_i \cap G_j$ for any $g \in G_J$. Since α is clearly type preserving, it follows that α is a homomorphism of geometries. In fact, it is not difficult to prove the following.

Proposition 2.3.5 ([5], Theorem 11.5.10)**.** *For any* $J \subseteq I$*, the map* α_J *is an injective geometric homomorphism from* $\Pi\left(G_J, \left\{G_{J\cup\{i\}} \ : \ i \in I\backslash J\right\}\right)$ *to Res*$\left(\mathcal{F}_J, \Pi\right)$*.*

In “nice” coset geometries (those which admit a flag transitive group action), α_J is in fact an isomorphism.

Theorem 2.3.6 ([5], Theorem 11.5.10)**.** *Let* $\Pi\left(G, (G_i)_{i\in I}\right)$ *be the coset incidence system of* G *over* $(G_i)_{i\in I}$*. The following are equivalent.*

(i) The group G *acts flag transitively on* Π*.*

(ii) For each $J \subseteq I$*, the map* α_J *is an isomorphism.*

(iii) For each $J \subseteq I$ *and for each* $i \in I\backslash J$*,*

$$G_J G_i = \bigcap_{j\in J} \left(G_j G_i\right).$$

Furthermore, if one (hence all) of the above hold, then Π *is a geometry.*

Corollary 2.3.7. *Suppose* $\Pi\left(G, (G_i)_{i\in I}\right)$ *is a coset incidence system on which* G *acts flag transitively. Then* Π *is thin if and only if*

$$\left[G_{I\backslash\{j\}} : G_I\right] = 2$$

for each $j \in I$*.*

Proof. By flag transitivity, Π is thin if and only if the residue of $\mathcal{F}_{I\backslash\{j\}}$ has cardinality two for each $j \in I$. By the theorem, any such residue is isomorphic to the geometry $\Pi(G_{I\backslash\{j\}}, G_I)$ and hence has precisely $\left[G_{I\backslash\{j\}} : G_I\right]$ elements. $\square$

We can also characterize connectedness in a coset geometry in terms of the properties of the underlying group. We have the following.

Proposition 2.3.8 ([5], Theorem 11.5.9)**.** *Suppose $\Pi\left(G, (G_i)_{i\in I}\right)$ is a coset incidence system on which G acts flag transitively. Then Π is connected if and only if $|I| \geq 2$ and $G = \langle G_i \mid i \in I\rangle$.*

We end this section by applying the construction of a Coset geometry to the specific case in which G is a Coxeter group. For the remainder of this section, n denotes a non-negative integer.

Definition 2.3.9. *Let (W, S) be a Coxeter system indexed over I_n. The* ***Coxeter geometry*** *of (W, S), written $\Pi(W, S)$, is the coset geometry of W over $\left(W^i\right)_{i\in I_n}$.*

Proposition 2.3.10. *If (W, S) is a Coxeter system indexed over I_n with rank $n+1 \geq 2$, then $\Pi(W, S)$ is a thin and connected geometry over I_n on which W acts flag transitively. If $J \subseteq I_n$, then the map*

$$\alpha_J : \Pi(W^J, S^J) \to Res(\mathcal{F}_J, \Pi),$$

given by $gG_{J\cup\{i\}} \mapsto gG_i$ for each $i \in I\backslash J$, is an isomorphism of geometries.

Proof. The statement of Proposition 2.2.12 is exactly condition (iii) of Theorem 2.3.6. Therefore, by the same Theorem, $\Pi(W, S)$ is a geometry and W acts flag transitively on Π (via left multiplication) as a group of automorphisms. Theorem 2.3.6 then implies the last statement of the Proposition.

Since

$$W = \left\langle W^i \ : \ i \in I_n \right\rangle,$$

hence Proposition 2.3.8 implies $\Pi(W, S)$ is connected. Finally, the geometry $\Pi(W, S)$ is thin by Corollary 2.3.7, coupled with the observation that for any $j \in I_n$,

$$W^{I_n\backslash\{j\}}/W^{I_n} = W_j/\{1\} = \{1, s_j\}.$$

$\square$

Proposition 2.3.11. *Let (W, S) be a Coxeter system indexed over I_n. The set of chambers of $\Pi(W, S)$ is a principal homogeneous space for the action of W.*

Proof. Let $\mathcal{C}$ denote the set of chambers of $\Pi(W, S)$. By Proposition 2.3.10, W acts transitively on $\mathcal{C}$ by left multiplication. Furthermore, the stabilizer of the fundamental chamber

$$C = \{W^0, W^1, \ldots, W^n\}$$

is given by

$$\{w \in W \ : \ wW^i = W^i \text{ for all } i \in I_n\} = \bigcap_{i \in I_n} W^i = \{1\};$$

where the final equality follows from Corollary 2.2.10. This establishes a bijection from W to $\mathcal{C}$ given by $w \mapsto wC$ which completes the proof. $\square$

Proposition 2.3.12. *Let (W_1, S_1) and (W_2, S_2) be indexed Coxeter systems over disjoint, nonempty sets I and J, respectively. Then*

$$\Pi(W_1 \times W_2, S_1 \sqcup_\times S_2) \simeq \Pi(W_1, S_1) \sqcup \Pi(W_2, S_2).$$

Proof. Define $\phi : \Pi(W_1 \times W_2, S_1 \sqcup_\times S_2) \to \Pi(W_1, S_1) \sqcup \Pi(W_2, S_2)$ by

$$(w_1, w_2)\,(W_1 \times W_2)^k \mapsto \begin{cases} w_1 W_1^k & \text{if } k \in I; \\ w_2 W_2^k & \text{if } k \in J. \end{cases}$$

Theorem 2.2.7 implies that $(W_1 \times W_2)^k = W_1^k \times W_2$ and $(W_1 \times W_2)^k = W_1 \times W_2^k$ whenever $k \in I$ and $k \in J$, respectively. This shows ϕ is well defined. Let $\mu_1 : \Pi(W_1, S_1) \to \Pi(W_1 \times W_2, S_1 \sqcup S_2)$ and $\mu_2 : \Pi(W_2, S_2) \to \Pi(W_1 \times W_2, S_1 \sqcup S_2)$ be the natural injections given by

$$wW_1^k \mapsto (w, 1)\,(W_1 \times W_2)^k$$

for any $k \in I$, and

$$wW_2^k \mapsto (1, w)\,(W_1 \times W_2)^k$$

for any $k \in J$, respectively. If $\mu = \mu_1 \sqcup \mu_2$, then μ is inverse to ϕ. By construction, both ϕ and μ preserve type. We claim both maps preserve incidence.

Let

$$x_i = (x_1, x_2)\,(W_1 \times W_2)^i\,,$$
$$y_j = (y_1, y_2)\,(W_1 \times W_2)^j$$

be elements of $\Pi(W_1 \times W_2, S_1 \sqcup S_2)$. If either $i, j \in I$ or $i, j \in J$, then, by definition, $x_i * y_j$ if and only if $\phi(x_i) *_\sqcup \phi(y_j)$. Assume therefore that $i \in I$ and $j \in J$. Then

$$(x_1, y_2) \equiv (x_1, 1) \equiv (x_1, x_2) \mod (W_1 \times W_2)^i\,,$$

and

$$(x_1, y_2) \equiv (1, y_2) \equiv (y_1, y_2) \bmod (W_1 \times W_2)^j\,.$$

Consequently, $(x_1, y_2) \in x_i \cap y_j$, so that $x_i * y_j$. On the other hand, since $i \in I$ and $j \in J$, we find $w_i W^i *_\sqcup w_j W^j$ by definition. This shows that $x_i * y_j$ if and only if $\phi(x_i) *_\sqcup \phi(y_j)$. $\square$

Part I

The Eisenstein Case

Chapter 3

The Eisenstein Topograph

Throughout this chapter $\mathcal{O}$ denotes the ring of **Eisenstein integers** $\mathbb{Z}[\omega]$, where $\omega = e^{2\pi i/3}$. We write $-$ for complex conjugation.

3.1 Units

The unit group $\mathcal{O}^\times = \{\pm 1, \pm\omega, \pm\omega^2\}$ is a cyclic group of order six generated by $-\omega$. The embedding of the unit group in $\mathbb{C}$ has the natural geometry of a hexagon. See Figure 3.1. Two units are said to be **adjacent** if they are incident to a common edge of the hexagon. Two units are adjacent if and only their difference is a unit.

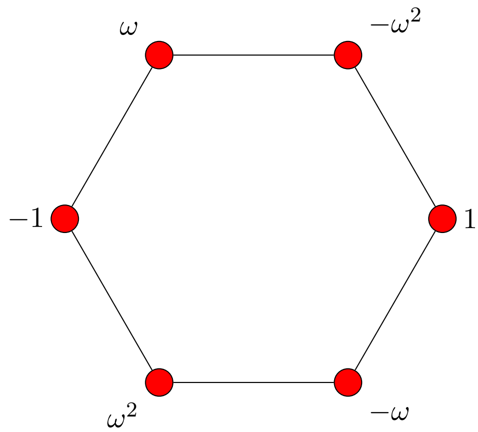


Figure 3.1: Adjacent units in $\mathcal{O}^\times$.

Since the set of squares $(\mathcal{O}^\times)^2$ forms an index two subgroup of $\mathcal{O}^\times$, the map $\{\pm 1\} \to \mathcal{O}^\times / (\mathcal{O}^\times)^2$ is an isomorphism.

3.2 Lax Vectors and Generalized Bases

Define

$$\mathcal{O}^2_{\text{prim}} := \left\{(x, y) \in \mathcal{O}^2 \ : \ \gcd(x, y) = 1\right\}.$$

If $u, v \in \mathcal{O}^2_{\text{prim}}$, then we write $u \sim v$ if there exists $\xi \in \mathcal{O}^\times$ such that $u = \xi \cdot v$.

Definition 3.2.1. *A **lax vector** is an element of the projective space $\mathbb{P}^1(\mathcal{O}) := \mathcal{O}^2_{prim}/\sim$.*

We write (x, y) for an element of $\mathcal{O}^2_{\text{prim}}$ and write $[x : y]$ for its equivalence class in $\mathbb{P}^1(\mathcal{O})$. Let **Tri** and **Tet** be the natural graph structure determined by the vertices and edges of a triangle and tetrahedron, respectively.

Definition 3.2.2. *Consider the undirected graph G whose vertex set is $\mathcal{O}^2$ and whose edge set is the set of unordered bases of the $\mathcal{O}$-module $\mathcal{O}^2$. A subset $B \subset \mathcal{O}^2$ is said to be a **tribasis** of $\mathcal{O}^2$, and is said to be an **tetrabasis** of $\mathcal{O}^2$, if the subgraph of G induced by B is isomorphic to Tri and Tet, respectively.*

*A subset $L \subset \mathbb{P}^1(\mathcal{O})$ is said to be a **lax basis** of $\mathcal{O}^2$, a **lax tribasis** of $\mathcal{O}^2$, and a **lax tetrabasis** of $\mathcal{O}^2$, if a (any) set of representatives of L in $\mathcal{O}^2$ is a basis of $\mathcal{O}^2$, a tribasis of $\mathcal{O}^2$, and a tetrabasis of $\mathcal{O}^2$, respectively.*

3.3 The Eisenstein Topograph

Definition 3.3.1. *Let*

$$\begin{aligned}
\mathcal{E}_0 &= \left\{\textit{Lax tetrabases of } \mathcal{O}^2\right\}, \\
\mathcal{E}_1 &= \left\{\textit{Lax tribases of } \mathcal{O}^2\right\}, \\
\mathcal{E}_2 &= \left\{\textit{Lax bases of } \mathcal{O}^2\right\}, \\
\mathcal{E}_3 &= \{\textit{Lax vectors}\} = \mathbb{P}^1(\mathcal{O}),
\end{aligned}$$

and let

$$\mathcal{E} = \bigsqcup_{i=0}^{3} \mathcal{E}_i.$$

Define an incidence relation $*$ on $\mathcal{E}$ by symmetric inclusion and define

$$\mathit{Typ} : \mathcal{E} \to I_3 = \{0, 1, 2, 3\}$$

by $\mathit{Typ}\,(\mathcal{E}_i) = i$, for each $i \in I_3$. The ***Eisenstein topograph*** *is the rank four incidence geometry $(\mathcal{E}, *, \mathit{Typ})$ over I_3.*

By a slight abuse of notation, we write $\mathcal{E}$ to denote both the set of elements of the Eisenstein topograph and the geometry $(\mathcal{E}, *, \mathrm{Typ})$ itself. The **dual** geometry to $\mathcal{E}$, written $\mathcal{E}^*$, is the rank four incidence geometry $(\mathcal{E}, *, \mathrm{Typ}')$, where $\mathrm{Typ}' = 3 - \mathrm{Typ}$. Note that the subscripts used in Definition 3.3.1 are labeled according to the function Typ rather than Typ'. Explicitly, we have the following correspondences:

$$\text{Vertices of } \mathcal{E} \leftrightarrow \mathcal{E}_0 \leftrightarrow \text{Lax tetrabases of } \mathcal{O}^2 \leftrightarrow \text{Cells of } \mathcal{E}^*,$$

$$\text{Edges of } \mathcal{E} \leftrightarrow \mathcal{E}_1 \leftrightarrow \text{Lax tribases of } \mathcal{O}^2 \leftrightarrow \text{Faces of } \mathcal{E}^*,$$

$$\text{Faces of } \mathcal{E} \leftrightarrow \mathcal{E}_2 \leftrightarrow \text{Lax bases of } \mathcal{O}^2 \leftrightarrow \text{Edges of } \mathcal{E}^*,$$

$$\text{Cells of } \mathcal{E} \leftrightarrow \mathcal{E}_3 \leftrightarrow \text{Lax vectors} \leftrightarrow \text{Vertices of } \mathcal{E}^*.$$

Definition 3.3.2. *For $0 \leq i \leq 3$, define pairwise incident elements $\Omega_i \in \mathcal{E}_i$:*

$$\begin{aligned}
\Omega_0 &= \{[1:0], [0:1], [1:1], [-w:1]\}\,, \\
\Omega_1 &= \{[1:0], [0:1], [1:1]\}\,, \\
\Omega_2 &= \{[1:0], [0:1]\}\,, \\
\Omega_3 &= \{[1:0]\}\,.
\end{aligned}$$

The set

$$\mathcal{C}_0 := \{\Omega_0, \Omega_1, \Omega_2, \Omega_3\}$$

is called the ***fundamental chamber*** *of $\mathcal{E}$. See Figure 3.2.*

We investigate the incidence relations of the Eisenstein topograph near the fundamental chamber. These results will be used in the subsequent section to identify $\mathcal{E}$ as a Coxeter geometry.

Proposition 3.3.3. *The lax basis Ω_2 is incident to (contained in) exactly six lax tribases, namely*

$$\{[1:0], [0:1], [\xi:1]\}\,,$$

for any $\xi \in \mathcal{O}^\times$.

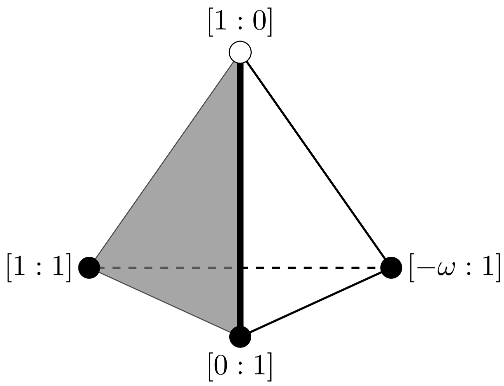


Figure 3.2: The dual of the fundamental chamber. The tetrahedron represents the lax tetrabasis Ω_0. The distinguished face, edge, and vertex represent the lax tribasis Ω_1, the lax basis Ω_2, and the lax vector Ω_3, respectively.

Proof. The set

$$\{[1,0],[0:1],[x:y]\}$$

forms a lax tribasis if and only if

$$\det\begin{pmatrix}1 & 0\\ x & y\end{pmatrix}\in\mathcal{O}^\times,\ \text{and}\ \det\begin{pmatrix}0 & 1\\ x & y\end{pmatrix}\in\mathcal{O}^\times$$

(independent of the choice of representative (x,y)). This is equivalent to the conditions $x\in\mathcal{O}^\times$ and $y\in\mathcal{O}^\times$. The observation $[x:y]=[y^{-1}x:1]$ completes the proof. □

Proposition 3.3.4. *The lax basis Ω_2 is incident to exactly six lax tetrabases, namely:*

$$\Big\{[1:0],[0:1],[(-\omega)^k:1],[(-\omega)^{k+1}:1]\Big\}$$

for any $k\in\{0,1,\dots,5\}$.

Proof. Each of the six sets above define a lax tetrabasis containing Ω_2. We prove this list is exhaustive. Figure 3.3 represents a lax tetrabasis containing Ω_2. From Proposition 3.3.3, we can take the adjacent lax vectors of the form $[s:1]$, $[t:1]$, with $s,t\in\mathcal{O}^\times$ and $s\neq t$. Since $\{[s:1],[t:1]\}$ is a lax basis,

$$\det\begin{pmatrix}s & 1\\ t & 1\end{pmatrix}=s-t\in\mathcal{O}^\times.$$

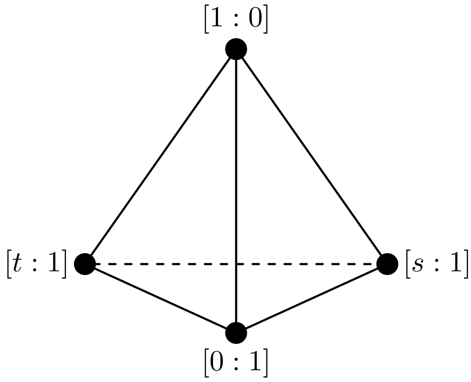


Figure 3.3: A lax tetrabasis incident to the lax basis Ω_2.

The only units whose difference is a unit are those which are adjacent in the hexagon of Figure 3.1. These have the form

$$\{s, t\} = \left\{(-\omega)^k, (-\omega)^{k+1}\right\}$$

for some $0 \le k \le 5$. □

Corollary 3.3.5. *The lax tribasis Ω_1 is incident to exactly two lax tetrabases, namely:*

$$\{[1:0],[0:1],[1:1],[-\omega:1]\},$$
$$\left\{[1:0],[0:1],[1:1],[-\omega^2:1]\right\}.$$

Proof. By the previous Proposition, the only tetrabases incident to

$$\Omega_1 = \left\{[1:0],[0:1],[(-\omega)^0:1]\right\}$$

are

$$\left\{[1:0],[0:1],[-\omega:1],[(-\omega)^1:1]\right\},$$
$$\left\{[1:0],[0:1],[-\omega:1],[(-\omega)^5:1]\right\}.$$

□

3.4 The Coxeter Geometry

The center of the general linear group $\mathrm{GL}_2(\mathcal{O})$ is naturally identified with $\mathcal{O}^\times$. The **projective general linear group** $\mathrm{PGL}_2(\mathcal{O})$ is defined

$$\mathrm{PGL}_2(\mathcal{O}) := \mathrm{GL}_2(\mathcal{O})/Z\left(\mathrm{GL}_2(\mathcal{O})\right) = \mathrm{GL}_2(\mathcal{O})/\mathcal{O}^\times.$$

We write

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for a matrix in $\mathrm{GL}_2(\mathcal{O})$ and

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

for its image in $\mathrm{PGL}_2(\mathcal{O})$. The determinant map $\det : \mathrm{GL}_2(\mathcal{O}) \to \mathcal{O}^\times$ descends to

$$\mathrm{pdet} : \mathrm{PGL}_2(\mathcal{O}) \to \mathcal{O}^\times/(\mathcal{O}^\times)^2 \equiv \{\pm 1\}.$$

The **projective semilinear group** is defined[1]

$$\mathrm{P\Gamma L}_2(\mathcal{O}) = \mathrm{PGL}_2(\mathcal{O}) \rtimes \{1, \sigma\},$$

where σ acts on $\mathrm{PGL}_2(\mathcal{O})$ by complex conjugation on the matrix entries.

The group $\mathrm{PGL}_2(\mathcal{O})$ acts transitively on $\mathbb{P}^1(\mathcal{O})$. There is a natural left action of $\mathrm{P\Gamma L}_2(\mathcal{O})$ on $\mathbb{P}^1(\mathcal{O})$ uniquely determined by

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, 1\right) \cdot [x : y] = [ax + by : cx + dy],$$
$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \sigma\right) \cdot [x : y] = [\overline{x} : \overline{y}].$$

The groups $\mathrm{PGL}_2(\mathcal{O})$ and $\mathrm{P\Gamma L}_2(\mathcal{O})$ act on the geometry $\mathcal{E}$ by transport of structure. Throughout this chapter, we use the notation

$$\Gamma := \mathrm{PGL}_2(\mathcal{O}),$$
$$\tilde{\Gamma} := \mathrm{P\Gamma L}_2(\mathcal{O}).$$

[1]In general, if $\mathcal{O}$ is the maximal order of a number field K, then the projective semilinear group $\mathrm{P\Gamma L}_n(\mathcal{O})$ is the semidirect product $\mathrm{PGL}_n(\mathcal{O}) \rtimes \mathrm{Gal}(K/\mathbb{Q})$.

Our first task is to show $\tilde{\Gamma}$ admits a Coxeter group presentation. The main result, Proposition 3.4.2 below, was published by Johnson and Weiss in [11]. We present a different proof and use an alternative generator set in the presentation.

Lemma 3.4.1. *The elements*

$$\rho_0 := \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \sigma \right), \qquad \rho_1 := \left(\begin{bmatrix} -\omega & 0 \\ 0 & 1 \end{bmatrix}, \sigma \right),$$
$$\rho_2 := \left(\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}, \sigma \right), \qquad \rho_3 := \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma \right),$$

generate $\tilde{\Gamma}$.

Proof. Consider the four matrices

$$E_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 1 & -\omega \\ 0 & 1 \end{pmatrix},$$
$$E_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad E_4 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since $\mathcal{O}$ is a Euclidean domain, $\mathrm{SL}_2(\mathcal{O})$ is generated by

$$\left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \ : \ x \in \mathcal{O} \right\} \cup \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}.$$

It follows that

$$\langle E_1, E_2, (E_3E_4) \rangle = \mathrm{SL}_2(\mathcal{O}).$$

Hence

$$\langle E_1, E_2, E_3, E_4 \rangle = \{ A \in \mathrm{GL}_2(\mathcal{O}) \ : \ \det(A) = \pm 1 \}.$$

Let $\tilde{E}_i$ denote the image of E_i in $\Gamma = \mathrm{PGL}_2(\mathcal{O})$. Since $\mathcal{O}^\times/(\mathcal{O}^\times)^2 = \{\pm 1\}$, it follows that

$$\left\langle \tilde{E}_1, \tilde{E}_2, \tilde{E}_3, \tilde{E}_4 \right\rangle = \Gamma.$$

Therefore,

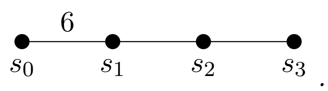
$$\tilde{\Gamma} = \mathrm{PGL}_2(\mathcal{O}) \rtimes \{1, \sigma\} = \left\langle (1, \sigma), (\tilde{E}_1, 1), (\tilde{E}_2, 1), (\tilde{E}_3, 1), (\tilde{E}_4, 1) \right\rangle.$$

The expressions:

$$\begin{aligned} \left(\tilde{E}_1, 1\right) &= \rho_2\rho_1(\rho_0\rho_1)^2, & \left(\tilde{E}_2, 1\right) &= \rho_1\rho_2(\rho_1\rho_0)^2, \\ \left(\tilde{E}_3, 1\right) &= \rho_0\rho_3, & \left(\tilde{E}_4, 1\right) &= (\rho_0\rho_1)^3, \end{aligned} \tag{3.4.1}$$

and the observation $(1, \sigma) = \rho_0$ complete the proof. $\square$

Proposition 3.4.2. *Let (W, S) be the indexed Coxeter system with diagram*



The map $\psi : S \to \tilde{\Gamma}$ given by

$$s_0 \mapsto \rho_0, \qquad s_1 \mapsto \rho_1, \qquad s_2 \mapsto \rho_2, \qquad s_3 \mapsto \rho_3;$$

extends to a group isomorphism $\Psi : W \xrightarrow{\sim} \tilde{\Gamma}$. Moreover, $\Gamma = PGL_2(\mathcal{O}) \simeq W^+$.

Proof. By the universal property of free groups and the relations

$$(\rho_i\rho_j)^{m(s_i,s_j)} = 1,$$

there exists a unique homomorphism $\Psi : W \to \tilde{\Gamma}$ such that $\Psi(s_i) = \psi(s_i)$ for each $0 \leq i \leq 3$. Lemma 3.4.1 shows that Ψ is surjective. We prove injectivity below.

Let V be the four dimensional real vector space of 2×2 complex Hermitian matrices. Define a quadratic form Q of signature $(1, 3)$ on V by

$$Q(M) = -\det(M).$$

Let $\phi : \tilde{\Gamma} \to \mathrm{GL}(V)$ be the representation uniquely determined by

$$[\phi(A, 1)]\,(M) = AMA^{\dagger},$$
$$[\phi(1, \sigma)]\,(M) = \overline{M};$$

where $A^{\dagger}$ represents the conjugate transpose of A and $\overline{M} = (M^{\dagger})^{\mathrm{t}}$ represents complex conjugation. For each $(A, g) \in \tilde{\Gamma}$ and each $M \in V$,

$$Q\left([\phi(A, g)]\,(M)\right) = \left(\det(A)\overline{\det(A)}\right) Q(M) = Q(M).$$

It follows that ϕ induces a Lie group homomorphism

$$\Phi : \tilde{\Gamma} \to O(V, Q).$$

Furthermore, each involution $\Phi(\rho_i)$ is a reflection in (V, Q) with corresponding root:

$$\alpha_0 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \qquad \alpha_1 = \begin{pmatrix} 0 & \omega \\ \overline{\omega} & 0 \end{pmatrix},$$
$$\alpha_2 = \begin{pmatrix} -1 & -1 \\ -1 & 0 \end{pmatrix} \qquad \alpha_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

If B is the bilinear form on V associated to Q, then for each $0 \le i, j \le 3$,

$$B(\alpha_i, \alpha_j) = -\cos\left(\frac{\pi}{m(s_i, s_j)}\right).$$

Consequently, by Theorem 2.2.2, there exists a monomorphism $\mu : W \to O(V, Q)$, determined by $s_i \mapsto \sigma_{s_i} := \Phi(\rho_i)$, such that the following diagram commutes:

$$\begin{tikzcd} W \arrow[r, hook, "\mu"] \arrow[d, "\Psi"'] & O(V,Q) \\ \tilde{\Gamma} \arrow[ur, "\Phi"'] & \end{tikzcd}.$$

It follows that Ψ is injective.

Finally, the isomorphism Ψ together with the expressions (3.4.1) for the generators of $\Gamma = \mathrm{PGL}_2(\mathcal{O})$ show that Γ embeds as a subgroup of W^+. Since $W \simeq \tilde{\Gamma}$ and $[W : W^+] = [\tilde{\Gamma} : \Gamma] = 2$, we find $\Gamma \simeq W^+$.

$\square$

Having established $\tilde{\Gamma}$ as the isomorphic image of the indexed Coxeter group (W, S), we adopt the notation introduced in Section 2.2. In particular, for $J \subseteq I_3 = \{0, 1, 2, 3\}$, we define

$$\tilde{\Gamma}_J := \langle \rho_j \ : \ j \in J \rangle,$$
$$\tilde{\Gamma}^J := \langle \rho_j \ : \ j \in I_3 \backslash J \rangle.$$

We follow our earlier convention of omitting the set notation when J has cardinality one.

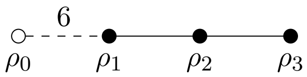
The isomorphism $W \simeq \tilde{\Gamma}$ in Proposition 3.4.2 is not canonical. The generators ρ_0, ρ_1, ρ_2, and ρ_3 were selected to reflect our choice of fundamental chamber $\mathcal{C}_0$. Specifically, we have the following.

Lemma 3.4.3. *For each $i \in I_3$,*

$$\tilde{\Gamma}^i = Stab_{\tilde{\Gamma}}(\Omega_i).$$

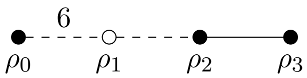
Proof. Let $i \in I_3$. If $j \in I_3$ and $j \neq i$, then ρ_j acts trivially on Ω_i. Hence $\tilde{\Gamma}^i$ is a subgroup of the stabilizer of Ω_i in $\tilde{\Gamma}$. We claim this embedding is surjective.

(Case $i = 0$.)



An element of $\mathrm{Stab}_{\tilde{\Gamma}}(\Omega_0)$ is determined by its action on the tetrahedron of Figure 3.2. Consequently $\#\left(\mathrm{Stab}_{\tilde{\Gamma}}(\Omega_0)\right)$ is bounded above by 24: the order of the automorphism group of the tetrahedron. On the other hand, Theorem 2.2.7 implies $\tilde{\Gamma}^0$ is isomorphic to the Coxeter group A_3, given by the restricted graph above. It is well known this group has order 24. Hence $\tilde{\Gamma}^0 = \mathrm{Stab}_{\tilde{\Gamma}}(\Omega_0)$.

(Case $i = 1$.)



The stabilizer of Ω_1 acts as a group of permutations on the set of lax vectors belonging to Ω_1. Labeling these elements:

$$[1:0] \leftrightarrow 1 \qquad [0:1] \leftrightarrow 2 \qquad [1:1] \leftrightarrow 3,$$

determines a homomorphism

$$\psi\colon \mathrm{Stab}_{\tilde{\Gamma}}(\Omega_1) \to S_3.$$

The images of

$$\rho_2, \rho_3 \in \tilde{\Gamma}^1 \leq \mathrm{Stab}_{\tilde{\Gamma}}(\Omega_1)$$

are the transpositions $(2,3)$ and $(1,2)$, respectively, and these permutations generate S_3. Consequently, there is a short exact sequence

$$1 \longrightarrow \mathrm{Stab}^{\mathrm{pw}}_{\tilde{\Gamma}}(\Omega_1) \longrightarrow \mathrm{Stab}_{\tilde{\Gamma}}(\Omega_1) \xrightarrow{\psi} S_3 \longrightarrow 1,$$

where the superscript "pw" refers to the pointwise stabilizer of Ω_1. Since

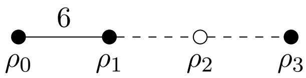
$$\mathrm{Stab}^{\mathrm{pw}}_{\tilde{\Gamma}}(\Omega_1) = \left\{ \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, 1 \right), \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \sigma \right) \right\} = \langle \rho_0 \rangle \,,$$

it follows that

$$\#\left(\mathrm{Stab}_{\tilde{\Gamma}}(\Omega_1)\right) = \#\left(\mathrm{Stab}^{\mathrm{pw}}_{\tilde{\Gamma}}(\Omega_1)\right) \cdot \#\left(S_3\right) = 12.$$

On the other hand, Theorem 2.2.7 implies $\tilde{\Gamma}^1 \simeq \{\pm 1\} \times A_2$. Therefore $\tilde{\Gamma}^1$ has order 12 and $\mathrm{Stab}(\Omega_1) = \tilde{\Gamma}^1$.

(Case $i = 2$.)

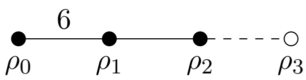


Since

$$\mathrm{Stab}_{\tilde{\Gamma}}(\Omega_2) = \left\{ \left(\begin{bmatrix} 1 & 0 \\ 0 & \xi \end{bmatrix}, g \right), \left(\begin{bmatrix} 0 & 1 \\ \xi & 0 \end{bmatrix}, g \right) \;:\; \xi \in \mathcal{O}^\times, g \in \{1, \sigma\} \right\},$$

we find $\#\left(\mathrm{Stab}_{\tilde{\Gamma}}(\Omega_2)\right) = 24$. Theorem 2.2.7 implies $\tilde{\Gamma}^2 \simeq G_2 \times \{\pm 1\}$. A comparison of orders gives the result.

(Case $i = 3$.)



We find

$$\mathrm{Stab}_{\tilde{\Gamma}}(\Omega_3) = \left\{ \left(\begin{bmatrix} 1 & b \\ 0 & \xi \end{bmatrix}, g \right) \;:\; b \in \mathcal{O}, \xi \in \mathcal{O}^\times, g \in \{1, \sigma\} \right\}.$$

Identifying this group with $((\mathcal{O}, +) \rtimes \mathcal{O}^\times) \rtimes \{1, \sigma\}$ shows $\mathrm{Stab}_{\tilde{\Gamma}}(\Omega_3)$ is generated by the elements

$$\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, 1 \right), \left(\begin{bmatrix} 1 & \omega \\ 0 & 1 \end{bmatrix}, 1 \right), \left(\begin{bmatrix} 1 & 0 \\ 0 & -\omega \end{bmatrix}, 1 \right), \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \sigma \right).$$

These generators may be expressed in terms of ρ_0, ρ_1, and ρ_2 by

$$\rho_2\rho_1\left(\rho_0\rho_1\right)^2, \quad \left(\rho_1\rho_0\right)^2\rho_2\left(\rho_1\rho_0\right)^2, \quad \rho_0\rho_1, \quad \rho_0,$$

respectively. Therefore $\tilde{\Gamma}^3 \subseteq \mathrm{Stab}_{\tilde{\Gamma}}(\Omega_3)$, completing the proof. $\square$

Lemma 3.4.4. *For each $i \in I_3$, the canonical image $\tilde{\Gamma}^+$ of Γ in $\tilde{\Gamma}$ acts transitively on $\mathcal{E}_i$.*

Proof. The group $\tilde{\Gamma}^+$ acts transitively on $\mathcal{E}_3$ since $\mathrm{PGL}_2(\mathcal{O})$ acts transitively on $\mathbb{P}^1(\mathcal{O})$. Similarly, $\tilde{\Gamma}^+$ acts transitively on $\mathcal{E}_2$ since $\mathrm{GL}_2(\mathcal{O})$ acts transitively on the set of bases of $\mathcal{O}^2$. Suppose $\Omega \in \mathcal{E}_0$ or $\Omega \in \mathcal{E}_1$. By the previous remarks we may assume $\Omega \in \mathrm{Res}(\Omega_2)$. Propositions 3.3.3 and 3.3.4 show that the tribases and tetrabases contained in $\mathrm{Res}(\Omega_2)$ correspond to the vertices and edges of the hexagon in Figure 3.1, respectively. The even subgroup $\tilde{\Gamma}^+_{0,1} = \langle \rho_0\rho_1 \rangle$ acts as the group of rotations of the Hexagon. This action is transitive on the vertices and edges.

□

Corollary 3.4.5. *If $i \in I_3$, then $\tilde{\Gamma}$ acts transitively on $\mathcal{E}_i$.*

Proof. Follows immediately from the previous Lemma. □

Theorem 3.4.6. *The Eisenstein topograph $\mathcal{E}$ is isomorphic to the Coxeter geometry $\Pi(\tilde{\Gamma}, \{\rho_0, \rho_1, \rho_2, \rho_3\})$.*

Proof. Given $i \in I_3$, define the function $\varphi_i : \mathcal{E}_i \to \tilde{\Gamma}/\tilde{\Gamma}^i$ as follows. For each $\Omega \in \mathcal{E}_i$, choose $\kappa \in \tilde{\Gamma}$ such that $\kappa \cdot \Omega_i = \Omega$ and set

$$\varphi_i\,(\Omega) = \kappa \bmod \tilde{\Gamma}^i.$$

By Lemma 3.4.3 and Corollary 3.4.5, φ_i is bijective. Clearly, the resulting map

$$\varphi : \mathcal{E} \to \Pi(\tilde{\Gamma}, \{\rho_0, \rho_1, \rho_2, \rho_3\}),$$
$$\varphi(\Omega) = \varphi_{\mathrm{Typ}(\Omega)}(\Omega)$$

is type preserving. It remains to prove that φ and φ^{-1} preserve incidence.

Fix $i, j \in I_3$ and let $\kappa_i, \kappa_j \in \tilde{\Gamma}$. We claim that $\kappa_i \cdot \Omega_i$ is incident to $\kappa_j \cdot \Omega_j$ in $\mathcal{G}$ if and only if $\kappa_i\tilde{\Gamma}^i$ is incident to $\kappa_j\tilde{\Gamma}^j$ in Π. Without loss of generality assume that $i \leq j$. Then $\kappa_i\Omega_i$ is incident to $\kappa_j\Omega_j$ if and only if $\kappa_j\Omega_j \subseteq \kappa_i\Omega_i$, or, equivalently, $\kappa_i^{-1}\kappa_j\Omega_j \subseteq \Omega_i$. Since $\tilde{\Gamma}^i$ acts transitively on the lax subsets of Ω_i, it follows that $\kappa_i^{-1}\kappa_j\Omega_j \subseteq \Omega_i$ if and only if there exists $\eta \in \tilde{\Gamma}^i$ such that $\eta\kappa_i^{-1}\kappa_j\Omega_j = \Omega_j$. Since $\mathrm{Stab}_{\tilde{\Gamma}}(\Omega_j) = \tilde{\Gamma}^j$, the condition $\eta\kappa_i^{-1}\kappa_j\Omega_j = \Omega_j$ is equivalent to $\kappa_j^{-1}\kappa_i\eta^{-1} \in \tilde{\Gamma}^j$. This shows $\kappa_i \cdot \Omega_i$ is incident to $\kappa_j \cdot \Omega_j$ if and only if there exists $\eta \in \tilde{\Gamma}^i$ such that

$$\kappa_i\eta^{-1} \in \kappa_i\tilde{\Gamma}^i \cap \kappa_j\tilde{\Gamma}^j.$$

This last statement is equivalent to $\kappa_i\tilde{\Gamma}^i \cap \kappa_j\tilde{\Gamma}^j \neq \emptyset$, which is precisely the condition that $\kappa_i\tilde{\Gamma}^i$ is incident to $\kappa_i\tilde{\Gamma}^i$ in Π.

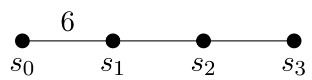
The claim shows that both φ and φ^{-1} are incidence preserving. Hence φ is an isomorphism of geometries. □

Corollary 3.4.7. *The Eisenstein topograph $\mathcal{E}$ is connected and thin. The group $\tilde{\Gamma}$ acts flag transitively on $\mathcal{E}$. The set of chambers of $\mathcal{E}$ is a principal homogeneous space for $\tilde{\Gamma}$.*

Proof. The first and second statements are immediate consequences of the previous Theorem and Proposition 2.3.10. The final statement follows by the Theorem and Proposition 2.3.11. □

3.5 Local Geometry

The isomorphism described by Theorem 3.4.6 allows us to study the geometry of the Eisenstein topograph $\mathcal{E}$ via the theory of Coxeter geometries outlined in the introduction. We consider the local geometry of $\mathcal{E}$ about an arbitrary vertex, edge, face, and cell. Throughout this section (W, S) represents the Coxeter system indexed over I_3 with diagram

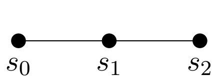


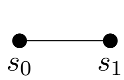
.

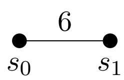
In analogy with the notation introduced for Coxeter groups, for each $i \in I_3$ we define

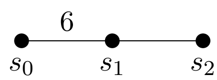
$$S^i := S \backslash \{s_i\}.$$

We make use of the following known constructions. The Coxeter geometries corresponding to the indexed Coxeter systems:



, 

, 

, and 

,

are isomorphic to the (rank three) tetrahedron, the (rank two) triangle, the (rank two) hexagon and the rank three edge to edge tiling of the plane by hexagons, respectively. See [8] for details.

Trivially, the Coxeter geometry of the Coxeter system with a single generator is the unique (up to type) rank one geometry consisting of two elements.

Proposition 3.5.1. *Let* Π *be the unique geometry consisting of two cells. Let* V*,* E*,* F*, and* C*, be a vertex, edge, face, and cell of* $\mathcal{E}$*, respectively.*

(i) The residue Res$(V, \mathcal{E}^*)$ *is isomorphic to the tetrahedron.*

(ii) The residue Res$(E, \mathcal{E}^*)$ *is isomorphic to the coproduct of the triangle and* Π*.*

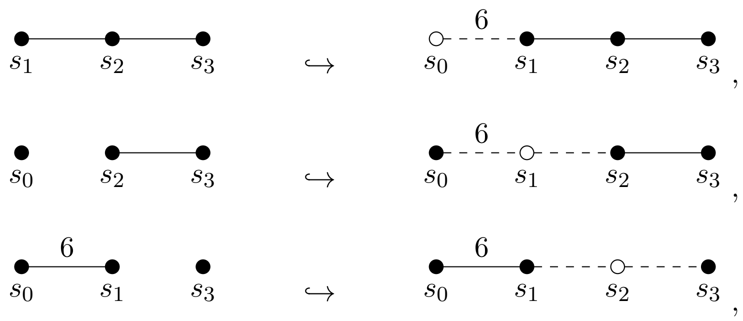
(iii) The residue Res$(F, \mathcal{E})$ *is isomorphic to the coproduct of the hexagon and* Π*.*

(iv) The residue Res$(C, \mathcal{E})$ *is isomorphic to the tiling of the plane by hexagons.*

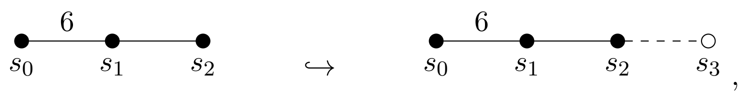
Proof. Since $\mathcal{E}$ is isomorphic to $\Pi(W, S)$, Proposition 2.3.10 implies that

$$\begin{aligned}
\mathrm{Res}(V, \mathcal{E}) &\simeq \Pi\left(W^0, S^0\right), \\
\mathrm{Res}(E, \mathcal{E}) &\simeq \Pi\left(W^1, S^1\right), \\
\mathrm{Res}(F, \mathcal{E}) &\simeq \Pi\left(W^2, S^2\right), \\
\mathrm{Res}(C, \mathcal{E}) &\simeq \Pi\left(W^3, S^3\right).
\end{aligned}$$

By Theorem 2.2.7, the indexed Coxeter systems (W^0, S^0), (W^1, S^1), (W^2, S^2), and (W^3, S^3) have Coxeter graphs:



and



respectively. The result then follows by Proposition 2.3.12 and the discussion above. □

If V is a lax tetrabasis of $\mathcal{O}$, then the lax vectors, lax bases, and lax tribases incident to V are exactly the vertices, edges, and faces of Res$(V, \mathcal{E}^*)$, respectively[2].

[2] Equivalently, these are the cells, faces, and edges of Res$(V, \mathcal{E})$, respectively.

Accordingly, under the identification described by part (i) of the proposition above, the four vertices, six edges, and four faces of the tetrahedron correspond to the $|V| = 4$ lax vectors, the $\binom{4}{2} = 6$ lax bases and $\binom{4}{3} = 3$ lax tribases incident to (contained in) V, respectively. See Figure 3.4.

If E is a lax tribasis of $\mathcal{O}$, then the lax vectors, the lax bases, and the lax tetrabases incident to E are exactly the vertices, edges, and cells[3] of $\mathrm{Res}(E, \mathcal{E}^*)$, respectively. Hence the vertices and edges of the triangle in part (ii) of Proposition 4.5.1 are identified with the $|E| = 3$ lax vectors and the $\binom{3}{2} = 3$ lax bases incident to (contained in) E, respectively. On the other hand, since the $\{3\}$-truncation of $\mathrm{Res}(E, \mathcal{E}^*)$ is identified with the two-cell geometry Π, it follows that E is incident to (is contained in) exactly two lax tetrabases. See Figure 3.5.

Similarly, if F is a lax basis of $\mathcal{O}$, then F contains exactly two lax vectors. These lax vectors are identified with the two cells of the geometry $\mathrm{Res}(F, \mathcal{E})$. The lax tetrabases and the lax tribases incident to F are the vertices and edges of the geometry $\mathrm{Res}(F, \mathcal{E})$, respectively. By the Proposition, the $\{0, 1\}$-truncation of $\mathrm{Res}(F, \mathcal{E})$ is isomorphic to a hexagon. It follows that each lax basis is incident to exactly six lax tetrabases and six lax tribases. See Figure 3.7.

Finally, the lax tetrabases, lax tribases, and lax bases incident to a lax vector C are exactly the vertices, edges, and faces of the geometry $\mathrm{Res}(C, \mathcal{E})$, respectively. By the Theorem, we may identify these with the vertices, edges and faces of a regular tiling of $\mathbb{E}^2$ by hexagons. See Figure 3.7. We have proved the following.

Corollary 3.5.2. *(i) Each lax tetrabasis is incident to exactly four lax vectors, six lax bases, and four lax tribases.*

(ii) Each lax tribasis is incident to exactly three lax vectors, three lax bases, and two lax tetrabases.

(iii) Each lax basis is incident to exactly two lax vectors, six lax tribases and six lax tetrabases.

(iv) Each lax vector is incident to infinitely many lax bases, infinitely many lax tribases, and infinitely many lax tetrabases.

[3] Equivalently, the cells, faces and vertices of $\mathrm{Res}(E, \mathcal{E})$, respectively.

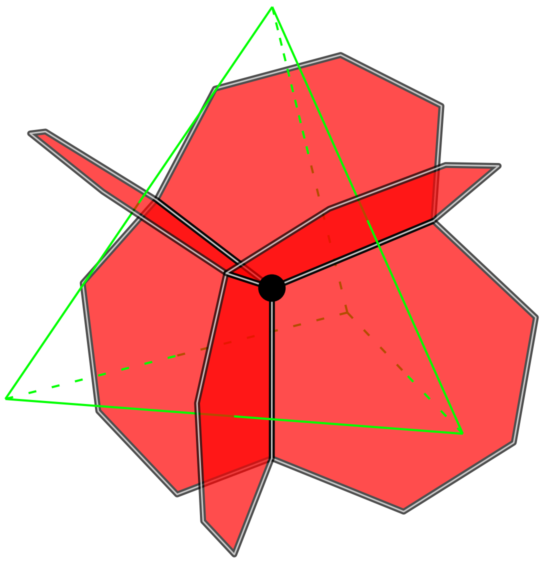


Figure 3.4: The incidence relations at a vertex V of $\mathcal{E}$. The residue $\operatorname{Res}(V, \mathcal{E}^*)$ is isomorphic to the tetrahedron, as illustrated in green.

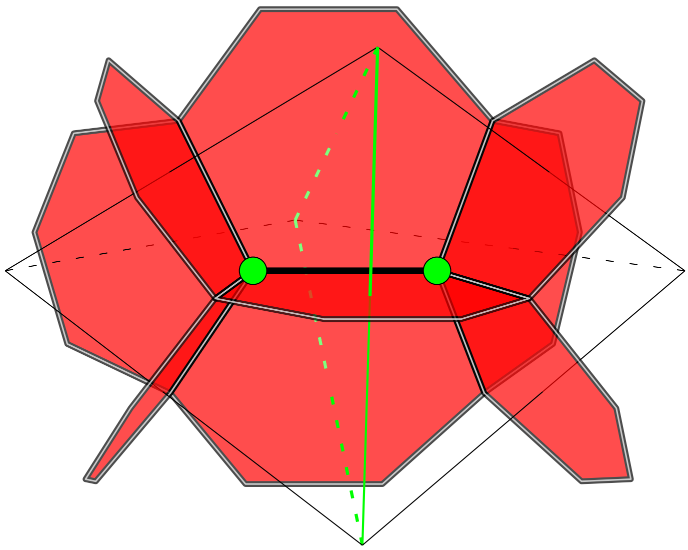


Figure 3.5: The incidence relations at an edge E of $\mathcal{E}$. The residue $\operatorname{Res}(E, \mathcal{E}^*)$ is isomorphic to the coproduct of the triangle and Π, as illustrated in green. The two green vertices represent the tetrahedral cells of Π (drawn black).

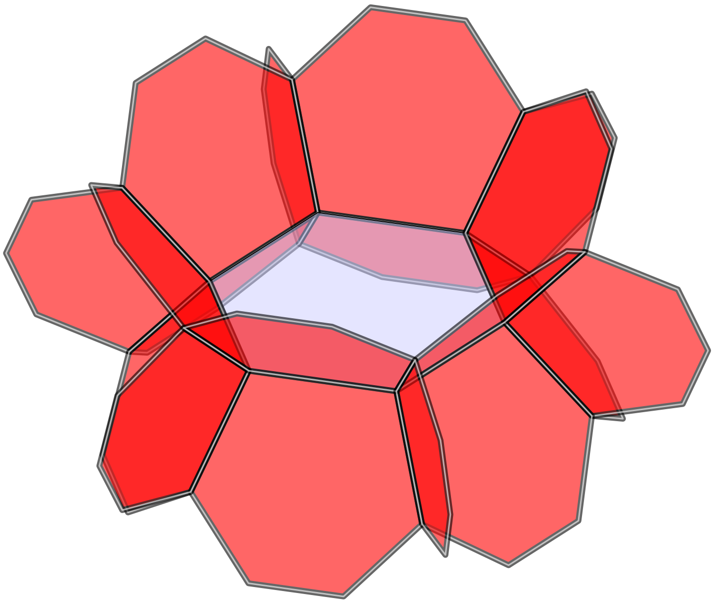


Figure 3.6: The incidence relations at a face of $\mathcal{E}$.

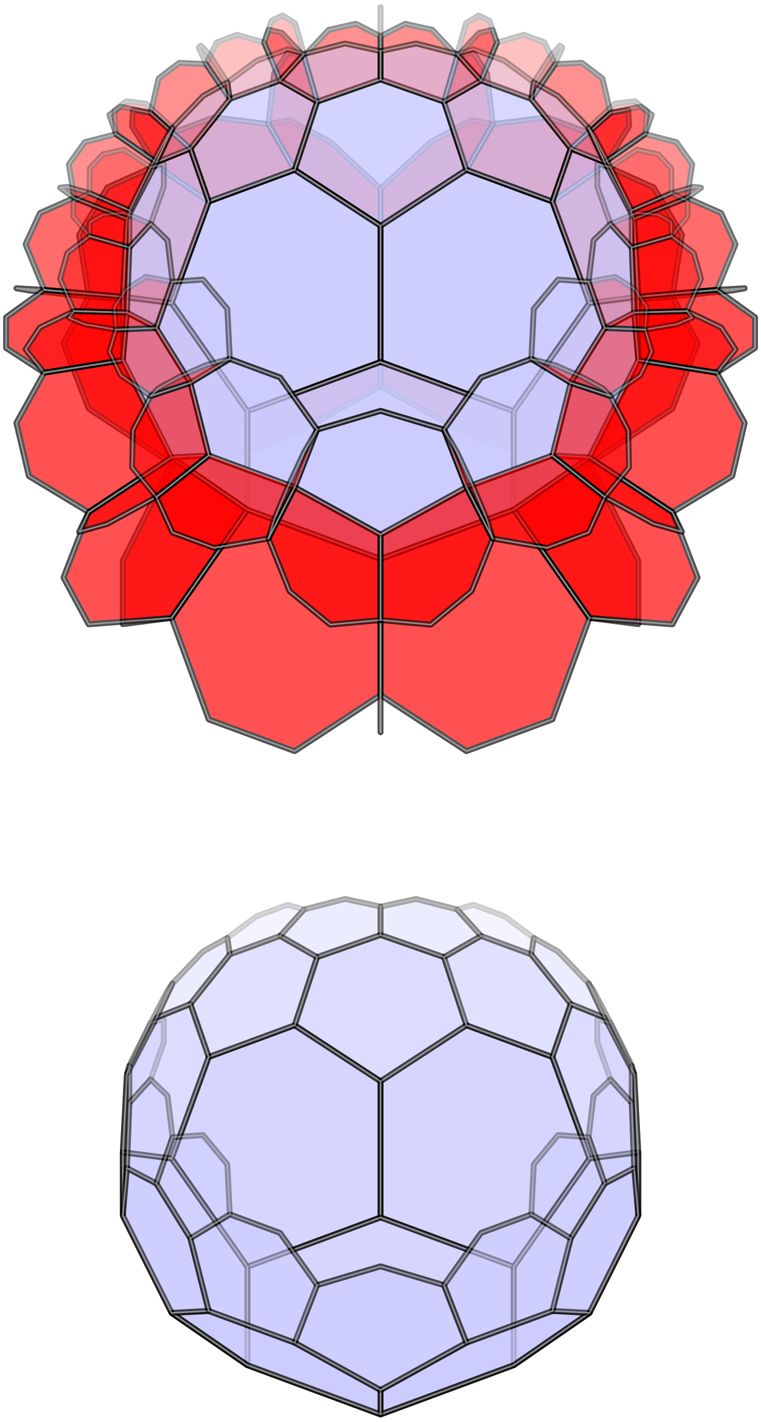


Figure 3.7: Above: a partial illustration of the incidence relations at a cell of $\mathcal{E}$. Below: the residue of a cell is isomorphic to a tiling of the plane by hexagons.

Part II

The Gaussian Case

Chapter 4

The Gaussian Topograph

This chapter is the analogy of Chapter 3 for the ring of **Gaussian integers** $\mathbb{Z}[i]$. The majority of the notation, results, and proofs carry over with only minor (obvious) revisions. Throughout this chapter $\mathcal{O}$ denotes the ring $\mathbb{Z}[i]$ and $-$ denotes complex conjugation.

4.1 Units

The units $\mathcal{O}^\times = \{1, -1, i, -i\}$ form a cyclic group of order four generated by $-i$. The map $\{\pm 1\} \to \mathcal{O}^\times / \left(\mathcal{O}^\times\right)^2$ determined by $-1 \mapsto i$ is an isomorphism. We may identify $\mathcal{O}^\times$ with the edges of the square whose vertices are $\pm 1 \pm i$. See Figure 4.1. Two units are said to be **adjacent** if the corresponding edges are incident to a common vertex of the square. We discuss the significance of the vertices in the following section but remark that given any vertex $k = \pm 1 \pm i$, the two edges incident to k are $k/(1+i)$ and $k/(1-i)$.

4.2 Lax Vectors and Generalized Bases

Define

$$\mathcal{O}^2_{\mathrm{prim}} = \left\{(x, y) \in \mathcal{O}^2 \ : \ \gcd(x, y) = 1\right\}.$$

If $a, b \in \mathcal{O}^2$, then we write $a \sim b$ if there exists $\xi \in \mathcal{O}^\times$ such that $a = \xi \cdot b$.

Definition 4.2.1. *We define a **lax vector** as an element of* $\mathbb{P}^1(\mathcal{O}) = \mathcal{O}^2_{prim}/\sim$.

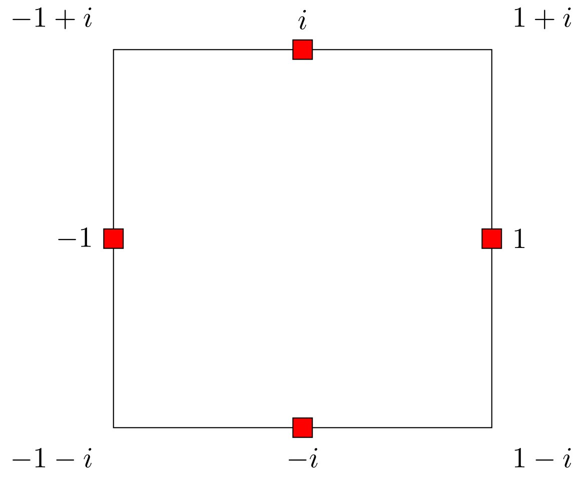


Figure 4.1: The adjacency relationships in $\mathcal{O}^\times$.

Following the notation of the previous chapter, we write $[x : y]$ for the equivalence class of $(x, y) \in \mathcal{O}^2$ in $\mathbb{P}^1(\mathcal{O})$. We write **Oct** for the natural graph structure arising from the vertices and edges of an Octahedron.

Definition 4.2.2. *Consider the undirected graph G whose vertex set is $\mathcal{O}^2$ and whose edge set is the set of (unordered) bases of the $\mathcal{O}$-module $\mathcal{O}^2$. A subset $B \subset \mathcal{O}^2$ is said to be a* ***tribasis*** *of $\mathcal{O}^2$, and is said to be an* ***octobasis*** *of $\mathcal{O}^2$, if the subgraph of G induced by B is isomorphic to Tri, and to Oct, respectively.*

A subset $L \subset \mathbb{P}^1(\mathcal{O})$ is said to be a ***lax basis*** *of $\mathcal{O}^2$, a* ***lax tribasis*** *of $\mathcal{O}^2$, and a* ***lax octobasis*** *of $\mathcal{O}^2$, if a (any) set of representatives of L in $\mathcal{O}^2$ is a basis of $\mathcal{O}^2$, a tribasis of $\mathcal{O}^2$, and a octobasis of $\mathcal{O}^2$, respectively.*

4.3 The Gaussian Topograph

Definition 4.3.1. *Let*

$$\begin{aligned}
\mathcal{G}_0 &= \left\{ \textit{Lax octobases of } \mathcal{O}^2 \right\}, \\
\mathcal{G}_1 &= \left\{ \textit{Lax tribases of } \mathcal{O}^2 \right\}, \\
\mathcal{G}_2 &= \left\{ \textit{Lax bases of } \mathcal{O}^2 \right\}, \\
\mathcal{G}_3 &= \{ \textit{Lax vectors} \} = \mathbb{P}^1(\mathcal{O}),
\end{aligned}$$

and let

$$\mathcal{G} = \bigsqcup_{i=0}^{3} \mathcal{G}_i.$$

Define an incidence relation $*$ on $\mathcal{G}$ by symmetric inclusion. Define

$$\mathit{Typ} : \mathcal{G} \to I_3 = \{0, 1, 2, 3\}$$

*by $\mathit{Typ}(\mathcal{G}_i) = i$, for each $i \in I_3$. The **Gaussian topograph** is the rank four incidence geometry $(\mathcal{G}, *, \mathit{Typ})$ over I_3.*

The **dual** geometry to $\mathcal{G}$, written $\mathcal{G}^*$, is the rank four incidence geometry $(\mathcal{G}, *, \mathrm{Typ}')$, where $\mathrm{Typ}' = 3 - \mathrm{Typ}$. We have the following correspondences:

$$\text{Vertices of } \mathcal{G} \leftrightarrow \mathcal{G}_0 \leftrightarrow \text{Lax octobases of } \mathcal{O}^2 \leftrightarrow \text{Cells of } \mathcal{G}^*,$$

$$\text{Edges of } \mathcal{G} \leftrightarrow \mathcal{G}_1 \leftrightarrow \text{Lax tribases of } \mathcal{O}^2 \leftrightarrow \text{Faces of } \mathcal{G}^*,$$

$$\text{Faces of } \mathcal{G} \leftrightarrow \mathcal{G}_2 \leftrightarrow \text{Lax bases of } \mathcal{O}^2 \leftrightarrow \text{Edges of } \mathcal{G}^*,$$

$$\text{Cells of } \mathcal{G} \leftrightarrow \mathcal{G}_3 \leftrightarrow \text{Lax vectors} \leftrightarrow \mathbb{P}^1(\mathcal{O}) \leftrightarrow \text{Vertices of } \mathcal{G}^*.$$

Definition 4.3.2. *For each $i \in I_3$, define pairwise incident elements $\Omega_i \in \mathcal{G}_i$:*

$$\begin{aligned}
\Omega_0 &= \{[1:0], [0:1], [1:1], [i:1], [1+i:1], [1:1-i]\}, \\
\Omega_1 &= \{[1:0], [0:1], [1:1]\}, \\
\Omega_2 &= \{[1:0], [0:1]\}, \\
\Omega_3 &= \{[1:0]\}.
\end{aligned}$$

We call the set

$$\mathcal{C}_0 := \{\Omega_0, \Omega_1, \Omega_2, \Omega_3\}$$

*the **fundamental chamber** of $\mathcal{G}$. See Figure 4.2.*

Proposition 4.3.3. *The lax basis Ω_2 is incident to (contained in) exactly four lax tribases, namely*

$$\{[1:0], [0:1], [\xi:1]\},$$

for any $\xi \in \mathcal{O}^\times$.

Proof. The same proof as Proposition 3.3.3 works, mutatis mutandis. □

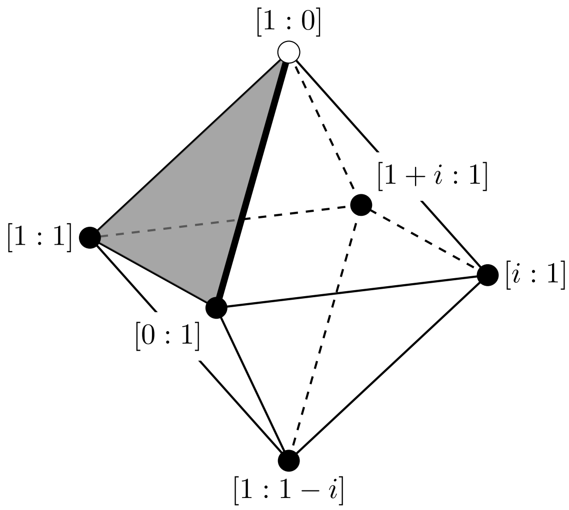


Figure 4.2: The dual of the fundamental chamber illustrating the lax octobasis Ω_0. The distinguished face, edge, and vertex represent the lax tribasis Ω_1, the lax basis Ω_2, and the lax vector Ω_3, respectively.

Proposition 4.3.4. *The lax basis Ω_2 is incident to exactly four lax octobases, namely:*

$$\left\{[1:0]\,,[0:1]\,,\left[\frac{k}{1+i}:1\right],\left[\frac{k}{1-i}:1\right],[k:1]\,,\left[1:\frac{2}{k}\right]\right\},$$

for any $k = \pm 1 \pm i$.

Proof. Each of the four sets are lax octobases, where all cardinality 2 subsets are lax bases except $\{[1:0],[1:2/k]\}$, $\{[0:1],[k:1]\}$, and $\{[k/(1+i):1],[k/(1-i):1]\}$. We argue these are the only lax octobases containing Ω_2. Figure 4.3 represents a lax octobasis containing Ω_2. By Proposition 4.3.3 we can take the adjacent lax vectors of the form $[\xi_1:1]$, $[\xi_2:1]$ with $\xi_1,\xi_2 \in \mathcal{O}^\times$ and $\xi_1 \neq \xi_2$. The edges $\{[1:0],[x:y]\}$ and $\{[0:1],[s:t]\}$ imply that

$$\det\begin{pmatrix}1 & 0\\ x & y\end{pmatrix},\ \det\begin{pmatrix}0 & 1\\ s & t\end{pmatrix} \in \mathcal{O}^\times.$$

Hence $y, s \in \mathcal{O}^\times$. On the other hand, since $\{[1:0],[s:t]\}$ and $\{[0:1],[x:y]\}$ do not belong to the edge set, it follows that

$$\det\begin{pmatrix}1 & 0\\ s & t\end{pmatrix} \notin \mathcal{O}^\times \text{ and } \det\begin{pmatrix}0 & 1\\ x & y\end{pmatrix} \notin \mathcal{O}^\times.$$

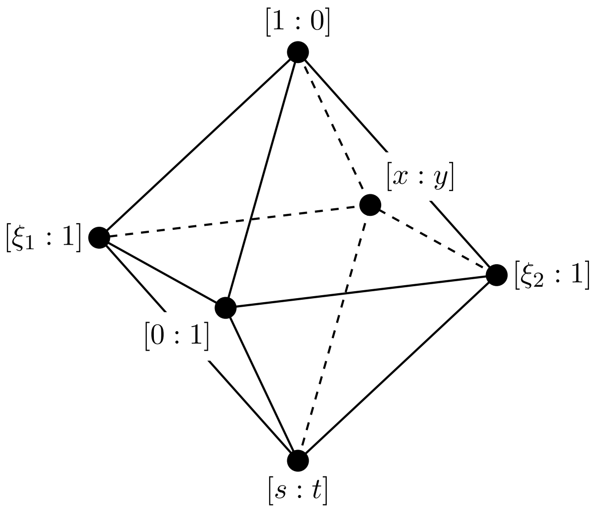


Figure 4.3: A lax octobasis containing Ω_2.

Hence neither x nor t are units. If $x' = y^{-1}x$ and $t' = s^{-1}t$, then the edge

$$\{[x : y], [s : t]\} = \big\{[x' : 1], [1 : t']\big\}$$

implies

$$\det \begin{pmatrix} x' & 1 \\ 1 & t' \end{pmatrix} \in \mathcal{O}^\times.$$

Consequently $x't' = 0,\ 2,\ 1 + i$, or $1 - i$. We can exclude $1 + i$ and $1 - i$ since both these elements are irreducible in $\mathcal{O}$, and neither x' nor t' are units. Similarly, if $x't' = 0$, then either $[x' : 1] = [0 : 1]$ or $[1 : t'] = [1 : 0]$; a contradiction. Hence $x't' = 2$. This leaves exactly four possible choices of x' and t', determined by the following exhaustive list of factorizations of 2 by an ordered product of non-units:

$$(x', t') \in \{(1 + i, 1 - i), (-1 - i, -1 + i), (-1 + i, -1 - i), (1 - i, 1 + i)\}.$$

Permuting $[\xi_1 : 1]$ and $[\xi_2 : 1]$ leaves the lax octobasis invariant. We are therefore done if we show that the unordered set $\{\xi_1, \xi_2\}$ is determined by x'. Noting that $[\xi_i : 1] = [1 : \xi_i^{-1}]$, the two edges $\{[x' : 1], [\xi_1 : 1]\}$ and $\{[x' : 1], [\xi_2 : 1]\}$ imply that

$$\det \begin{pmatrix} x' & 1 \\ 1 & \xi_1^{-1} \end{pmatrix} \in \mathcal{O}^\times \text{ and } \det \begin{pmatrix} x' & 1 \\ 1 & \xi_2^{-1} \end{pmatrix} \in \mathcal{O}^\times.$$

Hence $x'\xi_1^{-1}$ and $x'\xi_2^{-1}$ are distinct elements of $\{0, 2, 1+i, 1-i\}$. Since x' is non-zero, $x'\xi_i^{-1} \neq 0$. Similarly, since x' is not a unit multiple of 2 (else t' is a unit), $x'\xi_i^{-1} \neq 2$. Therefore

$$\{\xi_1^{-1}, \xi_2^{-1}\} = \left\{\frac{1+i}{x'}, \frac{1-i}{x'}\right\},$$

hence

$$\{\xi_1, \xi_2\} = \left\{\frac{x'}{1+i}, \frac{x'}{1-i}\right\},$$

as required. □

Corollary 4.3.5. *The lax tribasis Ω_1 is incident to exactly two lax octobases, namely:*

$$\{[1:0], [0:1], [1:1], [i:1], [1+i:1], [1:1-i]\},$$
$$\{[1:0], [0:1], [1:1], [-i:1], [1-i:1], [1:1+i]\}.$$

Proof. This is an immediate consequence of Proposition 4.3.4. □

Remark 4.3.6. *Proposition 4.3.4 demonstrates that a lax octobasis incident to Ω_2 is determined by a choice of vertex in Figure 4.1. Equivalently, a pair of adjacent units determine a lax octobasis of Res(Ω_2). A lax tribases incident to Ω_2 is determined by a choice of unit (edge). The two lax octobases incident to a lax tribasis $\Omega \in$ Res(Ω_1) correspond to the two vertices incident to the edge determined by Ω.*

4.4 The Coxeter Geometry

Notation follows Section 3.4. In particular, we define

$$\Gamma := \mathrm{PGL}_2(\mathcal{O}) = \mathrm{GL}_2(\mathcal{O})/\mathcal{O}^\times,$$
$$\tilde{\Gamma} := \mathrm{P\Gamma L}_2(\mathcal{O}) = \mathrm{PGL}_2(\mathcal{O}) \rtimes \{1, \sigma\};$$

where σ acts on $\mathrm{PGL}_2(\mathcal{O})$ by complex conjugation. The determinant map det : $\mathrm{GL}_2(\mathcal{O}) \to \mathcal{O}^\times$ descends to

$$\mathrm{pdet} : \mathrm{PGL}_2(\mathcal{O}) \to \mathcal{O}^\times / \left(\mathcal{O}^\times\right)^2 \equiv \{\pm 1\}.$$

The group $\Gamma = \mathrm{PGL}_2(\mathcal{O})$ acts transitively on $\mathbb{P}^1(\mathcal{O})$ by left multiplication, and the group $\tilde{\Gamma} = \mathrm{P\Gamma L}_2(\mathcal{O})$ acts on $\mathbb{P}^1(\mathcal{O})$ via the action defined in Section 3.4. The groups Γ and $\tilde{\Gamma}$ act on $\mathcal{G}$ by transport of structure.

We identify $\tilde{\Gamma}$ with a Coxeter group. The conclusion of Proposition 4.4.2 can be found in [11]. Our proof differs, as do our generators.

Lemma 4.4.1. *The elements*

$$\rho_0 := \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \sigma \right), \qquad \rho_1 := \left(\begin{bmatrix} i & 0 \\ 0 & 1 \end{bmatrix}, \sigma \right),$$

$$\rho_2 := \left(\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}, \sigma \right), \qquad \rho_3 := \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma \right),$$

generate $\tilde{\Gamma}$.

Proof. Consider the matrices

$$E_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix},$$

$$E_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad E_4 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let $\tilde{E}_i$ denote the canonical image of E_i in $\Gamma = \mathrm{PGL}_2(\mathcal{O})$. The same proof as Lemma 3.4.1 transfers, mutatis mutandis, to demonstrate that

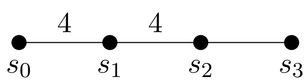
$$\tilde{\Gamma} = \left\langle (1,\sigma), \left(\tilde{E}_1, 1\right), \left(\tilde{E}_2, 1\right), \left(\tilde{E}_3, 1\right), \left(\tilde{E}_4, 1\right) \right\rangle.$$

The expressions

$$\begin{aligned} \left(\tilde{E}_1, 1\right) &= \rho_2\rho_1\rho_0\rho_1, & \left(\tilde{E}_2, 1\right) &= \rho_1\rho_2\rho_1\rho_0, \\ \left(\tilde{E}_3, 1\right) &= \rho_0\rho_3, & \left(\tilde{E}_4, 1\right) &= \rho_1\rho_0\rho_1\rho_0, \end{aligned} \tag{4.4.1}$$

and the observation $\rho_0 = (1, \sigma)$, complete the proof. $\square$

Proposition 4.4.2. *Let (W, S) be the indexed Coxeter system with diagram*



.

The map $\psi : S \to \tilde{\Gamma}$ given by

$$s_0 \mapsto \rho_0, \qquad s_1 \mapsto \rho_1, \qquad s_2 \mapsto \rho_2, \qquad s_3 \mapsto \rho_3,$$

extends to a group isomorphism $\Psi : W \xrightarrow{\sim} \tilde{\Gamma}$. Moreover, $\Gamma = PGL_2(\mathbb{Z}[i]) \simeq W^+$.

Proof. Let V be the four dimensional real vector space of 2×2 complex Hermitian matrices, and define a quadratic form Q on V by

$$Q(M) = -\det(M).$$

The same proof as Proposition 3.4.2 shows there is a Lie group homomorphism

$$\Phi : \tilde{\Gamma} \to O(V, Q).$$

Each involution $\Phi(\rho_i)$ is a reflection in (V, Q) with corresponding root:

$$\alpha_0 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \qquad \alpha_1 = \frac{\sqrt{2}}{2} \begin{pmatrix} 0 & 1-i \\ 1+i & 0 \end{pmatrix},$$
$$\alpha_2 = \begin{pmatrix} -1 & -1 \\ -1 & 0 \end{pmatrix}, \qquad \alpha_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

If B is the bilinear form on V associated to Q, then, for each $0 \leq i, j \leq 3$,

$$B(\alpha_i, \alpha_j) = -\cos\left(\frac{\pi}{m(s_i, s_j)}\right).$$

It follows by Theorem 2.2.2 that there exists a monomorphism $\mu : W \to O(V, Q)$, determined by $s_i \mapsto \sigma_{s_i} := \Phi(\rho_i)$, such that the following diagram commutes:

$$\begin{tikzcd} W \arrow[r, hook, "\mu"] \arrow[d, two heads, "\Psi"'] & O(V,Q) \\ \tilde{\Gamma} \arrow[ur, "\Phi"'] & \end{tikzcd}.$$

Hence Ψ is injective.

The final statements in the proof of Proposition 3.4.2 and the expressions (4.4.1) imply that $\Gamma \simeq W^+$. $\square$

Recall that for $J \subseteq I_3 = \{0, 1, 2, 3\}$, we define

$$\tilde{\Gamma}_J := \langle \rho_j \ : \ j \in J \rangle,$$
$$\tilde{\Gamma}^J := \langle \rho_j \ : \ j \in I_3 \backslash J \rangle.$$

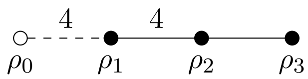
We omit the set notation when J has cardinality one.

Lemma 4.4.3. *For each $i \in I_3$,*

$$\tilde{\Gamma}^i = Stab_{\tilde{\Gamma}}(\Omega_i).$$

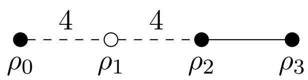
Proof. Let $i \in I_3$. If $j \in I_3$ and $i \neq j$, then ρ_j acts trivially on Ω_i. Hence $\tilde{\Gamma}^i$ is a subgroup of $\mathrm{Stab}_{\tilde{\Gamma}}(\Omega_i)$. We argue by cases.

(Case $i = 0$.)



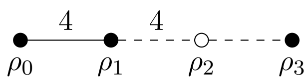
An element of $\mathrm{Stab}_{\tilde{\Gamma}}(\Omega_0)$ is determined by its action on the octahedron of Figure 4.4. Since the automorphism group of the octahedron has order 48, this gives an upper bound for $\#\left(\mathrm{Stab}_{\tilde{\Gamma}}(\Omega_0)\right)$. On the other hand, Theorem 2.2.7 implies $\tilde{\Gamma}^0$ is isomorphic to the Coxeter group BC_3, as indicated in the Coxeter graph above. It is well known that $\#(BC_3) = 48$, hence $\mathrm{Stab}_{\tilde{\Gamma}}(\Omega_0) = \tilde{\Gamma}^0$.

(Case $i = 1$.)



The proof of the corresponding case in Proposition 3.4.3 transfers verbatim.

(Case $i = 2$.)

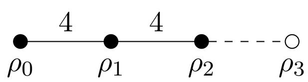


Since

$$\mathrm{Stab}_{\tilde{\Gamma}}(\Omega_2) = \left\{ \left(\begin{bmatrix} 1 & 0 \\ 0 & \xi \end{bmatrix}, g \right), \left(\begin{bmatrix} 0 & 1 \\ \xi & 0 \end{bmatrix}, g \right) \;:\; \xi \in \mathcal{O}^{\times}, g \in \{1, \sigma\} \right\},$$

we find $\#\left(\mathrm{Stab}_{\tilde{\Gamma}}(\Omega_2)\right) = 16$. On the other hand, $\tilde{\Gamma}^2 \simeq BC_2 \times \{\pm 1\}$, so that $\#(\tilde{\Gamma}^2) = 16$. It follows that $\mathrm{Stab}_{\tilde{\Gamma}}(\Omega_2) = \tilde{\Gamma}^2$.

(Case $i = 3$.)



The same proof as Proposition 3.4.3, Case ($i = 3$), demonstrates that $\mathrm{Stab}_{\tilde{\Gamma}}(\Omega_3)$ is generated by the elements

$$\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, 1\right), \left(\begin{bmatrix} 1 & i \\ 0 & 1 \end{bmatrix}, 1\right), \left(\begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}, 1\right), \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \sigma\right).$$

On the other hand, these generators may be expressed in terms of ρ_0, ρ_1, and ρ_2 as $\rho_2\rho_1\rho_0\rho_1$, $\rho_1\rho_2\rho_1\rho_0$, $\rho_1\rho_0$, and ρ_0, respectively. Consequently $\tilde{\Gamma}^3 \subset \mathrm{Stab}_{\tilde{\Gamma}}(\Omega_3)$, hence $\mathrm{Stab}_{\tilde{\Gamma}}(\Omega_3) \simeq \tilde{\Gamma}^3$. □

Lemma 4.4.4. *For each $i \in I_3$, the image $\tilde{\Gamma}^+$ of Γ in $\tilde{\Gamma}$ acts transitively on $\mathcal{G}_i$.*

Proof. The proofs of the cases $i = 2$ and $i = 3$ follow verbatim from the corresponding proof of Lemma 3.4.4. Let $\Omega \in \mathcal{G}_0$ or $\mathcal{G}_1$. Without loss of generality, we may assume $\Omega \in \mathrm{Res}(\Omega_2)$. The tribases and tetrabases incident to Ω_2 correspond to the vertices and edges of the square in Figure 3.1, respectively (see Remark 4.3.6). The rotation group of the square is the even subgroup $\tilde{\Gamma}^+_{0,1} = \langle \rho_0\rho_1 \rangle$. Clearly this group acts transitively on the vertices and edges.

□

Corollary 4.4.5. *If $i \in I_3$, then $\tilde{\Gamma}$ acts transitively on $\mathcal{G}_i$.*

Proof. Immediate from the Lemma above. □

Theorem 4.4.6. *The Gaussian topograph $\mathcal{G}$ is isomorphic to the Coxeter geometry $\Pi(\tilde{\Gamma}, \{\rho_0, \rho_1, \rho_2, \rho_3\})$.*

Proof. The same proof as Theorem 4.4.6 transfers, mutatis mutandis. □

Corollary 4.4.7. *The Gaussian topograph $\mathcal{G}$ is connected and thin. The group $\tilde{\Gamma}$ acts flag transitively on $\mathcal{G}$. The set of chambers of $\mathcal{G}$ is a principal homogeneous space for $\tilde{\Gamma}$.*

Proof. The previous Theorem and Proposition 2.3.10 imply the first and second statements. The Last statement follows by the Theorem and Proposition 2.3.11. □

4.5 Local Geometry

Throughout this section (W, S) denotes the Coxeter system indexed over I_3 with diagram

$$s_0 \overset{4}{\text{———}} s_1 \overset{4}{\text{———}} s_2 \text{———} s_3 \; .$$

Recall that the Coxeter geometry corresponding to the indexed Coxeter system

$$s_0 \text{———} s_1$$

is isomorphic to the triangle. The Coxeter geometries corresponding to the indexed Coxeter systems

$$s_0 \text{———} s_1 \overset{4}{\text{———}} s_2 \, , \quad s_0 \overset{4}{\text{———}} s_1 \, , \quad \text{and} \quad s_0 \overset{4}{\text{———}} s_1 \overset{4}{\text{———}} s_2 \, ,$$

are isomorphic to the (rank three) octahedron, the (rank two) square and the regular rank three edge to edge tiling of the plane by squares, respectively. See [8] for details. Recall that the Coxeter geometry of the Coxeter system with a single generator is the unique (up to type) rank one geometry consisting of two elements. We use these constructions to study the local geometry of $\mathcal{G}$. For $i \in I_3$, we write S^i for the set $S \backslash \{s_i\}$.

Proposition 4.5.1. *Let Π be the unique geometry consisting of exactly two cells. Let V, E, F, and C, represent a vertex, an edge, a face, and a cell of $\mathcal{G}$, respectively.*

(i) The residue Res$(V, \mathcal{G}^*)$ is isomorphic to an octahedron,

(ii) The residue Res$(E, \mathcal{G}^*)$ is isomorphic to the coproduct geometry of Π_2 and a triangle,

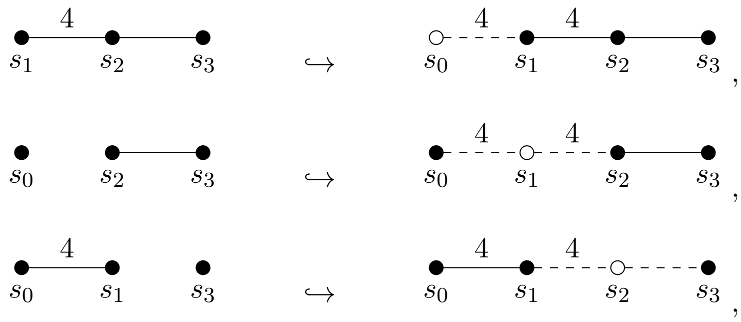
(iii) The residue Res$(F, \mathcal{G})$ is isomorphic to the coproduct geometry of Π_2 and a square,

(iv) The residue Res$(C, \mathcal{G})$ is isomorphic to the edge to edge tiling of the plane by squares.

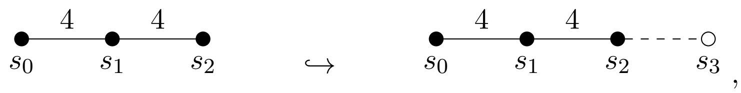
Proof. Since $\mathcal{G}$ is isomorphic to $\Pi(W, S)$, Proposition 2.3.10 implies

$$\begin{aligned}
\operatorname{Res}(V, \mathcal{G}) &\simeq \Pi\left(W^0, S^0\right), \\
\operatorname{Res}(E, \mathcal{G}) &\simeq \Pi\left(W^1, S^1\right), \\
\operatorname{Res}(F, \mathcal{G}) &\simeq \Pi\left(W^2, S^2\right), \\
\operatorname{Res}(C, \mathcal{G}) &\simeq \Pi\left(W^3, S^3\right).
\end{aligned}$$

By Theorem 2.2.7, the indexed Coxeter systems (W^0, S^0), (W^1, S^1), (W^2, S^2), and (W^3, S^3) have Coxeter graphs



and



respectively. The result then follows by Proposition 2.3.12 and the discussion above. □

If V is a lax octobasis of $\mathcal{O}$, then the lax vectors, lax bases, and lax tribases incident to V are exactly the vertices, edges, and faces of $\operatorname{Res}(V, \mathcal{G}^*)$, respectively[1]. Accordingly, under the identification described by part (i) of the proposition above, the four vertices, six edges, and four faces of the tetrahedron correspond to the 6 lax vectors, the 12 lax bases and 8 lax tribases incident to (contained in) V, respectively. See Figure 4.4.

If E is a lax tribasis of $\mathcal{O}$, then the lax vectors, the lax bases, and the lax octobases incident to E are exactly the vertices, edges, and cells[2] of $\operatorname{Res}(E, \mathcal{G}^*)$,

[1]Equivalently, these are the cells, faces, and edges of $\operatorname{Res}(V, \mathcal{G})$, respectively.
[2]Equivalently, the cells, faces and vertices of $\operatorname{Res}(E, \mathcal{G})$, respectively.

respectively. Accordingly, the vertices and edges of the triangle in part (ii) of Proposition 4.5.1 are identified with the $|E| = 3$ lax vectors and the $\binom{3}{2} = 3$ lax bases incident to (contained in) E, respectively. On the other hand, since the $\{3\}$-truncation of $\mathrm{Res}(E, \mathcal{G}^*)$ is identified with the two element geometry Π, it follows that E is incident to (is contained in) exactly two lax octobases. See Figure 4.5.

Similarly, if F is a lax basis of $\mathcal{O}$, then F contains exactly two lax vectors. These lax vectors are identified with the two cells of the geometry $\mathrm{Res}(F, \mathcal{G})$. The lax octobases and the lax tribases incident to F are the vertices and edges of the geometry $\mathrm{Res}(F, \mathcal{G})$, respectively. By the Proposition, the $\{0, 1\}$-truncation of $\mathrm{Res}(F, \mathcal{G})$ is isomorphic to the square. It follows that each lax basis is incident to exactly four lax octobases and four lax tribases. See Figure 4.7.

Finally, the lax octobases, lax tribases, and lax bases incident to a lax vector C are exactly the vertices, edges, and faces of the geometry $\mathrm{Res}(C, \mathcal{G})$, respectively. By the Theorem, we may identify these with the vertices, edges and faces of the edge to edge tiling of the plane by squares. See Figure 3.7. We have proved the following.

Corollary 4.5.2. *(i) Each lax octobasis is incident to exactly six lax vectors, twelve lax bases, and eight lax tribases.*

(ii) Each lax tribasis is incident to exactly three lax vectors, three lax bases and two lax octobases.

(iii) Each lax basis is incident to exactly two lax vectors, four lax tribases, and four lax octobases.

(iv) Each lax vector is incident to infinitely many lax bases, infinitely many lax tribases, and infinitely many lax octobases.

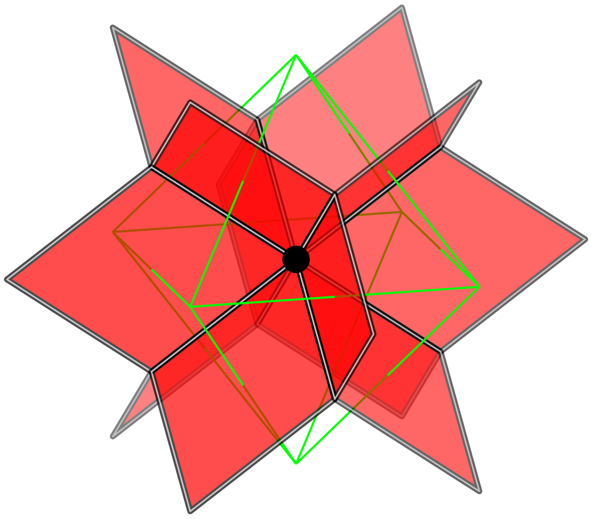


Figure 4.4: The incidence relations at a vertex V (drawn black) of $\mathcal{G}$. The dual residue $\mathrm{Res}(V, \mathcal{G}^*)$ is isomorphic to the octahedron, illustrated in green.

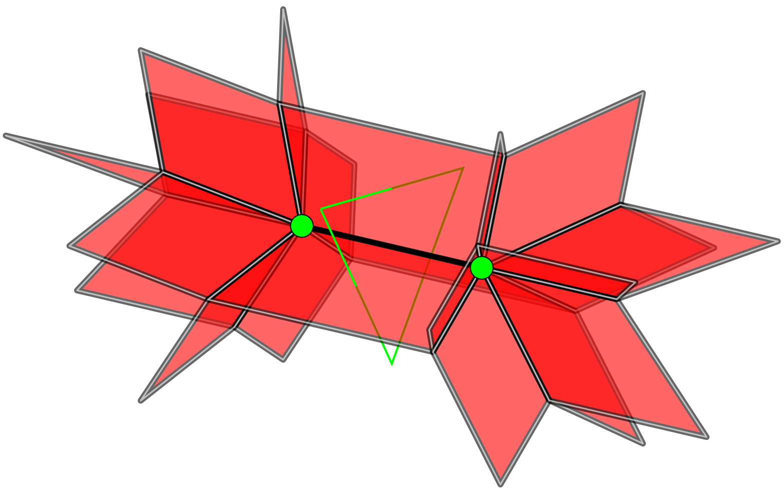


Figure 4.5: The incidence relations about an edge E (drawn black) of $\mathcal{G}$. The dual residue $\mathrm{Res}(E, \mathcal{G}^*)$ is isomorphic to the coproduct of the triangle and Π, illustrated in green. The vertices represent the two octahedral cells of $\mathrm{Res}(E, \mathcal{G}^*)$.

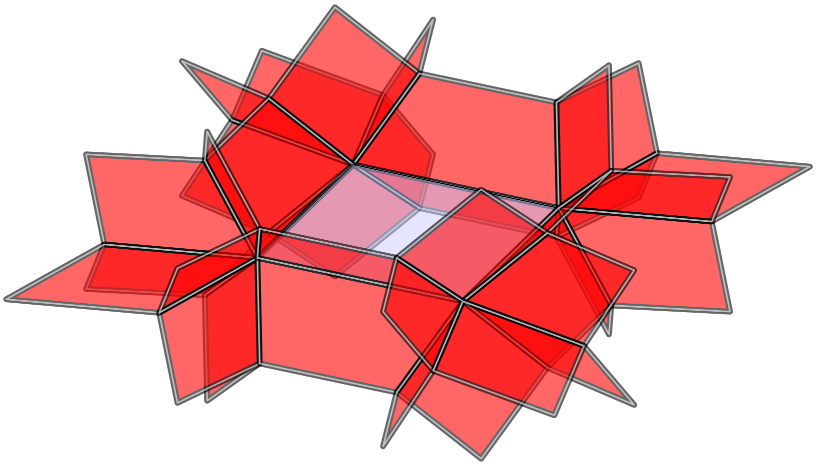


Figure 4.6: The incidence relations at a face of $\mathcal{G}$.

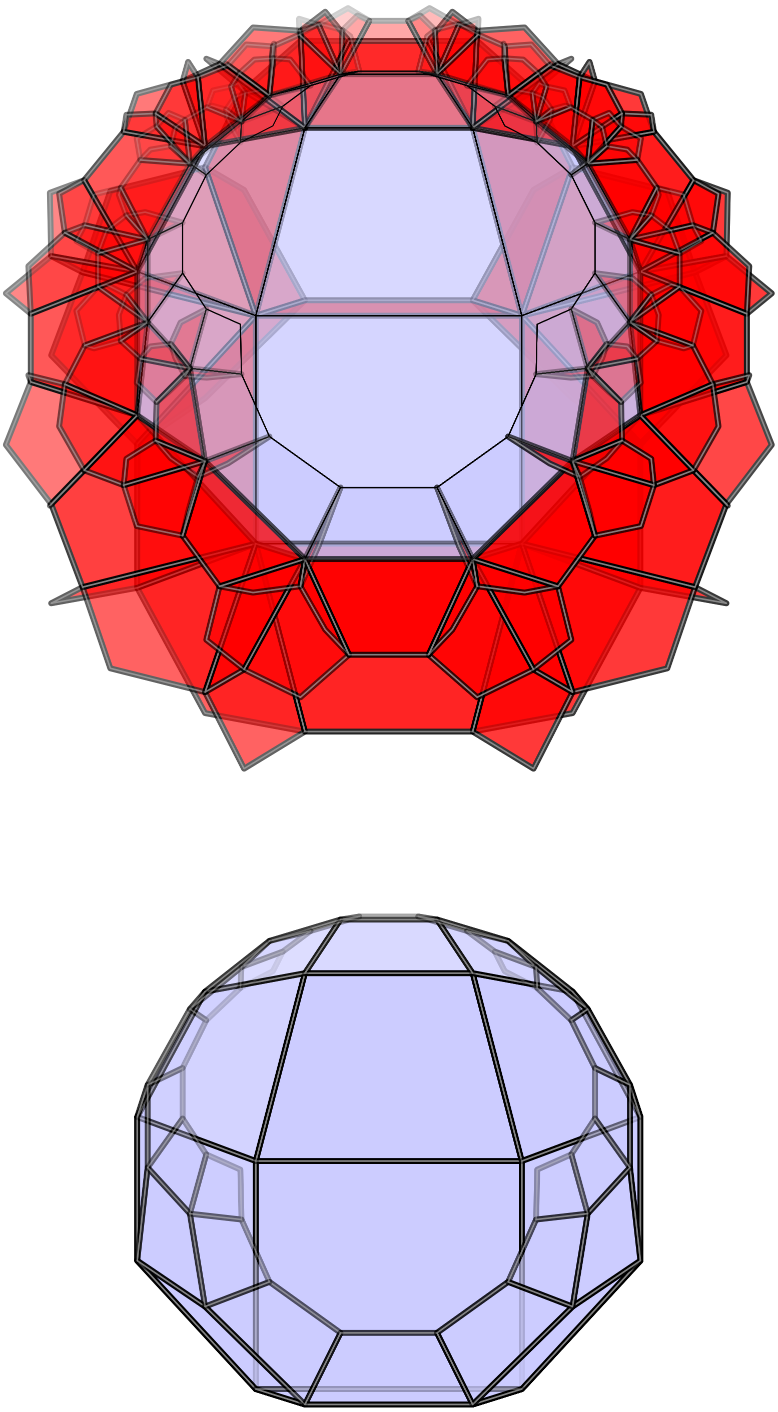


Figure 4.7: Above: a partial illustration of the incidence relations at a cell of $\mathcal{G}$. Below: the residue of a cell is isomorphic to the edge to edge tiling of the plane by squares.

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