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On Localizing Subcategories of the Derived Category of Smooth Mod-p Representations of a p-Adic Lie Group



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# On Localizing Subcategories of the Derived Category of Smooth Mod-$p$ Representations of a $p$-Adic Lie Group

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Mathematics

by

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2025

The dissertation of Bharatha Madusanka Rankothge is approved, and it is acceptable in quality and form for publication on microfilm and electronically.

University of California San Diego

2025

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DEDICATION

This thesis is dedicated to
my mother, Iresha Kumari;
my father, Priyantha Piyasiri;
and my partner, Zaithun Rafeeldeen.

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## VITA

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ABSTRACT OF THE DISSERTATION

**On Localizing Subcategories of the Derived Category of Smooth Mod-$p$ Representations of a $p$-Adic Lie Group**

by

Bharatha Madusanka Rankothge
Doctor of Philosophy in Mathematics

University of California San Diego, 2025

Professor Claus M. Sorensen, Chair

Understanding the category $\mathrm{Mod}_k(G)$ of the smooth representations of a $p$-adic Lie group $G$ over a field $k$ of characteristic $p$ is integral in developing the $p$-adic and mod-$p$ Langlands programs. However, the work of Peter Schneider, Matthew Emerton and others have suggested that we might need to shift our focus to the derived category $D(G)$ of $\mathrm{Mod}_k(G)$ in order to make further progress. Noting that $D(G)$ is a tensor triangulated category, we follow a common practice in studying tensor triangulated categories by attempting to classify the localizing tensor ideals of $D(G)$. In this thesis, we obtain such classifications when $G$ is compact and when $G$ is abelian with a Noetherian augmented Iwasawa algebra.

# Chapter 1

# Introduction

## 1.1 A Question and a Partial Answer

For the past few years, I have been interested in the following question.

Let $G$ be a $p$-adic Lie group and $k$ a field of characteristic $p$. Denote by $\mathrm{Mod}_k(G)$ the category of smooth $G$ representations over $k$. Let $D(G)$ be the unbounded derived category of $\mathrm{Mod}_k(G)$. What can we say about the localizing subcategories/tensor-ideals of $D(G)$?

While a general answer is yet to be found, at the moment, we can answer it in two interesting cases:

**Theorem 1.1.** *Let $G$ and $D(G)$ be as above.*

1. *When $G$ is compact, $D(G)$ does not have any localizing tensor-ideals.*

2. *When $G$ is abelian and the corresponding **augmented Iwasawa algebra** $k[[G]]$ is Noetherian, localizing subcategories of $D(G)$ are in bijection with the*

*subsets of the set of open prime ideals of* $k[[G]]$.

Before unpacking the question, first we discuss the motivation behind formulating this question in the first place.

## 1.2 Motivation

In this section we follow the conceptual evolution of our question.

### $p$-Adic Beginnings

Recall that for a prime $p$, the field of $p$-adic numbers, $\mathbb{Q}_p$, is the completion of $\mathbb{Q}$ with respect to the $p$-adic norm. Since they were introduced in early twentieth century by Kurt Hensel, partially as a framework to better explore local solutions to arithmetic equations, understanding $\mathbb{Q}_p$ and related $p$-adic Lie groups (which are groups with a manifold structure over $\mathbb{Q}_p$) like $GL_n(\mathbb{Q}_p)$ has played a fundamental role in number theory. This role has been further solidified by the current focus of the number theory community on the Langlands program.

### Langlands Connection

In 1967, Robert Langlands proposed a set of conjectures that sought to elucidate connections between number theory, representation theory, and algebraic geometry. The attempts to prove these conjectures in various settings, called Langlands program, have driven many of the advancements in modern number theory.

Given a finite extension $K$ of $\mathbb{Q}_p$, the local Langlands correspondence is a family of canonical bijections (for each $n$) between the set of isomorphism classes of irreducible admissible representations of $GL_n(K)$ over $\mathbb{C}$ and the set of isomorphism classes of $n$-dimensional Frobenius semi-simple Weil-Deligne representations of the Weil group of $K$ over $\mathbb{C}$ which preserves various analytic invariants. In the classical case, one considers representations over $\overline{\mathbb{Q}_l}$ for $l \neq p$ with the stipulation that the correspondence should be compatible with reduction modulo $l$. (See [7] for more details.)

## Mod-$p$ Complications

Problems arise when we try to replace the representations over $\mathbb{C}$ or $\mathbb{Q}_l$ by representations over $\mathbb{Q}_p$ or if we inquire about the mod-$p$ representations, i.e. representations over fields of characteristic $p$, in the above picture. It turns out that even the smooth mod-$p$ representations of $GL_2(K)$, when $K \neq \mathbb{Q}_p$ is much more complicated than expected (see [5]).

This is not entirely unexpected: even when you consider mod-$p$ representations of (finite) $p$-groups, basic results in complex representation theory like Maschke's theorem break down. In the case of mod-$p$ representations of $p$-adic groups, among other things, we lose a tool which is invaluable in the study of complex representations: there is no non-trivial $\mathbb{F}_p$-valued Haar measure ([14, Lemma 1.12])–which fundamentally changes the flavor of the subject.

## Need for a Derived Picture

It has been conjectured that in order to study the Langlands correspondence in the mod-$p$ setting, one might have to look at the derived picture (for example, in [13]). As we will see, Peter Schneider's work in [27] provides some supporting evidence for this.

Consider the complex case: Let $G$ be a unimodular $p$-adic group. By $C_c^\infty(G)$, we denote the space of functions $f : G \to \mathbb{C}$ which are locally constant and of compact support. By taking convolution (for some Haar measure $\mu$) as the binary operation, we can turn $C_c^\infty(G)$ into an associative $\mathbb{C}$-algebra called the Hecke algebra of $G$, denoted $\mathcal{H}(G)$. Given a compact open subgroup $I$ of $G$, we can define a function $e_I \in \mathcal{H}(G)$ as

$$e_I(x) = \begin{cases} \mu(I)^{-1} & \text{if } x \in I \\ 0 & \text{otherwise} \end{cases}$$

Now consider the subalgebra $\mathcal{H}(G, I) := e_I * \mathcal{H}(G) * e_I$ of $\mathcal{H}(G)$. Given a smooth representation $V$ of $G$, the $I$-invariants $V^I$ can be shown to be a $\mathcal{H}(G, I)$-module and it encapsulates significant information about the representation $V$ ([7, Proposition 4.3]):

**Proposition 1.2.** *The map $V \mapsto V^I$ induces a bijection between equivalence classes of irreducible smooth representations $V$ of $G$ such that $V^I \neq 0$ and isomorphism classes of simple $\mathcal{H}(G, I)$-modules.*

Now consider the general case (as explored in section 1 of [27]): Let $G$ be a $p$-adic Lie group with a compact open subgroup $I$. Let $\mathrm{Mod}_k(G)$ denote the category of smooth representations in $k$-vector spaces. Then,

$$\mathrm{ind}_I^G(1) := \{k\text{-valued functions with finite support on } G/I\}$$

is a compact object of $\mathrm{Mod}_k(G)$ which generates the full subcategory $\mathrm{Mod}_k^I(G)$ of all representations $V$ which are generated by the fixed vectors $V^I$. The Hecke algebra of $I$ is

$$\mathcal{H}_I := \mathrm{End}_{\mathrm{Mod}_k(G)}(\mathrm{ind}_I^G(1))^{\mathrm{op}}.$$

Let $\mathrm{Mod}(\mathcal{H}_I)$ denote the category of $\mathcal{H}_I$-modules. Consider the functor

$$\begin{aligned} H^0 : \mathrm{Mod}_k(G) &\longrightarrow \mathrm{Mod}(\mathcal{H}_I) \\ V &\longmapsto V^I \end{aligned}$$

If the characteristic of $k$ does not divide the pro-order of $I$ then the functor $H^0$ is exact. However, $H^0$ fails to be exact for any choice of $I$ when the characteristic of $k$ is $p$. This suggests that we should look at the derived picture of $H^0$ and consequently, at the appropriate derived categories of both sides.

Schneider shows that this is indeed worthwhile. Letting $D(G)$ be the unbounded derived category of $\mathrm{Mod}_k(G)$ and $D(\mathcal{H}_I^\bullet)$ be the derived category of differential graded left $\mathcal{H}_I^\bullet$-modules for the differential graded modification $\mathcal{H}_I^\bullet$ of $\mathcal{H}_I$, he proves that there is an equivalence between triangulated categories

$$D(G) \xrightarrow{\sim} D(\mathcal{H}_I^\bullet).$$

This establishes $D(G)$ as an object that is worth studying in its own.

## Tensor Triangulated Inspiration

Now, while $\mathrm{Mod}_k(G)$ is an abelian category, $D(G)$ is not abelian. However, by construction, any derived category of an abelian category is a triangulated category: exact triangles replace exact sequences. Furthermore, $D(G)$ is a monoidal category with a well-defined tensor product. So, it is an example of what is known as a tensor-triangulated category. Triangulated subcategories $\mathcal{P}$ of $\mathcal{C}$ which are closed under the tensor product (i.e. if $X \in \mathcal{P}$, for any $Y \in \mathcal{C}$, $X \otimes Y \in \mathcal{P}$) are called **tensor ideals**. If a tensor ideal is closed under taking direct summands, it is called **thick**. If it is closed under taking arbitrary direct sums, it is called **localizing**.

In his ICM address in 2010, Paul Balmer attempts to consolidate the study of various of tensor triangulated categories into a single framework. He does so by introducing a construction called the spectrum.

Given a tensor triangulated category, its *spectrum* is a topological space attached to it, in which each object would have a *support* that satisfy some desired properties. The existence of such a space is given by [3, Theorem 6]. One way to construct this space is using prime tensor ideals ([3, Construction 8]).

**Definition 1.1.** Let $\mathcal{C}$ be a tensor triangulated category. A thick tensor ideal $\mathcal{P} \subsetneq \mathcal{C}$ is called **prime** if it is proper ($\mathbb{1} \notin \mathcal{P}$) and if $a \otimes b \in \mathcal{P} \implies a \in \mathcal{P}$ or $b \in \mathcal{P}$. Then, the **spectrum** of $\mathcal{C}$ is the set of primes:

$$\mathrm{Spc}(\mathcal{C}) := \{\mathcal{P} \subsetneq \mathcal{C} \mid \mathcal{P} \text{ is prime}\}$$

and given $a \in \mathcal{C}$, the **support** of $a$ is

$$\operatorname{supp}(a) := \{\mathcal{P} \in \operatorname{Spc}(\mathcal{C} \mid a \notin \mathcal{P}\}.$$

The compliments $U(a) := \{\mathcal{P} \in \operatorname{Spc}(\mathcal{C} \mid a \in \mathcal{P}\}$, for all $a \in \mathcal{C}$, define an open basis of topology of $\operatorname{Spc}(\mathcal{C})$.

In [3, Remark 73], Balmer notes that $\operatorname{Spc}(\mathcal{C})$ is always the (algebraic) spectrum of *some* commutative ring. Furthermore, he shows that computation of $\operatorname{Spc}(\mathcal{C})$ is equivalent to the classification of thick tensor ideals of $\mathcal{C}$, via a bijection between *Thomason* subsets of the spectrum and *radical* thick tensor ideals of $\mathcal{C}$ ([3, Theorem 14]). Here, $Y \subset \operatorname{Spc}(\mathcal{C})$ is called **Thomason** if it is of the form $Y = \bigcup_{i \in I} Y_i$ with each complement $\operatorname{Spc}(\mathcal{C}) \backslash Y_i$ is open and quasi-compact. A thick tensor ideal $\mathcal{P}$ is called **radical**, if $a^{\otimes n} \in \mathcal{P} \implies a \in \mathcal{P}$.

While the set of localizing tensor ideals can be smaller than the set of thick tensor ideals, it can still inherit this connection to algebraic objects. The best example for this is the following result by Neeman ([23, Theorem 2.8]).

**Theorem 1.3.** *Let $R$ be a commutative Noetherian ring. Then there is a bijection*

$$\text{Localizing subcategories of } D(R) \rightleftarrows \text{Subsets of } \operatorname{Spec}(R)$$

This is the motivation behind our guiding question. Having established an interest in $D(G)$, we take an approach that is common in "tensor triangular geometry" and ask whether we can classify its localizing tensor ideals via a suitable algebraic structure.

## 1.3 Outline of the Thesis

We take two different approaches to classify localizing subcategories of $D(G)$ in the two cases we consider:

(i) $G$ is compact.

(ii) $G$ is abelian, possibly non-compact, with a Noetherian augmented Iwasawa algebra $k[[G]]$.

In the first case, we follow the strategy of David Benson et. al. in [4], where they show that $D(\mathrm{Mod}(k[G]))$ does not have any localizing tensor ideals when $k$ is a field of characteristic $p$ and $G$ is a finite group with $p$ dividing the order of $G$. We will also make use of a result by Peter Schneider in [27] which shows that $D(G)$ has a compact generator.

In the second case, we use Amnon Neeman's theorem from [23] which classifies the localizing subcategories of $D(\mathrm{Mod}(R))$ when $R$ is a commutative Noetherian ring. We will do so by identifying $D(G)$ as a full subcategory of $D(\mathrm{Mod}(k[[G]]))$ via a result of Matthew Emerton et. al. in [10]. Finally, we will show that there is a good class of examples that fall under case (ii).

Thus, the rest of the thesis is structured as follows.

Chapter 2: This chapter introduces the different kinds of categories (triangulated, derived, and tensor-triangulated) that we will be dealing with. We will also define what localizing subcategories/tensor-ideals are.

Chapter 3: This chapter continues setting up our question by introducing the category $\mathrm{Mod}_k(G)$ of smooth $k$-representations of a $p$-adic Lie group $G$. Then, using the results from the previous chapter, we construct $D(G)$. At the end of this chapter, all the terms in our guiding question are defined and explored.

Chapter 4: This chapter begins the work on the compact case (case (i) above) by discussing the case of a finite group. We also introduce the Iwasawa algebra $k[[G]]$ for a compact $p$-adic Lie group $G$.

Chapter 5: This chapter contains our proof of theorem 1.1 in the compact case.

Chapter 6: This chapter lays the groundwork for our work in case (ii). We first discuss the work of Neeman mentioned above. The rest of the chapter is devoted to defining the augmented Iwasawa algebra $k[[G]]$ for a general (potentially, non-compact) $p$-adic Lie group $G$ and topologizing it when $G$ is abelian.

Chapter 7: Here, we prove the theorem 1.1 in the case where $G$ is abelian with Noetherian $k[[G]]$ (case (ii) above) and exhibit some examples of groups that fall under this case.

Chapter 8: In the final chapter, we show that all $p$-adic tori belong to case (ii), i.e. we show that if $T$ is a torus over a $\mathbb{Q}_p$, then $k[[T(\mathbb{Q}_p)]]$ is Noetherian.

# Chapter 2

# A Category Theoretical Prelude

Our goal in this chapter is to introduce the category theoretical language that we will use throughout the thesis.

## 2.1 Triangulated Categories

We start by defining triangulated categories (following [18]) and their thick and localizing subcategories.

**Definition 2.1.** If a category $\mathcal{D}$ is endowed with a functor $T$ such that $T : \mathcal{D} \xrightarrow{\sim} \mathcal{D}$ is an equivalence of categories, $T$ is called a **translation functor** and $(\mathcal{D}, T)$ is called a **category with translation**.

**Definition 2.2.** Let $(\mathcal{D}, T)$ be an additive category with translation. A **triangle** in

$\mathcal{D}$ is a sequence of morphisms

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)\ .$$

A **morphism of triangles** is a commutative diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[r, "g"] \arrow[d, "\beta"'] & Z \arrow[r, "h"] \arrow[d, "\gamma"'] & T(X) \arrow[d, "T(\alpha)"'] \\ X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & T(X') \end{tikzcd}$$

**Definition 2.3.** An additive category with translation $(\mathcal{D}, T)$ is called a **triangulated category** if it is endowed with a class of triangles, called **distinguished triangles**, that satisfy the following axioms:

TR0 A triangle isomorphic to a distinguished triangle is also distinguished.

TR1 The triangle $X \xrightarrow{Id_X} X \to 0 \to T(X)$ is distinguished.

TR2 For all $f : X \to Y$, there exists a distinguished triangle $X \xrightarrow{f} Y \to Z \to T(X)$ (sometimes called Cone$(f)$).

TR3 A triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$ is distinguished if and only if $Y \xrightarrow{g} Z \xrightarrow{h} T(X) \xrightarrow{-f} T(Y)$ is distinguished.

TR4 Given two distinguished triangles $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$ and $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} T(X')$ with two morphisms $\alpha : X \to X'$ and $\beta : Y \to Y'$ such that $f' \circ \alpha = \beta \circ f$, there exists a morsphism $\gamma : Z \to Z'$ such that the following is a morphism of distinguished triangles:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[r, "g"] \arrow[d, "\beta"'] & Z \arrow[r, "h"] \arrow[d, "\gamma"'] & T(X) \arrow[d, "T(\alpha)"'] \\
X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & T(X')
\end{tikzcd}$$

TR5 Octahedral Axiom (see [18], Definition 10.1.6.): This states that given two cones, $\mathrm{Cone}(f)$ and $\mathrm{Cone}(f')$, there is a distinguished triangle relating them to $\mathrm{Cone}(f' \circ f)$.

**Definition 2.4.** Given two triangulated categories $(\mathcal{D}, T)$ and $(\mathcal{D}', T')$, An additive functor $F : \mathcal{D} \to \mathcal{D}'$ is called a **triangulated functor** if $F \circ T \simeq t' \circ F$ and $F$ sends distinguished triangles to distinguished triangles.

**Definition 2.5.** A subcategory $(\mathcal{C}, T')$ of a triangulated category $(\mathcal{D}, T)$ (here, $T'$ is the restriction of $T$ to $\mathcal{C}$) is called a **triangulated subcategory** if it is triangulated and the inclusion functor is triangulated.

**Definition 2.6.** A subcategory $\mathcal{C}$ of the category $\mathcal{D}$ is called

- **replete** if for any given isomorphism $f : X \xrightarrow{\sim} Y$ in $\mathcal{D}$, if $X$ is an object of $\mathcal{C}$, then so is $Y$ and $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$;
- **full** if for any two objects $X$ and $Y$ in $\mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(X, Y) = \mathrm{Hom}_{\mathcal{D}}(X, Y)$.

*Remark.* If $\mathcal{C}$ is a replete, full subcategory of $\mathcal{D}$, $\mathcal{C}$ is a triangulated subcategory of $\mathcal{D}$ if and only if

- $\mathcal{C}$ is closed under translation and

- for every distinguished triangle $X \to Y \to Z \to T(X)$ in $\mathcal{D}$, if two of $X, Y$, and $Z$ belong to $\mathcal{C}$, so does the third.

**Definition 2.7.** Let $\mathcal{D}$ be a triangulated category. A replete, full, triangulated subcategory of $\mathcal{D}$ is called **thick** if it is closed under taking direct summands.

**Definition 2.8.** Let $\mathcal{D}$ be a triangulated category that has arbitrary direct sums. A replete, full, thick, triangulated subcategory of $\mathcal{D}$ is called **localizing** if it is closed under taking direct sums.

Using "Eilenberg swindle", we can show that the thickness assumption in the above definition is unnecessary.

**Proposition 2.1.** *Let $\mathcal{D}$ be a triangulated subcategory that has arbitrary direct sums. If $\mathcal{C}$ is a replete, full, triangulated subcategory of $\mathcal{D}$ that is closed under taking direct sums, then $\mathcal{C}$ is thick.*

*Proof.* Let $X$ and $Y$ be objects in $\mathcal{D}$ such that $X \oplus Y \in \mathcal{C}$. Then, we have an isomorphism (Eilenberg swindle)

$$(X \oplus Y) \oplus (X \oplus Y) \oplus ... \simeq X \oplus (Y \oplus X) \oplus (Y \oplus X) \oplus ...$$

Observe that for any object $Z$ in $\mathcal{D}$, $X \to X \oplus Z \to Z \to T(X)$ is a distinguished triangle (Lemma 13.4.11 in [32]). Therefore, the above isomorphism gives rise to a distinguished triangle

$$X \to \bigoplus_{i \in \mathbb{N}} (X \oplus Y) \to \bigoplus_{j \in \mathbb{N}} (X \oplus Y) \to T(X)$$

in $\mathcal{D}$. Since the second and third terms of that triangle are in $\mathcal{C}$, so is the first.

□

## 2.2 Derived Categories

Given an abelian category $\mathcal{A}$, its homotopy category $K(\mathcal{A})$ and its derived category $D(\mathcal{A})$ are examples of triangulated categories. In this section, we briefly outline their constructions following [17] and [11]. Alternative expositions can be found in [18] and [20].

Let $\mathcal{A}$ be an abelian category.

**Definition 2.9.** A **complex** over $\mathcal{A}$ is a sequence of morphisms

$$X^\bullet : ... \to X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \to ...$$

such that $d^n \circ d^{n-1} = 0$ for all $n \in \mathbb{Z}$. Thus the complex $X^\bullet$ can be thought of as a graded object with **differential** $d$.

Given an object $Y$ in $\mathcal{A}$, denote by $Y[0]$ the complex $... \to 0 \to Y \to 0 \to ...$ with $Y$ at degree 0.

**Definition 2.10.** Given two complexes $X^\bullet$ and $Y^\bullet$ over $\mathcal{A}$, a **morphism of complexes** $f : X^\bullet \to Y^\bullet$ (also called a **chain map**) is a collection of morphisms $f^n : X^n \to Y^n$ such that $d_Y^n \circ f^n = f^{n+1} \circ d_X^n$, i.e. a commutative diagram

$$\begin{tikzcd}
X^\bullet : \arrow[d, "f"'] & \ldots \arrow[r] & X^{n-1} \arrow[r, "d_X^{n-1}"] \arrow[d, "f^{n-1}"'] & X^n \arrow[r, "d_X^n"] \arrow[d, "f^n"'] & X^{n+1} \arrow[r] \arrow[d, "f^{n+1}"'] & \ldots \\
Y^\bullet : & \ldots \arrow[r] & Y^{n-1} \arrow[r, "d_Y^{n-1}"] & Y^n \arrow[r, "d_Y^n"] & X^{n+1} \arrow[r] & \ldots
\end{tikzcd}$$

The complexes over $\mathcal{A}$ along with the chain maps as morphisms form the category $C(\mathcal{A})$ of complexes.

**Proposition 2.2.** *When $\mathcal{A}$ is abelian, so is $C(\mathcal{A})$.*

*Proof.* This is [17, Proposition 2.5]. □

We list some terminology describing various complexes.

**Definition 2.11.** A complex $X^\bullet$ is called **cyclic** if all its differentials $(d^n)$ are zero.

**Definition 2.12.** Given a complex $X^\bullet$, we define its **$i$-th cohomology group** as $H^i(X^\bullet) = \mathrm{Ker}(d^i)/\mathrm{Im}(d^{i-1})$.

**Definition 2.13.** A complex $X^\bullet$ is called **acyclic** if all its cohomologies are zero.

**Definition 2.14.** A chain map $f : X^\bullet \to Y^\bullet$ is called a **quasi-isomorphism** if the induced maps $H^i(f) : H^i(X^\bullet) \to H^i(Y^\bullet)$ are isomorphisms for each $i$.

The derived category of $\mathcal{A}$ can be defined as an category that embodies a universal property about quasi-isomorphisms as in [11, Definition/Theorem III.2.1]:

**Theorem 2.3.** *Let $\mathcal{A}$ be an abelian category and $C(\mathcal{A})$ its category of complexes. Then, there exists a category $D(\mathcal{A})$ and a functor $Q : C(\mathcal{A}) \to D(\mathcal{A})$ with the following properties:*

1. *For any quasi-isomorphism $f$, $Q(f)$ is an isomorphism.*

2. *Any functor $F : C(\mathcal{A}) \to \mathcal{D}$ transforming quasi-isomorphisms to isomorphisms can be uniquely factorized through $D(\mathcal{A})$.*

More concretely, we can construct $D(\mathcal{A})$ as the localization of the homotopy category $K(\mathcal{A})$ by quasi-isomorphisms.

**Definition 2.15.** Two chain maps $f, g : X^\bullet \to Y^\bullet$ are called **homotopic** (or **homotopy equivalent**) if there is a collection of morphisms $\rho^n : X^n \to Y^{n-1}$ such that $f^n - g^n = d_Y^{n-1} \circ \rho^n + \rho^{n+1} \circ d_X^n$.

$$\begin{tikzcd}
\cdots \arrow[r] & X^{n-1} \arrow[r, "d_X^n"] \arrow[d, shift right, "f^{n-1}"'] \arrow[d, shift left, "g^{n-1}"] & X^n \arrow[r, "d_X^{n+1}"] \arrow[dl, "\rho^n"'] \arrow[d, shift right, "f^n"'] \arrow[d, shift left, "g^n"] & X^{n+1} \arrow[r] \arrow[dl, "\rho^{n+1}"'] \arrow[d, shift right, "f^{n+1}"'] \arrow[d, shift left, "g^{n+1}"] & \cdots \\
\cdots \arrow[r] & Y^{n-1} \arrow[r, "d_Y^n"'] & Y^n \arrow[r, "d_Y^{n+1}"'] & Y^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

**Definition 2.16.** The **homotopy category** $K(\mathcal{A})$ is the category with same objects as $C(\mathcal{A})$ but with homotopy classes of chain maps as morphisms.

$K(\mathcal{A})$ can fail to be abelian even when $\mathcal{A}$ is abelian. An example is when $\mathcal{A}$ is the abelian category **Ab** of abelian groups (see [17, Example 2.6]). However, $K(\mathcal{A})$ is always a triangulated category. To describe the distinguished triangles in $K(\mathcal{A})$, we should first introduce shift and cone operations on complexes.

**Definition 2.17.** We define the **shift** functor in $C(\mathcal{A})$ as the functor taking a given complex $X^\bullet = (X^n, d_X^n)_{n\in\mathbb{Z}}$ to $X^\bullet[1]$ and a chain map $f = (f^n)_{n\in\mathbb{Z}}$ to $f[1]$, where

$$X^\bullet[1] := (X[1]^n, d_{X[1]}^n)_{n\in\mathbb{Z}} \quad \text{with} \quad X[1]^n = X^{n-1} \quad \text{and} \quad d_{X[1]}^n = -d_X^{n-1}$$

$$\text{and} \quad f[1] := (f[1]^n)_{n\in\mathbb{Z}} \quad \text{where} \quad f[1]^n = f^{n-1},$$

which is to say that it shifts $X^\bullet$ one degree to the left and reverses the sign of the differential.

**Definition 2.18.** Let $f : X^\bullet \to Y^\bullet$ be a chain map. The **mapping cone** $M(f)$ is the complex in $C(\mathcal{A})$ defined by

$$M(f)^n := X^{n-1} \oplus Y^n \quad \text{and} \quad d^n_{M(f)} = \begin{pmatrix} -d_X^{n-1} & 0 \\ f^{n-1} & d_Y^n \end{pmatrix},$$

i.e. $d^n_{M(f)}(x, y) = (-d_X^{n-1}(x), f^{n-1}(x) + d_Y^n(y))$.

**Definition 2.19.** A distinguished triangle in $K(\mathcal{A})$ is a triangle which is isomorphic in $K(\mathcal{A})$ to a triangle of the form

$$X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{\alpha(f)} M(f) \xrightarrow{\beta(f)} X^\bullet[1]$$

where $\alpha(f)^n := (0, \mathrm{id}_{Y^n})$ and $\beta(f)^n := (\mathrm{id}_{X^{n-1}}, 0)$.

**Theorem 2.4.** *With the above choice of distinguished triangles, $K(\mathcal{A})$ is a triangulated category.*

*Proof.* This is [17, Theorem 6.7]. □

To construct $D(\mathcal{A})$, let $S$ denote the class of quasi-isomorphisms in $K(\mathcal{A})$. Then, we have the following result.

**Theorem 2.5.** *$D(\mathcal{A})$ as defined in theorem 2.3 is canonically isomorphic to the category $K(\mathcal{A})[S^{-1}]$ which has the same objects as $K(\mathcal{A})$ and morphisms given by the equivalence classes of 'roofs' $(s, f)$ of the form*

$$\begin{tikzcd}
& A \arrow[dl, "s"'] \arrow[dr, "f"] & \\
X & & Y
\end{tikzcd}$$

*where $s$ and $f$ are morphisms in $K(\mathcal{A})$ and $s \in S$. They satisfy the following conditions.*

1. *Two roofs $(s, f)$ and $(t, g)$ are equivalent if and only if there exists a third roof forming a commutative diagram of the form*

$$\begin{tikzcd}
& & X''' \arrow[dl, "r"'] \arrow[dr, "h"] & & \\
& X' \arrow[dl, "s"'] \arrow[drrr, "f" near end] & & X'' \arrow[dlll, "t"' near end] \arrow[dr, "g"] & \\
X & & & & Y
\end{tikzcd}$$

2. *The composition of roofs $(s, f)$ and $(t, g)$ is the class of the roof $(st, gf)$ obtained as*

$$\begin{tikzcd}
& & X'' \arrow[dl, dashed, "t'"'] \arrow[dr, dashed, "f'"] & & \\
& X' \arrow[dl, "s"'] \arrow[dr, "f"] & & Y' \arrow[dl, "t"'] \arrow[dr, "g"] & \\
X & & Y & & Z
\end{tikzcd}
\quad \Longrightarrow \quad
\begin{tikzcd}
& X'' \arrow[dl, "st'"'] \arrow[dr, "gf'"] & \\
X & & Y
\end{tikzcd}$$

*Proof.* This follows from [11, Lemma III.2.8 and Proposition III.4.2 ] or [17, Proposition 7.13]. □

**Definition 2.20.** A triangle in $D(\mathcal{A})$ is distinguished if it is isomorphic in $D(\mathcal{A})$ to the image of a distinguished triangle from $K(\mathcal{A})$ under the localization functor $L : K(\mathcal{A}) \to D(\mathcal{A})$.

Finally, we have the following theorem which is [11, Corollary IV.2.7].

**Theorem 2.6.** *$D(\mathcal{A})$ is triangulated (with respect to the above choice of distinguished triangles).*

## 2.3 Tensor Triangulated Categories

In this section, we follow [3] for the definitions. In addition to [3], [33] also contains a good overview of the material.

**Definition 2.21.** A **monoidal category** is a category $\mathcal{K}$, equipped with a bifunctor $\otimes : \mathcal{K}\times\mathcal{K} \to \mathcal{K}$ (tensor product), an object $\mathbb{1}$ (tensor unit), and natural isomorphisms

- $\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$
- $\lambda_X : \mathbb{1} \otimes X \xrightarrow{\sim} X$
- $\rho_X : X \otimes \mathbb{1} \xrightarrow{\sim} X$ satisfying certain coherence conditions.

It is **symmetric** if there is a natural isomorphism $\beta_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$.

**Definition 2.22.** A **tensor triangulated category** $(\mathcal{K}, \otimes, \mathbb{1})$ is a triangulated category $\mathcal{K}$ equipped with a symmetric monoidal structure $\mathcal{K} \times \mathcal{K} \xrightarrow{\otimes} \mathcal{K}$ with a unit object $\mathbb{1}$ such that for any $K \in \mathcal{K}$, the endofunctors $- \otimes K$ and $K \otimes -$ are triangulated.

**Definition 2.23.** Let $\mathcal{K}$ be a tensor triangulated category. A full triangulated subcategory $\mathcal{J}$ of $\mathcal{K}$ is called a **tensor ideal** if for any $K \in \mathcal{K}$ and $X \in \mathcal{J}$, $K \otimes X$ is in $\mathcal{J}$.

We can define **thick tensor ideals** and **localizing tensor ideals** as tensor ideals that are thick and localizing (as triangulated subcategories) respectively.

# Chapter 3

# Mod-$p$ Representation Theory of $p$-Adic Lie Groups

With the category theoretical preliminaries out of the way, now we can introduce our main object of study: $D(G)$, where $G$ is $p$-adic Lie group. We will first define what we mean by a $p$-adic Lie group and then discuss the category $\mathrm{Mod}_k(G)$ of smooth mod-$p$ representations of $G$ over $k$. Lastly, we will define $D(G)$ as the derived category of $\mathrm{Mod}_k(G)$.

## 3.1 Structure of $p$-Adic Lie Groups

We follow [26] and [8] for the definitions and theorem statements. However, as those authors themselves acknowledge, many of the important results are due to Michel Lazard.

**Definition 3.1.** A **$p$-adic Lie group** is a manifold over $\mathbb{Q}_p$ which also carries the structure of a group such that the multiplication map

$$\begin{aligned} G \times G &\longrightarrow G \\ (g,h) &\longmapsto gh \end{aligned}$$

is locally analytic.

*Remark.* Definition of a Lie group usually requires the inverse map $g \mapsto g^{-1}$ to be locally analytic as well. But the analyticity of the inverse map can be obtained as a consequence of the above definition. See [26, Proposition 13.6].

**Definition 3.2.** Let $G$ be a $p$-adic Lie group. The **dimension** of $G$ is its dimension as a manifold over $\mathbb{Q}_p$, which is well-defined (see [26, Corollary 13.3]).

*Example* 3.1. The general linear group $GL_n(\mathbb{Q}_p) = \det^{-1}(\mathbb{Q}_p^\times)$ is a $\mathbb{Q}_p$-analytic group which is an open subset of $M_n(\mathbb{Q}_p)$ ($\cong \mathbb{Q}_p^{n^2}$), the space of $n \times n$ matrices over $\mathbb{Q}_p$. Therefore, it is a $p$-adic Lie group of dimension $n^2$.

Recall that a **profinite group** is a topological group that is compact, Hausdorff, and totally disconnected.

**Proposition 3.1.** *$p$-adic Lie groups are locally profinite.*

*Proof.* Given any $p$-adic Lie group $G$, it is locally homeomorphic to $\mathbb{Q}_p^n$ for some $n$. $\mathbb{Q}_p^n$ is Hausdorff, locally compact, and totally disconnected. Therefore, so is $G$, which means that $G$ is locally profinite. $\square$

**Corollary 3.2.** *Compact $p$-adic Lie groups are profinite.*

We have the following useful characterization of profinite groups ([8, Proposition 1.3]).

**Proposition 3.3.** *$G$ is a profinite group if and only if it is topologically isomorphic to the inverse limit $\varprojlim_{U \in \mathcal{N}} G/U$ where $\mathcal{N}$ is the set of all open normal subgroups of $G$.*

**Definition 3.3.** A profinite group is called **pro-$p$** if every normal open subgroup of it has index equal to a power of $p$.

Pro-$p$ groups play an important role in Lazard's characterization of $p$-adic Lie groups.

**Theorem 3.4.** *A topological group $G$ is a $p$-adic Lie group if and only if it has an open subgroup which is pro-$p$ group of finite rank.*

*Proof.* This is [8, Corollary 8.33]. □

Among other things, this has the following consequence that we will be using in chapter 5.

**Corollary 3.5.** *Given a $p$-adic Lie group $G$, one can always find an open subgroup $U$ which is pro-$p$ and torsion-free.*

*Proof.* This follows from the above theorem and [8, Propositions 4.3 and 4.5]. □

## 3.2 Smooth Representations

Proofs and references for all of the following results can be found in [14].

Let $G$ be a $p$-adic Lie group and $k$ a field of characteristic $p$.

**Definition 3.4.** We say a representation $V$ of $G$ over $k$ is **smooth** if any one of the following equivalent conditions hold:

1. For any $x \in V$, there exists an open subgroup $U$ of $G$ such that $x \in V^U$. (Here, $V^U := \{v \in V : u \cdot v = v \text{ for all } u \in U\}$.)

2. The action map $G \times V \longrightarrow V$ is continuous, with $V$ having discrete topology.

3. The stabilizer of any vector $v$ in $G$ is open.

4. $V = \bigcup_U V^U$, where $U$ runs through all the compact open subgroups of $G$.

**Definition 3.5.** Given any representation $V$ of $G$, we can define its smooth part as $V^\infty := \bigcup_U V^U$, where $U$ runs through all the compact open subgroups of $G$.

Let $V$ be a smooth $G$-representation (over $k$) endowed with discrete topology.

**Definition 3.6.** A function $f : G \longrightarrow V$ is **smooth** if there exists an open subgroup $C$ of $G$ such that for any $g \in G$, $f(cg) = f(g)$ for all $c \in C$.

When $G$ is compact, smooth functions and continuous functions are interchangeable.

**Lemma 3.6.** *Let $G$ be a compact $p$-adic Lie group and $V$ a smooth $G$-representation with discrete topology. Consider a function $f : G \longrightarrow V$. Then,*

$$f \text{ is smooth} \iff f \text{ is continuous}$$

The following result is sometimes useful.

**Lemma 3.7.** *Let $G$ be a compact $p$-adic Lie group, $k$ a field of characteristic $p$, and $V$ a smooth $G$-representation over $k$. Then,*

$$f : G \longrightarrow V \text{ is smooth } \implies f \text{ is locally constant and has finite range.}$$

Induced representations are important constructions in representation theory.

**Definition 3.7.** Let $G$ be a $p$-adic Lie group. Let $H$ be a closed subgroup of $G$ and let $V$ be a smooth representation of $H$.

1. The **smooth induction** is

   $$\mathrm{Ind}_H^G(V) := \left(\{f : G \to V : f(hg) = h \cdot f(g) \text{ for all } h \in H \text{ and } g \in G\}\right)^{\infty}.$$

   Concretely, this is the space of functions $f : G \to V$ such that

   - $f(hg) = h \cdot f(g)$ for all $h \in H$ and $g \in G$; and
   - there is an open subgroup $U$ of $G$, such that $f(gu) = f(g)$ for all $u \in U$ and $g \in G$.

   $G$ acts on $\mathrm{ind}_H^G(V)$ by left translation: $(g \cdot f)(\gamma) = f(g^{-1}\gamma)$.

2. The **compact induction** is

   $$\mathrm{ind}_H^G(V) := \{f \in \mathrm{Ind}_H^G(V) : \text{the image of } \mathrm{supp}(f) \text{ is compact in } H\backslash G\},$$

   where $\mathrm{supp}(f) = \{g \in G : f(g) \neq 0\}$.

Both $\mathrm{Ind}_H^G(V)$ and $\mathrm{ind}_H^G(V)$ are smooth representations of $G$.

We single out an important example.

**Definition 3.8.** For a smooth representation $V$ of $G$,

$$\mathrm{Ind}_e^G(V) = \{\text{all smooth functions } f : G \longrightarrow V\}.$$

When $V$ is the trivial representation, we get

$$\mathrm{Ind}_e^G(1) := \{\text{all smooth functions } f : G \longrightarrow k\}.$$

We make the following observations.

**Proposition 3.8.** *1. The functors* $\mathrm{Ind}_H^G$ *and* $\mathrm{ind}_H^G$ *are left exact.*

*2.* $\mathrm{ind}_H^G(V) \cong k[G] \otimes_{k[H]} V$*.*

*3. If* $H\backslash G$ *is compact, then* $\mathrm{Ind}_H^G = \mathrm{ind}_H^G$*.*

The following results are called **Frobenius reciprocity** and reflect adjunction properties.

**Proposition 3.9.** *Let* $H$ *be a closed subgroup of* $G$*,* $V$ *a smooth* $G$*-representation, and* $W$ *a smooth* $H$*-representation. Then, we have natural isomorphisms*

*1.* $\mathrm{Hom}_G(V, \mathrm{Ind}_H^G(W)) \cong \mathrm{Hom}_H(V|_H, W)$

*2. If* $H$ *is open in* $G$*,* $\mathrm{Hom}_G(\mathrm{ind}_H^G(W), V) \cong \mathrm{Hom}_H(W, V|_H)$*. Moreover,* $\mathrm{ind}_H^G$ *is exact in this case.*

The space of continuous functions also form a smooth representation of $G$.

**Definition 3.9.** For a smooth representation $V$ of $G$, define

$$C^\infty(G, V) := \{\text{all continuous functions } f : G \longrightarrow V\}.$$

This is a $G$-representation with the $G$-action

$$(g \cdot f)(\gamma) = g \cdot f(g^{-1}\gamma)$$

As expected, lemma 3.6 extends to an isomorphism of $G$-representations when $G$ is compact.

**Lemma 3.10.** *Given a smooth $G-$representation $V$,*

$$C^\infty(G, V) \xrightarrow{\sim} \mathrm{Ind}_e^G(V)$$

*(as $G$-representations).*

## 3.3 Mod($G$) and $D(G)$

Let $G$ be a $p$-adic Lie group and $k$ a field of characteristic $p$ as before.

**Definition 3.10.** We denote by $\mathrm{Mod}_k(G)$ the category of smooth representations of $G$ over $k$.

The construction of $D(G)$ from $\mathrm{Mod}_k(G)$ is discussed in [27] in detail. Following are some highlights.

**Proposition 3.11.** *(i)* $\mathrm{Mod}_k(G)$ *is abelian and has arbitrary limits and colimits.*

*(ii)* $\mathrm{Mod}_k(G)$ *has enough injective objects.*

*(iii)* $\mathrm{Mod}_k(G)$ *is a Grothendieck category.*

*Proof.* This is [27, Lemma 1]. The generator for (iii) is given by

$$Y = \bigoplus_j \mathrm{Ind}_J^G(1)$$

where $J$ runs over all the open subgroups of $G$. □

The previous result allows us to construct the derived category $D(\mathrm{Mod}_k(G))$ of unbounded complexes in $\mathrm{Mod}_k(G)$. Let $D(G) := D(\mathrm{Mod}_k(G))$. From [27, Remark 2] we have:

**Proposition 3.12.** *$D(G)$ has arbitrary direct sums.*

Let $G$ be a $p$-adic Lie group and $I$ a pro-$p$ torsion-free open subgroup. The following object plays an important role in describing the structure of $D(G)$.

**Definition 3.11.** In $\mathrm{Mod}_k(G)$, we have a representation

$$\mathrm{ind}_I^G(1) := \{k\text{-valued functions with finite support on } G/I\}$$

where $G$ acts by left translations.

Let $\mathbb{X} := \mathrm{ind}_I^G(1)[0]$ denote the complex in $D(G)$ with $\mathrm{ind}_I^G(1)$ at degree 0.

**Lemma 3.13.** *$\mathbb{X}$ is a compact object in $D(G)$.*

$\mathbb{X}$ generates $D(G)$ in the following sense.

**Proposition 3.14.** *Any localizing subcategory of $D(G)$ which contains $\mathbb{X}$ coincides with $D(G)$.*

*Proof.* This is [27, Proposition 6]. $\square$

Finally, we note that

**Proposition 3.15.** *$D(G)$ is a tensor triangulated category.*

*Proof.* $D(G)$ is triangulated by its construction as the derived category of an abelian category. It also has a closed symmetric monoidal structure as shown in [28, Corollary 3.2]. Consider the tensor product functor on $\mathrm{Mod}_k(G)$ with $G$ acting diagonally on the tensor product ($g \cdot v_1 \otimes v_2 = g \cdot v_1 \otimes g \cdot v_2$):

$$\begin{aligned} \mathrm{Mod}_k(G) \times \mathrm{Mod}_k(G) &\longrightarrow \mathrm{Mod}_k(G) \\ (V_1, V_2) &\longmapsto V_1 \otimes_k V_2. \end{aligned}$$

This is exact in both variables and therefore extends without derivation to $D(G)$. So, we have a tensor product functor

$$\begin{aligned} D(G) \times D(G) &\longrightarrow D(G) \\ (V_1^\bullet, V_2^\bullet) &\longmapsto \mathrm{tot}_{\oplus}(V_1^\bullet \otimes_k V_2^\bullet). \end{aligned}$$

The unit object $\mathbb{1}$ is given by the complex $k[0]$ with $k$ concentrated at degree 0. $\square$

With all the required terms defined and the context been established, we can now pose our guiding question.

## 3.4 Question

We are interested in the following question:

> Let $G$ be a $p$-adic Lie group and $k$ a field of characteristic $p$. Denote by $\mathrm{Mod}_k(G)$ the category of smooth $G$ representations over $k$. Let $D(G)$ be the unbounded derived category of $\mathrm{Mod}_k(G)$. Can we classify the localizing subcategories/tensor-ideals of $D(G)$?

First, we will answer this question when $G$ is compact.

# Chapter 4

# Compact Groups

Our strategy for studying the localizing subcategories of $D(G)$ when $G$ is compact follows from the work of David Benson et al. in [4] in the case of finite $G$. For them, it is just a preliminary step before going on to consider a different category called the stable module category.

## 4.1 Finite Groups

Let $G$ be a finite group with $p$ dividing its order and let $k$ be a field of characteristic $p$. Since $G$ is finite, it has discrete topology. Therefore, every $k$-representation of $G$ is smooth. So, letting $k[G]$ be the group ring of $G$ over $k$, we can canonically identify $\mathrm{Mod}_k(G)$ with $\mathrm{Mod}(k[G])$, the category of all $k[G]$ modules. Then, $D(G)$ is nothing other than $D(\mathrm{Mod}(k[G]))$. We then have the following proposition.

**Proposition 4.1.** $D(\mathrm{Mod}(k[G]))$ *does not have any non-zero proper localizing tensor ideals.*

*Proof.* This follows from the proof of [4, Proposition 2.2]. We will carefully go through their argument to isolate a strategy. The argument is made up of two steps:

Step 1 Every object in $D(\mathrm{Mod}(k[G]))$ is in the localizing subcategory generated by $k[G][0]$.

Step 2 Any nonzero localizing tensor ideal of $D(\mathrm{Mod}(k[G]))$ contains $k[G][0]$.

For the second step, they argue as follows. Given a nonzero $X$ in $D(\mathrm{Mod}(k[G]))$, it must have nonzero cohomology at some degree, say $i$. Now, we have that $H^i(X \otimes_k k[G])$ is isomorphic to $H^i(X) \otimes_k k[G]$. Since for any nonzero $k[G]$-module $M$, the $k[G]$-module $M \otimes_k k[G]$ is free and nonzero, $H^i(X \otimes_k k[G])$ is a nonzero direct sum of copies of $k[G]$. This implies that in $D(\mathrm{Mod}(k[G]))$, $X \otimes_k k[G]$ is isomorphic to a direct sum of copies of shifts of the complex $k[G][0]$. Therefore, $k[G][0]$ is in any localizing tensor ideal containing $X$. $\square$

We will see that the same strategy works when $G$ is compact. That is we will show that

Step 1 Every object in $D(G)$ is in the localizing subcategory generated by $\mathbb{X} := \mathrm{ind}_I^G(1)[0]$.

Step 2 Any nonzero localizing tensor ideal of $D(G)$ contains $\mathbb{X}$.

before delving into the proof however, we need to define a ring that captures the nature of a profinite group better than the usual group ring.

## 4.2 Iwasawa Algebra: $k[[G]]$

Recall that if $G$ is a compact $p$-adic group, then it is profinite. Therefore, it can be written as an inverse limit of finite quotient groups. The Iwasawa algebra is constructed by taking the inverse limit of the group rings of those quotients.

**Definition 4.1.** Let $G$ be a profinite group, with

$$G \cong \varprojlim_{U \in \mathrm{Op}(G)} G/U$$

where $\mathrm{Op}(G)$ denotes the set of open normal subgroups of $G$. Then, the **Iwasawa algebra** of $G$ (also called the **completed group ring** of $G$) is

$$k[[G]] := \varprojlim_{U \in \mathrm{Op}(G)} k[G/U].$$

Then $k[[G]]$ is a topological ring with inverse limit topology with each group ring $k[G/U]$ endowed with discrete topology. The following important result follows from Lazard's work.

**Theorem 4.2.** *Let $G$ be a compact $p$-adic Lie group and $k$ a field of characteristic $p$. Then, $k[[G]]$ is Noetherian.*

*Proof.* This is [9, Theorem 2.1.2]. $\square$

$k[[G]]$ modules come up naturally when studying smooth representations.

**Proposition 4.3.** *Let $G$ be a compact $p$-adic Lie group and $k$ a field of characteristic $p$. Any smooth $k$-representation of $G$ has a natural $k[[G]]$-module structure.*

*Proof.* Let $\mathcal{N}$ denote the set of open normal subgroups of $G$. Then, given a smooth $k$-representation $V$ of $G$, we can write $V = \varinjlim_{U \in \mathcal{N}} V^U$. Then, $k[[G]]$ acts on $V^U$, via the projection $k[[G]] \to k[G/U]$, which turn $V$ into a $k[[G]]$-module. $\square$

Finally, we note the following correspondence between $\mathrm{Mod}_k(G)$ and a subcategory of $k[[G]]$-modules. See section 1 of [19] for more details.

**Definition 4.2.** A topological $k$-vector space is called **pseudocompact** if it is the topological projective limit of finite dimensional $k$-vector spaces.

**Definition 4.3.** Let $\mathrm{Pct}_k(G)$ denote the category of pseudocompact $k$-vector spaces $M$ carrying a $k$-linear action of $G$ for which the action map $G \times M \to M$ is continuous with respect to the product topology on the left.

This category is related to $k[[G]]$ in the following way. First we recall a couple of definitions from [6].

**Definition 4.4.** A ring $A$ is called a **pseudocompact ring** if it is a complete Hausdorff topological ring which admits a system of open neighborhoods of 0 consisting of two-sided ideals $I$ for which $A/I$ is Artinian.

**Definition 4.5.** Let $A$ be a pseudocompact ring. Then, a complete Hausdorff topological $A$-module $M$ is called a **pseudocompact module** if it has a system of open neighborhoods of 0 consisting of submodules $N$ for which $M/N$ has finite length.

**Proposition 4.4.** *Let $G$ be a compact $p$-adic Lie group and $k$ a field of characteristic $p$. Then, $k[[G]]$ is a pseudocompact ring and $\mathrm{Pct}_k(G)$ is the category of pseudocompact $k[[G]]$-modules.*

*Proof.* This is [19, Corollary 1.4]. □

The promised correspondence, which is stated below, is called **Pontryagin duality**.

**Theorem 4.5.** *1. Given $V \in \mathrm{Mod}_k(G)$, if $(V_i)_{i\in I}$ is the family of its finite dimensional subspaces,*

$$V^\vee := \mathrm{Hom}_k(V,k) \cong \varprojlim_{i\in I} \mathrm{Hom}_k(V_i,k) \in \mathrm{Pct}_k(G).$$

*2. Given $M \in \mathrm{Pct}_k(G)$, let $M^\vee := \mathrm{Hom}_k^{cts}(M,k)$ be the space of all continuous $k$-linear forms on $M$, endowed with discrete topology. Then, $M^\vee \in \mathrm{Mod}_k(G)$.*

*3. The above two functors are mutually quasi-inverse equivalences between $\mathrm{Mod}_k(G)$ and $\mathrm{Pct}_k(G)$.*

*Proof.* This is [19, Theorem 1.5]. □

We also have the following important isomorphism.

**Proposition 4.6.** *Let $G$ and $k$ be as above. Then,*

$$k[[G]] \cong \mathrm{Hom}_k(C^\infty(G,k),k),$$

*where* $C^\infty(G,k) = \{\text{all continuous functions } G \to k\}$

*Proof.* This is [26, Lemma 21.1]. □

# Chapter 5

# Proof in the compact case

In this chapter, we prove the following theorem for compact $p$-adic Lie groups and its corollary for the pro-$p$, torsion-free groups.

**Theorem 5.1.** *Let $G$ be a compact $p$-adic Lie group, with a compact open subgroup that is pro-$p$ and torsion-free. Let $D(G)$ be the full unbounded derived category of the category of smooth mod-$p$ representations of $G$.*

*If $\mathcal{C}$ is a strict, fully triangulated, non-zero tensor ideal of $D(G)$ that is localizing, then $\mathcal{C} = D(G)$.*

**Corollary 5.2.** *Let $G$ be a compact $p$-adic Lie group that is pro-$p$ and torsion free. Then, if $\mathcal{C}$ is a strict, fully triangulated, localizing subcategory of $D(G)$, then $\mathcal{C} = D(G)$.*

## 5.1 Tensoring with the Induction

Fix a compact $p$-adic Lie group $G$, a field $k$ of characteristic $p$, and a smooth $G$-representation $W$.

Consider the $k$-vector space $W \bigotimes_k \mathrm{Ind}_e^G(1)$. This is a $G$-representation under the diagonal $G$-action:

$$g \cdot (w \otimes f) = g \cdot w \otimes g \cdot f.$$

It is actually a smooth representation: Given a pure tensor $w \otimes f$,

$$W \text{ is smooth } \implies \exists\, U_w \underset{\text{open}}{\leq} G \text{ such that } w \in W^{U_w}$$

$$\mathrm{Ind}_e^G(1) \text{ is smooth } \implies \exists\, U_f \underset{\text{open}}{\leq} G \text{ such that } f \in \mathrm{Ind}_e^G(1)^{U_f}$$

Take $U = U_w \cap U_f$. Then, $U \underset{\text{open}}{\leq} G$ and

$$w \otimes f \in \left( W \bigotimes_k \mathrm{Ind}_e^G(1) \right)^U .$$

(For a general element $x$ of $\mathrm{Ind}_e^G(W)$, we can write $x$ as a finite sum of pure tensors and take $U$ to be the (finite) intersection of open subgroups corresponding to those pure tensors.)

Next proposition is of immediate use.

**Proposition 5.3.** *As smooth $G$-representations,*

$$W \bigotimes_k \mathrm{Ind}_e^G(1) \xrightarrow{\sim} \mathrm{Ind}_e^G(W)$$

*Proof.* By lemma 3.10, it suffices to show that

$$W \bigotimes_k \mathrm{Ind}_e^G(1) \xrightarrow{\sim} C^\infty(G, W)$$

as $G$-representations. We claim that the map $\Psi : W \bigotimes_k \mathrm{Ind}_e^G(1) \longrightarrow C^\infty(G, W)$, given by

$$\Psi(w \otimes f)(g) = f(g)w$$

on pure tensors, is an isomorphism of $G$-representations.

<u>$\Psi$ is well-defined</u>: It is easy to show that the map

$$\begin{aligned} W \times \mathrm{Ind}_e^G(1) &\longrightarrow C^\infty(G, W) \\ (w, f) &\longmapsto [g \mapsto f(g)w] \end{aligned}$$

is $k$-balanced. So, by the universal property of the tensor product, there is a unique group homomorphism $\Psi : W \bigotimes_k \mathrm{Ind}_e^G(1) \longrightarrow C^\infty(G, W)$ that takes $w \otimes f$ to $[g \mapsto f(g)w]$. And it is clearly $k$-bilinear. So, $\Psi$ is well-defined as a $k$-module homomorphism.

<u>$\Psi$ is a $G$-map</u>: Given $(w \otimes f) \in W \bigotimes_k \mathrm{Ind}_e^G(1)$ and $g, h \in G$,

$$\Psi(h \cdot (w \otimes f))(g) = \Psi(h \cdot w \otimes h \cdot f)(g)$$

$$= [(h \cdot f)(g)](h \cdot w)$$
$$= f(h^{-1}g)(h \cdot w)$$

and

$$\begin{aligned}[h \cdot \Psi(w \otimes f)](g) &= h \cdot [\Psi(w \otimes f)(h^{-1}g)] \\ &= h \cdot f(h^{-1}g)w \\ &= f(h^{-1}g)(h \cdot w)\end{aligned}$$

noting that $f(h^{-1}g) \in k$. So,

$$\Psi(h \cdot (w \otimes f)) = h \cdot \Psi(w \otimes f)$$

It remains to show that $\Psi$ is a bijection. For this part, we can ignore the $G$-map structure and treat $\Psi$ as a map between $k$-vector spaces.

Let $\{x_i\}_{i \in I}$ be a (possibly infinite) basis for $W$ over $k$. For $w \in W$, let $c_i(w)$ denote the coefficient of $x_i$ when $w$ is expressed in terms of $\{x_i\}_{i \in I}$, i.e.

$$w = \sum_{i \in I} c_i(w) x_i.$$

(Note that for any $w \in W$, $c_i(w) = 0$ for all but finitely many $i$, since $\{x_i\}_{i \in I}$ is a basis.)

Under this setup, we have the following two $k$-module isomorphisms:

$$\begin{aligned} W \bigotimes_k \operatorname{Ind}_e^G(1) &\xrightarrow{\alpha} \left(\bigoplus_{i\in I} k\right) \bigotimes_k \operatorname{Ind}_e^G(1) \\ (w \otimes f) &\longmapsto (c_i(w))_{i\in I} \otimes f \end{aligned}$$

and

$$\begin{aligned} \left(\bigoplus_{i\in I} k\right) \bigotimes_k \operatorname{Ind}_e^G(1) &\xrightarrow{\beta} \bigoplus_{i\in I} \operatorname{Ind}_e^G(1) \\ (a_i)_{i\in I} \otimes f &\longmapsto (a_i f)_{i\in I} \end{aligned}.$$

We also have an isomorphism

$$\begin{aligned} C^\infty(G, W) &\xrightarrow{\gamma} C^\infty\left(G, \bigoplus_{i\in I} k\right) \\ f &\longmapsto \left[g \mapsto (c_i(f(g)))_{i\in I}\right] \end{aligned},$$

which is well-defined since $f(g) \in W \implies c_i(f(g)) = 0$ for all but finitely many $i$.

Finally, we can define a $k$-module homomorphism

$$\begin{aligned} \bigoplus_{i\in I} \operatorname{Ind}_e^G(1) &\xrightarrow{\Phi} C^\infty\left(G, \bigoplus_{i\in I} k\right) \\ (f_i)_{i\in I} &\longmapsto \left[g \mapsto (f_i(g))_{i\in I}\right] \end{aligned}.$$

**Claim 1:** The following diagram of $k$-vector spaces and their maps commutes.

$$\begin{tikzcd} W \bigotimes_k \operatorname{Ind}_e^G(1) \arrow[r, "\Psi"] \arrow[d, "\alpha"'] & C^\infty(G, W) \arrow[d, "\gamma"] \\ \left(\bigoplus_{i\in I} k\right) \bigotimes_k \operatorname{Ind}_e^G(1) \arrow[d, "\beta"'] & C^\infty\left(G, \bigoplus_{i\in I} k\right) \\ \bigoplus_{i\in I} \operatorname{Ind}_e^G(1) \arrow[ur, "\Phi"] & \end{tikzcd}$$

Indeed, given $w \otimes f \in W \bigotimes_k \operatorname{Ind}_e^G(1)$, we have

$$\begin{tikzcd} w \otimes f \arrow[r, mapsto, "\alpha"] & (c_i(w))_{i\in I} \otimes f \arrow[r, mapsto, "\beta"] & (c_i(w)f)_{i\in I} \arrow[d, mapsto, "\Phi"] \\ & & \left[g \mapsto (c_i(w)f(g))_{i\in I}\right] \end{tikzcd}$$

and

$$\begin{tikzcd} w \otimes f \arrow[r, mapsto, "\Psi"] & {[g \mapsto f(g)w]} \arrow[d, mapsto, "\gamma"] \\ & \left[g \mapsto (c_i(w)f(g))_{i\in I}\right] \end{tikzcd}$$

since

$$f(g) \in k \implies c_i(f(g)w) = c_i\left(\sum_{i\in I} f(g)c_i(w)x_i\right) = f(g)c_i(w) = c_i(w)f(g).$$

This computation extends naturally to the case where the element of $W \bigotimes_k \operatorname{Ind}_e^G(1)$ is a sum of pure tensors. Therefore, the diagram commutes.

Since the diagram commutes, in order to prove that $\Psi$ is a bijection, it suffices to prove that $\Phi$ is a bijection.

**Claim 2:** $\Phi$ is a bijection.

$\Phi$ is injective: Let $f = (f_i)_{i\in I} \in \bigoplus_{i\in I} \operatorname{Ind}_e^G(1)$. Then,

$$\begin{aligned} \Phi(f) = 0 &\implies \Phi(f)(g) = 0 \quad \forall g \in G \\ &\implies \forall i \in I, \quad f_i(g) = 0 \quad \forall g \in G \\ &\implies \forall i \in I, \quad f_i = 0 \end{aligned}$$

$$\implies f = 0$$

<u>$\Phi$ is surjective</u>: Given $\tilde{f} \in C^\infty\left(G, \bigoplus_{i\in I} k\right)$, i.e. a continuous function $\tilde{f} : G \longrightarrow \bigoplus_{i\in I} k$, by lemma 3.6, $\tilde{f}$ is smooth. So, there exists an open subgroup $C$ of $G$ such that $\tilde{f}(cg) = \tilde{f}(g)$ for all $g \in G$ and for all $c \in C$. Furthermore, by lemma 3.7, $\tilde{f}$ is locally constant and has finite range.

Define $f_i = \pi_i \circ \tilde{f}$ where $\pi_i : \bigoplus_{i\in I} k \longrightarrow k$ is the $i$-th projection map.

Then, $f_i : G \longrightarrow k$ is smooth: given $c \in C$ and $g \in G$,

$$f_i(cg) = \pi_i(\tilde{f}(cg)) = \pi_i(\tilde{f}(g)) = f_i(g)$$

And $f_i = 0$ for all but finitely many $i$, since $\tilde{f}$ takes only finitely many values in $\bigoplus_{i\in I} k$, and for each of those values, only finitely many coordinates are non-zero.

Therefore, $f := (f_i)_{i\in I}$ is an element of $\bigoplus_{i\in I} \mathrm{Ind}_e^G(1)$. And

$$\Phi(f)(g) = (f_i(g))_{i\in I} = \Big(\pi_i(\tilde{f}(g))\Big)_{i\in I} = \tilde{f}(g) \quad \forall g \in G \implies \Phi(f) = \tilde{f}.$$

So, $\Phi$ is surjective.

This completes the proof of the proposition. $\square$

*Remark.* Alternatively, you can argue as follows. Observe that $W \bigotimes_k \mathrm{Ind}_e^G(1) \cong W \bigotimes_k C^\infty(G,k)$ by lemma 3.10. But $W \bigotimes_k C^\infty(G,k) \cong C^\infty(G,W)$ by [27, Section 5.2: Claim (c)]. Finally, by lemma 3.10 again, we note that $C^\infty(G,W) \cong \mathrm{Ind}_e^G(W)$ which completes the proof.

## 5.2 Setup of the Proof

Now we can begin to prove our theorem. Let $G$ be a $p$-adic Lie group, $k$ a field of characteristic $p$, and denote by $D(G)$ the unbounded derived category of $\mathrm{Mod}_k(G)$, the category of smooth $G$-representations over $k$. Let $\mathcal{C}$ be a strict, thick, fully triangulated, nonzero tensor ideal of $D(G)$ that is also localizing.

Since $\mathcal{C}$ is nonzero, we can pick a non-acyclic $V^\bullet \in \mathrm{obj}(\mathcal{C})$. Say $\mathrm{h}^s(V^\bullet) \neq 0$ for some $s$.

The following lemma is immediate from the proposition 5.3 of the previous section:

**Lemma 5.4.**

$$V^\bullet \bigotimes_k \left(\mathrm{Ind}_e^G(1)\right)[0] \xrightarrow{\sim} \mathrm{Ind}_e^G(V^\bullet)$$

*where* $\left(\mathrm{Ind}_e^G(1)\right)[0]$ *is the complex*

$$\cdots \to 0 \to \mathrm{Ind}_e^G(1) \to 0 \to \cdots$$

*and* $\mathrm{Ind}_e^G(V^\bullet)$ *denotes the complex*

$$\cdots \to \mathrm{Ind}_e^G(V^{-1}) \to \mathrm{Ind}_e^G(V^0) \to \mathrm{Ind}_e^G(V^1) \to \cdots .$$

*Proof.* By definition, $V^\bullet \bigotimes_k \left(\mathrm{Ind}_e^G(1)\right)[0]$ is the chain complex

$$\cdots \to V^{-1} \bigotimes_k \mathrm{Ind}_e^G(1) \to V^0 \bigotimes_k \mathrm{Ind}_e^G(1) \to V^1 \bigotimes_k \mathrm{Ind}_e^G(1) \to \cdots$$

with the differential $d(v \otimes f) = dv \otimes f$. So, by proposition 5.3, we can write it as

$$\cdots \to \mathrm{Ind}_e^G(V^{-1}) \to \mathrm{Ind}_e^G(V^0) \to \mathrm{Ind}_e^G(V^1) \to \cdots ,$$

say $d'$ is the corresponding differential. $\square$

Let $\mathrm{Ch}(G)$ denote the category of chain complexes in $\mathrm{Mod}_k(G)$. We make the following observation.

**Lemma 5.5.** *There exists a cyclic complex $W^\bullet$ in $\mathrm{Ch}(G)$ such that*

$$\mathrm{Ind}_e^G(V^\bullet) \cong \bigoplus_{n \in \mathbb{Z}} \mathrm{Ind}_e^G(W^n)[-n]$$

*Proof.* Consider $V^\bullet$ as an element of $D(k)$. Since the category $\mathrm{Mod}_k$ of $k-$vector spaces is semisimple, by proposition 4 on [11, p146],

$$V^\bullet \cong W^\bullet$$

where $W^\bullet$ is the cyclic complex ($d = 0$) given by

$$W^n = h^n(V^\bullet).$$

Now, since $\mathrm{Mod}_k$ is semisimple and $\mathrm{Ind}_e^G : \mathrm{Mod}_k \to \mathrm{Mod}_k(G)$ is exact, the derived functor $\mathrm{Ind}_e^G : D(k) \to D(G)$ is also exact. Therefore,

$$\mathrm{Ind}_e^G(V^\bullet) \cong \mathrm{Ind}_e^G(W^\bullet).$$

But $\mathrm{Ind}_e^G(W^\bullet)$ is a cyclic complex. So,

$$\begin{aligned}\mathrm{Ind}_e^G(W^\bullet) = &\quad \cdots \xrightarrow{0} \mathrm{Ind}_e^G(W^n) \xrightarrow{0} \mathrm{Ind}_e^G(W^{n+1}) \xrightarrow{0} \cdots \\ &= \bigoplus_{n\in\mathbb{Z}} \mathrm{Ind}_e^G(W^n)[-n]\end{aligned}$$

□

Since $\mathcal{C}$ is a tensor ideal already containing $V^\bullet$, it also contains $V^\bullet \bigotimes_k \mathrm{Ind}_e^G(1)$. Then, since $\mathcal{C}$ is strictly full, by corollary 5.4 and lemma 5.5, it contains the complex $\bigoplus_{n\in\mathbb{Z}} \mathrm{Ind}_e^G(W^n)[-n]$. And since it is thick, it contains the complexes $\mathrm{Ind}_e^G(W^n)[-n]$ for each $n$. Pick an $N$ such that $W^N$ is non-zero. Then, since $\mathcal{C}$ is closed under shifts, it contains $\mathrm{Ind}_e^G(W^N)[0]$. Let $\{x_i\}_{i\in I}$ be a basis for $W^N$ over $k$. Then,

$$\mathrm{Ind}_e^G(W^N) = \mathrm{Ind}_e^G(\bigoplus_{i\in I} k) = \bigoplus_{i\in I} \mathrm{Ind}_e^G(k)$$

which implies that

$$\mathrm{Ind}_e^G(W^N)[0] = \bigoplus_{i\in I} \mathrm{Ind}_e^G(k)[0].$$

Since $\mathcal{C}$ is thick, we can conclude that $\mathcal{C}$ contains $\mathrm{Ind}_e^G(1)[0]$.

## 5.3 Resolution

Let $\mathbb{X} = \mathrm{Ind}_U^G(1)$. We recall [27, Proposition 6]:

**Proposition** (See proposition 3.14)**.** *If $\mathcal{D} \subset D(G)$ is a strict, full, triangulated subcategory such that*

- *$\mathcal{D}$ is localizing, i.e. $\mathcal{D}$ is closed under all coproducts $\bigoplus_{i\in I}$*

- $\mathbb{X} \in \mathrm{Obj}(\mathcal{D})$

*Then, $\mathcal{D} = D(G)$*

So, in order to show that $\mathcal{C} = D(G)$, it suffices to show that $\mathbb{X}[0] \in \mathrm{Obj}(\mathcal{C})$.

**Proposition 5.6.** *$\mathbb{X}$ has an injective resolution*

$$0 \to \mathbb{X} \to \mathrm{Ind}_e^G(1)^{\oplus n_0} \to \cdots \to \mathrm{Ind}_e^G(1)^{\oplus n_d} \to 0$$

*in $\mathrm{Mod}_k(G)$.*

*Proof.* Let $\Omega = k[[U]]$. Then, Lazard showed that $\Omega$ is Noetherian and has finite global dimension $d = \dim(U) = \dim(G)$.

Now let $\mathrm{Pct}_k(U)$ be the category of pseudocompact $U$-modules, i.e. the category of $k$-vector spaces with a $k$-linear continuous action $U \times W \to W$.

Since $U$ is compact, by proposition 4.4, $\mathrm{Pct}_k(U)$ is the same as the category of pseudocompact $\Omega$-modules.

So, we want to find a finite length projective resolution of $k$ of pseudocompact $\Omega$-modules. Since $k$ is finitely generated over $\Omega$ and $\Omega$ is Noetherian, there is a finite free module $\Omega^{\oplus n_1}$ such that

$$\Omega^{\oplus n_1} \xrightarrow{f_1} k \to 0$$

is exact. Now, since the kernel of $f_1$ is also finitely generated we can do the same to resolve that as

$$\Omega^{\oplus n_2} \to \Omega^{\oplus n_1} \to k \to 0.$$

We can continue this method to build a projective resolution of $k$ with finite free $\Omega$-modules.

However, the global dimension of $\Omega$ is $d$. Therefore, it should terminate with at most $d$ terms, leaving a finite length projective resolution of $k$ in $\mathrm{Pct}_k(U)$:

$$0 \to \Omega^{\oplus n_d} \to \Omega^{\oplus n_{d-1}} \to \cdots \to \Omega^{\oplus n_1} \to k \to 0$$

By Pontrayagin dulaity (theorem 4.5), this gives rise to an injective resolution in $\mathrm{Mod}_k(U)$:

$$0 \to \mathrm{Hom}_k^{\mathrm{cont}}(k,k) \to (\mathrm{Hom}_k^{\mathrm{cont}}(\Omega,k))^{\oplus n_1} \to \cdots \to (\mathrm{Hom}_k^{\mathrm{cont}}(\Omega,k))^{\oplus n_d} \to 0$$

We note that since

$$\mathrm{Hom}_k(C^\infty(U,k),k) = \Omega$$

by proposition 4.6, where $C^\infty(U,k) = \{\text{all continuous functions } U \to k\}$, by Pontryagin dulaity we have an equivalence

$$\mathrm{Hom}_k^{\mathrm{cont}}(\Omega,k) = C^\infty(U,k).$$

So, we can rewrite the injective resolution as:

$$0 \to k \to (C^\infty(U,k))^{\oplus n_1} \to \cdots \to (C^\infty(U,k))^{\oplus n_{d'}} \to 0$$

which, by lemma 3.10, is the same as:

$$0 \to k \to (\mathrm{Ind}_e^U(1))^{\oplus n_1} \to \cdots \to (\mathrm{Ind}_e^U(1))^{\oplus n_{d'}} \to 0.$$

Finally, we recall that when $G$ is compact, $\mathrm{Ind}_U^G(\ ) = \mathrm{ind}_U^G(\ )$ and $\mathrm{ind}_U^G(\ )$ is an exact functor.

So, after applying $\mathrm{Ind}_U^G(\ )$ to the above injective resolution and using the transitivity property of induction, we get the injective resolution we wanted to extabilish:

$$0 \to \mathrm{Ind}_U^G(1) \to (\mathrm{Ind}_e^G(1))^{\oplus n_1} \to \cdots \to (\mathrm{Ind}_e^G(1))^{\oplus n_{d'}} \to 0.$$

$\square$

Denote $0 \to \mathrm{Ind}_e^G(1)^{\oplus n_0} \to \cdots \to \mathrm{Ind}_e^G(1)^{\oplus n_d} \to 0$ by $I^\bullet$. Then, by the proposition 5.6, there exists a quasi-isomorphism $\mathbb{X}[0] \to I^\bullet$ in $\mathrm{Ch}(G)$, which implies that $\mathbb{X}[0] \cong I^\bullet$ in $D(G)$. We claim that $I^\bullet \in \mathrm{Obj}(\mathcal{C})$. Clearly $I^n[0]$ is in $\mathcal{C}$ for all $n$ since $\mathcal{C}$ is localizing and $\mathrm{Ind}_e^G(1)[0]$ is in $\mathcal{C}$ by the previous section. Then, the result follows from the following lemma.

**Lemma 5.7.** *If $I^0[0], ..., I^n[0]$ are all in $\mathcal{C}$, so is*

$$0 \to I^0 \xrightarrow{d^0} \cdots \xrightarrow{d^{n-1}} I^n \to 0$$

*Proof.* We proceed by induction. Let $I_t^\bullet$ denote

$$0 \to I^0 \xrightarrow{d^0} \cdots \xrightarrow{d^{t-1}} I^t \to 0$$

for some $0 \leq t < n$ and assume $I_t^\bullet \in \mathrm{Obj}(\mathcal{C})$. We want to show that $I_{t+1}^\bullet \in \mathrm{Obj}(\mathcal{C})$.

Now, since $\mathcal{C}$ is fully triangulated, it is closed under shifts and cones. So, $I_t^\bullet \in \mathrm{Obj}(\mathcal{C})$ implies that $I_t^\bullet[-1] \in \mathrm{Obj}(\mathcal{C})$ as well. And $I^{t+1}[0] \in \mathrm{Obj}(\mathcal{C})$ implies that $I^{t+1}[t+1] \in \mathrm{Obj}(\mathcal{C})$. Consider the chain map $f : I_t^\bullet[-1] \to I^{t+1}[t+1]$ given by

$$f^m = \begin{cases} d^{t+1} & \text{if } m = t+1 \\ 0 & \text{otherwise} \end{cases}$$

Then, it is easy to check that $I_{t+1}^\bullet = \mathrm{cone}(f) \in \mathrm{Obj}(\mathcal{C})$: if $\mathrm{cone}(f) = C^\bullet$ with differentials $\tilde{d}^m$,

$$C_m = \begin{cases} I^m \oplus 0 \cong I^m & \text{if } 0 \leq m \leq t \\ 0 \oplus I^{t+1} \cong I^{t+1} & \text{if } m = t+1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\begin{pmatrix} x \\ 0 \end{pmatrix} \overset{\tilde{d}^m}{\longmapsto} \begin{pmatrix} -(-d^m(x)) \\ 0+0 \end{pmatrix} = \begin{pmatrix} d^m(x) \\ 0 \end{pmatrix} \quad \text{when } 0 \leq m \leq t-1$$

$$\text{and} \qquad \begin{pmatrix} x \\ 0 \end{pmatrix} \overset{\tilde{d}^t}{\longmapsto} \begin{pmatrix} 0 \\ d^t(x)+0 \end{pmatrix} = \begin{pmatrix} 0 \\ d^t(x) \end{pmatrix}.$$

□

Therefore, $I^\bullet$ is in $\mathcal{C}$, and since $\mathcal{C}$ is strictly full, so is $\mathbb{X}[0]$.

This completes the proof of our theorem:

**Theorem** (5.1)**.** *Let $G$ be a compact $p$-adic Lie group, with a compact open subgroup* *that is pro-p and torsion-free. Let $D(G)$ be the full unbounded derived category of the category of smooth mod $p$ representations of $G$.*

*If $\mathcal{C}$ is a strict, fully triangulated, non-zero tensor ideal of $D(G)$ that is localizing, then $\mathcal{C} = D(G)$.*

We can remove the tensor ideal condition when $G$ is pro-p and torsion free.

**Corollary** (5.2)**.** *Let $G$ be a compact $p$-adic Lie group that is pro-$p$ and torsion free. Then, if $\mathcal{C}$ is a strict, fully triangulated, localizing subcategory of $D(G)$, then $\mathcal{C} = D(G)$.*

*Proof.* Under this setting, $D(G)$ is rigidly compactly generated by the tensor unit $\mathbf{1}$ [15, Proposition 2.3.19]. Therefore, every localizing subcategory of $D(G)$ is a localizing tensor ideal by [33, Lemma 3.13]. □

## 5.4 The Work of Heyer and Schneider

Around the same time as we finished our work on this problem, similar results to our theorem 5.1 and corollary 5.2 were discovered independently by Claudius Heyer and Peter Schneider. While many of our ideas are similar to the what appears in their pre-print [16], we highlight some subtle differences.

Heyer and Schneider work directly with pro-$p$ groups while we work in the general compact case first. Therefore, they do not state a version of our theorem 5.1. However, the proofs use similar approaches and their [16, Theorem 1] is stronger than our corollary 5.2, since they get rid of the torsion-free condition, by showing that any localizing subcategory of $D(G)$ must be a tensor ideal when $G$ is pro-$p$ [16, Lemma 4].

# Chapter 6

# Non-compact Case

## 6.1 A Category Theoretical Interlude

In Chapter 2, we introduced the concepts of thick and localizing tensor ideals of a tensor triangulated category. Classifying localizing tensor ideals of a given tensor triangulated category via an algebraic object that is related to the said category is a defining feature of many results in ”tensor triangulated geometry”.

*Example* 6.1. In [4, Theorem 1.1], Benson et. al. prove that if $G$ is a finite group and $k$ a field of characteristic $p$ with $p$ dividing the order of $G$, there is a bijection between nonzero localizing tensor ideals of $\mathrm{Mod}(k[G])$ and the subsets of $\mathrm{Proj}\, H^*(G,k)$, which is the set of homogeneous prime ideals in the cohomology ring $H^*(G,k)$ that are non-maximal.

The following theorem is another example of this phenomenon and we will use it in the next chapter to address our own classification problem of localizing subcategories of $D(G)$.

**Theorem 6.1** (Neeman)**.** *Let $R$ be a commutative, Noetherian ring. Let $D(R)$) be the derived category of the category of $R$-modules. Then there is a bijection*

$$\text{Localizing subcategories of } D(R) \underset{g}{\overset{f}{\rightleftarrows}} \text{Subsets of } \mathrm{Spec}(R)$$

*where*

$$f(L) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \exists X \in L \text{ with } X \otimes \kappa(\mathfrak{p}) \neq 0\}$$

*and*

$$g(P) = \mathrm{Loc}(\{\kappa(\mathfrak{p}) \ : \ \mathfrak{p} \in P\}).$$

*Here, for a collection of elements $A$, by* $\mathrm{Loc}(A)$, *we denote the localizing subcategory generated by all elements of $A$.*

*Proof.* This is [23, Theorem 2.8]. We note that in this case, all localizing subcategories are tensor ideals. □

## 6.2 Augmented Iwasawa Algebra

In chapter 4, we defined the completed group ring $k[[H]]$ of a compact $p$-adic group $H$. Kohlhaase extended this notion to all (locally compact) $p$-adic groups as follows.

**Proposition 6.2.** *Let $G$ be a locally compact $p$-adic group. Let $k$ be a field of characteristic $p$. Fix an open compact subgroup $H$ of $G$, and let $k[[H]]$ be its completed* *group ring. Then,*

$$k[[G]] := k[G] \bigotimes_{k[H]} k[[H]]$$

*has a unique $k$-algebra structure such that $k[G] \to k[[G]]$ and $k[[H]] \to k[[G]]$ are $k$-algebra homomorphisms. We call $k[[G]]$ the **augmented Iwasawa algebra** of $G$.*

*Proof.* This is proved in section 1 of [19] and [30, Poposition 3.2]. Here, we briefly review the multiplicative structure of $k[[G]]$, following [30].

The construction of $k[[G]]$ is independent of the choice of $H$, which means we have an isomorphism

$$\rho_{H,H'} : k[G] \bigotimes_{k[H]} k[[H]] \to k[G] \bigotimes_{k[H']} k[[H']].$$

If $H'' \subset H'$, then $\rho_{H,H''} = \rho_{H',H''} \circ \rho_{H,H'}$. So, we can construct the direct limit

$$k[[G]] = \varinjlim_{H} \left( k[G] \bigotimes_{k[H]} k[[H]] \right).$$

So, given $g, g' \in G$ with $g \otimes \alpha \in k[G] \bigotimes_{k[H]} k[[H]]$ and $g' \otimes \alpha' \in k[G] \bigotimes_{k[H']} k[[H']]$, we can assume that $H \subset g^{-1}H'g$ and define

$$(g \otimes \alpha)(g' \otimes \alpha') = gg' \otimes g^{-1}\alpha' g\alpha \in k[G] \bigotimes_{k[g^{-1}H'g]} k[[g^{-1}H'g]] = k[[G]]$$

□

*Remark.* The notation $k[[G]]$ is unambiguous since both definitions agree when $G$ is compact. Furthermore, as discussed above, when $G$ is non-compact, $k[[G]]$ is independent of the choice of the open compact subgroup $H$.

Recall that when $G$ is compact, a smooth representation of $G$ over $k$ has a natural $k[[G]]$-module structure. This is still true when $G$ is not compact and $k[[G]]$ is the augmented Iwasawa algebra.

**Proposition 6.3.** *Let $V$ be a smooth $k$-representation of $G$. There is a unique $k[[G]]$-module structure extending the $k[G]$-action on $V$.*

*Proof.* This is [34, Corollary 2.20]. Given $f \in K[G]$ and $\alpha \in k[[H]]$, the action of $(f \otimes \alpha) \in k[[G]]$ on a vector $v \in V$ is given by

$$(f \otimes \alpha) \cdot v = f \cdot (\alpha \cdot v).$$

(Recall that we have already shown in 4.3 that there is a natural action of $k[[H]]$ on $V$ when $H$ is compact.) $\square$

$k[[G]] \cong \bigoplus_g \left( g \bigotimes_{k[H]} k[[H]] \right)$ can be topologized by equipping it with the direct sum topology with respect to the topological abelian groups $g \bigotimes_{k[H]} k[[H]]$. The topology on $g \bigotimes_{k[H]} k[[H]]$ is determined by requiring the natural inclusions $k[H] \to g \bigotimes_{k[H]} k[[H]$ to be continuous.

In our work, we will be focusing on the case where $G$ is abelian, which allows us to topologize $k[[G]]$ differently by embedding it in a larger ring $\widehat{k[G]}$.

## 6.3 Completion in the Abelian Case: $\widehat{k[G]}$

**Definition 6.1.** Let $G$ be an abelian $p$-adic group and let $\mathrm{Op}(G)$ denote the set of open compact subgroups of $G$. By $\widehat{k[G]}$, we denote the completion

$$\widehat{k[G]} := \varprojlim_{U \in \mathrm{Op}(G)} k[G/U]$$

Naturally, we equip $\widehat{k[G]}$ with the inverse limit topology where each $k[G/U]$ is considered to be discrete, i.e. a neighborhood basis for zero is given by

$$\{V_U := \ker(\widehat{k[G]} \to k[G/U])\}_{U \in \mathrm{Op}(G)}.$$

*Remark.* This construction was introduced by Ardakov and Schneider in [1] for a general locally profinite group $G$ and any field $k$ as

$$k\widehat{[Z(G)]} := \varprojlim_{U \in \mathrm{Op}(Z(G))} k[Z(G)/U]$$

where $Z(G)$ is the center of $G$. They show that if $G$ is a locally profinite group containing an open pro-$p$ subgroup and no proper open centralizers and if $k$ is a field of characteristic $p$, then $k\widehat{[Z(G)]}$ is isomorphic to $\mathfrak{Z}(G)$, the Bernstein center of $\mathrm{Mod}_k(G)$ ([1, Theorem 1.1]). Here, $\mathfrak{Z}(G)$ is defined as the ring of natural endomorphisms of the identity functor on $\mathrm{Mod}_k(G)$.

We showed earlier that both the Iwasawa algebra and the augmented Iwasawa algebra have natural actions on smooth representations of $G$. This is true for $\widehat{k[G]}$

as well.

**Proposition 6.4.** *Let $G$ be an abelian $p$-adic Lie group and $k$ a field of characteristic $p$. Then, $\widehat{k[G]}$ acts naturally on every smooth $k$-representation $V$ of $G$.*

*Proof.* This is [1, Lemma 3.6]. We reproduce the proof here since we will use the idea in the proof of 7.3.

Let $V$ be a smooth representation of $G$. Given a compact open subgroup $H$ of $G$, by definition $H$ acts trivially on $V^H$. Therefore, if we define an action of $G/H$ on $V^H$ as $gH \cdot v = g \cdot v$, it is well defined:

$$gH = g'H \implies g^{-1}gH = H \implies g^{-1}g' \cdot v = v \implies gH \cdot v = g \cdot v = g' \cdot v = g'H \cdot v.$$

Then, via the projection $\widehat{k[G]} \to k[G/H]$ we can extends this to a $\widehat{k[G]}$ on $V^H$. So, every $V^H$ is naturally a $\widehat{k[G]}$-module.

Now, let $H'$ be another compact open subgroup $G$ such that $H' \subset H$. Then, $V^H \subset V^{H'}$ and both are $\widehat{k[G]}$-modules. Furthermore, the inclusion $V^H \to V^{H'}$ is $\widehat{k[G]}$-linear.

So, we have constructed a $\widehat{k[G]}$ action on $V^H$ for every compact open subgroup $H$ of $G$ and shown that these actions are compatible with the inclusions $V^H \to V^{H'}$ whenever $H' \subset H$. Therefore, there is a well-defined $\widehat{k[G]}$-action on the smooth representation $V = \bigcup_{H \in \mathrm{Op}(G)} V^H$.

$\square$

## 6.4 Topologizing $k[[G]]$ in the Abelian Case

**Proposition 6.5.** *There is an injection $k[[G]] \to \widehat{k[G]}$.*

*Proof.* Pick a compact open subgroup $H$ of $G$. Let $\mathrm{Op}(H)$ denote the set of open compact subgroups of $H$. Then,

$$k[[G]] = k[G] \bigotimes_{k[H]} \varprojlim_{U \in \mathrm{Op}(H)} k[H/U].$$

Consider the following map:

$$\begin{array}{ccc} G \times \varprojlim_{U \in \mathrm{Op}(H)} H/U & \xrightarrow{\varphi} & \varprojlim_{U \in \mathrm{Op}(H)} G/U \\ (g, (h_U U)_U) & \longmapsto & (gh_U U)_U \end{array}.$$

First we note that this map is compatible with both inverse systems $\{H/U, i_{UU'}, \mathrm{Op}(H)\}$ and $\{G/U, j_{UU'}, \mathrm{Op}(H)\}$. Indeed when $U_1 \supseteq U_2$,

$$\begin{aligned} i_{U_2U_1}(h_{U_2}U_2) = h_{U_1}U_1 &\iff h_{U_1}^{-1}h_{U_2} \in U_1 \\ &\iff (gh_{U_1})^{-1}(gh_{U_2}) \in U_1 \text{ for all } g \text{ in } G \\ &\iff j_{U_2U_1}(gh_{U_2}U_2) = gh_{U_1}U_1. \end{aligned}$$

$\varphi$ extends to the following map $\phi$ which is also compatible with the inverse systems:

$$\begin{array}{ccc} k[G] \times \varprojlim_{U \in \mathrm{Op}(H)} k[H/U] & \xrightarrow{\phi} & \varprojlim_{U \in \mathrm{Op}(H)} k[G/U] \\ (\sum_g a_g g, (\sum_{h_U} c_{h_U} h_U U)_U) & \longmapsto & (\sum_g \sum_{h_U} a_g c_{h_U} g h_U U)_U \end{array}.$$

This map is bilinear and $k[H]$-balanced:

$$\phi(gh,(h_U U)_U) = (ghh_U U)_U = \phi(g,(hh_U U)_U) \quad \text{for } h \in H.$$

Therefore, it extends to a $k[H]$-module map:

$$\Phi : k[G] \bigotimes_{k[H]} \varprojlim_{U\in \mathrm{Op}(H)} k[H/U] \longrightarrow \varprojlim_{U\in \mathrm{Op}(H)} k[G/U]$$

which maps pure tensors in the following way:

$$g \otimes (h_U)_U \mapsto (gh_U U)_U$$

$\Phi$ is an clearly a $k$-algebra map, since if $x = g \otimes (h_U)_U$ and $y = g' \otimes (h'_U)_U$,

$$\begin{aligned}\Phi(x)\Phi(y) = [(gh_U U)_U][(g'h'_U U)_U] &= (gg'h_U h'_U U)_U \\ &= \Phi(gg' \otimes (h_U h'_U U)_U) \\ &= \Phi(xy).\end{aligned}$$

Finally, we claim that $\Phi$ is injective. To show this, first pick a set of representatives $\{g_\beta H\}$ for $G/H$ and a set of representatives $\{h_1^U U, ..., h_{n_U}^U U\}$ for $H/U$ for each $U \in \mathrm{Op}(H)$. Then, we can write an element $x$ of $k[G] \bigotimes_{k[H]} \varprojlim_{U\in \mathrm{Op}(H)} k[H/U]$ as

$$x = (\sum_{\beta}{}' c_\beta g_\beta) \otimes \left( \sum_{i=1}^{n_U} d_i^U h_i^U U \right)_U$$

where $c_\beta, d_i^U \in k$ and $\sum'$ denotes that only finitely many $c_\beta$'s are non-zero. Now assume that

$$0 = \Phi(x) = \left( {\sum_\beta}' \sum_{i=1}^{n_U} c_\beta d_i^U g_\beta h_i^U U \right)_U .$$

Then, for any given $U$, we have that $\sum_\beta' \sum_{i=1}^{n_U} c_\beta d_i^U g_\beta h_i^U U = 0$. Since $\{g_\beta h_i^U U\}$ forms a set of representatives for $G/H$, this means that $c_\beta d_i^U = 0$ for all $\beta$ and $i$. Therefore, $c_\beta = 0$ for all $\beta$ or $d_i^U = 0$ for all $i$ and $U$. In either case, we have that $x = 0$, proving that $\Phi$ is injective.

Finally, we note that for any $N \in \mathrm{Op}(G)$, $N \cap H \in \mathrm{Op}(H)$. Therefore, $\mathrm{Op}(H)$ is cofinal in $\mathrm{Op}(G)$ and $\widehat{k[G]} \cong \varprojlim_{U \in \mathrm{Op}(H)} k[G/U]$. $\square$

While the map $\Phi$, constructed above, is not usually an isomorphism, $k[[G]]$ is embedded densely in $\widehat{k[G]}$.

**Proposition 6.6.** *$k[[G]]$ is dense in $\widehat{k[G]}$*

*Proof.* Observe that $k[G]$ is dense in $\widehat{k[G]}$ by construction. Then the result follows from the chain of embeddings $k[G] \hookrightarrow k[[G]] \hookrightarrow \widehat{k[G]}$. $\square$

Now we can equip $k[[G]]$ with the subspace topology deriving from $\widehat{k[G]}$.

# Chapter 7

# Abelian Groups

Our goal in this chapter is to use Neeman's theorem from the previous chapter to classify the localizing subcategories of $D(G)$ for some $p$-adic groups $G$ which satisfy certain conditions, but that are not necessarily compact. The following proposition by Emerton, Gee, and Hellman is our starting point [10, Proposition E.2.2].

**Proposition 7.1.** *Let $G$ be a $p$-adic Analytic group, $k$ a field of characteristic $p$. Let $D(k[[G]])$ denote the derived category of the category of $k[[G]]$-modules and let $D_{\mathrm{sm}}(k[[G]])$ denote the full subcategory of $D(k[[G]])$ consisting of objects whose cohomologies in every degree are smooth $G$-representations. Then, there is a canonical equivalence*

$$D(G) \xrightarrow{\sim} D_{\mathrm{sm}}(k[[G]]).$$

*Remark.* In [10], this is proved in the context of stable $\infty$-categories. However, we can identify the ordinary derived categories with the homotopy categories of the corresponding stable $\infty$-categories ([10, Section A.1], and also, [22, Example 1.1.1.11]).

And equivalences between stable $\infty$-categories induce equivalences between their homotopy categories [21, Remark 1.1.3.8.].

## 7.1 Noetherian, Commutative $k[[G]]$

Henceforth, let $G$ be an abelian $p$-adic analytic group whose augmented Iwasawa algebra $k[[G]]$ is Noetherian. Combining 6.1 and 7.1, we wish to prove the following.

**Theorem 7.2.** *Let $G$ be an abelian $p$-adic analytic group and $k$ a field of characteristic $p$. Assume that $k[[G]]$ is Noetherian. Let $\mathrm{Spf}(k[[G]])$ denote the set of open prime ideals of $k[[G]]$. Then, there is a bijection*

$$\text{Localizing subcategories of } D(G) \longleftrightarrow \text{Subsets of } \mathrm{Spf}(k[[G]]).$$

By 6.1, we already have a bijection

$$\text{Localizing subcategories of } D(k[[G]]) \longleftrightarrow \text{Subsets of } \mathrm{Spec}(k[[G]])$$

in this setting. Now observe that the full subacategory $D_{\mathrm{sm}}(k[[G]])$ of $D(k[[G]])$ is closed under arbitrary direct sums. Therefore, any localizing subcategory of $D_{\mathrm{sm}}(k[[G]])$ is a localizing subcategory of $D(k[[G]])$. So, all we need to do is to identify which subsets of $\mathrm{Spec}(k[[G]])$ correspond to the localizing subcategories of $D_{\mathrm{sm}}(k[[G]])$ under Neeman's bijection. We answer this question in the following lemma.

**Lemma 7.3.** *Given* $\mathfrak{p} \in \mathrm{Spec}(k[[G]])$*, let* $\kappa(\mathfrak{p})$ *denote the quotient field of* $k[[G]]/\mathfrak{p}$*. Then,* $\mathfrak{p}$ *is open in* $k[[G]]$ *(in the subspace topology derived from* $\widehat{k[G]}$*) if and only if* $\kappa(\mathfrak{p})$ *is smooth as a* $G$*-representation.*

*Proof.* It suffices to show that $\mathfrak{p}$ is open if and only if $k[[G]]/\mathfrak{p}$ is smooth.

( $\Longrightarrow$ ): Assume $\mathfrak{p}$ is open. Then, $\mathfrak{p} = V \cap k[[G]]$ for some $V$ open in $\widehat{k[G]}$. Since $0 \in \mathfrak{p} \subseteq V$, we can find an open compact subgroup $U$ of $G$ such that $0 \in \ker(\widehat{k[G]} \to k[G/U]) \subseteq V$. This means that for all $u \in U$, $u - 1 \in V$. However, for all $u \in U$, $u - 1 \in \mathfrak{p}$, since $u - 1 \in k[G] \cap V \subseteq k[[G]] \cap V = \mathfrak{p}$. Therefore, for any $\alpha \in k[[G]]$ and $u \in U$,

$$(u - 1) \cdot (\alpha + \mathfrak{p}) = 0 + \mathfrak{p}.$$

So, the open subgroup $U$ fixes all elements of $k[[G]]/\mathfrak{p}$, showing that $k[[G]]/\mathfrak{p}$ is smooth.

( $\Longleftarrow$ ): Assume that $V := k[[G]]/\mathfrak{p}$ is smooth. Then, for each of its elements there is an open subgroup of $G$ that fixes that elelement. In particular, we can find an open subgroup $H$ of $G$ which fixes $1+\mathfrak{p}$, i.e. $1+\mathfrak{p} \in V^H$. Now, $G/H$ acts naturally on $V^H$, and as discussed in the proof of lemma 6.4, this extends to a $\widehat{k[G]}$-action on $V^H$ via the map

$$\widehat{k[G]} \xrightarrow{\pi_H} k[G/H].$$

So, for any $\alpha \in I_H := \ker(\pi_H)$, $\alpha \cdot (1 + \mathfrak{p}) = \pi(\alpha) \cdot (1 + \mathfrak{p}) = 0 + \mathfrak{p}$. Now, since $I_H \cap k[[G]]$ kills $1 + \mathfrak{p}$ and $\mathfrak{p}$ is a prime ideal of $k[[G]]$, we have that $I_H \cap k[[G]] \subseteq \mathfrak{p}$. But $I_H$ is open in $\widehat{k[G]}$, therefore $I_H \cap k[[G]]$ is open in $k[[G]]$. And then, since $\mathfrak{p}$ contains an open ideal, it must be open in $k[[G]]$ as well. $\square$

Theorem 7.2 is now immediate.

*Proof.* We have

$$\begin{array}{ccccc} & & \begin{array}{c}\text{Localizing Subcategories}\\ \text{of } D(k[[G]])\end{array} & \overset{Thm.}{\underset{6.1}{\longleftrightarrow}} & \begin{array}{c}\text{Subsets of}\\ \text{Spec(k[[G]])}\end{array} \\ & & \cup\!\mid & & \cup\!\mid \\ \begin{array}{c}\text{Localizing Subcategories}\\ \text{of } D(G)\end{array} & \overset{Prop.}{\underset{7.1}{\longleftrightarrow}} & \begin{array}{c}\text{Localizing Subcategories}\\ \text{of } D_{\mathrm{sm}}(k[[G]])\end{array} & \overset{Lem.}{\underset{7.3}{\longleftrightarrow}} & \begin{array}{c}\text{Subsets of}\\ \text{Spf(k[[G]])}\end{array} \end{array}$$

$\square$

*Remark.* In [16], Heyer and Schneider prove the following result:

*Proposition* 7.4. *Let $G$ be a compact abelian $p$-adic Lie group. Then, there is a bijection*

$$\text{Localizing subcategories of } D(G) \longleftrightarrow \text{Subsets of maximal ideals of } k[[G]]$$

We can obtain this as a corollary to our theorem 7.2 by noting that when $G$ is compact, $k[[G]]$ is the regular non-augmented Iwasawa algebra and it is always Noetherian. All that remains to see is that open prime ideals of $k[[G]]$ in this case are exactly the maximal ones: Indeed, we know by [26, Lemma 19.8], that when $k[[G]]$ is Noetherian, its topology is the $\mathrm{Jac}(G)$-adic topology, where $\mathrm{Jac}(G)$ is the intersection of all maximal ideals of $k[[G]]$. Therefore, if $\mathfrak{p}$ is an open prime ideal of $k[[G]]$, it contains $\mathrm{Jac}(G)$. But by [2, Proposition 1.11.ii)], this means that $\mathfrak{p} \supseteq \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of $k[[G]]$. This forces $\mathfrak{p}$ to be equal to $\mathfrak{m}$, i.e. $\mathfrak{p}$ is maximal.

In the next section, we discuss some examples of non-compact $G$'s for which the theorem 7.2 holds. We do this by explicitly computing $k[[G]]$ for each $G$ and then showing that they are Noetherian.

## 7.2 Examples

We start by making the following simple but useful observation.

**Lemma 7.5.** *Let $G$ be a $p$-adic Lie group such that $G \cong H \times H_0$ with $H$ discrete and $H_0$ compact. Then, $k[[G]] \cong k[H] \bigotimes_k k[[H_0]]$.*

*Proof.* Since $\{e\}$ is open in $H$ (under discrete topology), $H_0 \cong \{e\} \times H_0$ is a compact open subgroup of $G$. Then, by definition,

$$\begin{aligned}
k[[G]] &= k[G] \bigotimes_{k[H_0]} k[[H_0]] \\
&\cong k[H \times H_0] \bigotimes_{k[H_0]} k[[H_0]] \\
&\cong \left( k[H] \bigotimes_{k} k[H_0] \right) \bigotimes_{k[H_0]} k[[H_0]] \\
&\cong k[H] \bigotimes_{k} \left( k[H_0] \bigotimes_{k[H_0]} k[[H_0]] \right) \\
&\cong k[H] \bigotimes_{k} k[[H_0]].
\end{aligned}$$

The penultimate equivalence follows from [2, Exercise 2.15] since $k[H]$ is a $(k, k[H])$-bimodule with compatible structures. $\square$

*Example* 7.1 ($\mathbb{Q}_p^\times$). Since $\mathbb{Q}_p^\times \cong p^{\mathbb{Z}} \times (1 + p\mathbb{Z}_p^\times) \cong \mathbb{Z} \times \mathbb{Z}_p^\times$ and $\mathbb{Z}_p^\times$ is compact, by lemma 7.5,

$$k[[\mathbb{Q}_p^\times]] \cong k[\mathbb{Z}] \bigotimes_k k[[\mathbb{Z}_p^\times]].$$

Now, $k[\mathbb{Z}]$ is isomorphic to the Laurent polynomial ring $k[X, X^{-1}]$. Since $\mathbb{Z}_p^\times \cong (\mathbb{Z}/(p-1)\mathbb{Z}) \times \mathbb{Z}_p \cong \mu_{p-1} \times \mathbb{Z}_p$ and $\mu_{p-1}$ is discrete, by remark 3.4 in [1], $k[[\mathbb{Z}_p^\times]] \cong k[\mu_{p-1}] \bigotimes_k k[[\mathbb{Z}_p]]$. And the group ring $k[\mu_{p-1}]$ is isomorphic to $k[Y]/(Y^p - 1)$. Finally, we note that the Iwasawa algebra $k[[\mathbb{Z}_p]]$ is isomorphic to the power series ring $k[[T]]$ by [26, Proposition 20.1]. Therefore,

$$\begin{aligned}
k[[\mathbb{Q}_p^\times]] &\cong k[\mathbb{Z}] \bigotimes_k k[[\mathbb{Z}_p^\times]] \\
&\cong k[\mathbb{Z}] \bigotimes_k \left( k[\mu_{p-1}] \bigotimes_k k[[\mathbb{Z}_p]] \right) \\
&\cong k[X, X^{-1}] \bigotimes_k k[Y]/(Y^p - 1) \bigotimes_k k[[T]] \\
&\cong \big( (k[[T]]) [X, X^{-1}][Y] \big) / (Y^p - 1)
\end{aligned}$$

Since $k$ is Noetherian, so is $k[[T]]$ by [32, Lemma 10.31.2]. Then, using Hilbert basis theorem and the fact that quotients and localizations of a Noetherian ring are Noetherian again [32, Lemma 10.31.1], we can show that $k[[T]][X, X^{-1}][Y]/(Y^p - 1)$ is Noetherian as well. Therefore, so is $k[[\mathbb{Q}_p^\times]]$.

*Example* 7.2 ($(\mathbb{Q}_p^\times)^2$). Observe that $(\mathbb{Q}_p^\times)^2 \cong \mathbb{Z}^2 \times (\mathbb{Z}_p^\times)^2$ and $(\mathbb{Z}_p^\times)^2$ is compact.

Therefore, the computation of $k[[(\mathbb{Q}_p^\times)^2]]$ is very similar to the previous example.

$$\begin{aligned}
k[[(\mathbb{Q}_p^\times)^2]] &\cong k[\mathbb{Z}^2] \bigotimes_k k[[(\mathbb{Z}_p^\times)^2]] \\
&\cong k[\mathbb{Z}^2] \bigotimes_k \left( k[\mu_{p-1}^2] \bigotimes_k k[[(\mathbb{Z}_p)^2]] \right) \\
&\cong k[X_1^{\pm 1}, X_2^{\pm 1}] \bigotimes_k k[Y_1, Y_2]/(Y_1^p - 1, Y_2^p - 1) \bigotimes_k k[[T_1, T_2]] \\
&\cong k[[T_1, T_2]][X_1^{\pm 1}, X_2^{\pm 1}, Y_1, Y_2]/(Y_1^p - 1, Y_2^p - 1)
\end{aligned}$$

*Example* 7.3 ($(\mathbb{Q}_p^\times)^n$). Extending the previous example, we obtain that

$$k[[(\mathbb{Q}_p^\times)^n]] \cong k[[T_1, ..., T_n]][X_1^{\pm 1}, ..., X_n^{\pm 1}, Y_1, ..., Y_n]/(Y_1^p - 1, ..., Y_n^p - 1)$$

which is still Noetherian. This estabilishes the following proposition.

**Proposition 7.6.** *Let $T$ be a split torus over $\mathbb{Q}_p$ and $k$ a field of characteristic $p$. Then, $k[[T]]$ is Noetherian.*

In the next couple of examples, we do not completely compute $k[[G]]$, but we can still show directly that they are Noetherian.

*Example* 7.4 ($F^\times$). Let $F$ be a finite extension of $\mathbb{Q}_p$. Pick a uniformizer $\pi$ for the maximal ideal of $\mathcal{O}_k$. Then, each $x \in F^\times$ can be uniquely written in the form $u\pi^v(x)$ with $u \in \mathcal{O}_F^\times$ and $v(x) \in \mathbb{Z}$. Therefore, $F^\times \cong \mathbb{Z} \times \mathcal{O}_F^\times$, and since $\mathcal{O}_F^\times$ is compact, we can use lemma 7.5 to compute $k[[F^\times]]$ as

$$\begin{aligned} k[[F^\times]] &\cong k[\mathbb{Z}] \bigotimes_k k[[\mathcal{O}_F^\times]] \\ &\cong k[X, X^{-1}] \bigotimes_k k[[\mathcal{O}_F^\times]] \\ &\cong k[[\mathcal{O}_F^\times]][X, X^{-1}] \end{aligned}$$

Now, while we have not explicitly computed the Iwasawa algebra $k[[\mathcal{O}_F^\times]]$, we know it is Noetherian by Lazard theory since $\mathcal{O}_F^\times$ is compact. And since $k[[F^\times]]$ is isomorphic to a ring of Laurent polynomials over this Noetherian ring, $k[[F^\times]]$ is also Noetherian.

*Example* 7.5. As in the example 8.3, we can extend the previous example of $F^\times$ to the case of $(F^\times)^n$. Since $(F^\times)^n \cong (\mathbb{Z} \times \mathcal{O}_F^\times)^n \cong \mathbb{Z}^n \times (\mathcal{O}_F^\times)^n$ and $(\mathcal{O}_F^\times)^n$ is compact,

$$k[[(F^\times)^n]] \cong k[[(\mathcal{O}_F^\times)^n]][X_1^{\pm 1}, ..., X_n^{\pm 1}]$$

is Noetherian.

All of the examples in this section are $p$-adic (algebraic) tori. So, it is natural to raise the question of whether theorem 7.2 holds for all tori. In the next chapter, we answer this question in the affirmative.

# Chapter 8

# Tori

The main result of this chapter is the following.

**Theorem 8.1.** *Let $T$ be an algebraic torus over $\mathbb{Q}_p$ and $k$ be a field of characteristic $p$. Then, the augmented Iwasawa algebra $k[[T(\mathbb{Q}_p)]]$ is Noetherian.*

We prove this by extending the work already done in the last section while discussing examples. But first we need to record some definitions and common results about tori over a $p$-adic field.

## 8.1 General Tori

Let $k$ be an algebraically closed field with a subfield $F$.

**Definition 8.1.** By an **algebraic group**, we mean an algebraic variety $G$ which is also a group such that the group structure maps $\mu : G \times G \to G$ and $i : G \to G$ with

$\mu(x, y) = xy$ and $i(x) = x^{-1}$ are morphisms of varieties. $G$ is a **linear algebraic group** is the underlying variety is affine.

The algebraic group $G$ is an $F$-**group** if $G$ is an $F$-variety, $\mu$ and $i$ are defined over $F$, and the identity element is an $F$-rational point. If $G$ is an $F$-group, denote the set of $F$-rational points by $G(F)$.

The following theorem is useful when working with linear algebraic groups.

**Theorem 8.2.** *A linear algebraic group $G$ is isomorphic to a closed subgroup of $GL_n$ for some $n$. If $G$ is an $F$-group, this isomorphism can be defined over $F$.*

*Proof.* This is [31, Theorem 2.3.7]. □

For example, if $G$ is isomorphic to an algebraic $F$-subgroup of $GL_n$, the topology of $GL_n(F)$ induces a topology on $G(F)$. In particular, if $G$ is a linear algebraic group defined over a $p$-adic field $F$, $G(F)$ is a $p$-adic Lie group. (See [24, Section 3.1].)

**Definition 8.2.** An algebraic group $G$ is called a torus if there is an isomorphism $G \cong (\mathbb{G}_m)^d$, where $d$ is the dimension of $G$.

**Definition 8.3.** An $F$-torus $T$ is called $F$-**split** if it is $F$-isomorphic to a product $(\mathbb{G}_m)^n$ for some $n$. $T$ is called $F$-**anisotropic** if it does not contain any non-trivial $F$-split subtori.

**Proposition 8.3.** *Let $F$ be a finite extension of $\mathbb{Q}_p$ and $T_0$ an $F$-anisotropic torus. Then $T_0$ is compact.*

*Proof.* This follows directly from [24, Theorem 3.1]. □

**Theorem 8.4.** *Let $G_1$ be an affine algebraic group scheme over a field $F$. Let $G_1 \to G_2$ be a quotient map with kernel $N$. Then there is a function $G_1(F) \to H^1(\overline{F}/F, N)$ for which the sequence*

$$1 \to N(F) \to G_1(F) \to G_2(F) \to H^1(\overline{F}/F, N) \to H^1(\overline{F}/F, G_1) \to H^1(\overline{F}/F, G_2)$$

*is exact. If $G_1$ is commutative, the maps are homomorphisms.*

*Proof.* This is [35, Theorem 18.1]. □

**Proposition 8.5.** *Let $F$ be a field. Given a $F$-torus $T$, we can find an $F$-split torus $T_1$ and $F$-anisotropic torus $T_0$ such that there is an isogeny (quotient map with finite kernel) from $T_0 \times T_1$ to $T$.*

*Proof.* This is [31, Proposition 13.2.4]. Take $T_1$ to be the unique maximal $F$-split subtorus of $T$ and $T_0$ to be the unique maximal $F$-anisotropic subtorus of $T$. □

Applying theorem 8.4 to the above proposition, we get the following important corollary which will form the base of our proof.

**Corollary 8.6.** *Given an $F$-torus $T$, we can find tori $T_0$ and $T_1$ that are $F$-anisotropic and $F$-split respectively such that we have an exact sequence*

$$M \to T_0(F) \times T_1(F) \xrightarrow{\varphi} T(F) \xrightarrow{f} H^1(\overline{F}/F, M)$$

*with $M$ finite.*

## 8.2 Proof

We start with a few lemmas.

**Lemma 8.7.** *Let $T_0$ and $T_1$ be $\mathbb{Q}_p$-tori. Assume that $T_0$ is $\mathbb{Q}_p$-anisotropic and $T_1$ is $\mathbb{Q}_p$-split. Then $k[[T_1(\mathbb{Q}_p) \times T_0(\mathbb{Q}_p)]]$ is Noetherian.*

*Proof.* Since $T_1$ is split over $\mathbb{Q}_p$, $T_1(\mathbb{Q}_p) \cong (\mathbb{Q}_p^\times)^n$ for some $n \geq 1$. So, we have

$$\begin{aligned} T_1(\mathbb{Q}_p) \times T_0(\mathbb{Q}_p) &\cong (\mathbb{Q}_p^\times)^n \times T_0(\mathbb{Q}_p) \\ &\cong (\mathbb{Z}\times\mathbb{Z}_p^\times)^n\times T_0(\mathbb{Q}_p) \\ &\cong \mathbb{Z}^n\times(\mu_{p-1}\times\mathbb{Z}_p)^n\times T_0(\mathbb{Q}_p) \\ &\cong (\mathbb{Z}^n\times\mu_{p-1}^n)\times(\mathbb{Z}_p^n\times T_0(\mathbb{Q}_p)) \end{aligned}$$

Now, $\mathbb{Z}^n\times\mu_{p-1}^n$ is discrete. And since $Z_p$ is compact and $T_0(\mathbb{Q}_p)$ is compact by proposition 8.3, by Tychonoff's theorem, $\mathbb{Z}_p^n\times T_0(\mathbb{Q}_p)$ is compact. So, we can apply lemma 7.5 to compute the augmented Iwasawa algebra.

$$\begin{aligned} k[[T_1(\mathbb{Q}_p) \times T_0(\mathbb{Q}_p)]] &\cong k[\mathbb{Z}^n\times\mu_{p-1}^n] \bigotimes_k k[[\mathbb{Z}_p^n\times T_0(\mathbb{Q}_p)]] \\ &\cong k[X_1^{\pm1}, ..., X_n^{\pm1}, Y_1, ..., Y_n]/(Y_1^p - 1, ..., Y_n^p - 1) \bigotimes_k k[[\mathbb{Z}_p^n\times T_0(\mathbb{Q}_p)]] \\ &\cong \left(k[[\mathbb{Z}_p^n\times T_0(\mathbb{Q}_p)]]\right) [X_1^{\pm1}, ..., X_n^{\pm1}, Y_1, ..., Y_n]/(Y_1^p - 1, ..., Y_n^p - 1) \end{aligned}$$

Now we can argue as in examples 9.4 and 9.5: Since $\mathbb{Z}_p^n\times T_0(\mathbb{Q}_p)$ is compact, $k[[\mathbb{Z}_p^n\times T_0(\mathbb{Q}_p)]]$ is Noetherian and hence, $\left(k[[\mathbb{Z}_p^n\times T_0(\mathbb{Q}_p)]]\right) [X_1^{\pm1}, ..., X_n^{\pm1}, Y_1, ..., Y_n]/(Y_1^p-$

$1, ..., Y_n^p - 1)$ is also Noetherian. This proves the lemma. $\square$

The next lemma is purely ring theoretic.

**Lemma 8.8.** *Let $R$ be a Noetherian ring and let $S$ be an $R$-algebra that is finite as an $R$-module. Then, $S$ is also Noetherian.*

*Proof.* This is [12, Proposition 1.6]. $\square$

The final lemma we need is about cohomology groups.

**Lemma 8.9.** *Let $E$ be a finite extension of $\mathbb{Q}_p$ and let $G_E$ denote* $\mathrm{Gal}(\overline{E}/E)$*. If $A$ is a finite $G_E$-module, then $H^n(\overline{E}/E, A)$ is finite for every $n$.*

*Proof.* This is [29, Chapter II, Proposition 14]. $\square$

We are now ready to prove theorem 8.1

*Proof.* Given a $\mathbb{Q}_p$-torus $T$, by corollary 8.6, we can find a $\mathbb{Q}_p$-split torus $T_0$ and a $\mathbb{Q}_p$-anisotropic torus $T_1$ such that there is an exact sequence

$$M \to T_0(\mathbb{Q}_p) \times T_1(\mathbb{Q}_p) \xrightarrow{\varphi} T(\mathbb{Q}_p) \xrightarrow{f} H^1(\overline{\mathbb{Q}_p}/Q_p, M) \tag{8.2.1}$$

with $M$ finite.

Assume that $k[[T(\mathbb{Q}_p)]]$ is a finite module over $k[[\mathrm{Im}(\varphi)]]$.

Now, the surjection $T_0(\mathbb{Q}_p) \times T_1(\mathbb{Q}_p) \to \mathrm{Im}(\varphi)$ induces a surejective $k$-algebra homomorphism $k[[T_0(\mathbb{Q}_p) \times T_1(\mathbb{Q}_p)] \to k[[\mathrm{Im}(\varphi)]]$ [34, Proposition 2.22]. Therefore, since $k[[T_0(\mathbb{Q}_p) \times T_1(\mathbb{Q}_p)]$ is Noetherian by lemma 8.7, $k[[\mathrm{Im}(\varphi)]]$ is also Noetherian.

Then finally, being a finite module over a Noetherian ring, $k[[T(\mathbb{Q}_p)]]$ is also Noetherian by lemma 8.8.

So, it suffices to show that $k[[T(\mathbb{Q}_p)]]$ is a finite module over $k[[\mathrm{Im}(\varphi)]]$ to complete the proof. We do this by finding a subgroup of $\mathrm{Im}(\varphi)$ that has finite index in $T(\mathbb{Q}_p)$.

**Claim 8.10.** *There exists an $N > 0$ such that $T(\mathbb{Q}_p)^N$ is a subgroup of* $\mathrm{Im}(\varphi)$*.*

Consider the exact sequence 8.2.1. Since $M$ is finite, so is $H^1(\overline{\mathbb{Q}_p}/\mathbb{Q}_p, M)$ by lemma 8.9. Let $N$ be its order. Then, given $\alpha \in T(\mathbb{Q}_p)$, $f(\alpha^N) = f(\alpha)^N = 1$. Thus, $\alpha^N \in \ker(f)$. However, since the sequence 8.2.1 is exact, $\ker(f) = \mathrm{Im}(\varphi)$. Therefore, $\alpha^N \in \mathrm{Im}(\varphi)$. Since $\alpha$ was an arbitrary element of $T(\mathbb{Q}_p)$, this shows that $T(\mathbb{Q}_p)^N \subseteq \mathrm{Im}(\varphi)$, proving the claim.

Next, we show the following.

**Claim 8.11.** *$T(\mathbb{Q}_p)^N$ has finite order in $T(\mathbb{Q}_p)$.*

We know that $T(\overline{\mathbb{Q}_p}) \cong \overline{\mathbb{Q}_p}^n$ for some $n$. So, we have an exact sequence

$$1 \to \mu_N^n \to T(\overline{\mathbb{Q}_p}) \xrightarrow{N} T(\overline{\mathbb{Q}_p}) \to 1$$

where $\mu_N$ denotes the $N$-th roots of unity in $\overline{\mathbb{Q}_p}$ and $N$ denotes the map $\alpha \mapsto \alpha^N$. This leads to the cohomology exact sequence

$$1 \to \mu_N^n(\mathbb{Q}_p) \to T(\mathbb{Q}_p) \xrightarrow{N} T(\mathbb{Q}_p) \xrightarrow{f'} H^1(\overline{\mathbb{Q}_p}/Q_p, \mu_N^n)$$

when we consider the $\mathbb{Q}_p$ points.

By lemma 8.9, $H^1(\overline{\mathbb{Q}_p}/Q_p, \mu_N^n)$ is finite. Therefore, $\mathrm{Im}(f')$ is also finite. And since

$$\mathrm{Im}(f') \cong T(\mathbb{Q}_p)/\ker(f') = T(\mathbb{Q}_p)/\mathrm{Im}(N) = T(\mathbb{Q}_p)/T(\mathbb{Q}_p)^N,$$

this establishes the claim.

Thus, we have found an $N$ such that $T(\mathbb{Q}_p)^N \leq \mathrm{Im}(\varphi) \leq T(\mathbb{Q}_p)$ and $[T(\mathbb{Q}_p) : T(\mathbb{Q}_p)^N]$ is finite. Therefore, $[T(\mathbb{Q}_p) : \mathrm{Im}(\varphi)]$ is finite as well . Say $s = [T(\mathbb{Q}_p) : \mathrm{Im}(\varphi)]$. Then, $k[T(\mathbb{Q}_p)]$ is a finite module over $k[\mathrm{Im}(\varphi)]$ and $k[T(\mathbb{Q}_p)] \cong \bigoplus_1^s k[\mathrm{Im}(\varphi)]$.

Furthermore, since $(T(\mathbb{Q}_p)/\mathrm{Im}(\varphi))$ is finite and therefore discrete, by [25, Proposition 1.4 (iv)], $\mathrm{Im}(\varphi)$ is open in $T(\mathbb{Q}_p)$. Choose a compact open subgroup $U$ of $\mathrm{Im}(\varphi)$. Then, $U$ is compact and open in $k[[T(\mathbb{Q}_p)]]$ as well and we obtain an isomorphism

$$\begin{aligned} k[[T(\mathbb{Q}_p)]] &= k[T(\mathbb{Q}_p)] \bigotimes_{k[U]} k[[U]] \\ &\cong \left(\bigoplus_1^s k[\mathrm{Im}(\varphi)]\right) \bigotimes_{k[U]} k[[U]] \\ &\cong \bigoplus_1^s \left(k[\mathrm{Im}(\varphi)] \bigotimes_{k[U]} k[[U]]\right) \\ &\cong \bigoplus_1^s k[[\mathrm{Im}(\varphi)]] \end{aligned}$$

So, $k[[T(\mathbb{Q}_p)]]$ is a finite module over $k[[\mathrm{Im}(\varphi)]]$ which is Noetherian. This completes our proof. □

*Remark.* All of our arguments above work with minimal modifications if we replace

$\mathbb{Q}_p$ with a finite extension $F$ of $\mathbb{Q}_p$. Thus, we can obtain the following statement: Let $T$ be an algebraic torus over a finite extension $F$ of $\mathbb{Q}_p$ and let $k$ be a field of characteristic $p$. Then, the augmented Iwasawa algebra $k[[T(F)]]$ is Noetherian.

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