Title
Topics in electroweak symmetry breaking

Authors
Uttayarat, Patipan
Uttayarat, Patipan

Publication Date
2012

Peer reviewed|Thesis/dissertation

UNIVERSITY OF CALIFORNIA, SAN DIEGO

Topics in Electroweak Symmetry Breaking

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Physics

by

Patipan Uttayarat

Committee in charge:

Professor Benjamín Grinstein, Chair
Professor Mark Gross
Professor Kenneth Intriligator
Professor Justin Roberts
Professor Avraham Yagil

2012

The dissertation of Patipan Uttayarat is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

____

____

____

____

____

Chair

University of California, San Diego

2012

EPIGRAPH

Religion is the opium of the masses.

—Karl Marx

TABLE OF CONTENTS

LIST OF FIGURES

ACKNOWLEDGEMENTS

There are many people whom I would like to acknowledge for contributing in various ways, both directly and indirectly, to this dissertation. First and foremost to my advisor, Ben Grinstein, who over the past six years has transformed me from a naive graduate student into a physicist. Through countless discussions, he has expanded my knowledge, pointed me to interesting projects and guided me through them. He has taught me the arts and crafts necessary to be successful in our field. All the accomplishments I have had (as well as future accomplishments, if any) are in no small part due to the advice and guidance he has provided. So thank you very much Ben.

My dissertation could never have been completed without my committee Mark Gross, Ken Intriligator, Justin Roberts and Avi Yagil. They have put in considerable time and effort and I thank them for that. I also want to acknowledge Nitu Kitchloo and Kate Okikiolu for serving on my advancement committee.

I have also benefited tremendously from interactions with Ken Intriligator, Julius Kuti and Aneesh Manohar. Through numerous casual conversations and formal lectures, I have learned a great amount of physics from them. It also happens to be a lot of fun hanging out with them. Thank you for making my time at UC San Diego enjoyable and productive.

I want to thank my teachers at UCSD: Micheal Fogler, George Fuller, Ben Grinstein, Terry Hwa, Ken Intriligator, Elizabeth Jenkins, Nitu Kitchloo, Aneesh Manohar, Tom O'Neil, Raj Pathria and Justin Roberts for all the stimulating classes they taught. The faculty at the University of Virginia also deserve thanks for teaching me the fundamentals and nurturing my curiosity in physics and mathematics. Here is the partial list of professors who have left a lasting impression on me: Peter Arnold, Kia-Uwe Bux, John Faulkner, Paul Fendley, Michael Fowler, Eugene Kolomeisky and Cass Sackett. Cass Sackett also deserves a special recognition for taking me into his lab and introducing me to scientific research.

Thanks to the particle theory graduate students and postdocs for numerous stimulating discussion and also for answering many of my stupid questions: Prarit Agarwal, Antonio Amarati, Grigor Aslanyan, Mannie Chiu, Casey Conger, Jeff Fortin, Andy Fuhrer, Randy Kelley, Chris Murphy, Dani Nogradi, David Pirtskhalava, Chris

Schroeder, Brian Shotwell, Andy Stergiou, David Stone, Ze'ev Surujon, Mike Trott and Wouter Waalewijn. Also special thanks to Chris Murphy, David Pirtskhalava, David Stone and Ze'ev Sorujon for patiently putting up with me while we work on many interesting projects.

There are many people I want to thanks for having non-physics interaction with me. First of all, to my friends, the Thai community in San Diego and my soccer buddies, thank you for all the wonderful activities we have. You guys are a lot of fun to hang out with. A few names deserve a special recognition for keeping me happy and/or sane: Kun, Noon, Namfon, Prae, Tee, Yhew, and most importantly Manow. I thank you all.

I also want to acknowledge my family for their unwavering support. My father and mother for shaping who I am and letting me choose the path I want to follow. I thank my sister for always looking out for me. I also thank my extended family for being so caring and loving to me.

Last but not least, the people at the Art of Espresso deserve an acknowledgment for brewing delicious espressos and keeping me function throughout the day. The Hare Krishna people also deserve a recognition for serving up great vegetarian food and desserts every Wednesday.

Finally, some technical acknowledgments. Chapter 2 and Appendix A comprise a reprint of the material as it appears in "Hidden fine tuning in the quark sector of little higgs models," B. Grinstein, R. Kelley and P. Uttayarat, JHEP **0909**, 040 (2009) [arXiv:0904.1622 [hep-ph]]. Chapter 3 is a reprint of material as it appears in "One Loop Renormalization of the Littlest Higgs Model," B. Grinstein, R. Kelley and P. Uttayarat, JHEP **1102**, 089 (2011) [arXiv:1011.0682 [hep-ph]]. Chapter 4 and Appendix B, C, and D, comprise a reprint of material as it appears in "Spontaneous CP Violation and Light Particles in The Littlest Higgs," Z. Surujon and P. Uttayarat, Phys. Rev. D **83**, 076010 (2011) [arXiv:1003.4779 [hep-ph]]. Chapter 5 is a reprint of material as it appears in "A Very Light Dilaton," B. Grinstein and P. Uttayarat, JHEP **1107** (2011) 038 [arXiv:1105.2370 [hep-ph]]. I was a co-author of all of these publications. This work was supported in part by UCSD grant DOE-FG03-97ER40546.

VITA

2006	B. S. in Physics, University of Virginia, Charlottesville
2007-2012	Teaching and Research Assistant, University of California, San Diego
2010-2012	Lecturer, University of California, San Diego
2011	M. S. in Physics, University of California, San Diego
2012	Ph. D. in Physics, University of California, San Diego

PUBLICATIONS

B. Grinstein and **P. U.**, "A Very Light Dilaton," JHEP **1107**, 038 (2011) [arXiv:1105.2370 [hep-ph]].

B. Grinstein, R. Kelley and **P. U.**, "One Loop Renormalization of the Littlest Higgs Model," JHEP **1102**, 089 (2011) [arXiv:1011.0682 [hep-ph]].

Z. Surujon and **P. U.**, "Spontaneous CP Violation and Light Particles in The Littlest Higgs," Phys. Rev. D **83**, 076010 (2011) [arXiv:1003.4779 [hep-ph]].

B. Grinstein, R. Kelley and **P. U.**, "Hidden fine tuning in the quark sector of little higgs models," JHEP **0909**, 040 (2009) [arXiv:0904.1622 [hep-ph]].

ABSTRACT OF THE DISSERTATION

Topics in Electroweak Symmetry Breaking

by

Patipan Uttayarat

Doctor of Philosophy in Physics

University of California, San Diego, 2012

Professor Benjamín Grinstein, Chair

In this dissertation, we discuss topics in electroweak symmetry breaking and related phenomena. A quick review of the Standard Model of particle physics is given in Chapter 1. There we also discuss its hierarchy problem and a possible solution– the Littlest Higgs model. In Chapter 2 and 3, we show that the Littlest Higgs Model and its variants do not successfully solve the hierarchy problem. In Chapter 4 we explore the possibility of spontaneous CP violation in the Littlest Higgs and related models. Another topic that we explore is the existence of a light dilaton, a pseudo-Nambu-Goldstone boson of a spontaneously broken scale symmetry, which is the subject of Chapter 5.

Chapter 1

Introduction

In this chapter I provide context for the work contained in the following chapters. I also review background material for lay-readers. Readers with basic knowledge of particle physics can skip to Section 1.1.4.

1.1 The Basics

One of the most important objective of science is to understand how nature works. On the quest to understand the fundamental laws of nature, we have come across many puzzles and their resolutions. Even with the considerable amount of progress made during the past century the universe is still a place full of mysteries. Resolving (some of) these mysteries is a necessary step for understanding how nature works. However, before discussing these unresolved puzzles, some of which are presented in this dissertation, it is instructive to review first the successes we have made in our attempt to understand the basic laws of nature.

Nature, at the most fundamental level, can be described by a set of building blocks (elementary particles), interactions (forces), and universal rules. Our universe, to a good approximation, follows the rules of relativity and quantum mechanics. We have identified 12 elementary particles, the quarks and leptons, and 4 types of force: gravity, electromagnetic, strong and weak. Gravity, the most familiar force from our everyday experience, is the universal force that every particle feels. Gravity successfully explains why we do not float around; why there is low tide and high tide; why the Earth goes around the sun, etc.. Even though it is the most familiar force, it remains the least understood. We do not understand how to incorporate gravity into a quantum theory. Luckily, due to the feeble nature of gravity, its effect on the interactions of elementary particles can be ignored. Thus, we will leave gravity out from the rest of this work.

Another familiar force from everyday experience is electromagnetism. It is the force that charged particles feel. The electromagnetic force is responsible for a wide range of phenomena: from static shock to lightning to the generation of electricity for our daily consumption. However, few people know that it is also responsible for light, or more generally, electromagnetic radiation. Unlike gravity, we do have a quantum theory of the electromagnetic interactions which successfully describes virtually all atomic phenomena. It is the most well-tested theory in all of physics.

The strong and weak forces are the least familiar. They operate at a subatomic level. The strong force is responsible for binding protons together inside the nucleus while the weak force is responsible for radioactivity. As is the case for the electromagnetic force, we have a quantum theory of strong and weak interactions. The quantum theory of the electromagnetic, strong and weak interactions is the cornerstone of our current understanding of the fundamental laws of nature, on which we will focus on for the rest of this chapter.

1.1.1 The Standard Model of Particle Physics

A fully relativistic and quantum mechanical theory that describes the electromagnetic, strong and weak interactions of quarks and leptons is known as the Standard Model (SM). Each type of interaction of quarks and leptons is a consequence of a symmetry. In the SM, the interaction is mediated by a force carrier, a gauge boson. Mathematically, the SM is a quantum field theory with a $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge group.

It is amazing that, at the fundamental level, much of the complex universe can be understood in terms of quarks and leptons and the three symmetries for the electromagnetic, strong and weak interactions. The SM has had tremendous successes when compared against experimental results. Prime examples are the prediction of an anomalous magnetic dipole moment of the electron and the invisible width of the Z boson. Despite its paramount successes there remain puzzles, both experimental and theoretical, that the SM cannot explain. On the experimental side, the observation of baryon asymmetry in the universe, dark matter, and the accelerated expansion of the universe cannot be accommodated within the SM. These observations suggest that the SM is an incomplete theory and needs to be extended to include new physics responsible for the above observations. SM also contains some theoretical puzzles. Most of the puzzles are in the electroweak sector, the $SU(2)_L \times U(1)_Y$ part responsible for the electromagnetic and weak interactions. The electroweak sector of the SM is the subject that we will turn to next.

1.1.2 The Electroweak Sector

The electroweak sector of the SM is the best verified sector experimentally. The gauge bosons responsible for the electroweak interaction are the $W^{\pm}$ and Z bosons, and the photon. Naively these gauge bosons, according to gauge theory, are expected to be massless. However, the photon is the only massless gauge boson we have observed so far. Both $W^{\pm}$ and Z are massive. To make matters worse, electroweak symmetry seems to constrain quarks and leptons to be massless in contradiction with experimental measurement. These apparent inconsistencies are resolved by the mechanism of spontaneous symmetry breakdown— the fundamental dynamics of the system respect the symmetry, but the ground state that the system settles into does not. Thus all the successful predictions of the theory remain valid while the $W^{\pm}$ and Z bosons, as well as the quarks and leptons, become massive.

In the SM, electroweak symmetry is broken by a hypothetical particles, the Higgs. The interaction of the Higgs with gauge bosons, quarks and leptons respect electroweak symmetry. However, the dynamics of the Higgs is such that it can settle into a ground state that is not invariant under electroweak symmetry. This is known as electroweak symmetry breaking. Once the Higgs is in this electroweak breaking ground state, the $W^{\pm}$ and Z bosons, as well as quarks and leptons, become massive due to their interaction with the Higgs— the Higgs mechanism. We parametrize the amount by which electroweak symmetry is broken by the vacuum expectation value (vev) of the Higgs, v. Experimentally, $v = 256$ GeV (about 250 times the mass of the proton). The scale v, which is often referred to as the electroweak scale, sets the scale of the gauge boson, quark and lepton masses.

The introduction of the Higgs particle successfully resolves the apparent inconsistency of the electroweak sector of the SM by the Higgs mechanism. It also leaves a remnant that could be detected at the Large Hadron Collider (LHC), the Higgs boson. Since the Higgs boson is tied to electroweak symmetry breaking (EWSB), the mass of the Higgs boson is expected to be around the electroweak scale. However, quantum effects will in general push the mass of the Higgs boson to a much higher scale associated with the new physics scale. Thus in order to have the Higgs boson at the electroweak scale, there must be a significant cancellation among quantum effects— also known as fine-tuning. For example, the scale of new physics expected from flavor physics experiments is around 10 TeV. This requires a fine-tuning at the level of 1 part in 10,000 in order to have to have the Higgs mass at around 100 GeV. The amount of fine-tuning gets worse for a higher scale of new physics (grand unification: 1 in 10^{26}, gravity: 1 in 10^{34}, etc.). The large amount of fine-tuning seems theoretically unsettling. The puzzle of why the electroweak scale is much lighter compared to other scales is known as the hierarchy problem. Solving the hierarchy problem has been one of the main focuses of recent theoretical particle physics research.

There is a large literature on possible solutions of the hierarchy problem. They can be classified into three categories: theories with extra-dimensions, supersymmetric theories, and little Higgs models. It is impossible to cover them all in this dissertation. Instead, we will focus on the little Higgs models which are described in Section 1.1.4. However, it is necessary to introduce first the concept of Nambu-Goldstone boson in order to understand the mechanism employed to solve the hierarchy problem in this type of models.

1.1.3 Consequence of Spontaneous Symmetry Breaking

In this subsection we discuss one important consequence of spontaneous symmetry breakdown— the appearance of massless particles known as Nambu-Goldstone bosons (NGBs). More concretely, for every generator of a continuous symmetry that is spontaneously broken, there is a massless particle associated with it. It is worth mentioning that when the spontaneously broken symmetries are only approximate symmetries, the resulting NGBs become massive. They are referred to as pseudo-Nambu-Goldstone bosons. The mass of the pseudo-Nambu-Goldstone boson is proportional to the amount by which the original symmetry is explicitly broken.

The NGBs are an important ingredient for the Higgs mechanism. The gauge boson corresponds to the spontaneously breakdown gauge symmetry can combine with the would-be NGB of the symmetry breaking to become massive. In the electroweak sector of the SM, for example, the $SU(2)_L \times U(1)_Y$ gauge symmetry is spontaneously broken down to $U(1)_{EM}$. The would be NGBs of this symmetry breaking merge with the otherwise massless $W^{\pm}$ and Z gauge bosons to give them masses.

1.1.4 The Little Higgs Models

We are now ready to discuss little Higgs models. As has been pointed out in the previous section, PNGBs of weakly broken symmetries can be generically light. In little Higgs models, the Higgs particles are PNGBs of some symmetries which are weakly broken at the new physics scale. However, this is usually not enough to ensure that the Higgs vev is much smaller than a new physics scale.

In order to have the Higgs mass much lighter than the new physics scale, little Higgs models usually employ a clever symmetry breaking pattern where the symmetry is broken only collectively (see Chapter 2 for more detail). This collective symmetry breaking protects the Higgs mass from leading quantum corrections. Thus only subleading quantum effects contribute to the Higgs mass and hence it can be, in general, much lighter than the new physics scale. However, this seemingly simple idea turns out to be difficult to implement consistently, and is the subject of the work discussed in this dissertation.

1.2 Outline and Conclusions

In Chapter 2 we consider the most widely studied little Higgs model, the Littlest Higgs. In this model, the collective symmetry needed to protect the Higgs mass from quantum effects is itself violated by quantum effects. Thus the protection is only possible when the quantum effects are fine-tuned so that they don't impede collective symmetry. Hence, the fine-tuning of the electroweak sector of the SM is moved to the fine-tuning of collective symmetry mechanism. We also proceed to show that such a fine-tuning is generic in the little Higgs model in which collective symmetry arises in the limit that some interactions are taken to be vanishing. The precise meaning of this statement will be made clear in Chaper 2.

In Chapter 3, we present a full analysis of leading quantum effects of the Littlest Higgs model. There, we introduce a technique for categorizing such effects in the model. With this study, it is easy to see that the Littlest Higgs model suffers from a fine-tuning problem.

In Chapter 4, we explore the possibility of accommodating spontaneous CP violation in the Littlest Higgs model. We find that this is impossible due to the minimal nature of the model. However, a simple extension of the model does allow for spontaneous CP violation. This CP violation could be useful in explaining the observed baryon asymmetry in the universe. Moreover, we show that such an extension of the Littlest Higgs model contains a new light PNGB which leads to interesting LHC phenomenology.

In Chapter 5, we study a slightly related topic regarding the spontaneous breakdown of the scale (or dilatation) symmetry. In a theory with a spontaneously broken approximate scale invariant, one would expect light dilaton— the PNGB of the dilatation symmetry. What is meant by light is that in the limit that the scale symmetry is exact, the dilaton becomes massless. However, quantum effects break dilatation symmetry explicitly and it becomes unclear if there is a dilaton at all. In fact, there has been a long standing debate whether there exist a quantum theory containing a light dilaton. In this Chapter, we construct a toy model in which, after quantization, there is a light dilaton in the spectrum of the theory. Moreover, we are able to verify, by direct computation, the properties of the dilaton often quoted in the literature obtained via indirect arguments. Possible motivations to study such a model are given in the chapter.

Chapter 2

Hidden Fine Tuning in the Quark Sector of Little Higgs Models

In little higgs models a collective symmetry prevents the higgs from acquiring a quadratically divergent mass at one loop. By considering first the littlest higgs model we show that this requires a fine tuning: the couplings in the model introduced to give the top quark a mass do not naturally respect the collective symmetry. We show the problem is generic: it arises from the fact that the would be collective symmetry of any one top quark mass term is broken by gauge interactions.

2.1 Introduction

Little Higgs (LH) models offer an alternative to the standard model in which no fundamental scalars need be introduced (for reviews see [1, 2, 3]). Generally, in LH models the Higgs is a composite particle, bound by interactions that become strong at a scale Λ. The mass of the Higgs is much less than Λ as the Higgs is a pseudo-Goldstone boson (PGB) of broken global symmetries in the theory of the new strong interaction.

The global "flavor" symmetry G_f of these models has a subgroup G_w that is weakly gauged. In the absence of this weak gauge force, the flavor symmetry is broken spontaneously to a subgroup H due to hyper-strong interactions at the scale Λ. As a result, there are massless Goldstone bosons that are coordinates on the G_f/H coset space. Since the weakly gauged G_w force breaks the flavor symmetry explicitly, including its effects leads to some of the Goldstone bosons (the would-be Goldstone bosons) being eaten by the Higgs mechanism and the rest becoming PGBs acquiring small masses of order Λ times a small symmetry breaking parameter, the weak gauge coupling constant. The Higgs is the lightest PGB in LH models, and its mass is naturally much less than Λ (and the other PGBs): due to the collective symmetry breaking mechanism its mass arises only at two loops.

Additional interactions must be included in LH models to account for quark and lepton masses. At low energies they reproduce the Yukawa couplings of the standard model. Since these interactions also break the flavor symmetry, they contribute to the masses of the PGBs. In order to ensure that the Higgs remains much lighter than Λ, the quark interactions are designed to implement the collective symmetry breaking mechanism again.

In the littlest higgs[4] model (L^2H) and variants a collective symmetry arises naturally in the gauge sector. Turning off some gauge couplings gives an enlarged symmetry group of the Lagrangian. This ensures that when those couplings are turned off the higgs remains an exact Goldstone boson. The implementation of the collective symmetry in the Yukawa couplings that give rise to the top quark mass is somewhat different. For example, in the L^2H the top-quark doublet is combined with a new quark singlet into a 'triplet' of collective $SU(3)$ symmetry. However, this is not automatically a symmetry of the Lagrangian, as it is not respected by gauge interactions. In other words, the couplings of the doublet and the singlet are a priori independent, but need to be equal in order to implement the collective symmetry mechanism. In this paper we investigate whether it is possible to construct a littlest higgs model for which collective symmetry in the top-quark sector arises naturally.

If one used generic couplings in the L^2H model for the doublet and the singlet of the top quark 'triplet', then loop diagrams induce unsuppressed, order $\Lambda/4\pi$, Higgs masses. Such generic couplings are not forbidden by any symmetry. As we will see, even if the coupling of the 'triplet' is taken to respect the collective $SU(3)$ symmetry at tree level, radiative effects split it into two terms. These, in fact, have different anomalous dimensions (they 'run' differently). The reader can view enforcing collective symmetry in the top quark sector as a fine tuning. Alternatively, one may argue that assuming the symmetry is consistent with the littlest higgs approach. Only an explicit UV completion can validate one view over the other. We will not investigate UV completions in this work, but enquire whether specific models avoid the issue.

Cannot collective symmetry arise naturally by gauging it? After all, if the symmetry is gauged then the restricted form of the quark coupling is a result of the gauge symmetry. For example one may construct a model based on $G_f/H = U(7)/O(7)$ with $G_w = SU(3) \times SU(2) \times U(1)^3$. The vacuum aligns[5, 6] so that G_w breaks to the electroweak subgroup $SU(2) \times U(1)$ at the scale Λ, and the spectrum has a light Higgs doublet plus many heavier PGBs. Could the gauged $SU(3)$ now play the role of the collective symmetry for the top quark mass? The problem with this is that the gauge symmetry is broken and the would be higgs is eaten. This model is higgsless. This is also generic: the collective symmetry must act nonlinearly on the higgs, and therefore it must be broken. Gauging it eats away the higgs.

In Sec. 2.2 we review and explain the problem in the L^2H model. The L^2H itself is phenomenologically disfavoured [7, 8, 9, 10, 11] by EWPD, and it is for this reason that alternatives, like models with custodial symmetry[12] or with T-parity[13, 14, 15], have been introduced. Rather than investigating these models individually we show in Sec. 2.3 that the problem is generic. We first give a very explicit proof for models with $SU(N)/SO(N)$ (and $SU(N)/Sp(N)$) vacuum manifold. We then generalize, which does not require much additional work. A brief recap is in Sec. 2.4.

2.2 Top-quark coupling fine tuning in the Littlest Higgs Model

2.2.1 Model Review

To establish notation we briefly review elements of the L^2H [4]. It has $G_f = SU(5)$, $H = SO(5)$ and $G_w = \prod_{i=1,2} SU(2)_i \times U(1)_i$. Symmetry breaking $SU(5) \to SO(5)$ is characterized by the Goldstone boson decay constant f. The embedding of G_w in G_f is fixed by taking the generators of $SU(2)_1$ and $SU(2)_2$ to be

$$Q_1^a = \begin{pmatrix} \frac{1}{2}\tau^a & 0_{2\times3} \\ 0_{3\times2} & 0_{3\times3} \end{pmatrix} \qquad \text{and} \qquad Q_2^a = \begin{pmatrix} 0_{3\times3} & 0_{3\times2} \\ 0_{2\times3} & -\frac{1}{2}\tau^{a*} \end{pmatrix} \tag{2.1}$$

and the generators of the $U(1)_1$ and $U(1)_2$

$$Y_1 = \frac{1}{10}\text{diag}(3,3,-2,-2,-2) \qquad \text{and} \qquad Y_2 = \frac{1}{10}\text{diag}(2,2,2,-3,-3). \tag{2.2}$$

The vacuum manifold is characterized by a unitary, symmetric 5×5 matrix Σ. We denote by g_i (g'_i) the gauge couplings associated with $SU(2)_i$ ($U(1)_i$). If one sets $g_1 = g'_1 = 0$ the model has an exact global $SU(3)$ symmetry (acting on the upper 3×3 block of Σ), while for $g_2 = g'_2 = 0$ it has a different exact global $SU(3)$ symmetry (acting on the lower 3×3 block). Either of these exact global $SU(3)$ would-be symmetries guarantee the Higgs remains exactly massless. Hence, the Higgs mass should vanish for either $g_1 = g'_1 = 0$ or $g_2 = g'_2 = 0$. The perturbative quadratically divergent correction to the Higgs mass must be polynomial in the couplings and can involve only one of the couplings at one loop order. Hence it must vanish at one loop. This is the collective symmetry mechanism that ensures the absence of 1-loop quadratic divergences in the higgs mass.

It is standard to introduce the top quark so that the collective symmetry argument still applies. The third generation doublet q_L is a doublet under $SU(2)_1$ and a singlet under $SU(2)_2$. Introduce additional $SU(2)_1\times SU(2)_2$-singlet spinor fields: q_R, u_L and u_R. The third generation right handed singlet is a linear combination of u_R and q_R. The charges of these under $U(1)\times U(1)$ are listed below, in (2.15). Their couplings are taken to be

$$\mathcal{L}_{\text{top}} = -\frac{1}{2}\lambda_1 f\, \bar{\chi}_{Li}\, \epsilon^{ijk}\, \epsilon^{xy}\, \Sigma_{jx}\, \Sigma_{ky}\; q_R - \lambda_2\, f\, \bar{u}_L\, u_R + \text{h.c.} \tag{2.3}$$

where the indexes i, j, k run over 1,2,3, the indexes x, y over $4, 5$ and the triplet χ_L is

$$\chi_L = \begin{pmatrix} i\tau^2 q_L \\ u_L \end{pmatrix}. \tag{2.4}$$

The collective symmetry argument now runs as follows. If $\lambda_2 = 0$ then $\mathcal{L}_{\text{top}}$ in (2.3) is constructed so that it exhibits an explicit global $SU(3)$ symmetry, a subgroup of $G_f = SU(5)$. Under this, the fields χ_L in (2.4) and Σ_{ix} transform as triplets (on $i = 1, 2, 3$). Since this would-be exact global symmetry is spontaneously broken it guarantees that the Higgs field remains an exactly massless Goldstone boson. Similarly, if $\lambda_1 = 0$ then there is no coupling of the quarks to the Goldstone bosons, which therefore remain massless. Hence, the mass term must vanish as either λ_1 or λ_2 are set to zero, and since the quadratic divergence is polynomial in the couplings, it can only arise at two loops.

The gauge and top-quark interactions generate an effective, Coleman-Weinberg potential which determines the vacuum orientation. If the gauge couplings are strong enough[16],

$$g_1'^2 + g_1^2 > \frac{2N_c}{3\pi^2 c}\lambda_1^2\lambda_2^2\left[\ln\left(\frac{\Lambda^2}{(\lambda_1^2+\lambda_2^2)f^2}\right)+\frac{\hat{c}'}{2}\right]. \tag{2.5}$$

where c and $\hat{c}'$ are unknown dynamical constants of order unity, the vacuum alignment is

$$\Sigma_{ew} = \begin{pmatrix} 0 & 0 & \mathbf{1}_{2\times 2} \\ 0 & 1 & 0 \\ \mathbf{1}_{2\times 2} & 0 & 0 \end{pmatrix}. \tag{2.6}$$

leading to the gauge-symmetry breaking into the electroweak subgroup, $\prod_{i=1,2} SU(2)_i \times U(1)_i \to SU(2)\times U(1)$.

2.2.2 The Hidden Fine Tuning

As we just saw, the top quark Lagrangian $\mathcal{L}_{\text{top}}$ in (2.3) is constructed so that it exhibits an explicit global $SU(3)$ symmetry. However, this is a symmetry of the Lagrangian only for $\lambda_2 = g_1 = g_1' = 0$.

There is in fact no symmetry reason for the fields in χ_L to combine into a triplet. Given that the effective Lagrangian is restricted only by the non-linear realization of the symmetry (by parametrizing G_f/H) and by the requirement of explicit gauge invariance under G_w, the coupling in (2.3) is more generally of the form

$$\mathcal{L}_{\text{top}} = -\lambda_1 f \bar{q}_L^{\,i}\epsilon^{xy}\Sigma_{ix}\Sigma_{3y}q_R - \frac{1}{2}\lambda_1' f\bar{u}_L\epsilon^{3jk}\epsilon^{xy}\Sigma_{jx}\Sigma_{ky}q_R - \lambda_2 f\bar{u}_L u_R + \text{h.c.} \tag{2.7}$$

Only when $\lambda_1' = \lambda_1$ (and $\lambda_2 = g_1 = g_1' = 0$) do we recover the global $SU(3)$ symmetry of the collective symmetry mechanism. The main observation of this work is that the relation $\lambda_1' = \lambda_1$, assumed throughout the little higgs literature, is unnatural. We refer to this as the hidden fine tuning problem. The reader may choose not to see this as a problem, that assuming collective symmetry in the Yukawa sector at tree level is in line with the littlest higgs approach. Only an explicit UV completion can validate one view over the other.

Although $\lambda_1' = \lambda_1$ is natural in the absence of the gauge interactions, these are already present in the UV completion. Below we comment in slightly more detail on how radiative effects explicitly introduce $SU(3)$ breaking into the Yukawa couplings.

It should be evident that for $\lambda_1' \neq \lambda_1$ the collective symmetry argument is spoiled. A straightforward computation gives a quadratically divergent correction to the higgs mass,

$$\delta m_h^2 = \frac{12}{16\pi^2}(\lambda_1^2 - \lambda_1'^2)\Lambda^2 \tag{2.8}$$

where Λ is a UV cut-off. The severity of the fine tuning can now be explored. If we insist that the Higgs mass should be naturally of order of 100 GeV, while $\Lambda \sim 10$ TeV, then, not surprisingly, $\lambda_1' - \lambda_1 \lesssim (4\pi m_h/\Lambda)^2 \sim 1\%$.

The Lagrangian in (3.8) is not the most general one consistent with symmetries to lowest order in the chiral expansion. If $SU(3)$ were a good symmetry one could add to the Lagrangian a term of the form

$$\bar{\chi}_{Li}\epsilon_{jkl}\epsilon_{xy}(\Sigma^*)^{ij}(\Sigma^*)^{kx}(\Sigma^*)^{ly}q_R \tag{2.9}$$

One can also freely replace $q_R \leftrightarrow u_R$ in Eqs. (2.3) and (2.9), and then, of course, split each $SU(3)$ invariant term into a sum of $SU(2) \times U(1)$ invariant terms. There is no reason a priori why these terms should be ignored, but they are not dangerous. In fact, they are inevitable, as they are generated radiatively, many of them already at one loop [17].

2.2.3 Radiatively induced $\lambda_1' \neq \lambda_1$

Imposing $\lambda_1' - \lambda_1 = 0$ is not only a fine tuning, it is unnatural. Since the symmetry is broken by marginal operators, the renormalization group evolution of the difference $\lambda_1' - \lambda_1$ takes it away from zero, even if it is chosen to be zero at some arbitrary renormalization point μ.

As a check we have computed explicitly the one loop renormalization group equations for these couplings (see Fig. 2.1):

$$\mu\frac{\partial}{\partial\mu}\ln\left(\frac{\lambda_1}{\lambda_1'}\right) = \left(\tfrac{2}{3} - y\right)\frac{3g_1'^2}{16\pi^2} \tag{2.10}$$

Here y is the charge of q_R under $U(1)_2$. Details of the calculation will be presented elsewhere[17]. If $\beta_{g_1'} = (b/16\pi^2)g_1'^3$ then we can write the solution in terms of the

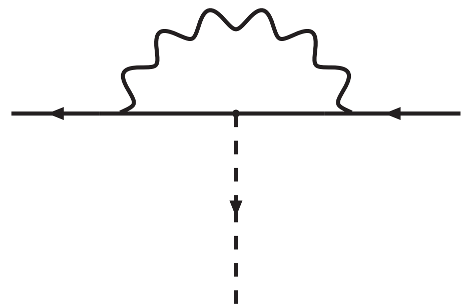


Figure 2.1: Feynman diagram that contributes to the renormalization of the Yukawa couplings λ_1 and λ'_1. The wavy line represents a gauge boson of $U(1)_1$ and the solid and doted lines a spinor and a PGB, respectively.

running coupling:

$$\frac{\lambda_1(\mu)}{\lambda'_1(\mu)} = \frac{\lambda_1(\Lambda)}{\lambda'_1(\Lambda)} \left(\frac{g'_1(\mu)}{g'_1(\Lambda)}\right)^{\frac{2-3y}{b}} \tag{2.11}$$

The numerical value for b can be obtained from the standard QED beta function (see [18, 19])

$$b = \frac{2}{3} \sum_{\text{Weyl fermion}} Y_{1i}^2 + \frac{1}{6} \sum_{\text{real scalar}} Y_{1i}^2 \tag{2.12}$$

To compute this, we need to introduce the Yukawa-type coupling for all the other standard model quarks. We will follow Perelstein [2] by noting that there is no need for implementing collective symmetry breaking for the other standard model quarks due to their small Yukawa couplings. Thus the other "up" type quarks Yukawa interaction can be introduced by

$$-\lambda_\alpha^u f \bar{q}_{\alpha L}^{\,i} \epsilon^{xy} \Sigma_{ix} \Sigma_{3y} q_{\alpha R} \tag{2.13}$$

where $\alpha = 1, 2$ is the quark family index. Similarly the other "down" type quark interactions can be introduced by

$$-\lambda_\alpha^d f \bar{q}_{\alpha L}^{\,i} \epsilon_{xy} (\Sigma^*)^{ix} (\Sigma^*)^{3y} d_{\alpha R} \tag{2.14}$$

here $\alpha = 1, 2, 3$. If we take $Y_2(q_R) = y$, then the Y_1 charge of all the particles involved are

	$q_{\alpha L}$	$q_{\alpha R}$	$d_{\alpha R}$	u_L	u_R	H	ϕ
Y_1	$\frac{11}{30} - y$	$\frac{2}{3} - y$	$\frac{1}{15} - y$	$\frac{13}{15} - y$	$\frac{13}{15} - y$	$1/4$	$1/2$
Y_2	$y - \frac{1}{5}$	y	$y - \frac{2}{5}$	$y - \frac{1}{5}$	$y - \frac{1}{5}$	$1/4$	$1/2$

(2.15)

Thus we get $b = \frac{1}{360}\left(2737 - 8832y + 10080y^2\right) \geq 46/105$. However, we note that the y can be arbitrary.

We do not dwell on the numerics, since there are too many adjustable parameters (the choice of y, the value of $U(1)$ couplings and $\lambda_{1,2}(\Lambda)$ which however must satisfy (2.5), the value of the cutoff Λ). We simply note that $1/16\pi^2 \log(\Lambda/m_h) \sim 1/16\pi^2 \log(100) \sim 3\%$. Hence, even fine tuning $\lambda_1(\Lambda) = \lambda'_1(\Lambda)$ generically produces a difference $\lambda_1(m_h) - \lambda'_1(m_h)$ in excess of 1%.

Note also that the same behavior must occur in the UV completion of the L^2H model. After all, the terms in the Lagrangian that break G_f-symmetry model the effects of symmetry breaking interactions at short distances, that is, in the UV completion. The interactions in the UV completion that are responsible for the quark Yukawa couplings cannot be taken to respect the $SU(3)$ symmetry required for the collective symmetry argument. The breaking of the $SU(3)$ symmetry in the UV completion is naturally much larger than in (2.11) since neither the $U(1)_1$ gauge coupling nor the Yukawa couplings are asymptotically free.

2.3 A no-go theorem

In this section we show the impossibility of constructing a theory that implements without fine tuning the collective symmetry mechanism on the terms responsible for quark and lepton masses. Let us begin by stating in general terms what is required in order to implement the collective symmetry mechanism. Any given term in the Lagrangian has to be symmetric under a subgroup G_c of the flavor group G_f under which the higgs field transforms non-linearly, and in particular, with a transformation that includes a constant shift.[1] In addition, there must not be any one loop divergent radiative corrections that involve the coupling constants for two different terms.

Of course there are additional requirements on each individual term in the Lagrangian. In particular any one term must be invariant under G_w, the gauged subgroup of G_f. We do not wish to specify this gauge group, since one could look for realizations of the collective symmetry mechanism in gauge groups other than the one of the L^2H. Below we will only need to use the fact that this group contains

[1]Different terms in the Lagrangian may be invariant under different collective symmetry groups G_c.

the electroweak gauge group, $G_{\rm ew} = SU(2) \times U(1)$, that this symmetry is linearly realized, *i.e.*, that $G_{\rm ew} \subset H$ so it remains unbroken at the scale at which G_f breaks to H, and that the higgs field must transform as a doublet with hypercharge 1/2 under the electroweak group.

The hidden fine tuning problem in the quark sector of the L^{2}H resulted from the fact that $G_c = SU(3)$ is not a symmetry of the Yukawa term, because G_c does not commute with G_w. The Yukawa term in the Lagrangian is actually a sum of terms that are separately invariant under the gauge group and the collection of terms can only be symmetric under G_c by fine tuning the separate coupling constants at one scale. There are two ways that immediately come to mind in which one could try to extend the L^{2}H model to get around this problem. Either extend the gauge group so that G_c itself is gauged or obtain G_c as an accidental symmetry. These, or other strategies cannot work: below we will prove in generality that the collective symmetry mechanism cannot work for terms other than the kinetic terms in the Lagrangian.

2.3.1 An $SU(7)/SO(7)$ example and its generalization to $SU(N)/SO(N)$

It is simpler to understand the general case by first looking at an explicit example. We can motivate this by the following observation. If the $SU(3)$ collective symmetry that acts on the first three rows and columns of Σ is elevated to a gauge symmetry, then the equality $\lambda_1' = \lambda_1$ is natural. Of course, in the L^{2}H model this won't work because the $SU(3)$ is broken at the scale Λ at which $SU(5)$ breaks to $SO(5)$, and the higgs is eaten at this scale. But perhaps one can construct a theory based on a larger G_f symmetry group with $SU(3)$ gauged and the higgs still transforming non-linearly under some G_c subgroup of G_f.

For example, one may consider a nonlinear sigma model based on $G_f/H = U(7)/O(7)$ (with spinor fields in non-trivial representations of the hyper-strong gauge group so that the $U(1)$ in $U(7) = SU(7) \times U(1)$ is non-anomalous). Assume the $U(7)$ is broken to $O(7)$ by a symmetric condensate, which transforms under $U(7)$ as $\Sigma \to V\Sigma V^T$. Now gauge a $G_w = SU(3)\times SU(2)\times U(1)^3$ subgroup of $U(7)$. The $SU(3)$ factor is precisely the gauged version of the top-block collective symmetry group, under which the royal triplet χ_L transforms as an actual triplet. It is a straightforward, if lengthy, exercise to show that the vacuum aligns correctly, that is, G_w breaks to the electroweak subgroup. One can identify Π_{i4}, and related entries, with the higgs doublet. By suitably choosing the generators of the gauged $U(1)^3$ symmetry one finds that the higgs field is the only light PGB.

Now introduce top quark couplings in a manner consistent with the collective symmetry and without fine tuning of Yukawa couplings. Just as in the L^{2}H model, in addition to the third generation quark doublet q_L and singlet q_R, introduce a pair of weak singlet Weyl fermions u_L and u_R that transform as $\mathbf{1}_{1/6}$ under $SU(2)_W \times U(1)_Y$. The singlet u_L is combined with the doublet q_L into a triplet of the gauged $SU(3)$, precisely as in (2.4). By suitably choosing the transformation properties under the $U(1)^3$ we can ensure that the most general Yukawa Lagrangian consistent with the symmetries, to lowest order in the chiral expansion, is

$$\mathcal{L}_{\text{top}} = -f\lambda_1 \bar{\chi}_{Li}(\Sigma^*)^{i4} q_R - \frac{1}{2} f\lambda_2 \bar{\chi}_{Li} \epsilon^{ijk} \epsilon^{xy} \Sigma_{jx} \Sigma_{ky} u_R + \text{h.c.} \tag{2.16}$$

where the indexes i, j, k run over $1, 2, 3$ and x, y over $5, 6$. The problem with this model is that the $SU(3)$ symmetry does not protect the higgs. The collective symmetry required is an $SU(4)$ acting on the top-left 4×4 block of Σ. This in turn requires enlarging the true triplet to a four-plet, which allows for more terms in the Lagrangian, which are related by the $U(4)$ symmetry. However, this is not a good symmetry of the Lagrangian and the added terms are related to the ones above only by imposing unnaturally a collective symmetry. This is precisely the same problem we encountered with the L^{2}H.

Let us generalize this to models with $SU(N)/SO(N)$ vacuum manifold, parametrized by the $N \times N$ symmetric unitary matrix Σ. We assume there is an $SU(2) \times U(1)$ gauged subgroup of $SO(N)$. Without loss of generality we can take its embedding in $SU(N)$ as follows:

$$\begin{aligned} Q^a &= \frac{1}{2} \begin{pmatrix} \tau^a & 0_{2\times(N-4)} & 0_{2\times 2} \\ 0_{(N-4)\times 2} & 0_{(N-4)\times(N-4)} & 0_{(N-4)\times 2} \\ 0_{2\times 2} & 0_{2\times(N-4)} & -\tau^{a*} \end{pmatrix} \\ Y &= \frac{1}{2} \text{diag}(1, 1, y_3, \ldots, y_{N-2}, -1, -1) \end{aligned} \tag{2.17}$$

with $\sum y_i = 0$. We assume further that the whatever other interactions exist they align the vacuum along (2.6) (with the proper interpretation for the dimensions of the

0 blocks and the center unit block). Then, as usual, $\Sigma = \exp(i\Pi/f)\Sigma_{ew}\exp(i\Pi^T/f) = \exp(2i\Pi/f)\Sigma_{ew}$, where in the last step we have chosen the broken generators to satisfy $\Pi\Sigma_{ew} = \Sigma_{ew}\Pi^T$. The $N-4$ doublets Π_{ix} with $i = 1,2$ and $x = 3,\ldots,N-2$, have hypercharge $1/2 + y_x$. So any one of these for which $y_x = 0$ is a prospective higgs doublet.

Under an infinitesimal $SU(N)$ transformation, $1 + i\epsilon^a T^a$, the matrix of goldstone bosons transforms as

$$\delta\Pi = \frac{f}{2}(T^a + \Sigma_{ew}T^{aT}\Sigma^{\dagger}_{ew}) + \cdots \tag{2.18}$$

where the ellipses stand for terms at least linear in the fields. We are interested in finding a subgroup G_c of $SU(N)$ under which the higgs field transformation includes a constant shift. However any such transformation does not commute with $SU(2)\times U(1)$. Without loss of generality we assume that the third entry has zero hypercharge, $y_3 = 0$, so that $\Pi_{i3} = \Pi^*_{3i} = \Pi_{(N-2)i} = \Pi^*_{i(N-2)}$ is the prospective higgs doublet. Then G_c must contain generators

$$X = \begin{pmatrix} 0_{2\times2} & x_{2\times1} & 0_{2\times(N-3)} \\ x^{\dagger}_{1\times2} & 0_{1\times1} & 0_{1\times(N-3)} \\ 0_{(N-3)\times2} & 0_{(N-3)\times1} & 0_{(N-3)\times(N-3)} \end{pmatrix} \tag{2.19}$$

or

$$X = \begin{pmatrix} 0_{(N-3)\times(N-3)} & 0_{(N-3)\times1} & 0_{(N-3)\times2} \\ 0_{1\times(N-3)} & 0_{1\times1} & x^{T}_{1\times2} \\ 0_{2\times(N-3)} & x^{*}_{2\times1} & 0_{2\times2} \end{pmatrix} \tag{2.20}$$

with x a complex two component vector. Both of these give the same linear shift on the prospective higgs field, as can be verified by computing $X + \Sigma_{ew}X^T\Sigma^{\dagger}_{ew}$. It follows that for either one of these generators we have

$$[Q^a, X] = X' \tag{2.21}$$

where X' is a generator of the form of X. This means that the X generators transform under $SU(2)$ as a tensor operator; they are in fact complex doublets with hypercharge 1/2, just like the higgs. Now, there are additional generators in G_c: at the very least it contains the $SU(3)$ subgroup generated by the top-left or bottom right 3×3 blocks. Together, X and these additional generators transform as a *reducible* representation of the electroweak subgroup. It follows that a gauge invariant term in the Lagrangian that is also invariant under G_c is a sum of terms that are individually gauge invariant. The only exception is when the term is constructed of fields that are separately $SU(2)$ invariant, as is the case of the λ_2 mass term, in (2.3), in the L^{2}H model. But it is unnatural to choose the coefficients of these various terms to make their sum G_c invariant. This is because the gauge interactions always break the symmetry. Gauge boson exchange Feynman diagrams like that of Fig. 2.1 give divergent corrections to these couplings, and the corrections do not preserve the G_c invariance.

We can relax one assumption above slightly. We do not need to assume the vacuum alignment Σ_{ew} is along (2.6). In order to have a collective symmetry argument that one can already apply in the gauge sector one needs the first and last two rows and columns to be as in (2.6). But the central $(N-4)\times(N-4)$ block does not have to be a diagonal matrix, only a unitary, symmetric matrix. However, the argument goes through as before: the components of Π that we identify with the higgs are changed in precisely the way that the shifts in (2.18) are modified and the rest of the argument goes through unchanged.

The explicit proof for the case $G_f/H = SU(N)/Sp(N)$ is completely analogous.

2.3.2 The general case

We turn now to the general case. We assume that G_w contains the electroweak gauge group $G_{\text{ew}} = SU(2)\times U(1)$, with $G_{ew}\subset H$. We further assume that a subset of goldstone bosons can be identified with the higgs field. We consider a term in the Lagrangian that is both symmetric under G_{ew} and has a collective symmetry G_c. We show in the appendix that we only need to consider semi-simple G_c, which we assume henceforth.

That the higgs transforms linearly under the electroweak gauge group means that there is a doublet h in Π that transforms as

$$\delta_\epsilon h = i\epsilon^a\frac{\tau^a}{2}h + i\epsilon\frac{1}{2}h \tag{2.22}$$

under $SU(2)\times U(1)$. Under a group $G_c \in G_f$ h transforms non-linearly,

$$\delta_\eta h = \eta^m x^m + \cdots \tag{2.23}$$

where the implicit sum over m is over all generators in G_c, for some two component complex vectors x^m and the ellipses stand for terms at least linear in h. One can redefine the basis of generators in G_c so that $x^m = 0$ for $m \geq 5$ and x^m for $m = 1, \cdots, 4$ are unit vectors, with $m = 1, 3$ real and $m = 2, 4$ purely imaginary. Now consider the commutator,

$$(\delta_\eta \delta_\epsilon - \delta_\epsilon \delta_\eta) h = i\epsilon^a \eta^m \frac{\tau^a}{2} x^m + i\epsilon \eta^m \frac{1}{2} x^m + \cdots \tag{2.24}$$

The commutator is again a non-linear transformation, a linear combination of the same four generators in G_c that shift the higgs. In terms of the Lie algebra of G_f, denoting these generators by X^i, with[2] $i = 1, 2$ and the generators of $G_{\rm ew}$ by Q^a and Y, we read off

$$[Q^a, X^i] = \frac{i}{2} (\tau^a)^{ij} X^j, \qquad [Y, X^i] = \frac{i}{2} X^i \tag{2.25}$$

This is precisely the statement in Eq. (2.21), derived there from the explicit form of matrices, that the generators transform as tensors of $G_{\rm ew}$ with the same quantum numbers as the higgs doublet, but we see now that it holds more generally, independently of those explicit matrix representations.

Since there is no semi-simple Lie algebra of rank 4, there must be additional generators, and $[X^i, X^j]$ must give some of these additional generators. Denote a non-vanishing commutator by $\hat{X}^{ij} = [X^i, X^j]$. Using the Jacobi identity we see that

$$[Q^a, \hat{X}^{ij}] = [Q^a, [X^i, X^j]] \tag{2.26}$$

$$= [X^i, [Q^a, X^j]] - [X^j, [Q^a, X^i]] \tag{2.27}$$

$$= \frac{i}{2} (\sigma^a)^{jk} X^{ik} - \frac{i}{2} (\sigma^a)^{ik} X^{jk} \tag{2.28}$$

So these generators also satisfy an equation like (2.21) but transform in a representation in the tensor product of two doublets. Continuing this way, considering commutators of the generators we have so far, we can eventually generate the complete Lie algebra and find that it breaks into sectors classified by irreducible representations under $G_{\rm ew}$.

We can use this to show that invariants under G_c break into a sum of terms separately invariant under $G_{\rm ew}$. Any non-trivial invariant must be a product of two combination of fields, one transforming in some irreducible representation R of G_c

[2] The index i runs over 1,2 because the hermitian matrices break into a symmetric and an anti-symmetric part, corresponding to the two real and two imaginary components of x^m, and also to the real and imagnary components of the higgs doublet.

and the other as the complex conjugate $\bar{R}$. But from the previous paragraph it follows that under $G_{\rm ew}$ the representation R breaks into a direct sum $R = r_1 \oplus r_2 \oplus \cdots$ of at least two irreducible representations of $G_{\rm ew}$. Therefore the product $R \times \bar{R}$, contains the sum of at least two invariants under $G_{\rm ew}$, $r_1 \times \bar{r}_1$ and $r_2 \times \bar{r}_2$. Since G_c is not a symmetry of the theory (because the kinetic energy term for the goldstone bosons is not invariant), the two (or more) $G_{\rm ew}$ invariants can be summed into a G_c invariant only by fine tuning coefficients in the Lagrangian. This completes the argument.

It may not be self-evident that any non-trivial representation of G_c breaks into two or more representations under $G_{\rm ew}$. This can be shown by noting that the roots of the Lie algebra, that is the weights of the adjoint representation, of G_c break into a sum of irreducible representations of $G_{\rm ew}$, precisely the same representations that the generators fall into.[3] Then by following the same procedure as in establishing branching rules for representations of Lie algebras, that is, introducing projection operators in weight space, and using the fact that the roots form irreducible representations, one obtains that every representation of G_c is decomposed into a sum of irreducible representations of $G_{\rm ew}$.

We remarked above that $U(1)$ factors in G_c are ignored. This requires some explanation. After all, one could conceivably take the four broken generators to generate a collective symmetry group of dimension 4, say $U(1)^4$ or $SU(2) \times U(1)$. But the $U(1)$ symmetries do not help insure the higgs remains massless. It is easy to see why by considering first the familiar $\mathrm{L}^2\mathrm{H}$ case. The λ_1 and λ_1' terms of (3.8) that need to be related by collective symmetry to obtain necessary cancellations in one loop graphs can be made separately invariant under several $U(1)$ symmetries. In fact, the situation is reversed from the semi-simple group case, where a representation R of G_c is a direct sum of at least two irreducible representation of $G_{\rm ew}$. Since the irreducible representations of $U(1)$ are one dimensional, it is $G_{\rm ew}$ that relates several irreducible representation of $U(1)$, and forces them together into a term in the Lagrangian.

[3]This follows form considering the standard map $T^A \to |T^A\rangle$ of the generators of G_f, with $T^A|T^B\rangle = |[T^A, T^B]\rangle$. Then $Q^a|X^i\rangle = i/2(\sigma^a)^{ij}|X^j\rangle$ and so on for the other generators of G_c.

2.4 Conclusions

It is easy to see that radiative effects break the collective symmetry of top quark couplings of the L^2H model. These effects must also be present in the underlying UV completion so they cannot be dismissed as small. Also, they are generically too large for successful phenomenology even if one chooses to enforce collective symmetry on the tree level top Yukawa couplings.

The problem cannot be circumvented by enlarging the model to one with a larger underlying flavor symmetry group. Gauging collective symmetry is not an option: it either gives a higgsless model or again requires imposing an unnatural symmetry at tree level to avoid quadratically divergent radiative corrections to the higgs mass. The reader may see this as a fine tuning problem, or may adopt the view that imposing the collective symmetry on the top quark sector is in keeping with the littlest higgs strategy.

We have shown that the collective symmetry argument cannot be implemented naturally on the Yukawa couplings of little higgs models. Of course, no-go theorems are only as good as its assumptions. We did not prove that no model exists that can both include top quarks and solve the little hierarchy problem. For example, one can presumably partially supersymmetrize the model to ensure the cancellation of top loop induced quadratic mass divergences, at least at one loop. In the absence of a novel mechanism to suppress the quadratic divergences in the top quark-induced radiative corrections to the higgs mass without fine tuning, it seems one must rely on a UV completion to explain the approximate collective symmetry of the model.

This chapter is a reprint of material as it appears in “Hidden fine tuning in the quark sector of little higgs models,” B. Grinstein, R. Kelley and P. Uttayarat, JHEP **0909**, 040 (2009) [arXiv:0904.1622 [hep-ph]], of which I was a co-author.

Chapter 3

One Loop Renormalization of the Littlest Higgs Model

In Little Higgs models a collective symmetry prevents the Higgs from acquiring a quadratically divergent mass at one loop. This collective symmetry is broken by weakly gauged interactions. Terms, like Yukawa couplings, that display collective symmetry in the bare Lagrangian are generically renormalized into a sum of terms that do not respect the collective symmetry except possibly at one renormalization point where the couplings are related so that the symmetry is restored. We study here the one loop renormalization of a prototypical example, the Littlest Higgs Model. Some features of the renormalization of this model are novel, unfamiliar form similar chiral Lagrangian studies.

3.1 Introduction

The Littlest Higgs (L^2H) model [4] is a realization of the idea that the Higgs field, responsible for electroweak symmetry breaking, is a pseudo-Goldstone boson, and as such its mass is automatically small (for some reviews see Ref. [1, 2, 3]). What is meant by "small" is that the Higgs mass can be made arbitrarily small compared to the scale of breaking of the symmetry that gives rise to this Goldstone boson. Earlier realizations of this idea faced difficulties, required additional fine tuning [20, 21]. In the L^2H model, as well as its many extensions, the absence of quadratically divergent radiative corrections to the Higgs mass is guaranteed, at one loop order, by the collective symmetry argument. The argument fails beyond one loop order, so the Higgs can be made naturally light only if its mass is no smaller than of the order of a two loop radiative correction with a cut-off at the scale of the new physics.

While there is a vast literature exploring the phenomenological effects of L^2H-type models, the renormalization structure of the model has been little explored. Computations have been presented that check that the collective symmetry argument does work; however, the structure of counterterms needed to subtract the divergences that do occur has not been studied. Furthermore, the renormalization group equations have not been determined.

Phenomenologically the L^2H model has fallen somewhat out of favor because of its difficulties simultaneously accommodating the electroweak precision constraints and in solving the little hierarchy problem [7, 9, 8, 10, 11]. However, its structure is prototypical of many models, like Littlest Higgs models with reduced gauge symmetry [22], or with custodial [12, 23] or T-parity [13, 14, 15] symmetries. Therefore, the methods we will introduce here should be directly applicable to the one loop renormalization of any of the models in this class.

It was noted in Ref. [24] that renormalization group running of the top Yukawa coupling in L^2H-models disrupts the collective symmetry. That is, in order for the collective symmetry argument to operate in the top-quark Yukawa sector, the coupling is built to satisfy an $SU(3)$ symmetry. However, this symmetry is broken by weak gauge interactions. The would be $SU(3)$ symmetric top-Yukawa coupling actually splits into two $SU(2)\times U(1)$ symmetric terms with coupling constants that run away from each other as they evolve under the renormalization group. This begs the question, what is the full renormalization group structure of the model? It is the purpose of this paper to address this question, at one loop order.

There are several energy scales associated with this model. In addition to the cutoff, Λ, there is the scale of masses of heavy vector bosons, gf where $f \sim \Lambda/4\pi$ is a Goldstone boson decay constant and g some gauge coupling, and the electroweak breaking scale v. We are largely interested in the cut-off dependence, so for our computations we will focus on the largest energies, above gf. Therefore to determine the ultraviolet behavior we retain the massive gauge vector bosons in our calculations and neglect their masses. On the other hand, the renormalization structure below the scale of these masses, gf, is well understood. The model reduces there to the standard electroweak model with one Higgs doublet supplemented by irrelevant operators.

The main result of this paper, the splitting of the Yukawa couplings responsible for the top quark mass, was already noted in Ref. [24]. There, a no-go theorem for the collective symmetry mechanism for Yukawa terms was proved. However, the details of the calculation of the running of Yukawa couplings were not given there since, as can be seen from this work, this merits a lengthy discussion that would have detracted from its main point. In fact, we have encountered several stumbling blocks, and corresponding solutions, along the way. Readers interested in questions of principle or practice, or both, in L^{2}H-type models, will hopefully find this work useful.

The paper is organized as follows. We first review the L^{2}H Model in Sec. 3.2. The 1-loop non-derivative counterterms formed only of scalar fields has been extensively studied in the context of determining the effective potential. We classify the remaining counterterms needed to renormalize the model at one loop in Sec. 3.3 and proceed to compute the renormalization constants and corresponding beta functions in Sec. 3.4. We offer some brief concluding remarks in Sec. 3.5.

3.2 The Model

The L^{2}H model is an effective low energy description of some incompletely specified shorter distance dynamics. The short distance dynamics has a global "flavor" symmetry $G_f = SU(5)$, of which a subgroup $G_w = SU(2) \times SU(2) \times U(1) \times U(1)$ is weakly gauged. In the absence of this weak gauge force, the flavor symmetry is broken spontaneously to a subgroup $H = SO(5)$ due to hyper-strong interactions at a scale Λ. As a result, there are massless Goldstone bosons that are coordinates on the G_f/H coset space. Since the weakly gauged G_w force breaks the flavor symmetry explicitly, including its effects leads to some of the Goldstone bosons (the would-be Goldstone bosons) being eaten by the Higgs mechanism and the rest becoming pseudo-Goldstone bosons (PGBs) acquiring small masses of order Λ times a small symmetry breaking parameter, a gauge coupling constant of the weakly gauged G_w. The Higgs is the lightest PGB in Little Higgs models, and its mass is naturally much less than Λ (and the other PGBs): due to the collective symmetry breaking mechanism a contribution of order Λ^2 to its mass arises only at two loops.

To establish notation we briefly review elements of the L^{2}H. Symmetry breaking $SU(5) \to SO(5)$ is characterized by the Goldstone boson decay constant f. The embedding of G_w in G_f is fixed by taking the gauge generators

$$Q_1^a = \begin{pmatrix} \tau^a/2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y_1 = \mathrm{diag}(3,3,-2,-2,-2)/10,$$
$$Q_2^a = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\tau^{a*}/2 \end{pmatrix}, \quad Y_2 = \mathrm{diag}(2,2,2,-3,-3)/10. \tag{3.1}$$

The vacuum manifold is characterized by a unitary, symmetric 5×5 matrix Σ, transforming as $\Sigma \to U\Sigma U^T$ under $U \in SU(5)$. A convenient parametrization of Σ in terms of the hermitian matrix of Goldstone bosons Π is

$$\Sigma = e^{2i\Pi/f}\Sigma_0, \quad \Sigma_0 = \begin{pmatrix} 0 & 0 & \mathbf{1}_{2\times2} \\ 0 & 1 & 0 \\ \mathbf{1}_{2\times2} & 0 & 0 \end{pmatrix}, \tag{3.2}$$

where

$$\Pi = \begin{pmatrix} \omega + \eta\mathbf{1}/\sqrt{20} & h/\sqrt{2} & \phi \\ h^\dagger/\sqrt{2} & -2\eta/\sqrt{5} & h^T/\sqrt{2} \\ \phi^* & h^*/\sqrt{2} & \omega^T + \eta\mathbf{1}/\sqrt{20} \end{pmatrix} \tag{3.3}$$

Here Σ_0 gives the dynamically determined direction in which the vacuum aligns[1] [5, 6] relative to the embedding of G_w in G_f given in Eq. (3.1). Fluctuations along broken symmetry directions are parametrized by fourteen fields in Π: ω and ϕ are 2×2 matrices satisfying $\omega^\dagger = \omega$ and $\phi^T = \phi$, h is an unrestricted 2×1 matrix and η is 1×1 and real. The vacuum spontaneously breaks $G_w \to SU(2) \times U(1)$, and the four fields in ω and η are eaten by the broken generators of gauge symmetries.

The covariant derivative is

$$D_\mu\Sigma = \partial_\mu\Sigma - i\sum_{j=1}^{2}[g_j W^a_{j\mu}(Q^a_j\Sigma + \Sigma Q^{aT}_j) + g'_j B_{j\mu}(Y_j\Sigma + \Sigma Y_j)], \tag{3.4}$$

[1] To ensure this alignment the weakly gauge coupling constant have to be strong enough; see Ref. [16].

where B_j and W_j^a are the $U(1)_j$ and $SU(2)_j$ gauge fields respectively. The $U(1)_j$ coupling constant is taken to be g'_j while the $SU(2)_j$ coupling constant is g_j.

The effective low energy theory has kinetic term

$$\mathcal{L}_{kin} = \frac{f^2}{8}\mathrm{Tr}(D_\mu\Sigma)(D^\mu\Sigma)^\dagger. \tag{3.5}$$

If one sets $g_1 = g'_1 = 0$ the model has an exact global $SU(3)$ symmetry (acting on upper 3×3 block of Σ), while for $g_2 = g'_2 = 0$ it has a different exact global $SU(3)$ symmetry (acting on the lower 3×3 block). Either of these exact global $SU(3)$ would-be symmetries guarantee the Higgs remains exactly massless. Hence, the Higgs mass should vanish for either $g_1 = g'_1 = 0$ or $g_2 = g'_2 = 0$. The perturbative quadratically divergent correction to the Higgs mass must be polynomial in the couplings and can involve only one of the couplings at a time at one loop order. Hence it must vanish at one loop. This is the collective symmetry mechanism that ensures the absence of 1-loop quadratic divergences in the Higgs mass.

For a top-quark sector introduce a pair of singlet Weyl fermions u_L and u_R with hypercharge 2/3. u_L is combined with the 3$^{\mathrm{rd}}$ generation doublet $q_L = (t_L, b_L)^T$ to form a "royal" triplet

$$\chi_L = \begin{pmatrix} i\tau^2 q_L \\ u_L \end{pmatrix}. \tag{3.6}$$

The top Yukawa interaction is obtained from coupling the fermions to the upper right 2×3 block of the Σ field,

$$\mathcal{L}_{top} = -\frac{1}{2}\lambda_1 f \bar{\chi}_{LI}\epsilon^{IJK}\epsilon^{xy}\Sigma_{Jx}\Sigma_{Ky}q_R - \lambda_2 f \bar{u}_L u_R + h.c. \tag{3.7}$$

Here and below implicit sums are over 1, 2, 3 for I, J, K, over 1, 2, for i, j, k and over 4, 5 for x, y.

There is in fact no symmetry reason for the fields in χ_L to combine into a triplet [24]. More generally the coupling is of the form

$$\mathcal{L}_{\mathrm{top}} = -\lambda_1 f \bar{\chi}_{Li}\epsilon^{ij}\epsilon^{xy}\Sigma_{jx}\Sigma_{3y}q_R - \frac{1}{2}\lambda'_1 f \bar{u}_L \epsilon^{jk}\epsilon^{xy}\Sigma_{jx}\Sigma_{ky}q_R - \lambda_2 f \bar{u}_L u_R + \mathrm{h.c.} \tag{3.8}$$

In this case, there is a quadratically divergent correction to the Higgs mass,

$$\delta m_h^2 = \frac{12}{16\pi^2}(\lambda_1^2 - \lambda_1'^2)\Lambda^2 \tag{3.9}$$

where Λ is a UV cut-off. As we will show below the relation $\lambda_1' = \lambda_1$ is unstable against radiative corrections. For $f \approx 1$ TeV and $m_h \approx 100$ GeV this requires a tuning $\delta\lambda_1 < 0.04\%$. Even if some unknown mechanism enforced $\lambda_1 = \lambda_1'$ at the cut-off Λ, running gives $\delta\lambda_1$ of order a few per cent at the scale of the Higgs mass; see Eq. (58). If $\lambda_1(\Lambda) = \lambda_1'(\Lambda)$ the correction to the Higgs mass is formally a two loop effect. However it is enhanced relative to the naive expectation by the large $\ln(4\pi f/m_h) \approx 5$ and and the numerical factor of 12 in Eq. (9).

3.3 General Structure of Counterterms

In this section we study the structure of counterterms induced at 1-loop order in the L^{2}H model (Eqs. (3.5) and (3.7)). We focus on counterterms which have been neglected in the literature. We omit any discussion of non-derivative counterterms formed of scalar fields only, since the scalars' effective potential has been studied extensively,[2] starting already with the original LLH paper [4].

3.3.1 Scalar Kinetic Energy Counterterms

Kinetic energy counterterms are normally introduced in field theory by rescaling the bare fields $\phi \to Z^{1/2}\phi$. In non-linear sigma models the self-interactions of Goldstone bosons require counterterms that are higher order in the derivative expansion, and no rescaling of fields is necessary. However, non-linear sigma models coupled to light gauge bosons and fermions do generally require counterterms quadratic in derivatives. We will see that in the L^{2}H model no rescaling $\phi \to Z^{1/2}\phi$ is needed. Instead new terms that are not symmetric under the full $SU(5)$ symmetry are required to completely subtract the model at one loop.

We begin our study of the structure of kinetic energy counterterms by considering the slightly simpler case $\lambda_1' = \lambda_1$. Working only to 1 loop, there is only one coupling constant present in each divergent self-energy diagram so the corresponding counterterm could just as well be computed setting all other coupling constants to zero. The Lagrangian with all but one couplings set to zero has an $SU(3) \times SU(2) \times U(1)$

[2] See Ref. [25] for a detailed study of the 1-loop scalars' effective potential.

symmetry. Since we can choose the regulator to respect this symmetry we demand the counterterms are invariant under $SU(3) \times SU(2) \times U(1)$.

Consider the possibility of partly subtracting the divergent graphs by rescaling the bare fields. In general we can choose a different wavefunction renormalization factor Z for each of the fourteen Goldstone boson fields in Π. Were the interaction and the regularization method to respect the full flavor symmetry ($SU(5)$), there would only be one common Z for all the fields in Π. The question becomes: what is the restriction that $SU(3) \times SU(2) \times U(1)$ imposes on the Z?

To answer this consider the expansion of the bare kinetic term

$$\frac{f^2}{8} \mathrm{Tr}\, \partial_\mu \Sigma^\dagger \partial^\mu \Sigma = \mathrm{Tr}\, \partial_\mu \phi^\dagger \partial^\mu \phi + \tfrac{1}{2} \partial_\mu \eta \partial^\mu \eta + \mathrm{Tr}\, \partial_\mu \omega \partial^\mu \omega + \partial_\mu h^\dagger \partial^\mu h + \ldots \tag{3.10}$$

where the ellipsis stand for terms quartic in the fields. Now we rescale each of the fourteen fields by an independent factor Z and ask what are the constraints from imposing $SU(3) \times SU(2) \times U(1)$. There is a $SU(2) \times U(1)$ subgroup that acts linearly and hence there are only four different Z factors:

$$Z_\phi \mathrm{Tr}\, \partial_\mu \phi^\dagger \partial^\mu \phi + \tfrac{1}{2} Z_\eta \partial_\mu \eta \partial^\mu \eta + Z_\omega \mathrm{Tr}\, \partial_\mu \omega \partial^\mu \omega + Z_h \partial_\mu h^\dagger \partial^\mu h + \ldots \tag{3.11}$$

We are led to consider the restrictions from $SU(3)$ on these four factors. It is a straightforward but laborious exercise to compute the transformation properties of the fields in Π under $SU(3)$. We take for definiteness the $SU(3)$ generated by the top-left 3×3 block. Of particular interest are transformations generated by the 4-7 Gell-Mann matrices

$$\sum_{a=4}^{7} \epsilon^a T^a = \sum_{a=4}^{7} \epsilon^a \begin{pmatrix} \lambda^a & 0_{3\times 2} \\ 0_{2\times 3} & 0_{2\times 2} \end{pmatrix} \equiv \begin{pmatrix} 0_{2\times 2} & \lambda & 0_{2\times 2} \\ \lambda^\dagger & 0 & 0_{1\times 2} \\ 0_{2\times 2} & 0_{2\times 1} & 0_{2\times 2} \end{pmatrix}, \tag{3.12}$$

where λ is a 2×1 complex matrix of order ϵ. The resulting nonlinear transformations, to first order in ϵ, are

$$\delta h = \frac{1}{\sqrt{2}} f \lambda + \frac{i}{\sqrt{2}} \left[-\omega \lambda - \frac{5}{\sqrt{20}} \eta \lambda + \phi \lambda^* \right] + \cdots \tag{3.13}$$

$$\delta \phi = \frac{i}{2\sqrt{2}} \left[h \lambda^T + \lambda h^T \right] + \cdots \tag{3.14}$$

$$\delta \eta = i \frac{\sqrt{10}}{4} \left[h^\dagger \lambda - \lambda^\dagger h \right] + \cdots \tag{3.15}$$

$$\delta \omega = \frac{i}{2\sqrt{2}} \left[\lambda h^\dagger - h \lambda^\dagger \right] - \frac{i}{4\sqrt{2}} \left[h^\dagger \lambda - \lambda^\dagger h \right] \mathbf{1} + \cdots \tag{3.16}$$

where the ellipses stand for terms of quadratic and higher order in the fields.

Applying this variation to the kinetic term in (3.11) and retaining only terms quadratic in the fields we obtain

$$\delta\mathcal{L} = \frac{1}{\sqrt{2}}(Z_\phi - Z_h)\mathrm{Tr}\,\partial_\mu\phi^\dagger\partial^\mu h\lambda^T + \mathrm{h.c.}$$
$$+ \frac{1}{\sqrt{2}}(Z_\omega - Z_h)\mathrm{Tr}\,\partial_\mu\omega\partial^\mu\left[\lambda h^\dagger - h\lambda^\dagger\right] + \frac{\sqrt{10}}{4}(Z_\eta - Z_h)\mathrm{Tr}\,\partial_\mu\eta\partial^\mu\left[h^\dagger\lambda - \lambda^\dagger h\right]$$

Hence invariance under $SU(3)$ requires $Z_h = Z_\phi = Z_\omega = Z_\eta \equiv Z$. The same conclusion is reached by consideration of other embeddings of the invariance subgroup.

Already in the special $SU(3)\times SU(2)\times U(1)$-symmetric case one sees that divergences in the self-energy diagrams cannot be subtracted with a single common Z factor. One must introduce counterterms invariant under $SU(3)\times SU(2)\times U(1)$, or more generally, under G_w, that are not invariant under $SU(5)$. We next turn to constructing the relevant counterterms.

Scalar Kinetic Counterterms from Gauge Interaction

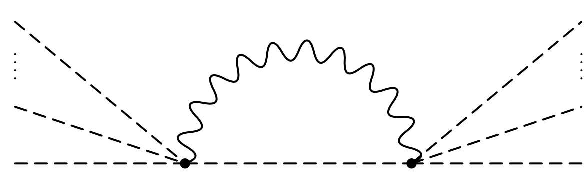


Figure 3.1: Scalar 2-point Function from gauge interaction with background pion fields.

Gauge interactions induce divergences in the scalar 2-point function with arbitrary background pion fields as shown in Fig. 3.1. We obtain the counterterms by the method of spurions. The gauge generators are promoted to spurions transforming in the adjoint representation of $SU(5)$, $T^a \to UT^aU^\dagger$. We list all the $SU(5)$ invariant counterterms with two T^a's and two derivatives. In the $SU(2)_1$ sector, with the generator Q_1^a defined in Eq. (3.1), we find

$$\begin{aligned}
\mathcal{O}_1^{g_1} &= \text{Tr}\,\left(Q_1^a Q_1^a\right) \text{Tr}\,\left(D_\mu \Sigma D^\mu \Sigma^*\right), \\
\mathcal{O}_2^{g_1} &= \text{Tr}\,\left(Q_1^a Q_1^a D_\mu \Sigma D^\mu \Sigma^*\right), \\
\mathcal{O}_3^{g_1} &= \text{Tr}\,\left(Q_1^a D_\mu \Sigma (Q_1^a)^T D^\mu \Sigma^*\right), \\
\mathcal{O}_4^{g_1} &= \text{Tr}\,\left(Q_1^a (D_\mu \Sigma)\Sigma^*\right) \text{Tr}\,\left(Q_1^a \Sigma D^\mu \Sigma^*\right), \\
\mathcal{O}_5^{g_1} &= \text{Tr}\,\left(Q_1^a (D_\mu \Sigma)\Sigma^* Q_1^a \Sigma D^\mu \Sigma^*\right), \\
\mathcal{O}_6^{g_1} &= \text{Tr}\,\left(Q_1^a \Sigma (Q_1^a)^T \Sigma^*\right) \text{Tr}\,\left(D_\mu \Sigma D^\mu \Sigma^*\right), \\
\mathcal{O}_7^{g_1} &= \text{Tr}\,\left(Q_1^a (D_\mu \Sigma)(D^\mu \Sigma^*)\Sigma (Q_1^a)^T \Sigma^*\right) + \text{h.c.}.
\end{aligned} \tag{3.17}$$

The counterterms for the $SU(2)_2$ sector are obtained from those in the $SU(2)_1$ sector by the replacements $g_1 \to g_2$ and $Q_1^a \to Q_2^a$. For the $U(1)_1$ sector, with the generator Y_1 defined in Eq. (3.1), we have

$$\begin{aligned}
\mathcal{O}_1^{g_1'} &= \text{Tr}\,\left(Y_1 Y_1\right) \text{Tr}\,\left(D_\mu \Sigma D^\mu \Sigma^*\right) \\
\mathcal{O}_2^{g_1'} &= \text{Tr}\,\left(Y_1 Y_1 D_\mu \Sigma D^\mu \Sigma^*\right) \\
\mathcal{O}_3^{g_1'} &= \text{Tr}\,\left(Y_1 D_\mu \Sigma Y_1 D^\mu \Sigma^*\right) \\
\mathcal{O}_4^{g_1'} &= \text{Tr}\,\left(Y_1 (D_\mu \Sigma)\Sigma^*\right) \text{Tr}\,\left(Y_1 \Sigma D^\mu \Sigma^*\right) \\
\mathcal{O}_5^{g_1'} &= \text{Tr}\,\left(Y_1 (D_\mu \Sigma)\Sigma^* Y_1 \Sigma D^\mu \Sigma^*\right) \\
\mathcal{O}_6^{g_1'} &= \text{Tr}\,\left(Y_1 \Sigma Y_1 \Sigma^*\right) \text{Tr}\,\left(D_\mu \Sigma D^\mu \Sigma^*\right) \\
\mathcal{O}_7^{g_1'} &= \text{Tr}\,\left(Y_1 (D_\mu \Sigma)(D^\mu \Sigma^*)\Sigma Y_1 \Sigma^*\right) + \text{h.c.}.
\end{aligned} \tag{3.18}$$

Similarly, the counterterms for the $U(1)_2$ sector can be obtained by substituting $g_1' \to g_2'$ and $Y_1 \to Y_2$ in the operators above.

Scalar Kinetic Counterterms from Yukawa Interaction

Yukawa interactions also induce divergences in the scalar 2-point function with arbitrary background pion fields as shown in Fig. 3.2. Just as was done for gauge generators, we treat the Yukawa couplings as $SU(5)$ breaking spurions. In doing so, we promote χ_L to a 5-plet

$$\mathcal{L}_{Yuk} = \bar{\chi}_{La} S^{abcde} \Sigma_{bc} \Sigma_{de} q_R + \text{h.c.}, \tag{3.19}$$

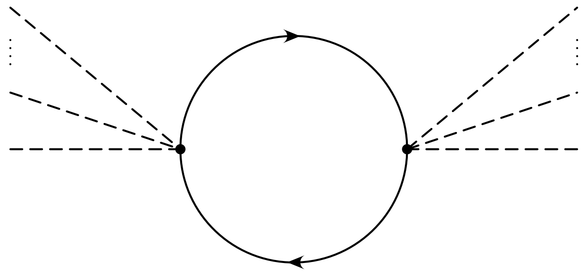


Figure 3.2: Scalar 2-point Function from Yukawa interaction with background pion fields.

with S symmetric in $\{b, c\}$, $\{d, e\}$ and the exchange of the pair $(b, c) \leftrightarrow (d, e)$. The spurion S is not arbitrary, but rather takes a fixed "vacuum expectation" value

$$< S^{abcde} >= \begin{cases} \frac{\lambda_1}{8}\epsilon^{abd45}\epsilon^{123ce} + \ldots & a = 1, 2 \\ \frac{\lambda_1'}{8}\epsilon^{3bd45}\epsilon^{123ce} + \ldots & a = 3 \end{cases} \tag{3.20}$$

where $+\ldots$ stands for symmetrization. Note that we can demand that $S \to S^*$ under CP, so $\mathcal{L}$ is invariant under CP. The counterterms will be also invariant under CP and hence hermitian. For notational compactness we define $\Psi^a = S^{abcde}\Sigma_{bc}\Sigma_{de}$. In terms of this, the counterterm is

$$\mathcal{O}_\Psi = D_\mu \Psi^{\dagger a} D^\mu \Psi_a. \tag{3.21}$$

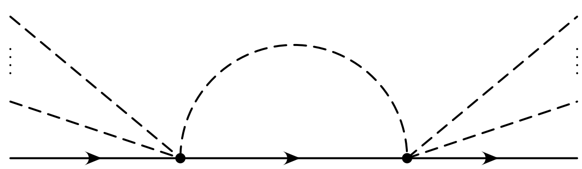


Figure 3.3: Fermion 2pt Function from Yukawa interaction with background pion fields

3.3.2 Fermion Kinetic Energy Counterterms

The divergence in the fermion self-energy is also present in the diagram with arbitrary number of pion fields at each of the Yukawa vertices, as shown in Fig. 3.3.

For notational compactness we defined $\Psi^{abc} = S^{abcde}\Sigma_{de}$ and $\xi_{abc} = S^*_{abcde}\Sigma^{*de}$. The counterterms for the q_R 2-point function are

$$\begin{aligned}
\mathcal{O}_{q_1} &= \bar{q}_R\bar{\Psi}_{abc}i\not{D}\Psi^{abc}q_R, \\
\mathcal{O}_{q_2} &= \bar{q}_R\bar{\Psi}_{abc}\Sigma^{*ce}i\not{D}\Sigma_{ed}\Psi^{abd}q_R, \\
\mathcal{O}_{q_3} &= \bar{q}_R\bar{\Psi}_{abc}\Sigma^{*bc}i\not{D}\Sigma_{de}\Psi^{ade}q_R, \\
\mathcal{O}_{q_4} &= \bar{q}_R\bar{\Psi}_{abc}\Sigma_{de}i\not{D}\Sigma^{*bc}\Psi^{ade}q_R, \\
\mathcal{O}_{q_5} &= \bar{q}_R\bar{\Psi}_{abc}\gamma^\mu\Psi^{ade}D_\mu\left(\Sigma^{*bc}\Sigma_{de}\right)q_R,
\end{aligned} \tag{3.22}$$

while the counterterms for χ_L 2-ponit function are

$$\begin{aligned}
\mathcal{O}_{\chi_1} &= \bar{\chi}_{La}\bar{\xi}^{abc}i\not{D}\xi_{a'bc}\chi_L^{a'}, \\
\mathcal{O}_{\chi_2} &= \bar{\chi}_{La}\bar{\xi}^{abc}\Sigma_{ce}i\not{D}\Sigma^{*ed}\xi_{a'bd}\chi_L^{a'}, \\
\mathcal{O}_{\chi_3} &= \bar{\chi}_{La}\bar{\xi}^{abc}\Sigma_{bc}i\not{D}\Sigma^{*de}\xi_{a'de}\chi_L^{a'}, \\
\mathcal{O}_{\chi_4} &= \bar{\chi}_{La}\bar{\xi}^{abc}\Sigma^{*de}i\not{D}\Sigma_{bc}\xi_{a'de}\chi_L^{a'}, \\
\mathcal{O}_{\chi_5} &= \bar{\chi}_{La}\bar{\xi}^{abc}\gamma^\mu\xi_{a'de}D_\mu\left(\Sigma^{*de}\Sigma_{bc}\right)\chi_L^{a'}.
\end{aligned} \tag{3.23}$$

3.3.3 Yukawa Vertex Counterterms

At 1-loop order, gauge interactions do not introduce a new counterterm. So we can subtract off the divergences with the Yukawa operator (*i.e.*, $\bar{\chi}_{La}S^{abcde}\Sigma_{bc}\Sigma_{de}q_R$+h.c.). This is not the case for Yukawa interactions which generate two new counterterms

$$\begin{aligned}
\mathcal{O}_{v_1} &= \bar{q}_R S^*_{abcde}\Sigma^{*de}S^{lmnop}\Sigma_{mn}\Sigma_{op}S^*_{lqrst}\Sigma^{*st}\Sigma^{*bc}\Sigma^{*qr}\chi_L^a + \text{h.c.}, \\
\mathcal{O}_{v_2} &= \bar{q}_R S^*_{abcde}\Sigma^{*de}S^{lmnop}\Sigma_{mn}\Sigma_{op}S^*_{lqrst}\Sigma^{*st}\Sigma^{*bq}\Sigma^{*cr}\chi_L^a + \text{h.c.}
\end{aligned} \tag{3.24}$$

3.3.4 Counterterms to counterterms: The general case

The counterterms displayed so far are appropriate to render all green functions finite if the only interactions in the model are those displayed in the Lagrangian given in Sec. 3.2. That is, the counterterms are appropriate to the case were the bare Lagrangian has the form given in Sec. 3.2. However, this cannot be maintained beyond 1-loop order. The 1-loop counterterms become interaction terms at 2-loops. This requires additional counterterms. And so on, as one moves to higher orders in the loop expansion. All terms consistent with the symmetries of the model will be generated by renormalization.

It is more natural to start with the complete set of interaction terms (formerly counterterms) and treat them all on an equal footing. However, this is not a viable program for this model since the complete set does not appear to be finite. The next best option is an organizing principle for a calculation that requires finite precision.

Before we make a specific proposal for one such organizing principle, we would like to contrast this with other models. Clearly the case of renormalizable theories is very different: only a finite number of terms is required to renormalize the theory to all orders in the loop expansion. More apropos, the case of chiral Lagrangians is different too. For these as one goes up in the loop expansion the counterterms involve accordingly more derivatives. Therefore the infinite set of counterterms are neatly organized by the number of derivatives which is tied to the loop expansion. In the $\mathrm{L}^2\mathrm{H}$ this is explicitly not the case already at 1-loop order: the counterterms generated are not suppressed by additional derivatives.

Suppose we are interested in processes that do not involve more than n PGBs. By expanding the Σ field in powers of the PGBs we will discover there is a finite number, $N(n, d)$ of linearly independent operators containing no more than d derivatives. Denote this basis of operators by $\widehat{\mathcal{O}}_i$. Then one can re-define the remaining (infinite set of) operators so that their expansion in PGBs starts at order higher than n, $\mathcal{O}_a \to \mathcal{O}_a - \sum_i c_a^i \widehat{\mathcal{O}}_i$ where the sum runs to $N(n, d)$. Given a desired precision for a calculation one can determine the order in the loop and momentum expansions required to achieve that precision. The latter gives us directly the required number of derivatives d to be retained. The number n of PGBs to be retained is a bit more complicated. For a process that involves k PGBs, an operator with $k + 2L$ PGBs can contribute at L-loop order. Therefore, for processes with no more than k PGBs that require L-loop precision and up to d powers of momenta, the basis with $N(k + 2L, d)$ operators should be used.

While the above algorithm is quite specific, we have not carried out that program of renormalization. The reason should be clear: the algorithm requires making a specific choice of process to study, or at least a restriction on the number of PGBs in the processes that will be considered. So, as explained at the top of this section, we have opted instead for the full 1-loop renormalization of the model of Sec. 3.2 assuming all other possible terms consistent with symmetries (an infinite set) is absent in the bare Lagrangian.

3.4 Renormalization

3.4.1 Generalities

The renormalized Lagrangian is

$$\mathcal{L} = \mathcal{L}_\phi + \mathcal{L}_\psi + \mathcal{L}_{\text{Yuk}} \tag{3.25}$$

where

$$\mathcal{L}_\phi = \frac{f^2}{8}\left[\text{Tr}\,\left(D_\mu\Sigma^\dagger D^\mu\Sigma\right) + \sum_{a,i}\zeta_a^{g_i} Z_a^{g_i}\mathcal{O}_a^{g_i} + \mu^{-\epsilon}\kappa_\Psi Z_\Psi \mathcal{O}_\Psi\right], \tag{3.26}$$

$$\begin{aligned}\mathcal{L}_\psi &= Z_{\chi_L}\bar{\chi}_{LI} i\not{D}\chi_L^I + Z_{q_R}\bar{q}_R i\not{D} q_R \\ &\quad + \mu^{-\epsilon}\sum_{a=1}^{5}\kappa_{\chi_a} Z_{\chi_a} Z_{\chi_L}\mathcal{O}_{\chi_a} + \mu^{-\epsilon}\sum_{b=1}^{5}\kappa_{q_b} Z_{q_b} Z_{q_R}\mathcal{O}_{q_b},\end{aligned} \tag{3.27}$$

$$\begin{aligned}\mathcal{L}_{\text{Yuk}} &= -f\mu^{\epsilon/2}\lambda_1 Z_\lambda (Z_{\chi_L} Z_{q_R})^{1/2}\bar{\chi}_{Li}\epsilon^{ij}\epsilon^{xy}\Sigma_{jx}\Sigma_{3y} q_R + \text{h.c.} \\ &\quad - \frac{f}{2}\mu^{\epsilon/2}\lambda_1' Z_{\lambda'}(Z_{\chi_L} Z_{q_R})^{1/2}\bar{u}_L\epsilon^{jk}\epsilon^{xy}\Sigma_{jx}\Sigma_{ky} q_R + \text{h.c.} \\ &\quad + f\mu^{-\epsilon}\sum_{a=1}^{2}\kappa_{v_a} Z_{v_a}\mathcal{O}_{v_a}.\end{aligned} \tag{3.28}$$

and the wavefunction renormalization of the Goldstone bosons is implicit in $\Sigma = \exp(2iZ^{1/2}\Pi/f)\Sigma_0$. Note that we have kept the bare f throughout, and it has dimension $1-\epsilon/2$ in dimensional regularization (with $d = 4-\epsilon$). This ensures that the coefficients of the power expansion of the kinetic terms do not run (or rather, they all run the same, just according to the wavefunction of the field Π). Since the spurion S^{abcde} includes the Yukawa coupling constants it has dimension $\epsilon/2$. Therefore, the bare couplings κ have dimension $-\epsilon$. We have ignored the λ_2 term in the Yukawa Lagrangian as it plays no role in renormalization.

The calculation will require we fix the $U(1)$ charges of all fields. For the Σ fields these are already determined by the transformation properties under G_f, and the fact that G_w is a subgroup of G_f. Since the hypercharges $Y = Y_1 + Y_2$ are fixed and the interactions are invariant under the gauge transformations, there is only freedom to choose the $U(1)$ charge of one quark field. We take q_R to have Y_2 charge y. Then the rest of the charges are fixed:

$$\begin{aligned} Y_1(q_R) &= 2/3 - y \quad & Y_2(q_R) &= y \\ Y_1(\chi) &= \tfrac{11}{30} - y \quad & Y_2(\chi_L) &= y - \tfrac{1}{5} \\ Y_1(u_L) &= \tfrac{13}{15} - y \quad & Y_2(u_L) &= y - \tfrac{1}{5} \end{aligned} \tag{3.29}$$

The ζ_a and κ_a terms modify the Lagrangian at tree level and these modifications should be included in our perturbative computations. However, we intend to take the bare parameters ζ_a and κ_a to vanish at the end of the calculation. This is because we want to study the radiative corrections that generate these terms, even if absent from the bare Lagrangian. Then, while ζ_a terms in the tree level Feynman rules can be neglected, the counterterms, of the form $\zeta_a(Z_a - 1)$, do not vanish as $\zeta_a \to 0$ (similarly for κ_a terms). The RGE for these couplings is derived through standard methods. We use a generic coupling ζ for the couplings of ζ, κ terms. Taking a log-derivative with respect to μ of $\zeta^{\text{bare}} = \mu^{\epsilon D_\zeta} Z\zeta$, where ϵD_ζ is the dimension of the bare coupling ζ, we have

$$\epsilon D_\zeta Z\zeta + \mu \frac{\partial \zeta}{\partial \mu}\left(Z + \zeta \frac{\partial Z}{\partial \zeta}\right) + \zeta\left(-\frac{1}{2}\epsilon g + \beta_g\right)\frac{\partial Z_a}{\partial g} = 0. \tag{3.30}$$

Here g stands for the collection of Yukawa and gauge coupling constants, and there is an implicit sum over these. Since $\mu\frac{\partial\zeta}{\partial\mu}$ has a finite limit as $\epsilon \to 0$ and Z can be written as

$$Z = 1 + \frac{a^{(1)}}{\zeta}\frac{1}{\epsilon} + O(\epsilon^{-2}),$$

where $a^{(1)} = a^{(1)}(g)$ is only a function of the couplings, we have

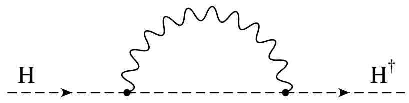
$$\begin{aligned} \mu\frac{\partial \zeta}{\partial \mu} &= -\epsilon D_\zeta \zeta + \beta_\zeta, \\ \beta_\zeta &= -D_\zeta a^{(1)} + \frac{1}{2} g \frac{\partial a^{(1)}}{\partial g}. \end{aligned} \tag{3.31}$$

We will determine the normalization factors Z_a in the next subsection.

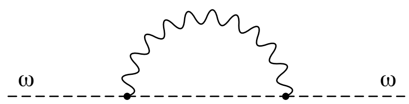
3.4.2 Matching Counterterms

Scalar 2-Point Functions with Arbitrary Scalar Background

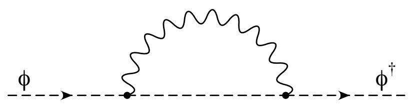
We first consider the $SU(2)_1$ gauge sector. The non-trivial 2-point functions for the scalars are



$$= -\frac{3}{4}g_1^2\frac{3}{4}\frac{1}{16\pi^2}\frac{2}{\epsilon}ip^2\delta_{ij} \tag{3.32}$$

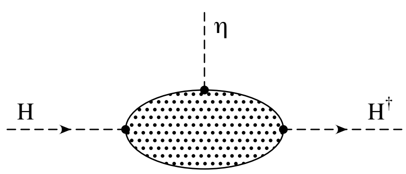


$$= -2g_1^2\frac{3}{4}\frac{1}{16\pi^2}\frac{2}{\epsilon}ip^2\delta^{bc} \tag{3.33}$$

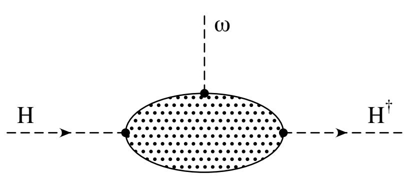


$$= -2g_1^2\frac{3}{4}\frac{1}{16\pi^2}\frac{2}{\epsilon}ip^2\delta^{bc} \tag{3.34}$$

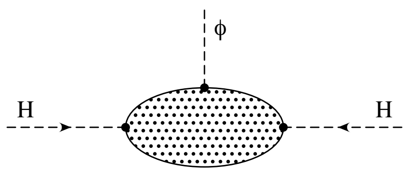
There is no η self-enrgy diagram because it is a singlet under the gauge group. For the 3-point functions, we denote by p_i the momentum of particle i starting from the left in the clockwise direction in the following diagrams



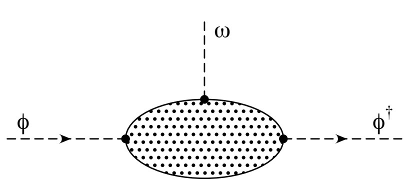
$$= \frac{15}{4\sqrt{20}}\frac{g_1^2}{f}\frac{3}{4}\frac{1}{16\pi^2}\frac{2}{\epsilon}\left(p_1^2 - p_3^2\right)\delta_{ij} \tag{3.35}$$



$$= \frac{5}{4}\frac{g_1^2}{f}\frac{3}{4}\frac{1}{16\pi^2}\frac{2}{\epsilon}\left(p_1^2 - p_3^2\right)\frac{\sigma^a_{ij}}{2} \tag{3.36}$$



$$= \frac{1}{8}\frac{g_1^2}{f}\frac{3}{4}\frac{1}{16\pi^2}\frac{2}{\epsilon}\left(7p_2^2 - 2p_1^2 - 2p_3^2\right) \times \left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) \tag{3.37}$$



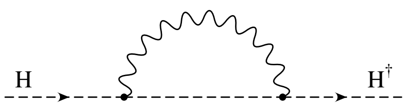
$$= 2\frac{g_1^2}{f}\frac{3}{4}\frac{1}{16\pi^2}\frac{2}{\epsilon}\left(p_1^2 - p_3^2\right)i\epsilon^{abc} \tag{3.38}$$

where we used $\omega = \omega^a\sigma^a/2$ and $\phi = \sigma^a\sigma^2\phi^a/\sqrt{2}$. We have also found that the 4-point function with two η's and two ϕ's vanishes. Cancelation of divergences in these diagrams, together with the absence of diagram with two η's and two ϕ's requires

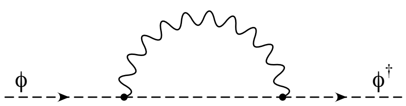
$$\begin{aligned} Z &= 1, & Z_4^{g_1} &= 1, \\ Z_1^{g_1} &= 1, & Z_5^{g_1} &= 1 - 3\frac{1}{\zeta_5^{g_1}}\frac{g_1^2}{16\pi^2}\frac{2}{\epsilon}, \\ Z_2^{g_1} &= 1 + 3\frac{1}{\zeta_2^{g_1}}\frac{g_1^2}{16\pi^2}\frac{2}{\epsilon}, & Z_6^{g_1} &= 1 + \frac{3}{20}\frac{1}{\zeta_6^{g_1}}\frac{g_1^2}{16\pi^2}\frac{2}{\epsilon}, \\ Z_3^{g_1} &= 1 + 3\frac{1}{\zeta_3^{g_1}}\frac{g_1^2}{16\pi^2}\frac{2}{\epsilon}, & Z_7^{g_1} &= 1 - \frac{3}{2}\frac{1}{\zeta_7^{g_1}}\frac{g_1^2}{16\pi^2}\frac{2}{\epsilon}, \end{aligned} \tag{3.39}$$

We can similarly determine $Z_i^{g_2}$ by nothing that $\mathcal{L}_{g_2} \to \mathcal{L}_{g_1}$ and $\mathcal{O}_i^{g_2} \to \mathcal{O}_i^{g_1}$ when $\Pi \to -\Pi$. Thus we have $Z_i^{g_1} = Z_i^{g_2}$.

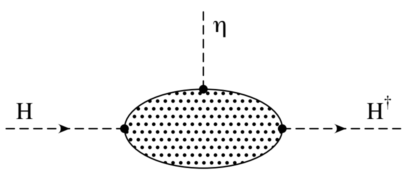
We next consider the $U(1)_1$ sector. The divergent 2-point and 3-point functions are



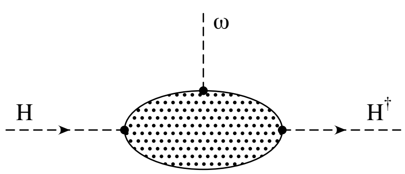
$$= \; -\frac{1}{4}g'^2\frac{3}{4}\frac{1}{16\pi^2}\frac{2}{\epsilon}ip^2\delta_{ij}, \tag{3.40}$$



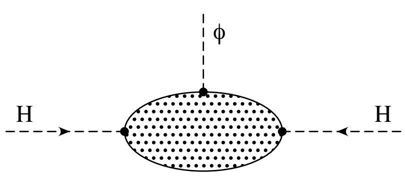
$$= \; -g'^2\frac{3}{4}\frac{1}{16\pi^2}\frac{2}{\epsilon}ip^2\delta^{ab}, \tag{3.41}$$



$$= \; \frac{\sqrt{5}}{8}\frac{g'^2}{f}\frac{3}{4}\frac{1}{16\pi^2}\frac{2}{\epsilon}\left(p_1^2 - p_3^2\right)\delta_{ij}, \tag{3.42}$$



$$= \; -\frac{1}{4}\frac{g'^2}{f}\frac{3}{4}\frac{1}{16\pi^2}\frac{2}{\epsilon}\left(p_1^2 - p_3^2\right)\frac{\sigma_{ij}^a}{2}, \tag{3.43}$$



$$= \; -\frac{3}{8}\frac{g'^2}{f}\frac{3}{4}\frac{1}{16\pi^2}\frac{2}{\epsilon}p_2^2\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right). \tag{3.44}$$

As in the case of $SU(2)$, the 4-point function with two η's and two ϕ's vanishes. Cancelation of divergences in these diagrams, together with the absence of a divergence in the diagram with two ηs and two ϕs, requires

$$\begin{aligned}
Z_1^{g_1'} &= 1 - \frac{1}{40}\frac{1}{\zeta_1^{g_1'}}\frac{g_1'^2}{16\pi^2}\frac{2}{\epsilon}, & Z_4^{g_1'} &= 1, \\
Z_2^{g_1'} &= 1 + \frac{3}{8}\frac{1}{\zeta_2^{g_1'}}\frac{g_1'^2}{16\pi^2}\frac{2}{\epsilon}, & Z_5^{g_1'} &= 1 - \frac{3}{8}\frac{1}{\zeta_5^{g_1'}}\frac{g_1'^2}{16\pi^2}\frac{2}{\epsilon}, \\
Z_3^{g_1'} &= 1 + \frac{3}{8}\frac{1}{\zeta_3^{g_1'}}\frac{g_1'^2}{16\pi^2}\frac{2}{\epsilon}, & Z_6^{g_1'} &= 1 - \frac{3}{80}\frac{1}{\zeta_6^{g_1'}}\frac{g_1'^2}{16\pi^2}\frac{2}{\epsilon}, \\
& & Z_7^{g_1'} &= 1 - \frac{3}{16}\frac{1}{\zeta_7^{g_1'}}\frac{g_1'^2}{16\pi^2}\frac{2}{\epsilon},
\end{aligned} \tag{3.45}$$

and $Z_i^{g_1'} = Z_i^{g_2'}$. The divergence in the H 2-point function from Yukawa interaction is

$$\frac{2\lambda_1^2}{16\pi^2}\frac{2}{\epsilon}$$

and in the η 2-point function is

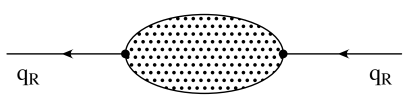
$$\frac{8}{5}\frac{\lambda_1'^2}{16\pi^2}\frac{2}{\epsilon}.$$

Thus we obtain

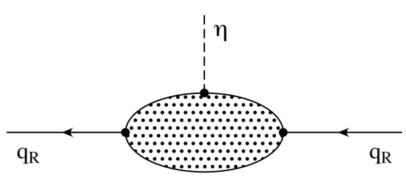
$$Z_\Psi = 1 - \frac{1}{\kappa_\psi}\frac{1}{16\pi^2}\frac{2}{\epsilon}. \tag{3.46}$$

Fermion 2-Point Functions

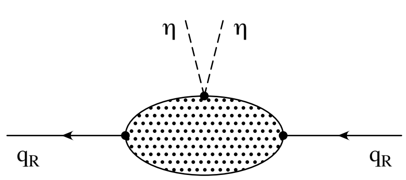
We first consider the q_R 2-point functions with arbitrary scalar background. The 1-loop diagrams are



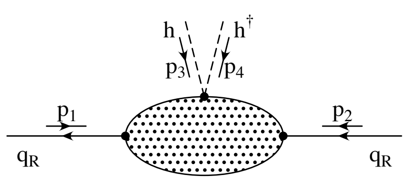
$$= \frac{1}{16\pi^2}\frac{2}{\epsilon}\left(2\lambda_1^2 + \frac{2}{5}\lambda_1'^2\right) i\not{p}, \tag{3.47}$$



$$= \frac{1}{16\pi^2}\frac{2}{\epsilon}\left(-\frac{1}{\sqrt{5}}\lambda_1^2 + \frac{4}{5\sqrt{5}}\lambda_1'^2\right) i\not{p}_\eta, \tag{3.48}$$



$$= 0, \tag{3.49}$$



$$= \frac{1}{16\pi^2}\frac{2}{\epsilon}\left[\frac{3i}{5}(\lambda_1^2 - \lambda_1'^2)(\not{p}_1 - \not{p}_2) + \frac{i\lambda_1^2}{10}(\not{p}_3 - \not{p}_4)\right]. \tag{3.50}$$

Matching counterterms yields

$$\begin{aligned} Z_{q_R} &= 1, \qquad Z_{q_2} = 1 - 8\frac{1}{\kappa_{q_2}}\frac{1}{16\pi^2}\frac{2}{\epsilon}, \qquad Z_{q_4} = 1, \\ Z_{q_1} &= 1, \qquad Z_{q_3} = 1 + \frac{8}{5}\frac{1}{\kappa_{q_3}}\frac{1}{16\pi^2}\frac{2}{\epsilon}, \qquad Z_{q_5} = 1. \end{aligned} \tag{3.51}$$

Similarly, for the χ_L 2-point functions, we find

$$\begin{aligned} Z_{\chi_L} &= 1, \qquad Z_{\chi_2} = 1 - 8\frac{1}{\kappa_{\chi_2}}\frac{1}{16\pi^2}\frac{2}{\epsilon}, \qquad Z_{\chi_4} = 1, \\ Z_{\chi_1} &= 1, \qquad Z_{\chi_3} = 1 + \frac{8}{5}\frac{1}{\kappa_{\chi_3}}\frac{1}{16\pi^2}\frac{2}{\epsilon}, \qquad Z_{\chi_5} = 1. \end{aligned} \tag{3.52}$$

Yukawa Vertex Counterterms

As we mentioned above, gauge interactions do not induce new operators but the Yukawa interaction do. Here we distinguish the $SU(2)$ and $U(1)$ part of the Yukawa in their action:

$$\begin{aligned} Z_{\lambda_1} &= 1 - 3\frac{1}{16\pi^2}\frac{2}{\epsilon}\left[\left(\frac{11}{30} - y\right)\left(\frac{2}{3} - y\right)g_1'^2 + \left(y - \frac{1}{5}\right)yg_2'^2\right], \\ Z_{\lambda_1'} &= 1 - 3\frac{1}{16\pi^2}\frac{2}{\epsilon}\left[\left(\frac{13}{15} - y\right)\left(\frac{2}{3} - y\right)g_1'^2 + \left(y - \frac{1}{5}\right)yg_2'^2\right]. \end{aligned} \tag{3.53}$$

We see that λ_1 and λ_1' are renormalized differently. The other renomalization factors are:

$$\begin{aligned} Z_{v_1} &= 1 + \frac{4}{5}\frac{1}{\kappa_{v_1}}\frac{1}{16\pi^2}\frac{2}{\epsilon}, \\ Z_{v_2} &= 1 + \frac{11}{5}\frac{1}{\kappa_{v_2}}\frac{1}{16\pi^2}\frac{2}{\epsilon}. \end{aligned} \tag{3.54}$$

3.4.3 β Functions

We have already pointed out that the coupling λ_1 and λ_1' run differently [24]. It is now straightforward to obtain their beta functions:

$$\frac{\beta_{\lambda_1}}{\lambda_1} = -\frac{3}{8\pi^2}\left[\left(\tfrac{11}{30} - y\right)\left(\tfrac{2}{3} - y\right)g_1'^2 + \left(y - \tfrac{1}{5}\right)y\, g_2'^2\right] \tag{3.55}$$

$$\frac{\beta_{\lambda_1'}}{\lambda_1'} = -\frac{3}{8\pi^2}\left[\left(\tfrac{13}{15} - y\right)\left(\tfrac{2}{3} - y\right)g_1'^2 + \left(y - \tfrac{1}{5}\right)y\, g_2'^2\right] \tag{3.56}$$

Note, in particular, that

$$\mu\frac{\partial}{\partial\mu}\ln\left(\frac{\lambda_1}{\lambda_1'}\right) = \left(\tfrac{2}{3} - y\right)\frac{3g_1'^2}{16\pi^2} \tag{3.57}$$

With $\beta_{g_1'} = (b/16\pi^2) g_1'^3$ we can write the solution in terms of the running coupling:

$$\frac{\lambda_1(\mu)}{\lambda_1'(\mu)} = \frac{\lambda_1(\Lambda)}{\lambda_1'(\Lambda)} \left(\frac{g_1'(\mu)}{g_1'(\Lambda)} \right)^{\frac{2-3y}{b}} \tag{3.58}$$

The β functions for the couplings ζ_a^g are determined using Eq. (3.31). We find

$$\begin{aligned}
\beta_{\zeta_1}^{g_{1(2)}} &= 0, & \beta_{\zeta_1}^{g'_{1(2)}} &= -\frac{1}{20}\frac{g'^2_{1(2)}}{16\pi^2}, \\
\beta_{\zeta_2}^{g_{1(2)}} &= 6\frac{g^2_{1(2)}}{16\pi^2}, & \beta_{\zeta_2}^{g'_{1(2)}} &= \frac{3}{4}\frac{g'^2_{1(2)}}{16\pi^2}, \\
\beta_{\zeta_3}^{g_{1(2)}} &= 6\frac{g^2_{1(2)}}{16\pi^2}, & \beta_{\zeta_3}^{g'_{1(2)}} &= \frac{3}{4}\frac{g'^2_{1(2)}}{16\pi^2}, \\
\beta_{\zeta_4}^{g_{1(2)}} &= 0, & \beta_{\zeta_4}^{g'_{1(2)}} &= 0, \\
\beta_{\zeta_5}^{g_{1(2)}} &= -6\frac{g^2_{1(2)}}{16\pi^2}, & \beta_{\zeta_5}^{g'_{1(2)}} &= -\frac{3}{4}\frac{g'^2_{1(2)}}{16\pi^2}, \\
\beta_{\zeta_6}^{g_{1(2)}} &= \frac{3}{10}\frac{g^2_{1(2)}}{16\pi^2}, & \beta_{\zeta_6}^{g'_{1(2)}} &= -\frac{3}{40}\frac{g'^2_{1(2)}}{16\pi^2}, \\
\beta_{\zeta_7}^{g_{1(2)}} &= -3\frac{g^2_{1(2)}}{16\pi^2}, & \beta_{\zeta_7}^{g'_{1(2)}} &= -\frac{3}{8}\frac{g'^2_{1(2)}}{16\pi^2}.
\end{aligned} \tag{3.59}$$

For κ the couplings are implicit in the operators so the β functions are pure number

$$\begin{aligned}
\beta_{\kappa_{q_1}} &= 0, & \beta_{\kappa_{\chi_1}} &= 0, \\
\beta_{\kappa_{q_2}} &= -16\frac{1}{16\pi^2}, & \beta_{\kappa_{\chi_2}} &= -16\frac{1}{16\pi^2}, \\
\beta_{\kappa_{q_3}} &= \frac{16}{5}\frac{1}{16\pi^2}, & \beta_{\kappa_{\chi_3}} &= +\frac{16}{5}\frac{1}{16\pi^2}, \\
\beta_{\kappa_{q_4}} &= 0, & \beta_{\kappa_{\chi_4}} &= 0, \\
\beta_{\kappa_{q_5}} &= 0, & \beta_{\kappa_{\chi_5}} &= 0,
\end{aligned} \tag{3.60}$$

and

$$\beta_\psi = -2\frac{1}{16\pi^2}, \quad \beta_{\kappa_{v_1}} = \frac{8}{5}\frac{1}{16\pi^2}, \quad \beta_{\kappa_{v_2}} = \frac{22}{5}\frac{1}{16\pi^2}. \tag{3.61}$$

3.5 Conclusions

We have studied the one loop renormalization of the Littlest Higgs Model. Phenomenologically this model has fallen somewhat out of favor because of its difficulties simultaneously accommodating electroweak precision constraints and solving the little hierarchy problem [7, 8, 9, 10, 11]. However, its structure is prototypical of many models, like Littlest Higgs models with reduced gauge symmetry [22], or with custodial [12, 23] or T-parity [13, 14, 15] symmetries. Therefore, the methods introduced here should be largely the same as those needed for one loop renormalization of any model in this class.

We have displayed explicit counterterms and their Z factors in dimensional regularization, in Landau gauge. These results are only of interest to understand the procedure, so they have been included here more for clarity of presentation. However, the beta functions of the couplings of all the terms in the Lagrangian are independent of gauge and scheme choice. They, together with the methods introduced, constitute the main result of this work and are displayed explicitly in Sec. 3.4.

One important result is that the coupling constants associated with the Yukawa coupling of the top quark run differently; see Eq. (3.57). As observed in Ref. [24] in the absence of fine tuning, the collective symmetry mechanism fails for Yukawa couplings in the Littlest Higgs model and its relatives. One can similarly conclude that the terms that were required as counterterms, all allowed by the symmetries and being of leading order in the derivative expansion, should have been included in the model from the start.

This chapter is a reprint of material as it appears in "One Loop Renormalization of the Littlest Higgs Model," B. Grinstein, R. Kelley and P. Uttayarat, JHEP **1102**, 089 (2011) [arXiv:1011.0682 [hep-ph]], of which I was a co-author.

Chapter 4

Spontaneous CP Violation and Light Particles in The Littlest Higgs Models

Little Higgs models often feature spontaneously broken extra global symmetries, which must also be explicitly broken in order to avoid massless Goldstone modes in the spectrum. We show that a possible conflict with collective symmetry breaking then implies light modes coupled to the Higgs boson, leading to interesting phenomenology. Moreover, spontaneous CP violation is quite generic in such cases, as the explicit breaking may be used to stabilize physical CP odd phases in the vacuum. We demonstrate this in an SU(2)×SU(2)×U(1) variant of the Littlest Higgs, as well as in an original SU(6)/SO(6) model. We show that even a very small explicit breaking may lead to large phases, resulting in new sources of CP violation in this class of models.

4.1 Introduction

Despite its impressive experimental success, the Standard Model (SM) is known to have several theoretical puzzles. One of these, the "hierarchy problem", is the apparent fine tuning associated with the electroweak scale. This paradigm has led to numerous hypotheses, such as supersymmetry, technicolor, extra-dimensions, and more. In order to eliminate the hierarchy problem, models based on these hypotheses often introduce new physics at the TeV scale. Unfortunately, such low scale new physics seem to generally spoil the success of the SM by introducing low energy effects which are tightly constrained by experimental data.

The tension between the need to solve the hierarchy problem and the above experimental constraints is known as the "little hierarchy problem". It may be solved by using the Little Higgs framework [26, 27], where physics beyond the SM appears only at $\Lambda \sim 10$ TeV instead of the generically expected 1 TeV. The SM Higgs field remains naturally light by serving as a pseudo Goldstone boson of multiple approximate global symmetries. Explicit breaking of this set of symmetries is "collective", *i.e.,* apparent only in the presence of at least two terms in the Lagrangian. This ensures that the only one-loop diagrams contributing to the Higgs mass are logarithmically divergent at most, thereby allowing for a cutoff at $\Lambda \sim (4\pi)^2 v$ instead of the generic $\Lambda \sim 4\pi v$.

In Little Higgs models, the electroweak gauge group is extended to a partially gauged global symmetry. The gauged generators are broken spontaneously to the electroweak gauge group. Some of the global generators are broken spontaneously too, but in a realistic model they must be also broken explicitly in order to avoid exact Goldstone bosons. Then, one has to make sure that the set of global symmetries which protect the Higgs is not broken non-collectively. Such non-collective breaking would destabilize the electroweak scale.

In this paper we discuss cases, such as the $SU(2)^2 \times U(1)$ Littlest Higgs variant [10, 22], where there is a tension between lifting the mass of the pseudo-Goldstone bosons and retaining collective symmetry breaking. A consequence of this is the presence of light particles with direct couplings to the SM Higgs, leading to interesting phenomenology. For example, there is a range of parameters for which a new decay channel for the Higgs opens up.

Another possible consequence is the appearance of spontaneous CP violation, *i.e.,* physical phases in the VEV. Such phases are rotated by field redefinitions. The generators of these transformations must obey some conditions if the vacuum indeed breaks CP invariance [28]. In particular, in order for a phase to be physical, the related generator must be both explicitly and spontaneously broken. In case there is a conflict between this requirement and that of collective symmetry breaking, one may expect that the effect of spontaneous CP violation is suppressed - from the same reason that the related pseudo-Goldstone bosons are light. However, as we will show, the CP violating phase may be $\mathcal{O}(1)$, even in the limit of small explicit breaking.

We begin by reviewing the Littlest Higgs model and its SU(2)$^2\times$U(1) variant, showing that it includes an exact Goldstone due to a spontaneously broken global U(1) which would be gauged in the original SU(2)$^2\times$U(1)^2 version of the littlest Higgs. We then show that lifting the exact Goldstone requires spoiling collective symmetry breaking, hence leading to a suppression of its mass. Once collective symmetry breaking is spoiled, even by a small parameter, it becomes possible for the vacuum to align with an $\mathcal{O}(1)$ CP-odd phase. We discuss how such CP violation arises in the low energy limit. In the SU(2)$^2\times$U(1) model it turns out to be suppressed, but we argue that this is a peculiarity of the minimal nature of the SU(5) structure, rather than a generic feature in Little Higgs models. In order to support this statement, we construct an original SU(6)/SO(6) model which accommodates an $\mathcal{O}(1)$ physical phase in the low energy limit.

4.2 Saving the SU(2)×SU(2)×U(1) Model

Here we will discuss the SU(2)×SU(2)×U(1), and how to make its Goldstone boson massive without destabilizing the electroweak scale. But before that, let us briefly review the original Littlest Higgs.

4.2.1 The Littlest Higgs

A very elegant implementation of the Little Higgs idea is the Littlest Higgs [4], whose lagrangian is described as an approximate SU(5)/SO(5) effective field theory. The vacuum manifold SU(5)/SO(5) may be parametrized as $\Sigma_0 = UU^T$, where U is a broken SU(5) transformation. The global SU(5) is explicitly broken by gauging an

$[\mathrm{SU}(2)\times\mathrm{U}(1)]^2$ subgroup, where the gauged generators are embedded in SU(5) as

$$T_1^i = \begin{pmatrix} \sigma^i/2 & & \\ & 0_{1\times 1} & \\ & & 0_{2\times 2} \end{pmatrix}, \quad Y_1 = \mathrm{diag}(3,3,-2,-2,-2)/10;$$
$$T_2^i = \begin{pmatrix} 0_{2\times 2} & & \\ & 0_{1\times 1} & \\ & & -\sigma^{a*}/2 \end{pmatrix}, \quad Y_2 = \mathrm{diag}(2,2,2,-3,-3)/10. \tag{4.1}$$

Once this explicit SU(5) breaking is included, the degeneracy is partially lifted, as a minimum energy vacuum appears at

$$\Sigma_0 = \begin{pmatrix} & & e^{i\delta}V \\ & e^{-4i\delta} & \\ e^{i\delta}V^T & & \end{pmatrix}. \tag{4.2}$$

Here, V is a 2×2 special unitary matrix and δ is a real parameter. Gauging the $[\mathrm{SU}(2)\times\mathrm{U}(1)]^2$ subgroup breaks explicitly all the SU(5) generators which are not gauged. The vacuum breaks the $[\mathrm{SU}(2)\times\mathrm{U}(1)]^2$ gauge group to the electroweak group, $\mathrm{SU}(2)_L\times\mathrm{U}(1)_Y$. One can then use the spontaneously broken generators to rotate the vacuum into the form

$$\Sigma_0 = \begin{pmatrix} & & \mathbf{1} \\ & 1 & \\ \mathbf{1} & & \end{pmatrix}. \tag{4.3}$$

By doing so, we have chosen a basis in which the electroweak gauge group is given by the diagonal subgroup of the full Little Higgs gauge group.

We follow the common formalism for chiral lagrangians [29, 30] and arrange the Goldstone bosons in a matrix,

$$\Sigma = e^{i\Pi/f}\Sigma_0 e^{i\Pi^T/f}, \qquad \Pi = \Pi^a X^a, \tag{4.4}$$

where X^a are the 14 broken generators of SU(5). We can always choose a basis where

$$X^a\Sigma_0 = \Sigma_0 X^{aT}. \tag{4.5}$$

In this basis, Eq.(4.4) simplifies to

$$\Sigma = e^{2i\Pi/f}\Sigma_0. \tag{4.6}$$

Four of the above fourteen degrees of freedom become the longitudinal components of the $W'^{\pm}$, Z' and γ', which correspond to the spontaneously broken gauged generators. The remaining ten pseudo-Goldstone bosons can be classified according to their SM quantum numbers as one complex doublet H, which we identify with the SM Higgs, and one complex triplet, ϕ, which carries one unit of hypercharge. These pseudo-Goldstone bosons are parametrized as follows:

$$\Pi = \begin{pmatrix} \text{eaten} & H/\sqrt{2} & \phi \\ H^\dagger/\sqrt{2} & \text{eaten} & H^T/\sqrt{2} \\ \phi^\dagger & H^*/\sqrt{2} & \text{eaten} \end{pmatrix}. \tag{4.7}$$

From the transformation law $\Sigma \to U\Sigma U^T$, it follows that the Higgs transforms nonlinearly under $SU(3)_1$ and $SU(3)_2$, which act on the (123) and (345) blocks, respectively. Note that the $SU(2)_1 \times U(1)_1$ gauge interactions break $SU(3)_1$ and conserve $SU(3)_2$, whereas $SU(2)_2 \times U(1)_2$ gauge interactions conserve $SU(3)_1$ and break $SU(3)_2$. However, the two (overlapping) groups $SU(3)_1$ and $SU(3)_2$ are fully broken only when both sets of gauge couplings are turned on, namely, they are collectively broken. Therefore, any diagram which contributes to the Higgs mass must involve both '1' and '2' gauge interactions. However, the only one-loop diagrams contributing to the Higgs mass involve two gauge boson propagators, leading to only a logarithmic dependence: $\delta m_H^2 \sim \left(\frac{gf}{4\pi}\right)^2 \log(\Lambda/f)$.

In order to maintain collective symmetry breaking also in the top quark sector, we introduce a new vector-like quark pair (t'_L, t'_R) which is $SU(2)_L$ singlet, and we define $\chi_L^i = (i\sigma^2 Q_L, t'_L)$, where $i = 1, 2, 3$ and Q_L is the SM third generation quark doublet. The top quark sector is taken to be

$$\mathcal{L} = \lambda f \overline{\chi}_{Li} \Omega^i t_R + \lambda' f \overline{t}'_L t'_R + \text{c.c.}, \tag{4.8}$$

where

$$\Omega^i = \epsilon^{ijk} \epsilon^{xy} \Sigma_{jx} \Sigma_{ky}. \tag{4.9}$$

Here, i, j, k run over $1, 2, 3$ and x, y over $4, 5$. The first term is invariant under $SU(3)_1$, but breaks $SU(3)_2$, whereas the second term breaks $SU(3)_1$ and preserves $SU(3)_2$. That this is the case can be seen by taking χ_L^i to be an $SU(3)_1$ triplet. Diagrams which contribute to the Higgs mass must involve both couplings, and are only logarithmically divergent at one-loop.

4.2.2 The Hypercharge Model

The Littlest Higgs model suffers from large corrections to electroweak precision observables, mainly due to the heavy gauge boson related to $U(1)'$. One solution to this problem is to impose T-parity [13, 14, 31], under which SM fields are even and new heavy fields are odd. This removes all the single heavy field exchange diagrams, effectively pushing many dangerous contributions to the electroweak precision observables to the loop level. In the same time, T-parity provides a WIMP dark matter which naturally gives the correct thermal relic abundance. Nevertheless, the multitude of new fields makes it potentially vulnerable to flavor problems [32, 33]. Moreover, it becomes difficult to find a simple UV completion to match it onto [34, 35, 36, 37].

Another solution was to gauge only one of the U(1) generators [10, 22]. It is this solution that we are considering here, although our lessons for model building and spontaneous CP violation are rather generic, and we expect them to hold whether or not T-parity is imposed. Let us define the following two combinations of U(1) generators:

$$\begin{aligned} Y &= \frac{Y_1 + Y_2}{2} = \frac{1}{2}\mathrm{diag}(1,1,0,-1,-1), \\ Y' &= \frac{Y_1 - Y_2}{2} = \frac{1}{10}\mathrm{diag}(1,1,-4,1,1). \end{aligned} \tag{4.10}$$

In this model which we denote as the *hypercharge model*, only the the SM hypercharge Y is gauged while Y' generates a global symmetry, which we denote as U(1)$'$. The pseudo-Goldstone bosons matrix now becomes (in terms of the uneaten fields)

$$\Pi = \begin{pmatrix} \eta/\sqrt{20}\,\mathbf{1} & H/\sqrt{2} & \phi \\ H^\dagger/\sqrt{2} & -2\eta/\sqrt{5} & H^T/\sqrt{2} \\ \phi^\dagger & H^*/\sqrt{2} & \eta/\sqrt{20}\,\mathbf{1} \end{pmatrix}. \tag{4.11}$$

While gauging U(1)$_Y$ alone eliminates the troublesome heavy gauge boson, it spoils collective symmetry breaking, since unlike U(1)$_1$ or U(1)$_2$ gauge interactions which conserve one SU(3) each, the hypercharge gauge interaction breaks explicitly both SU(3)$_1$ and SU(3)$_2$ via a single term in the Lagrangian. As was shown in [10, 22], this effect is suppressed by the smallness of the hypercharge coupling g', and we will not discuss it further.[1]

Another, more acute problem of the hypercharge model is that it introduces a new massless Goldstone boson η, which corresponds to the spontaneously broken $\mathrm{U}(1)'$. Note that so far, $\mathrm{U}(1)'$ is an exact symmetry which is only broken spontaneously. In the Littlest Higgs, this Goldstone boson is eaten by the corresponding gauge boson, which is absent in the hypercharge model. Therefore, in order for this model to be phenomenologically viable, the new Goldstone must acquire mass, requiring explicit breaking of $\mathrm{U}(1)'$. This has been recognized previously, but without providing explicit realization. For example, in [38], the phenomenology of η was studied, *assuming* a range of masses up to $m_\eta \sim v$. Below we show that any operator that gives mass to η is bound to introduce further non-collective symmetry breaking, thus constraining m_η to be roughly below the SM Higgs mass. The assumption on m_η in [38] is therefore consistent with our results.

4.2.3 A Realistic Hypercharge Model

Before proving that collective symmetry breaking must be spoiled by any term that breaks $\mathrm{U}(1)'$, let us state a generic condition any explicitly broken generator has to satisfy in order not to spoil collective symmetry breaking, namely, in order not to break the full set of symmetries which protect the Higgs by a single term in the lagrangian. In order to do that, denote the collection of groups under which the Higgs transforms non-homogeneously by $\{\mathcal{C}_i\}$. Each of these groups should be *minimal* in the sense that it does not contain a subgroup which protects the Higgs. The $\mathcal{C}_i$ may be disjoint (as in the Minimal Moose [39] or in the Simplest Little Higgs [40]) or overlapping (as in the Littlest Higgs, where we have $\mathcal{C}_1 = \mathrm{SU}(3)_1$ and $\mathcal{C}_2 = \mathrm{SU}(3)_2$).

Consider a generator X. First note that if X is a linear combination of gauged generators and a generator of $\mathcal{C}_i$ for a particular i, then breaking X explicitly requires breaking $\mathcal{C}_i$ explicitly too (since gauge invariance must be an exact symmetry). If this is true for all i, then any term in the lagrangian which breaks X explicitly would inevitably spoil collective symmetry breaking. We thus arrive at the following condition:

[1] Radiative corrections break $\mathrm{SU}(3)_1$ already in the original $[\mathrm{SU}(2)\times\mathrm{U}(1)]^2$ model [24, 17]. Here we will consider only tree-level breaking.

In order that a generator can be broken explicitly without spoiling collective symmetry breaking, it must not be expressible as any kind of the linear combinations above.

Failing to satisfy this condition would lead to non-collective breaking of the set $\{\mathcal{C}_i\}$, which may be allowed provided that the breaking is small enough, such that it does not destabilize the weak scale.

Applying the condition above to Y', we see that the generator Y' cannot be broken without spoiling collective symmetry since it can be expressed as:

$$5Y' = -Y + 2\sqrt{3}T^8_{\text{SU}(3)_1} = Y + 2\sqrt{3}T^8_{\text{SU}(3)_2}. \tag{4.12}$$

Thus any term which breaks $\text{U}(1)'$ and is allowed by gauge invariance must break both $\text{SU}(3)_1$ and $\text{SU}(3)_2$. A spurion which qualifies is $s = (0,0,1,0,0)^T$, transforming (formally) in the fundamental of SU(5). Its symmetry breaking pattern is SU(5) $\rightarrow$SU(4) which acts on the $(3,3)$ minor. The 9 broken generators include Y' and generators which are also broken by the gauging. In particular, any function of $\Sigma_{33} = s^\dagger \Sigma s$ would break Y' while maintaining gauge invariance. For example, consider

$$\delta\mathcal{L} = \varepsilon f^4 \Sigma_{33} + \text{c.c.}, \tag{4.13}$$

where ε is dimensionless. Expanding Σ, we obtain

$$\delta\mathcal{L} = 4\varepsilon f^2 \left[\frac{4}{5}\eta^2 + H^\dagger H + \ldots\right], \tag{4.14}$$

where we took ε to be real, such that no extra explicit CP violation is implied. As expected, m_H gets a tree-level contribution, since $\delta\mathcal{L}$ breaks explicitly both $\text{SU}(3)_1$ and $\text{SU}(3)_2$. In order not to destabilize the electroweak scale, we require $\varepsilon \sim (1/4\pi)^2$.

It follows that mass of η can be as large as the Higgs mass, but it seems equally reasonable (or equally unreasonable) to have a much lighter η.

A light η which couples directly to the Higgs [via both a renormalizable term $\sim \varepsilon\eta^2 H^\dagger H$ and derivative couplings such as $\sim \frac{1}{f^2}(\eta\partial^\mu\eta)(H^\dagger\partial_\mu H)$] would open a new decay channel $h \rightarrow \eta\eta$ for the SM Higgs (see fig. 4.1). Due to its sizable couplings, η would decay promptly at the collider, but depending on its dominant decay modes, it could lead to unusual signatures. For example, η can decay into a pair of light particles: e^+e^-, or $b\bar{b}$, as studied in [38]. In this case, the Higgs can decay into two

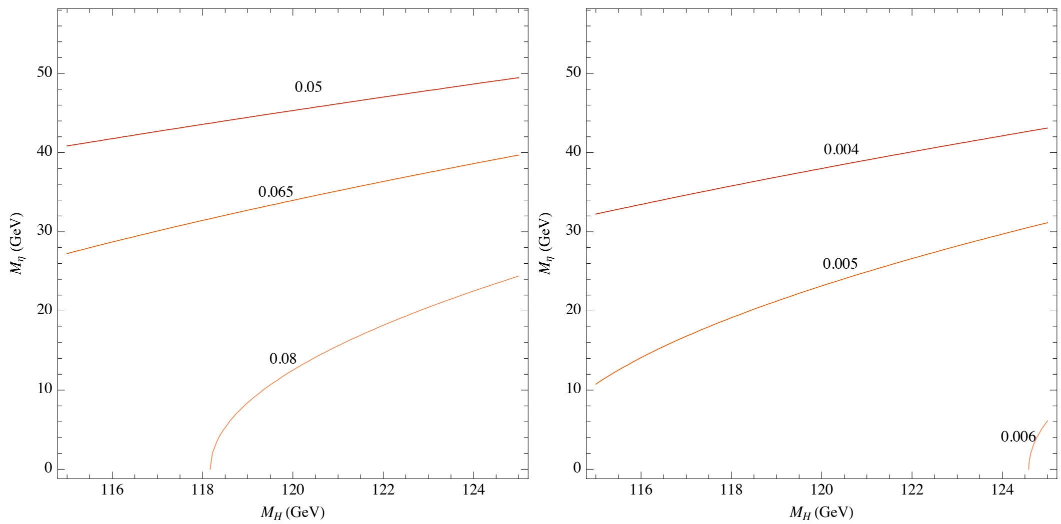


Figure 4.1: Contours of the branching ratio $BR(h \to \eta\eta)$ in the $m_\eta - m_H$ plane, for $f = 1.2$ TeV (left) and $f = 2.4$ TeV (right). While the dominant mode in this mass range is still $b\bar{b}$, the new mode $\eta\eta$ becomes significant throughout the parameter space.

pairs of boosted objects, such as $H \to \eta\eta \to (jj) + (\ell^+\ell^-)$, where the objects in the parentheses are collimated due to the large boost factor $\gamma \sim m_H/2m_\eta$. Another possibility is that the singlet η decays mainly via off-shell top quarks. Then, the Higgs will decay into two pairs of boosted tops. The viability of such unusual Higgs phenomenology deserves careful study, which we leave for future work.

The explicit breaking could as well be introduced in the Yukawa term, in which case the bound on ε comes from one-loop. For example, consider the following Yukawa term:

$$\mathcal{L}_Y = f\bar{\chi}_{Li}\Omega^i\left(\lambda + \varepsilon\Sigma_{33}\right)t_R + \text{c.c.}, \tag{4.15}$$

inducing a one-loop contribution of the form

$$\kappa\varepsilon^2 f^2 \left(\frac{\Lambda}{4\pi}\right)^2 \Omega^\dagger\Omega\left(\lambda^2 + 2\lambda\varepsilon\mathcal{R}e\Sigma_{33} + \varepsilon^2\left|\Sigma_{33}\right|^2\right), \tag{4.16}$$

where κ is an order one number depending on physics near the UV cutoff, and ε is real valued. This gives rise to

$$m_\eta \sim m_H \sim \varepsilon\lambda f. \tag{4.17}$$

Therefore ε may be as large as $\sim 1/4\pi$.

In the next section we discuss how such explicit breaking of U(1)′, when properly introduced, may give rise to spontaneous CP violation, by stabilizing the phase δ in Eq.(4.3), such that the VEV assumes the form

$$\Sigma_0 = e^{i\varphi} \begin{pmatrix} & & e^{i\delta}\mathbf{1} \\ & e^{-4i\delta} & \\ e^{i\delta}\mathbf{1} & & \end{pmatrix} . \tag{4.18}$$

4.3 Spontaneous CP Violation in the Hypercharge Model

4.3.1 Spontaneous CP Violation from Breaking U(1)′

We saw that the Hypercharge model has an explicitly broken U(1)′. One might expect that once this U(1)′ is broken (for example, by Σ_{33} insertions, like in the previous section), the related pseudo-Goldstone η acquires a VEV that breaks CP spontaneously. It turns out that there must be at least two terms which break U(1)′ explicitly, in order for spontaneous CP violation to occur [28]. It can be shown generically that as long as a U(1) is broken by one single term involving a single field, the U(1)-related phase in the VEV can be removed, by using a particular U(1) transformation. Because of the explicit breaking, the coupling flips its sign, but the phase is removed.[2] Once two explicit breaking terms are introduced, the phase gets generically stabilized at a non-zero value. Of course, more terms would be induced by loops, but the resulting phase would be also loop suppressed.

It is interesting to notice that the CP-odd phase may be $\mathcal{O}(1)$, even in the limit $\varepsilon \to 0$, where the explicit breaking vanishes. The reason for this non-analytical behavior of the phase as function of ε is that however small ε may be, it is the leading effect in lifting the degeneracy associated with the Goldstone direction. Nevertheless, any physical consequence of effect related to the phase is associated with some momentum scale p (for example, the mass of a particle whose decay exhibits direct CP violation). The effect will be negligible for $p > \varepsilon f \sim m_\eta$, since for such large characteristic momenta, the pseudo-Goldstone boson is effectively massless. [3]

[2]We thank H. Haber for pointing out such a possibility. See more examples in [28].

[3]We thank Richard Hill for raising this puzzle, and Ben Grinstein for his physical interpreation.

In the hypercharge model, there is one exact global $U(1)'$ generated by $Y' = \text{diag}(1, 1, -4, 1, 1)/10$, which is spontaneously broken. A single term is sufficient to lift the Goldstone boson mass, as we have discussed in the previous section. However, only in the presence of at least two different terms, a physical CP-odd phase would arise. A simple choice would then be the following:

$$\delta\mathcal{L}_{\text{SCPV}} = \varepsilon f^4 \left(a\Sigma_{33} + b\Sigma_{33}^2 \right) + \text{c.c.}, \tag{4.19}$$

where we take ε, a, b to be real, and a, b are $\mathcal{O}(1)$ whereas ε must be loop suppressed, as we have discussed in the previous section. This results in the following tree-level potential for η:

$$V_\eta = 2\varepsilon f^4 \left(a \cos \frac{2\eta}{\sqrt{5}f} + b \cos \frac{4\eta}{\sqrt{5}f} \right) + \dots \tag{4.20}$$

This potential is minimized for

$$\langle \eta \rangle = \frac{\sqrt{5}f}{2} \arccos \left(\frac{-a}{4b} \right) \qquad \text{if} \quad \left| \frac{a}{4b} \right| < 1, \tag{4.21}$$

which is of order one if we assume no hierarchy between a and b.

Two final comments are in order before we show how the above CP-odd phase shows up among SM fields. First, we note that η is odd under T-parity, and therefore a non-zero δ in the T-parity version of the model would have to be further suppressed, as it implies spontaneous breaking of T-parity. The second comment is about the possibility of CP violation from an overall phase, $\Sigma_0 \to e^{i\alpha}\Sigma_0$. This phase is related to an overall U(1) which commutes with SU(5). Therefore, none of the SU(5) Goldstone bosons transform and we conclude that the overall U(1) is not relevant as a symmetry transformation. This also means that there is no dynamical field whose VEV is related to that phase. CP violation from such phases is usually considered explicit, not spontaneous. Once we include the related Goldstone boson, $\eta' \equiv \text{Tr}\Pi$, the overall phase becomes related to spontaneous CP violation. This amounts to adding a U(1) factor, along with its Goldstone boson to the chiral lagrangian. We will not consider this issue further, since we find it unrelated to the rest of the discussion.

4.3.2 CP Violation in Non-renormalizable Couplings

Having found possible modifications of the Littlest Higgs which allow for spontaneous CP violation, it is worth asking what would be the effect of CP violation beyond the SM on the SM sector. Following [41], we will focus on CP violation in dimension-six couplings of the SM Higgs to quarks, ignoring other manifestations of CP violation on the SM sector. Assuming that the low energy effective theory is that of the SM, new CP violation involving the SM Higgs and quarks would arise predominantly in the dimension-six operators [41]

$$\begin{aligned}\Delta\mathcal{L} &= \frac{Z^u_{ij}}{f^2}\bar{Q}_i\tilde{H}u_jH^\dagger H + \frac{Z^d_{ij}}{f^2}\bar{Q}_iHd_jH^\dagger H + \frac{Z^\ell_{ij}}{f^2}\bar{L}_iH\ell_jH^\dagger H, \\ &+ \frac{Z^H}{f^2}(D_\mu H)^\dagger D^\mu(H\,H^\dagger H) + \text{c.c.},\end{aligned} \tag{4.22}$$

where f is the new physics scale, *i.e.,* the spontaneous symmetry breaking scale of the Little Higgs non-linear sigma model.[4]

The lagrangian (4.22) arises from Eq.(4.8) once we expand Σ in terms of the SM Higgs field. In this framework, new CP violation (*i.e.,* CP violation beyond the CKM phase) appears as relative phases between the new couplings and the SM Yukawas. In the Littlest Higgs, the expansion of Σ alone in the Yukawa term does not give rise to relative phases between the coefficients of H and $HH^\dagger H$. Therefore the only way a phase will show up, would be from two different Yukawa terms differing in both the expansion coefficients and an overall phase. However, as we show in appendix B, the Littlest Higgs model does not allow for different expansion coefficients, using any Yukawa term which preserves $SU(3)_1$. It follows that having a different expansion requires a Yukawa term which does not respect collective symmetry breaking. A qualifying Yukawa term is the standard one with a Σ_{33} insertion, just like the one discussed in the previous section. Such Yukawa term would have to be suppressed in order to keep the SM Higgs light.

We conclude that although we have shown how to get an $\mathcal{O}(1)$ physical phase δ in the Hypercharge model, the phase appearing in Higgs - SM fermions interactions is suppressed, of order $\varepsilon\delta$. This is a consequence of the constrained nature of the SU(5) structure, and is by no means generic. In order to confirm that, in the next section we present an SU(6)/SO(6) version of the Littlest Higgs.

[4] Note that the last term in Eq.(4.22) can always be shifted away by a non-linear field redefinition $H \to H\left(1 - \frac{Z^H}{f^2}H^\dagger H\right)$. To leading order, such field redefinition mimics replacing Z^f with $Z^f - Z^H$. Since the authors in [41] assume $Z^H = 0$, one has to replace Z^f by $Z^f - Z^H$ in their results in order to use them correctly.

4.4 An SU(6)/SO(6) variant

We have found that in SU(5)/SO(5) Little Higgs models, spontaneous CP violation requires spoiling collective symmetry breaking, However, this seems to be due to the minimal nature of the SU(5)/SO(5) Littlest Higgs. Once we consider a larger group, it becomes easier to find more global generators satisfying the two conditions. We illustrate this with an SU(6)/SO(6) version of the Littlest Higgs. Note that we do not attempt to give a full description of this model and its phenomenology. Rather, we give a preliminary analysis aimed at the basic features, namely, a successful mechanism for suppressing the electroweak scale, lifting all the Goldstone bosons, and a possible $\mathcal{O}(1)$ spontaneous CP violation.

We gauge an $[\mathrm{SU}(2)_1 \times \mathrm{U}(1)]^2$ subgroup of SU(6), generated by

$$\begin{aligned} T_1^i &= \begin{pmatrix} \sigma^i/2 & & \\ & \mathbf{0}_{2\times2} & \\ & & \mathbf{0}_{2\times2} \end{pmatrix}, \quad Y_1 = \mathrm{diag}(2,2,-1,-1,-1,-1)/6; \\ T_2^i &= \begin{pmatrix} \mathbf{0}_{2\times2} & & \\ & \mathbf{0}_{2\times2} & \\ & & -\sigma^{a*}/2 \end{pmatrix}, \quad Y_2 = \mathrm{diag}(1,1,1,1,-2,-2)/6. \end{aligned} \tag{4.23}$$

This gauging leaves an exact global $\mathrm{SU}(2)_M$ symmetry which acts on the (3,4) block. A vacuum which minimizes the effective potential generated by gauge interactions takes the form

$$\Sigma_0 = \begin{pmatrix} 0 & 0 & \mathbf{1} \\ 0 & V & 0 \\ \mathbf{1} & 0 & 0 \end{pmatrix}, \tag{4.24}$$

where V may be parametrized as

$$V = \begin{pmatrix} e^{i\alpha}\cos\theta & i\sin\theta \\ i\sin\theta & e^{-i\alpha}\cos\theta \end{pmatrix} = V_{1/2}V_{1/2}^T, \qquad V_{1/2} = e^{i\alpha\sigma^3/2}e^{i\theta\sigma^1/2}. \tag{4.25}$$

This VEV breaks spontaneously the exact global $\mathrm{SU}(2)_M$ to $\mathrm{SO}(2)_M$. The pseudo-Goldstone bosons are parametrized using

$$\Sigma = e^{2i\Pi/f}\Sigma_0, \tag{4.26}$$

where

$$\Pi = \begin{pmatrix} & \mathcal{H} & \phi \\ \mathcal{H}^\dagger & V_{1/2}\mathcal{E}V_{1/2}^\dagger & V\mathcal{H}^T \\ \phi^\dagger & \mathcal{H}^* V^\dagger & \end{pmatrix}, \quad \mathcal{H} = \frac{1}{\sqrt{2}}\,(H|K)\,, \quad \mathcal{E} = \begin{pmatrix} \sigma & \rho \\ \rho & -\sigma \end{pmatrix}. \tag{4.27}$$

Note that both H and K carry the quantum numbers of the SM Higgs, whereas σ and ρ are SM singlets, and ϕ is a complex triplet. The gauge interactions break collectively SU(4)$_1$ and SU(4)$_2$, which protect both doublets H and K from quadratically divergent mass parameters. They also leave SU(2)$_M$ unbroken, such that the SM singlets ρ and σ remain massless at this stage.

Note that a quartic coupling $(H^\dagger H)^2$ is not forbidden by collective symmetry breaking, since the field ϕ transforms in such a way that Tr $\left|\phi + i/(2f)\,HH^T + \ldots\right|^2$ remains invaraint. We compute the quadratic divergent part of the CW potential to verify this in Appendix D.

In the fermion sector, we introduce the following lagrangian:

$$\begin{aligned} -\mathcal{L}_f &= f\bar{\chi}_{Li}\left(\lambda_1\Omega_1^i + \lambda_2\Omega_2^i\right)t_R + \lambda' f \bar{t}'_L t'_R, \\ \Omega_1^i &= \bar{\Sigma}^{i4}\Sigma_{44}, \qquad \Omega_2^i = \epsilon_{jk\ell}\epsilon_{xy}\bar{\Sigma}^{ij}\bar{\Sigma}^{kx}\bar{\Sigma}^{\ell y}, \end{aligned} \tag{4.28}$$

where i, j, k, ℓ run through 1,2,3 and x, y from 5 to 6, and χ^i includes both the left handed quark doublet and t'_L, as usual - see section 4.2. The Yukawa terms and the mass term break SU(3)$_1$ and SU(4)$_2$ collectively, such that one doublet, H, remains light. The other doublet, K, becomes heavy since it is only protected by SU(4)$_{1,2}$ which are broken non-collectively by the Yukawa terms. The Yukawa terms also break SU(2)$_M$ which protects the SM singlet ρ, thus lifting its mass to $\mathcal{O}(f)$. Since there are two different spurions which break this symmetry, we expect spontaneous CP violation from $\theta \sim \mathcal{O}(1)$.

Note, however, that Eq.(4.28) cannot break the SU(2)$_M$ generator diag$(0, 0, 1, -1, 0, 0)$, since this generator violates the condition from the previous section: it is an SU(4)$_2$ generator which is also in the span of $\left\{T^8_{\mathrm{SU}(3)_1}, Y, Y'\right\}$. This is also manifest in the one-loop effective potential, whose quadratically divergent term is given by

$$V_{\mathrm{CW}}(\Sigma) = \frac{\kappa\Lambda^2}{16\pi^2}\mathrm{Tr}\,MM^\dagger, \tag{4.29}$$

where

$$M = M(\Sigma) = f\left(\lambda_1\Omega_1 + \lambda_2\Omega_2\right). \tag{4.30}$$

Here, the precise value of κ depends on unknown physics near the cutoff, and we have assumed real values for $\lambda_{1,2}$ in order to study the case of purely spontaneous CP violation. Using the explicit form of Σ_0 in Eq.(4.24), this yields

$$V_{\mathrm{CW}} = \kappa f^4 \cos^2\theta \left(\lambda_1^2 + 4\lambda_2^2 - \lambda_1^2\cos^2\theta\right). \tag{4.31}$$

We can distinguish between two cases:

1. For $\kappa > 0$, the minimum lies at $\theta = \pm\pi/2$. In that case there is no CP violation, since the phase can be removed by the field redefinition

   $$\Sigma \to \exp\left(\mp T^1_{\mathrm{SU}(2)_M}\pi/2\right), \tag{4.32}$$

   where $T^1_{\mathrm{SU}(2)_M} \equiv \mathrm{diag}\,(0, \sigma^1/2, 0)$. The potential becomes

   $$\begin{aligned} V_{\mathrm{CW}} \to \quad & \mp\kappa f^4 \sin^2\theta \left[\lambda_1^2 + 4\lambda_2^2 \pm \lambda_1^2 \sin^2\theta\right] \\ & = \mp\kappa f^4 \left(1-\cos^2\theta\right)\left[\lambda_1^2 + 4\lambda_2^2 \pm \lambda_1^2\left(1-\cos^2\theta\right)\right]. \end{aligned} \tag{4.33}$$

   Since this field redefinition is not a symmetry, the potential have changed, but using the same variable ($\cos^2\theta$), its coefficients remain real valued, while the minimum is now at $\theta = 0$, hence the field redefinition has removed the phase successfully from the lagrangian and it cannot be physical.

2. For $\kappa < 0$, the potential is minimized at

   $$\begin{aligned} \cos\theta &= \pm\sqrt{\frac{\lambda_1^2 + 4\lambda_2^2}{2\lambda_1^2}} \qquad && \text{if } |\lambda_1| \geq 2\,|\lambda_2|\,, \\ \theta &= 0, \pi && \text{if } |\lambda_1| \leq 2\,|\lambda_2|\,. \end{aligned} \tag{4.34}$$

   In the former case, there is a physical CP-odd phase in the vacuum, while in the latter, the phase can be removed by a field redefinition.

We conclude that if the UV completion is such that $\kappa < 0$, the loop effects are sufficient to generate a generically large CP-odd phase. In any case, a phase may be generated also at tree level by introducing a term of the same form as Eq.(4.29) with a negative coefficient.

As expected, the potential does not stabilize α and this will persist for all the terms in the effective potential, due to the unbroken U(1) symmetry generated by

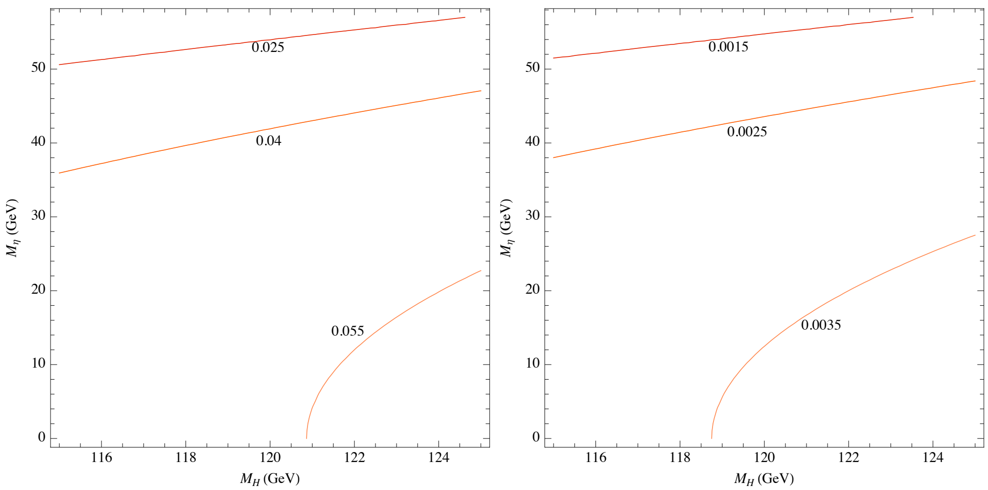


Figure 4.2: Contours of the branching ratio $BR(h \to \sigma\sigma)$ in the $m_\sigma - m_H$ plane, for $f = 1.2$ TeV (left) and $f = 2.4$ TeV (right). Again, the dominant mode in this mass range is still $b\bar{b}$, but the new mode $\sigma\sigma$ becomes significant throughout the parameter space.

diag$(0, 0, 1, -1, 0, 0)$. Stabilizing α can be done easily, by introducing a small non-collective breaking term, such as

$$\mathcal{L}_X = \varepsilon f^4 \bar{\Sigma}^{33} \Sigma_{44} + \text{c.c.}. \tag{4.35}$$

Similar to the hypercharge model, we will have to take $\varepsilon \lesssim 1/(4\pi)^2$, which fixes the mass of σ to be around or below the Higgs mass. The light singlet σ could alter the Higgs phenomenology, by playing a role which is similar to the role of the singlet η in the hypercharge model, although the branching ratio for $h \to \sigma\sigma$ at low Higgs mass is slightly lower than the corresponding branching ratio in the hypercharge model (see Fig. 4.2).

Unlike the hypercharge model, here an $\mathcal{O}(1)$ phase would show up in the dimension-six couplings. Expanding Eq.(4.28) in terms of H yields

$$\sqrt{2} e^{-i\alpha} \cos\theta (2i - \sin\theta) \left[1 + \left(\frac{1}{3} \frac{2i + \sin\theta}{2i - \sin\theta} \right) H^\dagger H \right] \overline{Q}_{3L} \tilde{H} t_R. \tag{4.36}$$

Note that only θ is manifested in the SM sector as a relative phase in the Z couplings, and moreover, the resulting phase in the low energy lagrangian is not suppressed by the small parameter ε. We conclude that the SU(6)/SO(6) model admits spontaneous CP violation from phases in the VEV. Unlike in the SU(5)/SO(5) hypercharge model, the resulting phase between $\tilde{H}$ and $\tilde{H} H^\dagger H$ can be $\mathcal{O}(1)$.

4.5 Conclusions and Further Implications

In this work we have discussed the tension between lifting Goldstone bosons and collective symmetry breaking. We showed that such tension is present in the SU(2)×SU(2)×U(1) model. This model has a Goldstone mode which acquires mass only via terms that spoil collective symmetry breaking. Such terms must be suppressed in order to keep the SM Higgs boson light, implying that the related pseudo-Goldstone bosons must be light too. This may lead to interesting collider phenomenology, such as non-standard Higgs decays.

Once collective symmetry breaking is spoiled, even by a small parameter, CP invariance may be broken spontaneously, inducing a large CP-odd phase. However, due to the minimal nature of the SU(5) structure, the phase which appears in interactions among SM fields is suppressed. We have shown that this difficulty is lifted in an SU(6)/SO(6) model, where an $\mathcal{O}(1)$ phase may give rise to observable effects in non-renormalizabe couplings of the SM Higgs to quarks.

The $\mathcal{O}(1)$ spontaneous CP violating phase in Eq.(4.36) would contribute to electric dipole moments (EDMs) and will be detectable in the next-generation of electric dipole moments experiments [41].

At this level of discussion, we did not suggest experimental ways to distinguish between spontaneous and explicit CP violation. In the case of continuous symmetry breaking, there are Goldstone bosons associated with the continuous set of vacua. Since CP is a Z_2, its spontaneous breaking implies the existence of two equivalent vacua. [5] Indeed, it is evident from Eq.(4.34) that there are two values for θ which lead to the same phase in Eq.(4.36). The doubling of vacua might imply the existence of domain walls in the universe, once the temperature had dropped below the breaking scale f. This poses a problem for the cosmology of the model, which can be avoided if the reheating temperature after inflation is lower than f, or if there is additional explicit CP violation - which would tilt the potential, making one of the vacua the true vacuum.

A related issue is whether the phase from spontaneous CP violation can contribute to successful electroweak baryogenesis. This depends on other features of the model,

[5]The two vacua cease to be equivalent once explicit CP violation is added. In our case, this is introduced by the usual Kobayashi-Maskawa phase.

such as the scale f and the sign and size of higher dimension terms in the effective potential. Note that unlike Nelson-Barr models which are renormalizable, in Little Higgs models the proximity of the UV completion does not permit predictive statements regarding this issue. The investigation of the above issues, as well as a detailed analysis of collider phenomenology, is left for future work.

This chapter is a reprint of material as it appears in "Spontaneous CP Violation and Light Particles in The Littlest Higgs," Z. Surujon and P. Uttayarat, Phys. Rev. D **83**, 076010 (2011) [arXiv:1003.4779 [hep-ph]], of which I was a co-author.

Chapter 5

A Very Light Dilaton

We present a completely perturbative model that displays behavior similar to that of walking technicolor. In one phase of the model RG-trajectories run towards an IR-fixed point but approximate scale invariance is spontaneously broken before reaching the fixed point. The trajectories then run away from it and a light dilaton appears in the spectrum. The mass of the dilaton is controlled by the "distance" of the theory to the critical surface, and can be adjusted to be arbitrarily small without turning off the interactions. There is a second phase with no spontaneous symmetry breaking and hence no dilaton, and in which RG trajectories do terminate at the IR-fixed point.

5.1 Introduction

The Nambu-Goldstone boson of spontaneously broken scale invariance is known as a dilaton. The name is also used to describe the pseudo Nambu-Goldstone boson, a massive state that appears when scale invariance is slightly broken. Classically this notion makes good sense. For example, take a scale invariant field theory, one with only dimensionless couplings,[1] with a flat direction for the minima of the potential for scalar fields. A dilaton follows from expanding about a non-zero field value. Adding arbitrarily small terms with dimensional couplings will generally give the dilaton a small mass. However, ordinarily the passage to the quantum case can destroy this picture. Quantum effects break scale invariance even in the absence of explicit mass terms. The state that before quantization would have been identified as a dilaton

[1] In this work we consider only field theories in four space-time dimensions.

acquires a mass that is not small. In fact, it is not clear one can uniquely identify a state with what would have been the dilaton. What is meant by a "small" mass is that it can be made arbitrarily small while keeping all the remaining spectrum roughly constant and interacting. However it is not easy to construct models displaying this behavior, that is, models of a very light dilaton.

In their celebrated analysis of the massless abelian $U(1)$ model Coleman and Weinberg find a scalar of mass m and a vector of mass M in the spectrum, with $m^2/M^2 = 3e^2/8\pi^2$ [42]. Since the model is classically scale invariant one is tempted to identify the only scalar with the pseudo Nambu-Goldstone boson of broken scale invariance. It is not clear that this identification makes sense. But even if we insist on it we see that the dilaton can only be made light by turning off the interactions, $e^2 \to 0$. Moreover, if we insist in keeping the scale of symmetry breaking fixed then in this limit the vector meson mass also approaches zero, albeit at a slower rate.

One may guess that a good search strategy for a light dilaton model is to take as a starting point an exactly conformal model. Then look to spontaneously break scale invariance and finally add small explicit scale symmetry breaking terms. But this strategy has proven ineffective. Consider, for example, $N = 4$ supersymmetric Yang-Mills theory, an exactly conformal interacting theory. The scalar potential has minimum energy flat directions and one can choose to expand about a non-trivial vacuum. Scale invariance is spontaneously broken and a massless dilaton must emerge. However, supersymmetry is not broken and a lot more massless stuff emerges too. As the vacuum breaks the Yang-Mills symmetry group G to one of its maximal subgroups H a full $N = 4$ H-gauge theory remains in the massless spectrum. The potential again has many zero energy flat directions and we are free to identify these with "dilatons." Of course, we could just as well have identified with dilatons the flat directions of the original theory, based on G. Moreover, adding perturbations will render the dilaton very heavy, calling into question the identification of any one state with the dilaton. A perturbation, either relevant or marginal, vitiates the cancellations that give vanishing beta functions and the theory runs to strong coupling in the infrared.

In this work we construct a model of a light dilaton. The strategy, construction of the model and the results of our analysis are easily summarized. We look for a light dilaton in an interacting field theory that displays a perturbative attractive infrared fixed point and contains scalars. The idea is to look for spontaneous symmetry breaking along a renormalization group trajectory headed towards the fixed point. For a specific model we take that of Banks and Zaks [43, 44] supplemented with scalars that are neutral under the gauge group. The scalars have quartic self-interactions and are Yukawa-coupled to the Banks-Zaks spinors. As the Yang-Mills gauge coupling runs toward the Banks-Zaks IR-fixed point, it drives the scalar and Yukawa couplings towards the non-trivial fixed point values too. Depending on the relative values of the coupling constants the Coleman-Weinberg effective potential for the scalar fields may develop a non-trivial minimum [42]. The parameter space of the theory is split according to whether scaling symmetry is spontaneously broken or not, and for couplings near the boundary between these regions the dilaton is very light in units of its decay constant. Yet the theory is fully interacting and the spectrum is non-trivial (and insensitive to the parameter adjustment required to make the dilaton arbitrarily light).

Our search for a model of a very light dilaton was partially motivated by recent work of Appelquist and Bai [45] (henceforth 'AB') and by Hashimoto and Yamawaki [46] rekindling and old debate on whether walking technicolor (WTC) may have a light dilaton in its spectrum [47, 48, 49, 50, 51, 52, 53, 54]. The idea of "walking" promises to solve many difficulties of technicolor (TC) theories. The conjectural behavior of the theory requires that (1) the TC coupling constant g evolves very slowly, (2) this occurs while at large value of the TC coupling constant, so that anomalous dimensions are large, and (3) the slowly running coupling eventually crosses a threshold, exceeding a critical value g_c for chiral symmetry breaking. The picture is that once the coupling crosses this threshold, techniquarks become massive, decouple and leave the technigluons to drive alone the running of the coupling constant (which from that point on grows quickly, much like in QCD). The condensate that results breaks electroweak symmetry giving masses to W and Z gauge bosons. The large anomalous dimension of the techniquark bi-linear insures that four-fermion operators induced by extended-TC interactions (ETC) give acceptable masses to all but the top quarks (and leptons) while effectively suppressing ETC mediated FCNCs. Moreover, the large anomalous dimensions of 4-techniquark operators also induced by the ETC tend to increase the masses of troublesome pseudo-Goldstone to acceptable levels. In this picture the slow evolution of the coupling constant can be viewed as an approach towards a would-be conformal fixed point, g_*. It is a "would-be" fixed point only because $g_c < g_*$, which triggers the fast QCD-like evolution of g once it exceeds the critical value g_c. AB argue, while Hashimoto and Yamawaki rebut, that a dilaton does appear and estimate that its mass is roughly determined by the value of the beta function at its closest approach to the would-be fixed point, $\beta(g_c)$.

The existence of a light dilaton in WTC is by no means obvious. The dilaton is in some respects similar to the η' in QCD. Were we to ignore the $U(1)_A$ anomaly it would be a pseudo Nambu-Goldstone boson, on par with the (π, K, η) octet. But the anomaly breaks the symmetry explicitly and because it involves the strong interactions this breaking is not a small perturbation. Beyond deciding whether the light dilaton appears in the spectrum of WTC, there are many other questions that arise. For example, what precisely is the meaning of the critical coupling g_c, what is the dilaton decay constant, etc.

Unfortunately, as of this writing there is no explicit realization of the WTC idea as a specific model. Numerous numerical studies are ongoing to determine whether QCD-like theories at the edge of the conformal window display the phenomenon of walking [55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77]. While a positive result from these studies may confirm the existence of models exhibiting the WTC idea, a negative result would not rule out the possibility that some non-QCD like theory behaves this way. In the mean time it would be useful to construct a toy model displaying some of the WTC behavior. One would like the toy model to be fully perturbative so that one may readily compute and resolve questions. In some ways our model fits the bill. It does have coupling constants that grow as they approach a fixed point, then walk for quite a long RG-time and finally swerve away. This change of behavior is triggered, much like in the WTC idea, by the analogue of chiral symmetry breaking, that is, the scalar fields acquiring a non-trivial expectation value, giving masses to the spinors through their Yukawa couplings. To be sure, the model fails to mimic WTC in important ways. By design it remains perturbative, and therefore anomalous dimensions remain small. And, as opposed to a would be WTC theory, our model is not asymptotically free; while the Banks-Zaks sector is, RG-running in the scalar sector encounters Landau poles. We do not see the latter of these difficulties as central. One can view this as a theory with a cut-off at a scale that is exponentially large compared to where the physics of the symmetry breaking takes place, or imagine that it is the low energy effective theory of a more complete model.

But the usefulness of an explicit model of a very light dilaton goes beyond that of being a toy WTC. Sundrum has remarked that the dilaton can serve as a scalar analog of the graviton. By studying the properties of the dilaton one can hope to gain insights into the theory of gravity and perhaps find the answer to the cosmological constant puzzle [78]. A dilaton is also likely to appear in the AdS/CFT dual of the Randal-Sundrum model [79] with the Goldberger-Wise mechanism stabilizing the extra-dimension [80]. In the 4-dimensional language, the theory is described not by a CFT but by a flow to a CFT fixed point which is however interrupted close to the fixed point by the expectation value of a field that measures the distance from the origin in moduli space [81, 82]. This is described effectively by a theory at the fixed point, a CFT Lagrangian, supplemented by small perturbations. The latter are made scale invariant by including couplings to the dilaton in the spirit of phenomenological Lagrangians [29, 30]. If the SM is embedded in such a scheme the dilaton may behave much like, but not exactly the same as, the higgs boson of the minimal standard model [83, 84, 85]. An amusing question that one can now ponder is the inverse AdS/CFT problem: given our perturbative model, what is the AdS dual (presumably a strongly interacting non-factorizable gravity model in 5 dimensions)?

Another area where the dilaton may play a role is in astrophysics and cosmology. By noting that the dilaton couples to the trace of the stress energy tensor, the authors in Ref. [86] propose to use a light dilaton as a force mediator between the SM particles and dark matter particles. Some authors also propose a light dilaton as a new dark matter candidate [87]. In all these cases an explicit computable model may be put to use in understanding issues currently clouded by our inability to compute at or near strongly interacting fixed points.

The paper is organized as follows. In Sec. 5.2 we introduce our model and show the existent of both the IR-fixed point and the non-trivial vacuum. In Sec. 5.3 we identify the state corresponding to the dilaton and we compute its mass. In Sec. 5.4

we discuss a phase structure of our model accessible in perturbation theory. We discuss our results briefly in Sec. 5.5.

5.2 The Model

We study a class of SU(N) gauge theories with $n_f = n_\chi + n_\psi = 2n_\chi$ flavors of spinors, ψ_i and χ_k, and two real scalars. The spinors are taken to be vector-like in the fundamental representation of the gauge group while the scalars are singlets. The most general Lagrangian that is classically scale invariant and also invariant under the discrete symmetry $\phi_1 \to \phi_1$, $\phi_2 \to -\phi_2$, $\psi \to \psi$, $\chi \to -\chi$, and the global simultaneous $SU(n_\chi)$ transformations $\psi \to U\psi$, $\chi \to U\chi$ is

$$\begin{aligned}\mathcal{L} = &-\frac{1}{2}\mathrm{Tr}\, F^{\mu\nu}F_{\mu\nu} + \sum_{j=1}^{n_\chi} \bar{\psi}^j i\not{D}\psi_j + \sum_{k=1}^{n_\chi} \bar{\chi}^k i\not{D}\chi_k + \frac{1}{2}(\partial_\mu\phi_1)^2 + \frac{1}{2}(\partial_\mu\phi_2)^2 \\ &- y_1\left(\bar{\psi}\psi + \bar{\chi}\chi\right)\phi_1 - y_2(\bar{\psi}\chi + \mathrm{h.c.})\phi_2 - \frac{1}{24}\lambda_1\phi_1^4 - \frac{1}{24}\lambda_2\phi_2^4 - \frac{1}{4}\lambda_3\phi_1^2\phi_2^2\,.\end{aligned} \tag{5.1}$$

Quantum effects will induce scalar masses of the order of the cut-off. We will at first fine-tune the masses to zero, much like Coleman and Weinberg [42]; after all, we are not interested in solving the hierarchy problem. Alternatively one can study this theory perturbatively in the continuum, using dimensional regularization. Later, in Sec 5.4.1 we consider the effect of adding a scalar mass.

For small number of families this model is very similar to QCD. The gauge sector will run to strong coupling in the infrared, the remaining parameters will only act as small perturbations. The chiral symmetry $SU(n_f) \times SU(n_f)$ is spontaneously broken to its diagonal subgroup with associated Nambu-Goldstone bosons in the spectrum.

We are interested in larger values of n_f for which the gauge coupling is still asymptotically free but behaves very differently in the infrared, as we now discuss.

5.2.1 Fixed Point Structure

We arrange the values of N and n_f so that the coefficient in the one-loop term of the gauge beta function is small, much as Banks and Zaks do for QCD [43]. The perturbative fixed point value in the gauge coupling appears from balancing the one and two loop terms against each other. To arrange for an arbitrarily small fixed point value we consider only large values of N and n_f. The coefficients of the one-loop terms of the beta functions for the remaining couplings are not small. Hence it suffices to retain only up to one loop order in the beta functions of Yukawa and scalar couplings, while, of course, retaining up to two loop order for that of the gauge coupling. The mass independent (*e.g.,* minimal subtraction) β-functions at large N and n_f are given by [88, 19, 89, 90]

$$
\begin{aligned}
(16\pi^2)\frac{\partial g}{\partial t} &= -\frac{\delta N}{3}g^3 + \frac{25N^2}{2}\frac{g^5}{16\pi^2} \\
(16\pi^2)\frac{\partial y_1}{\partial t} &= 4y_1y_2^2 + 11N^2y_1^3 - 3Ng^2y_1, \\
(16\pi^2)\frac{\partial y_2}{\partial t} &= 3y_1^2y_2 + 11N^2y_2^3 - 3Ng^2y_2 \\
(16\pi^2)\frac{\partial \lambda_1}{\partial t} &= 3\lambda_1^2 + 3\lambda_3^2 + 44N^2\lambda_1y_1^2 - 264N^2y_1^4, \\
(16\pi^2)\frac{\partial \lambda_2}{\partial t} &= 3\lambda_2^2 + 3\lambda_3^2 + 44N^2\lambda_2y_2^2 - 264N^2y_2^4, \\
(16\pi^2)\frac{\partial \lambda_3}{\partial t} &= \lambda_1\lambda_3 + \lambda_2\lambda_3 + 4\lambda_3^2 + 22N^2\lambda_3y_1^2 + 22N^2\lambda_3y_2^2 - 264N^2y_1^2y_2^2.
\end{aligned}
\tag{5.2}
$$

The number of families is taken to be fixed at $n_f = 11N/2(1 - \delta/11)$ and we drop the $\mathcal{O}(\delta)$ terms except in β_g. Even though N and n_f are integers, one can make δ arbitrarily small by taking N and n_f arbitrarily large.

These equations will play an important role in our discussion. The first step is to determine whether any non-trivial fixed points exist. To see that one does indeed run into the fix point we can argue as follows. First, there is no question that the gauge coupling flows in the IR towards it fixed point. All that is required is that it starts its flow from the UV at a value smaller than the fixed point. Then the Yukawa couplings' beta functions are dominated by the last term, which is negative and only linear in the y_i's. Hence they grow until the positive non-linear terms compensate against the last negative, linear term. And the story is then repeated for the scalars, but now having the Yukawa couplings drive the beta functions (the last terms in each of the three scalar coupling beta functions are negative and λ_i independent).

The mechanism that is driving the couplings towards theIR-fixed point values is mimicked by the process of determining their location. The gauge coupling has the same fixed point as in the Banks-Zaks model. This is used in the equations for the Yukawa couplings $y_{1,2}$ which are then trivially solved to leading order in $1/N$ accuracy. In turn these solutions are used in the equations for the scalar self-couplings.

To leading order in $1/N$ accuracy, the fixed point is at the following zeroes of the beta functions:

$$g_*^2 = 16\pi^2 \frac{2}{75}\frac{\delta}{N}\,, \qquad y_{1*}^2 = y_{2*}^2 = \frac{3}{11}\frac{g_*^2}{N}\,, \qquad \lambda_{1*} = \lambda_{2*} = \lambda_{3*} = 6y_{1*}^2 = \frac{18}{11}\frac{g_*^2}{N}. \quad (5.3)$$

Since δ is arbitrarily small while N is arbitrarily large the fixed point values of the couplings are all perturbative. It is easy to check that the terms omitted in the loop expansion of the beta functions are parametrically smaller.

This result may be surprising. Common lore, which of course cannot be documented, is that theories with scalars and fermions do not exhibit nontrivial IR-fixed points in 4 dimensions. While this is obviously false in $4-\epsilon$ dimensions, we see that it is also false in exactly four dimensions. The lore's intuition is vitiated here because it is the gauge coupling which is driving the remaining couplings toward the fixed point.

5.2.2 Vacuum Structure

We turn now to the physical content of our model. The first order of business is to understand its vacuum structure and determine the fate of the symmetries of the Lagrangian. At the classical level, the potential is trivially minimized, $\langle\phi_1\rangle = \langle\phi_2\rangle = 0$ and all symmetries are explicitly realized. However, this may change once quantum effects are included. The one-loop effective potential in the $\overline{\text{MS}}$ scheme is [42]

$$\begin{aligned} V_{\text{eff}} = &-\frac{1}{24}\lambda_1\phi_1^4 - \frac{1}{24}\lambda_2\phi_2^4 - \frac{1}{4}\lambda_3\phi_1^2\phi_2^2 \\ &- \frac{11N^2M_{f+}^4}{(64\pi^2)}\left(\ln\frac{M_{f+}^2}{2\mu^2} - \frac{3}{2}\right) - \frac{11N^2M_{f-}^4}{(64\pi^2)}\left(\ln\frac{M_{f-}^2}{2\mu^2} - \frac{3}{2}\right) \\ &+ \frac{M_{s+}^4}{(64\pi^2)}\left(\ln\frac{M_{s+}^2}{\mu^2} - \frac{3}{2}\right) + \frac{M_{s-}^4}{(64\pi^2)}\left(\ln\frac{M_{s-}^2}{2\mu^2} - \frac{3}{2}\right), \end{aligned} \quad (5.4)$$

where

$$\begin{aligned} M_{f\pm} &= y_1\phi_1 \pm y_2\phi_2, \\ M_{s\pm}^2 &= \frac{(\lambda_1+\lambda_3)\phi_1^2 + (\lambda_2+\lambda_3)\phi_2^2}{4} \\ &\quad \pm \frac{\sqrt{(\lambda_1-\lambda_3)^2\phi_1^4 + (\lambda_2-\lambda_3)^2\phi_2^4 - 2(\lambda_1\lambda_2 - \lambda_1\lambda_3 - \lambda_2\lambda_3 - 7\lambda_3^2)\phi_1^2\phi_2^2}}{4}. \end{aligned} \quad (5.5)$$

No mass terms have appeared because we have used dimensional regularization (in the $\overline{\text{MS}}$ scheme). As explained earlier, this is in keeping with Coleman and Weinberg who completely subtract the mass terms. We will return to this point in the discussion where we will argue that including small masses for the scalars and spinors of the model does not modify the main conclusions (but we have to wait until then to explain the meaning of "small.")

It is fairly difficult to search for the minimum of this function. We can however find some local minima easily, by searching only for a vacuum that preserves the discrete symmetry $\phi_1 \to \phi_1$, $\phi_2 \to -\phi_2$, $\psi \to \psi$ and $\chi \to -\chi$. The effective potential along the $\phi_2 = 0$ axis is much simplified:

$$\begin{aligned} V_{\text{eff}} = \frac{\lambda_1}{24}\phi_1^4 + \frac{(\lambda_1\phi_1^2)^2}{256\pi^2}\left(\ln\frac{\lambda_1\phi_1^2}{2\mu^2} - \frac{3}{2}\right) + \frac{(\lambda_3\phi_1^2)^2}{256\pi^2}\left(\ln\frac{\lambda_3\phi_1^2}{2\mu^2} - \frac{3}{2}\right) \\ - \frac{22N^2y_1^4\phi_1^4}{64\pi^2}\left(\ln\frac{y_1^2\phi_1^2}{\mu^2} - \frac{3}{2}\right). \end{aligned} \tag{5.6}$$

It is straightforward to find an extremum of this function,

$$\begin{aligned} &\frac{\partial}{\partial\phi_1}V_{\text{eff}}(\langle\phi_1\rangle) = 0 \\ &\quad\Longrightarrow -\frac{\lambda_1}{6} = \frac{\lambda_1^2}{64\pi^2}\left(\ln\frac{\lambda_1\langle\phi_1\rangle^2}{2\mu^2} - 1\right) + \frac{\lambda_3^2}{64\pi^2}\left(\ln\frac{\lambda_3\langle\phi_1\rangle^2}{2\mu^2} - 1\right) \\ &\qquad\qquad - \frac{88N^2y_1^4}{64\pi^2}\left(\ln\frac{y_1^2\langle\phi_1\rangle^2}{\mu^2} - 1\right). \end{aligned} \tag{5.7}$$

If the extremum is a minimum this equation determines the vacuum expectation $\langle\phi_1\rangle$ in terms of the coupling constants of the model. Alternatively one may eliminate one of the dimensionless parameters of the model in favor of the dimensional vacuum expectation value. This is the well known dimensional transmutation procedure. Since the expectation value sets the physical scale for the theory we adopt this approach here so in what follows the dimensionless parameter λ_1 is understood as a function of the couplings and the expectation value, given in (5.7). In order that the perturbative expansion of V_{eff} not be invalidated by large logs in higher orders we insist that $\lambda_1/16\pi^2 \ln(\langle\phi_1\rangle^2/\mu^2) \ll 1$. Then λ_1 is given by the last two terms in (5.7), and this condition becomes

$$\frac{\lambda_3^2 - 88N^2y_1^4}{(16\pi^2)^2}\ln^2\frac{\langle\phi_1\rangle^2}{\mu^2} \ll 1\,. \tag{5.8}$$

Since λ_1 has been eliminated in favor of $\langle\phi_1\rangle$, the conditions that the perturbative analysis is valid are that $\langle\phi_1\rangle$ satisfies (5.8) and that dimensionless couplings remain small. Next, we must check that the extremum is a local minimum and that it is of lower energy than that of the origin of field space.

We first verify that the extremum is a local minimum. To this end we need to check that the eigenvalues of the mass matrix are both positive. Owing to the discrete symmetry and the fact that we are on the $\phi_2 = 0$ axis, the mixed derivatives terms vanish at $\langle\phi_1\rangle$, $\frac{\partial^2}{\partial\phi_1\partial\phi_2}V_{\text{eff}}(\langle\phi_1\rangle, 0) = 0$. Hence the two eigenvalues are given by

$$\frac{\partial^2}{\partial\phi_1^2}V_{\text{eff}}(\langle\phi_1\rangle, 0) = \frac{\lambda_3^2 - 88N^2y_1^4}{32\pi^2}\langle\phi_1\rangle^2, \tag{5.9}$$

$$\begin{aligned}\frac{\partial^2}{\partial\phi_2^2}V_{\text{eff}}(\langle\phi_1\rangle, 0) = {} & \frac{\lambda_3}{2}\langle\phi_1\rangle^2 - \frac{\lambda_3(\lambda_2 + 4\lambda_3)\langle\phi_1\rangle^2}{64\pi^2}\left(\ln\frac{\lambda_3\langle\phi_1\rangle^2}{2\mu^2} + 1\right) \\ & - \frac{264N^2y_1^2y_2^2\langle\phi1\rangle^2}{64\pi^2}\left(\ln\frac{y_1^2\langle\phi_1\rangle^2}{\mu^2} - \frac{1}{3}\right).\end{aligned} \tag{5.10}$$

The first eigenvalue is positive provided

$$\varepsilon \equiv \lambda_3^2 - 88N^2y_1^4 > 0. \tag{5.11}$$

The second eigenvalue is generally positive provided we are in the regime where the one loop terms are small compared to the tree level term. This is generally the case in perturbation theory, although one could have one coupling, in this case λ_3 be small compared to the remaining couplings (and indeed this is the situation for λ_1 in the region of parameter space of interest).

We can now check that the effective potential at $\langle\phi_1\rangle$ is negative:

$$V_{\text{eff}}(\langle\phi_1\rangle) = -\frac{\lambda_3^2 - 88N^2y_1^4}{512\pi^2}\langle\phi_1\rangle^4 = -\frac{\varepsilon}{512\pi^2}\langle\phi_1\rangle^4. \tag{5.12}$$

Remarkably, the condition that this be negative is precisely the same as having the first eigenvalue of the mass matrix be positive, Eq. (5.11).

Here we take a small detour to discuss the role of ϕ_2. The readers might notice that ϕ_2 plays virtually no role in the above analysis of the vacuum. Moreover, we could have arranged for the non-trivial IR fixed-point with only one scalar. This can be seen by setting y_2, λ_2 and λ_3 to zero in Eq (5.2) and repeating the fixed-point analysis given above.[2] This begs the question – what is the purpose of ϕ_2? With only one scalar, ϕ_1, we can repeat the above analysis and reproduce Eqs. (5.8)–(5.12) with

[2]Similar result regarding the fixed-point in Banks-Zaks type theory with an extra scalar singlet has been independently obtained in [91, 92].

λ_3 set to 0. Clearly, the extremum becomes the maximum and the effective potential seems to be unbounded from below. The extra scalar field allow us to introduce more couplings and more importantly establish the non-trivial minimum via perturbative analysis.

Note that the conditions we have found for the non-trivial minimum of the effective potential are not satisfied at or in the vicinity of the IR-fixed point. But neither are the conditions for perturbative computability. In order to determine the vacuum structure near the IR fixed point we must re-sum the leading log expansion of the effective potential. Equivalently we can take any point in the vicinity of the fixed point and ask whether its RG-trajectory maps back at some large RG-time t to the region where the analysis above is valid. If that is the case we can further ask whether it gives a non-trivial minimum. This is the approach we adopt here. We will come back to this issue in Sec. 5.4 where we will discuss the phase structure of the model and integrate the RGEs numerically to verify the vacuum structure near the IR-fixed point. But even without numerical studies we can argue physically that there are points arbitrarily close to the IR-fixed points for which the vacuum is non-trivial and scale invariance is spontaneously broken.

Choose the parameters to satisfy (5.11) and to be small at some fixed renormalization scale μ_0. One can arrange for the allowed range of expectation values to be large, so that $\langle\phi_1\rangle \ll \mu_0$ is included, by choosing ε to be as small as necessary. The coupling constants will run as in the mass independent scheme until the scale μ reaches values comparable to the mass of the heaviest particle in the model. At that point the running is modified. The trajectory that would end at the IR-fixed point is modified before the fixed point is reached. However this modification to the trajectory occurs only for $\mu \lesssim \langle\phi_1\rangle$. That is, given a fixed starting point μ_0 we can choose to run as far as needed on the mass-independent trajectory, far enough that it gets arbitrarily close to the IR-fixed point; all that is required is that one starts with a small enough value of $\langle\phi_1\rangle$.

We have not been able to explore fully the landscape of our effective potential. Other, lower minima may exist outside the $\phi_2 = 0$ axis. If that is the case the minimum we have found describes only a metastable vacuum. The analysis that follows is still largely correct. But more importantly, an analogous analysis could be applied to the global minimum and the qualitative results will not be different. What is important here is that the non-trivial minimum found at one-loop spontaneously breaks the scale invariance of the classical Lagrangian. The scale invariance is explicitly broken at one-loop too, by a quantum mechanical anomaly. If the former effect is dominant then we expect to see a pseudo Nambu-Goldstone boson of spontaneously broken approximate scale invariance, while if the latter effect is dominant no such state will be seen. So we turn in the next section to determining the spectrum of the model.

5.2.3 Particle Spectrum

If the theory is in the symmetric phase, $\langle\phi_1\rangle = \langle\phi_2\rangle = 0$, then all the particles are massless. Here, we compute the spectrum in the broken phase, $\langle\phi_1\rangle = v, \langle\phi_2\rangle = 0$. We retain up to one-loop order in the computation of the spectrum so that we may later address questions of invariance of physical quantities under RG-evolution. This is important because on the one hand we determine the vacuum structure far away from the IR fixed point while on the other we are interested in the fate of scale invariance and hence want to study the RG flow towards, and eventually in the vicinity of, the IR-fixed point.

We first compute the fermion spectrum. For large N the leading contribution to the fermion self-energy is from the gauge interaction. We can parametrize the self-energy as

$$i\Sigma(\not{p}) = i(Am + B\not{p}). \tag{5.13}$$

We obtain, to one-loop order,

$$A = \frac{g^2}{16\pi^2}\frac{N}{2}\left(-3\ln\frac{y_1^2 v^2}{\mu^2} + 4\right), \quad B = 1\,, \tag{5.14}$$

in Landau gauge. Hence the masses of χ and ψ (poles in the respective propagators) are

$$M_\psi(\mu) = M_\chi(\mu) = y_1 v\left[1 - \frac{g^2}{16\pi^2}\frac{N}{2}\left(3\ln\frac{y_1^2 v^2}{\mu^2} - 4\right)\right]. \tag{5.15}$$

The pole masses of the scalar fields ϕ_1 and ϕ_2 can be computed in a similar manner. Schematically, to one-loop order, the mass is

$$M_\phi^2 = \frac{\lambda}{2}v^2 + \Pi(\lambda v^2/2). \tag{5.16}$$

Explicit computation yields

$$
\begin{aligned}
M_{\phi_1}^2 &= \frac{\lambda_1 v^2}{2} + \frac{3\lambda_1^2 v^2}{64\pi^2}\left(\ln\frac{\lambda_1 v^2}{2\mu^2} - \frac{5}{3} + \frac{2\pi}{3\sqrt{3}}\right) + \frac{3\lambda_3^2 v^2}{64\pi^2}\left(\ln\frac{\lambda_3 v^2}{2\mu^2} - \frac{1}{3} - \frac{2\lambda_1}{3\lambda_3}\right)\\
&\quad + \frac{22N^2 y_1^2}{16\pi^2}\Bigg[y_1^2 v^2 - \frac{\lambda_1 v^2}{12} - 3\left(y_1^2 v^2 - \frac{\lambda_1 v^2}{12}\right)\left(\ln\frac{y_1^2 v^2}{\mu^2}\right)\\
&\qquad\qquad - 3\int_0^1 \mathrm{d}x \left(y_1^2 v^2 - \frac{x(1-x)}{2}\lambda_1 v^2\right)\ln\left(1 - x(1-x)\frac{\lambda_1}{2y_1^2}\right)\Bigg], \qquad (5.17)\\
&= \frac{3\lambda_1^2 v^2}{64\pi^2}\left(-\frac{2}{3} + \frac{2\pi}{3\sqrt{3}}\right) + \frac{3\lambda_3^2 v^2}{64\pi^2}\left(\frac{2}{3} - \frac{2\lambda_1}{3\lambda_3}\right)\\
&\quad + \frac{22N^2 y_1^2}{16\pi^2}\Bigg[-2\left(y_1^2 v^2 - \frac{\lambda_1 v^2}{12}\right)\\
&\qquad\qquad - 3\int_0^1 \mathrm{d}x \left(y_1^2 v^2 - \frac{x(1-x)}{2}\lambda_1 v^2\right)\ln\left(1 - x(1-x)\frac{\lambda_1}{2y_1^2}\right)\Bigg],\\
&\simeq \frac{\lambda_3^2 - 88N^2 y_1^4}{32\pi^2} v^2,\\
&= \frac{\varepsilon}{32\pi^2} v^2,\\
M_{\phi_2}^2 &= \frac{\lambda_3 v^2}{2} + \frac{\lambda_1 \lambda_3 v^2}{64\pi^2}\left(\ln\frac{\lambda_1 v^2}{2\mu^2} - 1\right) + \frac{\lambda_2 \lambda_3 v^2}{64\pi^2}\left(\ln\frac{\lambda_3 v^2}{2\mu^2} - 1\right)\\
&\quad + \frac{\lambda_3^2 v^2}{16\pi^2}\left(\ln\frac{\lambda_3 v^2}{2\mu^2} + \int_0^1 \mathrm{d}x \ln\left(x^2 + (1-x)\frac{\lambda_1}{\lambda_3}\right)\right)\\
&\quad + \frac{22N^2 y_2^2}{16\pi^2}\Bigg[y_1^2 v^2 - \frac{\lambda_3 v^2}{12} - 3\left(y_1^2 v^2 - \frac{\lambda_3 v^2}{12}\right)\left(\ln\frac{y_1^2 v^2}{\mu^2}\right)\\
&\qquad\qquad - 3\int_0^1 \mathrm{d}x \left(y_1^2 v^2 - \frac{x(1-x)}{2}\lambda_3 v^2\right)\ln\left(1 - x(1-x)\frac{\lambda_3}{2y_1^2}\right)\Bigg], \qquad (5.18)\\
&\simeq \frac{\lambda_3 v^2}{2} + \frac{\lambda_2 \lambda_3 v^2}{64\pi^2}\left(\ln\frac{\lambda_3 v^2}{2\mu^2} - 1\right) + \frac{\lambda_3^2 v^2}{16\pi^2}\left(\ln\frac{\lambda_3 v^2}{2\mu^2} - 2\right)\\
&\quad + \frac{22N^2 y_2^2}{16\pi^2}\Bigg[y_1^2 v^2 - \frac{\lambda_3 v^2}{12} - 3\left(y_1^2 v^2 - \frac{\lambda_3 v^2}{12}\right)\left(\ln\frac{y_1^2 v^2}{\mu^2}\right)\\
&\qquad\qquad - 3\int_0^1 \mathrm{d}x \left(y_1^2 v^2 - \frac{x(1-x)}{2}\lambda_3 v^2\right)\ln\left(1 - x(1-x)\frac{\lambda_3}{2y_1^2}\right)\Bigg].
\end{aligned}
$$

The first lines of Eqs. (5.17) and (5.18) are the complete one-loop expressions for the pole masses, while the second line on Eq. (5.17) uses Eq. (5.7) and shows that the whole expression is of one-loop order and that it has no explicit μ dependence. The last line in both equations is further simplified using the approximation valid at μ_0

that λ_1 is small. Observe that these scalar masses differ from the curvature of the effective potential at the minimum. This is because the effective potential is computed at zero external momentum,while the pole mass is computed at a momentum equal to the pole mass itself.

It is instructive to check that these masses are RG-invariant. The important observation is that the vacuum expectation value, v, transforms under the RGE with the anomalous dimension of ϕ_1:

$$\frac{\partial v}{\partial t} = \gamma_{\phi_1} v = -\frac{11N^2 y_1^2}{16\pi^2} v\,. \tag{5.19}$$

Using this, the above expressions for the pole masses and the beta functions in Eq. (5.2), one can verify that

$$\frac{\partial M_\psi}{\partial t} = \frac{\partial M_\chi}{\partial t} = \frac{\partial M_{\phi_1}}{\partial t} = \frac{\partial M_{\phi_2}}{\partial t} = 0\,, \tag{5.20}$$

up to terms of order of two loops. This is of course expected, but the explicit computation gives a check of the above expressions. For this check we have not used the approximation that λ_1 is small. This approximation is only valid for $\mu \sim \mu_0$, but we will be examining shortly RG-trajectories that extend to $\mu \ll \mu_0$ where the approximation breaks down.

5.3 Dilaton

5.3.1 Dilatation Current

The dilatation current, $\mathcal{D}^\mu$ is related to the improved stress-energy tensor through $\mathcal{D}^\mu = x_\nu \Theta^{\mu\nu}$ [93]. There are two important properties of the improved energy momentum tensor. First, it is not renormalized, so it has no anomalous dimensions. And second, it is such that the divergence of the dilatation current is just the trace of the stress-energy tensor, $\partial_\mu \mathcal{D}^\mu = \Theta^\mu_\mu$. A simple way of computing this tensor is by re-writing the model in a general covariant fashion, with a background metric $g_{\mu\nu}$, taking $\Theta^{\mu\nu} = -2\frac{\delta}{\delta g_{\mu\nu}} S_m$ where S_m is the action integral (exclusive of the Hilbert-Einstein term) and then re-setting the metric to the trivial one $g_{\mu\nu} = \eta_{\mu\nu}$. From the

Lagrangian in (5.1) we have

$$\Theta^{\mu\nu} = -F^{a\mu\lambda}F^{a\nu}_{\lambda} + \frac{1}{2}\bar{\chi}i(\gamma^\mu D^\nu + \gamma^\nu D^\mu)\chi + \frac{1}{2}\bar{\psi}i(\gamma^\mu D^\nu + \gamma^\nu D^\mu)\psi + \partial^\mu\phi_i\partial^\nu\phi_i - \frac{1}{2}\kappa(\partial^\mu\partial^\nu - g^{\mu\nu}\partial^2)\phi_i^2 - g^{\mu\nu}\mathcal{L}\,. \quad (5.21)$$

The term proportional to κ is the improvement: it is automatically conserved and is itself a total derivative so its integral vanishes, leaving the generators of energy and momentum $\int d^3x\ \Theta^{0\mu}$ unmodified. The improved tensor corresponds to setting $\kappa = 1/3$.

Classically the trace of this tensor vanishes and therefore the divergence of the dilatation current vanishes too. The theory is classically scale invariant. As is famously known this is no longer the case once quantum effects are included. Instead one has a "trace anomaly:" [94, 95]

$$\Theta^\mu_\mu = \gamma_{\phi_1}\phi_1\partial^2\phi_1 + (4\gamma_{\phi_1}\lambda_1 - \beta_{\lambda_1})\frac{\phi_1^4}{24} + \ldots, \quad (5.22)$$

where we have kept only the terms involving ϕ_1 since these will play a role in our discussion below. The terms involving the field anomalous dimension γ_{ϕ_1} are often overlooked. They can be ignored when application of the equations of motion is valid but may play a role in off-shell matrix elements or Green functions.[3] There is a simple indirect indication that these additional terms must be included: since $\Theta^{\mu\nu}$ is not renormalized the trace anomaly must be an RG-invariant, and the γ_{ϕ_1}-terms are required for this purpose [97, 98].

5.3.2 Dilaton

As a pseudo-Nambu-Goldstone boson the dilaton state $|\sigma\rangle$ should be created by acting on the vacuum with the spontaneously broken dilatation current. In analogy with PCAC we define a dilaton decay constant f_σ and a dilaton mass M_σ so that

$$\langle 0|\partial_\mu\mathcal{D}^\mu|\sigma\rangle = \langle 0|\Theta^\mu_\mu|\sigma\rangle_{x=0} = -f_\sigma M_\sigma^2\,. \quad (5.23)$$

[3]There is an interesting technical subtlety here. The equations of motion that can and should be used are those for the bare fields [96]. The use of the equation of motion in Eq. (5.22) gives that the terms proportional to γ_{ϕ_1} cancel. On the other hand, the insertion of the anomaly into a matrix element would have us replace $-M^2_{\phi_1}$ for ∂^2 but since this mass starts only at one-loop order its product with γ_{ϕ_1} would give a higher order effect and spoil the cancellation against the rest of the γ_{ϕ_1} terms. We have verified by explicit computation that in fact the cancellation is not spoiled. To this end one must use the relation in Eq. (5.7) that effectively trades λ_1 for one-loop terms.

This equation contains a particular combination of decay constant and mass and we would like to be able to distinguish between them. The matrix element of the current itself (which in PCAC gives the decay constant directly) is not very useful because of the explicit coordinate dependence. Instead consider the energy momentum tensor, before taking the trace:

$$\langle 0|\Theta^{\mu\nu}(x)|\sigma\rangle = \frac{f_\sigma}{3}\left(p^\mu p^\nu - g^{\mu\nu}p^2\right)e^{ip\cdot x} \tag{5.24}$$

The form of this equation is fixed by conservation of the stress-energy tensor and that its trace is given by Eq. (5.23). Note that in Eq. (5.24) the momentum is on-shell, $p^2 = M_\sigma^2$.

In order to compute f_σ and M_σ we must first identify a state in the spectrum of our model as the dilaton. Were we in the exact symmetry limit there would be a unique one-particle state that couples to the stress energy tensor, making the identification of the dilaton straightforward. If the symmetry is not exact but approximate we expect the dilaton to be a spinless state that (1) couples most strongly to the stress energy tensor and (2) is the lightest state that does. It is easy to see that the state of mass M_{ϕ_1} fits the bill. First, it is the lightest of the two spinless one-particle states in the spectrum, which is clear since the perturbative expansion for its mass starts at one-loop order. To see that it couples more strongly, note that when expanding the fields about the vacuum $\langle\phi_1\rangle = v$ and $\langle\phi_2\rangle = 0$ in the stress energy tensor, the only field that appears linearly is ϕ_1. Therefore the only one-particle state that has tree level overlap with the stress energy tensor is the state created by ϕ_1.

With this identification we can now compute the decay constant to tree level. Shifting the fields in Eq. (5.21) and concentrating on terms that can give $p^\mu p^\nu$ in the matrix element, we have $\Theta^{\mu\nu} = -1/3v\partial^\mu\partial^\nu\phi_1 + \cdots$. The ellipsis stand for terms that contribute only at higher order than tree level. Hence we read off $f_\sigma = v$. And, of course, $M_\sigma = M_{\phi_1}$.

The anomaly equation gives us a non-trivial check of this identification. Going to shifted fields in the anomaly Eq. (5.22), we have

$$\Theta^\mu_\mu = \gamma_{\phi_1} v\partial^2\phi_1 + (4\gamma_{\phi_1}\lambda_1 - \beta_{\lambda_1})\frac{v^3\phi_1}{6} + \ldots \tag{5.25}$$

Taking the matrix element of this, working to lowest order (tree level in the graphs).

we obtain

$$\langle 0|\Theta^\mu_\mu|\sigma\rangle_{x=0} = -\gamma_{\phi_1} v p^2 - \frac{\lambda_1^2+\lambda_3^2-88N^2y_1^4}{32\pi^2}v^3 + \ldots. \tag{5.26}$$

This agrees with Eq. (5.23) if we use our identifications

$$f_\sigma = v \qquad \text{and} \qquad M_\sigma^2 = M_{\phi_1}^2 = \frac{\varepsilon}{32\pi^2}v^2. \tag{5.27}$$

We have dropped the $\gamma_{\phi_1} v M_\sigma^2$ and $\lambda_1^2 v^3$ terms for consistency.

Since the improved stress energy tensor is not renormalized the decay constant f_σ must be an RG-invariant quantity. M_σ is also RG-invariant as any physical mass must. The expressions we have found are not RG-invariant only because we have expressed them in lowest order of perturbation theory. The pole mass, which we have already discussed earlier, is explicitly seen to be RG-invariant to one-loop order for the trivial reason that it itself starts at one-loop order. On the other hand, the vacuum expectation value runs like the field, Eq. (5.19). If $Z(t)$ is the wave-function renormalization factor, $\partial Z/\partial t = 2\gamma_{\phi_1} Z$, $Z(0) = 1$, where $t = \ln(\mu/\mu_0)$, then $f_\sigma = v/Z^{1/2}$ is an RG-invariant, the RG-improved version of the previous result.

5.4 Phase Structure

We return here to the study of the phase structure of the model, posed earlier in Sec. 5.2.2. Let us recapitulate from there. A perturbative study of the vacuum structure of the theory requires that we limit our attention to a region of parameter space where λ_1 is small. Then the model possesses a new, non-trivial minimum provided (5.11) is satisfied. Neither of these conditions are satisfied in the neighborhood of the IR-fixed point. However, we can take any point in the vicinity of the fixed point and ask whether its RG-trajectory maps back at some large RG-time t to the region where a perturbative analysis of the effective potential is valid and gives a non-trivial minimum. In fact, by reversing the process, that is, by starting with a well chosen point at large RG-time t and then running towards the IR, we argued that there always exist points arbitrarily near the IR-fixed point for which the symmetry is spontaneously broken. Choose coupling constants at some renormalization scale μ_0 that give a non-trivial minimum and so that the expectation value is small $\langle\phi_1\rangle \ll \mu_0$. The coupling constants will run as in the mass independent scheme towards the IR-fixed point and will get closer the smaller the value of $\langle\phi_1\rangle$. At $\mu \sim \langle\phi_1\rangle$ the running will be modified and the trajectory will not hit the fixed point, but will have gotten very close.

Now let's complete the picture. When μ becomes of the order of the physical mass of the heaviest particles in the spectrum the running of the couplings is modified. For μ below the scale of that mass the beta function becomes effectively the one for the model in the absence of those massive particles, that is, the heavy particles are "integrated out." As μ is further decreased one sequentially integrates out all massive particles in the model. This all occurs near the fixed point so all couplings are still perturbative, but now all scalars and spinors are integrated out. The Yukawa and self-couplings stopped running and become uninteresting since the effective theory contains only massless Yang-Mills vectors. Now the beta function of this effective theory is very much like that of QCD: the coupling constant quickly runs to strong coupling,

$$g^2(\mu) \approx \frac{g_*^2}{1 + \frac{g_*^2}{16\pi^2}\frac{22N}{3}\ln\frac{\mu}{\langle\phi_1\rangle}} \tag{5.28}$$

The spectrum of the effective theory is that of a theory of pure glue, that is glueballs, of mass

$$M_g \sim \langle\phi_1\rangle e^{-\frac{3}{22N}\frac{16\pi^2}{g_*^2}} = \langle\phi_1\rangle e^{-225/44\delta} \tag{5.29}$$

So the spectrum of the model consists of two massive scalars and n_f massive fermions with masses given in Sec. 5.2.3 plus glueballs with masses M_g. The lighter scalar can be identified with the dilaton and its mass is given by Eq. (5.27).

We can repeat the analysis, only now starting from a set of coupling constants that does not satisfy the condition (5.11) at μ_0. The potential now remains positive up to large values of ϕ_1/μ_0 and one expects that by the time it starts decreasing perturbation theory ceases to be applicable. So we expect the true vacuum is at the origin of field space $\langle\phi_1\rangle = \langle\phi_2\rangle = 0$. There is no spontaneous scaling symmetry breaking, all particles are massless. As $t \to -\infty$ the RG-trajectories run into the IR-fixed point.

The following picture emerges: the theory has two phases. The parameter space of the model, which we identify with the space of couplings at a fixed renormalization scale μ_0, is split in two regions. In region I the spectrum is massless and all RG-trajectories run into the IR-fixed point. In region II there are no massless particles and RG-trajectories do not end at the IR-fixed point. There is a boundary between these phases, a hypersurface in the parameter space of the model. The fixed point lies on this surface.

The expectation value $\langle\phi_1\rangle$ vanishes in region I, but does not in region II. The transition is discontinuous: by dimensional transmutation, there is a non-trivial minimum of $V_{\rm eff}$ at an arbitrary[4] value of $\langle\phi_1\rangle$ provided $\lambda_3^2 - 88N^2y_1^4$ is positive, no matter how small. Since the physical content is preserved by flows we see that the surface itself is RG-invariant.

But perhaps we have rushed into conclusions. Firstly, when (5.11) is not satisfied the effective potential is unbounded from below as one moves along the ϕ_1 axis towards large values of ϕ_1. We stated without justification that at large ϕ_1 perturbation theory breaks down and one expects the potential stays bounded from below. But there is no guarantee of this, and even if the potential stays bounded it may develop a new global minimum at large ϕ_1. Perhaps none of region I is physical? And secondly, in order to reach the vicinity of the IR point, which is AB's prescription for obtaining a light dilaton, we argued we can choose $\langle\phi_1\rangle$ small enough that our RG-trajectory will get there. But how do we know that this does not occur only for such small $\langle\phi_1\rangle$ that the logs in the effective potential become too large, again invalidating the analysis?

Fortunately we can go a long way towards settling these issues by explicit computation. Inasmuch as the potential becomes one dimensional (the minimum or the unbounded direction both lie on the axis) we can use the RGE to re-sum the leading logs hence extending the region of validity of the computation to the whole space of perturbative parameters. For the effectively one dimensional case the effective potential is $V_{\rm eff} = \frac{1}{24}\bar{\lambda}_1(t,\lambda_1)Z(t)^2\phi_1^4$ [99]. Here $t = \ln(\phi_1/\mu_0)$, Z is a wave-function renormalization factor and $\bar{\lambda}_1(t,\lambda_1)$ is the running coupling constant, defined with boundary condition $\bar{\lambda}_1(0,\lambda_1) = \lambda_1$. The first objection above is settled as follows: for any RG-trajectory for which $\bar{\lambda}_1$ stays positive we can assert the minimum of $V_{\rm eff}$ is at the origin of field space and there is no symmetry breaking. The only caveat is that we cannot trust the calculation at very large t where the scalar couplings become non-perturbatively large. Recall the model has Landau poles so it either is considered as a cut-off model or as the low energy limit of a complete theory.

The second objection can also be settled by following the trajectory towards the

[4]Arbitrary, but not extreme: the logs of $\langle\phi_1\rangle/\mu_0$ cannot be too large if the perturbative analysis is to remain valid.

IR. If at any point along the trajectory the running coupling turns negative then there will be a minimum away from the origin in field space, symmetry will be broken and a pseudo Nambu-Goldstone boson associated with the breaking of scale invariance will appear in the spectrum. One can then follow the trajectory and determine how close it gets to the IR-fixed point. This is somewhat unnecessary, since we already established in the previous two sections that for small ε we get a light dilaton.

Although the model is perturbative, we do not know how to analytically integrate the RG trajectories. But it is quite straightforward to investigate them numerically. It is beyond the scope of this work to conduct an exhaustive study of the phase diagram numerically. Instead we follow the trajectories from some initial points at μ_0 to gain confidence the picture we have painted is not obviously flawed. We use $N = 20$, $n_f = 11N/2$, $\delta = 0.2$. First we take $g(\mu_0) = \frac{4}{9}g_*$, $y_1(\mu_0) = 0.45y_{1*}$, $y_2(\mu_0) = \frac{1}{5}y_{2*}$, $\lambda_1(\mu_0) = \frac{1}{30}\lambda_{1*}$, $\lambda_2(\mu_0) = 3\lambda_{2*}$, $\lambda_3(\mu_0) = 5.2\lambda_{3*}$.. This set of parameters does not satisfy (5.11). The effective potential doesn't develop a non-trivial minimum,the running coupling $\bar{\lambda}_1$ remains positive. The theory flows to the IR fixed point. Next we analyze the case when $g(\mu_0) = \frac{4}{9}g_*$, $y_1(\mu_0) = 0.32y_{1*}$, $y_2(\mu_0) = \frac{1}{5}y_{2*}$, $\lambda_1(\mu_0) = \frac{1}{30}\lambda_{1*}$, $\lambda_2(\mu_0) = 3\lambda_{2*}$, $\lambda_3(\mu_0) = 5.2\lambda_{3*}$. Naively, this theory seems to flow to the IR-fixed point as well. But in this case, the effective potential does develop a minimum at $\ln(v^2/\mu_0^2) \simeq -58$. We estimate the fractional correction to the effective potential from higher orders in the loop expansion to be of order

$$\left| \frac{Ng^2}{16\pi^2} \ln\left(\frac{y_1^2 v^2}{\mu^2}\right) \right| \simeq 0.2$$

Thus we can trust the minimum we find using perturbation theory. With this vev, the spectrum is $M_{\psi,\chi}/v \simeq 8.5 \times 10^{-3}, M_{\phi_1}/v \simeq 7.9 \times 10^{-4}, M_{\phi_2}/v \simeq 9.5 \times 10^{-2}$. The scale μ_0 is some 13 orders of magnitude larger than the vacuum expectation value v, but it is unphysical.

We have studied numerically the transition between these two parameters sets by varying $y_1(\mu_0)$ or $\varepsilon(\mu_0)$. When $y_1(\mu_0)$ is sufficiently large, or when ε turns negative, we change from a broken phase to the symmetric phase as expected from Eq. (5.12). Note that with our particular value of parameters, the theory is close to the boundary of the broken/symmetric phases.

5.4.1 Relevant Perturbations

Suppose we consider a modification of the model, one in which scale invariance is explicitly broken. This is accomplished by adding relevant perturbations. If the symmetries of the model are to be preserved only mass terms can be added. This enlarges the parameter space of the model. The origin of all the relevant-perturbation axes corresponds to the parameter space described in the previous paragraphs, and it is on that hyperplane that the IR-fixed point lies together with the two phases and the hypersurface separating them.

Far away from this hyperplane, a long ways along the relevant-perturbation axes, the physics is very simple: scalars and spinors have hard masses and below the scale of those masses they decouple so as to leave only light glueballs in the spectrum. A more interesting region of parameter space is the direction of large scalar masses and small spinor masses. Then the scalars decouple and one is left with a Banks-Zaks-like model. Only it does not run into an IR-fixed point because the spinors eventually decouple, the YM-coupling then runs strongly and glueballs form. Only at zero spinor mass do we see that our IR-fixed point is really part of an IR-fixed hyperline.

What is the fate of the two phases as one extends into the new axes? In the symmetric phase the addition of hard masses can only make the vacuum at the origin of field space more stable. The spectrum is modified, particles are massive now and there is no IR-fixed point (save for the zero spinor mass case).

Analysis of the broken symmetry phase is more subtle. Provided we stay very close to the origin of the new axes, so that the added mass terms are really small perturbations, much smaller than the masses obtained in the absence of the perturbations, then nothing changes qualitatively and the quantitative changes to the spectrum are small. As the strength of the relevant perturbations increase the model may remain in a broken phase, depending on the precise nature of the perturbations. But for large enough perturbations the dilaton will be unrecognizable as a pseudo Nambu-Goldstone boson.

Summarizing, the two phase diagram does extend into the larger parameter space. The fixed point becomes a (hyper) line of fixed points. For large perturbations the dilaton is gone.

5.5 Discussion, Conclusion and Open Questions

We have presented a model with an IR-fixed point, and demonstrated that the model has two phases. In phase I RG-trajectories run into the IR-fixed point (in infinite RG-time). The scale symmetry is approximate and explicitly realized and it becomes exact at the fixed point. In phase II scale symmetry is spontaneously broken. Of course, scale invariance is also explicitly broken by the trace anomaly. The trajectories don't reach the IR-fixed point but some get very close and for those the explicit, relative to spontaneous, breaking of scale invariance is small: A light dilaton appears in the spectrum.

Analytic evidence for this picture was presented at length but the numerical support was scant. This is clearly an interesting direction for future work. In particular, one could determine the actual location of the phase transition. Another direction for future work is to find generalizations of the model. We do not know how general this picture is or how difficult it may be to come about models that display arbitrarily light dilatons (we were not aware of any example prior to this work).

Among new models one may try to construct some with the Standard Model of electroweak interactions embedded in it. One could then test whether the setup in Ref. [83] works as advertised. The authors there considered the possibility that the standard model is embedded in an almost conformal, possibly strongly interacting field theory with spontaneously broken scale invariance. In the context of 4-dimensional strongly interacting near-CFTs obtained as AdS/CFT-like duals of 5-dimensional non-factorizable geometries (RS models) one encounters often the schematic Lagrangian describing the dynamics:

$$\mathcal{L} = \mathcal{L}_{\rm CFT} + \sum_n \lambda_n \mathcal{O}_n \,. \tag{5.30}$$

The first term is a CFT while the sum that follows is an attempt to capture the deviations ("deformations") from the CFT by adding small perturbations [81, 82]. Obviously this basic setup applies to our model, and because it is fully perturbative model one should be able to verify the validity of some general assertions. The deviations from conformality can be small in one of two ways, either the anomalous dimensions γ_n or the coefficients λ_n of the operators $\mathcal{O}_n$ are small. On general grounds one can show that for $|\gamma_n| \ll 1$ the effective potential for the field χ whose expectation value gives rise to the dilaton is [83]

$$V_{\text{eff}}(\chi) = \frac{M_\sigma^2}{4f_\sigma^2}\chi^4\left[\ln\left(\frac{\chi}{f_\sigma}\right) - \frac{1}{4}\right] + \mathcal{O}(\gamma^2)\,. \tag{5.31}$$

The case $|\lambda_n| \ll 1$ is more cumbersome. Only in the case that only one perturbation is added does one obtain a parameter-free effective potential

$$V_{\text{eff}}(\chi) = \frac{M_\sigma^2}{f_\sigma^2\gamma}\chi^4\left[\frac{1}{4+\gamma}\left(\frac{\chi}{f_\sigma}\right)^\gamma - \frac{1}{4}\right] + \mathcal{O}(\lambda^2)\,,$$

while for more than one perturbation occur one has the less restricted

$$V_{\text{eff}}(\chi) = \frac{M_\sigma^2}{f_\sigma^2}\chi^4\sum_n\left\{x_n\left[\frac{1}{4+\gamma_n}\left(\frac{\chi}{f_\sigma}\right)^{\gamma_n} - \frac{1}{4}\right]\right\} + \mathcal{O}(\lambda^2)\,,$$

where the coupling constants have been traded for constants x_n that are constrained by $\sum_n \gamma_n x_n = 1$.

Any model with a conformal fixed point g_* can be written in the fashion of Eq. (5.30)

$$\mathcal{L}(g) = \mathcal{L}(g_*) + (\mathcal{L}(g) - \mathcal{L}(g_*))$$

where g are coupling constants at arbitrary values. If g is sufficiently close to g_* one is in the case $|\lambda_n| \ll 1$ above, while if the region of couplings that includes g and g_* is perturbative one expects $|\gamma_n| \ll 1$. We need, in addition, that the model display spontaneous breaking of scale invariance in the vicinity of the fixed point. Our model furnished an explicit example. The analogue of χ is our field ϕ_1. Because it is perturbative one has $|\gamma_n| \ll 1$. Reassuringly, when the tree level term in the effective potential of Eq. (5.6) is eliminated by use of Eq. (5.7), and the expressions for dilaton mass and decay constant in Eq. (5.27) the resulting potential is *exactly* of the form of Eq. (5.31). To emphasize, the dependence on the many coupling constants of our model is completely contained now in only two parameters: M_σ and f_σ.

Finally, we address one of the central questions we set out to investigate: Is the AB estimate of the dilaton mass in walking technicolor scenarios correct? For AB, the dilaton mass is given by

$$M_\sigma^2 \simeq \frac{s(\alpha_* - \alpha_c)}{\alpha_c}\Lambda^2 \simeq \frac{N_f^c - N_f}{N_f^c}\Lambda^2, \tag{5.32}$$

where α_* is the coupling at the fixed point, N_f is the number of flavors and Λ is the scale of chiral symmetry breaking which occurs only if the critical coupling α_c is below the fixed point, $\alpha_c < \alpha_*$, which in turn corresponds to the number of flavors below a critical value, N_f^c. The middle expression in Eq. (5.32), relating the mass to the distance between the critical coupling and the fixed-point, does not carry over to our model. In our case, the role of the critical value of the coupling constant α_c is played by a critical surface, $\varepsilon = 0$, separating the symmetric and broken phases. But the mass of the dilaton is not proportional to the distance between this surface and the fixed point (however one defines distance): the fixed-point lies on the critical surface and the dilaton mass vanishes everywhere on the surface. The rightmost expression in Eq. (5.32), however, has a counterpart in our model. In that formula $(N_f^c - N_f)/N_f^c$ measures how far the theory is from the critical point. In our model ε plays the role of this quantity. It measures how far the theory is from the critical surface. Moreover, both $(N_f^c - N_f)/N_f^c$ and ε can be made arbitrarily small which in turn make the dilaton arbitrarily light compared to the scale of symmetry breaking. To the extent that one can arrange for arbitrarily small $(N_f^c - N_f)/N_f^c$, AB's estimate of a parametrically small dilaton mass is consistent with our analysis.

We thank F. Sannino for bringing to our attention Ref. [100] which arrives at a similar conclusion to that of [45]. We also thank L. Vecchi for bringing to our attention Ref. [101, 102] which, via a different method, also arrives at the same conclusion as [46].

This chapter is a reprint of material as it appears in “A Very Light Dilaton,” B. Grinstein and P. Uttayarat, JHEP **1107** (2011) 038 [arXiv:1105.2370 [hep-ph]], of which I was a co-author.

Appendix A

Proof that G_C is Semi-simple

We show that the four generators X^i in G_c that produce the non-linear transformations of the four real components of the higgs field are in a subalgebra that generates a semi-simple subgroup of G_c.

Our starting point are the commutation relations

$$[Q^a, X^i] = R(Q^a)^{ij} X^j, \qquad [Y, X^i] = R(Y)^{ij} X^j. \tag{A.1}$$

These are part of the algebra of G_f. We recall some basic facts about compact Lie algebras (we follow and use the notation of Ref. [103]). The Cartan subalgebra of G_f is the largest set of mutually commuting generators H_i, $i = 1, \ldots, r \equiv \text{rank}(G_f)$. In the adjoint representation define a vector space by the map $T^A \to |T^A\rangle$, where the T^A are generators of G_f, and define the action of generators on this vectors by $T^A|T^B\rangle = |[T^A, T^B]\rangle$. Moreover, define an inner product on this space by $\langle T^A|T^B\rangle = \text{Tr}(T^{A\dagger} T^B)$. Since the H_i are mutually commuting one can find a basis of the vector space $H_i|E_\alpha\rangle = \alpha_i|E_\alpha\rangle$. The states correspond to the rest of the generators, E_α. It follows that $[H_i, E_\alpha] = \alpha_i E_\alpha$ and $E_{-\alpha} = E_\alpha^\dagger$. Choose the generators of the Cartan subalgebra to satisfy $\langle H_i|H_j\rangle = \delta_{ij}$. It can be shown

$$[E_\alpha, E_{-\alpha}] = \sum_{i=1}^{r} \alpha_i H_i \tag{A.2}$$

We intend to show that there is a basis of of Cartan generators for which the four X^i (or a linear combination of them) correspond to two pairs $(E_\alpha, E_{-\alpha})$ that therefore do not commute among themselves.

We are free to take $H_1 = Q^3$ and $H_2 = Y$ as the first two members of the Cartan subalgebra. Now, the standard representation

$$R(Y) = \begin{pmatrix} -\tau^2/2 & 0 \\ 0 & -\tau^2/2 \end{pmatrix}, \qquad R(Q^3) = \begin{pmatrix} -\tau^2/2 & 0 \\ 0 & \tau^2/2 \end{pmatrix}, \tag{A.3}$$

$$R(Q^1) = \begin{pmatrix} 0 & -\tau^2/2 \\ -\tau^2/2 & 0 \end{pmatrix}, \qquad R(Q^2) = \begin{pmatrix} 0 & -i/2\,\mathbf{1}_{2\times 2} \\ i/2\,\mathbf{1}_{2\times 2} & 0 \end{pmatrix}, \tag{A.4}$$

and we can make a transformation $X^i \to U^{ij} X^j$ to diagonalize $R(Q^3)$ and $R(Y)$:

$$\begin{aligned} [Y, X^1] &= -\frac{1}{2}X^1 & [Q^3, X^1] &= -\frac{1}{2}X^1 \\ [Y, X^2] &= +\frac{1}{2}X^2 & [Q^3, X^2] &= +\frac{1}{2}X^2 \\ [Y, X^3] &= -\frac{1}{2}X^3 & [Q^3, X^3] &= +\frac{1}{2}X^3 \\ [Y, X^4] &= +\frac{1}{2}X^4 & [Q^3, X^4] &= -\frac{1}{2}X^4 \end{aligned} \tag{A.5}$$

The rest of the Cartan subalgebra can be chosen to commute with X^i, as we now show. Suppose

$$\begin{aligned} [H_i, X^1] &= -\frac{a_i}{2}X^1 \\ [H_i, X^2] &= +\frac{a_i}{2}X^2 \\ [H_i, X^3] &= -\frac{b_i}{2}X^3 \\ [H_i, X^4] &= +\frac{b_i}{2}X^4 \end{aligned} \tag{A.6}$$

Then the generators $H'_i = H_i - (a_i + b_i)/2\; Y - (a_i - b_i)/2\; Q^3$, commute with X^j.

Now, it is clear the the four $|X^i\rangle$ are among the states $|E_{\alpha'}\rangle$ that satisfy $H'_i|E_{\alpha'}\rangle = \alpha'_i|E_{\alpha'}\rangle$. Moreover the vectors α' for X^i are of the form $(\pm\frac{1}{2}, \pm\frac{1}{2}, 0, \ldots, 0)$. Equation (A.2) holds only provided the H_i that satisfy $\mathrm{Tr}(H_i H_j) = \delta^{ij}$. While the H_i satisfy this orthonormality condition, the new basis H'_i generally does not. Writing $H_i = V_{ij} H'_j$ gives an explicit set of eigenvectors of H_i,

$$H_i|E_{\alpha'}\rangle = V_{ij} H'_j|E_{\alpha'}\rangle = V_{ij}\alpha'_j|E_{\alpha'}\rangle$$

The eigenvectors are the same as those of H'_i, and hence the $|X^j\rangle$ are still among them, but the eigenvalues have changed, $\alpha_i = V_{ij}\alpha'_j$. But with this basis we can use

(A.2). Explicitly

$$[X^1, X^2] = -\frac{1}{2}\sum_{i=1}^{r}(V_{i1} + V_{i2})H_i \tag{A.7}$$

$$[X^3, X^4] = -\frac{1}{2}\sum_{i=1}^{r}(-V_{i1} + V_{i2})H_i \tag{A.8}$$

We see that both commutators are non-vanishing, as we set out to demonstrate.

This appendix is a reprint of the material as it appears in "Hidden fine tuning in the quark sector of little higgs models," B. Grinstein, R. Kelley and P. Uttayarat, JHEP **0909**, 040 (2009) [arXiv:0904.1622 [hep-ph]], of which I was a co-author.

Appendix B

Expansion Coefficients of Ω in Littlest Higgs and SU$(3)_1$

Here we will show that any two Yukawa terms which are invariant under $SU(3)_1$ have the same expansion coefficients for $H^\dagger$ vs. $H^\dagger H H^\dagger$ (up to an overall constant). The expression for Ω^i is given by

$$\begin{aligned}\Omega^I &= \delta^{i3}\left[a_0 + \frac{a_1}{f}\eta + \frac{a_2}{f^2}\eta^2 + \frac{a_3}{f^2}H^\dagger H + \frac{a_4}{f^2}\mathrm{Tr}\,(\phi^\dagger\phi) + \ldots\right] \\ &\quad + \delta^{i\alpha}\left[\frac{b_1}{f}H^\dagger + \frac{b_2}{f^2}\eta H^\dagger + \frac{b_3}{f^2}H^\dagger\omega + \frac{b_4}{f^2}h^T\phi^\dagger + \frac{b_5}{f^3}(H^\dagger H)H^\dagger + \ldots\right]^\alpha, \end{aligned} \tag{B.1}$$

where $i = 1, 2, 3$ and $\alpha = 1, 2$. Now we will show that the coefficient b_1 and b_5 are completely determined by a_0. Consider an $SU(3)_1$ transformation generated by

$$\Lambda = \begin{pmatrix} 0 & \lambda & 0 \\ \lambda^\dagger & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{B.2}$$

The Goldstone bosons transform nonlinearly under Λ. Let us define

$$\delta\Pi = \delta\Pi^{(0)} + \delta\Pi^{(1)} + \delta\Pi^{(2)} + \ldots, \tag{B.3}$$

where

$$\begin{aligned}\delta\Pi^{(0)} &= \frac{f}{2}\left(\Lambda + \Sigma_0\Lambda^T\Sigma_0\right), \\ \delta\Pi^{(1)} &= \frac{i}{2}\left(\Lambda\Pi - \Pi\Lambda + \Pi\Sigma_o\Lambda^T\Sigma_0 - \Sigma_0\Lambda^T\Sigma_0\Pi\right), \\ \delta\Pi^{(2)} &= \frac{1}{6f}\left(-\Pi^2\Lambda + 2\Pi\Lambda\Pi - \Lambda\pi^2 - \Pi^2\Sigma_0\Lambda^T\Sigma_0 + 2\Pi\Sigma_0\Lambda^T\Sigma_0\Pi - \Sigma_0\Lambda^T\Sigma_0\Pi^2\right).\end{aligned} \tag{B.4}$$

In terms of the component fields we have

$$\begin{aligned}
\delta H &= \frac{1}{\sqrt{2}} f\lambda + \frac{i}{\sqrt{2}}\left(-\omega\lambda + \phi\lambda^\dagger + \frac{5}{\sqrt{20}}\eta\lambda\right) \\
&\quad + \frac{1}{6\sqrt{2}f}\left[(H^\dagger\lambda + \lambda^\dagger H)H - 2(H^\dagger H)\lambda\right] + \dots, \\
\delta\phi &= \frac{i}{2\sqrt{2}}\left(\lambda H^T + H\lambda^T\right) + \dots, \\
\delta\eta &= -\frac{i\sqrt{10}}{4}\left(H^\dagger\lambda - \lambda^\dagger H\right) + \dots, \\
\delta\omega &= \frac{i}{2\sqrt{2}}\left(\lambda H^\dagger - H\lambda^\dagger\right) - \frac{i}{4\sqrt{2}}\left(H^\dagger\lambda - \lambda^\dagger H\right) + \dots
\end{aligned} \tag{B.5}$$

Applying $SU(3)$ transformation to Ω yields

$$\begin{aligned}
\delta\Omega^i &= -i\delta^{i\alpha}\left[a_0 + \frac{a_1}{f}\eta + \frac{a_2}{f^2}\eta^2 + \frac{a_3}{f^2}H^\dagger H + \frac{a_4}{f^2}\mathrm{Tr}\left(\phi^\dagger\phi\right)\right]\lambda^{\dagger\alpha} \\
&\quad -i\delta^{i3}\left[\frac{b_1}{f}H^\dagger + \frac{b_2}{f^2}\eta H^\dagger + \frac{b_3}{f^2}H^\dagger\omega + \frac{b_4}{f^2}H^T\phi^\dagger + \frac{b_5}{f^3}(H^\dagger H)H^\dagger\right]\lambda.
\end{aligned} \tag{B.6}$$

This must be the same as applying Eq.(B.5) to Eq.(B.1)

$$\begin{aligned}
\delta\mathcal{O}^i &= \delta^{i\alpha}\frac{b_1}{\sqrt{2}}\lambda^{\dagger\alpha} + \frac{1}{f}\Bigg\{\delta^{i3}\left[-\frac{a_1 i\sqrt{10}}{4}\left(H^\dagger\lambda - \lambda^\dagger H\right) + \frac{a_3}{\sqrt{2}}\left(H^\dagger\lambda + \lambda^\dagger H\right)\right] \\
&\quad + \delta^{i\alpha}\left[-i\frac{b_1}{\sqrt{2}}(-\lambda^\dagger\omega + \frac{5}{\sqrt{20}}\eta\lambda^\dagger + \lambda^T\phi^\dagger) + \frac{b_2}{\sqrt{2}}\eta\lambda^\dagger + \frac{b_3}{\sqrt{2}}\lambda^\dagger\omega + \frac{b_4}{\sqrt{2}}\lambda^T\phi^\dagger\right]^\alpha\Bigg\} \\
&\quad + \frac{1}{f^2}\delta^{i\alpha}\Bigg\{\frac{b_1}{6\sqrt{2}}\left[(H^\dagger\lambda + \lambda^\dagger H)H^\dagger - 2(H^\dagger H)\lambda^\dagger\right] - i\frac{b_2\sqrt{10}}{4}\left(H^\dagger\lambda - \lambda^\dagger H\right)H^\dagger \\
&\quad + i\frac{b_3}{4\sqrt{2}}H^\dagger\left(2\lambda H^\dagger - 2H\lambda^\dagger - H^\dagger\lambda + \lambda^\dagger H\right) - i\frac{b_4}{2\sqrt{2}}H^T\left(\lambda^* H^\dagger + H^*\lambda^\dagger\right) \\
&\quad + \frac{b_5}{\sqrt{2}}\left[(H^\dagger H)\lambda^\dagger + (\lambda^\dagger H + H^\dagger\lambda)H^\dagger\right]\Bigg\}^\alpha.
\end{aligned} \tag{B.7}$$

Matching the coefficients in Eq.(B.6) and Eq.(B.7), we get

$$b_1 = -a_0, \quad b_5 = \frac{i2\sqrt{2}}{3}a_0. \tag{B.8}$$

Thus any two Yukawa operators Ω_1 and Ω_2 will have the same ratio of the coefficients of $H^\dagger$ and $H^\dagger H H^\dagger$.

This appendix is a reprint of material as it appears in "Spontaneous CP Violation and Light Particles in The Littlest Higgs," Z. Surujon and P. Uttayarat, Phys. Rev. D **83**, 076010 (2011) [arXiv:1003.4779 [hep-ph]], of which I was a co-author.

Appendix C

Tree-level Decay Rate for $h \to \eta\eta$

The kinetic term for the Σ field includes the interaction

$$\frac{1}{f^2}\left[a\eta\partial^\mu\eta\left(H^\dagger\partial_\mu H+\partial_\mu H^\dagger H\right)-b\left(\partial_\mu\eta\right)^2 H^\dagger H\right]\to\frac{v}{f^2}\left[a\eta\partial^\mu\eta\partial_\mu h-b(\partial_\mu\eta)^2 h\right], \tag{C.1}$$

where v is the Higgs vev. The explicit breaking term, *e.g.* Eq.(4.13), includes the interaction

$$\frac{cM_\eta^2}{f^2}\eta\eta H^\dagger H\to\frac{cvM_\eta^2}{f^2}. \tag{C.2}$$

The decay amplitude due to these two terms is

$$i\mathcal{M}(h\to\eta\eta)=i\frac{v}{f^2}\left[ap^2+2b(q_1\cdot q_2)+cM_\eta^2\right], \tag{C.3}$$

where p is the momentum of the incoming h and q_i are momenta of the outgoing η's. In the Higgs rest frame, the amplitude reduces to

$$i\mathcal{M}(h\to\eta\eta)=i\frac{v}{f^2}M_H^2\left[a+b+\frac{M_\eta^2}{M_H^2}(c-2b)\right]. \tag{C.4}$$

Thus the rate for $h\to\eta\eta$ is

$$\Gamma(h\to\eta\eta)=\frac{1}{32\pi}\frac{\sqrt{1-4M_\eta^2/M_H^2}}{M_H}\frac{M_H^4}{f^4}v^2\left[a+b+\frac{M_\eta^2}{M_H^2}(c-2b)\right]^2. \tag{C.5}$$

We include the tree-level decay rate for $h\to b\bar{b}$ for completeness. The amplitude is

$$i\mathcal{M}(h\to b\bar{b})=3\frac{m_b^2}{v^2}\,\mathrm{Tr}\left(\not{p}_1+m_b\right)\left(\not{p}_2-m_b\right)=3\frac{m_b^2}{v^2}M_H^2\left(1-\frac{4m_b^2}{M_H^2}\right). \tag{C.6}$$

Thus the decay rate is

$$\Gamma(h \to b\bar{b}) = \frac{1}{16\pi}\frac{6m_b^2}{v^2} M_H \left(1 - \frac{8m_b^2}{M_H^2}\right)^{3/2} . \tag{C.7}$$

In the hypercharge model $a = b = 5/12$ and $c = 25\sqrt{2}/48$. For the SU(6)/SO(6) model with $\alpha = \theta = 0$, we get $a = b = 1/3$ and $c = 17/24$.

This appendix is a reprint of material as it appears in "Spontaneous CP Violation and Light Particles in The Littlest Higgs," Z. Surujon and P. Uttayarat, Phys. Rev. D **83**, 076010 (2011) [arXiv:1003.4779 [hep-ph]], of which I was a co-author.

Appendix D

1-loop Effective Potential in SU(6)/SO(6) Model

Here we give the one-loop effective potential in the case $\theta = \alpha = 0$ and retain only terms relevant for the quartic potential of the Higgs doublet. The contributions from gauge interactions are

$$\begin{aligned} V_{\text{gauge}} &= a(g_1^2 + g_1'^2) f^2 \,\text{Tr} \left| \phi + \frac{i}{2f} \left(HH^T + KK^T \right) \right|^2 \\ &\quad + a(g_2^2 + g_2'^2) f^2 \,\text{Tr} \left| \phi - \frac{i}{2f} \left(HH^T + KK^T \right) \right|^2 + \ldots, \end{aligned} \tag{D.1}$$

where a is an order one constant whose precise value depends on the UV completion. The contributions form the top quark loop are

$$V_{\text{top}} = -\kappa f^4 \left[\lambda_1^2 |\Omega_1|^2 + \lambda_2^2 |\Omega_2|^2 + 2\lambda_1\lambda_2 \text{Re}(\Omega_1^\dagger \Omega_2) \right],$$

$$\begin{aligned} |\Omega_1|^2 &= \frac{2}{f^2} \left(2\rho^2 + K^\dagger K \right) \\ &\quad - \frac{2i}{f^3} \left(\rho H^\dagger K - \rho K^\dagger H + K^T \phi^\dagger K - K^\dagger \phi K^* \right) + \mathcal{O}\left(\frac{1}{f^4} \right), \end{aligned} \tag{D.2}$$

$$\begin{aligned} |\Omega_2|^2 &= \frac{8}{f^2} \left(2\text{Tr}\, \phi\phi^\dagger + 2\rho^2 + K^\dagger K \right) \\ &\quad + \frac{8i}{f^3} \left(\rho H^\dagger K - \rho K^\dagger H - H^T \phi^\dagger H + H^\dagger \phi H^* \right) + \mathcal{O}\left(\frac{1}{f^4} \right), \end{aligned} \tag{D.3}$$

$$\text{Re}(\Omega_1^\dagger \Omega_2) = \mathcal{O}\left(\frac{1}{f^5} \right), \tag{D.4}$$

where κ is the order one constant depends on the UV completion.

This appendix is a reprint of material as it appears in "Spontaneous CP Violation and Light Particles in The Littlest Higgs," Z. Surujon and P. Uttayarat, Phys. Rev. D **83**, 076010 (2011) [arXiv:1003.4779 [hep-ph]], of which I was a co-author.

Bibliography

[1] M. Schmaltz and D. Tucker-Smith, *Little Higgs review*, *Ann.Rev.Nucl.Part.Sci.* **55** (2005) 229–270, [hep-ph/0502182].

[2] M. Perelstein, *Little Higgs models and their phenomenology*, *Prog.Part.Nucl.Phys.* **58** (2007) 247–291, [hep-ph/0512128].

[3] H. Georgi, *The Higgs as a pseudo-Goldstone boson*, *Comptes Rendus Physique* **8** (2007) 1029–1047.

[4] N. Arkani-Hamed, A. Cohen, E. Katz, and A. Nelson, *The Littlest Higgs*, *JHEP* **0207** (2002) 034, [hep-ph/0206021].

[5] J. Preskill, *Subgroup Alignment in Hypercolor Theories*, *Nucl.Phys.* **B177** (1981) 21–59.

[6] M. E. Peskin, *The Alignment of the Vacuum in Theories of Technicolor*, *Nucl.Phys.* **B175** (1980) 197–233.

[7] C. Csaki, J. Hubisz, G. D. Kribs, P. Meade, and J. Terning, *Big corrections from a little Higgs*, *Phys.Rev.* **D67** (2003) 115002, [hep-ph/0211124].

[8] J. L. Hewett, F. J. Petriello, and T. G. Rizzo, *Constraining the littlest Higgs*, *JHEP* **0310** (2003) 062, [hep-ph/0211218].

[9] T. Han, H. E. Logan, B. McElrath, and L.-T. Wang, *Phenomenology of the little Higgs model*, *Phys.Rev.* **D67** (2003) 095004, [hep-ph/0301040].

[10] C. Csaki, J. Hubisz, G. D. Kribs, P. Meade, and J. Terning, *Variations of little Higgs models and their electroweak constraints*, *Phys.Rev.* **D68** (2003) 035009, [hep-ph/0303236].

[11] R. Barbieri, A. Pomarol, R. Rattazzi, and A. Strumia, *Electroweak symmetry breaking after LEP-1 and LEP-2*, *Nucl.Phys.* **B703** (2004) 127–146, [hep-ph/0405040].

[12] S. Chang and J. G. Wacker, *Little Higgs and custodial SU(2)*, *Phys.Rev.* **D69** (2004) 035002, [hep-ph/0303001].

[13] H.-C. Cheng and I. Low, *TeV symmetry and the little hierarchy problem*, *JHEP* **0309** (2003) 051, [hep-ph/0308199].

[14] H.-C. Cheng and I. Low, *Little hierarchy, little Higgses, and a little symmetry*, *JHEP* **0408** (2004) 061, [hep-ph/0405243].

[15] H.-C. Cheng, I. Low, and L.-T. Wang, *Top partners in little Higgs theories with T-parity*, *Phys.Rev.* **D74** (2006) 055001, [hep-ph/0510225].

[16] B. Grinstein and M. Trott, *Top quark induced vacuum misalignment in little higgs models*, *JHEP* **0811** (2008) 064, [arXiv:0808.2814].

[17] B. Grinstein, R. Kelley, and P. Uttayarat, *One Loop Renormalization of the Littlest Higgs Model*, *JHEP* **1102** (2011) 089, [arXiv:1011.0682].

[18] T. Cheng, E. Eichten, and L.-F. Li, *Higgs Phenomena in Asymptotically Free Gauge Theories*, *Phys.Rev.* **D9** (1974) 2259.

[19] M. E. Machacek and M. T. Vaughn, *Two Loop Renormalization Group Equations in a General Quantum Field Theory. 1. Wave Function Renormalization*, *Nucl.Phys.* **B222** (1983) 83.

[20] D. B. Kaplan and H. Georgi, *SU(2) x U(1) Breaking by Vacuum Misalignment*, *Phys.Lett.* **B136** (1984) 183.

[21] D. B. Kaplan, H. Georgi, and S. Dimopoulos, *Composite Higgs Scalars*, *Phys.Lett.* **B136** (1984) 187.

[22] M. Perelstein, M. E. Peskin, and A. Pierce, *Top quarks and electroweak symmetry breaking in little Higgs models*, *Phys.Rev.* **D69** (2004) 075002, [hep-ph/0310039]. 28 pages, 6 figures: new ST fits (Fig. 3,4) Report-no: SLAC-PUB-10185.

[23] S. Chang, *A 'Littlest Higgs' model with custodial SU(2) symmetry*, *JHEP* **0312** (2003) 057, [hep-ph/0306034].

[24] B. Grinstein, R. Kelley, and P. Uttayarat, *Hidden fine tuning in the quark sector of little higgs models*, *JHEP* **0909** (2009) 040, [arXiv:0904.1622].

[25] J. Espinosa and J. No, *Scalar loops in little Higgs models*, *JHEP* **0701** (2007) 006, [hep-ph/0610255].

[26] N. Arkani-Hamed, A. G. Cohen, and H. Georgi, *Electroweak symmetry breaking from dimensional deconstruction*, *Phys.Lett.* **B513** (2001) 232–240, [hep-ph/0105239].

[27] N. Arkani-Hamed, A. G. Cohen, T. Gregoire, and J. G. Wacker, *Phenomenology of electroweak symmetry breaking from theory space*, *JHEP* **0208** (2002) 020, [hep-ph/0202089].

[28] H. E. Haber and Z. Surujon, *A Group-theoretic Condition for Spontaneous CP Violation*, arXiv:1201.1730.

[29] S. R. Coleman, J. Wess, and B. Zumino, *Structure of phenomenological Lagrangians. 1.*, *Phys.Rev.* **177** (1969) 2239–2247.

[30] J. Callan, Curtis G., S. R. Coleman, J. Wess, and B. Zumino, *Structure of phenomenological Lagrangians. 2.*, *Phys.Rev.* **177** (1969) 2247–2250.

[31] I. Low, *T parity and the littlest Higgs*, *JHEP* **0410** (2004) 067, [hep-ph/0409025].

[32] J. Hubisz, S. J. Lee, and G. Paz, *The Flavor of a little Higgs with T-parity*, *JHEP* **0606** (2006) 041, [hep-ph/0512169]. 29 pages, 7 figures.

[33] M. Blanke, A. J. Buras, A. Poschenrieder, S. Recksiegel, C. Tarantino, et al., *Another look at the flavour structure of the littlest Higgs model with T-parity*, *Phys.Lett.* **B646** (2007) 253–257, [hep-ph/0609284].

[34] C. T. Hill and R. J. Hill, *T^- parity violation by anomalies*, *Phys.Rev.* **D76** (2007) 115014, [arXiv:0705.0697].

[35] D. Krohn and I. Yavin, *Anomalies in Fermionic UV Completions of Little Higgs Models*, *JHEP* **0806** (2008) 092, [arXiv:0803.4202].

[36] C. Csaki, J. Heinonen, M. Perelstein, and C. Spethmann, *A Weakly Coupled Ultraviolet Completion of the Littlest Higgs with T-parity*, *Phys.Rev.* **D79** (2009) 035014, [arXiv:0804.0622].

[37] T. Brown, C. Frugiuele, and T. Gregoire, *UV friendly T-parity in the SU(6)/Sp(6) little Higgs model*, *JHEP* **1106** (2011) 108, [arXiv:1012.2060].

[38] W. Kilian, J. Reuter, and D. Rainwater, *Pseudo-axions in little Higgs models*, *eConf* **C050318** (2005) 0109, [hep-ph/0507081].

[39] N. Arkani-Hamed, A. Cohen, E. Katz, A. Nelson, T. Gregoire, et al., *The Minimal moose for a little Higgs*, *JHEP* **0208** (2002) 021, [hep-ph/0206020].

[40] M. Schmaltz, *The Simplest little Higgs*, *JHEP* **0408** (2004) 056, [hep-ph/0407143].

[41] S. J. Huber, M. Pospelov, and A. Ritz, *Electric dipole moment constraints on minimal electroweak baryogenesis*, *Phys.Rev.* **D75** (2007) 036006, [hep-ph/0610003].

[42] S. R. Coleman and E. J. Weinberg, *Radiative Corrections as the Origin of Spontaneous Symmetry Breaking*, *Phys.Rev.* **D7** (1973) 1888–1910.

[43] T. Banks and A. Zaks, *On the Phase Structure of Vector-Like Gauge Theories with Massless Fermions*, *Nucl.Phys.* **B196** (1982) 189.

[44] A. Belavin and A. Migdal, *Calculation of anomalous dimensions in non-abelian gauge field theories*, *Pisma Zh.Eksp.Teor.Fiz.* **19** (1974) 317–320.

[45] T. Appelquist and Y. Bai, *A Light Dilaton in Walking Gauge Theories*, *Phys.Rev.* **D82** (2010) 071701, [`arXiv:1006.4375`].

[46] M. Hashimoto and K. Yamawaki, *Techni-dilaton at Conformal Edge*, *Phys.Rev.* **D83** (2011) 015008, [`arXiv:1009.5482`].

[47] W. A. Bardeen, C. N. Leung, and S. Love, *The Dilaton and Chiral Symmetry Breaking*, *Phys.Rev.Lett.* **56** (1986) 1230.

[48] T. Clark, C. N. Leung, and S. Love, *PROPERTIES OF THE DILATON*, *Phys.Rev.* **D35** (1987) 997.

[49] C. N. Leung, S. Love, and W. A. Bardeen, *Aspects of Dynamical Symmetry Breaking in Gauge Field Theories*, *Nucl.Phys.* **B323** (1989) 493.

[50] K. Yamawaki, M. Bando, and K.-i. Matumoto, *Scale Invariant Technicolor Model and a Technidilaton*, *Phys.Rev.Lett.* **56** (1986) 1335.

[51] M. Bando, K.-i. Matumoto, and K. Yamawaki, *TECHNIDILATON*, *Phys.Lett.* **B178** (1986) 308.

[52] T. Nonoyama, T. B. Suzuki, and K. Yamawaki, *PHASE STRUCTURE AND DILATON IN QED WITH FOUR FERMION INTERACTION*, *Prog.Theor.Phys.* **81** (1989) 1238.

[53] B. Holdom and J. Terning, *A Light Dilaton in Gauge Theories?*, *Phys.Lett.* **B187** (1987) 357.

[54] B. Holdom and J. Terning, *NO LIGHT DILATON IN GAUGE THEORIES*, *Phys.Lett.* **B200** (1988) 338–342.

[55] Z. Fodor, K. Holland, J. Kuti, D. Nogradi, and C. Schroeder, *Chiral symmetry breaking in fundamental and sextet fermion representations of SU(3) color*, `arXiv:1103.5998`.

[56] Z. Fodor, K. Holland, J. Kuti, D. Nogradi, C. Schroeder, et al., *Twelve massless flavors and three colors below the conformal window*, *Phys.Lett.* **B703** (2011) 348–358, [`arXiv:1104.3124`].

[57] A. Deuzeman, M. Lombardo, and E. Pallante, *Evidence for a conformal phase in SU(N) gauge theories*, *Phys.Rev.* **D82** (2010) 074503, [`arXiv:0904.4662`].

[58] A. Hasenfratz, *Conformal or Walking? Monte Carlo renormalization group studies of SU(3) gauge models with fundamental fermions*, *Phys.Rev.* **D82** (2010) 014506, [`arXiv:1004.1004`].

[59] S. Catterall, J. Giedt, F. Sannino, and J. Schneible, *Probes of nearly conformal behavior in lattice simulations of minimal walking technicolor*, arXiv:0910.4387.

[60] A. J. Hietanen, J. Rantaharju, K. Rummukainen, and K. Tuominen, *Spectrum of SU(2) lattice gauge theory with two adjoint Dirac flavours*, *JHEP* **0905** (2009) 025, [arXiv:0812.1467].

[61] A. J. Hietanen, K. Rummukainen, and K. Tuominen, *Evolution of the coupling constant in SU(2) lattice gauge theory with two adjoint fermions*, *Phys.Rev.* **D80** (2009) 094504, [arXiv:0904.0864].

[62] F. Bursa, L. Del Debbio, D. Henty, E. Kerrane, B. Lucini, et al., *Improved Lattice Spectroscopy of Minimal Walking Technicolor*, *Phys.Rev.* **D84** (2011) 034506, [arXiv:1104.4301].

[63] F. Bursa, L. Del Debbio, L. Keegan, C. Pica, and T. Pickup, *Mass anomalous dimension in SU(2) with two adjoint fermions*, *Phys.Rev.* **D81** (2010) 014505, [arXiv:0910.4535].

[64] F. Bursa, L. Del Debbio, L. Keegan, C. Pica, and T. Pickup, *Mass anomalous dimension in SU(2) with six fundamental fermions*, *Phys.Lett.* **B696** (2011) 374–379, [arXiv:1007.3067].

[65] J. Andersen, O. Antipin, G. Azuelos, L. Del Debbio, E. Del Nobile, et al., *Discovering Technicolor*, *Eur.Phys.J.Plus* **126** (2011) 81, [arXiv:1104.1255].

[66] L. Del Debbio, B. Lucini, A. Patella, C. Pica, and A. Rago, *Mesonic spectroscopy of Minimal Walking Technicolor*, *Phys.Rev.* **D82** (2010) 014509, [arXiv:1004.3197].

[67] L. Del Debbio, B. Lucini, A. Patella, C. Pica, and A. Rago, *The infrared dynamics of Minimal Walking Technicolor*, *Phys.Rev.* **D82** (2010) 014510, [arXiv:1004.3206].

[68] T. DeGrand, Y. Shamir, and B. Svetitsky, *Infrared fixed point in SU(2) gauge theory with adjoint fermions*, *Phys.Rev.* **D83** (2011) 074507, [arXiv:1102.2843].

[69] T. DeGrand, Y. Shamir, and B. Svetitsky, *Running coupling and mass anomalous dimension of SU(3) gauge theory with two flavors of symmetric-representation fermions*, *Phys.Rev.* **D82** (2010) 054503, [arXiv:1006.0707].

[70] T. DeGrand, Y. Shamir, and B. Svetitsky, *Phase structure of SU(3) gauge theory with two flavors of symmetric-representation fermions*, *Phys.Rev.* **D79** (2009) 034501, [arXiv:0812.1427].

[71] J. Kogut and D. Sinclair, *Thermodynamics of lattice QCD with 2 flavours of colour-sextet quarks: A model of walking/conformal Technicolor*, *Phys.Rev.* **D81** (2010) 114507, [arXiv:1002.2988].

[72] D. Sinclair and J. Kogut, *New results with colour-sextet quarks*, *PoS* **LATTICE2010** (2010) 071, [arXiv:1008.2468].

[73] M. Hayakawa, K.-I. Ishikawa, Y. Osaki, S. Takeda, S. Uno, et al., *Running coupling constant of ten-flavor QCD with the Schródinger functional method*, *Phys.Rev.* **D83** (2011) 074509, [arXiv:1011.2577].

[74] N. Yamada, M. Hayakawa, K.-I. Ishikawa, Y. Osaki, S. Takeda, et al., *Study of the running coupling constant in 10-flavor QCD with the Schródinger functional method*, arXiv:1003.3288.

[75] H. Ohki, T. Aoyama, E. Itou, M. Kurachi, C.-J. Lin, et al., *Study of the scaling properties in SU(2) gauge theory with eight flavors*, *PoS* **LATTICE2010** (2010) 066, [arXiv:1011.0373].

[76] X.-Y. Jin and R. D. Mawhinney, *Lattice QCD with 8 and 12 degenerate quark flavors*, *PoS* **LAT2009** (2009) 049, [arXiv:0910.3216].

[77] E. Bilgici, A. Flachi, E. Itou, M. Kurachi, C.-J. Lin, et al., *Search for the IR fixed point in the twisted Polyakov loop scheme*, *PoS* **LAT2009** (2009) 063, [arXiv:0910.4196].

[78] R. Sundrum, *Gravity's scalar cousin*, hep-th/0312212.

[79] L. Randall and R. Sundrum, *A Large mass hierarchy from a small extra dimension*, *Phys.Rev.Lett.* **83** (1999) 3370–3373, [hep-ph/9905221].

[80] W. D. Goldberger and M. B. Wise, *Modulus stabilization with bulk fields*, *Phys.Rev.Lett.* **83** (1999) 4922–4925, [hep-ph/9907447].

[81] N. Arkani-Hamed, M. Porrati, and L. Randall, *Holography and phenomenology*, *JHEP* **0108** (2001) 017, [hep-th/0012148].

[82] R. Rattazzi and A. Zaffaroni, *Comments on the holographic picture of the Randall-Sundrum model*, *JHEP* **0104** (2001) 021, [hep-th/0012248].

[83] W. D. Goldberger, B. Grinstein, and W. Skiba, *Distinguishing the Higgs boson from the dilaton at the Large Hadron Collider*, *Phys.Rev.Lett.* **100** (2008) 111802, [arXiv:0708.1463].

[84] J. Fan, W. D. Goldberger, A. Ross, and W. Skiba, *Standard Model couplings and collider signatures of a light scalar*, *Phys.Rev.* **D79** (2009) 035017, [arXiv:0803.2040].

[85] L. Vecchi, *Phenomenology of a light scalar: the dilaton*, *Phys.Rev.* **D82** (2010) 076009, [arXiv:1002.1721].

[86] Y. Bai, M. Carena, and J. Lykken, *Dilaton-assisted Dark Matter*, *Phys.Rev.Lett.* **103** (2009) 261803, [arXiv:0909.1319].

[87] K.-Y. Choi, D. K. Hong, and S. Matsuzaki, *Techni-dilaton as Dark Matter*, *Phys.Lett.* **B706** (2011) 183–187, [arXiv:1101.5326].

[88] I. Jack and H. Osborn, *GENERAL TWO LOOP BETA FUNCTIONS FOR GAUGE THEORIES WITH ARBITRARY SCALAR FIELDS*, *J.Phys.A* **A16** (1983) 1101.

[89] M. E. Machacek and M. T. Vaughn, *Two Loop Renormalization Group Equations in a General Quantum Field Theory. 2. Yukawa Couplings*, *Nucl.Phys.* **B236** (1984) 221.

[90] M. E. Machacek and M. T. Vaughn, *Two Loop Renormalization Group Equations in a General Quantum Field Theory. 3. Scalar Quartic Couplings*, *Nucl.Phys.* **B249** (1985) 70.

[91] R. Rattazzi, V. S. Rychkov, E. Tonni, and A. Vichi, *Bounding scalar operator dimensions in 4D CFT*, *JHEP* **0812** (2008) 031, [arXiv:0807.0004].

[92] O. Antipin, M. Mojaza, C. Pica, and F. Sannino, *Magnetic Fixed Points and Emergent Supersymmetry*, arXiv:1105.1510.

[93] J. Callan, Curtis G., S. R. Coleman, and R. Jackiw, *A New improved energy - momentum tensor*, *Annals Phys.* **59** (1970) 42–73.

[94] S. R. Coleman and R. Jackiw, *Why dilatation generators do not generate dilatations?*, *Annals Phys.* **67** (1971) 552–598.

[95] S. L. Adler, J. C. Collins, and A. Duncan, *Energy-Momentum-Tensor Trace Anomaly in Spin 1/2 Quantum Electrodynamics*, *Phys.Rev.* **D15** (1977) 1712.

[96] H. D. Politzer, *Power Corrections at Short Distances*, *Nucl.Phys.* **B172** (1980) 349.

[97] B. Grinstein and L. Randall, *The Renormalization of g^2*, *Phys.Lett.* **B217** (1989) 335.

[98] D. G. Robertson, *Composite operator renormalization and the trace anomaly*, *Phys.Lett.* **B253** (1991) 143–148.

[99] M. J. Duncan, R. Philippe, and M. Sher, *THEORETICAL CEILING ON QUARK MASSES IN THE STANDARD MODEL*, *Phys.Lett.* **B153** (1985) 165.

[100] D. D. Dietrich, F. Sannino, and K. Tuominen, *Light composite Higgs from higher representations versus electroweak precision measurements: Predictions for CERN LHC*, *Phys.Rev.* **D72** (2005) 055001, [hep-ph/0505059].

[101] L. Vecchi, *Multitrace deformations, Gamow states, and Stability of AdS/CFT*, *JHEP* **1104** (2011) 056, [arXiv:1005.4921].

[102] L. Vecchi, *Technicolor at Criticality*, *JHEP* **1104** (2011) 127, [arXiv:1007.4573].

[103] H. Georgi, *Lie Algebras in Particle Physics.* Frontiers in Physics. Perseus Books, Advanced Book Program, 1999.