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By W. Michael Hanemann*

In his recent paper Jerry Hausman argues that the enormous interest aroused by Robert Willig's paper on approximations to the compensating and equivalent variations for a single price change has tended to obscure the fact that in many cases exact measures of these welfare concepts can readily be obtained. In particular, he derives exact formulas for the compensating variation implied by an ordinary demand function which (i) exhibits constant price and income elasticities, (ii) is linear in price and income, or (iii) is quadratic in prices and also in income. Using some examples he argues that these exact measures may differ by an order of magnitude from conventional Marshallian measures of consumer's surplus or deadweight loss. I made similar points in my own paper and developed the same formulas for the compensating variation in cases (i) and (ii). Here I will present formulas for some other demand functions, not mentioned by Hausman, which are commonly used in empirical demand analysis. These will be compared with the corresponding formulas for Marshallian consumer's surplus. Before presenting these results, however, I will review Hausman's approach and explain its relation to the standard exposition of integrability theory.

I. Integrability Techniques

Let $x_i(p, y)$, $i = 1, \ldots, N$ be a known set of ordinary demand functions, where p is a vector of N (non-normalized) prices and y is money income. I assume that these demand functions are each homogeneous of degree zero in prices and income and satisfy the adding-up condition, $\Sigma p_i x_i = y$. The standard textbook accounts of integrability — for example, Angus Deaton and John Muellbauer, pp. 49-50 — emphasize the role of the system of N partial differential equations

$$(1) \qquad \frac{\partial m}{\partial p_i} = x_i(p, m) \qquad i = 1, \ldots, N$$

whose solution, the income compensation function, $\mu(p)$, satisfies the initial condition $\mu(p^*) = y^*$. The income compensation function is the key tool of integrability theory. From it one can construct the indirect utility function, $v(p, y)$, and the expenditure function, $e(p, u)$, in terms of which the compensating and equivalent variations are defined, as well as the direct utility function, $u(x)$.

However, because the ordinary demand functions $x_i(p, y)$ possess the homogeneity and adding-up properties, the system of equations (1) can be reduced to an equivalent system of (N - 1) partial differential equations

$$(2) \qquad \frac{\partial \bar{m}}{\partial \pi_i} = \bar{x}_i(\pi, \bar{m}) \qquad i = 1, \ldots, N-1 .$$

Here $\bar{x}_i(\pi, w)$, $i = 1, \ldots, N-1$ are normalized ordinary demand functions; their arguments are the (N-1) relative prices, $\pi_i \equiv p_i/p_N$, $i = 1, \ldots, N-1$, and relative income, $w \equiv y/p_N$.[1] The solution of (2), denoted $\bar{\mu}(\pi)$, satisfies the initial condition $\bar{\mu}(\pi^*) = w^*$. From it one can construct the normalized indirect utility function, $\bar{v}(\pi, w)$, and the normalized expenditure function, $\bar{e}(\pi, u)$. The pairs of functions $\mu(p)$, $\bar{\mu}(\pi)$ and $e(p, u)$, $\bar{e}(\pi, u)$ satisfy Euler's relation for homogeneous functions: $\mu(p) = p_N \cdot \bar{\mu}(p_1/p_N, \ldots, p_{N-1}/p_N)$ and $e(p, u) = p_N \cdot \bar{e}(p_1/p_N, \ldots, p_{N-1}/p_N, u)$. Thus, as an alternative to solving the system (1) and obtaining $e(p, u)$ directly, one can solve the system (2) and then apply Euler's relation.

Some mathematical integrability conditions are required in order for the systems (1) and (2) to possess a solution. These involve a regularity condition on the demand functions and the symmetry of the Slutsky terms. In the case of the system (2), the regularity and symmetry apply to the normalized demand functions

$$(3) \qquad \frac{\partial \bar{x}_i}{\partial \pi_j} + \bar{x}_j \frac{\partial \bar{x}_i}{\partial w} = \frac{\partial \bar{x}_j}{\partial \pi_i} + \bar{x}_i \frac{\partial \bar{x}_j}{\partial w} \qquad i, j = 1, \ldots, N-1.$$

In the case of system (1), the regularity and symmetry apply to the non-normalized demand functions, $x_i(p, y)$. Given that $p_N > 0$, the one set of conditions implies the other. There are also what Leonid Hurwicz calls economic integrability conditions on the systems (1) and (2) which ensure that the underlying utility function is quasi-concave. These involve the negative semi-definiteness of the Slutsky terms. In the case of the system (2) the conditions are

$$(4) \qquad \frac{\partial \bar{x}_i}{\partial \pi_i} + \bar{x}_i \frac{\bar{x}_i}{\partial w} \leq 0 \qquad i = 1, \ldots, N-1.$$

For the special case where $N = 2$, however, the mathematical integrability problem can always be solved without imposing any conditions on the demand functions besides regularity, homogeneity and the adding-up property. This was pointed out by Paul Samuelson in the context of the integrability of the indirect normalized demand functions, which involves a system of partial differential equations dual to (2). In the context of direct integrability it can be seen in two ways. Any pair of non-normalized ordinary demand functions possessing the homogeneity and adding-up properties must satisfy the non-normalized Slutsky symmetry condition (see Donald Katzner, p. 68). Therefore, the pair of partial differential equations in (1) must possess a solution. Alternatively, when $N = 2$ the system (2) collapses to a single ordinary differential equation; the regularity condition on the normalized demand function $\bar{x}_1(\pi_1, w)$ ensures that this differential equation possesses a solution. Thus, when $N = 2$ the utility function can always be recovered from the ordinary demand functions either by solving the system of two partial differential equations (1) or by solving the single ordinary differential equation (2) and applying euler's relation.[2] Textbook examples of integrability for the case where $N = 2$, such as Karl-Goran Maler, pp. 123-25, generally adopt the first approach. The second approach, which is somewhat simpler, is the one that Hausman adopts. The distinction between these two approaches is only implicit in his paper because he sets $p_N = 1$, which leads the normalized and non-normalized expenditure functions to coincide. I emphasize it explicitly here in order to explain how the single ordinary differential equation which appears in his paper relates to the system of partial differential equations which appears in other accounts of integrability theory.

II. New Results on Exact Consumer's Surplus

It follows from the foregoing that, when $N = 2$, exact formulas for the compensating and equivalent variations can readily be obtained for many ordinary demand functions besides those discussed in Hausman's paper. Here I will consider four demand functions which are commonly found in empirical studies as alternatives to the linear and log-linear forms discussed by Hausman. The first is the semi-log form

$$(5) \qquad \bar{x}_1(\pi_1, w) = e^{\alpha\pi_1 + \delta w + \gamma z}$$

where z is a vector of socioeconomic variables.[3] The economic integrability condition implies the restriction that

$$(6) \qquad \alpha + \delta \bar{x}_1(\pi_1, w) \leq 0.$$

The ordinary differential equation corresponding to (2) has the solution

$$e^{-\delta\bar{\mu}(\pi_1)} = -(\frac{\delta}{\alpha})\, e^{\alpha\pi_1 + \gamma z} - \delta c$$

where c is the constant of integration. Taking this as the utility index, one obtains the normalized indirect utility function

$$(7) \qquad \bar{v}(\pi_1, w) = c = \frac{-e^{-\delta w}}{\delta} - \frac{e^{-\alpha\pi_1 + \gamma z}}{\alpha}$$

and the normalized expenditure function

$$\bar{e}(\pi_1, u) = -\frac{1}{\delta}\, \ell n\, [\, -\delta u - \frac{\delta}{\alpha} e^{\alpha\pi_1 + \gamma z}\,]\, .$$

By Euler's relation the non-normalized expenditure function is

$$(8) \qquad e(p_1, p_2, u) = -\frac{p_2}{\delta}\, \ell n\, [\, -\delta u - \frac{\delta}{\alpha} e^{\alpha(p_1/p_2) + \gamma z}\,]\, .$$

This can be used to calculate the exact compensating and equivalent variations associated with a change in one or both prices. Suppose that the individual's income is y^0 and the price of the first good changes from p_1^0 to p_1^1 while the price of the second good stays constant at p_2^0, which is the case considered by Hausman. From (7) the individual's original utility level is

$$(9) \qquad u^0 = \bar{v}(p_1^0/p_2^0, y^0/p_2^0) = -\frac{e^{-y^0/p_2^0}}{\delta} - \frac{e^{\alpha(p_1^0/p_2^0) + \gamma z}}{\alpha}\, .$$

Combining (8) and (9), the compensating variation for this price change is

$$(10) \qquad CV = e(p_1^1, p_2^0, u^0) - y^0$$

$$= -\frac{p_2^0}{\delta}\, \ell n[e^{-\delta y^0/p_2^0} + \frac{\delta}{\alpha} e^{\alpha(p_1^0/p_2^0) + \gamma z} - \frac{\delta}{\alpha} e^{\alpha(p_1^1/p_2^0) + \gamma z}\,] - y^0$$

$$= -\frac{p_2^0}{\delta}\, \ell n[\frac{\delta}{\alpha}(x_1^0 - x_1^1) + 1]$$

where $x_1^t = x_1(p_1^t, p_2^0, y^0)$, $t = 0, 1$. For this price change the conventional Marshallian measure of consumer's surplus is the quantity A, defined as

$$(11) \qquad A = \int_{p_1^0}^{p_1^1} x_1(p_1,\ p_2^0,\ y^0)dp_1$$

$$= \int_{p_1^0}^{p_1^1} \exp [\ \alpha(p_1/p_2^0) + \delta(y^0/p_2^0) + \gamma z\ ]\ dp_1$$

$$= \frac{p_2^0}{\alpha}(x_1^1 - x_1^0).$$

Substituting this into (10) yields the following relation between the exact compensating variation and the conventional Marshallian measure

$$(12) \qquad CV = -\frac{p_2^0}{\delta}\ \ell n[1 - \frac{A}{p_2^0}].$$

Two alternative demand functions which differ from the semi-log form in that the price elasticity and the income elasticity respectively are constant are

$$(13) \qquad \bar{x}_1(\pi_1,\ w) = \pi_1^{\alpha} e^{\delta w + \gamma z}$$

$$\bar{x}_1(\pi_1,\ w) = e^{\alpha\pi_1 + \gamma z} w^{\delta}, \qquad \delta \neq 1.$$

The respective economic integrability conditions are

$$(15) \qquad \alpha + \delta\pi_1\bar{x}_1(\pi_1,\ w) \leq 0$$

$$\alpha + \frac{\delta}{w}\ \bar{x}_1(\pi_1,\ w) \leq 0.$$

In the case of the demand function (13), integration of the ordinary differential equation corresponding to (2) yields

$$(17) \qquad \bar{e}(\pi_1,\ u) = -\frac{1}{\delta}\ \ell n[-\delta u - (\frac{\delta}{1+\alpha})\pi_1^{1+\alpha} e^{\gamma z}]\ .$$

Hence,

$$e(p_1,\ p_2,\ u) = -\frac{p_2}{\delta}\ \ell n\ [-\delta u - (\frac{\delta}{1+\alpha})(p_1/p_2)^{1+\alpha} e^{\gamma z}\ ].$$

For the price change described above

(18) $$CV = -\frac{p_2^0}{\delta}\ \ell n\ [(\frac{\delta}{1+\alpha})(\frac{p_1^0x_1^0 - p_1^1x_1^1}{p_2^0}) + 1]$$

whereas

(19) $$A = \frac{1}{1+\alpha}\ (p_1^1x_1^1 - p_1^0x_1^0).$$

Therefore the relationship between the compensating variation and the Marshallian measure is given by (12). For the demand function (14), integration of the ordinary differential equation corresponding to (2) yields

(20) $$\bar{e}(\pi_1,\ u) = [u + (\frac{1-\delta}{\alpha})e^{\alpha\pi_1 + \gamma z}]^{\frac{1}{1-\delta}}.$$

Hence

$$e(p_1,\ p_2,\ u) = [up_2^{1-\delta} + (\frac{1-\delta}{\alpha})p_2^{1-\delta}e^{\alpha\pi_1 + \gamma z}]^{\frac{1}{1-\delta}}.$$

Accordingly, the compensating variation for the price change is

(21) $$CV = y^0\{[(\frac{1-\delta}{\alpha})(\frac{p_2^0}{y^0})(x_1^1 - x_1^0) + 1]^{\frac{1}{1-\delta}} - 1\}$$

whereas the Marshallian measure is

(22) $$A = \frac{p_2^0}{\alpha}(x_1^1 - x_1^0).$$

Therefore the two measures are related by

(23) $$CV = y^0\{[(1-\delta)\frac{A}{y^0} + 1]^{\frac{1}{1-\delta}} - 1\},$$

which is in fact the general formula derived by Robert Willig for the case of a demand function with a constant non-unitary income elasticity.[4]

Lastly consider the following modification of the linear demand function

(24) $$\bar{x}_1(\pi_1,\ w) = \alpha\,\ell n\,\pi_1 + \delta w + \gamma z$$

for which the economic integrability condition is given by (15). Integration of the ordinary differential equation corresponding to (2) yields

$$\tilde{e}(\pi_1, u) = ue^{\delta\pi_1} - \frac{\alpha \ln \pi_1 + \gamma z}{\delta} + \frac{\alpha}{\delta} e^{\delta \pi_1} Ei(-\delta\pi_1) \tag{25}$$

where Ei(·) is the exponential-integral function.[5] Hence

$$e(p_1, p_2, u) = p_2 u e^{\delta(p_1/p_2)} - (\frac{p_2}{\delta})(\alpha \ln p_1 - \alpha \ln p_2 + \gamma z)$$

$$+ \frac{\alpha p_2}{\delta} e^{\delta(p_1/p_2)} Ei(-\delta p_1/p_2).$$

The resulting formula for the compensating variation is

$$CV = \frac{p_2^0}{\delta} \{x_1^0 \exp [(\frac{\delta}{p_2^0})(p_1^1 - p_1^0)] - x_1^1 + \alpha e^{\delta(p_1^1/p_2^0)} \int_{(p_1^0/p_2^0)}^{(p_1^1/p_2^0)} \frac{e^{-\delta t}}{t} dt \} \tag{26}$$

where the integral could be evaluated by the methods described in Milton Abramowitz and Irene Stegum, Chapter 5. By contrast, the Marshallian consumer's surplus for this demand function is given by

$$A = (p_1^1 x_1^1 - p_1^0 x_1^0) - \alpha(p_1^1 - p_1^0). \tag{27}$$

III. Conclusions and a Caveat

In this paper I have extended Hausman's results on the derivation of exact welfare measures for single price changes to some additional demand functions. The demand functions discussed here and in Hausman's paper probably account for most of the formulations employed in empirical demand studies. However other functional forms can in principle be treated in a similar manner, and some more general results can be obtained. For example, just as Willig shows that for any demand function of the form $\bar{x}_1(\pi_1, w) = a(\pi_1)w^\delta$ the relation between the compensating variation and the conventional

Marshallian consumer's surplus is given by (23), so too it can be shown that for any demand function of the form $\bar{x}_1(\pi_1, w) = a(\pi_1)e^{\delta w}$ this relation is given by (12).[6] Other general classes of demand functions may have the property that the ordinary differential equation corresponding to (2) has a known solution. Examples include $\bar{x}_1(\pi_1, w) = a(\pi_1)w^2 + b(\pi_1)w + c(\pi)$, which leads to Riccati's equation, and $\bar{x}_1(\pi_1, w) = a(\pi_1)w + b(\pi_1)w^{\delta}$, which leads to Bernoulli's equation. In many cases the solution to the differential equation will be available in closed form. In other cases, or where the demand function has a more complex form, the differential equation must be solved by numerical integration. It is in these cases that Willig's approximation results are most valuable. His formulas provide a first-order approximation to the income compensation function associated with an arbitrary demand function;[7] they can be applied even when the exact income compensation function would otherwise have to be obtained by numerical integration.

The method described above can also be employed when there are two goods (N = 2) and both prices change, or when N > 2 and only one price changes. As Hausman points out, the latter case can be treated by invoking Hicks' composite commodity theorem which reduces the multivariate utility function $u(x_1, \ldots, x_N)$ to an equivalent bivariate utility function $u^*(x_1, x_c)$, where $x_c = x_2 + \Sigma_3^N(p_i/p_2)x_i$. The general case where N > 2 and two or more prices change is much harder to treat by the methods described above, since then the symmetry conditions (3) represent a nontrivial constraint on the solution of the system of partial differential equations (2). In principle one may be able to apply Willig's approximations to a sequence of single price changes, as he suggested in his original unpublished technical report. However, it must be emphasized that if the ordinary demand functions do not satisfy the symmetry conditions, they cannot be shown to be generated by a conventional utility maximization process.

It is also important that the negative semi-definiteness conditions (4) be satisfied, although they are sometimes overlooked by practitioners of demand analysis. This applies whether one is dealing with a single price change or multiple price changes, and with exact welfare measures or Willig's approximations. If these conditions are violated over some or all of the price-income space in question, the exact or approximate welfare measures that one obtains are meaningless. To see this consider the single price change mentioned above.[8] There is logically an upper bound on the compensation required to offset the effects of this price change, namely the extra amount of money that would be needed to buy the original quantity of good 1 at the new price

$$CV \leq x_1^0(p_1^1 - p_1^0). \tag{28}$$

This restriction is, in fact, the Laspeyres upper bound on the true cost-of-living index rewritten in a slightly unfamiliar form.[9] If the compensated demand function for good 1 slopes downwards the restriction will be satisfied, since the compensating variation is simply the area under this compensated demand function between the prices p_1^1 and p_1^0. However, if the negative semi-definiteness condition (4) is violated, the compensated demand function has a positive slope. In that case, whether one calculates the compensating variation by the exact method described above or by Willig's approximation, it will violate the restriction in (28). This should be borne in mind before one engages in applied welfare analysis based on empirical ordinary demand functions.

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FOOTNOTES

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[1]I am assuming that $P_N > 0$. The equivalence of (1) and (2) is mentioned by Leonid Hurwicz, pp. 204-05.

[2]Note that the demand functions must still satisfy the negative semi-definiteness condition (4).

[3]In practice this function would usually be estimated by ordinary least squares in the form: $\ln x_1 = a\pi_1 + \delta w + \gamma z$.

[4]See Willig's equation (15). Note that the same formula applies to the log-linear demand function discussed by Hausman, $\bar{x}_1(\pi_1, w) = e^{\gamma z}\pi_1^{\alpha} w^{\delta}$.

[5]It is not possible, however, to obtain a closed form solution to the ordinary differential equation (2) generated by the alternative variant of the linear demand function $\bar{x}_1(\pi_1, w) = \alpha\pi_1 + \delta \ln w + \gamma z$.

[6]For this class of demand functions, integration of the ordinary differential equation corresponding to (2) yields

$$e(p_1^1, p_2^0, u^0) - y^0 = -\frac{p_2^0}{\delta} \ln[1 - \delta\int_{\pi_1^0}^{\pi_1^1} a(\pi_1) e^{\delta(y^0/p_2^0)} d\pi_1].$$

Making a change of variable from π_1 to p_1 produces the general formula (12).

[7]See his equation (19), in particular.

[8]Essentially the same argument applies to more general price changes.

[9]In the present context, the Laspeyres upper bound would conventionally be written in the form

$$\frac{e(p_1^1, p_2^0, y^0)}{y^0} \leq \frac{p_1^1 x_1^0 + p_2^0 x_2^0}{p_1^0 x_1^0 + p_2^0 x_2^0} .$$

Multiplying both sides by y^0 and then subtracting y^0 from both sides, remembering that $y^0 = \Sigma p_i^0 x_i^0$, yields (28).