Title
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Authors
Hung, Fan-Yun
Yu, Chia-Fu

Publication Date
2024-05-01

DOI
10.1016/j.jnt.2023.11.013

CLASS NUMBERS OF MULTINORM-ONE TORI

FAN-YUN HUNG AND CHIA-FU YU

Abstract. We present a formula for the class number of a multinorm one torus $T_{L/k}$ associated to any étale algebra L over a global field k. This is deduced from a formula for analogues of invariants introduced by T. Ono, which are interpreted as a generalization of Gauss genus theory. This paper includes the variants of Ono's invariant for arbitrary S-ideal class numbers and the narrow version, generalizing results of Katayama, Morishita, Sasaki and Ono.

1. Introduction

Let K/k be a finite extension of number fields. Ono [20, 21] introduced the following alternating products of class numbers

$$E(K/k) := \frac{h(K)}{h(k)\cdot h(R^{(1)}_{K/k}\mathbb{G}_{\mathrm{m},K})} \quad \text{and} \quad E^+(K/k) := \frac{h^+(K)}{h^+(k)\cdot h^+(R^{(1)}_{K/k}\mathbb{G}_{\mathrm{m},K})}, \tag{1.1}$$

where $R_{K/k}$ denotes the Weil restriction of scalars from K to k, $R^{(1)}_{K/k}\mathbb{G}_{\mathrm{m},K} \subset R_{K/k}\mathbb{G}_{\mathrm{m},K}$ is the norm one torus, and h (resp. h^+) denotes the class number (resp. the narrow class number). When the extension K/k is Galois, Ono computed these invariants in terms of cohomological invariants [21, Section 2, Theorem, p. 123] and thus gave a class number relation among $h(K)$, $h(k)$ and $h(R^{(1)}_{K/k}\mathbb{G}_{\mathrm{m},K})$, as well as their narrow variants. Particularly, this yields a formula for the class number $h(R^{(1)}_{K/k}\mathbb{G}_{\mathrm{m},K})$ of the norm one torus $R^{(1)}_{K/k}\mathbb{G}_{\mathrm{m},K}$. Restricted to the special case where K is a CM field and $k = K^+$ is its maximal totally real subfield, Ono's formula gives an alternative proof of the following class number formula (see [26, (16), p. 375])

$$h(R^{(1)}_{K/K^+}\mathbb{G}_{\mathrm{m},K}) = \frac{h_K}{h_{K^+}}\frac{1}{Q_{K/K^+}\cdot 2^{t-1}}, \tag{1.2}$$

where $Q_{K/K^+} = [O_K^\times : \mu_K O_{K^+}^\times]$ is the Hasse unit index and t is the number of finite places of K^+ ramified in K. This formula was applied to compute the number of polarized CM complex abelian varieties in [6].

Ono observed that the class number relation deduced from $E^+(K/k)$ generalizes Gauss's theorem on the genera of quadratic forms. For example, in the simplest case where K is any quadratic extension of $k = \mathbb{Q}$, Ono's formula for $E^+(K/k)$ reads

$$h^+(K) = h^*_K \cdot 2^{t-1},$$

where t is the number of rational primes ramified in K and h^*_K is the class number of any genus in the narrow ideal class group $\mathrm{Cl}^+(K)$. On the other hand, when K/k is any cyclic Kummer extension, Ono's formula shows a direct relation with the ambiguous class number for K/k; see [20, Equation (10)] for details. We refer for a few explorations of ambiguous class numbers to [12, Chapter XIII, Section 4], [5] and [14].

Date: July 24, 2023.
2010 *Mathematics Subject Classification.* 14K22, 11R29.
Key words and phrases. Class numbers, multinorm-one tori.

There are generalizations and extensions of Ono's work by several other authors. In [25] Sasaki gave a more direct proof of Ono's formulas which avoids K-theory. The formulas were generalized by Katayama [9, 10] for any finite extension K/k using Ono-Shyr's formula [26] for isogenous tori. Morishita [16] generalized Ono's formula to the S-arithmetical setting including the function field case (still, as Sasaki and Ono, assuming K/k Galois). He adopted a new approach using Nisnevich cohomology and gave a different approach. As another generalization, Morishita also showed a formula for the Ono invariant associated to the product $K_1 \times K_2$ of two linearly disjoint Galois extensions K_1 and K_2, relating to Hürlimann's result [8] on the Hasse norm principle for $K_1 \times K_2$.

In this paper we generalize the results of Ono, Sasaki, Katayama and Morishita to an arbitrary étale algebra over any global function k, including an arbitrary S-arithmetical setting (i.e. S is nonempty or not). Our approach is close to that of Sasaki, which does not rely on K-theory nor the Nisnevich cohomology and is more elementary.

Let $L = \prod_{i=1}^r K_i$ be an étale algebra over a global field k with finite separable field extensions K_i/k, and let $N_{L/k} = \prod_{i=1}^r N_{K_i/k}$ be the norm map from L to k. Set

$$T_{L/k} := \ker(N_{L/k} : T^L := R_{L/k}\mathbb{G}_{\mathrm{m},L} \to \mathbb{G}_{\mathrm{m},k}),$$

called the *multinorm-one torus* $T_{L/k}$ associated to L/k. For simplicity, we write N for $N_{L/k}$.

Let $\mathbb{A}_k$ and $\mathbb{A}_L := \prod_i \mathbb{A}_{K_i}$ be the adele rings of k and L, respectively. Let S be a nonempty finite set of places of k which contains all archimedean places if k is a number field. Let $\mathbb{A}_{k,S}$ and $\mathbb{A}_{L,S}$ be the S-adele rings of k and L, respectively; see (2.1). Let $O_{k,S} := k \cap \mathbb{A}_{k,S}$ and $O_{L,S} := L \cap \mathbb{A}_{L,S}$ be the S-rings of integers in k and L, respectively. Denote by $U_{k,S} := \mathbb{A}_{k,S}^\times$ (resp. $U_{L,S} := \mathbb{A}_{L,S}^\times$) the unit group of $\mathbb{A}_{k,S}$ (resp. $\mathbb{A}_{L,S}$).

For any k-torus T, let

$$\mathrm{Cl}_S(T) := \frac{T(\mathbb{A}_k)}{T(k)U_{T,S}} \quad \text{and } h_S(T) := |\,\mathrm{Cl}_S(T)|$$

denote the *S-class group* and *S-class number* of T, respectively; see (2.2). If k is a number field, we let $\mathrm{Cl}_S^+(T)$ and $h_S^+(T)$ denote the *narrow S-class group* and *narrow S-class number* of T, respectively; see (2.4). Following Ono, we define the following alternating products:

$$E_S(L/k) := \frac{h_S(L)}{h_S(k)h_S(T_{L/k})} \quad \text{and} \quad E_S^+(L/k) := \frac{h_S^+(L)}{h_S^+(k)h_S^+(T_{L/k})}, \tag{1.3}$$

where $h_S^{(+)}(L) := h_S^{(+)}(T^L)$ and $h_S^{(+)}(k) := h_S^{(+)}(\mathbb{G}_{\mathrm{m},k})$ are the (narrow) S-class numbers of L and k, respectively.

In the case where k is a global function field and $S = \emptyset$, the class group $\mathrm{Cl}_\emptyset(T) =: \mathrm{Cl}(T)$ of a k-torus T may be infinite. So instead we consider the class group $\mathrm{Cl}^0(T) \subset \mathrm{Cl}(T)$ of degree zero of T (see (3.1)) and the class number $h^0(T) := |\,\mathrm{Cl}^0(T)|$ of degree zero. Set

$$E^0(L/k) := \frac{h^0(L)}{h^0(k)h^0(T_{L/k})}, \tag{1.4}$$

where $h^0(L) := h^0(T^L)$, $h^0(k) := |\mathbb{A}_k^{\times,0}/k^\times \cdot U_k|$ and $U_k = \prod_v O_{k_v}^\times$. Let $U_L := \prod_{i=1}^r U_{K_i}^\times$ be the unit subgroup of $\mathbb{A}_L^\times$.

To describe our main results, we set some more notation. Let

$$Ш(L/k) := \frac{k^\times \cap N(\mathbb{A}_L^\times)}{N(L^\times)} \tag{1.5}$$

denote the Tate-Shafarevich group of L/k. For any map $\alpha : A \to B$ of abelian groups, the q-symbol of α is defined by

$$q(\alpha) := \frac{|\operatorname{coker}\alpha|}{|\ker\alpha|} \tag{1.6}$$

if both $\operatorname{coker}\alpha$ and $\ker\alpha$ are finite. If k is a number field, we refer to (2.3) for the definition of subgroups $\mathbb{A}_k^{\times,+} \subset \mathbb{A}_k^\times$ and $\mathbb{A}_L^{\times,+} \subset \mathbb{A}_L^\times$, and for any subgroup $A \subset \mathbb{A}_k^\times$ (resp. $A \subset \mathbb{A}_L^\times$), set $A^+ := A \cap \mathbb{A}_k^{\times,+}$ (resp. $A^+ := A \cap \mathbb{A}_L^{\times,+}$).

Our main results give formulas for the invariants $E_S(L/k)$, $E_S^+(L/k)$ and $E^0(L/k)$:

Theorem 1.1 (Theorems 2.6 and 3.5)**.** *Let L be an étale algebra over a global field k.*

(1) When S is nonempty, we have

$$E_S(L/k) = \frac{|\text{Ш}(L/k)|}{[L_{ab} : k]} \cdot \frac{[U_{k,S} : N(U_{L,S})]}{[O_{k,S}^\times : N(O_{L,S}^\times)]} \tag{1.7}$$

and

$$E_S^+(L/k) = \frac{|\text{Ш}(L/k)|}{[L_{ab} : k] \cdot q(\phi)} \cdot \frac{[U_{k,S} : N(U_{L,S})]}{[O_{k,S}^{\times+} : N(O_{L,S}^{\times+})]}, \tag{1.8}$$

where L_{ab} is the maximal abelian extension of k that is contained in all K_i and $\phi : k^{\times+}/N(L^{\times+}) \to k^\times/N(L^\times)$ is the canonical homomorphism.

(2) When k is a global function field and S is empty, we have

$$E^0(L/k) = \frac{|\text{Ш}(L/k)|}{[L_{ab} : k]} \cdot q(\phi^0) \cdot \frac{[U_k : N(U_L)]}{[\mathbb{F}_q^\times : N(\prod_i \mathbb{F}_{q_i}^\times)]}, \tag{1.9}$$

where $\phi^0 : \mathbb{A}_k^{\times,0}/N(\mathbb{A}_L^{\times,0}) \to \mathbb{A}_k^\times/N(\mathbb{A}_L^\times)$ is the canonical homomorphism.

When k is a number field and $S = \infty$ is the set of archimedean places, the class number $h_S(T)$ will be denoted by $h(T)$.

Corollary 1.2. *Let $L = \prod_{i=1} K_i$ be an étale algebra over a number field and $T_{L/k}$ be the associated multinorm-one k-torus. Then*

$$h(T_{L/k}) = \frac{h(L)}{h(k)} \cdot \frac{[L_{ab} : k]}{|\text{Ш}(L/k)|} \cdot \frac{[O_k^\times : N(O_L^\times)]}{[U_k : N(U_L)]}. \tag{1.10}$$

Using Ono's formula on Tamagawa numbers of tori [19, Main Theorem, p. 68] (also see [17, Chapter IV, Corollary 3.3, p. 56]), one observes that $[L_{ab} : k]/|\text{Ш}(L/k)|$ is equal to the Tamagawa number $\tau(T_{L/k})$ of $T_{L/k}$. The formula (1.10) can also be written as

$$h(T_{L/k}) = \frac{h(L)}{h(k)} \cdot \tau(T_{L/k}) \cdot \frac{[O_k^\times : N(O_L^\times)]}{[U_k : N(U_L)]}. \tag{1.11}$$

We give the proof of Theorem 1.1 in Sections 2 and 3. In Section 4, we explore the terms of our formulas in Theorem 1.1 and give a few examples. Some of them are classical computational problems, for example, computing the unit group $O_k^\times$ of O_k. We also discuss some very recent results on the group $\text{Ш}(L/k)$ and indicate particularly that the term $|\text{Ш}_k(T')|$ in the main theorem of Morishita [16, p. 135] is always equal to 1.

2. S-class numbers of multinorm-one tori

Let k be a global field and k_s a fixed separable closure of k. Let $L = \prod_{i=1}^r K_i$ be an étale k-algebra, where each K_i is a separable field extension of k in k_s. Let $\mathbb{A}_k$ and $\mathbb{A}_L = \prod_{i=1}^r \mathbb{A}_{K_i}$ be the adele rings of k and L, respectively. For each place v of k, denote by k_v the completion of k at v and set $L_v := L \otimes_k k_v = \prod_{w|v} L_w$. Here a place w of L is a place of K_i for some i and its completion L_w is simply $K_{i,w}$. If v is finite, let O_v be the valuation ring of k_v and O_{L_v} the maximal O_v-order of L_v, which is the product of the valuation rings O_{L_w} of L_w for all $w|v$.

Let S be a nonempty finite set of places of k which contains all archimedean places if k is a number field. The S-adele rings of k and L are

$$\mathbb{A}_{k,S} := \prod_{v\in S} k_v \times \prod_{v\notin S} O_v, \quad \text{and} \quad \mathbb{A}_{L,S} := \prod_{v\in S} L_v \times \prod_{v\notin S} O_{L_v}. \tag{2.1}$$

Let $O_{k,S} := k \cap \mathbb{A}_{k,S}$ and $O_{L,S} := L \cap \mathbb{A}_{L,S}$ be the S-rings of integers in k and L, respectively. Denote by $U_{k,S} := \mathbb{A}_{k,S}^\times$ (resp. $U_{L,S} := \mathbb{A}_{L,S}^\times$) the unit group of $\mathbb{A}_{k,S}$ (resp. $\mathbb{A}_{L,S}$). Let $N_{L/k} : \mathbb{A}_L^\times \to \mathbb{A}_k^\times$ be the norm map, and let $\mathbb{A}_L^{(1)} := \ker N_{L/k} \subset \mathbb{A}_L^\times$ be the norm one subgroup. For any subgroup $A \subset \mathbb{A}_L^\times$, denote by $A^{(1)} := A \cap \mathbb{A}_L^{(1)}$ its norm one subgroup.

The *S-class group* and *S-class number* of a k-torus T are defined as

$$\mathrm{Cl}_S(T) := \frac{T(\mathbb{A}_k)}{T(k)U_{T,S}}, \quad h_S(T) := |\mathrm{Cl}_S(T)|, \tag{2.2}$$

where $U_{T,S} := T(\mathbb{A}_{k,S}) = \prod_{v\in S} T(k_v) \times \prod_{v\notin S} T(O_v)$ is the S-unit subgroup of $T(\mathbb{A}_k)$.

When k is a number field, we let

$$T(\mathbb{A}_k)^+ \subset T(\mathbb{A}_k) \tag{2.3}$$

denote the subgroup consisting of elements (x_v), such that x_v lies in the neutral component $T(k_v)^0$ of the Lie group $T(k_v)$ for all real places v. For any subgroup $A \subset T(\mathbb{A}_k)$, define $A^+ := A \cap T(\mathbb{A}_k)^+$. The *narrow S-class group* and *narrow S-class number* of a k-torus T are defined as

$$\mathrm{Cl}_S^+(T) := \frac{T(\mathbb{A}_k)}{T(k)U_{T,S}^+}, \quad h_S^+(T) := |\mathrm{Cl}_S^+(T)|. \tag{2.4}$$

Denote by $\mathbb{A}_k^S := \prod'_{v\notin S} k_v$ the prime-to-S adele ring of k.

Lemma 2.1. *If k is a number field, we have $\mathrm{Cl}_S^+(T) \simeq T(\mathbb{A}_k)/T(k)^+U_{T,S}$.*

Proof. Clearly, $T(\mathbb{A}_k)/T(k)^+U_{T,S} = T(\mathbb{A}_k^S)/T(k)^+U_T^S$, where $U_T^S = \prod_{v\notin S} T(O_v)$ is the maximal open subgroup of $T(\mathbb{A}_k^S)$. Since T is connected, real approximation implies that $T(k) \subset T(k_\infty) = \prod_{v|\infty} T(k_v)$ is dense. Thus, $T(\mathbb{A}_k) = T(k)T(\mathbb{A}_k)^+$ and the surjective map $T(\mathbb{A}_k)^+ \to T(k)T(\mathbb{A}_k)^+/T(k)U_{T,S}^+$ induces an isomorphism $T(\mathbb{A}_k)^+/T(k)^+U_{T,S}^+ \xrightarrow{\sim} T(\mathbb{A}_k)/T(k)U_{T,S}^+$, while the left hand side is $T(\mathbb{A}_k^S)/T(k)^+U_T^S$. This proves the lemma. ∎

Let $I_{L,S} = \prod_{i=1}^r I_{K_i,S}$ (resp. $P_{L,S} = \prod_{i=1}^r P_{K_i,S}$, $P_{L,S}^+ = \prod_{i=1}^r P_{K_i,S}^+$) be the group of fractional ideals (resp. principal ideals, principal ideals generated by totally positive elements) of L which are prime to S. We set

$$\mathrm{Cl}_S(L) := \prod_{i=1}^r \mathrm{Cl}_S(K_i) = \prod_{i=1}^r I_{K_i,S}/P_{K_i,S}$$

(resp. $\mathrm{Cl}_S^+(L) := \prod_{i=1}^r \mathrm{Cl}_S^+(K_i) = \prod_{i=1}^r I_{K_i,S}/P_{K_i,S}^+$) to be the S-ideal class group (resp. narrow S-ideal class group) of L. We denote by

$$h_S(L) := |\,\mathrm{Cl}_S(L)| \ \text{ and } \ h_S^+(L) := |\,\mathrm{Cl}_S^+(L)|$$

the S-class number and narrow S-class number of L, respectively. Furthermore, we denote by $I^{(1)}_{L,S}$ the kernel of the norm map $N_{L/k}: I_{L,S} \to I_{k,S}$ and write $P^{(1)}_{L,S} = P_{L,S} \cap I^{(1)}_{L,S}$.

If $T = R_{L/k}\mathbb{G}_{\mathrm{m},L}$, then we have $\mathrm{Cl}_S(L) = \mathrm{Cl}_S(T) = \mathbb{A}_L^\times / L^\times U_{L,S}$ and $\mathrm{Cl}^+_S(L) = \mathrm{Cl}^+_S(T) = \mathbb{A}_L^\times / L^\times U^+_{L,S}$, the S-ideal class group of L and its narrow version, and we have $h_S(T) = h_S(L)$ and $h^+_S(T) = h^+_S(L)$. Recall that $T_{L/k} := \ker\left(N_{L/k} : R_{L/k}\mathbb{G}_{\mathrm{m},L} \to \mathbb{G}_{\mathrm{m},k}\right)$ denotes the multinorm-one torus associated to L/k. We have

$$h_S(T_{L/k}) = [\mathbb{A}^{(1)}_L : L^{(1)} U^{(1)}_{L,S}] \quad \text{and} \quad h^+_S(T_{L/k}) = [\mathbb{A}^{(1)}_L : L^{(1)} U^{(1)+}_{L,S}] = [\mathbb{A}^{(1)}_L : L^{(1)+} U^{(1)}_{L,S}],$$

where $U^{(1)+}_{L,S} = U^{(1)}_{L,S} \cap U^+_{L,S}$ and $L^{(1)+} = L^{(1)} \cap \mathbb{A}^{(1)+}_L$. Following Ono, we extend the definition of the invariants in (1.1) to L/k:

Definition 2.2. Let L/k and S be as above, the *Ono invariant* and its narrow version are defined as

$$E_S(L/k) := \frac{h_S(L)}{h_S(k) h_S(T_{L/k})} \quad \text{and} \quad E^+_S(L/k) := \frac{h^+_S(L)}{h^+_S(k) h^+_S(T_{L/k})}, \tag{2.5}$$

where the narrow version is defined only when k is a number field.

Proposition 2.3. *We have*

$$h_S(T_{L/k}) = \frac{[I^{(1)}_{L,S} : P^{(1)}_{L,S}][O^\times_{k,S} \cap N(L^\times) : N(O^\times_{L,S})]}{[U_{k,S} \cap N(\mathbb{A}^\times_L) : N(U_{L,S})]}; \tag{2.6}$$

$$h^+_S(T_{L/k}) = \frac{[I^{(1)}_{L,S} : P^{(1)+}_{L,S}][O^{\times+}_{k,S} \cap N(L^{\times+}) : N(O^{\times+}_{L,S})]}{[U_{k,S} \cap N(\mathbb{A}^\times_L) : N(U_{L,S})]}. \tag{2.7}$$

PROOF. We prove (2.7); the proof of (2.6) is the same where one deletes "+" from the terms. Consider the following two commutative diagrams

$$\begin{array}{ccccccccc} 1 & \longrightarrow & U_{L,S} & \longrightarrow & \mathbb{A}^\times_L & \longrightarrow & I_{L,S} & \longrightarrow & 1 \\ & & \downarrow N & & \downarrow N & & \downarrow N & & \\ 1 & \longrightarrow & U_{k,S} & \longrightarrow & \mathbb{A}^\times_k & \longrightarrow & I_{k,S} & \longrightarrow & 1; \end{array}$$

$$\begin{array}{ccccccccc} 1 & \longrightarrow & O^{\times+}_{L,S} & \longrightarrow & L^{\times+} & \longrightarrow & P^+_{L,S} & \longrightarrow & 1 \\ & & \downarrow N & & \downarrow N & & \downarrow N & & \\ 1 & \longrightarrow & O^{\times+}_{k,S} & \longrightarrow & k^{\times+} & \longrightarrow & P^+_{k,S} & \longrightarrow & 1. \end{array}$$

The snake lemma gives the following two exact sequences

$$1 \longrightarrow U^{(1)}_{L,S} \longrightarrow \mathbb{A}^{(1)}_L \longrightarrow I^{(1)}_{L,S} \xrightarrow{\delta_1} \frac{U_{k,S}}{N(U_{L,S})} \longrightarrow \frac{\mathbb{A}^\times_k}{N(\mathbb{A}^\times_L)} \longrightarrow \cdots$$

$$1 \longrightarrow O^{(1)+}_{L,S} \longrightarrow L^{(1)+} \longrightarrow P^{(1)+}_{L,S} \xrightarrow{\delta_2} \frac{O^{\times+}_{k,S}}{N(O^{\times+}_{L,S})} \longrightarrow \frac{k^{\times+}}{N(L^{\times+})} \longrightarrow \cdots$$

Clearly, $\mathrm{Im}\,\delta_1 = \left(U_{k,S} \cap N(\mathbb{A}^\times_L)\right)/N(U_{L,S})$ and $\mathrm{Im}\,\delta_2 = \left(O^{\times+}_{k,S} \cap N(L^{\times+})\right)/N(O^{\times+}_{L,S})$ and these two abelian groups are finite. Thus, we have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{A}_L^{(1)}/U_{L,S}^{(1)} & \longrightarrow & I_{L,S}^{(1)} & \longrightarrow & \left(U_{k,S}\cap N(\mathbb{A}_L^\times)\right)/N(U_{L,S}) & \longrightarrow & 1 \\ & & {\scriptstyle f'}\uparrow & & {\scriptstyle f}\uparrow & & {\scriptstyle f''}\uparrow & & \\ 1 & \longrightarrow & L^{(1)+}/O_{L,S}^{(1)+} & \longrightarrow & P_{L,S}^{(1)+} & \longrightarrow & \left(O_{k,S}^{\times +}\cap N(L^{\times +})\right)/N(O_{L,S}^{\times +}) & \longrightarrow & 1. \end{array}$$

Therefore,

$$\begin{aligned} [I_{L,S}^{(1)} : P_{L,S}^{(1)+}] &= q(f) = q(f')q(f'') \\ &= [\mathbb{A}_L^{(1)} : L^{(1)+}U_{L,S}^{(1)}]\frac{[U_{k,S}\cap N(\mathbb{A}_L^\times) : N(U_{L,S})]}{[O_{k,S}^{\times+}\cap N(L^{\times +}) : N(O_{L,S}^{\times +})]}. \end{aligned}$$

This proves the proposition. ∎

For any k-torus T, we denote by $\widehat{T}$ its *character group* $\mathrm{Hom}_{\overline{k}}(T,\mathbb{G}_{\mathrm{m},k})$. Also, for any finite commutative group G, let $G^\vee := \mathrm{Hom}(G,\mathbb{Q}/\mathbb{Z})$ denote the Pontryagin dual.

Theorem 2.4. *There are canonical isomorphisms*

$$\frac{\mathbb{A}_k^\times}{k^\times\cdot N(\mathbb{A}_L^\times)} \simeq H^1(\widehat{T}_{L/k})^\vee \simeq \mathrm{Gal}(L_{\mathrm{ab}}/k), \tag{2.8}$$

where L_{ab} is the maximal abelian extension of k that is contained in all K_i.

Proof. Let K be a finite Galois extension of k containing all K_i and $G=\mathrm{Gal}(K/k)$. Denoting by C_K the idele class group of K, one has the exact sequence

$$1 \longrightarrow K^\times \longrightarrow \mathbb{A}_K^\times \longrightarrow C_K \longrightarrow 1.$$

Let T be a k-torus splitting over K. Following Ono in [19], we apply $\mathrm{Hom}(\widehat{T},\cdot)$ to the above exact sequence with the canonical identifications,

$$T(K)=\mathrm{Hom}(\widehat{T},K^\times),\quad T(\mathbb{A}_K)=\mathrm{Hom}(\widehat{T},\mathbb{A}_K^\times)$$

and define

$$C_K(T) := T(\mathbb{A}_K)/T(K) \simeq \mathrm{Hom}(\widehat{T},C_K).$$

The short exact sequence

$$1\longrightarrow T(K)\longrightarrow T(\mathbb{A}_K)\longrightarrow C_K(T)\longrightarrow 1$$

induces the long exact sequence

$$1\to T(k)\to T(\mathbb{A}_k)\to C_K(T)^G\to H^1(T(K))\to H^1(T(\mathbb{A}_K))\to H^1(C_K(T))\to\cdots$$

We claim that for $T=R_{L/k}\mathbb{G}_{m,L}$ or $T=\mathbb{G}_{m,k}$, we have $C_K(T)^G=T(\mathbb{A}_K)/T(k)$. If $T=\mathbb{G}_{m,k}$, this assertion holds since $H^1(G,T(K))=H^1(k,\mathbb{G}_{m,k})=0$ by Hilbert's Theorem 90. If $T=R_{L/k}\mathbb{G}_{m,L}$, for each K_i, by Shapiro's lemma we have

$$H^1(k,R_{K_i/k}\mathbb{G}_{m,K_i})=H^1(K_i,\mathbb{G}_{m,K_i})=0$$

and hence $H^1(G,T(K))=H^1\left(k,\prod_{i=1}^r R_{K_i/K}\mathbb{G}_{m,K_i}\right)=\prod_{i=1}^r H^1(k,R_{K_i/K}\mathbb{G}_{m,K_i})=0$.

Now suppose we have the following exact sequence of k-tori splitting over K:

$$1\longrightarrow T' \xrightarrow{\ \iota\ } T \xrightarrow{\ N\ } T''\longrightarrow 1. \tag{2.9}$$

Putting in $K\hookrightarrow\mathbb{A}_K$, we obtain the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & T'(K) & \xrightarrow{\iota} & T(K) & \xrightarrow{N} & T''(K) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & T'(\mathbb{A}_K) & \xrightarrow{\iota} & T(\mathbb{A}_K) & \xrightarrow{N} & T''(\mathbb{A}_K) & \longrightarrow & 1. \end{array}$$

By the snake lemma, we have the short exact sequence

$$1 \longrightarrow C_K(T') \xrightarrow{\iota} C_K(T) \xrightarrow{N} C_K(T'') \longrightarrow 1,$$

which induces the long exact sequence

$$\cdots \longrightarrow C_K(T)^G \xrightarrow{N} C_K(T'')^G \longrightarrow H^1(G, C_K(T')) \xrightarrow{\iota} H^1(G, C_K(T)) \longrightarrow \cdots$$

and the long exact sequence

$$1 \longrightarrow \operatorname{coker} N \longrightarrow H^1(G, C_K(T')) \xrightarrow{\iota} H^1(G, C_K(T)) \longrightarrow \cdots. \tag{2.10}$$

Taking (2.9) to be

$$1 \longrightarrow T_{L/k} \xrightarrow{\iota} R_{L/k}\mathbb{G}_{m,L} \xrightarrow{N} \mathbb{G}_{m,k} \longrightarrow 1, \tag{2.11}$$

we find that

$$\operatorname{coker} N = \frac{\mathbb{A}_k^\times}{k^\times \cdot N(\mathbb{A}_L^\times)}.$$

We have

$$H^1(G, C_K(R_{L/k}\mathbb{G}_{m,L})) = \prod_{i=1}^r H^1(\mathrm{Gal}(K/k), C_K(R_{K_i/k}\mathbb{G}_{m,K_i}))$$

where for each i,

$$C_K(R_{K_i/k}\mathbb{G}_{m,K_i}) = \frac{(K_i \otimes \mathbb{A}_K)^\times}{(K_i \otimes K)^\times} = (\mathbb{A}_K^\times/K^\times)^{[K:K_i]}.$$

By Theorem 9.1 in [27], we have $H^1(\mathrm{Gal}(K/k), \mathbb{A}_K^\times/K^\times) = 0$, so $H^1(G, C_K(R_{L/k}\mathbb{G}_{m,L})) = 0$. Thus,

$$\operatorname{coker} N \xrightarrow{\sim} H^1(G, C_K(T_{L/k})) \simeq H^1(G, \widehat{T}_{L/k})^\vee,$$

where the second isomorphism is Nakayama's duality [22, Theorem 6.3].

Setting $T^L = R_{L/k}\mathbb{G}_{m,L}$ and taking duals from the exact sequence of k-tori (2.11) , we obtain

$$1 \longrightarrow \mathbb{Z} \xrightarrow{\widehat{N}} \widehat{T^L} \xrightarrow{\widehat{\iota}} \widehat{T}_{L/k} \longrightarrow 1.$$

Denoting the Galois group $\mathrm{Gal}(K_i/k)$ by H_i, we obtain

$$H^1(G, \widehat{T^L}) = \bigoplus_{i=1}^r H^1\left(G, \mathrm{Ind}_{H_i}^G \mathbb{Z}\right) = \bigoplus_{i=1}^r H^1(H_i, \mathbb{Z}) = 0.$$

We have the following long exact sequence

$$\begin{array}{ccccccc} 1 \longrightarrow H^1(G, \widehat{T}_{L/k}) & \longrightarrow & H^2(G, \mathbb{Z}) & \xrightarrow{\widehat{N}} & \bigoplus_{i=1}^r H^2(H_i, \mathbb{Z}) & \longrightarrow & \cdots \\ & & \downarrow \wr & & \downarrow \wr & & \\ & & \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{r} & \bigoplus_{i=1}^r \mathrm{Hom}(H_i, \mathbb{Q}/\mathbb{Z}) & & \end{array}$$

where the map r is $f \mapsto (f|_{H_i})_i$. We describe $\ker r$ explicitly:

$$\begin{aligned} H^1(G, \widehat{T}_{L/k}) = \ker r &= \{f : G \to \mathbb{Q}/\mathbb{Z} \mid f|_{H_i} = 0 \text{ for all } i\ \} \\ &= \{f : G \to \mathbb{Q}/\mathbb{Z} \mid f|_{[G,G]H_1 \cdots H_r} = 0\} \\ &\simeq \{f : G/([G,G]H_1 \cdots H_r) \to \mathbb{Q}/\mathbb{Z}\}. \end{aligned}$$

The fixed field of $[G,G]H_1 \cdots H_r$ is exactly $L_{\rm ab}$, the maximal abelian extension of k contained in all K_i. Thus

$$H^1(G, \widehat{T}_{L/k}) \simeq \mathrm{Hom}(\mathrm{Gal}(L_{\rm ab}/k), \mathbb{Q}/\mathbb{Z}) = \mathrm{Gal}(L_{\rm ab}, k)^\vee.$$

Altogether,

$$\frac{\mathbb{A}_k^\times}{k^\times \cdot N(\mathbb{A}_L^\times)} \simeq H^1(\widehat{T}_{L/k})^\vee \simeq \mathrm{Gal}(L_{\rm ab}/k),$$

where each isomorphism is canonical. ∎

Remark 2.5. We can show the consequence $\mathbb{A}_k^\times/k^\times \cdot N(\mathbb{A}_L^\times) \simeq \mathrm{Gal}(L_{\rm ab}/k)$ of Theorem 2.4 directly. Indeed, by class field theory, the maximal abelian subextension $K_{i,\rm ab}$ of K_i/k corresponds to the subgroup $k^\times \cdot N_{K_i/k}(\mathbb{A}_{K_i}^\times)$ of finite index. Thus, their intersection $\cap K_{i,\rm ab} = L_{\rm ab}$ corresponds to the subgroup $k^\times \cdot N(\mathbb{A}_L^\times)$ of finite index.

Theorem 2.6. *Let the notation be as above. We have*

$$E_S(L/k) = \frac{|Ш(L/k)|}{[L_{ab} : k]} \cdot \frac{[U_{k,S} : N(U_{L,S})]}{[O_{k,S}^\times : N(O_{L,S}^\times)]} \tag{2.12}$$

and

$$E_S^+(L/k) = \frac{|Ш(L/k)|}{[L_{ab} : k] \cdot q(\phi)} \cdot \frac{[U_{k,S} : N(U_{L,S})]}{[O_{k,S}^{\times +} : N(O_{L,S}^{\times +})]}, \tag{2.13}$$

where L_{ab} is the maximal abelian extension of k contained in all K_i and $\phi : k^{\times +}/N(L^{\times +}) \to k^\times/N(L^\times)$ is the canonical homomorphism.

Proof. We shall prove the formula (2.13). The proof of (2.12) is the same where one deletes "+" from the terms. Let $\tilde{a} : k^\times/N(L^\times) \to \mathbb{A}_k^\times/N(\mathbb{A}_L^\times)$ be the canonical homomorphism and put $a := \tilde{a} \circ \phi$. Consider the commutative diagram whose rows are exact:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & O_{k,S}^{\times +}/O_{k,S}^{\times +} \cap N(L_S^{\times +}) & \longrightarrow & k^{\times +}/N(L^{\times +}) & \longrightarrow & k^{\times +}/O_{k,S}^{\times +} N(L^{\times +}) & \longrightarrow & 1 \\ & & \downarrow a' & & \downarrow a & & \downarrow a'' & & \\ 1 & \longrightarrow & U_{k,S}/U_{k,S} \cap N(\mathbb{A}_L^\times) & \longrightarrow & \mathbb{A}_k^\times/N(\mathbb{A}_L^\times) & \longrightarrow & \mathbb{A}_k^\times/U_{k,S} N(\mathbb{A}_L^\times) & \longrightarrow & 1. \end{array}$$

From the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & O_{k,S}^{\times +} \cap N(L^{\times +})/N(O_{L,S}^{\times +}) & \longrightarrow & O_{k,S}^{\times +}/N(O_{L,S}^{\times +}) & \longrightarrow & O_{k,S}^{\times +}/O_{k,S}^{\times +} \cap N(L^{\times +}) & \longrightarrow & 1 \\ & & \downarrow b' & & \downarrow b & & \downarrow a' & & \\ 1 & \longrightarrow & U_{k,S} \cap N(\mathbb{A}_L^\times)/N(U_{L,S}) & \longrightarrow & U_{k,S}/N(U_{L,S}) & \longrightarrow & U_{k,S}/U_{k,S} \cap N(\mathbb{A}_L^\times) & \longrightarrow & 1 \end{array}$$

we have

$$q(a') = \frac{q(b)}{q(b')} = \frac{[U_{k,S} : N(U_{L,S})]}{[O_{k,S}^{\times +} : N(O_{L,S}^{\times +})]} \cdot \frac{[O_{k,S}^{\times +} \cap N(L^{\times +}) : N(O_{L,S}^{\times +})]}{[U_{k,S} \cap N(\mathbb{A}_L^\times) : N(U_{L,S})]}.$$

Now, the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & P_{L,S}^+ & \longrightarrow & I_{L,S} & \longrightarrow & \mathrm{Cl}_{L,S}^+ & \longrightarrow & 1 \\ & & \downarrow N' & & \downarrow N & & \downarrow N'' & & \\ 1 & \longrightarrow & P_{k,S}^+ & \longrightarrow & I_{k,S} & \longrightarrow & \mathrm{Cl}_{k,S}^+ & \longrightarrow & 1 \end{array}$$

induces the exact sequence

$$\begin{array}{ccccccccc} I_{L,S}^{(1)} \longrightarrow \ker N'' & \xrightarrow{\delta} & P_{k,S}^+/N'(P_{L,S}^+) & \xrightarrow{c} & I_{k,S}/N(I_{L,S}) & \twoheadrightarrow & \operatorname{coker} N'' & \longrightarrow & 1 \\ & & \wr\uparrow & & \wr\uparrow & & & & \\ & & k^{\times+}/O_{k,S}^{\times+}N(L^{\times+}) & \xrightarrow{a''} & \mathbb{A}_k^\times/U_{L,S}N(\mathbb{A}_L^\times). & & & & \end{array}$$

Then

$$\begin{aligned} q(a'') = q(c) = \frac{|\operatorname{coker} N''|}{|\ker c|} &= |\operatorname{coker} N''| \cdot \frac{[I_{L,S}^{(1)} : P_{L,S}^{(1)+}]}{|\ker N''|} \\ &= q(N'')[I_{L,S}^{(1)} : P_{L,S}^{(1)+}] = [I_{L,S}^{(1)} : P_{L,S}^{(1)+}] \cdot \frac{h_S^+(k)}{h_S^+(L)}. \end{aligned}$$

Thus using Proposition 2.3, we have

$$q(a) = q(a')q(a'') = \frac{[U_{k,S} : N(U_{L,S})]}{[O_{k,S}^{\times+} : N(O_{L,S}^{\times+})]} \cdot E_S^+(L/k)^{-1}.$$

On the other hand,

$$q(a) = q(\tilde{a})q(\phi) = \frac{[\mathbb{A}_k^\times : k^\times N(\mathbb{A}_L^\times)]}{[k^\times \cap N(\mathbb{A}_L^\times) : N(L^\times)]} \cdot q(\phi).$$

Using Theorem 2.4, we obtain the formula

$$E_S^+(L/k) = \frac{|\text{Ш}(L/k)|}{[L_{ab} : k] \cdot q(\phi)} \cdot \frac{[U_{k,S} : N(U_{L,S})]}{[O_{k,S}^{\times+} : N(O_{L,S}^{\times+})]}. \quad \blacksquare$$

3. Class numbers of degree zero

Let k be a global function field and T a k-torus. Here we fix a separable closure k_s over k and denote by Γ_k the Galois group $\mathrm{Gal}(k_s/k)$. Considering the degree map $\deg_k : \mathbb{A}_k^\times \to \mathbb{Z}$, we define $\mathbb{A}_k^{\times,0} := \ker(\deg_k)$. Suppose that $\chi_1, \dots, \chi_r$ is a $\mathbb{Z}$-basis for $\widehat{T}^{\Gamma_k}$. Consider the map

$$T(\mathbb{A}_k) \xrightarrow{\chi=(\chi_1,\dots,\chi_r)} (\mathbb{A}_k^\times)^r \xrightarrow{(\deg_k)_i} \mathbb{Z}^r \to 0.$$

We define $T(\mathbb{A}_k)^0 := \ker(\deg \circ \chi)$ and set the *class group of degree zero of* T as

$$\mathrm{Cl}^0(T) := \frac{T(\mathbb{A}_k)^0}{T(k)U_T}, \tag{3.1}$$

where $U_T = \prod_v T(\mathcal{O}_v)$ is the maximal open compact subgroup of $T(\mathbb{A}_k)$. The order of this class group is finite an d is denoted by $h^0(T) := |\mathrm{Cl}^0(T)|$, called the *class number of degree zero* of T.

Now let $L = \prod_{i=1}^r K_i$ be an étale k-algebra and $T^L = R_{L/k}\mathbb{G}_{m,L}$. We set

$$h^0(L) := h^0(T^L), \quad h^0(k) := |\mathbb{A}_k^{\times,0}/k^\times \cdot U_k|. \tag{3.2}$$

Recall that $T_{L/k} = \ker(N_{L/k} : T^L \to \mathbb{G}_{\mathrm{m},k})$ is the multinorm-one torus associated to L/k.

Definition 3.1. The *Ono invariant of degree zero* is defined as

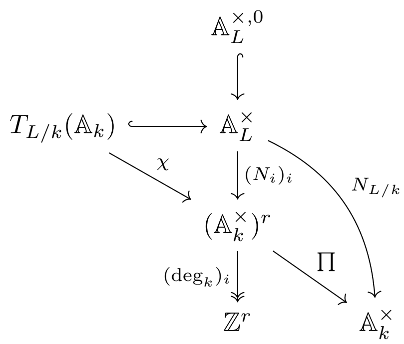
$$E^0(L/k) := \frac{h^0(T^L)}{h^0(k)h^0(T_{L/k})}. \tag{3.3}$$

Lemma 3.2. *If k is a global function field, we have*

$$h^0(T_{L/k}) = [\mathbb{A}_L^{(1)} \cap \mathbb{A}_L^{\times,0} : L^{(1)}U_L^{(1)}],$$

where $\mathbb{A}_L^{\times,0} := \prod_{i=1}^r \mathbb{A}_{K_i}^{\times,0}$.

PROOF. Denoting the norm map by $N_i = N_{K_i/k}$, we have the following commutative diagram



where $\prod(a_1,\dots,a_r) = \prod_{i=1}^r a_i$. We have

$$T_{L/k}(\mathbb{A}_k)^0 = \left\{ x = (x_i) \in \mathbb{A}_L^{\times} : \prod_{i=1}^r N_i(x_i) = 1, \deg \circ N_i(x_i) = 0 \text{ for all } i \right\} = \mathbb{A}_L^{(1)} \cap \mathbb{A}_L^{\times,0}.$$

Also, since $L^\times$ and U_L are already subsets of $\mathbb{A}_L^{\times,0}$, we have $T_{L/k}(k) = L^{(1)} \cap \mathbb{A}_L^{\times,0} = L^{(1)}$ and $U_{T_{L/k}} = U_L^{(1)} \cap \mathbb{A}_L^{\times,0} = U_L^{(1)}$. This completes the proof. So

$$\mathrm{Cl}^0(T_{L/k}) = T_{L/k}(\mathbb{A}_k)^0/T_{L/k}(k)U_{T_{L/k}} = (\mathbb{A}_L^{(1)} \cap \mathbb{A}_L^{\times,0})/L^{(1)}U_L^{(1)}.$$

This completes the proof. ∎

In the following, for an abelian subgroup $A \subset \mathbb{A}_L$, we set $A^0 := A \cap \mathbb{A}_L^{\times,0}$.

Let $\mathbb{F}_q$ be the constant subfield of k and X the geometrically connected projective smooth algebraic curve over $\mathbb{F}_q$ with $k = \mathbb{F}_q(X)$. Denote by $|X|$ the set of closed points of X. Recall that the group of divisors on X is

$$\mathrm{Div}(X) = \sum_{P\in|X|} n_P P, \quad n_P = 0 \text{ for all but finitely many } P \in |X|$$

and the degree map on $\mathrm{Div}(X)$,

$$\deg(D) = \sum_{P\in|X|} n_P \deg(P) \quad \text{for } D = \sum_{P\in|X|} n_P P, \text{ where } \deg(P) = [k(P) : k].$$

The group of divisors of degree zero is denoted as $\mathrm{Div}^0(X)$. Now for any rational function $f \in k^\times$ on X, we define

$$\mathrm{div}(f) := \sum_{P\in|X|} \mathrm{ord}_P(f)P$$

and the group of principal divisors

$$P(X) := \{\mathrm{div}(f) : f \in k^\times\}.$$

Since $P(X) \subset \mathrm{Div}^0(X)$, we can consider the quotient $\mathrm{Pic}^0(X) := \mathrm{Div}^0(X)/P(X)$. Finally we have the isomorphisms

$$\mathrm{Pic}^0(X) \simeq \frac{\mathbb{A}_k^{(1)}}{k^\times \prod_v O_v^\times}, \quad \mathrm{Div}^0(X) \simeq \frac{\mathbb{A}_k^{(1)}}{\prod_v O_v^\times}.$$

Recall $L = \prod_{i=1}^r K_i$. For each i, let $Y_i \to X$ be a finite cover over $\mathbb{F}_q$ with $K_i = \mathbb{F}_q(Y_i)$. Note that Y_i may not be geometrically connected over $\mathbb{F}_q$. Let $\mathbb{F}_{q_i} = \Gamma(Y_i, \mathcal{O}_{Y_i})$ be the constant subfield of K_i. We define similarly $\mathrm{Div}(Y_i)$ to be the group consisting of $\sum_{P\in|Y_i|} n_P P$. Although different definitions of the degree of a divisor on Y_i may differ by a constant, the notions of $\mathrm{Div}^0(Y_i)$ and $\mathrm{Pic}^0(Y_i)$ are still well-defined.

Similarly to the case when k is a number field, we shall write $I_k^0 = \mathrm{Div}^0(X)$ and $P_k = P(X)$. We also define

$$I_L^0 := \prod_{i=1}^r I_{K_i}^0, \quad P_L := \prod_{i=1}^r P_{K_i} \quad \text{and} \quad \mathrm{Cl}^0(L) := I_L^0/P_L = \prod_{i=1}^r \mathrm{Cl}^0(K_i).$$

Finally, the norm map $N : \mathbb{A}_L^{\times,0} \to \mathbb{A}_k^{\times,0}$ induces a map $N : I_L^0 \to I_k^0$, which we also call norm, and we denote by $I_L^{0(1)}$ its kernel and $P_L^{0(1)} := P_L^0 \cap I_L^{0(1)}$.

Proposition 3.3. *With the above notation, we have*

$$h^0(T_{L/k}) = \frac{[I_L^{(1)} : P_L^{(1)}][\mathbb{F}_q^\times \cap N(L^\times) : N(\prod_{i=1}^r \mathbb{F}_{q_i}^\times)]}{[U_k \cap N(\mathbb{A}_L^{\times,0}) : N(U_L)]}.$$

PROOF. Consider the two commutative diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U_L & \longrightarrow & \mathbb{A}_L^{\times,0} & \longrightarrow & I_L^0 & \longrightarrow & 0 \\ & & \downarrow N & & \downarrow N & & \downarrow N & & \\ 0 & \longrightarrow & U_k & \longrightarrow & \mathbb{A}_k^{\times,0} & \longrightarrow & I_k^0 & \longrightarrow & 0 \end{array}$$

and

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \prod_{i=1}^r \mathbb{F}_{q_i}^\times & \longrightarrow & L^\times & \xrightarrow{\mathrm{div}} & P_L & \longrightarrow & 1 \\ & & \downarrow N_{L/k} & & \downarrow N_{L/k} & & \downarrow N & & \\ 1 & \longrightarrow & \mathbb{F}_q^\times & \longrightarrow & k^\times & \xrightarrow{\mathrm{div}} & P_k & \longrightarrow & 1 \end{array}$$

where the norm map $N_{L/k} = \prod N_{K_i/k}$ restricted to $\prod_{i=1}^r \mathbb{F}_{q_i}^\times$ is given by

$$N_{L/k}((x_i)) = \prod_{i=1}^r N_{K_i/k}(x_i) = \prod_{i=1}^r \Big(N_{\mathbb{F}_{q_i}/\mathbb{F}_q}(x_i)\Big)^{[K_i/k]/[\mathbb{F}_{q_i}:\mathbb{F}_q]} \qquad \text{for } x_i \in \mathbb{F}_{q_i}$$

and we denote by $\big(\prod_{i=1}^r \mathbb{F}_{q_i}^\times\big)^{(1)}$ its kernel. By the snake lemma, each diagram gives an exact sequence, respectively:

$$0 \longrightarrow U_L^{(1)} \longrightarrow \mathbb{A}_L^{0(1)} \longrightarrow I_L^{(1)} \xrightarrow{\delta_1} U_k/N(U_L) \longrightarrow \mathbb{A}_k^{\times,0}/N(\mathbb{A}_L^{\times,0}) \longrightarrow \cdots;$$

$$1 \longrightarrow \big(\textstyle\prod_{i=1}^r \mathbb{F}_{q_i}^\times\big)^{(1)} \longrightarrow L^{(1)} \longrightarrow P_L^{(1)} \xrightarrow{\delta_2} \mathbb{F}_q^\times/N(\prod_{i=1}^r \mathbb{F}_{q_i}^\times) \longrightarrow k^\times/N(L^\times) \longrightarrow \cdots.$$

The images $\mathrm{Im}\,\delta_1$ and $\mathrm{Im}\,\delta_2$ are finite abelian groups. Thus we have

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \dfrac{\mathbb{A}_L^{\times,0}\cap \mathbb{A}_L^{(1)}}{U_L^{(1)}} & \longrightarrow & I_L^{(1)} & \longrightarrow & \dfrac{U_k\cap N(\mathbb{A}_L^{\times,0})}{N(U_L)} & \longrightarrow & 1\\
 & & {\scriptstyle f'}\Big\uparrow & & {\scriptstyle f}\Big\updownarrow & & {\scriptstyle f''}\Big\uparrow & & \\
1 & \longrightarrow & \dfrac{L^{(1)}}{\left(\prod_{i=1}^r \mathbb{F}_{q_i}^\times\right)^{(1)}} & \longrightarrow & P_L^{(1)} & \longrightarrow & \dfrac{\mathbb{F}_q^\times\cap N(L^\times)}{N(\prod_{i=1}^r \mathbb{F}_{q_i}^\times)} & \longrightarrow & 1.
\end{array}$$

Since $q(f)=q(f')q(f")$,

$$[I_L^{(1)}:P_L^{(1)}]=[\mathbb{A}_L^{\times,0}\cap \mathbb{A}_L^{(1)}:L^{(1)}U_L^{(1)}]\frac{[U_k\cap N(\mathbb{A}_L^{\times,0}):N(U_L)]}{[\mathbb{F}_q^\times\cap N(L^\times):N(\prod_{i=1}^r\mathbb{F}_{q_i}^\times)]}. \quad \blacksquare$$

Lemma 3.4. *We have*

$$\frac{[k^\times\cap N(\mathbb{A}_L^{\times,0}):N(L^{\times,0})]}{[\mathbb{A}_k^{\times,0}:k^\times N(\mathbb{A}_L^{\times,0})]}=\frac{|Ш(L/k)|}{[L_{ab}:k]}\cdot q(\phi^0),$$

where $\phi^0:\mathbb{A}_k^{\times,0}/N(\mathbb{A}_L^{\times,0})\to \mathbb{A}_k^\times/N(\mathbb{A}_L^\times)$ *is the canonical homomorphism.*

Proof. Noting that $\mathbb{A}_k^{\times,0}$ is the kernel of the degree map $\deg_k:\mathbb{A}_k^\times\to\mathbb{Z}$, we have

$$\ker\phi^0=\frac{\mathbb{A}_k^{\times,0}\cap N(\mathbb{A}_L)}{N(\mathbb{A}_L^{\times,0})},\quad \operatorname{coker}\phi^0=\frac{\mathbb{A}_k^\times}{\mathbb{A}_k^{\times,0}N(\mathbb{A}_L^\times)}\simeq\frac{\mathbb{Z}}{\deg_k(N(\mathbb{A}_L^\times))}$$

and then

$$q(\phi^0)=\frac{[\mathbb{Z}:\deg_k(N(\mathbb{A}_L^\times)])}{[\mathbb{A}_k^{\times,0}\cap N(\mathbb{A}_L^\times):N(\mathbb{A}_L^{\times,0})]}.$$

Let b be the canonical homomorphism $b:\mathbb{A}_k^{\times,0}/\left(k^\times N(\mathbb{A}_L^{\times,0})\right)\to\mathbb{A}_k^\times/\left(k^\times N(\mathbb{A}_L^\times)\right)$. Then we have

$$\ker b=\frac{k^\times\left(\mathbb{A}_k^{\times,0}\cap N(\mathbb{A}_L^\times)\right)}{k^\times N(\mathbb{A}_L^{\times,0})},\quad \operatorname{coker} b=\frac{\mathbb{A}_k^\times}{\mathbb{A}_k^{\times,0}k^\times N(\mathbb{A}_L^\times)}\simeq\frac{\mathbb{Z}}{\deg_k(N(\mathbb{A}_L^\times))}.$$

From the finiteness of domain and codomain of b and the exact sequence

$$1\longrightarrow\frac{k^\times\cap N(\mathbb{A}_L^\times)}{k^\times\cap N(\mathbb{A}_L^{\times,0})}\longrightarrow\frac{\mathbb{A}_k^{\times,0}\cap N(\mathbb{A}_L^\times)}{N(\mathbb{A}_L^{\times,0})}\longrightarrow\ker b\longrightarrow 1$$

we compute that

$$\begin{aligned}
\frac{[\mathbb{A}_k^\times:k^\times N(\mathbb{A}_L^\times)]}{[\mathbb{A}_k^{\times,0}:k^\times N(\mathbb{A}_L^{\times,0})]}=q(b)&=\frac{[\mathbb{Z}:\deg_k(N(\mathbb{A}_L^\times))]}{|\ker b|}\\
&\overset{(*)}{=}q(\phi^0)\cdot[k^\times\cap N(\mathbb{A}_L^\times):k^\times\cap N(\mathbb{A}_L^{\times,0})]\\
&=q(\phi^0)\cdot\frac{[k^\times\cap N(\mathbb{A}_L^\times):N(L^\times)]}{[k^\times\cap N(\mathbb{A}_L^{\times,0}):N(L^\times)]}.
\end{aligned}$$

In $(*)$ we use the fact that for abelian subgroups A, B, and C of an abelian group G with $B\subset A$, we have

$$[A:B]=[A\cap C:B\cap C][AC:BC].$$

This follows from considering the exact sequences

$$0 \longrightarrow (A\cap BC)/B \longrightarrow A/B \longrightarrow AC/BC \longrightarrow 0;$$
$$0 \longrightarrow B\cap C \longrightarrow A\cap C \longrightarrow (A\cap BC)/B \longrightarrow 0.$$

Finally, using Theorem 2.4 we obtain

$$\frac{[k^\times \cap N(\mathbb{A}_L^{\times,0}) : N(L^\times)]}{[\mathbb{A}_k^{\times,0} : k^\times N(\mathbb{A}_L^{\times,0})]} = q(\phi^0)\cdot \frac{[k^\times \cap N(\mathbb{A}_L^{\times}) : N(L^\times)]}{[\mathbb{A}_k^{\times} : k^\times N(\mathbb{A}_L^{\times})]} = q(\phi^0)\cdot \frac{|\text{Ш}(L/k)|}{[L_{ab}:k]}. \quad \blacksquare$$

Theorem 3.5. *Let the notation be as above. We have*

$$E^0(L/k) = \frac{|\text{Ш}(L/k)|}{[L_{ab}:k]}\cdot q(\phi^0)\cdot \frac{[U_k : N(U_L)]}{[\mathbb{F}_q^\times : N(\prod_i \mathbb{F}_{q_i}^\times)]}.$$

PROOF. Let $a : k^\times/N(L^\times) \to \mathbb{A}_k^{\times,0}/N(\mathbb{A}_L^{\times,0})$ be the canonical homomorphism. Consider the commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{F}_q^\times & \longrightarrow & k^\times/N(L^\times) & \longrightarrow & k^\times/\mathbb{F}_q^\times N(L^\times) & \longrightarrow & 1\\
 & & \downarrow{\scriptstyle a'} & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle a''} & & \\
1 & \longrightarrow & U_k/U_k\cap N(\mathbb{A}_L^{\times,0}) & \longrightarrow & \mathbb{A}_k^{\times,0}/N(\mathbb{A}_L^{\times,0}) & \longrightarrow & \mathbb{A}_k^{\times,0}/U_k N(\mathbb{A}_L^{\times,0}) & \longrightarrow & 1
\end{array}$$

From the commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{F}_q^\times \cap N(L^\times)/N(\prod_i \mathbb{F}_{q_i}^\times) & \longrightarrow & \mathbb{F}_q^\times/N(\prod_i \mathbb{F}_{q_i}^\times) & \longrightarrow & \mathbb{F}_q^\times/\mathbb{F}_q^\times \cap N(L^\times) & \longrightarrow & 1\\
 & & \downarrow{\scriptstyle b'} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle a'} & & \\
1 & \longrightarrow & U_k\cap N(\mathbb{A}_L^{\times,0})/N(U_L) & \longrightarrow & U_k/N(U_L) & \longrightarrow & U_k/U_k\cap N(\mathbb{A}_L^{\times,0}) & \longrightarrow & 1
\end{array}$$

we have

$$q(a') = \frac{q(b)}{q(b')} = \frac{[U_k : N(U_L)]}{[\mathbb{F}_q^\times : N(\prod_i \mathbb{F}_{q_i}^\times)]}\cdot \frac{[\mathbb{F}_q^\times \cap N(L^\times) : N(\prod_i \mathbb{F}_{q_i}^\times)]}{[U_k \cap N(\mathbb{A}_L^{\times,0}) : N(U_L)]}.$$

Now the diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & P_L & \longrightarrow & I_L^0 & \longrightarrow & \mathrm{Cl}^0(L) & \longrightarrow & 1\\
 & & \downarrow{\scriptstyle N'} & & \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle N''} & & \\
1 & \longrightarrow & P_k & \longrightarrow & I_k^0 & \longrightarrow & \mathrm{Cl}^0(k) & \longrightarrow & 1
\end{array}$$

induces the exact sequence

$$\begin{array}{ccccccccccc}
I_L^{0(1)} & \longrightarrow & \ker N'' & \xrightarrow{\ \delta\ } & P_k/N'(P_L) & \xrightarrow{\ c\ } & I_k^0/N(I_L^0) & \longrightarrow & \operatorname{coker} N'' & \longrightarrow & 1\\
 & & & & \uparrow\wr & & \uparrow\wr & & & & \\
 & & & & k^\times/\mathbb{F}_q^\times\cdot N(L^\times) & \xrightarrow{\ a''\ } & \mathbb{A}_k^{\times,0}/U_L\cdot N(\mathbb{A}_L^{\times,0}) & & & &
\end{array}$$

Then

$$\begin{aligned}
q(a'') = q(c) = \frac{|\operatorname{coker} N''|}{|\ker c|} &= |\operatorname{coker} N''|\cdot \frac{[I_L^{0(1)} : P_L]}{|\ker N''|}\\
&= q(N'')[I_L^{0(1)} : P_L] = [I_L^{0(1)} : P_L]\cdot \frac{h^0(k)}{h^0(L)}.
\end{aligned}$$

Using Proposition 3.3, we have

$$q(a) = q(a')q(a'') = \frac{[U_k : N(U_L)]}{[\mathbb{F}_q^\times : N(\prod_i \mathbb{F}_{q_i}^\times)]}\cdot E^0(L/k)^{-1}.$$

On the other hand from the definition of $q(a)$,

$$q(a) = \frac{[\mathbb{A}_k^{\times,0} : k^\times N(\mathbb{A}_L^{\times,0})]}{[k^\times \cap N(\mathbb{A}_L^{\times,0}) : N(L^\times)]}.$$

Using Lemma 3.4, we obtain the formula

$$E^0(L/k) = \frac{|\text{Ш}(L/k)|}{[L_{ab} : k]} \cdot q(\phi^0) \cdot \frac{[U_k : N(U_L)]}{[\mathbb{F}_q^\times : N(\prod_i \mathbb{F}_{q_i}^\times)]}. \quad \blacksquare$$

4. Exploration of terms in Theorem 2.6 and examples

In this section we explore the terms appearing in the class number relation arising from multinorm-one tori (Theorem 2.6).

4.1. Local norm indices $[U_{k,S} : N(U_{L,S})]$.

4.1.1. Suppose that K/k is a finite separable extension of local fields. We denote by $K_{\rm ab}$ and $K_{\rm ur}$ the maximal abelian and unramified subextensions of K/k, respectively. Local class field theory says that $N_{K/k}(K^\times) = N_{K_{\rm ab}/k}(K_{\rm ab}^\times)$ and we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & O_k^\times/N(O_{K_{\rm ab}}^\times) & \longrightarrow & k^\times/N(K_{\rm ab}^\times) & \longrightarrow & \mathbb{Z}/f\mathbb{Z} & \longrightarrow & 0 \\ & & \wr\downarrow & & \wr\downarrow{\rm Art} & & \wr\downarrow & & \\ 0 & \longrightarrow & \mathrm{Gal}(K_{\rm ab}/K_{\rm ab,ur}) & \longrightarrow & \mathrm{Gal}(K_{\rm ab}/k) & \longrightarrow & \mathrm{Gal}(\lambda/\kappa) & \longrightarrow & 0 \end{array}$$

where λ and κ are the residue fields of $K_{\rm ab}$ and k, respectively, and $f = [\lambda : \kappa]$ is the residue degree of $K_{\rm ab}$ over k. In particular, we have

$$[O_k^\times : N(O_K^\times)] = e(K_{\rm ab}/k), \tag{4.1}$$

the ramification index of $K_{\rm ab}$ over k.

4.1.2. Recall

$$U_{k,S} = \prod_{v\in S} k_v^\times \times \prod_{v\notin S} O_v^\times \quad \text{and} \quad U_{L,S} = \prod_{v\in S} L_v^\times \times \prod_{v\notin S} O_{L_v}^\times, \text{ where } L_v = \prod_{w|v} L_w,$$

and we have

$$[U_{k,S} : N(U_{L,S}] = \prod_{v\in S}[k_v^\times : N(L_v^\times)] \cdot \prod_{v\notin S}[O_v^\times : N(O_{L_v}^\times)].$$

Fix a separable closure $k_{v,s}$ of k_v and let $k_v^{\rm ab}$ and $k_v^{\rm ur}$ be the maximal abelian and unramified extensions of k_v in $k_{v,s}$, respectively. One has $\mathrm{Gal}(k_v^{\rm ab}/k_v^{\rm ur}) \simeq O_v^\times$. For each place $w|v$ of L, we choose an embedding $\iota : L_w \hookrightarrow k_{v,s}$ over k_v. Since $L_{w,\rm ab}$ is Galois over k_v, its image $\iota(L_{w,\rm ab})$ is independent of the choice of ι, which we denote by $L_{w,\rm ab}$ again for simplicity. Put

$$L_{v,\rm ab} := \bigcap_{w|v} L_{w,\rm ab} \tag{4.2}$$

and

$$\widetilde{L}_{v,\rm ab} := \text{the compositum of the abelian extensions } L_{w,\rm ab} \text{ of } k_v \text{ for all } w|v. \tag{4.3}$$

If v is a finite place of k, we choose a finite unramified extension k_v' of k_v which contains the maximal unramified subextension $(\widetilde{L}_{v,\rm ab})_{\rm ur}$ of $\widetilde{L}_{v,\rm ab}/k_v$ and set

$$L'_{w,\rm ab} := k_v' L_{w,\rm ab} \quad \text{and} \quad L'_{v,\rm ab} := \bigcap_{w|v} L'_{w,\rm ab}. \tag{4.4}$$

The degree $[L'_{v,\mathrm{ab}} : k'_v]$ does not depend on the choice of k'_v. If there is a place $w|v$ of L which is unramified in L/k, then $[L'_{v,\mathrm{ab}} : k'_v] = 1$. For this reason, we write

$$e_v(L/k) := [L'_{v,\mathrm{ab}} : k'_v]. \tag{4.5}$$

When L_v is an abelian field extension of k_v, the invariant $e_v(L/k)$ coincides with the ramification index of v in L/k.

Proposition 4.1. *(1) For any place v, we have $[k_v^\times : N(L_v^\times)] = [L_{v,\mathrm{ab}} : k_v]$.*
(2) For any finite place v, we have $[O_v^\times : N(O_{L_v}^\times)] = [L'_{v,\mathrm{ab}} : k'_v]$.

PROOF. (1) By local class field theory, we have

$$N_{L_{v,\mathrm{ab}}/k_v}(L_{v,\mathrm{ab}}^\times) = \prod_{w|v} N_{L_{w,\mathrm{ab}}/k_v}(L_{w,\mathrm{ab}}^\times) = \prod_{w|v} N_{L_w/k_v}(L_w^\times) = N(L_v^\times) \tag{4.6}$$

and get

$$[L_{v,\mathrm{ab}} : k_v] = [k_v^\times : N_{L_{v,\mathrm{ab}}/k_v}(L_{v,\mathrm{ab}}^\times)] = [k_v^\times : N(L_v^\times)]. \tag{4.7}$$

(2) The open subgroup corresponding to $L_{w,\mathrm{ab}}^{\mathrm{ur}} = k_v^{\mathrm{ur}} L_{w,\mathrm{ab}}$ is $U_w := N_{L_w/k_v}(O_{L_w}^\times)$. If we put $U_v = \cap_{w|v} U_w$, then it corresponds to the field extension $(\widetilde{L}_{v,\mathrm{ab}})^{\mathrm{ur}}$. We have

$$\mathrm{Gal}((\widetilde{L}_{v,\mathrm{ab}})^{\mathrm{ur}}/L_{w,\mathrm{ab}}^{\mathrm{ur}}) \simeq \mathrm{Gal}((\widetilde{L}_{v,\mathrm{ab}})'/L'_{w,\mathrm{ab}}) \simeq U_w/U_v.$$

and hence

$$\mathrm{Gal}((\widetilde{L}_{v,\mathrm{ab}})'/L'_{v,\mathrm{ab}}) \simeq \prod_{w|v} U_w/U_v = \left(\prod_{w|v} U_w\right)/U_v = N(O_{L_v}^\times)/U_w.$$

Therefore,

$$[L'_{v,\mathrm{ab}} : k'_v] = [O_v^\times : N(O_{L_v}^\times)]. \quad \blacksquare$$

Remark 4.2. In Proposition 4.1 we make an auxiliary base change of L_v by a sufficiently large unramified extension k'_v so that $\mathrm{Gal}(L'_{w,\mathrm{ab}}/k'_v) \simeq [O_v \times : N_{L_w/k_v}(O_{L_w}^\times)]$ and take the intersection $\cap_{w|v} L'_{w,\mathrm{ab}}$. This is necessary even when each field extension L_w/k_v is abelian and totally ramified. Indeed, one has $\mathrm{Gal}(L_w/k_v) \simeq O_v^\times/N_{L_w/k_v}(O_{L_w}^\times)$ and

$$\mathrm{Gal}(L_{v,ab}/k_v) \simeq O_v^\times/(N(L_v^\times) \cap O_v^\times).$$

However, we only have the inclusion $N(O_{L_v}^\times) \subset (N(L_v^\times) \cap O_v^\times)$ and no equality in general.

For example, let $k_v = \mathbb{Q}_\ell$ where ℓ is an odd prime. By local class field theory, there are exactly two ramified quadratic fields L_{w_1} and L_{w_2}. By weak approximation, there exist quadratic extensions K_1 and K_2 such that $(K_1)_v \simeq L_{w_1}$ and $(K_2)_v \simeq L_{w_1}$. Put $L = K_1 \times K_2$ and hence $L_v = L_{w_1} \times L_{w_2}$. We have $N(L_v^\times) = k_v^\times$ and $N(L_v^\times) \cap O_v^\times = O_v^\times$, while $N(O_{L_v}^\times) \subset O_v^\times$ is of index two.

4.2. **The global unit norm index $[O_{k,S}^\times : N(O_{L,S}^\times)]$.**

Proposition 4.3. *Let F_L be a free part of $O_{L,S}^\times$, that is, F_L is a finitely generated free subgroup such that $O_{L,S}^\times = \mu_L \times F_L$, where $\mu_L = \prod \mu_{K_i}$ is the group of roots of unity in L.*

(1) We can choose a free part F_k of $O_{k,S}^\times$ such that $O_{k,S}^\times = \mu_k \times F_k$ and $N(F_L) \subset F_k$. Then we have $O_{k,S}^\times/N(O_{L,S}^\times) \simeq \mu_k/N(\mu_L) \times F_k/N(F_L)$.

(2) Moreover, let $\{\xi_j\}$ be a system of fundamental units of $O_{L,S}^\times$ and let $\pi_d : F_k \to F_k/(F_k)^d$ be the natural projection, where d is the greatest common divisor of $[K_i : k]$, $1 \le i \le r$. Then $F_k/N(F_L) \simeq (F_k/(F_k)^d)/E_L$, where E_L is the subgroup generated by $\pi_d(\xi_j)$ for all j.

PROOF. (1) Note that $N(F_L)$ is a finitely generated free subgroup of $O_{k,S}^\times$. By Zorn's lemma, there exists a maximal finitely generated free subgroup F_k of $O_{k,S}^\times$ such that $N(F_L) \subset F_k$. Then

$$O_{k,S}^\times/N(O_{L,S}^\times) = \frac{\mu_k \times F_k}{N(\mu_L) \times N(F_L)} \simeq \mu_k/N(\mu_L) \times F_k/N(F_L).$$

(2) Note that the group $O_{L,S}^\times = \prod_{i=1}^r O_{K_i,S}^\times$ contains the subgroup $\prod_{i=1}^r O_k^\times$ and hence the free subgroup F_L contains the free subgroup $\prod_{i=1}^r F_k$. Therefore, we have

$$N(F_L) \supset N\left(\prod_{i=1}^r F_k\right) = F_k^{[K_1:k]} \cdots F_k^{[K_r:k]} = F_k^d, \tag{4.8}$$

and $F_k/N(F_L)$ is the quotient of the F_k/F_k^d by its image of $N(F_L)$. This shows statement (2). ∎

It is rather easy to compute the index of torsion part $[\mu_k : N(\mu_L)]$ for each given specific case. However, finding a system of fundamental units is a classical problem in number theory, which is known to be difficult in general. However, by Proposition 4.3, if the degrees $[K_i : k]$ have only the trivial common divisor, then $O_{k,S}^\times/N(O_{L,S}^\times) = \{1\}$, cf. (4.8).

If $k = \mathbb{Q}$ and S is the set of archimedean places, then we have that $[O_{k,S}^\times : N(O_{L,S}^\times)] = [\mathbb{Z}^\times : N(O_L^\times)]$ is 1 or 2. Moreover,

$$[O_{k,S}^\times : N(O_{L,S}^\times)] = 1 \iff -1 \in N_{K_i/k}(O_{K_i}^\times) \text{ for some } i.$$

In particular, if one of the degrees $[K_i : k]$ is odd, then $[O_k^\times : N(O_L^\times)] = 1$. When K is a real quadratic field, one can use continued fractions to compute the fundamental unit ϵ_K (cf. [18]) and determine whether $N(\epsilon_K) = -1$. In the case where $K = \mathbb{Q}(\sqrt{p})$ and p is a prime, we know (from [29, Lemma 2.4]) that

$$N(\epsilon_K) = \begin{cases} 1, & \text{if } p \equiv 3 \pmod 4; \\ -1, & \text{otherwise.} \end{cases} \tag{4.9}$$

However, there is no known direct determination of $N(\epsilon_K)$ from the discriminant of K as (4.9) in general.

4.3. **The Tate-Shafarevich group** $\text{Ш}(L/k)$**.** The most interesting and involved term in the class number formula for multinorm-one tori is $|\text{Ш}(L/k)|$. In this section we organize some results about computation of $\text{Ш}(L/k)$. In particular, we introduce a criterion for $\text{Ш}(L/k) = 0$.

First, we have an exact sequence of k-tori

$$1 \longrightarrow T_{L/k} \xrightarrow{\iota} R_{L/k}\mathbb{G}_{m,L} \xrightarrow{N} \mathbb{G}_{m,k} \longrightarrow 1.$$

By Hilbert's Theorem 90, we have an isomorphism

$$k^\times/N_{L/k}(L^\times) \xrightarrow{\sim} H^1(k, T_{L/k}). \tag{4.10}$$

Taking the kernel of the local-global map on each side, we have the isomorphisms

$$\text{Ш}(L/k) \simeq \text{Ш}^1(k, T_{L/k}) \simeq \text{Ш}^2(k, \widehat{T}_{L/k})^\vee, \tag{4.11}$$

where the second isomorphism is given by Poitou-Tate duality [22, Theorem 6.10].

Let

$$\text{Ш}^i_\omega(k, \widehat{T}) := \left\{ [C] \in H^i(k, \widehat{T}) \,|\, [C]_v = 0 \text{ for almost all places } v \text{ of } k \right\}, \tag{4.12}$$

where $[C]_v$ denotes the class in $H^i(k_v, \widehat{T})$ under the restriction map $H^i(k, \widehat{T}) \to H^i(k_v, \widehat{T})$. We shall utilize the following exact sequence

$$0 \longrightarrow Ш^2(k, \widehat{T}) \longrightarrow Ш^2_\omega(k, \widehat{T}) \longrightarrow A(T)^\vee \longrightarrow 0,$$

where $T(k) \hookrightarrow \prod_v T(k_v)$ embeds diagonally and $A(T) := \prod_v T(k_v)/\overline{T(k)}$ is the group measuring the defect of weak approximation of T, or its dual version

$$0 \longrightarrow A(T) \longrightarrow Ш^2_\omega(k, \widehat{T})^\vee \longrightarrow Ш^2(k, \widehat{T})^\vee \longrightarrow 0. \tag{4.13}$$

Here $Ш^2(k, \widehat{T})^\vee \simeq Ш^1(k, T)$ and $Ш^2_\omega(k, \widehat{T}) \simeq H^1(k, \mathrm{Pic}(\overline{T}))$, where $\overline{T}$ is a smooth compactification of T, by [28, Theorem 6]. By definition, the local-global principle for T holds if $Ш^1(k, T) = 0$, and *weak approximation* for T holds if $A(T) = 0$. From the second exact sequence (4.13) we see that

$$Ш^2_\omega(k, \widehat{T}) = 0 \Longleftrightarrow A(T) = 0 \text{ and } Ш^1(k, T) = 0.$$

Recall that *Hasse norm principle* (HNP) holds for the étale k-algebra L/k if $Ш(L/k) = 0$ or $Ш^2(k, \widehat{T}_{L/k}) = 0$ by (4.11).

In the following, we collect some results from the literature. For a finite separable extension K/k of global fields, we denote by K^c the Galois closure of K over k. We also write $Ш_\omega(L/k)$ for $Ш^1_\omega(k, T_{L/k})$.

Theorem 4.4. *Let $L = \prod_{i=1}^r K_i$ be an étale k-algebra and $T_{L/k}$ the associated multinorm-one torus. The Hasse Norm Principle (HNP) holds for L/k, that is, $Ш(L/k) = 0$, if one of the following conditions holds.*

Case 1. Let $r = 1$ and $L = K$.

- *(1.a) K/k is Galois with Galois group G such that $Ш^3(G, \mathbb{Z}) = 0$.*
- *(1.b) $[K : k]$ is a prime.*
- *(1.c) $[K : k] = n$ and $\mathrm{Gal}(K^c/k)$ is isomorphic to the dihedral group D_n.*
- *(1.d) $[K : k] = n$ and $\mathrm{Gal}(K^c/k)$ is isomorphic to the symmetric group S_n.*
- *(1.e) $[K : k] = n$ and $\mathrm{Gal}(K^c/k)$ is isomorphic to the alternating group A_n, for $n \geq 5$.*

Case 2. Let $r = 2$ and $L = K_1 \times K_2$. *We set $F = K_1 \cap K_2$ and let $Ш_\omega(F/k) = Ш^1_\omega(k, T_{F/k})$ denote the quotient of the subgroup of $k^\times$ which is a norm locally almost everywhere, by $N_{F/k}(F^\times)$.*

- *(2.a) K_1 is a cyclic extension and K_2 is an arbitrary finite separable extension.*
- *(2.b) $K_1^c \cap K_2^c = k$.*
- *(2.c) K_1 and K_2 are abelian extensions over k, and $Ш(F/k) = 0$.*
- *(2.d) $Ш_\omega(F/k) = 0$.*

Case 3. General $r \geq 2$.

- *(3.a) $K_1, \dots, K_r$ are Galois over of k, the field $K_1 \cdots K_i \cap (K_{i+1} \cdots K_r)$ equals to the intersection $F := \bigcap_{i=1}^r K_i$ for some $1 \leq i \leq r-1$, and $Ш_\omega(F/k) = 0$.*
- *(3.b) $K_1, \dots, K_r$ are Galois over of k, and $K_1 \cdots K_i \cap (K_{i+1} \cdots K_r) = k$ for some $1 \leq i \leq r-1$.*
- *(3.c) $K_1, \dots, K_r$ are distinct extensions over k of degree p, where p is a prime, with K_i is cyclic for some i, and either the composition $\widetilde{F} := K_1 \cdots K_r$ has degree $> p^2$ over k or one local degree of $\widetilde{F}$ is $> p$.*

Proof. *Case 1. $r = 1$ and $L = K$.*

- (1.a) This is a theorem of Tate, cf. [22, Theorem 6.11].
- (1.b) See Bartels [2, Lemma 4], cf. [22, Proposition 6.3].
- (1.c) See Bartels [1, Satz 1].
- (1.d) This is a result of B. Kunyavskii and V. Voskresenskii; see [11], cf. [15, p. 2].
- (1.e) See Macedo [15, Theorem 1.1].

We remark the references for (1.a), (1.b), (1.d) and (1.e) are stated for number fields. However, since the methods in loc. cit use Galois cohomology and group theory, the proofs also apply to the global function field case.

Case 2. $r = 2$ *and* $L = K_1 \times K_2$.

(2.a) The case where K_2/k is Galois is proved by Hürlimann in [8, Proposition 3.3] and the general case is proved in [3, Proposition 4.1].

(2.b) Pollio and Rapinchuk proved that this condition implies $Ш(L/k) = 0$ in [24].

(2.c) In [23], Pollio proved that if K_1 and K_2 are abelian extensions of k, then $Ш(L/k) = Ш(F/k)$.

(2.d) This follows from Demarche and Wei's work [4]. Applying [4, Theorem 6] to the case $I = \{1\}$ and $J = \{2\}$, we obtain $Ш_\omega(L/k) = Ш_\omega(F/k)$. In particular, $Ш_\omega(L/k) = 0$ implies $Ш(L/k) = 0$.

Case 3. General $r \geq 2$.

(3.a) In [4, Example 9], Demarche and Wei proved that if $K_1, \dots, K_r$ are Galois extensions of k and $(K_1 \cdots K_i) \cap (K_{i+1} \cdots K_r) = F = \bigcap_{i=1}^r K_i$ for some $1 \leq i \leq r$, then $Ш_\omega(L/k) \simeq Ш_\omega(F/k)$.

(3.b) This condition originates from [4, Theorem 1].

(3.c) This is an application of Bayer-Fluckiger, T.-Y. Lee and Parimala's [3, Proposition 8.5]. Note that when $r = 2$ this recovers condition (2.a). ∎

Remark 4.5. Theorem 4.4(2.b) implies that the term $|Ш_k(T')|$ in the main theorem of [16, p. 135] is equal to 1.

Corollary 4.6. *Let the notation be as in Theorem 2.6 and in Proposition 4.1. Assume one of the conditions in Theorem 4.4 holds. Then*

$$E_S(L/k) = \frac{\prod_{v\in S}[L_{v,\mathrm{ab}} : k_v] \cdot \prod_{v \in R(L/k)\smallsetminus S} e_v(L/k)}{[L_{ab} : k] \cdot [O^\times_{k,S} : N(O^\times_{L,S})]}, \tag{4.14}$$

$$E^+_S(L/k) = \frac{\prod_{v\in S}[L_{v,\mathrm{ab}} : k_v] \cdot \prod_{v \in R(L/k)\smallsetminus S} e_v(L/k)}{[L_{ab} : k] \cdot q(\phi) \cdot [O^{\times +}_{k,S} : N(O^{\times +}_{L,S})]}, \tag{4.15}$$

and

$$E^0(L/k) = \frac{q(\phi^0) \cdot \prod_{v \in R(L/k)} e_v(L/k)}{[L_{ab} : k] \cdot [\mathbb{F}_q^\times : N(\prod_i \mathbb{F}_{q_i}^\times)]}, \tag{4.16}$$

where $L_{v,\mathrm{ab}}$ *is a finite abelian extension of* k_v *defined in* (4.2), $e_v(L/k)$ *is defined in* (4.5), *and* $R(L/k)$ *is the finite set of finite places* v *of* k *for which none of the places* $w|v$ *of* L *is unramified in* L/k.

PROOF. This follows from Theorems 2.6 and 3.5, Proposition 4.1 and Theorem 4.4. ∎

As in Theorem 4.4, there have already been many affirmative results for determining the HNP for L/k. However, when the HNP for L/k fails, results for computing $Ш(L/k)$ are only sporadic.

For $r = 1$ and K/k a Galois extension with group G, a theorem of Tate gives us a general method for computing $Ш(K/k)$ through the canonical isomorphism

$$Ш(K/k) \simeq Ш^3(G, \mathbb{Z}).$$

Via the natural isomorphism $H^3(G,\mathbb{Z}) \simeq \mathrm{Hom}(H_2(G,\mathbb{Z}), \mathbb{Q}/\mathbb{Z})$, this reduces the problem to computing the Schur multiplier $M(G) \simeq H_2(G,\mathbb{Z})$ of G and computing the cokernel of the map

$$\bigoplus_v M(G_v) \to M(G),$$

where v runs through all places of k ramified in K and G_v denotes the ramification group of v.

For higher r, Bayer-Fluckiger, Lee and Parimala [3] made a breakthrough for computing $Ш(L/k)$ in the case where one factor of L is cyclic over k. Morever, when every factor of L is a cyclic extension of k, the authors gave a necessary and sufficient condition for $Ш(L/k) = 0$ under a mild condition. Extending the work [3], Lee [13] gave a general formula for computing $Ш(L/k)$ when all factors of L have p-power degrees. Combining Lee's result and a reduction result [3, Proposition 8.6], the group $Ш(L/k)$ is essentially known when all factors of L are cyclic.

As the last part of this article, we present briefly Lee's formula for $Ш(L/k)$, and describe a further result [7] for computing a certain class of $Ш(L/k)$. Let us write $L = \prod_{i=0}^m K_i$ and assume that each K_i/k is cyclic and that $\cap_{i=0}^r K_i = k$. By [3, Proposition 8.6], each p-primary subgroup $Ш(L/k)(p)$ is isomorphic to $Ш(L(p))$, where $L(p)$ is the maximal étale k-subalgebra of L of p-power degree. Thus, without loss of generality we may assume that each K_i/k has p-power degree, say degree p^{ϵ_i}.

For any $0 \le i, j \le m$, set

(i) $p^{e_{i,j}} = [K_i \cap K_j : k]$, and
(ii) $e_i = \epsilon_0 - e_{0,i}$.

We may assume that $e_i \ge e_{i+1}$ and that $\epsilon_0 = \min_{0 \le i \le m}\{\epsilon_i\}$. For $0 \le r \le \epsilon_0$, set

$$U_r := \{i \in \mathcal{I} |\ e_{0,i} = r\}.$$

Definition 4.7. (1) Let $i, j \in \mathcal{I} := \{1, \dots, m\}$ and l be a nonnegative integer. We say that i, j are *l-equivalent* , denoted by $i \underset{l}{\sim} j$, if $e_{i,j} \ge l$ or $i = j$. For any nonempty subset c of $\mathcal{I}$, let $n_l(c)$ be the number of l-equivalence classes of c.

(2) For each subset $c \subseteq \mathcal{I}$ with $|c| \ge 1$, the *level* of c is defined by

$$L(c) := \min\{e_{i,j} : i, j \in c\}.$$

In [13, Theorem 6.5], Lee proves the following general formula:

$$Ш(L/k) \cong \bigoplus_{r \in \mathcal{R} \smallsetminus \{0\}} \mathbb{Z}/p^{\Delta_r - r}\mathbb{Z} \bigoplus_{r \in \mathcal{R}} \bigoplus_{l \ge L(U_r)} \bigoplus_{c \in U_r/\underset{l}{\sim}} (\mathbb{Z}/p^{f_c - r}\mathbb{Z})^{n_{l+1}(c)-1}, \tag{4.17}$$

where $\mathcal{R} = \{0 \le r \le \epsilon_0 |\ U_r \ne \emptyset\}$. We refer to [13, Sections 4 and 5] (also see [7, Section 2]) for the definitions of the *patching degree* Δ_r of U_r and of the *degree of freedom* f_c of each l-equivalence class c.

In [7], Huang, Liang and the present authors investigate the invariants Δ_r and f_c in Lee's formula when $L = \prod_{i=0}^m K_i$ is assumed to be of Kummer type, namely each cyclic extension K_i is of the form $k(\alpha^{1/p^{\epsilon_i}})$ for some $\alpha \in k^\times$. A basic idea is to describe these invariants in terms of a combinatorial way. The authors also implemented computer programs for computing the $Ш(L/k)$ in the following cases:

- $k = \mathbb{Q}(\zeta_{p^n})$ is a p^nth cyclotomic field extension;
- $F := k(\ell_1^{1/p^n}, \ell_2^{1/p^n})$ is a bicyclic extension over k with distinct rational primes ℓ_1 and ℓ_2; and
- each K_i is a cyclic subextension of F, that is, $K_i = k(\ell_1^{a_i/p^n} \ell_2^{b_i/p^n})$ for some integers $0 \le a_i, b_i < p^n$.

The programs have input data: $p, n, \{a_i, b_i\}_{0 \le i \le m}$, and compute several invariants including Δ_r, c, $n_l(c)$ and f_c in Lee's formula (4.17). The programs use the mathematical software SageMath and can be found on

`https://github.com/hfy880916/Tate-Shafarevich-groups-of-multinorm-one-torus`.

Example 4.8. We put $p = 3$ and $n = 3$, so $k = \mathbb{Q}(\zeta_{27})$. Choose the primes $\ell_1 = 5$ and $\ell_2 = 19$. We consider the multinorm-one torus defined by the following extensions over k: $K_0 = k(\sqrt[27]{5})$, $K_1 = k(\sqrt[27]{5 \times 19})$, $K_2 = k(\sqrt[27]{5^2 \times 19^3})$, $K_3 = k(\sqrt[27]{5^3 \times 19^5})$, $K_4 = k(\sqrt[27]{5^5 \times 19^{11}})$. We list a_i and b_i as follows:

$$\begin{array}{lllll} a_0 = 1, & a_1 = 1, & a_2 = 2, & a_3 = 3, & a_4 = 5, \\ b_0 = 0, & b_1 = 1, & b_2 = 3, & b_3 = 5, & b_4 = 11. \end{array}$$

Using Lee's formula (4.17) and the computer program, we compute the Tate-Shafarevich group

$$Ш(L/k) \simeq (\mathbb{Z}/3\mathbb{Z})^3.$$

Example 4.9. Let $p, n, k, \ell_1, \ell_2, m$ be the same as in Example 4.8. Consider a different multinorm-one torus defined by the following field extensions: $K_0 = k(\sqrt[27]{5})$, $K_1 = k(\sqrt[27]{5 \times 19})$, $K_2 = k(\sqrt[27]{5^2 \times 19^3})$, $K_3 = k(\sqrt[27]{5^4 \times 19^9})$, $K_4 = k(\sqrt[27]{5^{10} \times 19^{19}})$. We list a_i and b_i as follows:

$$\begin{array}{lllll} a_0 = 1, & a_1 = 1, & a_2 = 2, & a_3 = 4, & a_4 = 10, \\ b_0 = 0, & b_1 = 1, & b_2 = 3, & b_3 = 9, & b_4 = 19. \end{array}$$

In this case we obtain $Ш(L/k) \simeq \mathbb{Z}/3\mathbb{Z}$.

In the first example K_i are linearly disjoint. Our computation result agrees with [13, Proposition 7.3]. In the second example some of K_i are not linearly disjoint so there are some contributions from U_r for $r \geq 1$. We refer the reader to [7] for the details.

Acknowledgments

The present project started from the authors' participation of an undergraduate research program in the National Center for Theoretical Sciences (NCTS). They acknowledge the NCTS for the stimulating environment. The authors thank Jiangwei Xue for helpful discussions and are grateful to Valentijn Karemaker for her careful reading of an earlier draft and helpful comments. Yu was partially supported by the NSTC grant 109-2115-M-001-002-MY3 and the Academia Sinica Investigator Award AS-IA-112-M01.

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(Hung) Department of Mathematics, UCLA, 520 Portola Plaza, Los Angeles, CA 90095-1555, USA

Email address: hfy880916@gmail.com

(Yu) Institute of Mathematics, Academia Sinica and NCTS, 6F Astronomy Mathematics Building, No. 1, Roosevelt Rd. Sec. 4 Taipei, Taiwan, 106319

Email address: chiafu@math.sinica.edu.tw