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Unipotent Radicals of the Standard Borel and Parabolic Subgroups in Quantum Special Linear Groups

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by

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by

Andrew Cecil Jaramillo

To my wife, Maree.

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Abstract

Unipotent Radicals of the Standard Borel and Parabolic Subgroups in Quantum Special Linear Groups

Andrew Cecil Jaramillo

In this dissertation we find noncommutative analogues of the coordinate rings of the unipotent radicals of the standard Borel and standard parabolic subgroups in quantum special linear groups. In each case, two subalgebras are defined, both of which can be considered quantizations of the unipotent radical of a standard Borel or a standard parabolic subgroup. Presentations are given for these algebras. It is also shown that these algebras arise as a coinvariant subalgebra of a natural comodule algebra action induced from the Hopf algebra structure on quantum special linear groups.

Contents

Introduction

Quantum groups were first discovered in the 1980's in mathematical physics while studying the quantum inverse scattering method. Since that time there has been much work done on these groups, studying their various properties. Still though, there is not yet a widely accepted axiomatic definition for them. (See [1, Problem II.10.1].) Nevertheless, this has not prevented many objects being given the title of "quantum group." What many of these groups have in common is that they are noncommutative deformations (in some sense) of classical k-algebras with a parameter q, with the property that as q approaches special values (e.g. $q \to 1$), we recover the classical k-algebra. Thus, a quantum group is a noncommutative object that "acts like" some classical object, and from the quantum group the classical structure is recovered.

For instance, the quantized coordinate ring of SL_{n+1} or quantum SL_{n+1} for short, is a noncommutative deformation of $O(\mathrm{SL}_{n+1})$, the coordinate ring of SL_{n+1}. If we take a presentation for quantum SL_{n+1} (see (1.19) – (1.23)), then as $q \to 1$ we get a presentation for $O(\mathrm{SL}_{n+1})$. Thus, we may think of quantum SL_{n+1}

as a coordinate ring of a "noncommutative space" which "limits to" $O(\mathrm{SL}_{n+1})$. Of course this can be made more precise (see [6]), but the basic idea is the following: quantum SL_{n+1} is the coordinate ring of a space that has "vanished" leaving behind the "shadow" of SL_{n+1}.

With this general framework of a quantum group in mind, the purpose of the present document is to define and study the properties of quantized coordinate rings for the unipotent radicals of standard Borel and standard parabolic groups of SL_{n+1}. To do this we remind the reader of the classical version of these groups, and show obstacles to finding quantum analogues of them.

Standard Borel Subgroups and their Unipotent Radicals in SL_{n+1}

In SL_{n+1} the **positive** (**negative**) **standard Borel subgroup** B^+ (respectively B^-) is the subgroup consisting of all upper (lower) triangular matrices in SL_{n+1}. The **positive** (**negative**) **unipotent radical** N^+ (respectively N^-) of B^+ (respectively B^-) is the subgroup of upper (lower) triangular unipotent matrices in B^+ (respectively B^-).

Since $B^\pm$ are closed subvarieties of SL_{n+1}, it follows that for the coordinate rings $O(B^\pm)$ we have

$$O(B^+) \cong O(\mathrm{SL}_{n+1})/\langle X_{ij} \mid i > j\rangle \quad \text{and} \quad O(B^-) \cong O(\mathrm{SL}_{n+1})/\langle X_{ij} \mid i < j\rangle.$$

Similarly, since $N^{\pm}$ are closed subvarieties of SL_{n+1}, for the coordinate rings $O(N^{\pm})$ we have

$$O(N^{\pm}) \cong O(B^{\pm})/\langle X_{ii} - 1 \mid i = 1, \ldots, n, n+1 \rangle.$$

We now wish to "quantize" these coordinate rings using the above general framework. Moreover, since $B^{\pm}$ are Poisson-algebraic subgroups of SL_{n+1} we also require the semiclassical limits, as in [6], of $O_q(B^{\pm})$ to be $O(B^{\pm})$ as Poisson-algebras. Having this property does not leave us much choice in defining $O_q(B^{\pm})$. Thus, following [25, Section 6.1] we define the **quantized coordinate rings of the standard Borel subgroups** or **quantum standard positive (negative) Borel subgroup** for short, to be

$$O_q(B^+) := O_q(\mathrm{SL}_{n+1})/\langle X_{ij} \mid i > j \rangle \quad \text{and} \quad O_q(B^-) := O_q(\mathrm{SL}_{n+1})/\langle X_{ij} \mid i < j \rangle.$$

Attempting to define the quantized coordinate rings of the unipotent radicals in the same way as in the classical case gives us

$$O_q(B^{\pm})/\langle X_{ii} - 1 \mid 1 \leq i \leq n+1 \rangle.$$

However, this would not be helpful to us since relation (1.19) in $O_q(\mathrm{SL}_{n+1})$ implies that for all $i \neq j$

$$X_{ij} = X_{ii}X_{ij} = qX_{ij}X_{ii} = qX_{ij}.$$

For $q \neq 1$ this implies $X_{ij} = 0$ for all $i \neq j$. Thus, $O_q(N^{\pm}) \cong k$ ([25, Remark 6.3]). Though this may be a nice algebra to study, it is not a particularly useful analogue to the classical setting. Therefore, we must try and define $O_q(N^{\pm})$ in another way.

There are (surprisingly) few definitions found in the literature although some authors (e.g. [4]) have defined $O_q(N^{\pm})$ to be $U_q^{\pm}(\mathfrak{sl}_{n+1})$; since when $q \to 1$ we recover $O(N^{\pm})$. This definition "quantizes" $O(N^{\pm})$ however, we would like to find an algebra more directly related to $O_q(B^{\pm})$. Since $N^{\pm}$ are *not* Poisson-algebraic subgroups of $B^{\pm}$ there are fewer requirements when defining quantized coordinate rings for them. In contrast to the classical case, there are no "natural" quotient algebras of $O_q(B^{\pm})$ that reduce to $O(N^{\pm})$ when $q \to 1$. (See [25, Remark 6.3.].) Nevertheless, there exist candidate subalgebras of $O_q(B^{\pm})$ that do have this property; some of which were defined in [8], and thus are good candidates for the definition of $O_q(N^{\pm})$.

Another possible way to define $O_q(N^{\pm})$ is to first note that $B^{\pm}$ is the semidirect product $T \ltimes N^{\pm}$ where T is the diagonal subgroup (standard maximal torus) of SL_{n+1}. Thus, $O(N^{\pm})$ is the algebra of coinvariants using the corresponding coaction of $O(T)$ on $O(B^{\pm})$. Hence, another possible definition for $O_q(N^{\pm})$ is as the subalgebra of coinvariants for a natural coaction of a quantized standard maximal torus, $O_q(T)$, on $O_q(B^{\pm})$ using the Hopf algebra structure of $O_q(\mathrm{SL}_{n+1})$.

Therefore, with this in mind we define *two* subalgebras in $O_q(\mathrm{SL}_{n+1})$ which can be considered **quantized coordinate rings for the unipotent radicals** of positive (negative) standard Borel subgroups. Presentations for these subalgebras are found and each of these subalgebras has the property that when $q \to 1$, the coordinate ring of the standard unipotent radical is recovered. Moreover, the following properties are also shown:

(i) All of the quantized coordinate rings for the unipotent radical of a standard Borel subgroup are isomorphic.

(ii) Each quantized coordinate ring for the unipotent radical of a standard Borel subgroup is the coinvariant subalgebra of a natural coaction of the quantized standard maximal torus on the quantum standard Borel subgroup.

(iii) The quantized coordinate ring of a standard Borel subgroup is a smash product of a quantized unipotent radical and the quantized coordinate ring of the standard maximal torus.

(iv) Each quantized coordinate ring for the unipotent radical of a standard Borel subgroup is isomorphic to the quantized universal enveloping algebra of the standard nilpotent subalgebra of $\mathfrak{sl}_{n+1}$.

It follows that all of the above possibilities for defining a quantized coordinate ring for the unipotent radical of a standard Borel will lead to isomorphic algebras.

Standard Parabolic Subgroups and their Unipotent Radicals in SL_{n+1}

In SL_{n+1} a **positive (negative) standard parabolic subgroup** is a subgroup that contains a positive (negative) standard Borel subgroup. In particular, standard Borel subgroups and SL_{n+1} are examples of standard parabolic subgroups.

In general though, positive (negative) standard parabolic subgroups are composed of matrices in SL_{n+1} that are "upper (lower) triangular block matrices." That is, the matrices in the subgroup can be written with $m \times m$ block matrices on the diagonal (with each block possibly a different size m) and 0 on the entries below (above) these blocks. The subgroup of matrices in SL_{n+1} that are nonzero on the "block diagonal" and zero elsewhere also form a subgroup called the **Levi subgroup** for the parabolic group. Moreover, the **positive (negative) unipotent radical** of a positive (negative) standard parabolic is composed of upper (lower) triangular matrices with identity matrices on the "block diagonal."

For instance, in SL_7, an example of a positive standard parabolic group is composed of matrices in SL_7 of the following form:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} \\ 0 & 0 & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} \\ 0 & 0 & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} \\ 0 & 0 & 0 & 0 & a_{55} & a_{56} & a_{57} \\ 0 & 0 & 0 & 0 & a_{65} & a_{66} & a_{67} \\ 0 & 0 & 0 & 0 & a_{75} & a_{76} & a_{77} \end{pmatrix}$$

The Levi subgroup is composed of matrices that are nonzero in the indicated diagonal blocks and zero elsewhere. In addition, the positive unipotent radical consists of matrices of the following form:

$$\begin{pmatrix} 1 & 0 & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} \\ 0 & 1 & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} \\ 0 & 0 & 1 & 0 & a_{35} & a_{36} & a_{37} \\ 0 & 0 & 0 & 1 & a_{45} & a_{46} & a_{47} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Just as with standard Borel subgroups, their unipotent radicals, and the standard maximal torus; standard parabolic groups, their unipotent radicals, and the Levi subgroup are closed subvarieties of SL_{n+1}.

When attempting to "quantize" these coordinate rings, again following [25, Section 6.1] a **quantized coordinate ring for a positive (negative) standard parabolic subgroups** or a **quantum positive (negative) standard parabolic subgroup**, can be defined in the same way as the quantized coordinate ring of a standard Borel was defined. That is, by direct analogy with the classical case. Similarly, we define the **quantized coordinate ring of the Levi subgroup** by direct analogy with the coordinate ring of the Levi subgroup.

However, just as with unipotent radicals of standard Borel subgroups, attempting to define quantized coordinate rings for the unipotent radicals as the natural quotient algebras will give us an algebra isomorphic to k. Therefore, we must find a quantized coordinate ring of the unipotent radical of a standard parabolic in another way.

Again, just as with the unipotent radical of a standard Borel, we define *two* **quantized coordinate rings for the unipotent radical** of a positive (negative) standard parabolic group, which are generated by products of quantum minors of a certain form. We then show the following:

(i) A quantized coordinate ring for the unipotent radical of a standard parabolic subgroup is the subalgebra of coinvariants of a natural coaction of the quantized standard Levi on the quantum standard parabolic subgroup.

(ii) The quantized coordinate ring of a standard parabolic subgroup is the smash product of a quantized unipotent radical of the standard parabolic subgroup and the quantized coordinate ring of the standard Levi.

(iii) A presentation for a quantum unipotent radical for a positive (negative) standard parabolic subgroup is found, and has the property that when $q \to 1$ the coordinate ring of the standard unipotent radical is recovered.

(iv) A quantized coordinate ring for the unipotent radical of a standard parabolic subgroup is a noncommutative UFD and satisfies the Dixmier-Moeglin Equivalence.

We note that a quantum standard Borel is a special instance of a quantum standard parabolic, hence many of the theorems from Chapter 2 are special instances of theorems from Chapter 3.

Chapter 1

Background

1.1 Notation and Conventions

Let k be a field. Unless otherwise noted, we make no further assumptions on k. For two k-vector spaces V, W we use $V \otimes K$ to denote $V \otimes_k W$. All rings will contain 1. All ring homomorphisms have the property that $f(1) = 1$.

1.2 Algebras, Coalgebras, Bialgebras, and Hopf Algebras

Definition 1.1. *A* **k-algebra** *is a ring, A, with a ring homomorphism $\eta : k \to A$,whose image is contained in the center of A. The map η is called the* **structure map** *for the algebra.*

Using the map η, A is a k-vector space by the following rule:

$$ra = \eta(r)a$$

for all $r \in k$ and $a \in A$.

A **k-algebra homomorphism** or **a morphism of algebras** is a ring homomorphism $f : A \to A'$ so that $f(\eta(r)) = \eta'(r)$ for all $x, y \in A$ and $r \in k$ where η' is the structure map of A'.

Changing our point of view slightly, multiplication in the ring A may be thought of as a k-linear map $\mu : A \otimes A \to A$ which satisfies certain axioms. Specifically, a k-algebra A is a k-vector space with k-linear maps η and μ that make the following diagrams commute:

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "\mu\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\mu"'] & A \otimes A \arrow[d, "\mu"] \\
A \otimes A \arrow[r, "\mu"'] & A
\end{tikzcd} \tag{1.1}$$

$$\begin{tikzcd}
 & A & \\
k \otimes A \arrow[ur, "l"] \arrow[r, "\eta\otimes\mathrm{id}"'] & A \otimes A \arrow[u, "\mu"] & A \otimes k \arrow[ul, "r"'] \arrow[l, "\mathrm{id}\otimes\eta"]
\end{tikzcd} \tag{1.2}$$

where l, r are the canonical isomorphisms defined by $l(s \otimes a) = sa$ and $r(a \otimes s) = as$ for all $s \in k$ and $a \in A$.

We may also reinterpret a k-algebra homomorphism by using using the maps, μ and η. Specifically, f is a k-algebra homomorphism if and only if

$$f \circ \mu = \mu' \circ (f \otimes f) \qquad \text{and} \qquad f \circ \eta = \eta'$$

where μ' is multiplication map and η' is the structure map on A'.

For any k-algebra A we may form the **opposite algebra** denoted A^{op} whose underlying vector space is the same as A and whose structure map is the same as A, but where the multiplication map, μ^{op}, is defined by

$$\mu^{\mathrm{op}}(a \otimes b) := \mu(b \otimes a)$$

for all $a, b \in A$. Notice, A is a commutative algebra if and only if $\mu = \mu^{\mathrm{op}}$.

A k-linear map $f : A \to B$ so that $f(ab) = f(b)f(a)$ for all $a, b \in A$ is referred to as an **anti-homomorphism**. We note that an anti-homomorphism is a morphism $f : A \to B^{\mathrm{op}}$.

Definition 1.2. *A* k-**coalgebra** *is a* k*-vector space,* C*, equipped with* k*-linear maps* $\Delta : C \to C \otimes C$ *and* $\epsilon : C \to k$ *that make the following diagrams commute:*

$$\begin{tikzcd} C \arrow[r, "\delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\mathrm{id}\otimes\Delta"] \\ C \otimes C \arrow[r, "\Delta\otimes\mathrm{id}"'] & C \otimes C \otimes C \end{tikzcd} \tag{1.3}$$

$$\begin{tikzcd} k \otimes C & C \otimes C \arrow[l, "\epsilon\otimes\mathrm{id}"'] \arrow[r, "\mathrm{id}\otimes\epsilon"] & C \otimes k \\ & C \arrow[ul, "\lambda"] \arrow[u, "\Delta"] \arrow[ur, "\rho"'] & \end{tikzcd} \tag{1.4}$$

where λ, ρ are the canonical isomorphisms defined by $\lambda(c) = 1 \otimes c$ and $\rho(c) = c \otimes 1$ for all $c \in C$.

The map Δ is referred to as the **comultiplication map** *and ϵ is called the* **counit**.

Note that a coalgebra dualizes the definition of an algebra. That is, diagram (1.3) reverses the arrows of diagram (1.1) and diagram (1.4) reverses the arrows of diagram (1.2). For this reason the axiom diagram (1.3) is referred to as **coassociativity** and the axiom diagram (1.4) is referred to as the **counital axiom**.

A **coalgebra homomorphism** or a **morphism of coalgebras** is a k-linear map $f : C \to C'$ such that

$$(f \otimes f) \circ \Delta = \Delta' \circ f \qquad \text{and} \qquad \epsilon = \epsilon' \circ f$$

where Δ', ϵ' are the comultiplication and counit of C', respectively. Again note that a morphism of coalgebras dualizes the definition for a k-algebra homomorphism.

Let $\mathrm{op} : C \otimes C \to C \otimes C$, be the k-linear map defined by $\mathrm{op}(c \otimes c') = c' \otimes c$ for $c, c' \in C$. For any coalgebra C we may form the **co-opposite coalgebra** denoted C^{cop} whose underlying vector space is the same as C, but where the comultiplication map, Δ^{op}, is defined by

$$\Delta^{\mathrm{op}} := \mathrm{op} \circ \Delta.$$

A coalgebra is **co-commutative** if $\Delta = \Delta^{\mathrm{op}}$. A morphism $f : C \to D^{\mathrm{op}}$ is referred to as an **coalgebra anti-homomorphism from C to D**.

Sweedler's Notation

For all x in a coalgebra C we have

$$\Delta(x) = \sum_{i=1}^{n} x_i' \otimes x_i''.$$

In order to make what follows more readable, we adopt **Sweedler's sigma notation**. That is, we will denote the above expression by

$$\Delta(x) = \sum_{(x)} x_1 \otimes x_2$$

where x_1 and x_2 stand for the first and second components of the pure tensors in $\Delta(x)$. Using this notation, coassociativity of Δ becomes

$$\sum_{(x)} \left(\sum_{(x_1)} (x_1)_1 \otimes (x_1)_2 \right) \otimes x_2 = \sum_{(x)} x_1 \otimes \left(\sum_{(x_2)} (x_2)_1 \otimes (x_2)_2 \right).$$

Since this is still a bit unwieldy, simplifying notation further we denote either side of this equation by

$$\sum_{(x)} x_1 \otimes x_2 \otimes x_3.$$

In this notation f is a coalgebra homomorphism if and only if

$$\sum_{(x)} f(x_1) \otimes f(x_2) = \sum_{(f(x))} f(x)_1 \otimes f(x)_2$$

for all $x \in C$. Finally, we note that

$$\Delta^{\text{op}}(x) = \sum_{(x)} x_2 \otimes x_1.$$

Bialgebras and Hopf Algebras

Definition 1.3. *A* **bialgebra** *is a k-algebra, H, that is also a coalgebra so that the comultiplication Δ and counit ϵ are morphisms of algebras. Equivalently, from [17, Theorem III.2.1] H is a bialgebra if the maps μ and η are coalgebra morphisms.*

A **morphism of bialgebras** is a map that is both a morphism of algebras and a morphism of coalgebras.

For a bialgebra H, let H^{op} denote the vector space that has the same underlying k-vector space as H, but with structure map η, multiplication μ^{op}, comultiplication Δ, and counit ϵ. Similarly, H^{cop} has structure map η, multiplication

μ, comultiplication Δ^{op}, and counit ϵ, From [17, Proposition II.2.2] if H is a bialgebra, then so are H^{op} are H^{cop}.

Let C be a coalgebra and A a k-algebra.

Definition 1.4. *The* **convolution product,** $\star$**,** *of two k-linear maps $f, g : C \to A$ is defined as the linear map $f \star g : C \to A$ such that*

$$(f \star g)(x) = \sum_{(x)} f(x_1)g(x_2).$$

Notice, for any linear map $f : C \to A$ that $f \star \eta\epsilon = \eta\epsilon \star f = f$. Hence, C^* and $\text{Hom}_k(C, A)$ are k-algebras with multiplication $\star$ and unit, $\eta\epsilon$.

Definition 1.5. *Let $\gamma : C \to A$ be a k-linear map where $\gamma(1) = 1$. Then γ is* **convolution invertible** *if there exists a k-linear map $\overline{\gamma} : C \to A$ so that*

$$\gamma \star \overline{\gamma} = \overline{\gamma} \star \gamma = \eta \circ \epsilon.$$

Let H be a bialgebra. An **antipode, S,** is an endomorphism of H so that

$$\text{id} \star S = S \star \text{id} = \eta \circ \epsilon.$$

That is, the identity map on H is convolution invertible with convolution inverse S.

Using Sweedler's notation, this can be rewritten as

$$\sum_{(x)} S(x_1)x_2 = \sum_{(x)} x_1 S(x_2) = \epsilon(x) \cdot 1$$

for $x \in H$.

Definition 1.6. *A* **Hopf algebra** H *is a bialgebra with an antipode,* S.

From [17, Theorem II.3.4] we have that S is an algebra anti-homomorphism. A **morphism of Hopf algebras** is a map $f : H \to H'$ that is a morphism between the underlying bialgebras so that $f \circ S = S' \circ f$ where S and S' are the antipodes of H and H', respectively.

From [17, Corollary II.3.5] H^{opcop} is a a Hopf algebra with antipode S. Moreover, if S is an anti-isomorphism, then H^{op} and H^{cop} are also Hopf algebras with antipode S^{-1}.

Definition 1.7. *A* **Hopf ideal** I *in a Hopf algebra* H *is an ideal of* H *so that* $I \subset \ker \epsilon$ *and* $\Delta(I) \subset H \otimes I + I \otimes H$ *and* $S(I) \subset I$.

If I is a Hopf ideal then H/I is also a Hopf algebra with comultiplication $\overline{\Delta}$, counit $\overline{\epsilon}$, and antipode $\overline{S}$ induced from Δ, ϵ, and S [23, Section 1.1]. We say H/I is the **quotient or factor Hopf algebra** induced from the Hopf ideal I.

The **Hopf dual** for H is the subspace of H^* defined by

$$H^\circ := \{f \in H^* \mid f(I) = 0 \text{ some ideal } I \text{ with } \dim_k(H/I) < \infty\}.$$

We note that H° is a subalgebra of H^*. Also, H° is a Hopf algebra using the transpose of the comultiplication map in H.

1.3 Modules, Comodules, and Comodule Algebras

Definition 1.8. *Let A be a k-algebra with structure map η and multiplication map μ. A* **(left) A-module** *M, is a k-vector space with a linear map $\mu_M : A \otimes M \to M$ such that the following diagrams commute*

$$\begin{tikzcd} A \otimes A \otimes M \arrow[r, "\mu \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes \mu_M"'] & A \otimes M \arrow[d, "\mu_M"] \\ A \otimes M \arrow[r, "\mu_M"'] & M \end{tikzcd} \tag{1.5}$$

$$\begin{tikzcd} & A \\ k \otimes A \arrow[ur, "l_\mu"] \arrow[r, "\eta \otimes \mathrm{id}"'] & A \otimes M \arrow[u, "\mu_M"'] \end{tikzcd} \tag{1.6}$$

where l_M is the canonical isomorphism defined by $l_\mu(s \otimes a) = sa$ for $s \in k$ and $a \in A$. The map μ_A is called the **structure map** *for the A-module M or the* **action** *of A on the module. A* **right A-module** *is defined similarly.*

A **morphism of A-modules** is a linear map $f : M \to M'$ so that $\mu_{M'} \circ (\mathrm{id} \otimes f) = f \circ \mu_M$. For $a \in A$ and $m \in M$ we will often denote $\mu_M(a \otimes m)$ by am or $a.m$. Similar conventions also hold for a right module.

Modules over a Hopf Algebra

Let H be a bialgebra. If M and M' are both H-modules, then $M \otimes M'$ is also an H-module with H-action defined by

$$h.(m \otimes m') := \sum_{(h)} h_1.m \otimes h_2.m'$$

for all $m \in M$, $m' \in M'$, and $h \in H$.

Now suppose that H is a Hopf algebra. For M an H-module, the **dual H-module**, denoted M^*, is a left H-module on the dual space (i.e. the space of linear functions $f : M \to k$) with H-action defined by

$$(h.f)(m) = f(S(h).m)$$

for all $f \in M^*$, $h \in H$, and $m \in M$. In addition, we also define a right H-module on M^* given by

$$(f.h)(x) = f(h.x)$$

for all $h \in H$, $f \in M^*$ and $x \in M$.

Comodules, Comodule Algebras, and Module Algebras

Definition 1.9. *Let C be a coalgebra. A* **(left) C-comodule,** M, *is a k-vector space with a k-linear map $\Delta_M : M \to C \otimes M$ such that the following diagrams* *commute:*

$$\begin{tikzcd} M \arrow[r, "\Delta_M"] \arrow[d, "\Delta_M"'] & C \otimes M \arrow[d, "\mathrm{id}\otimes\Delta_M"] \\ C \otimes M \arrow[r, "\Delta\otimes\mathrm{id}"'] & C \otimes C \otimes M \end{tikzcd} \tag{1.7}$$

$$\begin{tikzcd} k \otimes M & C \otimes M \arrow[l, "\epsilon\otimes\mathrm{id}"'] \\ & M \arrow[ul, "\lambda_M"] \arrow[u, "\Delta_M"'] \end{tikzcd} \tag{1.8}$$

where λ_M is the canonical isomorphism defined by $\lambda_M(m) = 1 \otimes m$ for all $m \in M$. The map μ_M is called the **coaction** *of the coalgebra. A* **right C-comodule** *is defined similarly.*

A **morphism of C-comodules** is a linear map $f : M \to M'$ so that $(\mathrm{id} \otimes f) \circ \Delta_M = \Delta_{M'} \circ f$.

Note the diagrams in the definition of a comodule dualizes the definition of a module. That is, diagram (1.7) reverse the arrows of diagram (1.5) and diagram (1.8) reverses the arrows of diagram (1.6). We note that C is a C-comodule using the comultiplication Δ.

Definition 1.10. *Let H be a bialgebra and A an algebra. We say A is an* **H-comodule algebra (on the left)** *if*

(i) A is a left H-comodule with coaction $\Delta_A : A \to H \otimes A$;

(ii) Δ_A is a morphism of algebras.

An **H-comodule algebra on the right** *is defined similarly.*

It follows from [17, Proposition III.7.2] that if $\mu_A : A \otimes A \to A$ and $\eta_A : k \to A$ are the structure maps for the algebra A and A is a left H-comodule, then A is an H-comodule algebra if and only if μ_A and η_A are H-comodule morphisms.

Definition 1.11. *Let H be a bialgebra and A an algebra. We say A is an* **H-module algebra (on the left)** *if*

(i) A is an H-module;

(ii) The multiplication map and structure map for A are morphisms of H-modules.

An **H-module algebra on the right** *is defined similarly.*

1.4 Smash Product and Coinvariants

Definition 1.12. *Let A be a left H-module algebra. The* **(right) smash product** *denoted $A\#H$ is defined as follows,*

(i) As a k-vector space $A\#H = A \otimes H$ and we write $a\#h$ for the elements $a \otimes h$;

(ii) Multiplication is defined by

$$(a\#h)(b\#k) = \sum_{(h)} a(h_1.b)\#h_2k$$

for all $a, b \in A$ and $h, k \in H$.

Similarly, if A is a right H-module algebra the **left smash product** *$H\#A$ has underling k-vector space $H \otimes A$ and multiplication defined by*

$$(h\#a)(k\#b) = \sum_{(k)} hk_1\#(a.k_2)b$$

for all $h, k \in H$ and $a, b \in A$.

Definition 1.13. *Let H be a bialgebra. A* **(left) coinvariant** *for a left H-comodule algebra B is any element $b \in B$ so that $\Delta_B(b) = 1 \otimes b$. Similarly, a* **right coinvariant** *for a right comodule algebra is an element b so that $\Delta_B(b) = b \otimes 1$.*

Since Δ_B is an algebra homomorphism, it follows that if $a, b \in B$ are left (right) coinvariants then ab is also a left (right) coinvariant. Therefore, the left (right) coinvariants of B form a subalgebra that we denote by $B^{\mathrm{co}\,\Delta_B}$.

Definition 1.14. *Let H be a Hopf algebra, B a right H-comodule algebra, and $A = B^{\mathrm{co}\,\Delta_B}$. The comodule algebra B is called a* **(right) H-cleft extension** *of A if there is an H-comodule morphism $\gamma : H \to B$ where $\gamma(1) = 1$ that is convolution invertible (with convolution inverse $\overline{\gamma}$). A* **left H-cleft extension** *is defined similarly for a left H-comodule algbera.*

According to a result of [2], shown in [23, Proposition 7.2.3], for B an H-cleft extension with $\sigma : H \otimes H \to A$ defined by

$$\sigma(h,k) := \sum_{(h),(k)} \gamma(h_1)\gamma(k_1)\overline{\gamma}(h_2k_2)$$

there is a left H-action on A is given by

$$h.a := \sum_{(h)} \gamma(h_1)a\overline{\gamma}(h_2),$$

and there is a well-defined multiplication on $A \otimes H$ given by

$$(a \otimes h)(b \otimes k) := \sum_{(h),(k)} a(h_1.b)\sigma(h_2,k_1) \otimes h_3k_2 \tag{1.9}$$

for all $a, b \in A$ and $h, k \in H$.

Similarly, if B' is a left H-cleft extension of A' with

$$\sigma'(h,k) := \sum_{(h),(k)} \overline{\gamma}(h_1k_1)\gamma(h_2)\gamma(k_2)$$

there is a right H-action on A' is given by

$$a.h := \sum_{(h)} \overline{\gamma}(h_1)a\gamma(h_2),$$

and there is a well-defined multiplication on $H \otimes A'$ given by

$$(h \otimes a)(k \otimes b) := \sum_{(h),(k)} h_1 k_1 \otimes \sigma'(h_2, k_2)(a.h_3)b \tag{1.10}$$

for all $a, b \in A'$ and $h, k \in H$.

Definition 1.15. *The k-algebra with multiplication defined by equation* (1.9) *is called a* **(right) crossed product** *of A and H. It is denoted $A\#_\sigma H$ and the vectors $a \otimes h$ are denoted by $a\#h$. Similarly, $H\#_{\sigma'}A'$ is the* **left crossed product** *of H and A' with multiplication defined by equation* (1.10)*.*

For the remainder of this section H will be a Hopf algebra, B an H-comodule algebra, and $A = B^{\mathrm{co}\,\Delta_B}$.

Definition 1.16. *Let B be an H-cleft extension of A with map $\gamma : H \to B$ and B' be an H'-cleft extension of A' with map $\gamma' : H' \to B'$. Let $\tau : H \to H'$ be a Hopf algebra homomorphism and let $f : A \to A'$ be a k-algebra homomorphism. The maps τ and f are* **H-cleft intertwining** *if they have the property that $\gamma'\tau = f\gamma$ and $\overline{\gamma'}\tau = f\overline{\gamma}$. Similar definitions also hold for left H-cleft extensions.*

Proposition 1.17. *Let $\tau : H \to H'$ be a Hopf algebra homomorhpism and $f : B \to B'$ a k-algebra homomorphism with $f(A) \subseteq A'$. If τ and f are H-cleft intertwining maps then the map $f \otimes \tau : A\#_\sigma H \to A'\#_{\sigma'}H'$ is a k-algebra homomorphism.*

Similar statements also hold for left crossed products.

Proof. First note that for all $h, k \in H$ we have

$$f(\sigma(h,k)) = f\left(\sum_{(h),(k)} \gamma(h_1)\gamma(k_1)\overline{\gamma}(h_2k_2)\right) = \sum_{(h),(k)} f(\gamma(h_1))f(\gamma(k_1))f(\overline{\gamma}(h_2k_2))$$

$$= \sum_{(h),(k)} \gamma'(\tau(h_1))\gamma'(\tau(k_1))\overline{\gamma'}(\tau(h_2k_2))$$

$$= \sum_{(\tau(h)),(\tau(k))} \gamma'(\tau(h)_1)\gamma'(\tau(k)_1)\overline{\gamma'}(\tau(h)_2\tau(k)_2)$$

$$= \sigma'(\tau(h),\tau(k)).$$

Moreover, for all $h \in H$ and $a \in A$ we have

$$f(h.a) = f\left(\sum_{(h)} \gamma(h_1)a\overline{\gamma}(h_2)\right) = \sum_{(h)} f(\gamma(h_1))f(a)f(\overline{\gamma}(h_2))$$

$$= \sum_{(h)} \gamma'(\tau(h_1))f(a)\overline{\gamma'}(\tau(h_2))$$

$$= \sum_{(\tau(h))} \gamma'(\tau(h)_1)f(a)\overline{\gamma'}(\tau(h)_2) = \tau(h).f(a).$$

Since $f \otimes \tau$ is a k-linear map, we need only check that it is an algebra homomorphism. Indeed, for all $h, k \in H$ and $a, b \in A$ we have

$$(f \otimes \tau)((a\#h)(b\#k)) = (f \otimes \tau)\left(\sum_{(h)(k)} a(h_1.b)\sigma(h_2,k_1)\#h_3k_2\right)$$

$$= \sum_{(h)(k)} f(a(h_1.b))f(\sigma(h_2,k_1))\#\tau(h_3k_2)$$

$$= \sum_{(h)(k)} f(a)f(h_1.b)\sigma'(\tau(h_2),\tau(k_1))\#\tau(h_3)\tau(k_2)$$

$$= \sum_{(h)(k)} f(a)(\tau(h_1).f(b))\sigma'(\tau(h_2),\tau(k_1))\#\tau(h_3)\tau(k_2)$$

$$= \sum_{(\tau(h))(\tau(k))} f(a)(\tau(h)_1.f(b))\sigma'(\tau(h)_2,\tau(k)_1)\#\tau(h)_3\tau(k)_2$$

$$= (f(a)\#\tau(h))(f(b)\#\tau(k)). \qquad \square$$

We may denote the map $f \otimes \tau$ by $f\#\tau$. Moreover, if τ and f are isomorphisms, so is $f\#\tau$.

If γ is a k-algebra homomorphism then

$$\sigma(h,k) = \sum_{(h),(k)} \gamma(h_1k_1)\overline{\gamma}(h_2k_2) = (\gamma \star \overline{\gamma})(hk) = \epsilon(hk) \cdot 1$$

for all $h, k \in H$. In this case, we note that multiplication becomes

$$(a\#h)(b\#k) = \sum_{(h)} a(h_1.b)\#h_2k$$

for all $a, b \in A$ and $h, k \in H$. That is, $A\#_\sigma H$ is exactly the (right) smash product from Definition 1.12. Similarly, if γ is a homomorphism, the left crossed product becomes the left smash product.

Theorem 1.18 ([23, Proposition 7.2.3])**.** *Let H be a Hopf algebra, and B an H-cleft extension of A. There is a k-algebra isomorphism $\Phi : A\#_\sigma H \to B$ given by $\Phi(a\#h) = a\gamma(h)$.*

Similarly, for B' a left H-cleft extension of A' there is a k-algebra isomorphism $\Psi : H'\#_{\sigma'} A' \to B'$ defined by $\Psi(h\#a) = \gamma(h)a$.

1.5 Prime Ideals and H-prime ideals

Let R be a ring.

Definition 1.19. *A* **prime ideal** *P in R is any proper ideal of R such that whenever I and J are ideals of R with $IJ \subseteq P$ then either $I \subseteq P$ or $J \subseteq P$.*

The set of all prime ideals for R is denoted by spec(R). A **completely prime ideal** P is an ideal so that R/P is a domain. Note, if P is completely prime then P is prime. (The converse is not true, generally.)

Let H be a group acting by automorphisms on R.

Definition 1.20. *An H-**ideal** K is an ideal of R so that $h.K = K$ for all $h \in H$. An H-**prime ideal** K is a proper H-ideal so that whenever I, J are H-ideals of R so that $IJ \subseteq K$ then either $I \subseteq K$ or $J \subseteq K$.*

The set of all H-prime ideals for R is denoted by H-spec(R).

Suppose that the group H acting on R is an affine algebraic group over k. The action of H on A is **rational** if A is the directed union of finite dimensional H-

invariant k-subspaces V_i, so that the restriction maps $H \to \mathrm{GL}(V_i)$ are morphisms of algebraic varieties.

1.6 Gelfand-Kirillov Dimension

Let R be a k-algebra. If V and W are subspaces of R, let VW denote the set of all finite sums of products vw where $v \in V$ and $w \in W$. Moreover, we denote $V^0 = k$ and for any natural number n denote $V^n = VV^{n-1}$.

Definition 1.21. *A* **(nonnegative, exhaustive) filtration** *for R is some indexed family of subspaces $\{R_i \mid i \in \mathbb{Z}_{\geq 0}\}$ so that*

(i) $R_i \subseteq R_j$ *for* $i < j$*;*

(ii) $R_i R_j \subset R_{i+j}$ *for all* i, j*;*

(iii) $\bigcup_{i=0}^{\infty} R_i = R$*.*

The filtration is **standard** if $R_i = R_1^i$ for all i. The filtration is **finite** if $\dim R_n < \infty$ for all n.

Definition 1.22. *A k-algebra R is* **affine** *if it is generated as a k-algebra by a finite set of elements.*

If R is affine then R is necessarily generated by a finite dimensional vector space V called a **generating subspace for** R. Moreover, R has a standard finite filtration generated by V, given by $R_0 = V^0 = k$ and $R_n = \sum_{i=0}^{n} V^i$ for each $n \in \mathbb{N}$.

For a generating subspace V with standard filtration $\{R_i\}$ as above, let

$$g_V := \limsup_{i\to\infty}((\log \dim R_i)/\log i).$$

Lemma 1.23 ([22, Lemma 8.1.10])**.** *Let V, V' be generating subspaces for standard filtrations $\{R_i\}$ and $\{R'_i\}$ on a ring R. Then $g_V = g_{V'}$.*

Using this Lemma, we have a well-defined invariant of the ring R.

Definition 1.24. *The* **Gelfand-Kirillov dimension** *or* **GK dimension** *for an affine k-algebra R is*

$$\text{GK.dim}\,(R) = g_V$$

for any choice of generating subspace V.

Recall the definitions and notation of Section 1.4. We have the following Lemma.

Lemma 1.25. *Let H be a Hopf algebra and A an H-module algebra. Suppose $U \subseteq W\#V$ is a generating subspace for $A\#H$ where V is a finite dimensional subspace of H and W is a finite dimensional subspace of A. Assume $H.W \subseteq W$ and $\Delta(V) \subseteq H \otimes V$. Then*

$$\sup(\text{GK.dim}\,(A), \text{GK.dim}\,(H)) \leq \text{GK.dim}\,(A\#H) \leq \text{GK.dim}\,(A) + \text{GK.dim}\,(H).$$

Proof. The first inequality follows from the fact that A and H naturally imbed into $A\#H$ as the the subalgebras $A\#1$ and $1\#H$, respectively.

Let n be a natural number and suppose $U^n \subseteq W^n\#V^n$. We first observe that $H.W^n \subseteq (H.W)W^{n-1} \subseteq W^n$. Suppose $b \in W^n$ and $k \in V^n$ with $a \in W$ and $h \in V$. Since $h_2 \in V$ and $h_1.b \in W^n$ and by how multiplication on $A\#H$ is defined, we have

$$(a\#h)(b\#k) = \sum_{(h)} a(h_1.b)\#h_2k \in W^{n+1}\#V^{n+1}.$$

Hence, $U^{n+1} \subseteq W^{n+1}\#V^{n+1}$. Therefore $U^k \subseteq W^k\#V^k$ for all $k \in \mathbb{N}$.

Next, let S and T be the subalgebras of A and H generated by W and V, respectively. From what we have just shown, it follows that $\dim(A\#H)_k \leq (\dim S_k)(\dim(T_k)$ for all nonnegative integers k. Thus, by [22, Lemma 8.1.7 (ii)] the proposition follows. □

We note that Lemma 1.25 also holds for a right smash product $H\#A$ making the appropriate changes.

1.7 Skew Polynomial Rings

Let R be a ring, α an automorphism of R, and δ a map on R. The map δ is an **α-derivation** if

$$\delta(rs) = \alpha(r)\delta(s) + \delta(r)s \qquad \text{and} \qquad \delta(r+s) = \delta(r) + \delta(s)$$

for all $r, s \in R$.

Definition 1.26. *A* **skew polynomial ring over** R*, denoted* $S = R[x; \alpha, \delta]$*, is a ring satisfying the following conditions:*

(i) S *is a ring containing* R *as a subring;*

(ii) $x \in S$*;*

(iii) S *is a free left* R*-module with basis* $\{1, x, x^2, \ldots\}$*;*

(iv) α *is an automorphism of* R *and* δ *is an* α*-derivation;*

(v) $xr = \alpha(r)x + \delta(r)$ *for all* $r \in R$*.*

Definition 1.27. *A* **skew polynomial k-algebra** *is a skew polynomial ring* $S = R[x; \alpha, \delta]$ *so that* R *is a* k*-algebra and* α, δ *are* k*-linear. It follows that* S *is a* k*-algebra.*

We say

$$S = R[x_1; \alpha_1, \delta_1][x_2; \alpha_2, \delta_2] \cdots [x_n; \alpha_n, \delta_n]$$

is an **iterated skew polynomial ring** if

$$(R[x_1; \alpha_1, \delta_1][x_2; \alpha_2, \delta_2] \cdots [x_{i-1}; \alpha_{i-1}, \delta_{i-1}])\,[x_i; \alpha_i, \delta_i]$$

is a skew polynomial ring over $R[x_1; \alpha_1, \delta_1][x_2; \alpha_2, \delta_2] \cdots [x_{i-1}; \alpha_{i-1}, \delta_{i-1}]$ for each $i = 1, 2, \ldots, n$.

Lemma 1.28. *Let $S = R[x; \alpha, \delta]$ be a skew polynomial k-algebra and let V be a generating subspace of R. Suppose δ' is a an α-derivation of R. If $\delta(v) = \delta'(v)$ for all $v \in V$ then $\delta = \delta'$.*

Proof. By hypothesis, for all $v \in V$ we have $\delta(v) = \delta'(v)$. Proceeding inductively, suppose for all $w \in V^n$ that $\delta(w) = \delta'(w)$. Let $z \in V^{n+1}$. Then $z = \sum_i v_i w_i$ where $v_i \in V$ and $w_i \in V^n$. Hence,

$$\begin{aligned}\delta(z) = \delta\left(\sum_i v_i w_i\right) &= \sum_i \left(\alpha(v_i)\delta(w_i) + \delta(v_i)w_i\right) \\ &= \sum_i \left(\alpha(v_i)\delta'(w_i) + \delta'(v_i)w_i\right) = \delta'\left(\sum_i v_i w_i\right) = \delta'(z).\end{aligned}$$

Thus $\delta(z) = \delta'(z)$ for all $z \in V^{n+1}$. Therefore, for all $N \in \mathbb{N}$ if $z \in V^N$ then $\delta(z) = \delta'(z)$. Since every $z \in R$ is in V^N for some N large enough, it follows that $\delta = \delta'$. $\square$

Definition 1.29. *Let λ a nonzero element of k with $\lambda \neq \pm 1$. For all $m \in \mathbb{N}$ define*

$$[m]_\lambda := \frac{\lambda^m - \lambda^{-m}}{\lambda - \lambda^{-1}}.$$

We define $[0]_\lambda ! = 1$ and for $m \in \mathbb{N}$ we define

$$[m]_\lambda ! := [m]_\lambda [m-1]_\lambda \cdots [1]_\lambda.$$

Finally, for $m \in \mathbb{Z}_{\geq 0}$ and $l = 0, 1, \ldots, m$ we define

$$\binom{m}{l}_\lambda := \frac{[m]_\lambda !}{[l]_\lambda ! [m-l]_\lambda !}.$$

Lemma 1.30. *Let $S = R[x; \alpha, \delta]$ be a skew polynomial k-algebra. Suppose that $\alpha\delta = \lambda\delta\alpha$ for λ a nonzero, non root of unity in k For all $m \in \mathbb{N}$ and $r, s \in R$ we have*

$$\delta^m(rs) = \sum_{k=0}^{m} \binom{m}{k}_\lambda \alpha^{m-k}\delta^k(r)\delta^{m-k}(s).$$

Proof. See [1, Section I.8.4]. $\square$

Definition 1.31. *Let $S = R[x; \alpha, \delta]$ be a skew polynomial ring. The α-derivation δ is* **locally nilpotent** *if for each $r \in R$ there exists an $N \in \mathbb{N}$ so that $\delta^N(r) = 0$.*

Lemma 1.32. *Let $S = R[x; \alpha, \delta]$ be a skew polynomial k-algebra and let V be a generating subspace for R. Suppose that $\alpha\delta = \lambda\delta\alpha$ for some nonzero, non root of* *unity scalar λ and that there exists an integer m so that $\delta^m(v) = 0$ for all $v \in V$. Then δ is locally nilpotent.*

Proof. By hypothesis, there exists an $m \in \mathbb{N}$ so that $\delta^m(v) = 0$ for all $v \in V$. Proceeding inductively, suppose that for all $w \in V^n$ that there is an integer s so that $\delta^s(w) = 0$. Let $z \in V^{n+1}$. Then $z = \sum_i v_i w_i$ where $v_i \in V$ and $w_i \in V^n$. From Lemma 1.30 we have

$$\delta^{s+m}(z) = \delta^{s+m}\left(\sum_i v_i w_i\right) = \sum_i \sum_{k=0}^{s+m} \binom{s+m}{k}_\lambda \alpha^{s+m-k}\delta^k(v_i)\delta^{s+m-k}(w_i).$$

If $k \geq m$ then $\alpha^{s+m-k}\delta^k(v_i) = 0$ by assumption. Moreover, if $k < m$ then $\delta^{s+m-k}(w_i) = 0$ by our hypothesis. Hence, for all $z \in V^{n+1}$ we have $\delta^{s+m}(z) = 0$. Since every element of R is in V^N for large enough N, it follows that δ is locally nilpotent. □

CGL Extensions

Definition 1.33. *A* **CGL extension** *is an iterated skew polynomial k-algebra*

$$R = k[x_1][x_2; \tau_2, \delta_2] \ldots [x_m; \tau_m, \delta_m]$$

equipped with a rational action of a k-torus H by algebra automorphisms, which satisfies the following conditions:

(i) For all $1 \leq j < k \leq m$, $\tau_k(x_j) = \lambda_{kj} x_j$ for some $\lambda_{kj} \in k^*$.

(ii) *For* $2 \leq k \leq m$, δ_k *is a locally nilpotent* τ_k*-derivation on*

$$R_{k-1} = k[x_1][x_2; \tau_2, \delta_2] \ldots [x_{k-1}; \tau_{k-1}, \delta_{k-1}].$$

(iii) *The elements* $x_1, \cdots, x_m$ *are* H*-eivgenvectors.*

(iv) *For every* $1 \leq k \leq m$ *there is an* $h_k \in H$ *so that* $h_k. \mid_{R_{k-1}} = \tau_k$ *and* $h_k.x_k = \lambda_k x_k$ *where* $\lambda_k \in k^*$ *is not a root of unity.*

1.8 Root Systems

Let V be a vector space over k.

Definition 1.34. *A* **nondegenerate bilinear form** *on a vector space* V *is a map* $(-,-) : V \times V \to k$ *so that*

(i) $(-,-)$ *is linear in both arguments;*

(ii) *If* $(x, y) = 0$ *for all* $y \in V$ *then* $x = 0$*;*

(iii) *If* $(x, y) = 0$ *for all* $x \in V$ *then* $y = 0$.

To emphasize the vector space V *we may write* $(-,-)$ *as* $(-,-)_V$.

Definition 1.35. *Let* V *be a a finite-dimensional space with nondegenerate bilinear form* $(-,-)$. *A* **root system** Φ *for* V *is a set* $\Phi \subset V$ *so that*

(i) Φ *is finite,* Φ *spans* V , $0 \notin \Phi$*;*

(ii) If $\alpha \in \Phi$, the only multiples of α in Φ are $\pm\alpha$;

(iii) For all $\alpha, \beta \in \Phi$ we have $2\frac{(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$;

(iv) For all $\alpha, \beta \in \Phi$ we have $\beta - 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha \in \Phi$.

The elements of Φ are called **roots***.*

For each $\alpha \in \Phi$ we define a transformation s_α of V by

$$s_\alpha(v) := v - 2\frac{(v,\alpha)}{(\alpha,\alpha)}\alpha \tag{1.11}$$

for all $v \in V$.

If Φ is a root system for V and Ψ is a root system of W then the two **root systems are equivalent** if there is an invertible linear transformation $f : V \to W$ that sends Φ to Ψ so that for all $\alpha \in \Phi$ we have

$$fs_\alpha = s_{f(\alpha)}f.$$

Definition 1.36. *A* **base** *for a root system Φ is a set $\Pi \subset \Phi$ so that*

(i) Π is a basis for V;

(ii) Each root $\alpha \in \Phi$ can be expressed as a linear combination of elements of Π such that the coefficients are either all negative or all positive integers.

The roots for which the coefficients are nonnegative (relative to Π) are called the **positive roots** for Φ . The roots for which the coefficients are nonpositive are called the **negative roots** for Φ. The elements in Π are called the **positive simple roots**. From [14, Theorem 10.1], every root system Φ has a base.

Definition 1.37. *Let $\{\alpha_1, \ldots, \alpha_n\}$ be the set of positive simple roots for a root system Φ. The* **Cartan matrix** *of Φ is the matrix C so that $C_{ij} = 2\frac{(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)}$.*

Theorem 1.38 ([14, Proposition 11.1])**.** *If Φ a root system for V with base $\{\alpha_1, \ldots, \alpha_t\}$ with Cartan matrix C and Φ' a root systems for W with base $\{\alpha'_1, \ldots, \alpha'_t\}$ with Cartan matrix C', and $C_{ij} = C'_{ij}$ for all i, j, then the bijection $\alpha_i \mapsto \alpha'_i$ extends to an equivalency of the root systems Φ and Φ'.*

Theorem 1.38 essentially says that the Cartan matrix of Φ determines Φ up to equivalence.

Weyl Group

Definition 1.39. *The* **Weyl group** *W of Φ is the subgroup of $\mathrm{GL}(V)$ generated by the s_α for $\alpha \in \Phi$ from equation* (1.11)*.*

From [14, Theorem 10.3] the Weyl group is generated by the $s_i := s_{\alpha_i}$ where α_i is a positive simple root. A **reduced word** in the Weyl group is an irreducible representation of a element of $w \in W$ as the product of the s_i. The **length** of w is the shortest possible length of a reduced word for w. There is a unique element of longest length in the Weyl group called the **longest word**, which is denoted by w_0 [1, Section I.5.1].

Irreducible Root Systems

Let Φ be a root system for V and Ψ a root system for W. The vector space $V \oplus W$ has a natural bilinear form induced from the bilinear forms on V and W given by

$$(v_1 + w_1, v_2 + w_2)_{V \oplus W} = (v_1, v_2)_V + (w_1, w_2)_W$$

for all $v_i \in V$ and $w_i \in W$. If follows from [11, Proposition 8.3] that $\Phi \sqcup \Psi$ is a root system for $V \oplus W$, which we denote by $\Phi \oplus \Psi$.

Definition 1.40. *A root system Φ is* **reducible** *if $\Phi = \Phi_1 \oplus \Phi_2$ where $\Phi_1, \Phi_2 \neq \emptyset$. A root system is* **irreducible** *if it is not reducible.*

Theorem 1.41 ([14, Proposition 11.3])**.** *Every root system Φ can be decomposed into*

$\Phi_1 \oplus \Phi_2 \oplus \cdots \oplus \Phi_t$ where each Φ_i is an irreducible nonempty root system. Moreover, this decomposition is unique.

From [14, Lemma 10.4 C] if Φ is an irreducible root system, at most two root lengths occur in Φ. In this case we refer to the roots as **long** or **short** depending on the length. If all roots are the same length, then all roots are considered short.

For an irreducible root system the bilinear form may be normalized so that $(\alpha, \alpha) = 2$ for short roots. It follows this root system is equivalent to our original one.

Convention 1. *Let Φ be an irreducible root system. We fix some choice of positive simple roots $\Pi = \{\alpha_1, \cdots, \alpha_n\}$ with $C = (c_{ij})$ the Cartan matrix. The bilinear form is assumed to have the property that $(\alpha, \alpha) = 2$ for short roots.*

It follows from Convention 1 that for all positive roots $\alpha_i, \alpha_j \in \Pi$ and α_j a short root, $(\alpha_i, \alpha_j) = c_{ij}$.

Dynkin Diagrams

Definition 1.42. *A* **Dynkin diagram** *for a root system Φ with Cartan matrix C, is a diagram so that*

(i) Each vertex is labeled by a simple root of $\alpha_i \in \Pi$ and each simple root is labeled by some vertex;

(ii) For each pair of distinct $\alpha_i, \alpha_j \in \Pi$ there are $C_{ij}C_{ji}$ edges drawn between the vertices labeled α_i and α_j;

(iii) If the length of α_i is shorter then the length of α_j , then an arrow is drawn pointing from the vertex labeled α_j to the vertex labeled α_i.

A Dynkin diagram is said to be an **irreducible** if is connected. Hence, a Dynkin diagram is irreducible if and only if the root system is irreducible. Two Dynkin diagrams are **equivalent** if there is a bijective map of the vertices that preserves the number of edges and direction of the arrows between vertices.

Theorem 1.43 ([14, Theorem 11.4] and [11, Theorem 8.27]). *Every irreducible Dynkin diagram is equivalent to one of the following:*

$$A_n(n \geq 1) \qquad \underset{\alpha_1}{\circ} \text{ — } \underset{\alpha_2}{\circ} \text{ — } \cdots \text{ — } \underset{\alpha_{n-1}}{\circ} \text{ — } \underset{\alpha_n}{\circ}$$

$$B_n(n \geq 2) \qquad \underset{\alpha_1}{\circ} \text{ — } \underset{\alpha_2}{\circ} \text{ — } \cdots \text{ — } \underset{\alpha_{n-1}}{\circ} \Rightarrow \underset{\alpha_n}{\circ}$$

$$C_n(n \geq 3) \qquad \underset{\alpha_1}{\circ} \text{ — } \underset{\alpha_2}{\circ} \text{ — } \cdots \text{ — } \underset{\alpha_{n-1}}{\circ} \Leftarrow \underset{\alpha_n}{\circ}$$

$$D_n(n \geq 4) \qquad \begin{array}{ccccccccc} & & & & & & \circ & & \\ & & & & & & \vert\ \alpha_n & & \\ \underset{\alpha_1}{\circ} & \text{—} & \underset{\alpha_2}{\circ} & \text{—} & \cdots & \text{—} & \underset{\alpha_{n-2}}{\circ} & \text{—} & \underset{\alpha_{n-1}}{\circ} \end{array}$$

$$E_6 \qquad \begin{array}{ccccccccc} & & & & \circ & & & & \\ & & & & \vert\ \alpha_6 & & & & \\ \underset{\alpha_1}{\circ} & \text{—} & \underset{\alpha_2}{\circ} & \text{—} & \underset{\alpha_3}{\circ} & \text{—} & \underset{\alpha_4}{\circ} & \text{—} & \underset{\alpha_5}{\circ} \end{array}$$

$$E_7 \qquad \begin{array}{ccccccccccc} & & & & \circ & & & & & & \\ & & & & \vert\ \alpha_7 & & & & & & \\ \underset{\alpha_1}{\circ} & \text{—} & \underset{\alpha_2}{\circ} & \text{—} & \underset{\alpha_3}{\circ} & \text{—} & \underset{\alpha_4}{\circ} & \text{—} & \underset{\alpha_5}{\circ} & \text{—} & \underset{\alpha_6}{\circ} \end{array}$$

$$E_8 \qquad \begin{array}{ccccccccccccc} & & & & & & & & \circ & & & & \\ & & & & & & & & \vert\ \alpha_8 & & & & \\ \underset{\alpha_1}{\circ} & \text{—} & \underset{\alpha_2}{\circ} & \text{—} & \underset{\alpha_3}{\circ} & \text{—} & \underset{\alpha_4}{\circ} & \text{—} & \underset{\alpha_5}{\circ} & \text{—} & \underset{\alpha_6}{\circ} & \text{—} & \underset{\alpha_7}{\circ} \end{array}$$

$$F_4 \qquad \underset{\alpha_1}{\circ} \text{ — } \underset{\alpha_2}{\circ} \Rightarrow \underset{\alpha_3}{\circ} \text{ — } \underset{\alpha_4}{\circ}$$

$$G_2 \qquad \underset{\alpha_1}{\circ} \Rrightarrow \underset{\alpha_2}{\circ}$$

Theorem 1.44 ([14, Proposition 14.1]). *There is a bijective correspondence between the isomorphism classes of indecomposable finite dimensional complex semisimple Lie algebras and irreducible root systems.*

Root and Weight Lattices

Definition 1.45. *The groups*

$$\mathbb{Q}\Pi := \left\{ \sum_{i=1}^{n} r_i \alpha_i \mid r_i \in \mathbb{Q},\ \alpha_i \in \Pi \right\} \quad \text{and} \quad \mathbb{Z}\Pi := \left\{ \sum_{i=1}^{n} k_i \alpha_i \mid k_i \in \mathbb{Z},\ \alpha_i \in \Pi \right\}$$

are called the **rational root lattice** *and* **root lattice** *respectively.*

For all $\alpha \in \Phi$ set $d_\alpha = \frac{(\alpha,\alpha)}{2}$. From [11, Section 8.10] there exist $\omega_i \in \mathbb{Q}\Pi$ called the **fundamental weights** with the property that $(\omega_i, \alpha_j) = d_{\alpha_j}\delta_{ij}$ for all $\alpha_j \in \Pi$. The set of fundamental weights is denoted by Ω.

Definition 1.46. *The semigroups*

$$\Lambda := \left\{ \sum_{i=1}^{n} k_i \omega_i \mid k_i \in \mathbb{Z},\ \omega_i \in \Omega \right\} \quad \text{and} \quad \Lambda^+ := \left\{ \sum_{i=1}^{n} k_i \omega_i \mid k_i \in \mathbb{Z}_{\geq 0},\ \omega_i \in \Omega \right\}$$

are called the **weight lattice** *and* **positive weight lattice**, *respectively.*

It is straightforward to see that $\mathbb{Z}\Pi \subset \Lambda$ and that for $\lambda \in \Lambda$ and $\mu \in \mathbb{Z}\Pi$ we have $(\lambda, \mu) \in \mathbb{Z}$.

We may give Λ a partial order by defining $\lambda \geq 0$ if there exists $k_i \in \mathbb{Z}_{\geq 0}$ so that $\lambda = k_1\alpha_1 + k_2\alpha_2 + \cdots + k_n\alpha_n$. If $\mu, \lambda \in \Lambda$ we define $\mu \geq \lambda$ if and only if $\mu - \lambda \geq 0$.

1.9 Quantum Universal Enveloping Algebras

Let $q \in k^{\times}$ with q not a root of unity. Let $\mathfrak{g}$ be the irreducible semisimple Lie algebra with corresponding irreducible root system Φ. Keeping in mind convention 1, for each $\alpha \in \Phi$ denote $q_\alpha = q^{d_\alpha} = q^{\frac{(\alpha,\alpha)}{2}}$. Recall Definition 1.29. For all $m \in \mathbb{N}$ and $\alpha \in \Phi$ we use the notation

$$[m]_\alpha := [m]_{q_\alpha}$$

and

$$[m]_\alpha! := [m]_{q_\alpha}!.$$

Finally, for $m \in \mathbb{Z}_{\geq 0}$ and $l = 0, 1, \ldots, m$ we set

$$\binom{m}{l}_\alpha := \binom{m}{l}_{q_\alpha}.$$

Definition 1.47. *The* **quantized universal enveloping algebra** $U_q(\mathfrak{g})$ *is the k-algebra with generators E_α, F_α, $K_\lambda^{\pm 1}$ for $\alpha, \lambda \in \Pi$ presented with the following relations: for all $\alpha, \beta, \lambda, \mu \in \Pi$*

$$K_\lambda K_\lambda^{-1} = K_\lambda^{-1} K_\lambda = 1 \tag{1.12}$$

$$K_\lambda K_\mu = K_\mu K_\lambda \tag{1.13}$$

$$K_\lambda E_\alpha = q^{(\lambda,\alpha)} E_\alpha K_\lambda \tag{1.14}$$

$$K_\lambda F_\alpha = q^{-(\lambda,\alpha)} F_\alpha K_\lambda \tag{1.15}$$

$$[E_\alpha, F_\beta] = \delta_{\alpha\beta} \frac{K_\alpha - K_\alpha^{-1}}{q_\alpha - q_\alpha^{-1}} \tag{1.16}$$

$$\sum_{l=0}^{1-a_{\alpha\beta}} (-1)^l \binom{1-a_{\alpha\beta}}{l}_\alpha E_\alpha^{1-a_{\alpha\beta}-l} E_\beta E_\alpha^l = 0 \tag{1.17}$$

$$\sum_{l=0}^{1-a_{\alpha\beta}} (-1)^l \binom{1-a_{\alpha\beta}}{l}_\alpha F_\alpha^{1-a_{\alpha\beta}-l} F_\beta F_\alpha^l = 0 \tag{1.18}$$

where $\delta_{\alpha\beta}$ is the Kronecker delta and $a_{\alpha\beta} = 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}$.

The **quantized positive Borel subalgebra**, denoted $U_q^{\geq 0}(\mathfrak{g})$, is the subalgebra generated by the E_α and the $K_\lambda^{\pm 1}$. Similarly, The **quantized negative Borel subalgebra**, denoted $U_q(\mathfrak{g})^{\leq 0}$, is the subalgebra generated by the F_α and the $K_\lambda^{\pm 1}$. Moreover, the **positive nilpotent subalgebra** of $U_q(\mathfrak{g})$, denoted $U_q^+(\mathfrak{g})$, is the subalgebra generated by all the E_α and the **negative nilpotent subalgebra**, denoted by $U_q^-(\mathfrak{g})$, is the subalgebra generated by all the F_α. Finally, the the **quantum torus**, denoted by $U_q^0(\mathfrak{g})$, is the subalgebra generated by the $K_\lambda^{\pm 1}$.

Convention 2. *We use the notation $E_i = E_{\alpha_i}$, $F_i = F_{\alpha_i}$ and $K_i^{\pm 1} = K_{\alpha_i}^{\pm 1}$. Furthermore, we also adopt the convention that if $I = (i_1, i_2, \cdots i_{N-1}, i_N)$ where each $i_k \in \{1, 2, \ldots, n\}$ then*

$$E_I = E_{i_1} E_{i_2} \cdots E_{i_{N-1}} E_{i_N} \qquad \text{and} \qquad F_I = F_{i_1} F_{i_2} \cdots F_{i_{N-1}} F_{i_N}.$$

Moreover, $E_\emptyset = 1$ *and* $F_\emptyset = 1$.

Let $K_{\pm k\alpha_i} = (K_i^{\pm 1})^k$ *for* $k \in \mathbb{Z}_{\geq 0}$ *and* $\alpha_i \in \Pi$. *Since these elements commute, for any* $\lambda \in \mathbb{Z}\Pi$ *where* $\lambda = k_1\alpha_1 + \cdots + k_n\alpha_n$ *we also use the convention that*

$$K_\lambda = K_{k_1\alpha_1} K_{k_2\alpha_2} \cdots K_{k_n\alpha_n}.$$

We note, it follows from this convention that for $\lambda \in \mathbb{Z}\Pi$ and $1 \leq i \leq n$ we have

$$K_\lambda E_i = q^{(\lambda,\alpha_i)} E_i K_\lambda \qquad \text{and} \qquad K_\lambda F_i = q^{-(\lambda,\alpha_i)} F_i K_\lambda.$$

Hopf algebra Structure

$U_q(\mathfrak{g})$ is a Hopf algebra with comultiplication Δ, counit ϵ, and antipode S determined by the following:

$$\begin{aligned}
\Delta(E_i) &= K_i \otimes E_i + E_i \otimes 1 & \epsilon(E_i) &= 0 & S(E_i) &= -K_i^{-1}E_i \\
\Delta(F_i) &= F_i \otimes K_i^{-1} + 1 \otimes F_i & \epsilon(F_i) &= 0 & S(F_i) &= -F_iK_i \\
\Delta(K_\lambda) &= K_\lambda \otimes K_\lambda & \epsilon(K_\lambda) &= 1 & S(K_\lambda) &= K_\lambda^{-1}
\end{aligned}$$

for $i = 1, 2, \ldots, n$ and $\lambda \in \Pi$.

The Quantized Universal Enveloping Algebra $\check{U}_q(\mathfrak{g})$

Denote by $\check{U}_q(\mathfrak{g})$ the algebra generated by the E_α, F_α with $\alpha \in \Pi$ and K_λ for $\lambda \in \Omega$ instead of just $\lambda \in \Pi$. It is presented with relations as in equations (1.12) – (1.18), except we now allow $\lambda, \mu \in \Omega$. It is also a Hopf algebra with the same description as the Hopf algebra structure for $U_q(\mathfrak{g})$, again allowing $\lambda \in \Omega$.

The positive (resp. negative) Borel subalgebras of $\check{U}_q(\mathfrak{g})$ denoted by $\check{U}_q^{\geq 0}(\mathfrak{g})$ (resp. $\check{U}_q^{\leq 0}(\mathfrak{g})$) are defined analogously as above. Note that the positive and negative nilpotent subalgebras for $\check{U}_q(\mathfrak{g})$ are the same as for $U_q(\mathfrak{g})$.

1.10 The Algebra of Matrix Coefficients

Weights and Weight Spaces

Continue with $\mathfrak{g}$ as in Section 1.9. Let V be a left $U_q(\mathfrak{g})$-module.

Definition 1.48. *A nonzero vector $v \in V$ is a* **weight vector** *if it has that has the property that there is a $\lambda \in \Lambda$ and a homomorphism $\sigma : \Pi \to \{\pm 1\}$ such that*

$$K_\mu v = \sigma(\mu) q^{(\lambda,\mu)} v$$

for all $\mu \in \Pi$. The element λ is called the **weight** *for v and λ is said to be of* **type** *σ.*

The set of weights for V is denoted by $\Omega(V)$. The set of all weight vectors with weight λ of type σ, together with 0, is a subspace of V which we denote by $V_{(\lambda,\sigma)}$ and call it a **weight space** of V. If λ is of type $\mathbf{1}$ we denote $V_{(\lambda,\mathbf{1})}$ by V_λ.

If a weight vector $v \in V$ has the property that $E_i v = 0$ for all the $E_i \in U_q(\mathfrak{g})$ then v is called a **maximal weight vector** for V. Analogously, a weight vector $v \in V$ is a **minimal weight vector** if $F_i v = 0$ for all the $F_i \in U_q(\mathfrak{g})$.

The Category $\mathscr{C}_q(\mathfrak{g})$

Let V_σ denote the sum of the weight spaces of type σ for a module V. From [15, Section 5.2] we have

$$V = \bigoplus V_\sigma.$$

We say a finite dimensional $U_q(\mathfrak{g})$-module V has type σ if $V = V_\sigma$. In particular, irreducible modules have a well-defined type. From, [15, Section 5.2 - 5.4] the class of finite dimensional modules of type $\mathbf{1}$ is closed under direct sum, tensor product, and duals. Therefore, we denote by $\mathscr{C}_q(\mathfrak{g})$ the full subcategory whose objects are finite dimensional left $U_q(\mathfrak{g})$-modules of type $\mathbf{1}$. We note that from [15, Section 5.2] there is an equivalence of categories between $\mathscr{C}_q(\mathfrak{g})$ and the category of all finite dimensional $U_q(\mathfrak{g})$-modules of type σ, for any σ.

We collect some basic facts about $\mathscr{C}_q(\mathfrak{g})$ which can be found in [15, Chapters 5, 6].

For V and W objects in $\mathscr{C}_q(\mathfrak{g})$ and V_λ and W_μ weight spaces, $V_\lambda \otimes W_\mu \subseteq (V \otimes W)_{\lambda+\mu}$ and $(V_\lambda)^* \cong (V^*)_{-\lambda}$ where V^* has the left action of $U_q(\mathfrak{g})$ from Section 1.3. (See [15, Section 5.3].)

Theorem 1.49. *If $V \in \mathscr{C}_q(\mathfrak{g})$ then V is the direct sum of its weight spaces.*

Theorem 1.50. *Let $V \in \mathscr{C}_q(\mathfrak{g})$ be irreducible.*

(i) V contains a maximal weight vector v_λ with weight $\lambda \in \Lambda^+$.

(ii) $\dim V_\lambda = 1$.

(iii) If ν is any other weight for V then $\nu < \lambda$.

(iv) $U_q(\mathfrak{g})^{\leq 0} v_\lambda = V$.

Similarly, V also contains a minimal weight vector $v_{-\mu}$ with weight $-\mu \in -\Lambda^+$ with analogous properties.

From Theorem 1.50, if λ is the weight of a maximal weight vector v_λ then we call v_λ **a highest weight vector** for V and λ **the highest weight** for V. Analogously $v_{-\mu}$ is called **a lowest weight vector** for V and $-\mu$ called **the lowest weight** for V.

Theorem 1.51. *For every $\lambda \in \Lambda^+$ there is an irreducible module $V \in \mathscr{C}_q$ which has highest weight λ. Moreover, if W is any other irreducible module in $\mathscr{C}_q$ with highest weight μ, then $V \cong W$ if and only if $\mu = \lambda$.*

Taking account of Theorem 1.51 we denote an irreducible module with highest weight λ by $V(\lambda)$. Using this notation, we note $V(\mu) \cong V(\lambda)$ if and only if $\mu = \lambda$.

Definition and Structure of $C_q(\mathfrak{g})$

Definition 1.52. *For $V \in \mathscr{C}_q(\mathfrak{g})$ and $f \in V^*$ and $v \in V$, a* **coordinate function** *or a* **matrix coefficient** *is the linear functional $c^V_{f,v}$ in $U_q(\mathfrak{g})^*$ defined by*

$$c^V_{f,v}(u) := f(uv)$$

for all $u \in U_q(\mathfrak{g})$. The set of all coordinate functions is denoted by $C_q(\mathfrak{g})$.

Since the annihilator of V is contained in the kernel of $c^V_{f,v}$ and since V is finite dimensional, $c^V_{f,v} \in U_q(\mathfrak{g})^\circ$. Therefore, $C_q(\mathfrak{g}) \subseteq U_q(\mathfrak{g})^\circ$.

Note that using the standard addition and multiplication in $U_q(\mathfrak{g})^\circ$, we have

$$c^V_{f,v} + c^W_{g,w} = c^{V\oplus W}_{f\oplus g, v\oplus w} \qquad \text{and} \qquad c^V_{f,v} c^W_{g,w} = c^{V\otimes W}_{f\otimes g, v\otimes w}.$$

Since $\mathscr{C}_q(\mathfrak{g})$ is closed under finite direct sums and tensor products it follows that $C_q(\mathfrak{g})$ is a k-subalgebra of $U_q(\mathfrak{g})^\circ$.

Finally, $C_q(\mathfrak{g})$ is both a left $U_q(\mathfrak{g})$-module and a right $U_q(\mathfrak{g})$-module with actions given by

$$u.c^V_{f,v} = c^V_{f,u.v} \qquad \text{and} \qquad (c^V_{f,v}).u = c^V_{f.u,v}.$$

Convention 3. *For the irreducible modules $V(\mu)$ in $\mathscr{C}_q(\mathfrak{g})$, we will often denote $c^{V(\mu)}_{f,v}$ simply by $c^{\mu}_{f,v}$. Moreover, for weight vectors $v \in V(\mu)$ and $f \in V(\mu)^*$ with weights ν and $-\lambda$ respectively, we may denote $c^{\mu}_{f,v}$ by either $c^{\mu}_{f,\nu}$ or $c^{\mu}_{-\lambda,\nu}$. Therefore, any proposition written with $c^{\mu}_{-\lambda,\nu}$ will be independent of any choice of weight vectors v or f.*

1.11 Some Quantized Coordinate Rings

Define $\widehat{q} = q - q^{-1}$.

Definition 1.53. *The* **quantized coordinate ring for** M_{n+1} *or* **quantum** $(n+1) \times (n+1)$ **matrices**, *denoted $O_q(\mathrm{M}_{n+1})$, is the k-algebra generated by $\{X_{ij} \mid 1 \le i, j \le n+1\}$ presented with the following relations:*

$$X_{ij}X_{im} = qX_{im}X_{ij} \text{ for } j < m \tag{1.19}$$

$$X_{ij}X_{lj} = qX_{lj}X_{ij} \text{ for } i < l \tag{1.20}$$

$$X_{ij}X_{lm} = X_{lm}X_{ij} \text{ for } i < l \text{ and } j > m \tag{1.21}$$

$$X_{ij}X_{lm} - X_{lm}X_{ij} = \widehat{q}X_{im}X_{lj} \text{ for } i < l \text{ and } j < m. \tag{1.22}$$

We note this presentation is from [25, Section 3.5] but replacing q^{-1} there with q in our relations here. When $q \to 1$ we recover exactly the usual presentation for $O(\mathrm{M}_{n+1})$.

Definition 1.54. *Let $I, J \subset \{1, 2, \ldots, n+1\}$ with $|I| = |J| = k \neq 0$ and $I = \{i_1, \ldots, i_k\}$ with $i_1 < i_2 < \cdots < i_k$ and $J = \{j_1, \ldots, j_k\}$ with $j_1 < j_2 < \cdots < j_k$. The $k \times k$* **quantum minor** *$[I \mid J]$ is the element of $O_q(\mathrm{M}_{n+1})$ defined by*

$$[I \mid J] := \sum_{\sigma \in \mathrm{Sym}_k} (-q)^{\ell(\sigma)} X_{i_1, j_{\sigma(1)}} \cdots X_{i_k, j_{\sigma(k)}}.$$

In particular we set

$$D := [1, 2, \ldots, n+1 \mid 1, 2, \ldots, n+1]. \tag{1.23}$$

The element D is often referred to as the **quantum determinant** of $O_q(\mathrm{M}_{n+1})$ and from [19, 9.2 Proposition 9] belongs to the center of $O_q(\mathrm{M}_{n+1})$.

Definition 1.55. *The* **quantized coordinate ring for** GL_{n+1} *or* **quantum** GL_{n+1} *is the k-algebra*

$$O_q(\mathrm{SL}_{n+1}) := O_q(\mathrm{M}_{n+1})[D^{\pm 1}].$$

Definition 1.56. *The* **quantized coordinate ring for** SL_{n+1} *or* **quantum** SL_{n+1} *is the k-algebra*

$$O_q(\mathrm{SL}_{n+1}) := O_q(\mathrm{M}_{n+1}) / \langle D - 1 \rangle.$$

We note that when $q \to 1$ we recover exactly the usual presentation for $O(\mathrm{M}_{n+1})$.

We will often abuse notation in $O_q(\mathrm{SL}_{n+1})$ and $O_q(\mathrm{GL}_{n+1})$ and will refer to the coset which contains X_{ij} simply by X_{ij}. Similarly, we refer to the coset which contains $[I \mid J]$ simply by $[I \mid J]$.

Algebraic Torus Actions

Let

$$H^+ = (k^*)^{n+1} \times (k^*)^{n+1} \tag{1.24}$$

and

$$H := \{(u,v) \in (k^*)^{n+1} \times (k^*)^{n+1} \mid u_1 \cdots u_{n+1} v_1 \cdots v_{n+1} = 1\}.$$

From [1, II.1.16] there is a rational H-action on $O_q(\mathrm{SL}_{n+1})$ by algebra automorphisms given by

$$(u,v).X_{ij} = u_i v_j X_{ij} \tag{1.25}$$

for all $(u.v) \in H$ and $X_{ij} \in O_q(\mathrm{SL}_{n+1})$. Similarly, from [1, II.1.15] there is a rational H^+-action by algebra automorphisms on $O_q(\mathrm{M}_{n+1})$ and $O_q(\mathrm{GL}_n)$ given by $(u,v).X_{ij} = u_i v_j X_{ij}$.

Hopf Algebra Structure

In [19, 9.2.3 Proposition 10] it is shown that $O_q(\mathrm{M}_{n+1})$ is a coalgebra. Specifically, the comultiplication and the counit are defined by

$$\Delta(X_{ij}) := \sum_{k=1}^{n+1} X_{ik} \otimes X_{kj} \qquad \text{and} \qquad \epsilon(X_{ij}) := \delta_{ij}.$$

We note that from [19, 9.2.2 Proposition 7(ii)]

$$\Delta([I \mid J]) = \sum_{K} [I \mid K] \otimes [K \mid J]. \tag{1.26}$$

Moreover, $O_q(\mathrm{SL}_{n+1})$ is a Hopf algebra with comultiplication and counit carried over from $O_q(\mathrm{M}_{n+1})$ and with the antipode defined by

$$S(X_{ij}) = q^{i-j}[1, 2, \cdots, \widehat{j}, \cdots, n+1 \mid 1, 2, \cdots, \widehat{i}, \cdots, n+1].$$

In fact, from [19, 9.2.3 Proposition 10] we have

$$S^2(X_{ij}) = q^{2i-2j} X_{ij}.$$

It follows that S is an anti-isomorphism.

1.12 Quantized Universal Enveloping Algebra for $\mathfrak{sl}_{n+1}$

We now specialize Section 1.9 to $\mathfrak{g} = \mathfrak{sl}_{n+1}$.

From [14, 11.4 Table 1] the Cartan matrix is the matrix (C_{ij}) so that

$$C_{ij} = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } |i-j| = 1 \\ 0 & \text{otherwise} \end{cases} .$$

Moreover, from [14, 13.2 Table 1] the fundamental weights are given by

$$\omega_i = \frac{1}{n+1}\left((n-i+1)\sum_{t=1}^{i-1} t\alpha_t + i\sum_{t=i}^{n}(n-t+1)\alpha_t \right).$$

Using the convention that $\omega_0 = \omega_{n+1} = 0$, we also have

$$\alpha_i = -\omega_{i-1} + 2\omega_i - \omega_{i+1}. \tag{1.27}$$

Specializing Definition 1.47 we have the following presentation for $U_q(\mathfrak{sl}_{n+1})$. The generators are the $K_i^{\pm 1}$ and the E_i, F_j where $i, j \in \{1, 2, \ldots, n\}$ presented with the following relations: for $i, j \in \{1, 2, \ldots, n\}$

$$K_i K_i^{-1} = 1 \qquad\qquad K_i K_j = K_j K_i$$

$$K_jE_i = q^{(\alpha_j,\alpha_i)}E_iK_j \qquad K_jF_i = q^{-(\alpha_j,\alpha_i)}F_iK_j$$

$$E_iE_j = E_jE_i \text{ for } |i-j| > 2 \qquad F_iF_j = F_jF_i \text{ for } |i-j| > 2$$

$$[E_i, F_j] = \delta_{ij}\frac{K_i - K_i^{-1}}{q - q^{-1}}$$

$$E_i^2E_j - (q+q^{-1})E_iE_jE_i + E_jE_i^2 = 0 \text{ for } |i-j| = 1$$

$$F_i^2F_j - (q+q^{-1})F_iF_jF_i + F_jF_i^2 = 0 \text{ for } |i-j| = 1.$$

Example: The Module $V(\omega_1)$

Since the module $V(\omega_1)$ is so integral for what follows, we examine it a bit more closely.

Using [15, Section 5.15] we have that $\dim V(\omega_1) = n+1$ and there exists a basis $\{e_1, e_2, \ldots, e_{n+1}\}$ so that each e_j is a weight vector with weight $\omega_1 - \alpha_1 + \cdots - \alpha_{j-1}$. Moreover, we may select the e_j so that $E_ie_j = \delta_{i,j-1}e_{j-1}$ and $F_ie_j = \delta_{i,j}e_{j+1}$ for all the $E_i, F_i \in U_q(\mathfrak{sl}_{n+1})$. Consequently, e_1 has highest weight ω_1 and this basis is well-ordered with respect to weight, that is wt $e_i \geq$ wt e_j if and only if $i \leq j$.

Keeping in mind convention 2 if $I = (i_1, i_2, \ldots, i_N)$ then

$$E_Ie_j = \delta_{i_1,j-N+1}\delta_{i_2,j-N+2}\cdots\delta_{i_{N-1},j-2}\delta_{i_N,j-1}e_{i_1}.$$

Therefore, for $i < j$ we have $E_I e_j = e_i$ if and only if $I = (i, i+1, \cdots, j-2, j-1)$. Furthermore, if I is any other nonempty finite sequence of elements from $\{1, 2, \ldots, n\}$ then $E_I e_j = 0$ for all j. Finally, $E_I e_j = e_j$ if and only if $I = \emptyset$.

By (1.27) for each $i = 1, 2, \ldots, n$ we have the weight of e_j is $\beta_j := -\omega_{j-1} + \omega_j$. Hence for $\lambda \in \mathbb{Z}\Pi$ and $1 \leq j \leq n+1$ we get

$$K_\lambda e_j = q^{(\beta_j, \lambda)} e_j. \tag{1.28}$$

Denote the natural dual basis $\{f^1, f^2, \ldots, f^{n+1}\}$ for $V(\omega_1)^*$ corresponding to $\{e_1, e_2, \ldots, e_{n+1}\}$ so that $f^i(e_j) = \delta_{ij}$.

Finally, we note that there is an isomorphism between quantum SL_{n+1} and the quantum algebra of matrix coefficients $C_q(\mathfrak{sl}_{n+1})$.

Theorem 1.57 ([12, Theorem 1.4.1])**.** $\left\{ c^{\omega_1}_{f^i, e_j} \mid i, j = 1, 2, \ldots n+1 \right\}$ *is a generating set for the algebra* $C_q(\mathfrak{sl}_{n+1})$*. Moreover, there exists a* k*-algebra isomorphism* $\kappa : O_q(\mathrm{SL}_{n+1}) \to C_q(\mathfrak{sl}_{n+1})$ *so that* $\kappa(X_{ij}) = c^{\omega_1}_{f^i, e_j}$.

Using the isomorphism from Theorem 1.57 we will often abuse notation and denote $c^{\omega_1}_{f^i, e_j}$ by X_{ij}.

Chapter 2

Quantum Unipotent Radicals of the Standard Borel Subgroups

2.1 Quantum Standard Borel Subgroups

Definition 2.1. *The* **quantized coordinate ring of the positive (negative) standard Borel subgroup** *or the* **quantum positive (negative) standard Borel subgroup** *of quantum* SL_{n+1} *is*

$$O_q(B^+) := O_q(\mathrm{SL}_{n+1})/\langle X_{ij} \mid i > j\rangle \quad \text{and} \quad O_q(B^-) := O_q(\mathrm{SL}_{n+1})/\langle X_{ij} \mid i < j\rangle.$$

We will often abuse notation and denote simply by X_{ij} the coset containing X_{ij}. We note that these algebras are quantizations of the standard Borel subgroup since as $q \to 1$ for the above subgroup, we recover the coordinate ring of the standard Borel.

Definition and Structure of $O_q(N^{\pm})$ and $O_q(N^{\pm})'$

Note that in $O_q(B^+)$ (resp. $O_q(B^-)$) that $X_{ij} = 0$ for $i > j$ (resp. $i < j$). Moreover, since the quantum determinant is 1, relation (1.23) in $O_q(B^{\pm})$ simplifies to

$$X_{11} \cdots X_{n+1,n+1} = 1.$$

Moreover, by relation (1.22) in $O_q(B^{\pm})$ we have $X_{ii}X_{jj} = X_{jj}X_{ii}$. Taking these two facts together we can conclude that for all $i = 1, 2, \ldots, n+1$ the elements X_{ii} are in fact, invertible in $O_q(B^{\pm})$. Therefore we may define in $O_q(B^+)$ *two* subalgebras

$$O_q(N^+) := k\left\langle X_{ii}^{-1}X_{ij} \mid 1 \leq i < j \leq n+1 \right\rangle$$

$$O_q(N^+)' := k\left\langle X_{ij}X_{jj}^{-1} \mid 1 \leq i < j \leq n+1 \right\rangle.$$

These are natural choices because they both become isomorphic to $O_q(N^+)$ when $q \to 1$. We may analogously define subalgebras in $O_q(B^-)$ by

$$O_q(N^-) := k[\left\langle X_{jj}^{-1}X_{ji} \mid 1 \leq i < j \leq n+1 \right\rangle$$

$$O_q(N^-)' := k\left\langle X_{ji}X_{ii}^{-1} \mid 1 \leq i < j \leq n+1 \right\rangle.$$

Having defined the algebras $O_q(N^{\pm})$ and $O_q(N^{\pm})'$, we now analyze their structure.

Lemma 2.2. *Define for* $1 \leq i < j \leq n+1$ *the elements* $y_{ij} = X_{ii}^{-1}X_{ij}$ *and* $z_{ij} = X_{ij}X_{jj}^{-1}$ *in* $O_q(B^+)$. *The following are defining relations for* $O_q(N^+)$ *and* $O_q(N^+)'$ *respectively:*

$$y_{ij}y_{im} = qy_{im}y_{ij} \qquad (j < m) \tag{2.1}$$

$$y_{ij}y_{lj} = qy_{lj}y_{ij} \qquad (i < l) \tag{2.2}$$

$$y_{ij}y_{lm} = y_{lm}y_{ij} \qquad (i < l,\ j > m) \tag{2.3}$$

$$y_{ij}y_{lm} = \begin{cases} y_{lm}y_{ij} & \text{if } j < l \\ q^{-1}y_{lm}y_{ij} + q^{-1}\widehat{q}y_{im} & \text{if } j = l \\ y_{lm}y_{ij} + \widehat{q}y_{im}y_{lj} & \text{if } j > l \end{cases} \qquad (i < l,\ j < m) \tag{2.4}$$

$$z_{ij}z_{im} = qz_{im}z_{ij} \qquad (j < m) \tag{2.5}$$

$$z_{ij}z_{lj} = qz_{lj}z_{ij} \qquad (i < l) \tag{2.6}$$

$$z_{ij}z_{lm} = z_{lm}z_{ij} \qquad (i < l,\ j > m) \tag{2.7}$$

$$z_{ij}z_{lm} = \begin{cases} z_{lm}z_{ij} & \text{if } j < l \\ q^{-1}z_{lm}z_{ij} + q^{-1}\widehat{q}z_{im} & \text{if } j = l \\ z_{lm}z_{ij} + \widehat{q}z_{im}z_{lj} & \text{if } j > l. \end{cases} \qquad (i < l,\ j < m) \tag{2.8}$$

Proof. First we show that the generators y_{ij} of $O_q(N^+)$ satisfy the relations (2.1) – (2.4) above. Note, that from the relations (1.21) and (1.22) in $O_q(\mathrm{SL}_{n+1})$, the elements X_{ii} commute with X_{lm} in $O_q(B^+)$ whenever $l, m \neq i$.

For $j < m$ and $i = l$

$$\begin{aligned} y_{ij}y_{im} &= X_{ii}^{-1}X_{ij}X_{ii}^{-1}X_{im} = q^{-1}X_{ii}^{-1}X_{ij}X_{im}X_{ii}^{-1} \\ &= X_{ii}^{-1}X_{im}X_{ij}X_{ii}^{-1} = qX_{ii}^{-1}X_{im}X_{ii}^{-1}X_{ij} = qy_{im}y_{ij}. \end{aligned}$$

For $i < l$ and $j = m$

$$\begin{aligned} y_{ij}y_{lj} &= X_{ii}^{-1}X_{ij}X_{ll}^{-1}X_{lj} = X_{ii}^{-1}X_{ll}^{-1}X_{ij}X_{lj} = qX_{ii}^{-1}X_{ll}^{-1}X_{lj}X_{ij} \\ &= qX_{ll}^{-1}X_{ii}^{-1}X_{lj}X_{ij} = qX_{ll}^{-1}X_{lj}X_{ii}^{-1}X_{ij} = qy_{lj}y_{ij}. \end{aligned}$$

For $i < l$ and $j > m$

$$\begin{aligned} y_{ij}y_{lm} &= X_{ii}^{-1}X_{ij}X_{ll}^{-1}X_{lm} = X_{ii}^{-1}X_{ll}^{-1}X_{ij}X_{lm} = X_{ii}^{-1}X_{ll}^{-1}X_{lm}X_{ij} \\ &= X_{ll}^{-1}X_{ii}^{-1}X_{lm}X_{ij} = X_{ll}^{-1}X_{lm}X_{ii}^{-1}X_{ij} = y_{lm}y_{ij}. \end{aligned}$$

For $i < l$, $j < m$, and $j < l$

$$\begin{aligned} y_{ij}y_{lm} &= X_{ii}^{-1}X_{ij}X_{ll}^{-1}X_{lm} = X_{ii}^{-1}X_{ll}^{-1}X_{ij}X_{lm} = X_{ii}^{-1}X_{ll}^{-1}X_{lm}X_{ij} \\ &= X_{ll}^{-1}X_{ii}^{-1}X_{lm}X_{ij} = X_{ll}^{-1}X_{lm}X_{ii}^{-1}X_{ij} = y_{lm}y_{ij}. \end{aligned}$$

For $i < l$, $j < m$, and $j = l$

$$
\begin{aligned}
y_{ij}y_{lm} &= X_{ii}^{-1}X_{ij}X_{ll}^{-1}X_{lm} \\
&= q^{-1}X_{ii}^{-1}X_{ll}^{-1}X_{ij}X_{lm} = q^{-1}X_{ii}^{-1}X_{ll}^{-1}(X_{lm}X_{ij} + \widehat{q}X_{im}X_{lj}) \\
&= q^{-1}X_{ii}^{-1}X_{ll}^{-1}X_{lm}X_{ij} + q^{-1}\widehat{q}X_{ii}^{-1}X_{ll}^{-1}X_{im}X_{lj} \\
&= q^{-1}X_{ll}^{-1}X_{ii}^{-1}X_{lm}X_{ij} + q^{-1}\widehat{q}X_{ii}^{-1}X_{im}X_{ll}^{-1}X_{lj} \\
&= q^{-1}X_{ll}^{-1}X_{lm}X_{ii}^{-1}X_{ij} + q^{-1}\widehat{q}X_{ii}^{-1}X_{im} = q^{-1}y_{lm}y_{ij} + q^{-1}\widehat{q}y_{im}.
\end{aligned}
$$

For $i < l$, $j < m$, and $j > l$

$$
\begin{aligned}
y_{ij}y_{lm} &= X_{ii}^{-1}X_{ij}X_{ll}^{-1}X_{lm} = X_{ii}^{-1}X_{ll}^{-1}X_{ij}X_{lm} = X_{ll}^{-1}X_{ii}^{-1}X_{ij}X_{lm} \\
&= X_{ll}^{-1}X_{ii}^{-1}(X_{lm}X_{ij} + \widehat{q}X_{im}X_{lj}) = X_{ll}^{-1}X_{ii}^{-1}X_{lm}X_{ij} + \widehat{q}X_{ll}^{-1}X_{ii}^{-1}X_{im}X_{lj} \\
&= X_{ll}^{-1}X_{lm}X_{ii}^{-1}X_{ij} + \widehat{q}X_{ii}^{-1}X_{im}X_{ll}^{-1}X_{lj} = y_{lm}y_{ij} + \widehat{q}y_{im}y_{lj}.
\end{aligned}
$$

We now show that the above relations are a defining set of relations for $O_q(N^+)$. Let B be the algebra generated by $\{b_{ij} \mid 1 \leq i < j \leq n+1\}$ presented with relations analogous to those in (2.1)-(2.4) above but replacing y_{ij} with b_{ij}. Let ψ be the k-algebra homomorphism $\psi : B \to O_q(N^+)$ defined by $\psi(b_{ij}) = y_{ij}$.

Order the X_{ij} lexicographically in $O_q(B^+)$ omitting $X_{n+1,n+1}$. As asserted in the proof of [8, Lemma 2.8] , the set of monomials ordered in this way is linearly independent.

This is still true if we allow ordered monomials with negative exponents on X_{ii}. Moreover since the X_{ii} commute up to scalars with every X_{lm} in $O_q(B^+)$ then the set of ordered monomials in the y_{ij} is linearly independent, hence forms a basis for $O_q(N^+)$.

Now the monomials in the b_{ij} form a spanning set for B. Hence ψ maps a spanning set of B to a basis of $O_q(N^+)$. Therefore ψ is an isomorphism.

It can be similarly verified that the relations (2.5)-(2.8) give a presentation of $O_q(N^+)'$. $\square$

Theorem 2.3. *The algebras $O_q(N^\pm)$ and $O_q(N^\pm)'$ are all isomorphic.*

Proof. From Lemma 2.2 it is immediate that $O_q(N^+) \cong O_q(N^+)'$ since the algebras have the same presentation.

From [25, Proposition 3.7.1] there exists a transpose homomorphism $\tau : O_q(\mathrm{SL}_{n+1}) \to O_q(\mathrm{SL}_{n+1})$ so that $\tau(X_{ij}) = X_{ji}$ for all $i, j \in \{1, 2, \ldots, n+1\}$. This is an automorphism of $O_q(\mathrm{SL}_{n+1})$ that maps $\langle X_{ij} \mid i > j\rangle$ onto $\langle X_{ij} \mid i < j\rangle$. Therefore there is an induced isomorphism $\overline{\tau} : O_q(B^+) \to O_q(B^-)$.

Observe that

$$\overline{\tau}(X_{ii}^{-1}X_{ij}) = X_{ii}^{-1}X_{ji} = q^{-1}X_{ji}X_{ii}^{-1}$$

and

$$\overline{\tau}(X_{ij}X_{jj}^{-1}) = X_{ji}X_{jj}^{-1} = q^{-1}X_{jj}^{-1}X_{ji}$$

for all $i < j$. Hence, $\overline{\tau}$ maps $O_q(N^+)$ onto $O_q(N^-)'$ and so $O_q(N^+) \cong O_q(N^-)'$. Similarly $\overline{\tau}$ maps $O_q(N^+)'$ onto $O_q(N^-)$. Therefore $O_q(N^+)' \cong O_q(N^-)$ □

Using this theorem we will refer to either $O_q(N^+)$ or $O_q(N^+)'$ as a **positive quantized unipotent subgroup** or positive standard Borel. Similarly, $O_q(N^-)$ or $O_q(N^-)'$ are a **negative quantized unipotent subgroup.** Notice that as $q \to 1$ we have a commutative k-algebra which matches the usual presentation for $O(N^{\pm})$.

2.2 The Algebras of Coinvariants for $O_q(B^{\pm})$

$O_q(B^{\pm})$ as Hopf Algebras

Define $I^- := \langle X_{ij} \mid i > j\rangle$ and $I^+ := \langle X_{ij} \mid i < j\rangle$ in $O_q(\mathrm{SL}_{n+1})$. We now show that $I^{\pm}$ are Hopf ideals.

Lemma 2.4. *$I^{\pm}$ are Hopf ideals in $O_q(\mathrm{SL}_{n+1})$.*

Proof. It is clear that $I^+ \subset \ker\epsilon$. If $i < j$ then $\sum_{k=1}^{j-1} X_{ik} \otimes X_{kj} \in O_q \otimes I^+$ and $\sum_{k=j}^{n+1} X_{ik} \otimes X_{kj} \in I^+ \otimes O_q$. Hence $\Delta(X_{ij}) \in O_q \otimes I^+ + I^+ \otimes O_q$. Since Δ is an algebra homomorphism we have $\Delta(I^+) \subset I^+ \otimes O_q + O_q \otimes I^+$.

Now since $i < j$ then each monomial in M_{ji} is of the form

$$q^{\ell(\sigma)} X_{1,\sigma(1)} \cdots X_{i-1,\sigma(i-1)} X_{i,\sigma(i+1)} \cdots X_{j-1,\sigma(j)} X_{j+1,\sigma(j+1)} \cdots X_{n+1,\sigma(n+1)}$$

where is σ is some permutation on $I = \{1, 2, \ldots, \widehat{i}, \ldots n+1\}$. If σ is not the identity there is a $k \in I$ so that $\sigma(k) > k$. Hence the above monomial is in I^+ since it contains either the term $X_{k,\sigma(k)}$ or $X_{k-1,\sigma(k)}$, both of which are in I^+. If σ is the identity then again the above monomial is in I^+ since it contains the term $X_{i,i+1}$ which again is in I^+.

Thus for every σ, the monomials terms of M_{ji} are in I^+. Hence $S(X_{ij}) = q^{i-j} M_{ji} \in I^+$. Since S is an antihomomorphism then $S(I^+) \subset I^+$. Therefore I^+ is a Hopf ideal.

A similar argument also shows that I^- is a Hopf ideal. $\square$

Since $O_q(B^{\pm}) = O_q(\mathrm{SL}_{n+1})/I^{\mp}$ then $O_q(B^{\pm})$ are Hopf algebras induced from the Hopf algebra structure of $O_q(\mathrm{SL}_{n+1})$. We denote the comultiplication, counit, and antipode of the Hopf algebra of $O_q(B^{\pm})$ by $\Delta_{B^{\pm}}$, $\epsilon_{B^{\pm}}$, and $S_{B^{\pm}}$ respectively, when emphasis is needed, otherwise we will retain the standard notation Δ, ϵ, and S. Specifically we note that for $X_{ij} \in O_q(B^+)$ and $X_{rs} \in O_q(B^-)$ we have

$$\Delta_{B^+}(X_{ij}) = \sum_{i \le k \le j} X_{ik} \otimes X_{kj} \qquad \text{and} \qquad \Delta_{B^-}(X_{rs}) = \sum_{r \ge k \ge s} X_{rk} \otimes X_{ks}.$$

The Quantum Standard Maximal Torus

In the classical setting the coordinate ring of the standard maximal torus of $O(\mathrm{SL}_{n+1})$ is

$$O(T) \cong O(\mathrm{SL}_{n+1})/\langle X_{ij} \mid i \neq j \rangle.$$

We may therefore define the **quantized coordinate ring of the standard maximal torus for** SL_{n+1} or **quantum standard maximal torus** by

$$O_q(T) := O_q(\mathrm{SL}_{n+1})/\langle X_{ij} \mid i \neq j \rangle.$$

Denote by Y_{ii} the coset containing X_{ii} in $O_q(T)$. We first note that $O_q(T)$ is generated by the Y_{ii} where $i = 1, 2, \dots, n+1$. It is straightforward to check that $Y_{ii}Y_{jj} = Y_{jj}Y_{ii}$ for all $i, j \in \{1, 2, \cdots, n+1\}$. Thus $O_q(T)$ is actually a commutative algebra. Moreover since the quantum determinant is 1 in $O_q(\mathrm{SL}_{n+1})$ this implies that

$$Y_{1,1} \cdots Y_{n+1,n+1} = 1$$

That is, each of the Y_{ii} are invertible. Therefore we have, in fact, $O_q(T) \cong O(T)$. Moreover, since $O_q(T) = O_q(\mathrm{SL}_{n+1})/(I^+ + I^-)$ this implies that $O_q(T)$ is also a Hopf algebra induced from $O_q(\mathrm{SL}_{n+1})$. We denote the comultiplication, counit, and antipode by Δ_T, ϵ_T, and S_T respectively, when emphasis is needed.

Specifically, we note that for $Y_{ii} \in O_q(T)$ we have

$$\Delta_T(Y_{ii}) = Y_{ii} \otimes Y_{ii}.$$

$O_q(T)$-coactions on $O_q(B^{\pm})$

There are natural projection homomorphisms $p^{\pm} : O_q(B^{\pm}) \to O_q(T)$ defined by $p^{\pm}(X_{ii}) = Y_{ii}$ and $p^{\pm}(X_{ij}) = 0$ for $i \neq j$. Therefore using these maps as well as the comultiplication maps Δ on $O_q(B^{\pm})$ we define the maps $\eta^{\pm} : O_q(B^{\pm}) \to O_q(B^{\pm}) \otimes O_q(T)$ by

$$\eta^{\pm} := (id \otimes p^{\pm})\Delta.$$

Since $p^{\pm}$ and Δ are k-algebra homomorphisms, so are $\eta^{\pm}$. Similarly the maps $\theta^{\pm} : O_q(B^{\pm}) \to O_q(T) \otimes O_q(B^{\pm})$ defined by $\theta^{\pm} := (p^{\pm} \otimes id)\Delta$ are k-algebra homomorphisms. Specifically, since $p^{\pm}(X_{ij}) = 0$ for $i \neq j$ and $p^{\pm}(X_{ii}) = Y_{ii}$ this implies that for $X_{ij} \in O_q(B^{\pm})$ we have

$$\eta^{\pm}(X_{ij}) = X_{ij} \otimes Y_{jj} \qquad\qquad \theta^{\pm}(X_{ij}) = Y_{ii} \otimes X_{ij}.$$

In fact $\eta^{\pm}$ and $\theta^{\pm}$ are comodule homomorphisms which we now show.

Lemma 2.5. *$\eta^{\pm}$ and $\theta^{\pm}$ are right (resp. left) $O_q(T)$-comodule maps. Hence, $O_q(B^{\pm})$ are right (resp. left) $O_q(T)$-comodule algebras.*

Proof. We need to verify that $(\rho^{\pm} \otimes id)\rho^{\pm} = (id \otimes \Delta_T)\rho^{\pm}$. It suffices to verify it on the $X_{ij} \in O_q(B^{\pm})$ since $\rho^{\pm}$ and Δ_T are k-algebra homomorphisms. Note that

$$(\rho^{\pm} \otimes id)\rho^{\pm}(X_{ij}) = (\rho^{\pm} \otimes id)(X_{ij} \otimes Y_{jj}) = X_{ij} \otimes Y_{jj} \otimes Y_{jj}$$

and

$$(id \otimes \Delta_T)\rho^{\pm}(X_{ij}) = (id \otimes \Delta_T)(X_{ij} \otimes Y_{jj}) = X_{ij} \otimes Y_{jj} \otimes Y_{jj}.$$

Hence $\theta^{\pm}$ is a right $O_q(T)$-comodule map. In the same way we can show $\eta^{\pm}$ are left $O_q(T)$-comodule maps.

Since $\theta^{\pm}$ is also a k-algebra homomorphism we have $O_q(B^{\pm})$ are right $O_q(T)$-comodule algebras with these structure maps. Similarly, $\eta^{\pm}$ will also make $O_q(B^{\pm})$ into left $O_q(T)$-comodule algebras. □

$O_q(N^{\pm})$ as Coinvariants

Theorem 2.6. *Let $A^{\pm} = O_q(B^{\pm})^{\mathrm{co}\,\eta^{\pm}}$ then $A^{\pm}\#O_q(T) \cong O_q(B^{\pm})$. Similarly letting*

$C^{\pm} = O_q(B^{\pm})^{\mathrm{co}\,\theta^{\pm}}$ then $O_q(T)\#C^{\pm} \cong O_q(B^{\pm})$.

Proof. Let $r^{\pm} : O_q(T) \to O_q(B^{\pm})$ be the k-algebra homomorphism such that $r^{\pm}(Y_{ii}) = X_{ii}$. Define $\overline{r^{\pm}} : O_q(T) \to O_q(B^{\pm})$ by $\overline{r^{\pm}} = r^{\pm}S_T$. Then

$$(r^{\pm} * \overline{r^{\pm}})(Y) = \sum_{(Y)} r^{\pm}(Y_1)\overline{r^{\pm}}(Y_2) = r^{\pm}\left(\sum_{(Y)} Y_1 S_T(Y_2)\right) = r^{\pm}(\epsilon_T(Y) \cdot 1) = \epsilon(Y) \cdot 1$$

for all $Y \in O_q(T)$. Similarly, $(\overline{r^\pm} * r^\pm)(Y) = \epsilon(Y) \cdot 1$. Hence, $\overline{r^\pm}$ is the convolution inverse of $r^\pm$.

To check that $r^\pm$ is a right $O_q(T)$-comodule map we need to show that $\eta^\pm r^\pm = (r^\pm \otimes id)\Delta_T$. Since $r^\pm$ and $\eta^\pm$ are k-algebra homomorphisms, it is sufficient to show the equality holds on the $Y_{ii} \in O_q(T)$. This holds because

$$\eta^\pm r^\pm(Y_{ii}) = \eta^\pm(X_{ii}) = X_{ii} \otimes Y_{ii}$$

and

$$(r^\pm \otimes id)\Delta_T(Y_{ii}) = (r^\pm \otimes id)(Y_{ii} \otimes Y_{ii}) = X_{ii} \otimes Y_{ii}.$$

Hence, $r^\pm$ are right $O_q(T)$-comodule homomorphisms. Therefore $O_q(B^\pm)$ is an H-cleft extension. Moreover, by the above discussion, using the result of [2] there is a k-algebra isomorphism $\Phi^\pm : A^\pm \# O_q(T) \to O_q(B^\pm)$ where $\Phi^\pm(X\#Y) = Xr^\pm(Y)$.

Similarly, by making the appropriate changes to the above proof, there is a k-algebra isomophrism $\Psi^\pm : O_q(T)\#C^\pm \to O_q(B^\pm)$ defined by $\Psi(Y\#X) = r^\pm(Y)X$. $\square$

It is natural to ask what are the coinvariants for $O_q(B^\pm)$ using the structure maps $\eta^\pm$ or $\theta^\pm$? Note that for any $X_{jj}^{-1}, X_{ij} \in O_q(B^\pm)$

$$\eta^\pm(X_{ij}X_{jj}^{-1}) = \eta^\pm(X_{ij})\eta^\pm(X_{jj}^{-1}) = (X_{ij} \otimes Y_{jj})(X_{jj}^{-1} \otimes Y_{jj}^{-1}) = X_{ij}X_{jj}^{-1} \otimes 1.$$

That is $X_{ij}X_{jj}^{-1} \in O_q(B^{\pm})^{\mathrm{co}\,\eta^{\pm}}$. Similarly, $X_{ii}^{-1}X_{ij} \in O_q(B^{\pm})^{\mathrm{co}\,\theta^{\pm}}$. Since $\eta^{\pm}$ and $\theta^{\pm}$ are algebra homomorphism this implies that $O_q(N^{\pm})'$ is a subalgebra of $O_q(B^{\pm})^{\mathrm{co}\,\eta^{\pm}}$ and $O_q(N^{\pm})$ is a subalgebra of $O_q(B^{\pm})^{\mathrm{co}\,\theta^{\pm}}$. In fact these algebras are exactly the coinvariants for $\eta^{\pm}$ and $\theta^{\pm}$ which we now show.

Theorem 2.7. *$O_q(N^{\pm})' = O_q(B^{\pm})^{\mathrm{co}\,\eta^{\pm}}$ and $O_q(N^{\pm}) = O_q(B^{\pm})^{\mathrm{co}\,\theta^{\pm}}$.*

Proof. From our discussion we have shown that $O_q(N^{\pm})' \subseteq O_q(B^{\pm})^{\mathrm{co}\,\eta^{\pm}}$. Since the map $\Phi^{\pm}$ from Theorem 3.19 is an isomorphism of $A^{\pm} \otimes O_q(T)$ onto $O_q(B^{\pm})$, it is sufficient to show that $\Phi^{\pm}$ maps $O_q(N^{\pm})' \otimes O_q(T)$ onto $O_q(B^{\pm})$.

Notice that

$$\Phi^{\pm}(O_q(N^{\pm})' \otimes O_q(T)) = O_q(N^{\pm})' r^{\pm}(O_q(T)) \subseteq O_q(B^{\pm}).$$

Therefore we need only show that $O_q(N^{\pm})r^{\pm}(O_q(T))$ is a subalgebra that contains all the X_{ij} to prove the proposition.

Let $L^{\pm}$ be the subalgebra generated by $\{X_{ii}^{\pm 1} \mid 1 \leq i \leq n+1\}$ in $O_q(B^{\pm})$. Note that the image of the maps $r^{\pm}$ from Theorem 3.19 is $L^{\pm}$. Using the projection homomorphism $p^{\pm}$, it is straightforward to check that $p^{\pm}r^{\pm} = id_{O_q(T)}$ and $r^{\pm}p^{\pm} = id_{L^{\pm}}$. Hence $r^{\pm}$ is an isomorphism from $O_q(T)$ to $L^{\pm}$. Since each of the X_{kk} commutes up to a scalar with each of the $X_{ij} \in O_q(B^{\pm})$ we have that $O_q(N^{\pm})' r^{\pm}(O_q(T))$ is a subalgebra of $O_q(B^{\pm})$.

Now for $X_{jj}^{-1}, X_{ij} \in O_q(B^{\pm})$ with $i \neq j$ we have

$$\Phi^{\pm}(X_{ij}X_{jj}^{-1} \otimes Y_{jj}) = X_{ij}X_{jj}^{-1}r^{\pm}(Y_{jj}) = X_{ij}X_{jj}^{-1}X_{jj} = X_{ij}.$$

Hence, $O_q(N^{\pm})'r^{\pm}(O_q(T)) = O_q(B^{\pm})$. Therefore $O_q(N^{\pm})' = O_q(B^{\pm})^{\mathrm{co}\,\eta^{\pm}}$.

In the same way one can use the isomorphism $\Psi^{\pm}$ to show $O_q(N^{\pm}) = O_q(B^{\pm})^{\mathrm{co}\,\theta^{\pm}}$.

$\square$

Finally we have the following corollary.

Corollary 2.8. *$O_q(B^{\pm}) \cong O_q(T)\#O_q(N^{\pm}) \cong O_q(N^{\pm})\#O_q(T)$.*

Proof. The first isomorphism follows directly from Theorem 3.19 and Theorem 2.7. For the second isomorphism we note that Theorem 3.19 and Theorem 2.7 imply that $O_q(B^{\pm}) \cong O_q(N^{\pm})'\#O_q(T)$. Since from Theorem 2.3 we have $O_q(N^{\pm}) \cong O_q(N^{\pm})'$ the second isomorphism holds. $\square$

2.3 The Restriction Map and $C_q(\mathfrak{b}^{\pm})$

Having investigated $O_q(B^{\pm})$ and $O_q(N^{\pm})$ we now switch to the quantum function algebra or algebra of matrix coefficients $C_q(\mathfrak{sl}_{n+1})$. In order to do this though we first need some background.

For any element $c \in C_q(\mathfrak{sl}_{n+1})$ we may restrict the domain of c to the subalgebra $U_q^{+}(\mathfrak{sl}_{n+1})$ (resp. $U_q(\mathfrak{b}^{-})$). This induces a well-defined k-algebra homomorphism from $C_q(\mathfrak{sl}_{n+1})$ to $U_q(\mathfrak{b}^{-})^*$ (resp. $U_q(\mathfrak{b}^{-})^*$). Let ρ^{+} (resp. ρ^{-}) denote this restriction homomorphism and denote im ρ^+ by $C_q(\mathfrak{b}^+)$ (resp. im ρ^- by $C_q(\mathfrak{b}^-)$). For $c \in C_q(\mathfrak{sl}_{n+1})$ we denote $\rho^+(c)$ by $\bar{c}$. From this convention we note that $\bar{c} \neq 0$ if and only if $c(u) \neq 0$ for some $u \in U_q^{\geq 0}(\mathfrak{sl}_{n+1})$.

We now wish to better understand $\rho^+(c^{\omega_1}_{f^i,e_j})$ which, using the conventions above, we denote by $\overline{X}_{ij}$.

Example: Matrix Coefficients for $V(\omega_1)$ in $C_q(\mathfrak{b}^\pm)$

We wish to understand $C_q(\mathfrak{b}^\pm)$ and to do so we will start trying to understand the "basic" elements $\overline{X}_{ij} \in C_q(\mathfrak{b}^\pm)$. Since $U_q^{\geq 0}$ (resp. $U_q^{\leq 0}$) is spanned by monomials of the form $K_\lambda E_I$ (resp. $K_\lambda F_I$) where $\lambda \in \mathbb{Z}\Phi^+$ and $I \in P^+$, to understand any $\overline{X}_{ij}$ it is sufficient to understand where it maps each of these elements.

Lemma 2.9. *For $\lambda \in \mathbb{Z}\Pi$ and I a finite sequence of elements from $\{1, 2, \ldots, n+1\}$ we have the following:*

For $i \leq j$,

$$\overline{X}_{ij}(K_\lambda E_I) = \begin{cases} q^{(\beta_i,\lambda)} & \text{if } I = \emptyset \text{ and } i = j \\ q^{(\beta_i,\lambda)} & \text{if } I = (i, i+1, \cdots, j-2, j-1) \text{ and } i < j \\ 0 & \text{otherwise.} \end{cases}$$

where $\beta_i = -\omega_{i-1} + \omega_i$.

For $i > j$, then $\overline{X}_{ij}(K_\lambda E_I) = 0$.

Proof. By equation (1.12) if $i < j$ we have $E_I e_j = e_i$ if and only if $I = (i, i+1, \cdots, j-2, j-1)$ and $E_I e_j = e_j$ if and only if $I = \emptyset$. Furthermore, if I is any other finite sequence of elements from $\{1, 2, \ldots, n+1\}$ then $E_I e_j = 0$ for all $j = 1, 2, \ldots, n+1$. Hence by equation (1.28) if $i < j$ and $I = (i, \cdots, j-1)$ we have $K_\lambda E_I e_j = q^{(\beta_i, \lambda)} e_i$ and if $I = \emptyset$ then $K_\lambda E_I e_j = q^{(\beta_i, \lambda)} e_j$. Thus if $i \leq j$ then $\overline{X_{ij}}(K_\lambda E_I)$ is as indicated.

Note that for all the E_i we have $E_i e_j$ is either 0 or a weight vector of lower weight. Hence, for $i > j$ there is no E_I so that $E_I e_j = e_i$. Therefore for $i > j$ we have $f^i(E_I e_j) = 0$ for all E_I. Hence, $\overline{X}_{ij}(K_\lambda E_I) = 0$ for all $\lambda \in \mathbb{Z}\Phi^+$ and I a finite sequence with entries in $\{1, 2, \ldots, n+1\}$. $\square$

2.4 The Dual Pairing

By [3] Corollary 3.3, if $\sqrt[n+1]{q} \in k$ there exists a unique nondegenerate bilinear pairing $(-,-) : \check{U}_q^{\leq 0} \times \check{U}_q^{\geq 0} \to k$ defined by the following properties: for all $u, u' \in \check{U}^{\geq 0}$ and all $v, v' \in \check{U}^{\leq 0}$ and all $\mu, \nu \in \Lambda$ and $\alpha, \beta \in \Pi$

$$(u, vv') = (\Delta(u), v' \otimes v) \qquad (uu', v) = (u \otimes u', \Delta(v))$$

$$(K_\mu, K_\nu) = q^{-(\mu,\nu)} \qquad (F_\alpha, E_\beta) = -\delta_{\alpha\beta}\widehat{q}^{-1}$$

$$(K_\mu, E_\beta) = 0 \qquad (F_\alpha, K_\mu) = 0$$

We note there is a pairing on $U_q^{\geq 0}(\mathfrak{sl}_{n+1}) \times U_q^{\geq 0}(\mathfrak{sl}_{n+1})$ defined in [15, p. 6.12] as well as [16, p. 9.2.10] with the same properties as above.

The bilinear pairing above can be restricted to a bilinear pairing on $\check{U}_q^{\leq 0} \times U_q^{\geq 0}$. In fact the assumption that $\sqrt[n+1]{q} \in k$ is not needed to have a well defined dual pairing on $\check{U}_q^{\leq 0} \times U_q^{\geq 0}$. Since we will primarily be studying this pairing, we remove this assumption. We now show that the pairing in nondegenerate by slightly modifying the proof in [3, Corollary 3.3].

Theorem 2.10. *The dual pairing on $\check{U}_q^{\leq 0} \times U_q^{\geq 0}$ is nondegenerate.*

Proof. By [15, p. 4.7] , U_q^+ and U_q^- are $\mathbb{Z}\Pi$-graded with $\deg E_\alpha = \alpha$ and $\det F_{-\alpha} = -\alpha$ for $\alpha \in \Pi$. For each $\mu \in \mathbb{Z}\Pi$ denote a basis of U_μ^+ by $\{u_i^\mu\}$. By [15, Corollary 8.30] , the pairing when restricted to $U_{-\mu}^- \times U_\mu^+$ with $\mu \in \mathbb{Z}\Pi$ is nondegenerate. Therefore, we may select a corresponding dual basis of $U_{-\mu}^-$, which we denote by $\{v_i^{-\mu}\}$, with the property that $(v_i^{-\mu}, u_j^\mu) = \delta_{ij}$.

Suppose that $y \in \check{U}_q^{\leq 0}$ so that $(y, x) = 0$ for all $x \in U_q^{\geq 0}(\mathfrak{sl}_{n+1})$. We may write

$$y = \sum_{\mu' \in \mathbb{Z}\Pi, i} v_i^{-\mu'} p_{\mu',i}(K)$$

where

$$p_{\mu',i}(K) = \sum_{\lambda' \in \Lambda} c_{\lambda'}^{\mu',i} K_{\lambda'}$$

for some scalars $c_{\lambda'}^{\mu',i}$. It follows from [15, 6.10(3) and 6.10(4)] that if $\mu \in \mathbb{Z}\Pi$ then for all $\eta \in \mathbb{Z}\Pi$ and all j we have

$$0 = (y, u_j^{\mu} K_{\eta}) = \sum_{\lambda' \in \Lambda} c_{\lambda'}^{\mu,j} q^{-(\eta,\lambda')}.$$

Notice that for all $\lambda' \in \Lambda$ there exist scalars $\lambda'_1, \dots, \lambda'_n \in \mathbb{Z}$ so that $\lambda' = \lambda'_1\omega_1 + \cdots + \lambda'_n\omega_n$. Therefore, for each $m = (m_1, \dots, m_n) \in \mathbb{Z}^n$, setting $\eta = -(m_1\alpha_1 + \cdots + m_n\alpha_n)$ we get

$$0 = \sum_{\lambda' \in \Lambda} c_{\lambda'}^{\mu,j} q^{-(\eta,\lambda')} = \sum_{\lambda' \in \Lambda} c_{\lambda'}^{\mu,j} q^{m_1\lambda'_1} \cdots q^{m_n\lambda'_n} = p_{\mu,j}(q^{m_1}, \dots, q^{m_n}) = p_{\mu,j}(q^m)$$

where

$$p_{\mu,j}(x_1, \dots, x_n) = \sum_{\lambda' \in \Lambda} c_{\lambda'}^{\mu,j} x_1^{\lambda'_1} \cdots x_n^{\lambda'_n}.$$

Since q is not a root of unity, it follows from [3, Lemma 3.2] that $p_{\mu,j} = 0$ for all $\mu \in \mathbb{Z}\Pi$ and all j. Hence, $y = 0$.

Suppose for $x \in U_q^{\geq 0}(\mathfrak{sl}_{n+1})$ that $(y, x) = 0$ for all $y \in \check{U}_q^{\leq 0}$. Since $x \in \check{U}_q^{\geq 0}$ it follows from the nondegeneracy of $(-,-)$ on $\check{U}_q^{\leq 0} \times \check{U}_q^{\geq 0}$ that $x = 0$. $\square$

Using this dual pairing we define the map $\phi : \check{U}_q^{\leq 0} \to (U_q^{\geq 0})^*$ by

$$\phi(u)(v) = (u, v).$$

The map ϕ is a k-algebra homomorphism and since the bilinear form is nondegenerate by Theorem 2.10, ϕ is injective.

For $I = (i_1, i_2, \ldots, i_N)$ define wt $I = \alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_N}$. We say that $J \subseteq I$ if J is a subsequence of I. For $J \subseteq I$ we denote $I - J$ to be the sequence remaining when the subsequence J is removed.

It is shown in [15, 6.8(8) and 6.12] that $(F_I, E_J) = 0$ unless wt $I =$ wt J. If follows that $\ker(\phi(F_I))$ has finite codimension for every I. Similarly, $\ker \phi(K_\mu)$ has codimension 1. Hence $\phi(u) \in (U_q^{\geq 0}(\mathfrak{sl}_{n+1}))^\circ$ for all $u \in \check{U}_q^{\leq 0}$.

Lemma 2.11. *For the map ϕ we have $\phi(K_{-\beta_i+\alpha_i}F_i) = -q^{-1}\widehat{q}^{-1}\overline{X}_{ii+1}$ and $\phi(K_{\mp\beta_i}) = \overline{X}_{ii}^{\pm 1}$.*

Proof. From [15, 6.9 (2)] it follows for all $\mu, \lambda \in \Lambda$ that $(K_\mu, K_\lambda E_I) = 0$ if and only if $I \neq \emptyset$. Using this fact we get

$$\begin{aligned}
\phi(K_{-\beta_i+\alpha_i}F_i)(K_\lambda E_i) &= (K_{-\beta_i+\alpha_i}F_i, K_\lambda E_i) = (K_{-\beta_i+\alpha_i} \otimes F_i, \Delta(K_\lambda E_i)) \\
&= (K_{-\beta_i+\alpha_i} \otimes F_i, K_{\lambda+\alpha_i} \otimes K_\lambda E_i + K_\lambda E_i \otimes K_\lambda) \\
&= (K_{-\beta_i+\alpha_i}, K_{\lambda+\alpha_i})(F_i, K_\lambda E_i) + (K_{-\beta_i+\alpha_i}, K_\lambda E_i)(F_i, K_\lambda) \\
&= (K_{-\beta_i+\alpha_i}, K_{\lambda+\alpha_i})(F_i, K_\lambda E_i) \\
&= (K_{-\beta_i+\alpha_i}, K_{\lambda+\alpha_i})(\Delta(F_i), E_i \otimes K_\lambda) \\
&= (K_{-\beta_i+\alpha_i}, K_{\lambda+\alpha_i})(F_i \otimes K_{-\alpha_i} + 1 \otimes F_i, E_i \otimes K_\lambda) \\
&= (K_{-\beta_i+\alpha_i}, K_{\lambda+\alpha_i})\left((F_i, E_i)(K_{-\alpha_i}, K_\lambda) + (1, E_i)(F_i, K_\lambda)\right)
\end{aligned}$$

$$= (K_{-\beta_i+\alpha_i}, K_{\lambda+\alpha_i})(F_i, E_i)(K_{-\alpha_i}, K_\lambda)$$

$$= -q^{-(-\beta_i+\alpha_i,\lambda+\alpha_i)}\widehat{q}^{-1}q^{(\alpha_i,\lambda)} = -q^{-1}\widehat{q}^{-1}q^{(\beta_i,\lambda)}.$$

Now suppose $I \neq (i)$. It follows from [15, 6.8 (1)] , for $\lambda \in \mathbb{Z}\Pi$ and I a finite sequence of elements in $\{1, 2, \ldots, n\}$

$$\Delta(K_\lambda E_I) = \sum_{J \subseteq I} c_{I,J} K_\lambda K_J E_{I-J} \otimes K_\lambda E_J$$

for some scalars, $c_{I,J}$. Hence, we have

$$\begin{aligned}
\phi(K_{-\beta_i+\alpha_i}F_i)(K_\lambda E_I) &= (K_{-\beta_i+\alpha_i}F_i, K_\lambda E_I) = (K_{-\beta_i+\alpha_i} \otimes F_i, \Delta(K_\lambda E_I)) \\
&= \left(K_{-\beta_i+\alpha_i} \otimes F_i, \sum_{J \subseteq I} c_{I,J} K_\lambda K_J E_{I-J} \otimes K_\lambda E_J\right) \\
&= \sum_{J \subseteq I} c_{I,J}(K_{-\beta_i+\alpha_i}, K_\lambda K_J E_{I-J})(F_i, K_\lambda E_J).
\end{aligned}$$

It follows from [15, 6.8 (7)] that $(F_i, K_\lambda E_J) = 0$ if and only if $J \neq (i)$. Moreover, if $J = (i)$ then since $I \neq (i)$ we have that $I - J \neq \emptyset$. This implies $(K_{-\beta_i+\alpha_i}, K_\lambda K_J E_{I-J}) = 0$. Therefore, we have $\phi(K_{-\beta_i+\alpha_i}F_i)(K_\lambda E_I) = 0$. Therefore, we have

$$\phi(K_{-\beta_i+\alpha_i}F_i)(K_\lambda E_I) = \begin{cases} -q^{-1}\widehat{q}^{-1}q^{(\beta_i,\lambda)} & \text{if } I = (i) \\ 0 & \text{if } I \neq (i). \end{cases}$$

Similarly,

$$\phi(K_{\mp\beta_i})(K_\lambda) = (K_{\mp\beta_i}, K_\lambda) = q^{\pm(\beta_i,\lambda)}$$

Moreover, if $\lambda \in \mathbb{Z}\Pi$ and I is a finite sequence of elements from $\{1, 2, \ldots, n\}$ and $I \neq \emptyset$ we have

$$\begin{aligned} \phi(K_{\mp\beta_i})(K_\lambda E_I) &= (K_{\mp\beta_i}, K_\lambda E_I) \\ &= (K_{\mp\beta_i} \otimes K_{\mp\beta_i}, E_I \otimes K_\lambda) \\ &= (K_{\mp\beta_i}, E_I)(K_{\mp\beta_i}, K_\lambda) \\ &= 0. \end{aligned}$$

Hence,

$$\phi(K_{\mp\beta_i})(K_\lambda E_I) = \begin{cases} q^{\pm(\beta_i,\lambda)} & \text{if } I = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

By Lemma 2.9,

$$\overline{X}_{i,i+1}(K_\lambda E_I) = \begin{cases} q^{(\beta_i,\lambda)} & \text{if } I = (i) \\ 0 & \text{otherwise} \end{cases}$$

and

$$\overline{X}_{ii}^{\pm 1}(K_\lambda E_I) = \begin{cases} q^{\pm(\beta_i,\lambda)} & \text{if } I = \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Since $U_q^{\geq 0}(\mathfrak{sl}_{n+1})$ is spanned by the elements of the form $K_\lambda E_I$ where I is a finite sequence of elements from $\{1, 2, \ldots, n\}$ and $\lambda \in \mathbb{Z}\Pi$ then $\phi(K_{-\beta_i+\alpha_i}F_i) = -q^{-1}\widehat{q}^{-1}\overline{X}_{i,i+1}$ and $\phi(K_{\mp\beta_i}) = \overline{X}_{ii}^{\pm 1}$. $\square$

Lemma 2.12. *$C_q(\mathfrak{b}^+)$ is generated by $\left\{\overline{X}_{ii}^{\pm 1},\ \overline{X}_{j,j+1} \mid 1 \leq i \leq n+1,\ 1 \leq j \leq n\right\}$.*

Proof. Denote $\left\{X_{ii}^{\pm 1}, X_{i,i+1} \mid 1 \leq i \leq n+1,\ 1 \leq j \leq n\right\}$ in $O_q(B^+)$ by S. Since for all $1 \leq i \leq n-1$, $O_q(B^+)$ has the relation

$$X_{i,i+1,}X_{i+1,i+2} - X_{i+2,i+1}X_{i,i+1} = \widehat{q}X_{i+1,i+1}X_{i,i+2}.$$

By multiplying both sides by $X_{i+1,i+1}^{-1}$ we see $X_{i,i+2} \in S$ for all $1 \leq i \leq n-1$. Continuing inductively if $X_{i,i+k} \in S$ for all $1 \leq i \leq n-k$ since

$$X_{i,i+k}X_{i+k,i+k+1} - X_{i+k,i+k+1}X_{i,i+k} = \widehat{q}X_{i+k,i+k}X_{i,i+k+1}$$

multiplying by $X_{i+k,i+k}^{-1}$ we get $X_{i,i+k+1} \in S$ for all $1 \leq i \leq n-k$. Therefore $X_{ij} \in S$ for all $1 \leq i \leq j \leq n+1$. Hence S generates $O_q(B^+)$. Using the isomorphism induced from Corollary **??**, the proposition follows. $\square$

The following theorem can be found in [16, p. 9.2.12] however, in the present case we may prove it more simply.

Theorem 2.13. *The map ϕ is an isomorphism of $\check{U}_q^{\leq 0}$ onto $C_q(\mathfrak{b}^+)$.*

Proof. Note that $\mp\beta_1 = \mp\omega_1$ so then

$$\phi(K_{\mp\omega_1}) = \phi(K_{\mp\beta_1}) = \overline{X}_{11}^{\pm 1} \in C_q(\mathfrak{b}^+).$$

Proceeding inductively, we see that if $\phi(K_{\mp\omega_{i-1}}) \in C_q(\mathfrak{b}^+)$ then since $\phi(K_{\mp\beta_i}) = \overline{X}_{ii}^{\pm 1}$ we have

$$\phi(K_{\mp\omega_i}) = \phi(K_{\mp\beta_i})\phi(K_{\mp\omega_{i-1}}) = \overline{X}_{ii}^{\pm 1}\phi(K_{\mp\omega_{i-1}}) \in C_q(\mathfrak{b}^+)$$

It then follows that $\phi(K_\mu) \in C_q(\mathfrak{b}^+)$ for all $\mu \in \Lambda$.

Finally, we note that $\beta_i - \alpha_i = \beta_{i+1}$ for $1 \leq i \leq n$. Therefore we have for $1 \leq i \leq n$ that

$$\phi(F_i) = \phi(K_{\beta_i - \alpha_i} K_{-\beta_i + \alpha_i} F_i) = \phi(K_{\beta_{i+1}})\phi(K_{-\beta_i + \alpha_i} F_i) \in C_q(\mathfrak{b}^+).$$

Since the K_μ with $\mu \in \Lambda$ and the F_i generate $\check{U}_q^{\leq 0}$, then im $\phi \subseteq C_q(\mathfrak{b}^+)$. Conversely, since the $\overline{X}_{ii}^{\pm 1}$ and $\overline{X}_{i,i+1}$ generate $C_q(\mathfrak{b}^+)$ it follows that im $\phi = C_q(\mathfrak{b}^+)$. Since the dual pairing is nondegenerate, ϕ is injective. Hence, ϕ is a k-algebra isomorphism. $\square$

Corollary 2.14. *If $\omega : \check{U}_q^{\geq 0} \to \check{U}_q^{\leq 0}$ is the Cartan homomorphism then $\phi \circ \omega$ restricted to U_q^+ induces an isomorphism onto $\rho^+(\kappa(O_q(N^+)'))$.*

Proof. The Cartan homomorphism is defined by $\omega(K_\lambda) = K_\lambda$ and $\omega(E_i) = F_i$. It is clear that ω is an isomorphism, so $\phi\circ\omega$ is an isomorphism. Hence $\phi\circ\omega$ restricted to U_q^+ is an injective k-algebra homomorphism. Moreover, from Theorem 2.13 for $i = 1, 2, \ldots, n$

$$\phi\circ\omega(E_i) = \phi(K_{\beta_{i+1}})\phi(K_{-\beta_i+\alpha_i}F_i) = \overline{X}_{i+1,i+1}^{-1}\left(-q^{-1}\widehat{q}^{-1}\overline{X}_{i,i+1}\right) = -\widehat{q}^{-1}\overline{X}_{i,i+1}\overline{X}_{i+1,i+1}^{-1}.$$

Since $\rho^+(\kappa(O_q(N^+)'))$ is generated by the $\overline{X}_{i,i+1}\overline{X}_{i+1,i+1}^{-1}$ for $i = 1, 2, \ldots n$ then $\phi\circ\omega$ when restricted to U_q^+ is an isomorphism. $\square$

Theorem 2.15. *There is a k-algebra isomorphism $\psi : U_q^+ \to O_q(N^+)'$ such that $\psi(E_i) = -\widehat{q}^{-1}X_{i,i+1}X_{i+1,i+1}^{-1}$ for $1 \leq i \leq n$.*

Proof. This follows from Corollary 2.14 and the fact that, ρ^+ and κ are isomorphisms. $\square$

We note that since the Cartan automorphism ω is an isomorphism of U_q^+ onto U_q^+ it also follows form Theorem 2.15 that $U_q^- \cong O_q(N^+)'$.

In conclusion, we have shown that the algebras $O_q(N^\pm) = O_q(B^\pm)^{\mathrm{co}\,\theta^\pm}$, $O_q(N^\pm)' = O_q(B^\pm)^{\mathrm{co}\,\eta^\pm}$, and $U_q^\pm$ are all isomorphic from Theorem 2.3, Theorem 2.7, and Theorem 2.15 .

Chapter 3

Quantum Unipotent Radicals of Standard Parabolic Subgroups

Let n be a fixed positive integer.

3.1 Standard Parabolic Subgroups

Recall from the Introduction that a standard parabolic subgroup of SL_n is a group that contains a standard Borel subgroup, $B^{\pm}$. This implies that the matrices in a standard parabolic subgroup are in "upper (lower) block diagonal form." Since parabolic subgroups are closed subvarieties of SL_n, we first describe the coordinate ring of a standard parabolic, and then "quantize" this to define quantum standard parabolic subgroups of SL_n.

Definition 3.1. *A* **partition of** n *is a set* $P = \{P_1, \dots, P_k\}$ *where each* P_i *is a nonempty subset of* $\{1, \dots, n\}$ *so that*

(a) $\bigcup_{i=1}^{k} P_i = \{1, \dots, n\}$

(b) $P_i \cap P_j = \emptyset$ *for* $i \neq j$

(c) For each $i = 1, \dots, k-1$ *each element of* P_i *is less than every element of* P_{i+1}.

The idea behind Definition 3.1, is that the partition of n determines a "block diagonal" for matrices in SL_n. Using this block diagonal, a standard parabolic subgroup of SL_n is a subgroup that is composed of matrices that are zero below (above) the block diagonal.

Also note that this definition for a partition of n is different from the usual definition found in combinatorics. In that definition a partition of n is a sequence of positive integers $(n_1, \dots, n_k)$ so that $n_1 + \cdots n_k = n$. However, the two are related – for if P is a partition of n (in the sense of Definition 3.1) then $(|P_1|, \dots, |P_k|)$ is a partition of n in the combinatorial sense.

For each $j \in \{1, 2, \dots, |P|\}$ denote the sets

$$P_{>j} := \bigsqcup_{l>j} P_l \qquad \text{and} \qquad P_{\geq j} := \bigsqcup_{l \geq j} P_l.$$

Similarly, define $P_{<j}$ and $P_{\leq j}$. For convenience if $|P| = k$ we adopt the convention that $P_{k+1} = \emptyset$ and $P_0 = \emptyset$.

For P a partition of n, with $|P| = k$, define the following subsets of $\{1, 2, \dots, n\} \times \{1, 2, \dots, n\}$:

$$\mathcal{C}_P^+ := \bigcup_{i=1}^{k} (P_i \times P_{>i}) \qquad \text{and} \qquad \mathcal{C}_P^- := \bigcup_{i=1}^{k} (P_{>i} \times P_i).$$

It easy to check that $\mathcal{C}_P^+ = \bigcup_{i=1}^{k} (P_{<i} \times P_i)$ and $\mathcal{C}_P^- = \bigcup_{i=1}^{k} (P_i \times P_{<i})$.

Therefore a parabolic group relative to a partition P is

$$P_P^{\pm} = \{(a_{ij}) \in \mathrm{SL}_n \mid (i, j) \in \mathcal{C}_P^{\pm}\}.$$

Let x_{ij} be the coordinate functions in $O(\mathrm{SL}_n)$ and define the following ideals in $O(\mathrm{SL}_n)$:

$$T_P^+ := \left\langle x_{ij} \in O(\mathrm{SL}_n) \mid (i, j) \in \mathcal{C}_P^+ \right\rangle \quad \text{and} \quad T_P^- := \left\langle x_{ij} \in O(\mathrm{SL}_n) \mid (i, j) \in \mathcal{C}_P^- \right\rangle.$$

For the **coordinate rings of the standard parabolic subgroups of** SL_n **relative to** P we have

$$O(P_P^{\pm}) \cong O(\mathrm{SL}_n)/T_P^{\mp}.$$

Moreover, for the **coordinate ring of the standard Levi subgroup of** SL_n **relative to** P we have

$$O(L_P) \cong O(\mathrm{SL}_n)/(T_P^+ + T_P^-).$$

3.2 Quantum Standard Parabolic Subgroups

We may now define the quantized coordinate ring of the standard parabolic in direct analogy with the classical case.

Let P be a partition of n and define the following ideals in $O_q(\mathrm{SL}_n)$:

$$I_P^+ := \left\langle X_{ij} \mid (i,j) \in \mathcal{C}_P^+ \right\rangle \qquad \text{and} \qquad I_P^- := \left\langle X_{ij} \mid (i,j) \in \mathcal{C}_P^- \right\rangle .$$

Definition 3.2. *For P a partition of n, define the* **quantized coordinate rings of the standard parabolic subgroups of SL_n relative to** P *by*

$$O_q(P_P^{\pm}) := O_q(SL_n)/I_P^{\mp}.$$

We will often abuse notation and refer to the coset which contains X_{ij} in $O_q(P_P^{\pm})$ simply by X_{ij}. Similarly, we will refer to the coset which contains the quantum minor $[I \mid J]$ in $O_q(P_P^{\pm})$ simply by $[I \mid J]$.

We note that this definition for quantum parabolic groups is not standard; however, it is equivalent to the definition of quantum parabolic groups given in [25, Section 6.1].

Examples

i. If $P = \{\{1\}, \{2\}, \dots, \{n\}\}$ then $I_P^+ = \langle X_{ij} \mid i < j \rangle$ and $I_P^- = \langle X_{ij} \mid i > j \rangle$. Therefore, $O_q(P_P^\pm) = O_q(B^\pm)$ from Chapter 2.

ii. If $P = \{\{1, \dots, n\}\}$ then $I_P^\pm = 0$. Therefore, $O_q(P_P^\pm) = O_q(Sl_n)$.

Lemma 3.3. *Let P be a partition of n. For $1 \leq i < s \leq n$ and $1 \leq j < t \leq n$ if (i,j) or (s,t) in $\mathcal{C}_P^\pm$ then (i,t) or (s,j) is in $\mathcal{C}_P^\pm$.*

Proof. Let $i \in P_l$. If $(i,j) \in \mathcal{C}_P^+$ then $j \in P_{>l}$. Since $t > j$ then $t \in P_{>l}$ and so $(i,t) \in \mathcal{C}_P^+$. Similarly, if $(s,t) \in \mathcal{C}_P^+$, then $s \in P_r$ for some $r \geq l$ and $t \in P_{>r}$. It follows that $t \in P_{>l}$ and therefore $(i,t) \in \mathcal{C}_P^+$.

A similar proof also works for $\mathcal{C}_P^-$. $\square$

Theorem 3.4. *Let P be a partition of n. The algebras $O_q(P_P^\pm)$ are noetherian domains.*

Proof. Since $O_q(\mathrm{SL}_n)$ is noetherian, it follows that $O_q(P_P^\pm)$ is as well.

Define the following ideals of $O_q(\mathrm{M}_n)$:

$$J_P^+ := \left\langle X_{ij} \mid (i,j) \in \mathcal{C}_P^+ \right\rangle \qquad\qquad J_P^- := \left\langle X_{ij} \mid (i,j) \in \mathcal{C}_P^- \right\rangle .$$

Using Lemma 3.3, if follows from [9, Lemma 3.2] that $J_P^\pm$ are completely prime ideals in $O_q(\mathrm{M}_n)$. Let D be the quantum determinant of $O_q(\mathrm{M}_n)$. It is clear that $D^m \notin J_P^\pm$ for all positive integers m. Therefore, by [7, Theorem 10.20] the extension ideal, $(J_P^{\pm})^e$ in $O_q(\mathrm{GL}_n)$ is a prime ideal. In fact, $(J_P^{\pm})^e$ is in H^+-spec, where H^+ is the natural torus action on $O_q(\mathrm{GL}_n)$ (See (1.24)). From [1, Lemma II.5.16] there is a bijection between the H^+-spec of $O_q(\mathrm{GL}_n)$ and the H-spec of $O_q(\mathrm{SL}_n)$. Using this bijection it follows that $I_P^{\pm}$ is an H-prime of $O_q(\mathrm{SL}_n)$. Finally, from [1, Lemma II.5.17] we have that $I_P^{\pm}$ is completely prime. Therefore, $O_q(P_P^{\pm})$ is a domain. □

Preliminary Relations in $O_q(P_P^{\pm})$

Lemma 3.5. *Let P be a partition of n with $P_i \in P$ and $J \subset \{1, \ldots, n\}$ so that $|P_i| = |J|$.*

(i) If $J \cap P_{<i} \neq \emptyset$ then $[P_i \mid J] \in I_P^-$. Hence, $[P_i \mid J] = 0$ in $O_q(P_P^+)$.

(ii) If $J \cap P_{>i} \neq \emptyset$ then $[J \mid P_i] \in I_P^-$. Hence, $[J \mid P_i] = 0$ in $O_q(P_P^+)$.

(iii) If $J \cap P_{>i} \neq \emptyset$ then $[P_i \mid J] \in I_P^+$. Hence, $[P_i \mid J] = 0$ in $O_q(P_P^-)$.

(iv) If $J \cap P_{<i} \neq \emptyset$ then $[J \mid P_i] \in I_P^+$. Hence, $[J \mid P_i] = 0$ in $O_q(P_P^-)$.

Proof. Let $j \in P_{<i} \cap J$. Note for all $i_\alpha \in P_i$ that $\langle X_{i_\alpha,j} \rangle \subset I_P^-$. Since each monomial term of $[P_i \mid J]$ contains an element of $\langle X_{i_\alpha,j} \rangle$ for some $i_\alpha \in P_i$, it follows that $[P_i \mid J] \in I_P^-$. Hence, $[P_i \mid J] = 0$ in $O_q(P_P^+)$.

The other cases are proven similarly. □

For index sets $I, J \subseteq \{1, 2, \ldots, n\}$ define

$$\ell(I, J) := |\{(i, j) \mid i \in I, j \in J, i > j\}|\,.$$

Recall from [24, Proposition 1.1] the following q-Laplace relation in $O_q(\mathrm{SL}_n)$:

Theorem 3.6 ([24, Proposition 1.1])**.** *Let $I, J \subseteq \{1, 2, \ldots, n\}$ with $|I| = |J|$. For $I_1, I_2 \subset I$ with $|I_1| + |I_2| = |I|$ then*

$$\sum_{\substack{J_1 \sqcup J_2 = J \\ |J_l| = |I_l|}} (-q)^{\ell(J_1, J_2)} [I_1 \mid J_1][I_2 \mid J_2] = \begin{cases} (-q)^{\ell(I_1, I_2)} [I \mid J] & \text{if } I_1 \cap I_2 = \emptyset \\ 0 & \text{if } I_1 \cap I_2 \neq \emptyset. \end{cases}$$

For each $P_i \in P$ denote

$$D_{P_i} := [P_i \mid P_i].$$

Theorem 3.7. *For $P = \{P_1, P_2, \ldots, P_k\}$ a partition of n, we have*

$$D_{P_1} D_{P_2} \cdots D_{P_{k-1}} D_{P_k} = D = 1$$

in $O_q(P_P^{\pm})$.

Proof. Let $i \in \{1, 2, \ldots k + 1\}$. Setting $I_1 = P_{<i-1}$ and $I_2 = P_{i-1}$ it follows from Theorem 3.6 that

$$[P_{<i} \mid P_{<i}] = \sum_{J_1 \sqcup J_2 = P_{<i}} (-q)^{\ell(J_1, J_2)} [P_{<i-1} \mid J_1][P_{i-1} \mid J_2]$$

where $|J_1| = |P_{<i-1}|$ and $|J_2| = |P_{i-1}|$. However, by Lemma 3.5 if $J_2 \cap P_{<i-1} \neq \emptyset$ then

$[P_{i-1} \mid J_2] = 0$ in $O_q(P_P^+)$. Therefore, the only nonzero term on the right hand side of the equality occurs when $J_2 = P_{i-1}$. Hence, $J_1 = P_{<i-1}$. Since $\ell(P_{<i-1}, P_{i-1}) = 0$, we get the following relation in $O_q(P_P^+)$:

$$[P_{<i} \mid P_{<i}] = [P_{<i-1} \mid P_{<i-1}][P_{i-1} \mid P_{i-1}].$$

In particular,

$$D = [P_{<k} \mid P_{<k}][P_k \mid P_k] \qquad \text{and} \qquad [P_{<2} \mid P_{<2}] = [P_1 \mid P_1].$$

Thus,

$$\begin{aligned} D &= [P_{<k} \mid P_{<k}][P_k \mid P_k] = [P_{<k-1} \mid P_{<k-1}][P_{k-1} \mid P_{k-1}][P_k \mid P_k] \\ &= \cdots = [P_1 \mid P_1][P_2 \mid P_2] \cdots [P_{k-1} \mid P_{k-1}][P_k \mid P_k] = D_{P_1} D_{P_2} \cdots D_{P_k - 1} D_{P_k} \end{aligned}$$

in $O_q(P_P^+)$. A similar proof also works in $O_q(P_P^-)$. $\square$

Theorem 3.8. *Let P be a partition of n with $P_i, P_j \in P$ and $i < j$. The following relation holds in $O_q(P_P^{\pm})$:*

$$D_{P_i} D_{P_j} = D_{P_j} D_{P_i}.$$

Proof. For $i_\alpha, i_\beta \in P_i$ and $j_\gamma, j_\delta \in P_j$ relation (1.22) simplifies to $X_{i_\alpha i_\beta} X_{j_\gamma j_\delta} = X_{j_\gamma j_\delta} X_{i_\alpha i_\beta}$ in $O_q(P_P^\pm)$. The relation follows easily from this. □

Corollary 3.9. *For P a given partition of n we have D_{P_i} is invertible in $O_q(P_P^\pm)$ for all $P_i \in P$.*

Proof. This follows directly from Theorems 3.7 and 3.8. □

For P a partition of n and $P_i \in P$ where the elements of P_i are placed in increasing order, we denote the α^{th} term of P_i by i_α. Similarly, denote the β^{th} term of $P_{>i}$ by i_β. For each $i_\alpha \in P_i$ and $i_\beta \in P_{>i}$ denote the quantum minor

$$M_{i_\alpha i_\beta}^{P_i} := [P_i \mid (P_i \setminus i_\alpha) \cup \{i_\beta\}] .$$

Theorem 3.10. *Let P a partition of n and $P_i \in P$. For $i_\alpha \in P_i$ and $i_\beta \in P_{>i}$, the following relation holds in $O_q(P_P^+)$:*

$$X_{i_\alpha i_\beta} = \sum_{i_\gamma \in P_i} - \left(-q^{\gamma - |P_i| - 1}\right) X_{i_\alpha i_\gamma} M_{i_\gamma i_\beta}^{P_i} D_{P_i}^{-1}.$$

Proof. Set $I = J = P_i \cup \{i_\beta\}$ and $I_1 = \{i_\alpha\}$ with $I_2 = P_i$. Since $I_1 \cap I_2 \neq \emptyset$, Theorem 3.6 implies

$$\sum_{j \in J} (-q)^{\ell(\{j\},(J \setminus j))} [i_\alpha \mid j][P_i \mid J \setminus j] = 0. \tag{3.1}$$

If $j \in P_i$ then $j = i_\gamma$ for some γ. In this case it is clear that $\ell(\{i_\gamma\}, J \setminus i_\gamma) = \gamma - 1$. Similarly, if $j = i_\beta$ then $\ell(\{i_\beta\}, \{J \setminus i_\beta\}) = \ell(\{i_\beta\}, P_i) = |P_i|$. Thus, (3.1) becomes

$$\sum_{i_\gamma \in P_i} (-q)^{\gamma-1}[i_\alpha \mid i_\gamma][P_i \mid J \setminus i_\gamma] + (-q)^{|P_i|}[i_\alpha \mid i_\beta][P_i \mid P_i] = 0.$$

Since $[i_\alpha \mid i_\gamma] = X_{i_\alpha i_\gamma}$ and $[P_i \mid J \setminus i_\gamma] = M^{P_i}_{i_\alpha i_\gamma}$ as well as $[i_\alpha \mid i_\beta] = X_{i_\alpha i_\beta}$ and $[P_i \mid P_i] = D_{P_i}$, Corollary 3.9 completes the proof. □

Theorem 3.11. *For P a partition of n with $P_i \in P$ where $i_\alpha \in P_i$ and $i_\beta \in P_{>i}$, the following relation holds in $O_q(P_P^+)$:*

$$D_{P_i} M^{P_i}_{i_\alpha i_\beta} = q M^{P_i}_{i_\alpha i_\beta} D_{P_i}. \tag{3.2}$$

Proof. Let $i_\lambda, i_\mu \in P_i$. From [8, Lemma 5.2] we have $X_{i_\lambda i_\mu} D_{P_i} = D_{P_i} X_{i_\lambda i_\mu}$. Moreover, from [25, Lemma 4.5.1], following the proof in [8, Lemma 5.2] we have

$$D_{P_i} X_{i_\lambda i_\beta} - q X_{i_\lambda i_\beta} D_{P_i} = \widehat{q} \sum_{\substack{i_\gamma \in P_i \\ i_\gamma < i_\lambda}} (-q)^\bullet X_{i_\lambda i_\gamma}[(P_i \setminus i_\gamma) \cup \{i_\beta\} \mid P_i].$$

Since $(P_i \setminus i_\gamma) \cup \{i_\beta\} \cap P_{>i} \neq \emptyset$ we have from Lemma 3.5 that $D_{P_i} X_{i_\lambda i_\beta} = q X_{i_\lambda i_\beta} D_{P_i}$ in $O_q(P_P^+)$. Since $M^{P_i}_{i_\alpha i_\beta}$ is the sum of monomials that contain exactly one $X_{i_\lambda i_\beta}$ for some $i_\lambda \in P_i$ and the other terms are $X_{i_\lambda i_\mu}$, it follows that $M^{P_i}_{i_\alpha i_\beta} D_{P_i} = q D_{P_i} M^{P_i}_{i_\alpha i_\beta}$. □

Theorem 3.12. *Let P a partition of n with $P_i, P_j \in P$ where $i < j$. For $j_\gamma \in P_j$ and $j_\delta \in P_{>j}$ with $i_\alpha \in P_i$ and $i_\beta \in P_{>i}$, the following relations hold in $O_q(P_P^+)$:*

$$D_{P_i} M^{P_j}_{j_\gamma j_\delta} = M^{P_j}_{j_\gamma j_\delta} D_{P_i} \tag{3.3}$$

$$M^{P_i}_{i_\alpha i_\beta} D_{P_j} = q D_{P_j} M^{P_i}_{i_\alpha i_\beta} \text{ if } i_\beta \in P_j \tag{3.4}$$

$$M^{P_i}_{i_\alpha i_\beta} D_{P_j} = D_{P_j} M^{P_i}_{i_\alpha i_\beta} \text{ if } i_\beta \notin P_j. \tag{3.5}$$

Proof.

(i) Let $i_\lambda, i_\mu \in P_i$ and $j_\eta \in P_j$. For all $j_\theta \in P_{\geq j}$ relation (1.22) simplifies to $X_{i_\lambda i_\mu} X_{j_\eta j_\theta} = X_{j_\eta j_\theta} X_{i_\lambda i_\mu}$ in $O_q(P_P^+)$. Since D_{P_i} is the sum of monomials that contain $X_{i_\lambda i_\mu}$ and $M^{P_j}_{j_\gamma j_\delta}$ is the sum of monomials that contain $X_{j_\eta j_\theta}$, it follows that D_{P_i} and $M^{P_j}_{j_\gamma j_\delta}$ commute and (3.3) follows.

(ii) Let $i_\lambda, i_\mu \in P_i$. It is straightforward to check that $X_{i_\lambda i_\mu} D_{P_j} = D_{P_j} X_{i_\lambda i_\mu}$. (See the proof of Theorem 3.8.) Moreover, for $i_\beta \in P_j$, it follows from [25, Lemma 4.5.1] following the proof in [8, Lemma 5.2] that

$$X_{i_\lambda i_\beta} D_{P_j} - q D_{P_j} X_{i_\lambda i_\beta} = \widehat{q} \sum_{\substack{j_\gamma \in P_j \\ j_\gamma < \imath_\lambda}} (-q)^{\bullet} [P_j \mid (P_j \setminus j_\gamma) \cup \{i_\lambda\}] X_{j_\gamma i_\beta}.$$

Since $((P_j \setminus j_\gamma) \cup \{i_\lambda\}) \cap P_{<j} \neq \emptyset$, we have from Lemma 3.5 that

$$X_{i_\lambda i_\beta} D_{P_j} = q D_{P_j} X_{i_\lambda i_\beta}$$

in $O_q(P_P^+)$. Since $M_{i_\alpha i_\beta}^{P_i}$ is the sum of monomials that contain exactly one $X_{i_\lambda i_\beta}$ for some $i_\lambda \in P_i$ and the other terms are $X_{i_\lambda i_\mu}$, it follows that $M_{i_\alpha i_\beta}^{P_i} D_{P_j} = q D_{P_j} M_{i_\alpha i_\beta}^{P_i}$.

(iii) Let $i_\lambda, i_\mu \in P_i$. It is straightforward to check that $X_{i_\lambda i_\mu} D_{P_j} = D_{P_j} X_{i_\lambda i_\mu}$. Moreover, for $i_\beta \notin P_j$ we have from relations (1.22) and (1.21) for all $j_\gamma, j_\delta \in P_j$ that $X_{i_\lambda i_\beta} X_{j_\gamma j_\delta} = X_{j_\gamma j_\delta} X_{i_\lambda i_\beta}$ in $O_q(P_P^+)$. It follows that $X_{i_\lambda i_\beta} D_{P_j} = D_{P_j} X_{i_\lambda i_\beta}$ in $O_q(P_P^+)$. Since $M_{i_\alpha i_\beta}^{P_i}$ is the sum of monomials that contain exactly one $X_{i_\lambda i_\beta}$ and the other terms are $X_{i_\lambda i_\mu}$, it follows that $M_{i_\alpha i_\beta}^{P_i} D_{P_j} = D_{P_j} M_{i_\alpha i_\beta}^{P_i}$. $\square$

More Relations in $O_q(P_P^\pm)$

Let P be a partition of n. For all $P_i \in P$, $i_\delta \in P_i$, and $i_\gamma \in P_{<i}$ define in $O_q(P_P^+)$

$$W_{i_\gamma i_\delta}^{P_i} := [\{i_\gamma\} \cup (P_i \setminus i_\delta) \mid P_i].$$

Similarly, for all $P_i \in P$, $i_\gamma \in P_{<i}$, and $i_\delta \in P_i$ define in $O_q(P_P^-)$

$$m_{i_\gamma i_\delta}^{P_i} := [P_i \mid \{i_\gamma\} \cup (P_i \setminus i_\delta)].$$

Finally, for all $P_i \in P$, $i_\beta \in P_{>i}$, and $i_\alpha \in P_i$ define in $O_q(P_P^-)$

$$w^{P_i}_{i_\alpha i_\beta} := [(P_i \setminus i_\alpha) \cup \{i_\beta\} \mid P_i].$$

We note that there are relations in $O_q(P_P^{\pm})$ analogous to those in Theorems 3.10, 3.11 and 3.12, involving $W^{P_i}_{i_\gamma i_\delta}$, $m^{P_i}_{i_\gamma i_\delta}$, or $w^{P_i}_{i_\alpha i_\beta}$. The proofs are similar to the proofs in these theorems and we simply state the results explicitly below.

Theorem 3.13. *Let P be a partition of n with $P_i, P_j \in P$ where $i < j$.*

If $i_\alpha \in P_{<i}$ and $i_\beta \in P_i$ with $j_\gamma \in P_{<j}$ and $j_\delta \in P_j$, the following relations hold in $O_q(P_P^+)$:

$$W^{P_j}_{j_\gamma j_\delta} D_{P_j} = q^{-1} D_{P_j} W^{P_j}_{j_\gamma j_\delta}$$

$$W^{P_i}_{i_\alpha i_\beta} D_{P_j} = D_{P_j} W^{P_i}_{i_\alpha i_\beta}$$

$$D_{P_i} W^{P_j}_{j_\gamma j_\gamma} = q^{-1} W^{P_j}_{j_\gamma j_\delta} D_{P_i} \text{ if } j_\gamma \in P_i$$

$$D_{P_i} W^{P_j}_{j_\gamma j_\delta} = W^{P_j}_{j_\gamma j_\delta} D_{P_i} \text{ if } j_\gamma \notin P_i$$

If $i_\alpha \in P_{<i}$ and $i_\beta \in P_i$ with $j_\gamma \in P_{<j}$ and $j_\delta \in P_j$, the following relations hold in $O_q(P_P^-)$:

$$m^{P_j}_{j_\gamma j_\delta} D_{P_j} = q^{-1} D_{P_j} m^{P_j}_{j_\gamma j_\delta}$$

$$m^{P_i}_{i_\alpha i_\beta} D_{P_j} = D_{P_j} m^{P_i}_{i_\alpha i_\beta}$$

$$D_{P_i} m^{P_j}_{j_\gamma j_\gamma} = q^{-1} m^{P_j}_{j_\gamma j_\delta} D_{P_i} \text{ if } j_\gamma \in P_i$$

$$D_{P_i} m^{P_j}_{j_\gamma j_\delta} = m^{P_j}_{j_\gamma j_\delta} D_{P_i} \text{ if } j_\gamma \notin P_i$$

If $i_\alpha \in P_i$ and $i_\beta \in P_{>i}$ with $j_\gamma \in P_j$ and $j_\delta \in P_{>j}$, the following relations hold in $O_q(P^-_P)$:

$$D_{P_i} w^{P_i}_{i_\alpha i_\beta} = q w^{P_i}_{i_\alpha i_\beta} D_{P_i}$$

$$D_{P_i} w^{P_j}_{j_\gamma j_\delta} = w^{P_j}_{j_\gamma j_\delta} D_{P_i}$$

$$w^{P_i}_{i_\alpha i_\beta} D_{P_j} = q D_{P_j} w^{P_i}_{i_\alpha i_\beta} \text{ if } i_\beta \in P_j$$

$$w^{P_i}_{i_\alpha i_\beta} D_{P_j} = D_{P_j} w^{P_i}_{i_\alpha i_\beta} \text{ if } i_\beta \notin P_j$$

Theorem 3.14. *Let P be a partition of n and $P_i \in P$.*

For $i_\alpha \in P_{<i}$ and $i_\beta \in P_i$, the following relation holds in $O_q(P^+_P)$:

$$X_{i_\alpha i_\beta} = \sum_{i_\gamma \in P_i} - \left(-q^{\gamma - |P_i| - 1}\right) D^{-1}_{P_i} W^{P_i}_{i_\alpha i_\gamma} X_{i_\gamma i_\beta}.$$

For $i_\alpha \in P_i$ and $i_\beta \in P_{<i}$, the following relation holds in $O_q(P^-_P)$:

$$X_{i_\alpha i_\beta} = \sum_{i_\gamma \in P_i} - \left(-q^{\gamma - |P_i| - 1}\right) X_{i_\alpha i_\gamma} m^{P_i}_{i_\beta i_\gamma} D^{-1}_{P_i}.$$

For $i_\alpha \in P_{>i}$ and $i_\beta \in P_i$, the following relation holds in $O_q(P_P^-)$:

$$X_{i_\alpha i_\beta} = \sum_{i_\gamma \in P_i} - \left(-q^{\gamma - |P_i| - 1}\right) D_{P_i}^{-1} w_{i_\gamma i_\alpha}^{P_i} X_{i_\gamma i_\beta}.$$

3.3 The Algebras of Coinvariants for $O_q(P_P^{\pm})$

Proposition 3.15. *$I_P^{\pm}$ are Hopf ideals.*

Proof. If $|P| = 1$ then $I_P^{\pm} = 0$ and the result is trivial. Therefore, we may assume that $|P| > 1$.

It is clear that $I_P^+ \subset \ker \epsilon$. For each $X_{ij} \in I_P^+$ there exists an r so that $i \in P_r$ and $j \in P_{>r}$. Comultiplication of X_{ij} becomes

$$\Delta(X_{ij}) = \sum_{t=1}^{n} X_{it} \otimes X_{tj} = \sum_{t \in P_{<r}} X_{it} \otimes X_{tj} + \sum_{t \in P_r} X_{it} \otimes X_{tj} + \sum_{t \in P_{>r}} X_{it} \otimes X_{tj}. \tag{3.6}$$

But for $t \in P_{\leq r}$ since $j \in P_{>r}$ then $X_{tj} \in I_P^+$. Moreover for $t \in P_{>r}$ since $i \in P_r$ then $X_{it} \in I_P^+$. Therefore $\Delta(X_{ij}) \in I_P^+ \otimes O_q(\mathrm{SL}_n) + O_q(\mathrm{SL}_n) \otimes I_P^+$.

Let $L = \{1, 2, \ldots, \widehat{j}, \ldots, n\}$ and $M = \{1, 2, \ldots, \widehat{i}, \ldots, n\}$. Ordering the elements of L and M sequentially, let l_k and m_k be the k^{th} terms of L and M, respectively. From the definition of the antipode we have

$$S(X_{ij}) = q^{i-j}[L \mid M] = \sum_{\sigma \in \mathrm{Sym}_{n-1}} (-q)^{\ell(\sigma)} X_{l_1 m_{\sigma(1)}} \cdots X_{l_{n-1} m_{\sigma(n-1)}}.$$

Note, for any $\sigma \in \mathrm{Sym}_{n-1}$ we may interpret σ acting on M by $\sigma(m_k) = m_{\sigma(k)}$. Thus, to show that $S(X_{ij}) \in I_P^+$, it is sufficient to show that for any $\sigma \in \mathrm{Sym}_{n-1}$ there is a $k \in \{1, 2, \ldots, n-1\}$ so that $X_{l_k \sigma(m_k)} \in I_P^+$. To do this, we need to show that for all $\sigma \in \mathrm{Sym}_{n-1}$ there is a $k \in \{1, 2, \cdots, n-1\}$ so that $l_k \in P_{\leq r}$ and $\sigma(m_k) \in P_{>r}$.

Since $X_{ij} \in I_P^+$, from above there is a natural number r so that $i \in P_r$ and $j \in P_{>r}$. It follows that $i < j$ and so $l_k \leq m_k$ for all k. In fact, $i < m_k \leq j$ if and only if $l_k < m_k$. This implies that if $m_k \in P_{\leq r}$ then $l_k \in P_{\leq r}$.

There are two possibilities for $\sigma \in \mathrm{Sym}_{n-1}$ to consider. Either there is an $m_k \in M$ so that $m_k \in P_{\leq r}$ and $\sigma(m_k) \in P_{>r}$ or there is not. If the former, then from the previous discussion $l_k \in P_{\leq r}$ and we are done. If σ is in the latter case then for all $m_k \in P_{\leq r}$ we must have $\sigma(m_k) \in P_{\leq r}$. Consequently, for all $m_k \in P_{>r}$ we have $\sigma(m_k) \in P_{>r}$. Let $m_t = \min P_{>r}$. Then $i < m_t \leq j$. This implies that $l_t < m_t$. Since m_t is $\min P_{>r}$ then $l_t \in P_{\leq r}$. However, since $\sigma(m_t) \in P_{>r}$ again we are done.

It can similarly be shown that I_P^- is a Hopf ideal. $\square$

It follows from Theorem 3.15 that $O_q(P_P^{\pm})$ are Hopf algebras induced from the Hopf algebra structure of $O_q(\mathrm{SL}_n)$. For clarity we may denote the counit, comultiplication, and antipode in $O_q(P_P^{\pm})$ by $\epsilon_{P^{\pm}}$, $\Delta_{P^{\pm}}$, and $S_{P^{\pm}}$ respectively, otherwise we will keep the usual notation of ϵ, Δ, and S.

We note from (3.6) that if $i_\alpha \in P_i$ then

$$\Delta_{P^+}(X_{i_\alpha i_\beta}) = \sum_{i_\gamma \in P_i} X_{i_\alpha i_\gamma} \otimes X_{i_\gamma i_\beta} + \sum_{i_\gamma \in P_{>i}} X_{i_\alpha i_\gamma} \otimes X_{i_\gamma i_\beta}$$

and

$$\Delta_{P^-}(X_{i_\alpha i_\beta}) = \sum_{i_\gamma \in P_{<i}} X_{i_\alpha i_\gamma} \otimes X_{i_\gamma i_\beta} + \sum_{i_\gamma \in P_i} X_{i_\alpha i_\gamma} \otimes X_{i_\gamma i_\beta}.$$

We also note that for $P_i \in P$ and $i_\beta \in P_{>i}$ then from (1.26) we have

$$\Delta\left(M^{P_i}_{i_\alpha i_\beta}\right) = \Delta\left([P_i \mid (P_i \setminus i_\alpha) \cup \{i_\beta\}]\right) \;\; = \sum_{\substack{|J|=|P_i| \\ J \subset \{1,2,\ldots,n\}}} [P_i \mid J] \otimes [J \mid (P_i \setminus i_\alpha) \cup \{i_\beta\}].$$

However, by Lemma 3.5, we have that if $J \cap P_{<i} \neq \emptyset$ then $[P_i \mid J] = 0$ in $O_q(P^+_P)$.

Thus, we have

$$\Delta_{P^+}\left(M^{P_i}_{i_\alpha, i_\beta}\right) = \sum_{\substack{|J|=|P_i| \\ J \subset P_{\geq i}}} [P_i \mid J] \otimes [J \mid (P_i \setminus i_\alpha) \cup \{i_\beta\}].$$

Similarly,

$$\Delta_{P^+}\left(W^{P_i}_{i_\gamma i_\delta}\right) = \sum_{\substack{|J|=|P_i| \\ J \subset P_{\leq i}}} [\{i_\gamma\} \cup (P_i \setminus i_\delta) \mid J] \otimes [J \mid P_i]$$

$$\Delta_{P^-}\left(m^{P_i}_{i_\gamma i_\delta}\right) = \sum_{\substack{|J|=|P_i| \\ J \subset P_{\leq i}}} [P_i \mid J] \otimes [J \mid \{i_\gamma\} \cup (P_i \setminus i_\delta)]$$

$$\Delta_{P^-}\left(w^{P_i}_{i_\alpha i_\beta}\right) = \sum_{\substack{|J|=|P_i|\\ J\subset P_{\geq i}}} [(P_i \setminus i_\alpha) \cup \{i_\beta\} \mid J] \otimes [J \mid P_i].$$

Finally, we note that Lemma 3.5 also implies that $\Delta_{P^\pm}(D_{P_i}) = D_{P_i} \otimes D_{P_i}$ for all $P_i \in P$.

Levi Subalgebra $O_q(L_P)$

Definition 3.16. *The* **quantized (standard) Levi algebra of** $O_q(\mathrm{SL}_n)$ **relative to** P *is the algebra defined by*

$$O_q(L_P) := O_q(\mathrm{SL}_n)/(I^+_P + I^-_P).$$

For $i_\alpha, i_\beta \in P_i$ we denote the coset containing $X_{i_\alpha i_\beta}$ in $O_q(L_P)$ by $Y_{i_\alpha i_\beta}$. It is clear that $O_q(L_P)$ is generated by $\{Y_{i_\alpha i_\beta} \mid P_i \in P,\ i_\alpha,\ i_\beta \in P_i\}$. We will also abuse notation and denote by D_{P_i} the quantum minor $[P_i \mid P_i]$ in $O_q(L_P)$.

Since $I^\pm_P$ are Hopf ideals, $O_q(L_P)$ is also a Hopf algebra induced from $O_q(\mathrm{SL}_n)$. We denote the comultiplication, counit, and antipode by Δ_L, ϵ_L, and S_L respectively, when emphasis is needed. Specifically, for all $P_i \in P$ we note that for $i_\alpha, i_\beta \in P_i$

$$\Delta_L(Y_{i_\alpha i_\beta}) = \sum_{i_\gamma \in P_i} Y_{i_\alpha i_\gamma} \otimes Y_{i_\gamma i_\beta} \tag{3.7}$$

and

$$\Delta_L(D_{P_i}) = D_{P_i} \otimes D_{P_i}.$$

We also define the subalgebras $O_q(S_P^{\pm})$ in $O_q(P_P^{\pm})$ to be the algebras generated by

$\{X_{ij} \mid P_k \in P,\ i,j \in P_k\}$.

$O_q(L_P)$-coactions on $O_q(P_P^{\pm})$

There exist natural projection homomorphisms $p^{\pm} : O_q(P_P^{\pm}) \to O_q(L_P)$ such that

$$p^{\pm}(X_{i_\alpha i_\beta}) = \begin{cases} Y_{i_\alpha i_\beta} & \text{if } i_\alpha, i_\beta \in P_i \ \text{ some } P_i \in P \\ 0 & \text{otherwise.} \end{cases} \tag{3.8}$$

It is clear that $p^{\pm}$ is surjective when restricted to $O_q(S^{\pm})$. Lemma 3.5 implies that

$$p^{\pm}([P_i \mid J]) = \begin{cases} D_{P_i} & \text{if } J = P_i \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad p^{\pm}([J \mid P_i]) = \begin{cases} D_{P_i} & \text{if } J = P_i \\ 0 & \text{otherwise.} \end{cases}$$

Using these maps, as well as the comultiplication maps, Δ and identity maps, $\mathrm{id}_{P^{\pm}}$ on $O_q(P_P^{\pm})$, we define the maps $\eta^{\pm} : O_q(P_P^{\pm}) \to O_q(P_P^{\pm}) \otimes O_q(L_P)$ and

$\theta^{\pm} : O_q(P_P^{\pm}) \to O_q(L_P) \otimes O_q(P_P^{\pm})$ by

$$\eta^{\pm} := (\mathrm{id}_{P^{\pm}} \otimes p^{\pm})\Delta \qquad \text{and} \qquad \theta^{\pm} := (p^{\pm} \otimes \mathrm{id}_{P^{\pm}})\Delta.$$

Since $p^{\pm}$ and Δ are k-algebra homomorphisms, so too are $\eta^{\pm}$ and $\theta^{\pm}$. We note for $i_\alpha \in P_i$ and $j_\beta \in P_j$ we have

$$\eta^{\pm}(X_{i_\alpha j_\beta}) = \sum_{j_\gamma \in P_j} X_{i_\alpha j_\gamma} \otimes X_{j_\gamma j_\beta} \qquad \theta^{\pm}(X_{i_\alpha j_\beta}) = \sum_{i_\gamma \in P_i} X_{i_\alpha i_\gamma} \otimes X_{i_\gamma j_\beta}.$$

We also note that

$$\eta^{+}(W^{P_i}_{i_\gamma i_\alpha}) = W^{P_i}_{i_\gamma i_\beta} \otimes D_{P_i} \qquad \theta^{+}(M^{P_i}_{i_\alpha i_\beta}) = D_{P_i} \otimes M^{P_i}_{i_\alpha i_\beta}$$

$$\eta^{-}(w^{P_i}_{i_\alpha i_\beta}) = w^{P_i}_{i_\alpha i_\beta} \otimes D_{P_i} \qquad \theta^{-}(m^{P_i}_{i_\gamma i_\alpha}) = D_{P_i} \otimes m^{P_i}_{i_\gamma i_\alpha}.$$

Finally,

$$\eta^{\pm}(D_{P_i}) = D_{P_i} \otimes D_{P_i} \qquad \text{and} \qquad \theta^{\pm}(D_{P_i}) = D_{P_i} \otimes D_{P_i}.$$

It is easily checked that $\eta^{\pm}$ and $\theta^{\pm}$ are comodule structure maps, i.e., $O_q(P_P^{\pm})$ is a right $O_q(L_P)$-comodule algebra via $\eta^{\pm}$ and similarly, $O_q(P_P^{\pm})$ is a left $O_q(L_P)$-comodule algebra via $\theta^{\pm}$.

Lemma 3.17. *Let P be a partition of n with $|P| = k$. Let $M = \bigotimes_{i=1}^{k} O_q(\mathrm{M}_{|P_i|})$. There exists a surjective algebra homomorphism $\phi : M \to O_q(L_P)$ so that $1 \otimes \cdots 1 \otimes X_{i_l j_l} \otimes 1 \cdots \otimes 1 \mapsto Y_{(|P_{<l}|+i_l)(|P_{<l}|+j_l)}$ for all $X_{i_l j_l} \in O_q(\mathrm{M}_{P_l})$.*

Proof. For each $P_i \in P$, define a map $\phi_i : O_q(\mathrm{M}_{P_i}) \to O_q(L_P)$ by $X_{lm} \mapsto Y_{(|P_{<i}|+l)(|P_{<i}|+m)}$. It is straightforward to check that these maps are k-algebra homomorphisms.

Define the map $\overline{\phi} : \prod_{i=1}^{k} O_q(\mathrm{M}_{|P_i|}) \to O_q(L_P)$ defined by $\overline{\phi}(a_1, \ldots, a_k) = \phi_1(a_1) \cdots \phi_k(a_k)$. Again, it is straightforward to check that $\overline{\phi}$ is a surjective multilinear map. Moreover, if $i_s, j_s \in P_s$ and $l_t, m_t \in P_t$ with $s \neq t$ then $Y_{i_s j_s}$ and $Y_{l_t m_t}$ commute. It follows that

$$\begin{aligned} &\overline{\phi}(1, \cdots, X_{i_s j_s}, \cdots, 1)\overline{\phi}(1, \cdots, X_{l_t m_t}, \cdots, 1) \\ =& \overline{\phi}(1, \cdots, X_{l_t m_t}, \cdots, 1)\overline{\phi}(1, \cdots, X_{i_s j_s}, \cdots, 1) \end{aligned}$$

for all $X_{i_s j_s} \in O_q(\mathrm{M}_{|P_s|})$ and all $X_{l_t m_t} \in O_q(\mathrm{M}_{|P_t|})$ with $s \neq t$. By the universal property for the tensor products of algebras, there exists an algebra homomorphism ϕ described above. $\square$

Lemma 3.18. *The map $p^{\pm}$ is an isomorphism of $O_q(S_P^{\pm})$ onto $O_q(L_P)$.*

Proof. Let $\phi : M \to O_q(L_P)$ be the homomorphism from Lemma 3.17 and let $p_S^{\pm} = p^{\pm} \mid_{O_q(S_P^{\pm})}$.

It follows from the proof of Theorem 3.40 that GK.dim M = GK.dim $O_q(L_P)+1$. Notice that there exists a surjective algebra homomorphism $\phi' : M \to O_q(S_P^+)$ so that $p_S^+\phi' = \phi$. Since $D_1 \otimes \cdots D_n - 1 \in \ker \phi'$ it follows from [20, Proposition 3.15] that GK.dim $O_q(S^+) \leq$ GK.dim $M - 1$ = GK.dim $O_q(L_P)$. However, since $O_q(L_P) = p_S^+(O_q(S_P^+))$ it is not possible for GK.dim $O_q(S^+) < O_q(L_P)$. Therefore, $\ker p_S^+ = 0$.

Similarly results hold for p_S^-.

$\square$

Therefore, by Lemma 3.18 we may define homomorphisms $r^{\pm} : O_q(L_P) \to O_q(P_P^{\pm})$ by $r^{\pm} := (p^{\pm})^{-1}$ where $p^{\pm}$ is restricted to $O_q(S^{\pm})$. Notice that

$$r^{\pm}(Y_{i_\alpha i_\beta}) = X_{i_\alpha i_\beta}. \tag{3.9}$$

Note, from Corollary 3.9 we have that the $D_{P_i} \in O_q(L_P)$ are invertible which implies that $\Delta_L(D_{P_i}^{-1}) = D_{P_i}^{-1} \otimes D_{P_i}^{-1}$.

$O_q(P_P^{\pm})$ as the Smash Product of Coinvariants

Theorem 3.19. *Let $A^{\pm} = O_q(P_P^{\pm})^{\mathrm{co}\,\eta^{\pm}}$ then $A^{\pm} \# O_q(L_P) \cong O_q(P_P^{\pm})$. Similarly letting $C^{\pm} = O_q(P_P^{\pm})^{\mathrm{co}\,\theta^{\pm}}$ then $O_q(L_P) \# C^{\pm} \cong O_q(P_P^{\pm})$.*

Proof. Let $r^{\pm}$ be the homomorphism from (3.9). Define $\overline{r^{\pm}} : O_q(L_P) \to O_q(P_P^{\pm})$ by $\overline{r^{\pm}} = r^{\pm} S_L$. Note we have

$$(r^{\pm} * \overline{r^{\pm}})(Y) = \sum_{(Y)} r^{\pm}(Y_1)\overline{r^{\pm}}(Y_2) = r^{\pm}\left(\sum_{(Y)} Y_1 S_L(Y_2)\right) = r^{\pm}(\epsilon_L(Y)\cdot 1) = \epsilon_L(Y)\cdot 1$$

for all $Y \in O_q(L_P)$. Similarly, $(\overline{r^{\pm}} * r^{\pm})(Y) = \epsilon_L(Y)\cdot 1$. Since $\overline{r^{\pm}}(1) = 1$ then $\overline{r^{\pm}}$ is the convolution inverse of $r^{\pm}$.

To check that $r^{\pm}$ is a right $O_q(L_P)$-comodule map we need to show that $\eta^{\pm} r^{\pm} = (r^{\pm} \otimes id)\Delta_L$, but since $r^{\pm}$ and $\eta^{\pm}$ are k-algebra homomorphisms, it is sufficient to show the equality holds on the $Y_{ij} \in O_q(L_P)$. Indeed,

$$\eta^{\pm} r^{\pm}(Y_{i_\alpha i_\beta}) = \eta^{\pm}(X_{i_\alpha i_\beta}) = \sum_{i_\gamma \in P_i} X_{i_\alpha i_\gamma} \otimes Y_{i_\gamma i_\beta}$$

and

$$(r^{\pm} \otimes id)\Delta_L(Y_{i_\alpha i_\beta}) = (r^{\pm} \otimes id)\left(\sum_{i_\gamma \in P_i} Y_{i_\alpha i_\gamma} \otimes Y_{i_\gamma i_\beta}\right) = \sum_{i_\gamma \in P_i} X_{i_\alpha i_\gamma} \otimes Y_{i_\gamma i_\beta}$$

for all $P_i \in P$, $i_\alpha, i_\beta \in P_i$. Hence, $r^{\pm}$ are right $O_q(L_P)$-comodule homomorphisms. Therefore we have $O_q(P_P^{\pm})$ is an H-cleft extension. Moreover, using Theorem 1.18 we have a k-algebra isomorphism $\Phi^{\pm} : A^{\pm} \# O_q(L_P) \to O_q(P_P^{\pm})$ where $\Phi^{\pm}(X \# Y) = X r^{\pm}(Y)$.

Similarly, by making the appropriate changes to the above proof, there is a k-algebra isomorphism $\Psi^{\pm} : O_q(L_P)\#C^{\pm} \to O_q(P_P^{\pm})$ defined by $\Psi^{\pm}(Y\#X) = r^{\pm}(Y)X$. $\square$

Quantized Standard Unipotent Subgroups of $P_P^{\pm}$

It is natural to ask what are the coinvariants for $O_q(P_P^{\pm})$ using the structure maps $\eta^{\pm}$ or $\theta^{\pm}$. Note, for the elements $D_{P_i}^{-1}M_{i_\alpha i_\beta}^{P_i} \in O_q(P_P^{+})$ we have

$$\theta^{+}(D_{P_i}^{-1}M_{i_\alpha i_\beta}^{P_i}) = \theta^{+}(D_{P_i}^{-1})\theta^{+}(M_{i_\alpha i_\beta}^{P_i}) = (D_{P_i}^{-1} \otimes D_{P_i}^{-1})(D_{P_i} \otimes M_{i_\alpha i_\beta}^{P_i}) = 1 \otimes D_{P_i}^{-1}M_{i_\alpha i_\beta}^{P_i}.$$

That is, $D_{P_i}^{-1}M_{i_\alpha i_\beta}^{P_i} \in O_q(P_P^{+})^{\mathrm{co}\,\theta^{+}}$. Similarly, $w_{i_\alpha i_\beta}^{P_i}D_{P_i}^{-1} \in O_q(P_P^{-})^{\mathrm{co}\,\eta^{-}}$, $W_{i_\gamma i_\alpha}^{P_i}D_{P_i}^{-1} \in O_q(P_P^{+})^{\mathrm{co}\,\eta^{+}}$, and $D_{P_i}^{-1}m_{i_\gamma i_\alpha}^{P_i} \in O_q(P_P^{-})^{\mathrm{co}\,\theta^{-}}$.

Define the following subalgebras of $O_q(P_P^{+})$

$$O_q(N_{>P}^{+}) := k\left\langle D_{P_i}^{-1}M_{i_\alpha i_\beta}^{P_i} \mid P_i \in P,\, i_\alpha \in P_i,\, i_\beta \in P_{>i}\right\rangle \tag{3.10}$$

$$O_q(N_{<P}^{+}) := k\left\langle W_{i_\gamma i_\delta}^{P_i}D_{P_i}^{-1} \mid P_i \in P,\, i_\gamma \in P_{<i},\, i_\delta \in P_i\right\rangle. \tag{3.11}$$

Similarly, define the subalgebras of $O_q(P_P^{-})$

$$O_q(N_{<P}^{-}) := k\left\langle D_{P_i}^{-1}m_{i_\gamma i_\delta}^{P_i} \mid P_i \in P,\, i_\gamma \in P_{<i},\, i_\delta \in P_i\right\rangle \tag{3.12}$$

$$O_q(N_{>P}^{-}) := k\left\langle w_{i_\alpha i_\beta}^{P_i}D_{P_i}^{-1} \mid P_i \in P,\, i_\alpha \in P_{>i},\, i_\beta \in P_i\right\rangle. \tag{3.13}$$

Since $\eta^{\pm}$ and $\theta^{\pm}$ are algebra homomorphisms, from the above discussion $O_q(N^+_{>P})$ and $O_q(N^+_{<P})$ are subalgebras of $O_q(P^+_P)^{\mathrm{co}\,\theta^+}$ and $O_q(P^+_P)^{\mathrm{co}\,\eta^+}$. Similarly, $O_q(N^-_{<P})$ and $O_q(N^-_{>P})$ are subalgebras of $O_q(P^-_P)^{\mathrm{co}\,\theta^-}$ and $O_q(P^-_P)^{\mathrm{co}\,\eta^-}$. In fact, these algebras are exactly the coinvariants for $\eta^{\pm}$ and $\theta^{\pm}$ which we now show.

Theorem 3.20. *$O_q(N^+_{>P}) = O_q(P^+_P)^{\mathrm{co}\,\theta^+}$ and $O_q(N^-_{<P}) = O_q(P^-_P)^{\mathrm{co}\,\theta^-}$. Similarly, $O_q(N^+_{<P}) = O_q(P^+_P)^{\mathrm{co}\,\eta^+}$ and $O_q(N^-_{>P}) = O_q(P^-_P)^{\mathrm{co}\,\eta^-}$.*

Proof. From the discussion above it follows that $O_q(N^+_{>P}) \subseteq O_q(P^+_P)^{\mathrm{co}\,\theta^+}$. The map Ψ^+ from the proof of Theorem 3.19 is an isomorphism of $O_q(L_P)\#\, C^+$ onto $O_q(P^+_P)$, thus it is sufficient to show that Ψ^+ maps $O_q(L_P)\#O_q(N^+_{>P})$ onto $O_q(P^+_P)$.

Notice that

$$\Psi^+(O_q(L_P)\#O_q(N^+_{>P})) = r^+(O_q(L_P))O_q(N^+_{>P}) \subseteq O_q(P^+_P).$$

It follows that $r^+(O_q(L_P))O_q(N^+_{>P})$ is a subalgebra of $O_q(P^+_P)$. Therefore, we need only show that this subalgebra contains all the X_{ij} that generate $O_q(P^+_P)$ to prove the Theorem.

For $i_\alpha, i_\beta \in P_i$ we note that

$$\Psi^+(Y_{i_\alpha i_\beta}\#1) = r^+(Y_{i_\alpha i_\beta}) = X_{i_\alpha i_\beta}.$$

Moreover, for $i_\alpha \in P_i$ and $i_\beta \in P_{>i}$ the relation from Theorem 3.10 and Theorem 3.11 gives us

$$X_{i_\alpha i_\beta} = \sum_{i_\gamma \in P_i} -(-q^{\gamma - |P_i|}) X_{i_\alpha i_\gamma} D_{P_i}^{-1} M_{i_\gamma i_\beta}^{P_i}.$$

Hence,

$$\begin{aligned}\Psi^+ \left(\sum_{i_\gamma \in P_i} -(-q^{\gamma-|P_i|}) Y_{i_\alpha i_\gamma} \# D_{P_i}^{-1} M_{i_\gamma i_\beta}^{P_i} \right) &= \sum_{i_\gamma \in P_i} -(-q^{\gamma-|P_i|}) r^+(Y_{i_\alpha i_\gamma}) D_{P_i}^{-1} M_{i_\gamma i_\beta}^{P_i} \\ &= \sum_{i_\gamma \in P_i} -(-q^{\gamma-|P_i|}) X_{i_\alpha i_\gamma} D_{P_i}^{-1} M_{i_\gamma i_\beta}^{P_i} \\ &= X_{i_\alpha i_\beta}.\end{aligned}$$

Therefore, the generators of $O_q(P_P^+)$ are contained in the image of Ψ^+. Hence, $O_q(N_{>P}^+) = O_q(P_P^+)^{\mathrm{co}\,\theta^+}$.

By a similar argument using Ψ^- from from the proof of Theorem 3.19, and Theorems 3.13 and 3.14 we have $O_q(P_P^-)^{\mathrm{co}\,\theta^-} = O_q(N_{<P}^-)$. Finally, using $\Phi^\pm$ from the proof of Theorem 3.19, and Theorems 3.13 and 3.14, one can also show that $O_q(N_{<P}^+) = O_q(P_P^+)^{\mathrm{co}\,\eta^+}$ and $O_q(N_{>P}^-) = O_q(P_P^-)^{\mathrm{co}\,\eta^-}$. $\square$

Corollary 3.21. *For P a partition of n we have the following k-algebra isomorphisms*

$$O_q(P_P^+) \cong O_q(L_P) \# O_q(N_{>P}^+) \cong O_q(N_{<P}^+) \# O_q(L_P)$$

$$O_q(P_P^-) \cong O_q(L_P)\#O_q(N_{<P}^-) \cong O_q(N_{>P}^-)\#O_q(L_P).$$

Proof. This follows directly from Theorem 3.19 and Theorem 3.20. $\square$

Definition 3.22. *The algebras from* (3.10) *and* (3.11) *are called* **quantized standard positive unipotent subgroups of** P_P^+*. Similarly, the algebras from* (3.12) *and* (3.13) *are called* **quantized standard negative unipotent subgroups of** P_P^-*. Collectively, they are called the* **quantized standard unipotent subgroups***.*

3.4 Isomorphisms of Unipotent Radical Subalgebras

Let w_0 be the longest element in Sym_n. For P a partition of n where $|P| = k$, denote $w_0(P)$ to be the partition of n such that $w_0(P)_{k-i+1} = w_0(P_i)$. Note, for any $i, j \in \{1, 2, \ldots, n\}$ with $i < j$ then $w_0(i) > w_0(j)$.

From [25, Section 3.7] there exist a transpose automorphism $\tau : O_q(M_n) \to O_q(M_n)$ defined by $\tau(X_{ij}) = X_{ji}$ and an anti-automorphism $\xi : O_q(M_n) \to O_q(M_n)$ defined by $\xi(X_{ij}) = X_{w_0(j),w_0(i)}$. It is clear that $\tau^2 = \mathrm{id}$ and since $w_0^2 = 1$ then $\xi^2 = \mathrm{id}$. We note from [25, Lemma 4.3.1] that for index sets

$I, J \subset \{1, 2, \ldots, n\}$ where $|I| = |J|$ then

$$\tau([I \mid J]) = [J \mid I] \qquad \text{and} \qquad \xi([I \mid J]) = [w_0(J) \mid w_0(I)]. \tag{3.14}$$

It is clear that $\tau(D) = D$. Moreover, since $w_0(\{1, 2, \ldots, n\}) = \{1, 2, \ldots, n\}$ it follows that $\xi(D) = D$. Hence, τ induces an automorphism and ξ induces an anti-automorphism of $O_q(\mathrm{SL}_n)$, respectively. We will abuse notation and continue to denote these maps by τ and ξ.

Lemma 3.23. *Let P be a partition of n.*

(i) The map τ induces algebra isomorphisms $\tau_P^{\mp} : O_q(P_P^{\pm}) \to O_q(P_P^{\mp})$.

(ii) The map τ induces a Hopf algebra isomorphism $\tau_L : O_q(L_P) \to O_q(L_P)^{\mathrm{cop}}$.

(iii) The map ξ induces algebra isomorphisms $\xi_P^{\pm} : O_q(P_P^{\pm}) \to O_q\left(P_{w_0(P)}^{\pm}\right)^{\mathrm{op}}$.

(iv) The map ξ induces a Hopf algebra isomorphism $\xi_L : O_q(L_P) \to O_q(L_P)^{\mathrm{op}}$.

Proof. If $X_{ij} \in I_P^{\pm}$ then $\tau(X_{ij}) = X_{ji} \in I_P^{\mp}$ and so $\tau(I_P^{\pm}) \subseteq I_P^{\mp}$. Conversely, since $\tau^2 = \mathrm{id}$, then $I_P^{\pm} \subseteq \tau(I_P^{\mp})$. Thus, $I_P^{\mp} \subseteq \tau(I_P^{\pm}) \subseteq I_P^{\mp}$ and so $\tau(I_P^{\pm}) = I_P^{\mp}$. Hence, τ induces algebra homomorphisms $\tau_P^{\mp} : O_q(P_P^{\pm}) \to O_q(P_P^{\mp})$. Moreover, for all $X_{ij} \in O_q(P_P^{+})$ and all $X_{kl} \in O_q(P_P^{-})$ we get

$$\tau_P^{+}\tau_P^{-}(X_{ij}) = \tau_P^{+}(X_{ji}) = X_{ij} \qquad \text{and} \qquad \tau_P^{-}\tau_P^{+}(X_{kl}) = \tau_P^{+}(X_{lk}) = X_{kl}.$$

That is $\tau_P^{+}\tau_P^{-} = \mathrm{id}_{P^+}$ and $\tau_P^{-}\tau_P^{+} = \mathrm{id}_{P^-}$. Therefore, $\tau_P^{\pm}$ are isomorphisms.

From [25, Proposition 3.7.1 (2)] we have that τ is a Hopf algebra automorphism from $O_q(\mathrm{SL}_n) \to O_q(\mathrm{SL}_n)^{\mathrm{cop}}$. It follows from the above discussion that $\tau(I_P^+ + I_P^-) = I_P^+ + I_P^-$. Hence, from Proposition 3.15 the induced homomorphism $\tau_L : O_q(L_P) \to O_q(L_P)^{\mathrm{cop}}$ is also a Hopf algebra homomorphism. Since $\tau_L^2 = \mathrm{id}_L$ we have τ_L is a Hopf algebra isomorphism.

Similar proofs also hold for $\xi_P^{\pm}$ and ξ_L noting that ξ is an anti-automorphism of $O_q(\mathrm{SL}_n)$ and $\xi(I_P^{\pm}) = I_P^{\pm}$. $\square$

Proposition 3.24. *The maps $\tau_P^{\mp}$ from Lemma 3.23 restricted to $O_q(N_{>P}^{\pm})$ give isomorphisms onto $O_q(N_{>P}^{\mp})$. Also, the maps $\tau_P^{\mp}$ restricted to $O_q(N_{<P}^{\pm})$ give isomorphisms onto $O_q(N_{<P}^{\mp})$. Finally, $\xi_P^{\pm}$ restricted to $O_q(N_{>P}^{\pm})$ gives anti-isomorphisms onto $O_q(N_{<w_0(P)}^{\pm})$ and $\xi_P^{\pm}$ restricted to $O_q(N_{<P}^{\pm})$ gives anti-isomorphisms onto $O_q(N_{>w_0(P)}^{\pm})$.*

Proof. Let $P_i \in P$. We first note that for all $D_{P_i} \in O_q(P_P^+)$ we have $\tau_P^-(D_{P_i}) = D_{P_i}$ .We also note that

$$\tau_P^-\left(M_{i_\alpha i_\beta}^{P_i}\right) = \tau_P^-\left([P_i \mid (P_i \setminus i_\alpha) \cup \{i_\beta\}]\right) = [(P_i \setminus i_\alpha) \cup \{i_\beta\} \mid P_i] = w_{i_\alpha i_\beta}^{P_i}$$

for all $M_{i_\alpha i_\beta}^{P_i} \in O_q(P_P^+)$. Hence, for $D_{P_i}, M_{i_\alpha i_\beta}^{P_i} \in O_q(N_{>P}^+)$ it follows from Theorem 3.11 that

$$\tau_P^-\left(D_{P_i}^{-1} M_{i_\alpha i_\beta}^{P_i}\right) = \tau_P^-\left(q^{-1} M_{i_\alpha i_\beta}^{P_i} D_{P_i}^{-1}\right) = \tau_P^-\left(q^{-1} M_{i_\alpha i_\beta}^{P_i}\right) \tau_P^-\left(D_{P_i}^{-1}\right) = q^{-1} w_{i_\alpha i_\beta}^{P_i} D_{P_i}^{-1}.$$

Hence $\tau_P^-(O_q(N_{>P}^+)) \subseteq O_q(N_{>P}^-)$.

Conversely, for $w_{i_\alpha i_\beta}^{P_i} D_{P_i}^{-1} \in O_q(N_{>P}^-)$, we have

$$\tau_P^-(qD_{P_i}^{-1} M_{i_\alpha i_\beta}^{P_i} D_{P_i}^{-1}) = w_{i_\alpha i_\beta}^{P_i} D_{P_i}^{-1}.$$

Therefore, the generators of $O_q(N_{>P}^-)$ are contained in the image of τ_P^- and the inclusion becomes an equality. Since τ_P^- is an isomorphism on $O_q(P_P^+)$, it follows from the above that the restriction to $O_q(N_{>P}^+)$ is an isomorphism onto $O_q(N_{>P}^-)$.

The other statements are proven similarly. □

Theorem 3.25. *For P a partition of n we have the following k-algebra isomorphisms*

$$O_q(P_P^+) \cong O_q(L_P)^{\mathrm{cop}} \# O_q(N_{>P}^-) \cong O_q(N_{<P}^-) \# O_q(L_P)^{\mathrm{cop}}$$
$$O_q(P_P^-) \cong O_q(L_P)^{\mathrm{cop}} \# O_q(N_{<P}^+) \cong O_q(N_{>P}^+) \# O_q(L_P)^{\mathrm{cop}}.$$

Proof. Let $P_i \in P$. We note that for all $i_\alpha, i_\beta \in P_i$, if $r^\pm$ is the map defined in equation (3.9) we have

$$r^\mp(\tau_L(Y_{i_\alpha i_\beta})) = r^\mp(Y_{i_\beta i_\alpha}) = X_{i_\beta i_\alpha} = \tau_P^\mp(X_{i_\alpha i_\beta}) = \tau_P^\mp(r^\pm(Y_{i_\alpha i_\beta})).$$

Hence, $r^-\tau_L = \tau_P^- r^+$. Since $\tau_L : O_q(L_P) \to O_q(L_P)^{\mathrm{cop}}$ is a Hopf algebra isomorphism and since the antipode for $O_q(L_P)^{\mathrm{cop}}$ is S_L^{-1}, from Theorem 3.23 we have

$$\overline{r^-}\tau_L = r^- S_L \tau_L = r^- \tau_L S_L^{-1} = \tau_P^- r^+ S_L^{-1} = \tau_P^- \overline{r^+}.$$

Therefore, the maps $\tau_P^{\mp}$ and τ_L are H-cleft intertwining maps.

From Proposition 3.24 the restriction of τ_P^- is an isomorphism of, $O_q(N_{<P}^+)$ onto $O_q(N_{<P}^-)$. Thus, the map

$$(\tau_P^- \# \tau_L) : O_q(N_{<P}^+) \# O_q(L_P) \to O_q(N_{<P}^-) \# O_q(L_P)^{\mathrm{cop}}$$

is an algebra isomorphism from Proposition 1.17. It follows from Corollary 3.21 that $O_q(P_P^+) \cong O_q(N_{<P}^-) \# O_q(L_P)^{\mathrm{cop}}$.

The other statements follow similarly. $\square$

Theorem 3.26. *All of the algebras from Theorem 3.25 and Corollary 3.21 are isomorphic to $O_q(P_P^{\pm})$ and hence, are isomorphic to each other. Moreover, all of these algebras are anti-isomorphic to $O_q\left(P_{w_0(P)}^{\pm}\right)$.*

Proof. This follows from Lemma 3.23. $\square$

A succinct way to state Theorem 3.26 (ignoring some details) is that the quantized standard parabolic group is the smash product of the quantized standard Levi algebra and a quantized standard unipotent subgroup.

3.5 Structure of $O_q(N^+_{>P})$

Next, we will show that $O_q(N^+_{>P})$ is a CGL extension, and hence can give an explicit presentation for this algebra. To do this we first need relations involving the quantum minors $D_{P_i}, M^{P_i}_{i_\alpha i_\beta} \in O_q(P^+_P)$.

Relations with Quantum Minors in $O_q(P^+_P)$

Lemma 3.27. *Let P be a partition of n with $P_i \in P$. For $i_\alpha, i_\gamma \in P_i$ with $i_\alpha < i_\gamma$ and $i_\beta, i_\delta \in P_{>i}$ with $i_\beta < i_\delta$, the following relation holds in $O_q(P^+_P)$:*

$$D_{P_i}[P_i \mid (P_i \setminus \{i_\alpha, i_\gamma\}) \cup \{i_\beta, i_\delta\}] = qM^{P_i}_{i_\gamma i_\beta} M^{P_i}_{i_\alpha i_\delta} - q^2 M^{P_i}_{i_\alpha i_\beta} M^{P_i}_{i_\gamma i_\delta}.$$

Proof. We first note that from [18, Theorem 2.1] for $J_1, J_2, K \subset P_i \cup \{i_\beta, i_\delta\}$ with $|J_1|, |J_2| \leq r$ and $|K| = 2r - |J'| - |J''| > r$ then we have the following relation in $O_q(P^+_P)$:

$$\sum_{K' \sqcup K'' = K} (-q)^{\ell(J_1,K')+\ell(K',K'')+\ell(K'',J_2)} [P_i \mid K' \sqcup J_1][P_i \mid J_2 \sqcup K''] = 0. \tag{3.15}$$

Case (i) $|P_i| = 2$ that is, $P_i = \{i_\alpha, i_\gamma\}$.

Set

$$J_1 = \emptyset \qquad J_2 = \{i_\delta\} \qquad K = \{i_\alpha, i_\gamma, i_\beta\}$$

and $r = 2$. Applying (3.15) the only possibilities for K' and K'' are

$$K'_1 = \{i_\alpha, i_\gamma\} \qquad K'_2 = \{i_\alpha, i_\beta\} \qquad K'_3 = \{i_\gamma, i_\beta\}$$

$$K''_1 = \{i_\beta\} \qquad K''_2 = \{i_\gamma\} \qquad K''_3 = \{i_\alpha\}$$

and for these possibilities,

$$\ell(J_1, K'_1) + \ell(K'_1, K''_1) + \ell(K''_1, J_2) = 0$$

$$\ell(J_1, K'_2) + \ell(K'_2, K''_2) + \ell(K''_2, J_2) = 1$$

$$\ell(J_1, K'_3) + \ell(K'_3, K''_3) + \ell(K''_3, J_2) = 2.$$

Hence, (3.15) becomes

$$[P_i \mid P_i][P_i \mid \{i_\beta, i_\delta\}] - q[P_i \mid \{i_\alpha, i_\beta\}][P_i \mid \{i_\gamma, i_\delta\}] + q^2[P_i \mid \{i_\gamma, i_\beta\}][P_i \mid \{i_\alpha, i_\delta\}] = 0.$$

It follows that $D_{P_i}[P_i \mid i_\beta, i_\delta] = qM^{P_i}_{i_\gamma i_\beta} M^{P_i}_{i_\alpha i_\delta} - q^2 M^{P_i}_{i_\alpha i_\beta} M^{P_i}_{i_\gamma i_\delta}$.

Case (ii) $|P_i| \geq 3$.

Set $J = (P_i \setminus \{i_\alpha, i_\gamma\})$ with

$$J_1 = J \setminus \min J \qquad J_2 = \{\min J, i_\delta\} \qquad K = P_i \cup \{i_\beta\}$$

and $r = |P_i|$ and apply (3.15). Since $K' \sqcup K'' = K$ and $K' \sqcup J_1$ and $K'' \sqcup J_2$, the only possibilities for K' and K'' are

$$
\begin{array}{lll}
K_1' = \{\min J, i_\alpha, i_\gamma\} & K_2' = \{\min J, i_\alpha, i_\beta\} & K_3' = \{\min J, i_\gamma, i_\beta\} \\
K_1'' = J_1 \cup \{i_\beta\} & K_2'' = J_1 \cup \{i_\gamma\} & K_3'' = J_1 \cup \{i_\alpha\}.
\end{array}
$$

To compute $\ell(J_1, K_1')$ we note that every element in J_1 is greater than $\min J$. Therefore, there are are $r-3$ elements greater than $\min J$. Similarly, there are $r-\alpha-1$ elements in J_1 greater than i_α and $r-\gamma$ elements in J_1 greater than i_γ. Hence

$$\ell(J_1, K_1') = (r-3) + (r-\alpha-1) + (r-\gamma) = 3r - \alpha - \gamma - 4.$$

In the same way we also get

$$\ell(K_1', K_1'') = \alpha + \gamma - 5 \qquad \text{and} \qquad \ell(K_1'', J_2) = r - 2.$$

Therefore,

$$\ell(J_1, K_1') + \ell(K_1', K_1'') + \ell(K_1'', J_2) = 4r - 11.$$

In a similar way, we have

$$\ell(J_1, K_2') + \ell(K_2', K_2'') + \ell(K_2'', J_2) = 4r - 10$$

$$\ell(J_1, K_3') + \ell(K_3', K_3'') + \ell(K_3'', J_2) = 4r - 9.$$

Since

$$\begin{aligned} &J_1 \cup K_1' = P_i && J_2 \cup K_1'' = (P_i \setminus \{i_\alpha, i_\gamma\}) \cup \{i_\beta, i_\delta\} \\ &J_1 \cup K_2' = (P_i \setminus i_\gamma) \cup \{i_\beta\} && J_2 \cup K_2'' = (P_i \setminus i_\alpha) \cup \{i_\delta\} \\ &J_1 \cup K_3' = (P_i \setminus i_\alpha) \cup \{i_\beta\} && J_2 \cup K_3'' = (P_i \setminus i_\gamma) \cup \{i_\delta\} \end{aligned}$$

equation (3.15) simplifies to

$$(-q)^{4r-11} D_{P_i}[P_i \mid (P_i \setminus \{i_\alpha, i_\gamma\}) \cup \{i_\beta, i_\delta\}] + (-q)^{4r-10} M^{P_i}_{i_\gamma i_\beta} M^{P_i}_{i_\alpha i_\delta}$$
$$+(-q)^{4r-9} M^{P_i}_{i_\alpha i_\beta} M^{P_i}_{i_\gamma i_\delta} = 0.$$

It follows that $D_{P_i}[P_i \mid (P_i \setminus \{i_\alpha, i_\gamma\}) \cup \{i_\beta, i_\delta\}] = q M^{P_i}_{i_\gamma i_\beta} M^{P_i}_{i_\alpha i_\delta} - q^2 M^{P_i}_{i_\alpha i_\beta} M^{P_i}_{i_\gamma i_\delta}$. $\square$

We may give the set of all nonempty subsets of $\{1, 2, \ldots, n\}$ with the same cardinality a partial order by the following rule: for $I, J \in \{1, 2, \ldots, n\}$ with $I = \{i_1 < i_2 < \cdots < i_k\}$ and $J = \{j_1 < j_2 < \cdots < j_k\}$ we set $I \leq J$ if and only if $i_t \leq j_t$ for $t = 1, 2, \ldots, k$.

Let $I, J, M, N \subset \{1, 2, \ldots n\}$ so that $|I| = |J|$ and $|M| = |N|$. Define the family

$$\{< I||M\} := \{S \subset I \cup M \mid S \supset I \cap M \text{ with } |S| = |I| \text{ and } S < I\}$$

and

$$\{> J||N\} := \{T \subset J \cup N \mid T \supset J \cap N \text{ with } |T| = |J| \text{ and } T > J\}.$$

For $T \in \{> J||N\}$ and $S \in \{< I||M\}$ define

$$T^{\natural} := (J \cup N) \setminus (T \cup (J \cap N)) \qquad \text{and} \qquad S^{\natural} := (I \cup M) \setminus (S \cup (I \cap M)).$$

Define

$$\mathscr{L}(S, I, M) := \ell((S \setminus S^{\natural}) \cup (I \setminus M), S \setminus I) - \ell((S \setminus S^{\natural}) \cup (I \setminus M), I \setminus S)$$
$$\mathscr{L}^{\natural}(T, J, N) := \ell((T^{\natural} \setminus T) \cup (J \setminus N), T \setminus J) - \ell((T^{\natural} \setminus T) \cup (J \setminus N), J \setminus T).$$

For any set X we denote $X \setminus \{x\}$ simply by $X \setminus x$.

For any nonnegative integer d recall from Definition 1.29 that

$$[d]_{-q} = \frac{(-q)^d - (-q)^{-d}}{(-q)^1 - (-q)^{-1}}.$$

For each $l = 1, 2, \ldots, k$ set $d_l = |[1, i_l] \cap J| - l + 1$. Using this, define

$$\xi_q(I; J) = [d_1]_{-q}[d_2]_{-q} \cdots [d_k]_{-q}.$$

Since many of of our calculations will use [5, Theorem 6.2, and Corollary 6.3] we remind the reader of these theorems.

Theorem 3.28 (Theorem 6.2 [5])**.** *For $I, J, M, N \subset \{1, 2, \ldots, n\}$ with $|I| = |J|$ and $|M| = |N|$, then*

$$\begin{aligned} q^{|I\cap M|}[I \mid J][M \mid N] + q^{|I\cap M|} \sum_{S\in\{<I||M\}} \lambda_S[S \mid J][S^\natural \mid N] \\ = q^{|J\cap N|}[M \mid N][I \mid J] + q^{|J\cap N|} \sum_{T\in\{>J||N\}} \mu_T[M \mid T^\natural][I \mid T] \end{aligned}$$

where

$$\lambda_S = \widehat{q}^{|I\setminus S|}(-q)^{\mathscr{L}(S,I,M)}\xi_q(I \setminus S; S \setminus I) \quad \mu_T = \widehat{q}^{|T\setminus J|}(-q)^{\mathscr{L}^\natural(T,J,N)}\xi_q(T \setminus J; J \setminus T).$$

Theorem 3.29 (Corollary 6.3 [5])**.** *For $I, J, M, N \subset \{1, 2, \ldots, n\}$ with $|I| = |J|$ and $|M| = |N|$, then*

$$\begin{aligned} q^{|J\cap N|}[I \mid J][M \mid N] + q^{|J\cap N|} \sum_{S\in\{<J||N\}} \lambda_S[I \mid S][M \mid S^\natural] \\ = q^{|I\cap M|}[M \mid N][I \mid J] + q^{|I\cap M|} \sum_{T\in\{>I||M\}} \mu_T[T^\natural \mid N][T \mid J] \end{aligned}$$

where

$$\lambda_S = \widehat{q}^{|J\setminus S|}(-q)^{\mathscr{L}(S,J,N)}\xi_q(J\setminus S; S\setminus J) \quad \mu_T = \widehat{q}^{|T\setminus I|}(-q)^{\mathscr{L}^\natural(T,I,M)}\xi_q(T\setminus I; I\setminus T).$$

Theorem 3.30. *Let P be a partition of n with $P_i \in P$. For $i_\alpha, i_\gamma, i_\delta \in P_i$ and $i_\beta \in P_{>i}$, the following relations hold in $O_q(P_P^+)$:*

$$M^{P_i}_{i_\delta i_\beta} X_{i_\gamma i_\delta} = q^{-1} X_{i_\gamma i_\delta} M^{P_i}_{i_\delta i_\beta} + \widehat{q}\sum_{\eta=1}^{\delta-1} q^{\eta-\delta} X_{i_\gamma i_\eta} M^{P_i}_{i_\eta i_\beta} \tag{3.16}$$

$$M^{P_i}_{i_\alpha i_\beta} X_{i_\gamma i_\delta} = X_{i_\gamma i_\delta} M^{P_i}_{i_\alpha i_\beta} \text{ for } i_\alpha \neq i_\delta \tag{3.17}$$

$$D_{P_i} X_{i_\gamma i_\delta} = X_{i_\gamma i_\delta} D_{P_i}. \tag{3.18}$$

Proof.

(i) Set $I = P_i$ and $J = (P_i \setminus i_\delta) \cup \{i_\beta\}$ with $M = \{i_\gamma\}$ and $N = \{i_\delta\}$. Since $I \cap M = \{i_\gamma\}$ and $I \cup M = P_i$, there is no set S with $I \cup M \supset S \supset I \cap M$ with $|S| = |I|$ so that $S < I$. Hence, $\{< I||M\} = \emptyset$. Moreover, since $J \cup N = P_i \cup \{i_\beta\}$ and $J \cap N = \emptyset$ the only sets T with $T \subset J \cup N$ and $T \supset J \cap N$ with $|T| = |J|$ and $T > J$ are

$$T_\eta := (P_i \setminus i_\eta) \cup \{i_\beta\}$$

where $\eta = 1, 2, \ldots, \delta - 1$. This implies $T^\natural_\eta = \{i_\eta\}$. To calculate λ_{T_η} we note that $J \setminus N = J$ and

$$T_\eta \setminus J = \{i_\delta\} \qquad J \setminus T_\eta = \{i_\eta\} \qquad T^\natural_\eta \setminus T_\eta = \{i_\eta\} \qquad T_\eta \setminus T^\natural_\eta = T_\eta.$$

Hence,

$$\mathscr{L}^\natural(T_\eta, I, J) = \ell(J, \{i_\delta\}) - \ell(J, \{i_\eta\}) = \mu - \delta + 1 - (\mu - \eta) = \eta - \delta + 1.$$

Finally, since

$$\xi_q\left(\{i_\delta\}; \{i_\eta\}\right) = [1]_{-q} = 1$$

we get from Theorem 3.28 that

$$q[I \mid J][M \mid N] = [M \mid N][I \mid J] + \sum_{\eta=1}^{\delta-1} \widehat{q} q^{\eta-\delta+1}[M \mid T^\natural_\eta][I \mid T_\eta].$$

That is, $M^{P_i}_{i_\delta i_\beta} X_{i_\gamma i_\delta} = q^{-1} X_{i_\gamma i_\delta} M^{P_i}_{i_\delta i_\beta} + \widehat{q} \sum_{\eta=1}^{\delta-1} q^{\eta-\delta} X_{i_\gamma i_\eta} M^{P_i}_{i_\eta i_\beta}$.

(ii) Set $I = P_i$ and $J = (P_i \setminus i_\alpha) \cup \{i_\beta\}$ with $M = \{i_\gamma\}$ and $N = \{i_\delta\}$. Then $I \cap M = \{i_\gamma\}$ and $I \cup M = P_i$. Note that there is no set S such that $S \supset \{i_\gamma\}$ and $S \subset P_i$ with $S < I$. Hence, $\{< I||M\} = \emptyset$. Similarly, since $J \cap N = \{i_\delta\}$ and $J \cup N = J$ there is no set T where $T \supset \{i_\delta\}$ with $T \subset J$

so that $T > J$. Hence, $\{> J||N\} = \emptyset$. Therefore, from Theorem 3.28 we have

$$q[I \mid J][M \mid N] = q[M \mid N][I \mid J].$$

That is, $M^{P_i}_{i_\alpha i_\beta} X_{i_\gamma i_\delta} = X_{i_\gamma i_\delta} M^{P_i}_{i_\alpha i_\beta}$.

(iii) Since D_{P_i} is a central element in $\mathrm{M}_{|P_i|}$, it follows that $D_{P_i} X_{i_\gamma i_\delta} = X_{i_\gamma i_\delta} D_{P_i}$.

□

Theorem 3.31. *Let P be a partition of n and $P_i, P_j \in P$ with $i < j$. For $i_\alpha, i_\beta, i_\gamma, i_\delta \in P_i$ with $j_\alpha \in P_j$ and $j_\beta \in P_{>j}$, the following relations hold in $O_q(P^+_P)$:*

$$M^{P_j}_{j_\alpha j_\beta} X_{i_\gamma i_\delta} = X_{i_\gamma i_\delta} M^{P_j}_{j_\alpha j_\beta} \tag{3.19}$$

$$D_{P_j} X_{i_\alpha i_\beta} = X_{i_\alpha i_\beta} D_{P_j}. \tag{3.20}$$

Proof.

(i) Set $I = P_j$ and $M = \{i_\gamma\}$. Since $i < j$ then $I \cup M = P_j \cup \{i_\gamma\}$ and $I \cap M = \emptyset$. Note, for every $j_\eta \in P_i$ we have $j_\eta > i_\gamma$. Hence, there is no set S so that $S \subset I \cup M$ and $S \supset I \cap M$ with $S < I$. Therefore, $\{< I||M\} = \emptyset$.

Set $J = (P_j \setminus j_\alpha) \cup \{j_\beta\}$ and $N = \{i_\delta\}$, then $J \cup N = (P_j \setminus j_\alpha) \cup \{j_\beta, i_\delta\}$ and $J \cap N = \emptyset$. Note for any $T > J$ with $T \subset J \cup N$ and $T \supset J \cap N$ we must have $T^\natural \cap P_j \neq \emptyset$. Hence, $[i_\gamma \mid T^\natural] \in I^-_P$. Therefore, by Theorem 3.28

we have in $O_q(P_P^+)$

$$[I \mid J][M \mid N] = [M \mid N][I \mid J].$$

That is, $M^{P_j}_{j_\alpha j_\beta} X_{i_\gamma i_\delta} = X_{i_\gamma i_\delta} M^{P_j}_{j_\alpha j_\beta}$.

(ii) We note that for any $j_\eta, j_\theta \in P_j$ that $X_{i_\alpha i_\beta} X_{j_\eta j_\theta} = X_{j_\eta j_\theta} X_{i_\alpha i_\beta}$ in $O_q(P_P^+)$. Consequently, $D_{P_j} X_{i_\alpha i_\beta} = X_{i_\alpha i_\beta} D_{P_j}$.

□

Theorem 3.32. *Let P be a partition of n and $P_i, P_j \in P$ with $i < j$. For $j_\gamma, j_\delta \in P_j$ and $i_\alpha \in P_i$ and $i_\beta \in P_{>i}$ the following relations hold in $O_q(P_P^+)$:*

$$M^{P_i}_{i_\alpha i_\beta} X_{j_\gamma j_\delta} = X_{j_\gamma j_\delta} M^{P_i}_{i_\alpha i_\beta} \text{ for } i_\beta \in P_{<j} \tag{3.21}$$

$$M^{P_i}_{i_\alpha i_\beta} X_{j_\gamma j_\delta} = X_{j_\gamma j_\delta} M^{P_i}_{i_\alpha i_\beta} + \widehat{q} X_{j_\gamma i_\beta} M^{P_i}_{i_\alpha j_\delta} \text{ for } i_\beta \in P_j \text{ and } i_\beta < j_\delta \tag{3.22}$$

$$M^{P_i}_{i_\alpha i_\beta} X_{j_\gamma j_\delta} = q X_{j_\gamma j_\delta} M^{P_i}_{i_\alpha i_\beta} \text{ for } i_\beta \in P_j \text{ and } i_\beta = j_\delta \tag{3.23}$$

$$M^{P_i}_{i_\alpha i_\beta} X_{j_\gamma j_\delta} = X_{j_\gamma j_\delta} M^{P_i}_{i_\alpha i_\beta} \text{ for } i_\beta > j_\delta \tag{3.24}$$

$$D_{P_i} X_{j_\gamma j_\delta} = X_{j_\gamma j_\delta} D_{P_i}. \tag{3.25}$$

Proof. Set $I = P_i$ and $M = \{j_\gamma\}$. Since $i < j$ then $I \cup M = P_i \cup \{j_\gamma\}$ and $I \cap M = \emptyset$. Note for every $i_\eta \in P_i$ that $i_\eta < j_\gamma$. Hence, there is no set S so that $S \subset I \cup M$ and $S \supset I \cap M = \emptyset$ with $S < I$. Therefore, $\{< I || M\} = \emptyset$. Thus, Theorem 3.28 simplifies to

$$[I \mid J][M \mid N] = q^{|J \cap N|}[M \mid N][I \mid J] + q^{|J\cap N|} \sum_{T \in \{>J||N\}} \lambda_T [M \mid T^\natural][I \mid T].$$

We now consider each case separately.

(i) Set $J = (P_i \setminus i_\alpha) \cup \{i_\beta\}$ and $N = \{j_\delta\}$, then $J \cup N = (P_i \setminus i_\alpha) \cup \{i_\beta, j_\delta\}$ and $J \cap N = \emptyset$. Notice that since $i_\beta \in P_{<j}$ then $i_\beta < j_\delta$. Therefore, any $T > J$ with $T \subset J \cup N$ and $T \supset J \cap N$ must have $T^\natural \cap P_{<j} \neq \emptyset$. Hence, $[i_\alpha \mid T^\natural] \in I_P^-$. That is, in $O_q(P_P^+)$ the relation from Theorem 3.28 simplifies to

$$[I \mid J][M \mid N] = [M \mid N][I \mid J],$$

i.e., $M^{P_i}_{i_\alpha i_\beta} X_{j_\gamma j_\delta} = X_{j_\gamma j_\delta} M^{P_i}_{i_\alpha i_\beta}$.

(ii) Set $J = (P_i \setminus i_\alpha) \cup \{i_\beta\}$ and $N = \{j_\delta\}$. For each $i_\eta \in J$ denote

$$T_\eta = (J \setminus i_\eta) \cup \{j_\delta\}$$

It is clear that $T_\eta > J$ for all $i_\eta \in J$ and that $\{> J||N\}$ consists of exactly these T_η. Now, for $i_\eta \neq i_\beta$ we have $T_\eta^\natural = \{i_\eta\}$ and $T_\eta^\natural \cap P_{<j} \neq \emptyset$. Hence, $[j_\gamma \mid T_\eta^\natural] \in I_P^-$. Therefore, the relation from Theorem 3.28 simplifies in

$O_q(P_P^+)$ to

$$[I \mid J][M \mid N] = [M \mid N][I \mid J] + \lambda_{T_\beta}[M \mid T_\beta^\natural][I \mid T_\beta]$$

where $T_\beta = (J \setminus i_\beta) \cup \{j_\delta\}$. To calculate λ_{T_β} we note that $J \setminus N = J$ and

$$T_\beta \setminus J = \{j_\delta\} \qquad J \setminus T_\beta = \{i_\beta\} \qquad T_\beta^\natural \setminus T_\beta = \{i_\beta\} \qquad T_\beta \setminus T_\beta^\natural = T_\beta$$

Therefore, $\lambda_{T_\beta} = \widehat{q}$ since

$$\mathscr{L}^\natural(T_\beta, J, N) = \ell(J; \{j_\delta\}) - \ell(J; \{i_\beta\}) = 0 \quad \text{and} \quad \xi_q(\{j_\delta\}; \{i_\beta\}) = [1]_{-q} = 1.$$

It follows that $M^{P_i}_{i_\alpha i_\beta} X_{j_\gamma j_\delta} = X_{j_\gamma j_\delta} M^{P_i}_{i_\alpha i_\beta} + \widehat{q} X_{j_\gamma i_\beta} M^{P_i}_{i_\alpha j_\delta}$.

(iii) Set $J = (P_i \setminus i_\alpha) \cup \{i_\beta\}$ and $N = \{j_\delta\}$. Hence, $J \cup N = (P_i \setminus i_\alpha) \cup \{i_\beta\}$ and $J \cap N = \{i_\beta\}$. We note that for any set T with $T \subset J \cup N$ and $T \supset J \cap N = \{i_\beta\}$ so that $T > J$ we have $T^\natural \cap P_{<j} \neq \emptyset$. Hence, $[i_\alpha \mid T^\natural] \in I_P^-$. Therefore, in $O_q(P_P^+)$ the relation from Theorem 3.28 simplifies to

$$[I \mid J][M \mid N] = q[M \mid N][I \mid J].$$

That is, $M^{P_i}_{i_\alpha i_\beta} X_{j_\gamma j_\delta} = q X_{j_\gamma j_\delta} M^{P_i}_{i_\alpha i_\beta}$.

(iv) Set $J = (P_i \setminus i_\alpha) \cup \{i_\beta\}$ and $N = \{j_\delta\}$. Hence, $J \cup N = (P_i \setminus i_\alpha) \cup \{j_\delta, i_\beta\}$ and $J \cap N = \emptyset$. We note that for any set T with $T \subset J \cup N$ and $T \supset J \cap N$ so that $T > J$ we have $T^\natural \cap P_{<j} \neq \emptyset$. Hence, $[i_\alpha \mid T^\natural] \in I_P^-$. Therefore, in $O_q(P_P^+)$ the relation from Theorem 3.28 simplifies to

$$[I \mid J][M \mid N] = [M \mid N][I \mid J].$$

That is, $M^{P_i}_{i_\alpha i_\beta} X_{j_\gamma j_\delta} = X_{j_\gamma j_\delta} M^{P_i}_{i_\alpha i_\beta}$.

(v) Since D_{P_i} is a central element of $\mathrm{M}_{|P_i|}$ it follows that $D_{P_i} X_{j_\gamma j_\delta} = X_{j_\gamma j_\delta} D_{P_i}$.

□

Theorem 3.33. *Let P be a partition of n with $P_i \in P$. For $i_\alpha, i_\gamma \in P_i$ with $i_\alpha < i_\gamma$ and $i_\beta, i_\delta \in P_{>i}$ with $i_\beta < i_\delta$, the following relations hold in $O_q(P_P^+)$:*

$$M^{P_i}_{i_\alpha i_\beta} M^{P_i}_{i_\alpha i_\delta} = q M^{P_i}_{i_\alpha i_\delta} M^{P_i}_{i_\alpha i_\beta} \tag{3.26}$$

$$M^{P_i}_{i_\alpha i_\beta} M^{P_i}_{i_\gamma i_\beta} = q^{-1} M^{P_i}_{i_\gamma i_\beta} M^{P_i}_{i_\alpha i_\beta} \tag{3.27}$$

$$M^{P_i}_{i_\alpha i_\beta} M^{P_i}_{i_\gamma i_\delta} = M^{P_i}_{i_\gamma i_\delta} M^{P_i}_{i_\alpha i_\beta} \tag{3.28}$$

$$M^{P_i}_{i_\alpha i_\delta} M^{P_i}_{i_\gamma i_\beta} = M^{P_i}_{i_\gamma i_\beta} M^{P_i}_{i_\alpha i_\delta} - \widehat{q} M^{P_i}_{i_\alpha i_\beta} M^{P_i}_{i_\gamma i_\delta}. \tag{3.29}$$

Proof. Set $I = M = P_i$. We first note that since $\{< I||M\} = \emptyset$, Theorem 3.28 simplifies to

$$q^{|P_i|}[I \mid J][M \mid N] = q^{|J\cap N|}[M \mid N][I \mid J] + q^{|J\cap N|} \sum_{T\in\{>J||N\}} \mu_T[M \mid T^{\natural}][I \mid T].$$

(i) Set $J = (P_i \setminus i_\alpha) \cup \{i_\beta\}$ and $N = (P_i \setminus i_\alpha) \cup \{i_\delta\}$. Note that $J \cap N = (P_i \setminus i_\alpha)$ and $J \cup N = (P_i \setminus i_\alpha) \cup \{i_\beta, i_\delta\}$. Hence, the only possibilities for T so that $|T| = |J|$ with $T \subset J \cup N$ and $T \supset J \cap N$ are J and N. Since $N > J$, the relation above simplifies to

$$q^{|P_i|}[I \mid J][M \mid N] = q^{|P_i|-1}[M \mid N][I \mid J] + q^{|P_i|-1}\widehat{q}[M \mid J][I \mid N].$$

However, since $N \cap P_{>i} \neq \emptyset$ then $[I \mid N] = 0$ in $O_q(P_P^+)$ by Lemma 3.5. It follows that $M^{P_i}_{i_\alpha i_\beta} M^{P_i}_{i_\alpha i_\delta} = q M^{P_i}_{i_\alpha i_\delta} M^{P_i}_{i_\alpha i_\beta}$.

(ii) Set $J = (P_i \setminus i_\alpha) \cup \{i_\beta\}$ and $N = (P_i \setminus i_\gamma) \cup \{i_\beta\}$. Note that $J \cap N = (P_i \setminus \{i_\alpha, i_\gamma\}) \cup \{i_\beta\}$ and $J \cup N = P_i \cup \{i_\beta\}$. The only possibilities for T so that $|T| = |J|$ with $T \subset J \cup N$ and $T \supset J \cap N$ are J and N. None of these sets have the property that $T > J$ so $\{> J||N\} = \emptyset$. Hence, by Theorem 3.28 we get

$$q^{|P_i|}[I \mid J][M \mid N] = q^{|P_i|-1}[M \mid N][I \mid J].$$

It follows that $M^{P_i}_{i_\alpha i_\beta} M^{P_i}_{i_\gamma i_\delta} = q^{-1} M^{P_i}_{i_\gamma i_\beta} M^{P_i}_{i_\alpha i_\beta}$.

(iii) Set $J = (P_i \setminus i_\alpha) \cup \{i_\beta\}$ and $N = (P_i \setminus i_\gamma) \cup \{i_\delta\}$. Note that $J \cap N = (P_i \setminus \{i_\alpha, i_\gamma\})$ and $J \cup N = P_i \cup \{i_\beta, i_\delta\}$. Hence, the only possibilities for T so that $|T| = |J|$ with $T \subset J \cup N$ and $T \supset J \cap N$ with the property that $T > J$ are

$$T_1 = (P_i \setminus i_\alpha) \cup \{i_\delta\} \qquad \text{or} \qquad T_2 = (P_i \setminus \{i_\alpha, i_\gamma\}) \cup \{i_\beta, i_\delta\}.$$

To compute λ_{T_i} we note that $J \setminus N = \{i_\gamma, i_\beta\}$ and

$$\begin{aligned} T_1^\natural &= (P_i \setminus i_\gamma) \cup \{i_\beta\} & T_2^\natural &= P_i \\ T_1^\natural \setminus T_1 &= \{i_\alpha, i_\beta\} & T_2^\natural \setminus T_2 &= \{i_\alpha, i_\gamma\} \\ T_1 \setminus J &= \{i_\delta\} & T_2 \setminus J &= \{i_\delta\} \\ J \setminus T_1 &= \{i_\beta\} & J \setminus T_2 &= \{i_\gamma\}. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathscr{L}^\natural(T_1, J, N) &= \ell(\{i_\alpha, i_\gamma, i_\beta\}, \{i_\delta\}) - \ell(\{i_\alpha, i_\gamma, i_\beta\}, \{i_\beta\}) = 0 + 0 = 0 \\ \mathscr{L}^\natural(T_2, J, N) &= \ell(\{i_\alpha, i_\gamma, i_\beta\}, \{i_\delta\}) - \ell(\{i_\alpha, i_\gamma, i_\beta\}, \{i_\gamma\}) = 0 - 1 = -1. \end{aligned}$$

Similarly,

$$\xi_q(T_1 \setminus J; J \setminus T_1) = 1 \qquad \text{and} \qquad \xi_q(T_2 \setminus J; J \setminus T_2) = 1.$$

Thus, $\lambda_{T_1} = \widehat{q}$ and $\lambda_{T_2} = \widehat{q}(-q)^{-1}$. Therefore, by Theorem 3.28 we have

$$\begin{aligned} q^{|P_i|}[I \mid J][M \mid N] = {} & q^{|P_i|-2}[M \mid N][I \mid J] \\ & + q^{|P_i|-2}\widehat{q}[M \mid T_1^{\natural}][I \mid T_1] + q^{|P_i|-2}(-q^{-1})\widehat{q}[M \mid T_2^{\natural}][I \mid T_2]. \end{aligned}$$

That is,

$$M^{P_i}_{i_\alpha i_\beta} M^{P_i}_{i_\gamma i_\delta} = q^{-2} M^{P_i}_{i_\gamma i_\delta} M^{P_i}_{i_\alpha i_\beta} + q^{-2}\widehat{q} M^{P_i}_{i_\gamma i_\beta} M^{P_i}_{i_\alpha i_\delta} + q^{-2}(-q^{-1})\widehat{q} D_{P_i}[P_i \mid T_2].$$

However, by Lemma 3.27 we have $D_{P_i}[P_i \mid T_2] = q M^{P_i}_{i_\gamma i_\beta} M^{P_i}_{i_\alpha i_\delta} - q^2 M^{P_i}_{i_\alpha i_\beta} M^{P_i}_{i_\gamma i_\delta}$. Therefore

$$\begin{aligned} M^{P_i}_{i_\alpha i_\beta} M^{P_i}_{i_\gamma i_\delta} &= q^{-2} M^{P_i}_{i_\gamma i_\delta} M^{P_i}_{i_\alpha i_\beta} + q^{-2}\widehat{q} M^{P_i}_{i_\gamma i_\beta} M^{P_i}_{i_\alpha i_\delta} \\ &+ q^{-2}(-q^{-1})\widehat{q}\left(q M^{P_i}_{i_\gamma i_\beta} M^{P_i}_{i_\alpha i_\delta} - q^2 M^{P_i}_{i_\alpha i_\beta} M^{P_i}_{i_\gamma i_\delta}\right) \\ &= q^{-2} M^{P_i}_{i_\gamma i_\delta} M^{P_i}_{i_\alpha i_\beta} + q^{-2}\widehat{q} M^{P_i}_{i_\gamma i_\beta} M^{P_i}_{i_\alpha i_\delta} - q^{-2}\widehat{q} M^{P_i}_{i_\gamma i_\beta} M^{P_i}_{i_\alpha i_\delta} \\ &+ q^{-1}\widehat{q} M^{P_i}_{i_\alpha i_\beta} M^{P_i}_{i_\gamma i_\delta} \\ &= q^{-2} M^{P_i}_{i_\gamma i_\delta} M^{P_i}_{i_\alpha i_\beta} + M^{P_i}_{i_\alpha i_\beta} M^{P_i}_{i_\gamma i_\delta} - q^{-2} M^{P_i}_{i_\alpha i_\beta} M^{P_i}_{i_\gamma i_\delta}. \end{aligned}$$

It follows that $M^{P_i}_{i_\alpha i_\beta} M^{P_i}_{i_\gamma i_\delta} = M^{P_i}_{i_\gamma i_\delta} M^{P_i}_{i_\alpha i_\beta}$.

(iv) Set $J = (P_i \setminus i_\alpha) \cup \{i_\delta\}$ and $N = (P_i \setminus i_\gamma) \cup \{i_\beta\}$. Then $J \cup N = P_i \cup \{i_\beta, i_\delta\}$ and $J \cap N = P_i \setminus \{i_\alpha, i_\gamma\}$. The only possibility for T so that $|T| = |J|$ with

$T \subset J \cup N$ and $T \supset J \cap N$ with the property that $T > J$ is

$$T = (P_i \setminus \{i_\alpha, i_\gamma\}) \cup \{i_\beta, i_\delta\}.$$

To compute λ_T we note $J \setminus N = \{i_\gamma, i_\delta\}$ and

$$T^\natural = P_i \qquad T^\natural \setminus T = \{i_\alpha, i_\gamma\} \qquad T \setminus J = \{i_\beta\} \qquad J \setminus T = \{i_\gamma\}.$$

Therefore we get

$$\mathscr{L}^\natural(T, I, J) = \ell(\{i_\alpha, i_\gamma, i_\delta\}, \{i_\beta\}) - \ell(\{i_\alpha, i_\gamma, i_\beta\}, \{i_\gamma\}) = 1 - 1 = 0.$$

Similarly, we have

$$\xi_q(T \setminus J; J \setminus T) = 1.$$

Therefore, $\lambda_T = \widehat{q}$ and we have the relation

$$q^{|P_i|}[I \mid J][M \mid N] = q^{|P_i|-2}[M \mid N][I \mid J] + q^{|P_i|-2}\widehat{q}[M \mid T^\natural][I \mid T].$$

That is,

$$M^{P_i}_{i_\alpha i_\delta} M^{P_i}_{i_\gamma i_\beta} = q^{-2} M^{P_i}_{i_\gamma i_\beta} M^{P_i}_{i_\alpha i_\delta} + q^{-2}\widehat{q} D_{P_i}[P_i \mid T].$$

However, by Lemma 3.27 we have $D_{P_i}[P_i \mid T] = qM^{P_i}_{i_\gamma i_\beta}M^{P_i}_{i_\alpha i_\delta} - q^2M^{P_i}_{i_\alpha i_\beta}M^{P_i}_{i_\gamma i_\delta}$. Substituting this into the equation, we have

$$\begin{aligned} M^{P_i}_{i_\alpha i_\delta}M^{P_i}_{i_\gamma i_\beta} &= q^{-2}M^{P_i}_{i_\gamma i_\beta}M^{P_i}_{i_\alpha i_\delta} + q^{-2}\widehat{q}\left(qM^{P_i}_{i_\gamma i_\beta}M^{P_i}_{i_\alpha i_\delta} - q^2M^{P_i}_{i_\alpha i_\beta}M^{P_i}_{i_\gamma i_\delta}\right) \\ &= q^{-2}M^{P_i}_{i_\gamma i_\beta}M^{P_i}_{i_\alpha i_\delta} + q^{-1}\widehat{q}M^{P_i}_{i_\gamma i_\beta}M^{P_i}_{i_\alpha i_\delta} - \widehat{q}M^{P_i}_{i_\alpha i_\beta}M^{P_i}_{i_\gamma i_\delta} \\ &= M^{P_i}_{i_\gamma i_\beta}M^{P_i}_{i_\alpha i_\delta} - \widehat{q}M^{P_i}_{i_\alpha i_\beta}M^{P_i}_{i_\gamma i_\delta}. \qquad \square \end{aligned}$$

For $P_i, P_j \in P$ with $i_\alpha \in P_i$ and $j_\delta, j_\gamma \in P_j$ denote

$$M^{P_j\ P_i}_{(i_\alpha j_\delta), j_\gamma} := \left(\sum_{\eta=1}^{|P_j|-\gamma}(-q)^{1-\eta}M^{P_j}_{j_{\gamma+\eta}, j_\delta}M^{P_i}_{i_\alpha j_{\gamma+\eta}} + (-q)^{\gamma-|P_j|}D_{P_j}M^{P_i}_{i_\alpha j_\delta}\right).$$

Theorem 3.34. *Let P be a partition of n and $P_i, P_j \in P$ with $i < j$. For $i_\alpha \in P_i$ with $i_\beta \in P_{>i}$ and $j_\gamma \in P_j$ with $j_\delta \in P_{>j}$, the following relations hold in $O_q(P^+_P)$:*

$$M^{P_i}_{i_\alpha i_\beta}M^{P_j}_{j_\gamma j_\delta} = M^{P_j}_{j_\gamma j_\delta}M^{P_i}_{i_\alpha i_\beta} \text{ for } i_\beta \in P_{<j} \tag{3.30}$$

$$M^{P_i}_{i_\alpha i_\beta}M^{P_j}_{j_\gamma j_\delta} = qM^{P_j}_{j_\gamma j_\delta}M^{P_i}_{i_\alpha i_\beta} \text{ for } i_\beta \in P_j \text{ but } i_\beta \neq j_\gamma \tag{3.31}$$

$$M^{P_i}_{i_\alpha i_\beta}M^{P_j}_{j_\gamma j_\delta} = M^{P_j}_{j_\gamma j_\delta}M^{P_i}_{i_\alpha i_\beta} + \widehat{q}M^{P_j\ P_i}_{(i_\alpha j_\delta), j_\gamma} \text{ for } i_\beta \in P_j \text{ and } i_\beta = j_\gamma \tag{3.32}$$

$$M^{P_i}_{i_\alpha i_\beta}M^{P_j}_{j_\gamma j_\delta} = M^{P_j}_{j_\gamma j_\delta}M^{P_i}_{i_\alpha i_\beta} + \widehat{q}M^{P_j}_{j_\gamma i_\beta}M^{P_i}_{i_\alpha j_\delta} \text{ for } i_\beta \in P_{>j} \text{ and } i_\beta < j_\delta \tag{3.33}$$

$$M^{P_i}_{i_\alpha i_\beta}M^{P_j}_{j_\gamma j_\delta} = qM^{P_j}_{j_\gamma j_\delta}M^{P_i}_{i_\alpha i_\beta} \text{ for } i_\beta \in P_{>j} \text{ and } i_\beta = j_\delta \tag{3.34}$$

$$M^{P_i}_{i_\alpha i_\beta}M^{P_j}_{j_\gamma j_\delta} = M^{P_j}_{j_\gamma j_\delta}M^{P_i}_{i_\alpha i_\beta} \text{ for } i_\beta \in P_{>j} \text{ and } i_\beta > j_\delta. \tag{3.35}$$

Proof. Set $I = P_i$ and $M = P_j$. Since $i < j$ it follows that $\{< I||M\} = \emptyset$.

Set $J = (P_i \setminus i_\alpha) \cup \{i_\beta\}$ and $N = (P_j \setminus j_\gamma) \cup \{j_\delta\}$. For $T \in \{> J||N\}$ since $P_i \subset P_{<j}$, if $P_i \cap T^\natural \neq \emptyset$ then $[P_j \mid T^\natural] = 0$ in $O_q(P_P^+)$ by Lemma 3.5 (i). Since we are concerned with relations in $O_q(P_P^+)$ we need only consider the $T \in \{> J||N\}$ so that $P_i \cap T^\natural = \emptyset$, which is equivalent to $T \supset J \cap P_i$. Hence, Theorem 3.28 simplifies in $O_q(P_P^+)$ to

$$[I \mid J][M \mid N] = q^{|J \cap N|}[M \mid N][I \mid J] + q^{|J \cap N|} \sum_{\substack{T \in \{> J||N\} \\ T \supset J \cap P_i}} \lambda_T [M \mid T^\natural][I \mid T]. \quad (3.36)$$

We now consider each case individually.

(i) First note that $J \cap N = \emptyset$ and $J \cap P_i = P_i \setminus i_\alpha$. If $i_\beta \notin T$ then $i_\beta \in T^\natural$. Since $i_\beta \in P_{<j}$ then $T^\natural \cap P_{<j} \neq \emptyset$, hence $[M \mid T^\natural] = 0$ in $O_q(P_P^+)$ by Theorem 3.5 (i). If $i_\beta \in T$ then since we are only considering $T \supset J \cap P_i$ then this implies that $T = J$. Since $T \not> J$ equation (3.36) simplifies to

$$[I \mid J][M \mid N] = [M \mid N][I \mid J].$$

That is, $M^{P_i}_{i_\alpha i_\beta} M^{P_j}_{j_\gamma j_\delta} = M^{P_j}_{j_\gamma j_\delta} M^{P_i}_{i_\alpha i_\beta}$.

(ii) Since $i_\beta \in P_j$ then $i_\beta = j_\eta$ for some $j_\eta \in P_j$ except $i_\beta \neq j_\gamma$. Hence, $J \cup N = (P_i \setminus i_\alpha) \cup (P_j \setminus j_\gamma) \cup \{j_\delta\}$ and $J \cap N = \{i_\beta\}$. Notice, since we need only consider $T \supset J \cap P_i = P_i \setminus i_\alpha$ and since $T \supset \{i_\beta\}$ then we must have

$T = J$. However, since $T \not> J$ equation (3.36) simplifies to

$$[I \mid J][M \mid N] = q[M \mid N][I \mid J].$$

That is, $M^{P_i}_{i_\alpha i_\beta} M^{P_j}_{j_\gamma j_\delta} = q M^{P_j}_{j_\gamma j_\delta} M^{P_i}_{i_\alpha i_\beta}$.

(iii) Note that $J \cup N = (P_i \setminus i_\alpha) \cup P_j \cup \{j_\delta\}$ and $J \cap N = \emptyset$. It is straightforward to verify that the only $T \in \{> J||N\}$ with $T \supset J \cap P_i$ are

$$T_\eta := (P_i \setminus i_\alpha) \cup \{j_{\gamma+\eta}\}$$

where $\eta = 1, 2, \ldots, |P_j| - \gamma$ and

$$T_\delta := (P_i \setminus i_\alpha) \cup \{j_\delta\}.$$

To compute λ_{T_η} we note $J \setminus N = J$ and

$$T^\natural_\eta = (P_j \setminus j_{\gamma+\eta}) \cup \{j_\delta\} \quad T^\natural_\eta \setminus T_\eta = T_\eta \quad T_\eta \setminus J = \{j_{\gamma+\eta}\} \quad J \setminus T_\eta = \{j_\gamma\}.$$

Therefore,

$$\begin{aligned} \mathscr{L}^\natural(T_\eta, J, N) = \ell((P_i \setminus i_\alpha) \cup (P_j \setminus j_{\gamma+\eta}) \cup \{j_\delta\}\{j_{\gamma+\eta}\}) \\ - \ell(((P_i \setminus i_\alpha) \cup P_j \setminus j_{\gamma+\eta}) \cup \{j_\delta\}, \{j_\gamma\}) \end{aligned}$$

$$= |P_j| - (\gamma + \eta) + 1 - (|P_j| - \gamma) = 1 - \eta$$

and $\xi_q(T_i \setminus J; J \setminus T_i) = 1$. Thus, $\lambda_{T_\eta} = \widehat{q}(-q)^{1-\eta}$. Similarly, to compute λ_{T_δ} we note that

$$T_\delta^\natural = P_j \qquad T_\delta^\natural \setminus T_\delta = P_j \qquad T_\delta \setminus J = \{j_\delta\} \qquad J \setminus T_\delta = \{j_\gamma\}.$$

Therefore,

$$\begin{aligned} \mathscr{L}^\natural(T_\delta, I, J) &= \ell((P_i \setminus i_\alpha) \cup P_j, \{j_\delta\}) - \ell((P_i \setminus i_\alpha) \cup P_j, \{j_\gamma\}) \\ &= 0 - (|P_j| - \gamma) = \gamma - |P_j|. \end{aligned}$$

and $\xi_q(T_\delta \setminus J; J \setminus T_\delta) = 1$. Thus, $\lambda_{T_\delta} = \widehat{q}(-q)^{\gamma - |P_j|}$. Therefore, equation (3.36) becomes

$$\begin{aligned} &[I \mid J][M \mid N] = [M \mid N][I \mid J] \\ &+ \sum_{\eta=1}^{|P_j|-\gamma} \widehat{q}(-q)^{1-\eta}[M \mid T_\eta^\natural][I \mid T_\eta] + \widehat{q}(-q)^{\gamma-|P_j|}[M \mid T_\delta^\natural][I \mid T_\delta]. \end{aligned}$$

That is,

$$M^{P_i}_{i_\alpha i_\beta} M^{P_j}_{j_\gamma j_\delta} = M^{P_j}_{j_\gamma j_\delta} M^{P_i}_{i_\alpha i_\beta}$$

$$+\,\widehat{q}\left(\sum_{\eta=1}^{|P_j|-\gamma}(-q)^{1-\eta}M^{P_j}_{j_{\gamma+\eta},j_\delta}M^{P_i}_{i_\alpha j_{\gamma+\eta}}+(-q)^{\gamma-|P_j|}D_{P_j}M^{P_i}_{i_\alpha j_\delta}\right)$$

$$=M^{P_j}_{j_\gamma j_\delta}M^{P_i}_{i_\alpha i_\beta}+\widehat{q}M^{P_j\ P_i}_{(i_\alpha j_\delta),j_\gamma}.$$

(iv) We have $J\cup N=(P_i\setminus i_\alpha)\cup(P_j\setminus j_\gamma)\cup\{i_\alpha,j_\beta\}$ and $J\cap N=\emptyset$. Since $i<j$ then $i_\beta<j_\epsilon$ for all $j_\epsilon\in P_j$. Similarly, since $i_\beta\in P_{>i}$ then $i_\beta>i_\eta$ for all $i_\eta\in P_i$. Since $i_\beta<j_\delta$ the only element in $J\cup N$ larger than i_β is j_δ. Since we need only consider $T\in\{>J||N\}$ such that $T\supset J\cap P_i$ it is clear that the only such T is $T=(P_i\setminus i_\alpha)\cup\{j_\delta\}$.

Now, to compute λ_T we note

$$T^\natural=P_j\setminus\{i_\beta\}\qquad T^\natural\setminus T=T\qquad T\setminus J=\{j_\delta\}\qquad J\setminus T=\{i_\beta\}.$$

Hence,

$$\begin{aligned}\mathscr{L}^\natural(T,I,J)&=\ell((P_i\setminus i_\alpha)\cup(P_j\setminus j_\gamma)\cup\{i_\beta\},\{j_\delta\})\\&\quad-\ell((P_i\setminus i_\alpha)\cup(P_j\setminus j_\gamma)\cup\{i_\beta\},\{i_\beta\})=0\end{aligned}$$

and $\xi_q(T\setminus J;J\setminus T)=1$. Therefore, $\lambda_T=\widehat{q}$. Thus, equation (3.36) simplifies to become

$$[I\mid J][M\mid N]=[M\mid N][I\mid J]+\widehat{q}[M\mid T^\natural][I\mid T].$$

That is, $M^{P_i}_{i_\alpha i_\beta} M^{P_j}_{j_\gamma j_\delta} = M^{P_j}_{j_\gamma j_\delta} M^{P_i}_{i_\alpha i_\beta} + \widehat{q} M^{P_j}_{j_\gamma i_\beta} M^{P_i}_{i_\alpha j_\delta}$.

(v) Since $j_\beta = l_\delta$ we have $J \cup N = (P_i \setminus i_\alpha) \cup (P_j \setminus j_\gamma) \cup \{j_\delta\}$ and $J \cap N = \{j_\delta\}$. Since $J \cap P_i = P_i \setminus i_\alpha \subset T$ and since we must have $j_\delta \in T$ then $T = J$. Therefore, equation (3.36) simplifies to

$$[I \mid J][M \mid N] = q[M \mid N][I \mid J].$$

That is, $M^{P_i}_{i_\alpha i_\beta} M^{P_j}_{j_\gamma j_\delta} = q M^{P_j}_{j_\gamma j_\delta} M^{P_i}_{i_\alpha i_\beta}$.

(vi) We have $J \cup N = (P_i \setminus i_\alpha) \cup (P_j \setminus j_\gamma) \cup \{j_\delta, i_\beta\}$ and $J \cap N = \emptyset$. Since $i < j$ and $i_\beta > j_\delta$ then i_β is the largest element in $J \cup N$. Since $i_\beta \in J$ if $i_\beta \notin T$ then $T \not> J$. Hence, $i_\beta \in T$. Moreover, since we need only consider $T \in \{> J||N\}$ so that $J \cap P_i \subset T$ and $i_\beta \in T$ with $|T| = |J|$ then $T = J$. Therefore, equation (3.36) above simplifies to

$$[I \mid J][M \mid N] = [M \mid N][I \mid J].$$

That is, $M^{P_i}_{i_\alpha i_\beta} M^{P_j}_{j_\gamma j_\delta} = M^{P_j}_{j_\gamma j_\delta} M^{P_i}_{i_\alpha i_\beta}$. □

Relations in $O_q(N^+_{>P})$

Let P be a partition of n. Now that we have some relations in $O_q(P^+_P)$ involving the quantum minors D_{P_i} and $M^{P_i}_{i_\alpha i_\beta}$, we can deduce some relations in $O_q(N^+_{>P})$.

For $P_i \in P$ with $i_\alpha \in P_i$ and $i_\beta \in P_{>i}$ denote the generators of $O_q(N^+_{>P})$ by

$$x^{P_i}_{i_\alpha i_\beta} := D^{-1}_{P_i} M^{P_i}_{i_\alpha i_\beta}.$$

Moreover, for $P_i, P_j \in P$ with $i_\alpha \in P_i$ and $j_\delta, j_\gamma \in P_j$ denote

$$x^{P_j\ P_i}_{(i_\alpha j_\delta),j_\gamma} := D^{-1}_{P_j} D^{-1}_{P_i} M^{P_j\ P_i}_{(i_\alpha j_\delta),j_\gamma}.$$

It follows from (3.3) and Theorem 3.8 that

$$x^{P_j\ P_i}_{(i_\alpha j_\delta),j_\gamma} = \sum_{\eta=1}^{|P_j|-\gamma} (-q)^{1-\eta} x^{P_j}_{j_{\gamma+\eta} j_\delta} x^{P_i}_{i_\alpha j_{\gamma+\eta}} + (-q)^{\gamma-|P_j|} x^{P_i}_{i_\alpha j_\delta}.$$

Theorem 3.35. *Let P be a partition of n with $P_i \in P$.*

For $i_\alpha, i_\gamma \in P_i$ with $i_\alpha < i_\gamma$ and $i_\beta, i_\delta \in P_{>i}$ with $i_\beta < i_\delta$, the following relations hold in $O_q(N^+_{>P})$:

$$x^{P_i}_{i_\alpha i_\beta} x^{P_i}_{i_\alpha i_\delta} = q x^{P_i}_{i_\alpha i_\delta} x^{P_i}_{i_\alpha i_\beta} \tag{3.37}$$

$$x^{P_i}_{i_\alpha i_\beta} x^{P_i}_{i_\gamma i_\beta} = q^{-1} x^{P_i}_{i_\gamma i_\beta} x^{P_i}_{i_\alpha i_\beta} \tag{3.38}$$

$$x^{P_i}_{i_\alpha i_\beta} x^{P_i}_{i_\gamma i_\delta} = x^{P_i}_{i_\gamma i_\delta} x^{P_i}_{i_\alpha i_\beta} \tag{3.39}$$

$$x^{P_i}_{i_\alpha i_\delta} x^{P_i}_{i_\gamma i_\beta} = x^{P_i}_{i_\gamma i_\beta} x^{P_i}_{i_\alpha i_\delta} - \widehat{q} x^{P_i}_{i_\alpha i_\beta} x^{P_i}_{i_\gamma i_\delta}. \tag{3.40}$$

Let $P_j \in P$ with $i < j$ let $i_\alpha \in P_i$ with $i_\beta \in P_{>i}$ and $j_\gamma \in P_j$ with $j_\delta \in P_{>j}$. The following relations hold in $O_q(N^+_{>P})$:

$$x^{P_i}_{i_\alpha i_\beta} x^{P_j}_{j_\gamma j_\delta} = x^{P_j}_{j_\gamma j_\delta} x^{P_i}_{i_\alpha i_\beta} \text{ for } i_\beta \in P_{<j} \tag{3.41}$$

$$x^{P_i}_{i_\alpha i_\beta} x^{P_j}_{j_\gamma j_\delta} = x^{P_j}_{j_\gamma j_\delta} x^{P_i}_{i_\alpha i_\beta} \text{ for } i_\beta \in P_j \text{ but } i_\beta \neq j_\gamma \tag{3.42}$$

$$x^{P_i}_{i_\alpha i_\beta} x^{P_j}_{j_\gamma j_\delta} = q^{-1} x^{P_j}_{j_\gamma j_\delta} x^{P_i}_{i_\alpha i_\beta} + q^{-1} \widehat{q} x^{P_j\, P_i}_{(i_\alpha j_\delta), j_\gamma} \text{ for } i_\beta \in P_j \text{ and } i_\beta = j_\gamma \tag{3.43}$$

$$x^{P_i}_{i_\alpha i_\beta} x^{P_j}_{j_\gamma j_\delta} = x^{P_j}_{j_\gamma j_\delta} x^{P_i}_{i_\alpha i_\beta} + \widehat{q} x^{P_j}_{j_\gamma i_\beta} x^{P_i}_{i_\alpha j_\delta} \text{ for } i_\beta \in P_{>j} \text{ and } i_\beta < j_\delta \tag{3.44}$$

$$x^{P_i}_{i_\alpha i_\beta} x^{P_j}_{j_\gamma j_\delta} = q x^{P_j}_{j_\gamma j_\delta} x^{P_i}_{i_\alpha i_\beta} \text{ for } i_\beta \in P_{>j} \text{ and } i_\beta = j_\delta \tag{3.45}$$

$$x^{P_i}_{i_\alpha i_\beta} x^{P_j}_{j_\gamma j_\delta} = x^{P_j}_{j_\gamma j_\delta} x^{P_i}_{i_\alpha i_\beta} \text{ for } i_\beta \in P_{>j} \text{ and } i_\beta > j_\delta. \tag{3.46}$$

Proof. For $i_\alpha, i_\gamma \in P_i$ with $i_\alpha < i_\gamma$ and $i_\beta, i_\delta \in P_{>i}$ with $i_\beta < i_\delta$ by Theorem 3.11, D_{P_i} commutes with $M^{P_i}_{i_\alpha i_\beta}$, $M^{P_i}_{i_\alpha i_\gamma}$,$M^{P_i}_{i_\gamma i_\beta}$, and $M^{P_i}_{i_\gamma i_\delta}$ Hence, relations (3.37) – (3.40) follow from (3.26) – (3.29).

Now, for $P_i, P_j \in P$ with $i < j$ and $i_\alpha \in P_i$ with $i_\beta \in P_{>i}$ and $j_\gamma \in P_j$ with $j_\delta \in P_{>j}$ by Theorem 3.11 and relation (3.5) we have that D_{P_i} commutes with $M^{P_j}_{j_\gamma j_\delta}$ and D_{P_j} commutes with $M^{P_i}_{i_\alpha i_\beta}$. Moreover, by Theorem 3.8, D_{P_i} commute with D_{P_j}. Hence, relations (3.41) follows from (3.30) and relations (3.44) – (3.46) follow from (3.33) – (3.35).

(i) For relation (3.42) using relations (3.31), (3.3), and (3.4) along with Theorem 3.8 we have

$$\begin{aligned}
x_{i_\alpha i_\beta}^{P_i} x_{j_\gamma j_\delta}^{P_j} &= D_{P_i}^{-1} M_{i_\alpha i_\beta}^{P_i} D_{P_j}^{-1} M_{j_\gamma j_\delta}^{P_j} \\
&= q^{-1} D_{P_i}^{-1} D_{P_j}^{-1} M_{i_\alpha i_\beta}^{P_i} M_{j_\gamma j_\delta}^{P_j} \\
&= q^{-1} D_{P_j}^{-1} D_{P_i}^{-1} M_{i_\alpha i_\beta}^{P_i} M_{j_\gamma j_\delta}^{P_j} = D_{P_j}^{-1} D_{P_i}^{-1} M_{j_\gamma j_\delta}^{P_j} M_{i_\alpha i_\beta}^{P_i} \\
&= D_{P_j}^{-1} M_{j_\gamma j_\delta}^{P_j} D_{P_i}^{-1} M_{i_\alpha i_\beta}^{P_i} = x_{j_\gamma j_\delta}^{P_j} x_{i_\alpha i_\beta}^{P_i}.
\end{aligned}$$

(ii) For relation (3.43) using relations (3.32), (3.3), and (3.4) along with Theorem 3.8 we have

$$\begin{aligned}
x_{i_\alpha i_\beta}^{P_i} x_{j_\gamma j_\delta}^{P_j} &= D_{P_i}^{-1} M_{i_\alpha i_\beta}^{P_i} D_{P_j}^{-1} M_{j_\gamma j_\delta}^{P_j} = q^{-1} D_{P_i}^{-1} D_{P_j}^{-1} M_{i_\alpha i_\beta}^{P_i} M_{j_\gamma j_\delta}^{P_j} \\
&= q^{-1} D_{P_j}^{-1} D_{P_i}^{-1} M_{i_\alpha i_\beta}^{P_i} M_{j_\gamma j_\delta}^{P_j} \\
&= q^{-1} D_{P_j}^{-1} D_{P_i}^{-1} \left(M_{j_\gamma j_\delta}^{P_j} M_{i_\alpha i_\beta}^{P_i} + \widehat{q} M_{(i_\alpha j_\delta), j_\gamma}^{P_j\ P_i} \right) \\
&= q^{-1} D_{P_j}^{-1} D_{P_i}^{-1} M_{j_\gamma j_\delta}^{P_j} M_{i_\alpha i_\beta}^{P_i} + q^{-1} \widehat{q} D_{P_j}^{-1} D_{P_i}^{-1} M_{(i_\alpha j_\delta), j_\gamma}^{P_j\ P_i} \\
&= q^{-1} D_{P_j}^{-1} M_{j_\gamma j_\delta}^{P_j} D_{P_i}^{-1} M_{i_\alpha i_\beta}^{P_i} + q^{-1} \widehat{q} D_{P_j}^{-1} D_{P_i}^{-1} M_{(i_\alpha j_\delta), j_\gamma}^{P_j\ P_i} \\
&= q^{-1} x_{j_\gamma j_\delta}^{P_j} x_{i_\alpha i_\beta}^{P_i} + q^{-1} \widehat{q} x_{(i_\alpha j_\delta), j_\gamma}^{P_j\ P_i}. \qquad \square
\end{aligned}$$

Action of $O_q(L_P)$ on $O_q(N^+_{>P})$

Recall that there is a natural right $O_q(L_P)$ action on $O_q(N^+_{>P})$ given by

$$x.Y = \sum_{(Y)} r^+(S_L(Y_1))xr^+(Y_2)$$

for $Y \in O_q(L_P)$ and $x \in O_q(N^+_{>P})$.

Theorem 3.36. *Let P be a partition of n with $P_k, P_i, P_j \in P$ with $k < i < j$. For $i_\alpha, i_\gamma, i_\delta \in P_i$ and $i_\beta, \in P_{>i}$ and $k_\lambda, k_\mu \in P_k$ and $j_\sigma, j_\theta \in P_j$, the right $O_q(L_P)$ action on $O_q(N^+_{>P})$ satisfies the following:*

$$x^{P_i}_{i_\alpha i_\beta}.Y_{k_\lambda k_\mu} = \delta_{k_\lambda k_\mu} x^{P_i}_{i_\alpha i_\beta} \tag{3.47}$$

$$x^{P_i}_{i_\alpha i_\beta}.Y_{i_\gamma i_\delta} = \delta_{i_\gamma i_\delta} x^{P_i}_{i_\alpha i_\beta} \text{ for } i_\alpha \neq i_\delta \tag{3.48}$$

$$x^{P_i}_{i_\alpha i_\beta}.Y_{i_\gamma i_\delta} = q^{-1}\delta_{i_\gamma i_\delta} x^{P_i}_{i_\alpha i_\beta} + \widehat{q}\sum_{\epsilon=1}^{\delta-1} q^{\epsilon-\delta}\delta_{i_\gamma i_\epsilon} x^{P_i}_{i_\epsilon i_\delta} \text{ for } i_\alpha = i_\delta \tag{3.49}$$

$$x^{P_i}_{i_\alpha i_\beta}.Y_{j_\sigma j_\theta} = \delta_{j_\sigma j_\theta} x^{P_i}_{i_\alpha i_\beta} \text{ for } i_\beta \in P_{<j} \tag{3.50}$$

$$x^{P_i}_{i_\alpha i_\beta}.Y_{j_\sigma j_\theta} = \delta_{j_\sigma j_\theta} x^{P_i}_{i_\alpha i_\beta} + \widehat{q}\delta_{j_\sigma i_\beta} x^{P_i}_{i_\alpha j_\theta} \text{ for } i_\beta \in P_j \text{ and } i_\beta < j_\theta \tag{3.51}$$

$$x^{P_i}_{i_\alpha i_\beta}.Y_{j_\sigma j_\theta} = q\delta_{j_\sigma j_\theta} x^{P_i}_{i_\alpha i_\beta} \text{ for } i_\beta \in P_j \text{ and } i_\beta = j_\theta \tag{3.52}$$

$$x^{P_i}_{i_\alpha i_\beta}.Y_{j_\sigma j_\theta} = \delta_{j_\sigma j_\theta} x^{P_i}_{i_\alpha i_\beta} \text{ for } i_\beta > j_\theta. \tag{3.53}$$

Proof. From Theorem (3.31) and from the comultiplication (3.7) we have

$$
\begin{aligned}
x^{P_i}_{i_\alpha i_\beta}.Y_{k_\lambda k_\mu} &= \sum_{k_\eta \in P_k} r^+(S_L(Y_{k_\lambda k_\eta}))D^{-1}_{P_i}M^{P_i}_{i_\alpha i_\beta}r^+(Y_{k_\eta k_\mu}) \\
&= \sum_{k_\eta \in P_k} r^+(S_L(Y_{k_\lambda k_\eta}))D^{-1}_{P_i}M^{P_i}_{i_\alpha i_\beta}X_{k_\eta k_\mu} \\
&= \sum_{k_\eta \in P_k} r^+(S_L(Y_{k_\lambda k_\eta}))X_{k_\eta k_\mu}D^{-1}_{P_i}M^{P_i}_{i_\alpha i_\beta} \\
&= \sum_{k_\eta \in P_k} r^+(S_L(Y_{k_\lambda k_\eta}))r^+(Y_{k_\eta k_\mu})D^{-1}_{P_i}M^{P_i}_{i_\alpha i_\beta} \\
&= r^+(\epsilon_L(Y_{k_\lambda k_\mu}))x^{P_i}_{i_\alpha i_\beta} = \delta_{k_\lambda k_\mu}x^{P_i}_{i_\alpha i_\beta}.
\end{aligned}
$$

If $i_\alpha \neq i_\delta$, from equations (3.17) and (3.18) and from the comultiplication (3.7) we have

$$
\begin{aligned}
x^{P_i}_{i_\alpha i_\beta}.Y_{i_\gamma i_\delta} &= \sum_{i_\eta \in P_i} r^+(S_L(Y_{i_\gamma i_\eta}))D^{-1}_{P_i}M^{P_i}_{i_\alpha i_\beta}r^+(Y_{i_\eta i_\delta}) \\
&= \sum_{i_\eta \in P_i} r^+(S_L(Y_{i_\gamma i_\eta}))D^{-1}_{P_j}M^{P_i}_{i_\alpha i_\beta}X_{i_\eta i_\delta} \\
&= \sum_{i_\eta \in P_i} r^+(S_L(Y_{i_\gamma i_\eta}))X_{i_\eta i_\delta}D^{-1}_{P_i}M^{P_i}_{i_\alpha i_\beta} \\
&= \sum_{i_\eta \in P_i} r^+(S_L(Y_{i_\gamma i_\eta}))r^+(Y_{i_\eta i_\delta})D^{-1}_{P_i}M^{P_i}_{i_\alpha i_\beta} \\
&= r^+(\epsilon_L(Y_{i_\gamma i_\delta}))x^{P_i}_{i_\alpha i_\beta} = \delta_{i_\gamma i_\delta}x^{P_i}_{i_\alpha i_\beta}.
\end{aligned}
$$

If $i_\alpha = i_\delta$, from equations (3.16) and (3.18) and from the comultiplication (3.7) we have

$$\begin{aligned}
x^{P_i}_{i_\alpha i_\beta}.Y_{i_\gamma i_\delta} &= \sum_{i_\eta \in P_i} r^+(S_L(Y_{i_\gamma i_\eta}))D^{-1}_{P_i} M^{P_i}_{i_\alpha i_\beta} X_{i_\eta i_\delta} \\
&= \sum_{i_\eta \in P_i} r^+(S_L(Y_{i_\gamma i_\eta}))D^{-1}_{P_i} \left(q^{-1} X_{i_\eta i_\delta} M^{P_i}_{i_\alpha i_\beta} + \widehat{q} \sum_{\epsilon=1}^{\delta-1} q^{\epsilon-\delta} X_{i_\eta i_\epsilon} M^{P_i}_{i_\epsilon i_\beta} \right) \\
&= \sum_{i_\eta \in P_i} \left(q^{-1} r^+(S_L(Y_{i_\gamma i_\eta})) X_{i_\eta i_\delta} D^{-1}_{P_i} M^{P_i}_{i_\alpha i_\beta} \right) \\
&+ \sum_{i_\eta \in P_i} \left(\widehat{q} \sum_{\epsilon=1}^{\delta-1} q^{\epsilon-\delta} r^+(S_L(Y_{i_\gamma i_\eta})) X_{i_\eta i_\epsilon} D^{-1}_{P_i} M^{P_i}_{i_\epsilon i_\beta} \right) \\
&= \sum_{i_\eta \in P_i} q^{-1} r^+(S_L(Y_{i_\gamma i_\eta})) X_{i_\eta i_\delta} D^{-1}_{P_i} M^{P_i}_{i_\alpha i_\beta} \\
&+ \widehat{q} \sum_{\epsilon=1}^{\delta-1} q^{\epsilon-\delta} \sum_{i_\eta \in P_i} r^+(S_L(Y_{i_\gamma i_\eta})) X_{i_\eta i_\epsilon} D^{-1}_{P_i} M^{P_i}_{i_\epsilon i_\beta} \\
&= r^+(\epsilon_L(Y_{i_\gamma i_\delta})) q^{-1} D^{-1}_{P_i} M^{P_i}_{i_\alpha i_\beta} + \widehat{q} \sum_{\epsilon=1}^{\delta-1} q^{\epsilon-\delta} r^+(\epsilon_L(Y_{i_\gamma i_\epsilon})) D^{-1}_{P_i} M^{P_i}_{i_\epsilon i_\beta} \\
&= \delta_{i_\gamma i_\delta} q^{-1} x^{P_i}_{i_\alpha i_\beta} + \widehat{q} \sum_{\epsilon=1}^{\delta-1} q^{\epsilon-\delta} \delta_{i_\gamma i_\epsilon} x^{P_i}_{i_\epsilon i_\delta}.
\end{aligned}$$

If $i_\beta \in P_{<j}$, using equations (3.21) and (3.25) and from the comultiplication (3.7) we have

$$\begin{aligned}
x^{P_i}_{i_\alpha i_\beta}.Y_{j_\sigma j_\theta} &= \sum_{j_\eta \in P_j} r^+(S_L(Y_{j_\sigma j_\eta}))D^{-1}_{P_i} M^{P_i}_{i_\alpha i_\beta} X_{j_\eta j_\theta} = \sum_{j_\eta \in P_j} r^+(S_L(Y_{j_\sigma j_\eta})) X_{j_\eta j_\theta} D^{-1}_{P_i} M^{P_i}_{i_\alpha i_\beta} \\
&= r^+(\epsilon_L(Y_{j_\sigma j_\theta}))D^{-1}_{P_i} M^{P_i}_{i_\alpha i_\beta} = \delta_{j_\sigma j_\theta} x^{P_i}_{i_\alpha i_\beta}.
\end{aligned}$$

If $i_\beta \in P_j$ and $i_\beta < j_\theta$, using equations (3.22) and (3.25) and from the comultiplication (3.7) we have

$$\begin{aligned}
x^{P_i}_{i_\alpha i_\beta}.Y_{j_\sigma j_\theta} &= \sum_{j_\eta \in P_j} r^+(S_L(Y_{j_\sigma j_\eta}))D^{-1}_{P_i} M^{P_i}_{i_\alpha i_\beta} X_{j_\eta j_\theta} \\
&= \sum_{j_\eta \in P_j} r^+(S_L(Y_{j_\sigma j_\eta}))D^{-1}_{P_i} \left(X_{j_\eta j_\theta} M^{P_i}_{i_\alpha i_\beta} + \widehat{q} X_{j_\eta i_\beta} M^{P_i}_{i_\alpha j_\theta} \right) \\
&= \sum_{j_\eta \in P_j} \left(r^+(S_L(Y_{j_\sigma j_\eta}))X_{j_\eta j_\theta} D^{-1}_{P_i} M^{P_i}_{i_\alpha i_\beta} + \widehat{q} r^+(S_L(Y_{j_\sigma j_\eta}))X_{j_\eta i_\beta} D^{-1}_{P_i} M^{P_i}_{i_\alpha j_\theta} \right) \\
&= r^+(\epsilon_L(Y_{j_\sigma j_\theta}))D^{-1}_{P_i} M^{P_i}_{i_\alpha i_\beta} + \widehat{q} r^+(\epsilon_L(Y_{j_\sigma i_\beta}))D^{-1}_{P_i} M^{P_i}_{i_\alpha j_\theta} \\
&= \delta_{j_\sigma j_\theta} x^{P_i}_{i_\alpha i_\beta} + \widehat{q} \delta_{j_\sigma i_\beta} x^{P_i}_{i_\alpha j_\theta}.
\end{aligned}$$

If $i_\beta \in P_j$ and $i_\beta = j_\theta$, using equations (3.23) and (3.25) and from the comultiplication (3.7) we have

$$\begin{aligned}
x^{P_i}_{i_\alpha i_\beta}.Y_{j_\sigma j_\theta} &= \sum_{j_\eta \in P_j} r^+(S_L(Y_{j_\sigma j_\eta}))D^{-1}_{P_i} M^{P_i}_{i_\alpha i_\beta} X_{j_\eta j_\theta} = \sum_{j_\eta \in P_j} q r^+(S_L(Y_{j_\sigma j_\eta}))X_{j_\sigma j_\theta} D^{-1}_{P_i} M^{P_i}_{i_\alpha i_\beta} \\
&= q r^+(\epsilon_L(Y_{j_\sigma j_\theta}))D^{-1}_{P_i} M^{P_i}_{i_\alpha i_\beta} = q \delta_{j_\sigma j_\theta} x^{P_i}_{i_\alpha i_\beta}.
\end{aligned}$$

If $i_\beta > j_\theta$, using equations (3.24) and (3.25) and from the comultiplication (3.7) we have

$$\begin{aligned}
x^{P_i}_{i_\alpha i_\beta}.Y_{j_\sigma j_\theta} &= \sum_{j_\eta \in P_j} r^+(S_L(Y_{j_\sigma j_\eta}))D^{-1}_{P_i} M^{P_i}_{i_\alpha i_\beta} X_{j_\eta j_\theta} = \sum_{j_\eta \in P_j} r^+(S_L(Y_{j_\sigma j_\eta}))X_{j_\eta j_\theta} D^{-1}_{P_i} M^{P_i}_{i_\alpha i_\beta} \\
&= r^+(\epsilon_L(Y_{j_\sigma j_\theta}))D^{-1}_{P_i} M^{P_i}_{i_\alpha i_\beta}
\end{aligned}$$

$= \delta_{j_\sigma j_\theta} x^{P_i}_{i_\alpha i_\beta}.\square$

GK-dimension for $O_q(N^+_{>P})$

Having found some relations in $O_q(N^+_{>P})$ we wish to find the GK-dimension for $O_q(N^+_{>P})$. This will help us in showing that $O_q(N^+_{>P})$ is a CGL extension. To find the GK-dimension, we first need to return to $O_q(P^\pm_P)$ and find bounds on the GK-dimension for $O_q(P^\pm_P)$ as well as finding the GK-dimension for $O_q(L_P)$.

We may define an ordering on the generators of $O_q(\mathrm{SL}_n)$ by the following rule: for $X_{ij}, X_{kl} \in O_q(\mathrm{SL}_n)$ we have $X_{kl} <_{\mathrm{rlex}} X_{ij}$ if and only if $k < i$ or $k = i$ and $l > j$. We may similarly define another ordering by $X_{kl} <_{\mathrm{clex}} X_{ij}$ if and only if $l < j$ or $l = j$ and $k > i$. Note that $<_{\mathrm{rlex}}$ and $<_{\mathrm{clex}}$ are in fact well-orderings on the generators of $O_q(\mathrm{SL}_n)$. Let S^+_P be the sequence of generators from I^+_P ordered sequentially using $<_{\mathrm{rlex}}$. Similarly, let S^-_P be the sequence of generators from I^-_P ordered sequentially using $<_{\mathrm{clex}}$. Observe that

$$|S^+_P| = \sum_{i=1}^{k} |P_i||P_{>i}| \qquad \text{and} \qquad |S^-_P| = \sum_{i=1}^{k} |P_i||P_{<i}|. \tag{3.54}$$

Definition 3.37 (Definition 4.1.13 [22])**.** *A sequence of elements $a_1, \ldots, a_N$ in a ring R is a* **normalizing sequence** *if for each $j \in \{0, 1, 2, \ldots, N-1\}$ the image of a_{j+1} in $R/\sum_{i=1}^{j} a_i R$ is normal and $\sum_{i=1}^{N} a_i R \neq R$. The ideal generated by such a sequence is called* **polynormal***.*

Let $R = O_q(\mathrm{SL}_n)$. For each element X_{ij} of S_P^+, let

$$R_{ij}^+ := \sum_{X_{kl} <_{\mathrm{rlex}} X_{ij}} X_{kl} R.$$

We note that $<_{\mathrm{rlex}}$ induces an inclusion on the right ideals R_{ij} from Lemma 3.38. That is, $R_{ij}^+ \subset R_{st}^+$ if and only if $X_{ij} <_{\mathrm{rlex}} X_{st}$. We may similarly define right ideals R_{ij}^- from S_P^- using $<_{\mathrm{clex}}$.

Lemma 3.38. *Let P be a partition of n. The ideals $I_P^{\pm}$ are polynormal. Moreover, $I_P^+ + I_P^-$ is also a polynormal ideal.*

Proof. If $|P| = 1$ then $S_P^{\pm} = \emptyset$. Since $I_P^{\pm} = 0$ the statement holds trivially. Therefore, we suppose that $|P| > 1$.

To show S_P^+ is a normalizing sequence, we need to show that the image of each $X_{ij} \in S_P^+$ is normal in $O_q(\mathrm{SL}_n)/R_{ij}^+$. To do this, it sufficient to show that X_{ij} is normal modulo R_{ij}^+ with respect to each of the generators of $O_q(\mathrm{SL}_n)$.

Let $X_{st} \in O_q(\mathrm{SL}_n)$. If $i \leq s$ and $j \geq t$ or $i \geq s$ and $j \leq t$, then X_{ij} commutes or q-commutes with X_{st} and X_{ij} normalizes X_{st}. Therefore, we only need to consider the case where X_{ij} and X_{st} are "NW" or "SE" from each other. That is, if $i < s$ and $j < t$ or $i > s$ and $j > t$.

If $i < s$ and $j < t$ then $X_{ij}X_{st} = X_{st}X_{ij} + \widehat{q}X_{it}X_{sj}$. In this case we have $X_{it} <_{\mathrm{rlex}} X_{ij}$ and so $X_{it}X_{sj} \in R_{ij}^+$. Similarly, if $i > s$ and $j > t$ then again $X_{st}X_{ij} = X_{ij}X_{st} + \widehat{q}X_{sj}X_{it}$ and $X_{sj} <_{\mathrm{rlex}} X_{ij}$ and so $X_{sj}X_{it} \in R_{ij}^+$. It follows that

X_{ij} is normal modulo R_{ij}^+ with respect to any generator of $O_q(\mathrm{SL}_n)$. Therefore, S^+ is a normalizing sequence and I_P^+ is a polynormal ideal.

A similar proof also works for S_P^- replacing $<_{\text{rlex}}$ with $<_{\text{clex}}$ and R_{ij}^+ with R_{ij}^-.

Finally, denote $S_P^+ \cup S_P^-$ the sequence of elements of S_P^+ followed by those of S_P^-. To show $S_P^+ \cup S_P^-$ is a normalizing sequence, since S_P^+ and S_P^- are already normalizing sequences, it is sufficient to show that the lowest element of S_P^-, X_{n1}, is normal modulo the right ideal generated by S_P^+ to prove the Lemma. However, since X_{n1} is normalizes all the generators of $O_q(\mathrm{SL}_n)$ this is trivial. $\square$

Let $A_P^\pm$ be the right ideals in $O_q(\mathrm{SL}_n)$ generated by $S_P^\pm$.

Proposition 3.39. *Let P be a partition of n. All the $R_{ij}^\pm$ are two-sided ideals. Moreover, the $A_P^\pm$ are two-sided ideals equal to $I_P^\pm$.*

Proof. If $|P| = 1$ then $S_P^\pm = \emptyset$. It follows that the right ideal generated by $S_P^\pm$ and $I_P^\pm$ are both 0 and the theorem holds. Therefore we suppose that $|P| > 1$.

We proceed inductively to show that all the right ideals R_{ij}^+ are two-sided. It is trivial that R_{1n}^+ is a two-sided ideal. Let $X_{i'j'}$ be the immediate successor to X_{ij} in S_P^+ and suppose that R_{ij}^+ is a two-sided ideal. We wish to show that $R_{i'j'}^+$ is also a two-sided ideal. Since $X_{i'j'}$ is the immediate successor of X_{ij} we have $R_{i'j'}^+ = X_{ij}R + R_{ij}^+$ from the definition of $R_{i'j'}^+$. Moreover, it follows from Lemma 3.38 that $RX_{ij} \subseteq X_{ij}R + R_{ij}^+$. Therefore, since R_{ij}^+ is a two-sided ideal we have

$$RR^+_{i'j'} = R(X_{ij}R + R^+_{ij}) \subseteq (RX_{ij})R + R^+_{ij} \subseteq (X_{ij}R + R^+_{ij})R + R^+_{ij} \subseteq R^+_{i'j'}.$$

That is, $R^+_{i'j'}$ is a two-sided ideal. Hence, all the R^+_{ij} are two sided ideals. A similar proof replacing $<_{\text{rlex}}$ with $<_{\text{clex}}$ can also show that the R^-_{ij} are two-sided ideals.

It is clear that

$$A^+_P = X_{\max P_{<k} \max P_k} R + R^+_{\max P_{<k} \max P_k}.$$

In a similar way to above, since $R^+_{\max P_{<k} \max P_k}$ is a two-sided ideal, we have that A^+_P is also a two-sided ideal. A similar proof also shows A^-_P is a two-sided ideal.

Since $A^\pm_P$ are two-sided ideals and since $I^\pm_P$ are the ideals generated by $S^\pm_P$, they are equal. $\square$

Theorem 3.40. *For P a partition of n with $|P| = k$ we have*

$$\begin{aligned} \text{GK.dim}\left(O_q(P^+_P)\right) &\geq (n^2 - 1) - \sum_{i=1}^{k} |P_i||P_{<i}| \\ \text{GK.dim}\left(O_q(P^-_P)\right) &\geq (n^2 - 1) - \sum_{i=1}^{k} |P_i||P_{>i}|. \end{aligned}$$

Moreover,

$$\text{GK.dim}\,(O_q(L_P)) = (n^2-1) - \sum_{i=1}^{k} |P_i||P_{<i}| - \sum_{i=1}^{k} |P_i||P_{>i}|.$$

Proof. Since the quantum determinant D is a central element of $O_q(\mathrm{SL}_n)$ [19, 9.2 Proposition 9], it follows from [22, Theorem 4.1.13] and [1, Corollary II.9.18] that $\text{GK.dim}\,(O_q(\mathrm{SL}_n)) = n^2 - 1$.

By Lemma 3.38 and [22, Theorem 4.1.13] we have

$$\text{height}\,(I_P^{\pm}) \leq \text{height}\,(I_P^{\pm}/A_P^{\pm}) + |S_P^{\pm}|.$$

However, by Proposition 3.39 we have that $A_P^{\pm} = I_P^{\pm}$. Therefore,

$$\text{height}\,(I_P^{-}) \leq \sum_{i=1}^{k} |P_i||P_{<i}| \qquad \text{and} \qquad \text{height}\,(I_P^{+}) \leq \sum_{i=1}^{k} |P_i||P_{>i}|.$$

Since, by [1, Corollary II.9.18] we have that $I_P^{\pm}$ satisfies Tauvel's height formula, i.e.

$$\text{GK.dim}\,(O_q(\mathrm{SL}_n)) = \text{height}\,(I_P^{\pm}) + \text{GK.dim}\,(O_q(\mathrm{SL}_n/I_P^{\pm})),$$

it immediately follows that

$$\text{GK.dim}\,(O_q(P_P^+)) \geq (n^2-1) - \sum_{i=1}^{k} |P_i||P_{<i}|$$
$$\text{GK.dim}\,(O_q(P_P^-)) \geq (n^2-1) - \sum_{i=1}^{k} |P_i||P_{>i}|.$$

Moreover, using Lemma 3.38 and [22, Theorem 4.1.13] we have

$$\text{height}\ (I_P^+ + I_P^-) \leq |S_P^+ \cup S_P^-|.$$

By a similar argument to above, it follows that

$$\text{GK.dim}\,(O_q(L_P)) \geq (n^2-1) - \sum_{i=1}^{k} |P_i||P_{<i}| - \sum_{i=1}^{k} |P_i||P_{>i}|. \tag{3.55}$$

Let ϕ be the map described in Lemma 3.17. Let D_i be the quantum determinant of $O_q(\mathrm{M}_{|P_i|})$. Using Theorem 3.7 it is easy to verify that that $D_1 \otimes \cdots \otimes D_k - 1 \in \ker\phi$. Moreover, since each $O_q(\mathrm{M}_{|P_i|})$ is a domain we have that $D_1 \otimes \cdots \otimes D_k - 1$ is a regular element. Therefore, from [20, Proposition 3.15] we have

$$\text{GK.dim}\,(M/\ker\phi) + 1 \leq \text{GK.dim}\,(M).$$

Since $O_q(M_{|P_i|})$ is a finitely generated iterated skew polynomial ring, it follows from [13, Lemma 2.2] that $\text{GK.dim}\, O_q(M_{|P_i|}) = |P_i|^2$. Hence, [20, Lemma 3.10] implies

$$\text{GK.dim}\,(M) \leq \sum_{i=1}^{k} |P_i|^2.$$

Since $O_q(L_P) \cong M/\ker\phi$ we have

$$\text{GK.dim}\,(O_q(L_P)) \leq \sum_{i=1}^{k} |P_i|^2 - 1.$$

Since

$$\sum_{i=1}^{k} |P_i|^2 = n^2 - \sum_{i=1}^{k} |P_i||P_{<i}| - \sum_{i=1}^{k} |P_i||P_{>i}|$$

it follows that

$$\text{GK.dim}\,(O_q(L_P)) \leq (n^2 - 1) - \sum_{i=1}^{k} |P_i||P_{<i}| - \sum_{i=1}^{k} |P_i||P_{>i}|. \tag{3.56}$$

Combining (3.55) and (3.56) gives us the desired result. $\square$

Corollary 3.41. *Let P be a partition of n. We have that*

$$\text{GK.dim}\,(O_q(N^+_{>P})) \geq \sum_{i=1}^{k} |P_i||P_{>i}|.$$

Proof. Let W be the subspace spanned by the $Y_{i_\alpha i_\beta} \in O_q(L_P)$ and 1. Note that W is a generating subspace for $O_q(L_P)$. Similarly, if V is the subspace spanned by the generators $x^{P_i}_{i_\alpha i_\beta}$ of $O_q(N^+_{>P})$ and 1, then V is a generating subspace for $O_q(N^+_{>P})$. From Theorem 3.36 we have $V.O_q(L_P) \subseteq V$. Moreover, from (3.7) we have $\Delta(W) \subseteq W \otimes O_q(L_P)$. Hence, Lemma 1.25 implies

$$\mathrm{GK.dim}\,(O_q(N^+_{>P})) \geq \mathrm{GK.dim}\,(O_q(P^+_P)) - \mathrm{GK.dim}\,(O_q(L_P)).$$

Theorem 3.40 then gives the desired result. $\square$

If $|P| = 1$ then $O_q(P^{\pm}_P) \cong O_q(\mathrm{SL}_n)$ and therefore the quantum unipotent subgroups are trivial. From now on, unless indicated otherwise, we assume $|P| > 1$.

We may give the set of generators of $O_q(N^+_{>P})$ a total ordering, $<_{\mathrm{N}^+}$ by the following rule: $x^{P_i}_{i_\alpha i_\beta} <_{\mathrm{N}^+} x^{P_j}_{j_\gamma j_\delta}$ if and only if $i < j$, or $i = j$ and $i_\alpha > j_\gamma$, or $i = j$ and $i_\alpha = j_\gamma$ and $i_\beta < j_\delta$.

For $P_j \in P$ with $j_\gamma \in P_j$ and $j_\delta \in P_{>j}$ define

$$R^{P_j}_{j_\gamma j_\delta} := k\left\langle x^{P_i}_{i_\alpha i_\beta} \mid x^{P_i}_{i_\alpha i_\beta} <_{\mathrm{N}^+} x^{P_j}_{j_\gamma j_\delta} \right\rangle \quad \text{and} \quad \overline{R}^{P_j}_{j_\gamma j_\delta} := k\left\langle x^{P_i}_{i_\alpha i_\beta} \mid x^{P_i}_{i_\alpha i_\beta} \leq_{\mathrm{N}^+} x^{P_j}_{j_\gamma j_\delta} \right\rangle.$$

We note that $x^{P_i}_{i_\alpha i_\beta} <_{\mathrm{N}^+} x^{P_j}_{j_\gamma j_\delta}$ if and only if $\overline{R}^{P_i}_{i_\alpha i_\beta} \subset \overline{R}^{P_j}_{j_\gamma j_\delta}$. It follows that all the algebras $\overline{R}^{P_j}_{j_\gamma j_\delta}$ form an ascending chain of subalgebras of $O_q(N^+_{>P})$. Denote this chain by $\overline{R}_P$. We note that if $\overline{R}_i = \overline{R}^{P_j}_{j_\gamma j_\delta}$ is the i^{th} term in the chain, then $\overline{R}_{i-1} = R^{P_j}_{j_\gamma j_\delta}$ and

$$\overline{R}_i = \overline{R}_{i-1}\left\langle x^{P_j}_{j_\gamma j_\delta} \right\rangle.$$

Consequently, the length of this chain is $\sum_{P_i \in P} |P_i||P_{>i}|$. We also note that $\overline{R}_1$ is a polynomial ring in one variable and the highest element in the chain is $O_q(N^+_{>P})$.

Lemma 3.42. *Let P be a partition of n. Let $P_j \in P$ and $j_\gamma \in P_j$ and $j_\delta \in P_{>j}$. For each generator $x^{P_i}_{i_\alpha i_\beta}$ of $R^{P_j}_{j_\gamma j_\delta}$ there exist scalars $\lambda^{P_i}_{i_\alpha, i_\beta}$ so that*

$$x^{P_j}_{j_\gamma j_\delta} x^{P_i}_{i_\alpha i_\beta} - \lambda^{P_i}_{i_\alpha, i_\beta} x^{P_i}_{i_\alpha i_\beta} x^{P_j}_{j_\gamma j_\delta} \in R^{P_j}_{j_\gamma j_\delta}.$$

Proof. Let $x^{P_i}_{i_\alpha i_\beta}$ be a generator of $R^{P_j}_{j_\gamma j_\delta}$. We note that for $i < j$ and $i_\alpha \in P_i$ that $x^{P_i}_{i_\alpha j_\delta} \in R^{P_j}_{j_\gamma j_\delta}$. Moreover, for $\eta \in \{1, \dots, |P_j| - \gamma\}$ we have that $x^{P_j}_{j_{\gamma+\eta} j_\delta}, x^{P_i}_{i_\alpha j_{\gamma+\eta}} \in R^{P_j}_{j_\gamma j_\delta}$. It follows that $x^{P_j\ P_j}_{(i_\alpha j_\delta), j_\gamma} \in R^{P_j}_{j_\gamma j_\delta}$. Therefore, by (3.43) we have

$$x^{P_j}_{j_\gamma j_\delta} x^{P_i}_{i_\alpha i_\beta} - q x^{P_i}_{i_\alpha i_\beta} x^{P_j}_{j_\gamma j_\delta} \in R^{P_j}_{j_\gamma j_\delta}.$$

For $i_\alpha, i_\beta \in P_i$ with $i_\alpha > j_\gamma$ and $i_\beta < j_\delta$ we have that $x^{P_i}_{j_\gamma i_\beta}, x^{P_i}_{i_\alpha j_\delta} \in R^{P_j}_{j_\gamma j_\delta}$. Therefore, by (3.40) we have $x^{P_j}_{j_\gamma j_\delta} x^{P_i}_{i_\alpha i_\beta} - x^{P_i}_{i_\alpha i_\beta} x^{P_j}_{j_\gamma j_\delta} \in R^{P_j}_{j_\gamma j_\delta}$.

For $i < j$ with $i_\alpha \in P_i$ and $i_\beta \in P_{>j}$ with $i_\beta < j_\delta$ we have that $x^{P_i}_{i_\alpha j_\delta}, x^{P_j}_{j_\gamma i_\beta} \in R^{P_j}_{j_\gamma j_\delta}$. Therefore, by (3.44) we have $x^{P_j}_{j_\gamma j_\delta} x^{P_i}_{i_\alpha i_\beta} - x^{P_i}_{i_\alpha i_\beta} x^{P_j}_{j_\gamma j_\delta} \in R^{P_j}_{j_\gamma j_\delta}$.

Finally, for all other cases it follows from Theorem 3.35 that there exist scalars $\lambda^{P_i}_{i_\alpha, i_\beta}$ so that

$$x^{P_j}_{j_\gamma j_\delta} x^{P_i}_{i_\alpha i_\beta} - \lambda^{P_i}_{i_\alpha, i_\beta} x^{P_i}_{i_\alpha i_\beta} x^{P_j}_{j_\gamma j_\delta} = 0 \in R^{P_j}_{j_\gamma j_\delta}. \qquad \square$$

Theorem 3.43. *Let P be a partition of n. We have that*

$$\text{GK.dim}\left(O_q(N^+_{>P})\right) = \sum_{i=1}^{k} |P_i||P_{>i}|.$$

Moreover, for all $P_j \in P$ and $j_\gamma \in P_j$ and $j_\delta \in P_{>j}$ we have

$$\text{GK.dim}\left(\overline{R}^{P_j}_{j_\gamma j_\delta}\right) = 1 + \text{GK.dim}\left(R^{P_j}_{j_\gamma j_\delta}\right).$$

Proof. We note that using Lemma 3.42, the proof of [22, Proposition 8.6.7] can be modified to show that

$$\text{GK.dim}\left(\overline{R}^{P_j}_{j_\gamma j_\delta}\right) \leq 1 + \text{GK.dim}\left(R^{P_j}_{j_\gamma j_\delta}\right).$$

Hence, each subalgebra in the chain $\overline{R}_P$ has GK-dimension at most one more than the subalgebra immediately preceding it. Since the lowest element of the chain $\overline{R}_P$ is a polynomial ring in one variable, the GK-dimension of this algebra is 1. Moreover, since the highest element of the chain is $O_q(N^+_{>P})$, it follows that the $1 \leq \text{GK.dim}\,(O_q(N^+_{>P}) \leq \sum_{i=1}^{k} |P_i||P_{>i}|$. By, Corollary 3.41 we then have that $\text{GK.dim}\,(O_q(N^+_{>P})) = \sum_{i=1}^{k} |P_i||P_{>i}|$. It immediately follows that

$$\text{GK.dim}\left(\overline{R}^{P_j}_{j_\gamma j_\delta}\right) = 1 + \text{GK.dim}\left(R^{P_j}_{j_\gamma j_\delta}\right). \qquad \square$$

H-action on $O_q(N^+_{>P})$

Lemma 3.44. *Let P be a partition of n. The H-action on $O_q(\mathrm{SL}_n)$ from* (1.25) *induces a rational H-action by algebra automorphisms on $O_q(P^+_P)$. Moreover, this action has the property that*

$$(u,v).D_{P_i} = \left(\prod_{i_\lambda \in P_i} u_{i_\lambda} v_{i_\lambda}\right) D_{P_i}$$
$$(u,v).M^{P_i}_{i_\alpha i_\beta} = \left(\prod_{i_\lambda \in P_i} u_{i_\lambda} \prod_{\substack{i_\mu \in P_i \\ i_\mu \neq i_\alpha}} v_{i_\mu}\right) v_{i_\beta} M^{P_i}_{i_\alpha i_\beta}$$

for $(u,v) \in H$ and for all $P_i \in P$ with $i_\alpha \in P_i$ and $i_\beta \in P_{>i}$.

Proof. We first note that $I^\pm_P$ is generated by H-eigenvectors. It follows that there are induced actions of H on $O_q(P^\pm_P)$. From [1, Theorem II.2.7], the induced action H-action on $O_q(P^\pm_P)$ is rational.

If $|P_i| = k$, we have

$$\begin{aligned}
(u,v).D_{P_i} &= (u,v).[P_i \mid P_i] = (u,v).\left(\sum_{\sigma \in \mathrm{Sym}_k} (-q)^{\ell(\sigma)} X_{i_1 i_{\sigma(1)}} \cdots X_{i_k i_{\sigma(k)}}\right) \\
&= \sum_{\sigma \in \mathrm{Sym}_k} (-q)^{\ell(\sigma)} u_{i_1} v_{i_{\sigma(1)}} X_{i_1 i_{\sigma(1)}} \cdots u_{i_k} v_{i_{\sigma(k)}} X_{i_k i_{\sigma(k)}} \\
&= \sum_{\sigma \in \mathrm{Sym}_k} \left(\prod_{j=1}^{k} u_{i_j} v_{i_{\sigma(j)}}\right) (-q)^{\ell(\sigma)} X_{i_1 i_{\sigma(1)}} \cdots X_{i_k i_{\sigma(k)}}.
\end{aligned}$$

However, since for any $\sigma \in \mathrm{Sym}_k$

$$\prod_{j=1}^{k} u_{i_j} v_{i_{\sigma(j)}} = \prod_{j=1}^{k} u_{i_j} v_{i_j} = \prod_{i_\lambda \in P_i} u_{i_\lambda} v_{i_\lambda}$$

we have

$$\sum_{\sigma \in \mathrm{Sym}_k} \left(\prod_{j=1}^{k} u_{i_j} v_{i_{\sigma(j)}} \right) (-q)^{\ell(\sigma)} X_{i_1 i_{\sigma(1)}} \cdots X_{i_k i_{\sigma(k)}} = \left(\prod_{i_\lambda \in P_i} u_{i_\lambda} v_{i_\lambda} \right) D_{P_i}.$$

In the same way we also have,

$$(u, v).M^{P_i}_{i_\alpha i_\beta} = \left(\prod_{i_\lambda \in P_i} u_{i_\lambda} \prod_{\substack{i_\mu \in P_i \\ i_\mu \neq i_\alpha}} v_{i_\mu} \right) v_{i_\beta} M^{P_i}_{i_\alpha i_\beta}.$$ □

Proposition 3.45. *Let P be a partition of n. For $P_i \in P$ with $i_\alpha \in P_i$ and $i_\beta \in P_{>i}$ then*

$$(u, v).x^{P_i}_{i_\alpha i_\beta} = v^{-1}_{i_\alpha} v_{i_\beta} x^{P_i}_{i_\alpha i_\beta}$$

for all $(u, v) \in H$.

Proof. From Lemma 3.44 it follows that

$$(u, v).D^{-1}_{P_i} = \left(\prod_{i_\lambda \in P_i} u_{i_\lambda} v_{i_\lambda} \right)^{-1} D^{-1}_{P_i}.$$

Therefore, we have

$$\begin{aligned}(u,v).x^{P_i}_{i_\alpha i_\beta} &= (u,v).D^{-1}_{P_i} M^{P_i}_{i_\alpha i_\beta} \\ &= \left(\prod_{i_\lambda \in P_i} u_{i_\lambda} v_{i_\lambda}\right)^{-1} D^{-1}_{P_i} \left(\prod_{i_\lambda \in P_i} u_{i_\lambda} \prod_{\substack{i_\mu \in P_i \\ i_\mu \neq i_\alpha}} v_{i_\mu}\right) v_{i_\beta} M^{P_i}_{i_\alpha i_\beta} \\ &= v^{-1}_{i_\alpha} v_{i_\beta} D^{-1}_{P_i} M^{P_i}_{i_\alpha i_\beta} = v^{-1}_{i_\alpha} v_{i_\beta} x^{P_i}_{i_\alpha i_\beta}. \qquad \square\end{aligned}$$

Let P be a partition of n. For each $h \in H$ define the map $\tau_h : O_q(N^+_{>P}) \to O_q(P^+_P)$ by

$$\tau_h(x) := h.x \tag{3.57}$$

for all $x \in O_q(N^+_{>P})$. Since the H-action is by algebra homomorphisms, τ_h is an algebra homomorphism. Since each generator of $O_q(N^+_{>P})$ is an H-eigenvector, τ_h is an automorphism of $O_q(N^+_{>P})$.

For $P_j \in P$ and $j_\gamma \in P_j$ and $j_\delta \in P_{>j}$ define $h^{P_j}_{j_\gamma j_\delta} = (u,v) \in H$ where

$$u_k := \begin{cases} q^{-2} & \text{if } k = j_\gamma \\ 1 & \text{if } k = j_\delta \\ q^{-1} & \text{otherwise} \end{cases} \qquad \text{and} \qquad v_k := \begin{cases} q^{2} & \text{if } k = j_\gamma \\ 1 & \text{if } k = j_\delta \\ q & \text{otherwise.} \end{cases}$$

We note that the restriction of $\tau_{h^{P_j}_{j_\gamma j_\delta}}$ to $R^{P_j}_{j_\gamma j_\delta}$ is an automorphism of $R^{P_j}_{j_\gamma j_\delta}$. We denote this restriction by $\tau^{P_j}_{j_\gamma j_\delta}$. Using Proposition 3.45 it is straightforward to verify for each generator $x^{P_i}_{i_\alpha i_\beta} \in R^{P_j}_{j_\gamma j_\delta}$ that

$$\tau^{P_j}_{j_\gamma j_\delta}(x^{P_i}_{i_\alpha i_\beta}) = \begin{cases} q^{-1}x^{P_i}_{i_\alpha i_\beta} \text{ if } i = j \text{ and } i_\alpha = j_\gamma \\ q^{-1}x^{P_i}_{i_\alpha i_\beta} \text{ if } i = j \text{ and } i_\beta = j_\delta \\ x^{P_i}_{i_\alpha i_\beta} \quad \text{ if } i = j \text{ and } i_\alpha > j_\gamma \text{ and } i_\beta > j_\delta \\ x^{P_i}_{i_\alpha i_\beta} \quad \text{ if } i = j \text{ and } i_\alpha > j_\gamma \text{ and } i_\beta < j_\delta \\ x^{P_i}_{i_\alpha i_\beta} \quad \text{ if } i < j \text{ and } i_\beta \in P_{<j} \\ x^{P_i}_{i_\alpha i_\beta} \quad \text{ if } i < j \text{ and } i_\beta \in P_j \text{ but } i_\beta \neq j_\gamma \\ qx^{P_i}_{i_\alpha i_\beta} \quad \text{ if } i < j \text{ and } i_\beta \in P_j \text{ and } i_\beta = j_\gamma \\ x^{P_i}_{i_\alpha i_\beta} \quad \text{ if } i < j \text{ and } i_\beta \in P_{>j} \text{ and } i_\beta < j_\delta \\ q^{-1}x^{P_i}_{i_\alpha i_\beta} \text{ if } i < j \text{ and } i_\beta \in P_{>j} \text{ and } i_\beta = j_\delta \\ x^{P_i}_{i_\alpha i_\beta} \quad \text{ if } i < j \text{ and } i_\beta \in P_{>j} \text{ and } i_\beta > j_\delta. \end{cases} \tag{3.58}$$

Define the map $\delta^{P_j}_{j_\gamma j_\delta}$ on $R^{P_j}_{j_\gamma j_\delta}$ by

$$\delta^{P_j}_{j_\gamma j_\delta}(x) := x^{P_j}_{j_\gamma j_\delta}x - \tau^{P_j}_{j_\gamma j_\delta}(x)x^{P_j}_{j_\gamma j_\delta} \tag{3.59}$$

for all $x \in R^{P_j}_{j_\gamma j_\delta}$. Since $\tau^{P_j}_{j_\gamma j_\delta}$ is defined using the H-action, it is straightforward to check that $\delta^{P_j}_{j_\gamma j_\delta}$ is a $\tau^{P_j}_{j_\gamma j_\delta}$-derivation and from the relations of Theorem 3.35 that

$$\delta^{P_j}_{j_\gamma j_\delta}(x^{P_i}_{i_\alpha i_\beta}) = \begin{cases} 0 & \text{if } i = j \text{ and } i_\alpha = j_\gamma \\ 0 & \text{if } i = j \text{ and } i_\beta = j_\delta \\ 0 & \text{if } i = j \text{ and } i_\alpha > j_\gamma \text{ and } i_\beta > j_\delta \\ -\widehat{q}x^{P_i}_{j_\gamma i_\beta}x^{P_i}_{i_\alpha j_\delta} & \text{if } i = j \text{ and } i_\alpha > j_\gamma \text{ and } i_\beta < j_\delta \\ 0 & \text{if } i < j \text{ and } i_\beta \in P_{<j} \\ 0 & \text{if } i < j \text{ and } i_\beta \in P_j \text{ but } i_\beta \neq j_\gamma \\ -\widehat{q}x^{P_j\ P_i}_{(i_\alpha j_\delta),j_\gamma} & \text{if } i < j \text{ and } i_\beta \in P_j \text{ and } i_\beta = j_\gamma \\ -\widehat{q}x^{P_i}_{i_\alpha j_\delta}x^{P_j}_{j_\gamma i_\beta} & \text{if } i < j \text{ and } i_\beta \in P_{>j} \text{ and } i_\beta < j_\delta \\ 0 & \text{if } i < j \text{ and } i_\beta \in P_{>j} \text{ and } i_\beta = j_\delta \\ 0 & \text{if } i < j \text{ and } i_\beta \in P_{>j} \text{ and } i_\beta > j_\delta \end{cases} \tag{3.60}$$

for $x^{P_i}_{i_\alpha i_\beta} \in R^{P_j}_{j_\gamma j_\delta}$.

Having defined an automorphism $\tau^{P_j}_{j_\gamma j_\delta}$ and a $\tau^{P_j}_{j_\gamma j_\delta}$-derivation $\delta^{P_j}_{j_\gamma j_\delta}$ on $R^{P_j}_{j_\gamma j_\delta}$, it should not be surprising that $\overline{R}^{P_j}_{j_\gamma j_\delta}$ is a skew polynomial ring over $R^{P_j}_{j_\gamma j_\delta}$.

Theorem 3.46. *Let P be a partition of n. For $P_j \in P$ and $j_\gamma \in P_j$ and $j_\delta \in P_{>j}$ then*

$$\overline{R}^{P_j}_{j_\gamma j_\delta} = R^{P_j}_{j_\gamma j_\delta}[x^{P_j}_{j_\gamma j_\delta}; \tau^{P_j}_{j_\gamma j_\delta}, \delta^{P_j}_{j_\gamma j_\delta}].$$

Proof. Since $\tau^{P_j}_{j_\gamma j_\delta}$ is an automorphism and $\delta^{P_j}_{j_\gamma j_\delta}$ is a $\tau^{P_j}_{j_\gamma j_\delta}$-derivation, it follows that there exists a skew polynomial ring $R^{P_j}_{j_\gamma j_\delta}[b^{P_j}_{j_\gamma j_\delta}; \tau^{P_j}_{j_\gamma j_\delta}, \delta^{P_j}_{j_\gamma j_\delta}]$. Since $R^{P_j}_{j_\gamma j_\delta}$ has a finite dimensional generating subspace V such that $\tau^{P_j}_{j_\gamma j_\delta}(V) = V$ and $\delta^{P_j}_{j_\gamma j_\delta})(V) \subseteq V^2$, it follows from [13, Lemma 2.2] that

$$\text{GK.dim}\,(R^{P_j}_{j_\gamma j_\delta}[b^{P_j}_{j_\gamma j_\delta}; \tau^{P_j}_{j_\gamma j_\delta}, \delta^{P_j}_{j_\gamma j_\delta}]) = \text{GK.dim}\,(R^{P_j}_{j_\gamma j_\delta}) + 1. \tag{3.61}$$

From (3.59) there exists an algebra homomorphism $\phi : R^{P_j}_{j_\gamma j_\delta}[b^{P_j}_{j_\gamma j_\delta}; \tau^{P_j}_{j_\gamma j_\delta}, \delta^{P_j}_{j_\gamma j_\delta}] \to \overline{R}^{P_j}_{j_\gamma j_\delta}$ which is the identity on $R^{P_j}_{j_\gamma j_\delta}$ and $b^{P_j}_{j_\gamma j_\delta} \mapsto x^{P_j}_{j_\gamma j_\delta}$.

It is clear that ϕ is surjective. Suppose $\ker\phi \neq 0$. In view of (3.61), [20, Proposition 3.15] implies that

$$\text{GK.dim}\,(\overline{R}^{P_j}_{j_\gamma j_\delta}) \leq \text{GK.dim}\,(R^{P_j}_{j_\gamma j_\delta}).$$

Since this contradicts Theorem 3.43, we must have that $\ker\phi = 0$. It follows that ϕ is an isomorphism. $\square$

Properties of $\delta^{P_j}_{j_\gamma j_\delta}$ and $\tau^{P_j}_{j_\gamma j_\delta}$

Lemma 3.47. *Let P be a partition of n. For $P_j \in P$ with $j_\gamma \in P_j$ and $j_\delta \in P_{>j}$ we have*

$$\delta^{P_j}_{j_\gamma j_\delta}\tau^{P_j}_{j_\gamma j_\delta} = q^2\tau^{P_j}_{j_\gamma j_\delta}\delta^{P_j}_{j_\gamma j_\delta}.$$

Proof. Since $\tau^{P_j}_{j_\gamma j_\delta}$ is an automorphism, this is equivalent to $(\tau^{P_j}_{j_\gamma j_\delta})^{-1}\delta^{P_j}_{j_\gamma j_\delta}\tau^{P_j}_{j_\gamma j_\delta} = q^2\delta^{P_j}_{j_\gamma j_\delta}$. Since both sides of the equation are $\tau^{P_j}_{j_\gamma j_\delta}$-derivations, we can apply Lemma 1.28, and therefore need to show equality on the generators of $R^{P_j}_{j_\gamma j_\delta}$. Let $x^{P_i}_{i_\alpha i_\beta}$ a generator of $R^{P_j}_{j_\gamma j_\delta}$. For $i = j$ with $i_\alpha > j_\gamma$ and $i_\beta < j_\delta$, from (3.58) and (3.60)

$$\begin{aligned} q^2\tau^{P_j}_{j_\gamma j_\delta}\delta^{P_j}_{j_\gamma j_\delta}(x^{P_i}_{i_\alpha i_\beta}) &= q^2(-\widehat{q})\tau^{P_j}_{j_\gamma j_\delta}(x^{P_i}_{j_\gamma i_\beta})\tau^{P_j}_{j_\gamma j_\delta}(x^{P_i}_{i_\alpha j_\delta}) \\ &= q^2q^{-2}(-\widehat{q})x^{P_i}_{j_\gamma i_\beta}x^{P_i}_{i_\alpha j_\delta} = \delta^{P_j}_{j_\gamma j_\delta}\tau^{P_j}_{j_\gamma j_\delta}(x^{P_i}_{i_\alpha i_\beta}). \end{aligned}$$

Similarly, for $i < j$ and $i_\beta \in P_{>j}$ then

$$\begin{aligned} q^2\tau^{P_j}_{j_\gamma j_\delta}\delta^{P_j}_{j_\gamma j_\delta}(x^{P_i}_{i_\alpha i_\beta}) &= q^2(-\widehat{q})\tau^{P_j}_{j_\gamma j_\delta}(x^{P_i}_{i_\alpha j_\delta})\tau^{P_j}_{j_\gamma j_\delta}(x^{P_j}_{j_\gamma i_\beta}) \\ &= q^2q^{-2}(-\widehat{q})x^{P_i}_{i_\alpha j_\delta}x^{P_j}_{j_\gamma i_\beta} = \delta^{P_j}_{j_\gamma j_\delta}\tau^{P_j}_{j_\gamma j_\delta}(x^{P_i}_{i_\alpha i_\beta}). \end{aligned}$$

Moreover, for $i < j$ and $i_\beta \in P_j$ and $i_\beta = j_\gamma$ then

$$\begin{aligned}
&q^2\tau^{P_j}_{j_\gamma j_\delta}\delta^{P_j}_{j_\gamma j_\delta}(x^{P_i}_{i_\alpha i_\beta})\\
&\quad = q^2(-\widehat{q})\left(\sum_{\eta=1}^{|P_j|-\gamma}(-q)^{1-\eta}\tau^{P_j}_{j_\gamma j_\delta}(x^{P_j}_{j_{\gamma+\eta} j_\delta})\tau^{P_j}_{j_\gamma j_\delta}(x^{P_i}_{i_\alpha j_{\gamma+\eta}}) + (-q)^{\gamma-|P_j|}\tau^{P_j}_{j_\gamma j_\delta}(x^{P_i}_{i_\alpha j_\delta})\right)\\
&\quad = q^2(-\widehat{q})q^{-1}\left(\sum_{\eta=1}^{|P_j|-\gamma}(-q)^{1-\eta}x^{P_j}_{j_{\gamma+\eta} j_\delta}x^{P_i}_{i_\alpha j_{\gamma+\eta}} + (-q)^{\gamma-|P_j|}x^{P_i}_{i_\alpha j_\delta}\right)\\
&\quad = \delta^{P_j}_{j_\gamma j_\delta}(qx^{P_i}_{i_\alpha i_\beta}) = \delta^{P_j}_{j_\gamma j_\delta}\tau^{P_j}_{j_\gamma j_\delta}(x^{P_i}_{i_\alpha i_\beta}).
\end{aligned}$$

Finally, for any other $x^{P_i}_{i_\alpha i_\beta} \in V$, since $\tau^{P_j}_{j_\gamma j_\delta}(x^{P_i}_{i_\alpha i_\beta})$ is a scalar multiple of $x^{P_i}_{i_\alpha i_\beta}$ we have $q^2\tau^{P_j}_{j_\gamma j_\delta}\delta^{P_j}_{j_\gamma j_\delta}(x^{P_i}_{i_\alpha i_\beta}) = 0$ and $\delta^{P_j}_{j_\gamma j_\delta}\tau^{P_j}_{j_\gamma j_\delta}(x^{P_i}_{i_\alpha i_\beta}) = 0$. □

Proposition 3.48. *Let P be a partition of n. For all $P_j \in P$ and $j_\gamma \in P_j$ and $j_\delta \in P_{>j}$ we have that $\delta^{P_j}_{j_\gamma j_\delta}$ is locally nilpotent on $R^{P_j}_{j_\gamma j_\delta}$.*

Proof. By Lemma 1.32, it is sufficient to show that $(\delta^{P_j}_{j_\gamma j_\delta})^2(x^{P_i}_{i_\alpha i_\beta}) = 0$ for all $x^{P_i}_{i_\alpha i_\beta}$ a generator of $R^{P_j}_{j_\gamma j_\delta}$. If $i = j$ and $i_\alpha > j_\gamma$ and $i_\beta < j_\delta$ then from (3.60) we have

$$\delta^{P_j}_{j_\gamma j_\delta}(x^{P_i}_{j_\gamma i_\beta}x^{P_i}_{i_\alpha j_\delta}) = \tau^{P_j}_{j_\gamma j_\delta}(x^{P_i}_{j_\gamma i_\beta})\delta^{P_j}_{j_\gamma j_\delta}(x^{P_i}_{i_\alpha j_\delta}) + \delta^{P_j}_{j_\gamma j_\delta}(x^{P_i}_{j_\gamma i_\beta})x^{P_i}_{i_\alpha j_\delta} = 0.$$

Similarly, if $i < j$ and $i_\beta \in P_{>j}$ and $i_\beta < j_\delta$ then

$$\delta^{P_j}_{j_\gamma j_\delta}(x^{P_i}_{i_\alpha j_\delta}x^{P_j}_{j_\gamma i_\beta}) = \tau^{P_j}_{j_\gamma j_\delta}(x^{P_i}_{i_\alpha j_\delta})\delta^{P_j}_{j_\gamma j_\delta}(x^{P_i}_{j_\gamma i_\beta}) + \delta^{P_j}_{j_\gamma j_\delta}(x^{P_i}_{i_\alpha j_\delta})x^{P_j}_{j_\gamma i_\beta} = 0.$$

Moreover, if $i < j$ and $i_\beta \in P_j$ with $i_\beta = j_\gamma$ then

$$\begin{aligned}
\delta^{P_j}_{j_\gamma j_\delta}(x^{P_j\ P_i}_{(i_\alpha j_\delta),j_\gamma}) &= \delta^{P_j}_{j_\gamma j_\delta}\left(\sum_{\eta=1}^{|P_j|-\gamma}(-q)^{1-\eta}x^{P_j}_{j_{\gamma+\eta}j_\delta}x^{P_i}_{i_\alpha j_{\gamma+\eta}}\right) + (-q)^{\gamma-|P_j|}\delta^{P_j}_{j_\gamma j_\delta}(x^{P_i}_{i_\alpha j_\delta}) \\
&= \sum_{\eta=1}^{|P_j|-\gamma}(-q)^{1-\eta}(\tau^{P_j}_{j_\gamma j_\delta}(x^{P_j}_{j_{\gamma+\eta}j_\delta})\delta^{P_j}_{j_\gamma j_\delta}(x^{P_i}_{i_\alpha j_{\gamma+\eta}}) + \delta^{P_j}_{j_\gamma j_\delta}(x^{P_j}_{j_{\gamma+\eta}j_\delta})x^{P_i}_{i_\alpha j_{\gamma+\eta}}) \\
&= 0.
\end{aligned}$$

Thus $(\delta^{P_j}_{j_\gamma j_\delta})^2(x^{P_i}_{i_\alpha i_\beta}) = 0$ in the above cases. Since, $\delta^{P_j}_{j_\gamma j_\delta}(x^{P_i}_{i_\alpha i_\beta}) = 0$ for all other $x^{P_i}_{i_\alpha i_\beta}$, we are done. □

$O_q(N^+_{>P})$ is a CGL Extension

Theorem 3.49. *Let P be a partition of n. Let X^+_P be the sequence of generators of $O_q(N^+_{>P})$ ordered sequentially using $<_{\mathrm{N}^+}$. Let $x_i = x^{P_j}_{j_\gamma j_\delta}$ be the i^{th} term in the sequence with $\tau_i = \tau^{P_j}_{j_\gamma j_\delta}$ and $\delta_i = \delta^{P_j}_{j_\gamma j_\delta}$. The ring $O_q(N^+_{>P})$ is a CGL extension equal to*

$$k[x_1][x_2;\tau_2,\delta_2]\dots[x_m;\tau_m,\delta_m].$$

Proof. Let $R_i = R^{P_j}_{j_\gamma j_\delta}$ and $\overline{R}_i = \overline{R}^{P_j}_{j_\gamma j_\delta}$. Since $R_i = \overline{R}_{i-1}$, Theorem 3.46 gives us that $\overline{R}_i = \overline{R}_{i-1}[x_i;\tau_i,\delta_i]$. Since $O_q(N^+_{>P}) = \overline{R}_m$, repeatedly applying the theorem gives us that $O_q(N^+_{>P})$ is the skew polynomial ring above.

Notice that

$$h^{P_j}_{j_\gamma j_\delta}.x^{P_j}_{j_\gamma j_\delta} = q^{-2}x^{P_j}_{j_\gamma j_\delta}.$$

The fact that $O_q(N^+_{>P})$ is a CGL extension follows from this fact, the definition of τ_i, (3.58), Proposition 3.45, and Proposition 3.48. □

Theorem 3.50. *The relations from Theorem 3.35 give a presentation of* $O_q(N^+_{>P})$.

Proof. This is a direct consequence of Theorem 3.49. □

Notice that as $q \to 1$ we recover the relations for the coordinate ring for the unipotent radical of a a standard parabolic group in SL_n.

Definition 3.51. *Let R be a domain. A* **prime element** *in R is any nonzero normal element $p \in R$ such that Rp is a completely prime ideal. A* **noncommutative unique factorization domain (UFD)** *is a domain R such that each nonzero prime ideal of R contains a prime element.*

The fact that $O_q(N^+_{>P})$ is a GCL extension gives us many useful properties for this ring. For instance, from [21, Theorem 3.7] it follows that $O_q(N^+_{>P})$ is a noncommutative UFD. It also follows from [10, Theorem 4.7] that $O_q(N^+_{>P})$ satisfies the Dixmier-Moeglin Equivalence.

In fact, using the isomorphisms from Section 3.4, we have the following result.

Theorem 3.52. *Let P be a partition of n. The rings $O_q(N^\pm_{>P})$ and $O_q(N^\pm_{<P})$ are noncommutative UFDs and satisfy the Dixmier-Moeglin Equivalence.*

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