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ABSTRACT

Apparent ambiguities of definitions of masses and lifetimes of unstable particles that depend either on the introduction of unperturbed Hamiltonians and their eigenstates, or on an assumed correspondence between resonances and elementary fields, are noted. The S-matrix definition is unambiguous; the positions of poles in the first unphysical sheets are given by the zeros of the Fredholm denominator function, which is a function only of an appropriate center-of-mass energy. The mass and lifetime of a particle are consequently independent of the variables of the scattering process or of the particular process to which the particle contributes. The invariance of the Fredholm denominator under charge conjugation, which is a consequence of CPT invariance, ensures the equality of masses and lifetimes of relatively conjugate anti-particles.

Unstable particles are closely akin to stable ones; by the factorization of the residues of unstable-particle poles, unstable-particle scattering functions quite analogous to ordinary scattering functions can be unambiguously defined. Like ordinary scattering functions they are defined only on the mass shell, the fixed masses of the unstable particles being well-defined complex numbers. The needed factorizability of the residue is an immediate consequence of Fredholm's second theorem. The continuation, by means of unitarity, through the multiparticle physical cuts onto unphysical sheets is discussed.

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ON THE MASSES AND LIFETIMES OF UNSTABLE PARTICLES*

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I. INTRODUCTION

Lüders and Zumino[1] have given a proof that the mass and lifetime of a particle are the same as those of its conjugate antiparticle. Their proof is based on an examination of the positions of poles in matrix elements of the formal resolvent of the exact Hamiltonian. The matrix elements used are those corresponding to single-particle eigenstates of an unperturbed Hamiltonian. That such quantities exist, in a rigorous sense, is somewhat doubtful. In field theory exact and unperturbed Hamiltonians appear usually to act in mutually orthogonal subspaces, the action of one Hamiltonian on eigenstates of the other being undefined.[2-6] As a consequence, methods that avoid the use of unperturbed eigenstates are now generally employed in rigorous work. Whether these rigorous methods can provide answers to questions regarding lifetimes of unstable particles is not yet known.

Beyond this technical difficulty there is the practical question of whether the definition of the mass and lifetime introduced by these authors is unique. Does it depend, for instance, on the choice of unperturbed Hamiltonians or on the choice of eigenstate? Zumino has proposed elsewhere[7] that one should use the unperturbed vacuum state rather than the unperturbed one-particle state.

A number of other definitions for the mass and lifetimes of unstable particles have been proposed and studied.[8-15] The most prominent general definition is the one of Peierls, who suggests that the mass and lifetime be

defined by the position of a pole in the one-particle propagator, the two-point Green's function. This definition has the advantage of obviously not depending on an arbitrary separation into parts of the exact Hamiltonian. However, it introduces the very obscure question of the connection between fields and particles: Which field, if any, is it that corresponds to a particular observed resonance? The work of Zimmerman[16] and Nishijima[17] has emphasized that a stable particle need not be associated with a fundamental field, and Schwinger[14] has stressed that a fundamental field need not possess a stable particle. As the situation is probably the same for unstable particles, the entire question of what relations, if any, exist between the basic fields and the observed stable and unstable particles (resonances) becomes an acute basic problem for all of field theory.[18] In the present context, since the Green's function depends on which field is used, the propagator definition apparently becomes ambiguous unless a unique correspondence between unstable particles and fields can be established.

In this paper questions involving the masses and lifetimes of unstable particles are examined in the framework of S-matrix theory. An important virtue of this approach is the direct and unambiguous manner in which unstable particles are treated. That S-matrix theory should be well adapted to the treatment of unstable particles might at first appear surprising, for in S-matrix theory, much more than in field theory, the stable particles have, at the outset, a manifestly preferred status. Yet in spite of this it develops that a natural framework for the discussion of unstable particles is provided, one in which stable and unstable particles are closely related and are treated on a quite similar footing.

In an S-matrix approach the mass and lifetime of an unstable particle are defined by the position of a pole of the S matrix in a center-of-mass energy variable.[19,20]

A first requirement on an acceptable definition of the mass and lifetime of an unstable particle is that it provide unique, unambiguous universal constants that do not depend on the particular process in which the particle occurs, or on other free variables.[21] Accordingly, our first task is to verify that the position in the center-of-mass energy plane of the pole corresponding to an unstable particle depends neither on the other variables of the S matrix, nor on the process in which it occurs. In doing this we shall obtain an equation defining the position of the pole. The use of CPT invariance allows this to be shown invariant under charge conjugation, which proves that the mass and lifetime of a particle are equal to those of its conjugate antiparticle. Further analysis shows a quite close relationship between stable and unstable particles. A general proof of the factorizability of residues of simple poles of the S matrix is given, and the continuation of the S matrix through multiparticle cuts by means of unitarity is discussed.

II. FUNDAMENTAL EQUATIONS

The basis of the analysis will be the S-matrix formalism developed in reference 22. A general process is described by a function $M(K', -\tilde{K}")$, which is a covariant generalization of the scattering function $R = S - 1$. In this formalism the covariant unitarity relation takes the form

$$M(K', -\tilde{K}") + M^*(K", -\tilde{K}') = - \Sigma \int_K M(K', -\tilde{K}) \, K\cdot\tilde{\sigma} \, M^*(K", -\tilde{K}). \tag{2.1}$$

Neglecting for the moment contributions associated with disconnected parts, which are discussed in Section V, we write the M function in the center-of-mass frame in terms of the matrix $M(E_+)$ defined by

$$M(K', -\tilde{K}'')\Big|_{c.m.} = \langle\, \Omega', \Lambda', \Gamma' \,|\, M(E_+) \,|\, \Omega'', \Lambda'', \Gamma'' \,\rangle\, N(E) \qquad (2.2a)$$

where $N(E)$ is the normalization factor,[23]

$$N(E) = [\, (2\pi)^4\, \delta(E' - E)\, \delta(\underset{\sim}{P}')\, (2\pi)^4\, \delta(E'' - E)\, \delta(\underset{\sim}{P}'')\,]^{1/2} . \qquad (2.2b)$$

Here E'' and E' are the total c.m. energies of the initial and final particles, and $\underset{\sim}{P}'$ and $\underset{\sim}{P}''$ are the corresponding total momenta. The symbols Λ, Ω, and Γ represent sets of variables describing spin states, angle variables, and the energies of various subsystems of particles, respectively. The plus on E_+ indicates we are considering $M(E_+)$ to be the limit from above the real energy axis of a function defined there. Type variables have been suppressed on the right of (2.2a).

In terms of the matrix $M(E)$ the unitarity relations take the form

$$M(E_+) + M^\dagger(E_+) = -M(E_+)\, \rho(E_+)\, M^\dagger(E_+) \,, \qquad (2.3)$$

where the dagger represents Hermitian conjugation and $\rho(E)$ is the density-of-states factor defined by

$$\prod_i \frac{d^4 k_i}{(2\pi)^4}\, 2\pi\, \theta(k_i^{\,o})\, 2m\, \delta(k_i^{\,2} - m_i^{\,2})\, K \cdot \tilde{\sigma}$$

$$= d\Omega\, d\Gamma (2\pi)^{-4}\, dE\, d\underset{\sim}{P}\, \rho(E, \Omega, \Gamma) \,, \qquad (2.4)$$

$\rho(E, \Omega, \Gamma)$ being the diagonal elements of the matrix $\rho(E)$. Matrix multiplication is understood to mean integrations over $d\Omega$ and $d\Gamma$, together with sums over the spin states and the various particle combinations (configurations) that are energetically allowed at the specified energy.

The symbol $M(E_-)$ is defined to be the limit from below the real energy axis of the matrix defined there by[24]

$$M(E) = -M^{\dagger}(E^*) \qquad (\mathrm{Im}\, E < 0) . \qquad (2.5)$$

The unitarity relation, Eq. (2.3), can then be written in the alternative form

$$M(E_+) - M(E_-) = M(E_+)\, \rho(E_+)\, M(E_-) . \qquad (2.3a)$$

The matrix $\rho(E)$ has a different form in each interval of the energy axis, a new subspace being added at each threshold energy. We shall represent by $\rho^N(E)$ the expression for $\rho(E)$ that is valid in the Nth interval. The continuation of $M(E_+)$ clockwise through the Nth interval will be represented by $M_N(E)$, and $M_{N-1}(E)$ will represent the counterclockwise continuation of $M(E_-)$ through the Nth interval. In terms of these values on sheets adjacent to the physical sheet Eqs. (2.3) and (2.3a) can be written, for E in the Nth interval, in the forms

$$M(E_+) + M_N^{\dagger}(E_-) = - M(E_+)\, \rho^N(E_+)\, M_N^{\dagger}(E_-) \qquad (2.6)$$

and

$$M(E_+) - M_{N-1}(E_+) = M(E_+)\, \rho^N(E_+)\, M_{N-1}(E_+) . \qquad (2.6a)$$

By virtue of the postulated analyticity[25] of $M(E)$ on the boundary of the physical sheet, these relations, valid for real E, imply that the equations

$$M(E) + M_N^{\dagger}(E^*) = -M(E)\, \rho^N(E)\, M_N^{\dagger}(E^*) \qquad (2.7)$$

and

$$M(E) - M_{N-1}(E) = M(E)\, \rho^N(E)\, M_{N-1}(E) , \qquad (2.7a)$$

considered as functions of the complex variable E, are valid in a neighborhood of the interior points of the Nth interval.

Multiplying Eqs. (2.7) and (2.7a) by $(\rho^N(E))^{1/2}$ on both the right and left, and solving, one obtains, formally,

$$(\rho^N(E))^{1/2}\ M_N^{\dagger}(E^*)\ (\rho^N(E))^{1/2} = \frac{-1}{S^N(E)}\ (\rho^N(E))^{1/2}\ M(E)(\rho^N(E))^{1/2} \tag{2.8}$$

and

$$(\rho^N(E))^{1/2}\ M_{N^{-1}}(E)\ (\rho^N(E))^{1/2} = \frac{1}{S^N(E)}\ (\rho^N(E))^{1/2}\ M(E)\ (\rho^N(E))^{1/2}\ , \tag{2.8a}$$

where the quantity

$$S^N(E) \equiv I^N + (\rho^N(E))^{1/2}\ M(E)\ (\rho^N(E))^{1/2} \tag{2.9}$$

has been introduced. Here I^N is the unit matrix in the subspace where $\rho^N \not\equiv 0$. In terms of $S^N(E)$ alone these equations become

$$(S_N^N(E^*))^{\dagger} = \frac{1}{S^N(E)} \tag{2.10}$$

and

$$\epsilon^{1/2}\ (S_{N^{-1}}^N(E) - I^N)\epsilon^{1/2} = \frac{1}{S^N(E)}\ (S^N(E) - I^N)\ , \tag{2.10a}$$

where we have used the relation

$$(\rho_N^N(E^*))^* = \rho^N(E)\ , \tag{2.11}$$

which follows from the reality of $\rho^N(E)$ in the Nth interval, and the quantity $\epsilon^{1/2}$ defined by

$$\epsilon^{1/2}\ (\rho_{N^{-1}}^N(E))^{1/2} = (\rho^N(E))^{1/2}\ . \tag{2.12}$$

Equation (2.10) is the simple expression of unitarity. The general relativistic kinematic factors for the multiparticle processes are displayed in (2.9). The quantity ϵ is a matrix that gives plus or minus unity when acting on states of odd or even numbers of particles respectively. If only even numbers of particles occur in the configurations associated with an interval N, Eq. (2.10a) simplifies to

$$S_{N-1}^{N}(E) = \frac{1}{S^{N}(E)} \quad , \tag{2.13}$$

a result which is well known.[26] Equation (2.13) says that continuation through an interval associated with only even numbers of particles gives two-sheeted Riemann surfaces.

The sheets obtained by continuing clockwise from the physical sheet through some physical interval of the energy axis will be called first unphysical sheets. These are the unphysical sheets whose points are closest, on the Riemann surface, to the physical points, and whose poles, consequently, if close to the physical interval give the usual resonance effects. The value of $S(E)$ on these sheets is given directly by

$$(S_{N}^{N}(E^{*}))^{\dagger} = S^{N}(E)^{-1} \quad , \tag{2.14}$$

where $S^{N}(E)$ is evaluated on the physical sheet; S^{N} on the first unphysical sheet is the Hermitian adjoint of the inverse of S^{N} at the conjugate point on the physical sheet.

$S^{N}(E)$ will have various cuts on the physical sheet and these will evidently be reflected as cuts on the first unphysical sheets. The only other singularities on the first unphysical sheets will be at points E^{*} for which the inverse of $S^{N}(E)$ fails to exist.

If $S^N(E)$ were a matrix of finite order on discrete indices, the expression for the inverse would be

$$(S^N(E))^{-1} = \frac{\text{adj } S^N(E)}{\det S^N(E)} \quad . \tag{2.15}$$

Both the numerator and denominator, being finite combinations of matrix elements of $S^N(E)$, would be regular at regular points of $S^N(E)$. Hence the only possible singularities of the inverse in a domain of regularity of $S^N(E)$ would be points where the determinant in the denominator vanishes. This determinant could vanish only at isolated points of any domain of regularity of $S^N(E)$ that included the physical points, since otherwise it would vanish identically, which would conflict with unitarity at the physical points. Thus the only singularities in the images of the interior of the physical sheet would be isolated poles. The positions of such a pole, being at a zero of $\det S(E)$, a function of E alone, would be independent of the other variables. It would consequently occur at the same point in any matrix element in which it occurred, not wander about as a function of the remaining variables of S.

Our matrix $S^N(E)$ is generally not of finite order in discrete indices. It is a function of the many continuous variables represented by the sets Ω and Γ. The direct generalization of the expression (2.15), used above, to the case of continuous variables is provided by Fredholm theory. So the immediate task is to verify that the conditions needed for the validity of Fredholm theory are satisfied, and to thereby establish rigorously the analogs of the properties described above.

III. APPLICABILITY OF FREDHOLM THEORY

The standard Fredholm equation is

$$\phi(x) = f(x) + \lambda \int_R K(x,y)\, \phi(y) dy \quad . \tag{3.1}$$

In matrix notation the case $\lambda = -1$ becomes

$$(I + K)\phi \equiv S\,\phi = f \quad , \tag{3.2}$$

the formal solution of which is

$$\phi = (I + K)^{-1} f \equiv S^{-1} f \quad . \tag{3.3}$$

Fredholm theory gives an explicit expression for the inverse operator. This we write as

$$S^{-1} = \frac{\text{adj } S}{\det S} \quad , \tag{3.4}$$

where

$$\det S \equiv I + \text{Tr } K + \frac{(\text{Tr})^2}{2!} K^{(2)} + \frac{(\text{Tr})^3 K^{(3)}}{3!} + \cdots \tag{3.5}$$

and

$$\text{adj } S = I \det S - \left[K + \text{Tr } K^{(2)} + \frac{(\text{Tr})^2}{2!} K^{(3)} + \cdots \right] \quad . \tag{3.6}$$

Here $K^{(n)}$ is an operator in a space that is an n-fold tensor product of copies of the original space. It is defined in terms of the determinant of an n-by-n matrix of K's by the equation

$$\langle x_1, x_2, \cdots, x_n \,|K^{(n)}|\, y_1, y_2, \cdots, y_n \rangle \equiv \begin{vmatrix} \langle x_1 |K| y_1 \rangle \langle x_1 |K| y_2 \rangle \cdots \langle x_1 |K| y_n \rangle \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \langle x_n |K| y_1 \rangle \cdots\cdots\cdots\cdots \langle x_n |K| y_n \rangle \end{vmatrix} \quad . \tag{3.7}$$

The symbol Tr stands for trace, the sum (integral) over equated corresponding initial and final variables.

To apply the formula to our case we make the identification (see Eq. 2.9)

$$K = (\rho^N(E))^{1/2} M(E) (\rho^N(E))^{1/2} . \tag{3.8}$$

A sufficient condition[27] for the existence of the numerator and denominator in (3.4) is the boundedness of K over the region R, which will be chosen to be bounded.

The postulate of maximal analyticity states that $M(E)$ is a meromorphic function of all its variables at all values of these variables corresponding to points on the physical sheet, including its boundary points, except at certain boundary points called singularities required by unitarity, which are the generalized branch points terminating the cuts that bound the physical sheet, and except also at points at which the expressions $k_i(\Omega, \Gamma, E)$ for the individual momentum-energy four-vectors are not regular. The only poles allowed on the physical sheet are those associated with the stable particles. The points of the physical sheet are points in the space of reduced variables $(\Omega', \Gamma', E, \Omega'', \Gamma'')$ such that the set of points $\{ k_i(\Omega', \Gamma', E), k_j(\Omega'', \Gamma'', E)\}$ is a point of the physical sheet constructed in the manner given in reference 22.

Consider first values of E^* such that the points represented by $(\Omega', \Gamma', E, \Omega'', \Gamma'')$ are regular points of the physical sheet for all (real) physical values of the (Ω, Γ) in R. For these E^* the $M(E)$ is analytic hence continuous, over the closed region R. Thus $M(E)$ is bounded over R, and the numerator and denominator on the right of

$$(S_N^N(E^*))^* = (S^N(E))^{-1} = \frac{\text{adj } S^N(E)}{\text{def } S^N(E)} \tag{3.9}$$

are well-defined functions, provided the $(\rho^N(E))^{1/2}$ in

$$S^N(E) = I^N + (\rho^N(E))^{1/2} M(E)(\rho^N(E))^{1/2} \tag{3.10}$$

are bounded over R.

The matrix $\rho^N(E)$ is a tensor product extending over the density-of-states matrices $\rho_c(E)$ of the various configurations that are available in interval N. Specifically, $\rho^N(E)$ is given by

$$\rho^N(E) = \bigotimes_c \theta_c^N \rho_c(E) \equiv P^N \bigotimes_c \rho_c(E) , \tag{3.11}$$

where $\rho_c(E)$ acts in the subspace corresponding to the cth configuration, θ_c^N is unity if the cth configuration is available (energetically allowed) in the Nth interval, otherwise zero, and P^N is the operator that projects onto the subspace of configurations available in the Nth interval. To show the existence of the numerator and denominator of (3.9) it is sufficient to show that the variables Ω and Γ can be chosen so that the physical region R is a bounded region and the corresponding functions $\rho(\Omega, \Gamma, E)$ bounded over R.

The density matrix $\rho(E)$ is easily computed using the relativistic formulas (see Appendix A)

$$\prod_{i=0}^{N} \frac{d^3\underset{\sim}{k}_i}{\omega_i} = \frac{d^3\underset{\sim}{P}}{E} \prod_{n=1}^{N} \frac{d^3\underset{\sim}{q}_n}{\omega_n^r} = \frac{d^3\underset{\sim}{P}}{E} \prod_{n=1}^{N} q_n \, dE_n \, d\Omega_n .$$

Here (k_i, ω_i) are the momentum-energy vectors of the $(N + 1)$ particles, $(\underset{\sim}{P}, E)$ is the total energy-momentum and E_n is the center-of-mass energy of the system Σ_n consisting of particles zero through n. The vector $\underset{\sim}{q}_n$ is the momentum of particle n measured in the rest frame of the system Σ_n, and Ω_n and q_n represent its angles and magnitude. The ω_n^r is the reduced energy

$$\omega_n^{\,r} = \left(\sqrt{q_n^{\,2} + m_n^{\,2}} \quad \sqrt{q_n^{\,2} + (E_{n-1})^2}\right) \Big/ E_n \ ,$$

where

$$E_n = \sqrt{q_n^{\,2} + m_n^{\,2}} + \sqrt{q_n^{\,2} + (E_{n-1})^2} \ .$$

Making the identifications

$$d\Omega \equiv \prod_{n=1}^{N} d\Omega_n \ ,$$

$$d\Gamma = \prod_{n=1}^{N-1} dE_n/2\pi \ ,$$

we have, using the definition (2.4), the relativistic multiparticle phase-space factor

$$\rho = \frac{1}{E_N} \prod_{n=1}^{N} \frac{q_n}{(2\pi)^2} \prod_{i=0}^{N} m_i \, K\cdot\tilde{\sigma} \ .$$

The boundedness of ρ is apparent; in fact it vanishes like q_n on the boundary where the nth relative momentum vanishes and like

$$(E - \Sigma m_i)^{(3(N+1)-5)/2}$$ at the energy threshold.

Inspection of the Fredholm solution shows the uniqueness of the definition of the masses and lifetimes associated with the poles of the type we have been discussing; the zeros of the $\det S^N(E)$ can occur only at isolated points of the domain of analyticity of $S^N(E)$, and the associated poles cannot wander about as functions of the remaining variables. It is, of course, not necessary that the pole be present in every matrix element of M, as there can be compensating zeros in the numerator. Indeed, if the nonvanishing matrix elements of M were to group into small submatrices along

the diagonal, in some representation--as would happen if there were conservation laws--then the vanishing of the determinant associated with a submatrix would produce poles only in this submatrix, at the image point in the unphysical sheet. This accords with the physical expectation that a resonance will appear in all reactions having the same quantum numbers, but not in others.

So far we have considered only those E^* such that for all (Ω, Γ) in R the points $(\Omega', \Gamma', E, \Omega'', \Gamma'')$ are on the physical sheet. The denominator, $\det S^N(E)$, is of course a function of E alone so that a zero of this function persists when Ω and Γ are continued outside of R. (The integrations in Ω and Γ will continue to be over R, of course. It is the free variables in adj S that will be freed.) For the original values of E^* the Fredholm formula will continue to define the function so long as the kernels remain regular. As we continue in the various variables boundary points of the domain of validity of the Fredholm formula will be reached when singularities of the kernels reach the region of integration R. The usual situation will be that a single singularity comes to some interior point of R. But if this occurs one can distort the contour of integration away from the singularity and thereby extend the domain of validity of the Fredholm theory. As remarked by Polkinghorne, the situation is very similar to what he and others have been doing using Feynman type formulas. One can continue to use the Fredholm formula unless a singularity of the kernel comes to an end point of the integration region; or pinches the contour against another of its singularities; or retreats to infinity carrying the contour with it. One would expect that a detailed analysis would show that the limits of the domain of applicability would be just at the second sheet cuts given by the Landau equations, with both first and second type singularities included. But the unstable particle poles, and their associated branch

points, would now come out automatically from the vanishing of the Fredholm denominator and from the pinching of the contours against these poles. The pursuit of these questions is outside the scope of this paper.

There are similar expressions defining the scattering function in the sheets obtained by counterclockwise continuation up from the bottom of the energy cuts [see (2.10a)]. Using the expressions obtained for the values on the various unphysical sheets, one can continue the right-hand side of (3.9) through cuts onto unphysical sheets and establish the existence of $S^N(E)$ on second-order unphysical sheets, and so on.

The methods discussed here allow continuation only through the energy cuts associated with physical processes, where the simple unitarity relations are valid; what types of singularities lurk behind other cuts we do not know. However, it is reasonable to apply the name "particle poles" to the simple poles of the type we have been discussing, which are associated with the vanishing of a Fredholm denominator $\Delta(E)$. If this terminology is adopted the masses and lifetimes of the unstable particles will be fixed constants of nature, just like those of the stable particles. By reversing Eq. (3.9) and considering it an expression for S on the physical sheet in terms of S on the first unphysical sheet, the stable particle poles are seen to be a special case of the particle poles defined above; they correspond to the vanishing of the Fredholm determinant on the first unphysical sheet (see below).

In view of these similarities between stable and unstable particles, the idea of introducing unstable-particle scattering functions presents itself. These functions would, like ordinary scattering functions, be defined only on a manifold consistent with mass and conservation-law constraints, but the masses would now be fixed complex numbers, the fixed masses of the unstable particles.

These generalized scattering functions can be defined by factorization of residues of particle poles. That is, for ordinary M functions the unitarity relations imply that the residue of a stable-particle pole of an M function is a product of two other M functions, provided the class of M functions is extended to include also the three-particle functions usually called coupling constants. Accordingly, let us apply the name "generalized M function" to each of the two factors of the residue of a single-particle pole of any (connected) generalized M function, where a single-particle pole is defined to be the pole associated with a simple zero of a Fredholm denominator $\Delta(E)$.

That the residue of a pole associated with a simple zero of the Fredholm determinant is just a product of two factors is an immediate consequence of Fredholm's second theorem. This theorem says that if $\det S(E_r) = 0$, then for any V' the quantity

$$V = \text{adj } S(E_r)V' \tag{3.12}$$

is a solution of the equation

$$S(E_r)V = 0 \ . \tag{3.13}$$

Moreover, it says that aside from multiplicative constants, this solution is unique, provided $\text{adj } S(E_r)$ is not identically zero. The immediate consequence is factorizability:

$$\langle \alpha \,|\text{adj } S(E_r)|\, \beta \rangle = V_\alpha W_\beta^\dagger \ . \tag{3.14}$$

If $\text{adj } S(E_r)$ were identically zero, then, by virtue of its analyticity in E, each term in $\text{adj } S(E)$ would contain at least one power of $(E - E_r)$. But then if $\det S(E)$ had a simple zero the inverse

$$S^{-1}(E) = \frac{\text{adj } S(E)}{\det S(E)} \tag{3.15}$$

would be regular at $E = E_r$, and there would be no pole.

The argument can be reversed to show that if the residue is factorizable then the determinant of the image point has a simple zero. This justifies the statement made earlier that the stable-particle poles are a special case of "particle poles".

It will be observed that Fredholm theory not only gives a simple proof of factorizability, but also provides an explicit formula for the product of the two factors and hence also, aside from an indeterminant scale factor, for the individual M functions: every M function will be expressed explicitly in terms of others by means of Fredholm formulas. This gives, for example, explicit expressions for coupling constants in terms of the scattering functions at image points. The expressions also provide a basis for the analysis of the properties of the generalized M functions. For instance, from the anti-Hermitian analyticity property of M functions,

$$\langle \alpha \, | M_c(E) | \, \beta \rangle \equiv M_c(\alpha; E; \beta) = -M_c^{\dagger}(\alpha^*; E^*; \beta^*) , \tag{3.16}$$

and the relation

$$\lim_{E \to E_r} (E - E_r) \, M_c(\alpha; E; \beta) \equiv \text{Res } M_c(\alpha; E_r; \beta) \tag{3.17}$$

$$= -iM_1(\alpha; E_r; \gamma) \, \rho \, M_1(\gamma; E_r; \beta) , \tag{3.18}$$

which defines the generalized M function M_1, one concludes that, with the appropriate choice of the indeterminant scale factor, M_1 is also anti-Hermitian analytic. In (3.18) the variable γ corresponds to one particle of mass E_r.

The usual spinor transformation properties are easily established for generalized M functions, which may play an important role in the development of S-matrix theory. We have mentioned them here to emphasize the close kinship of stable and unstable particles.

IV. EQUALITY OF THE MASSES AND LIFETIMES OF CONJUGATE ANTIPARTICLES

The mass and lifetime of a particle are determined by the position of the zero of a Fredholm denominator, $\Delta(E)$. That a particle has the same mass and lifetime as its conjugate antiparticle follows from the invariance of Δ under charge conjugation. This, as will now be shown, is a consequence of invariance under CPT. Consider first the first term in the formula for $\det S^N(E)$. It may be written

$$\mathrm{Tr}\ K(E, T) = \sum_{K'K''} \int M^N(K', -\tilde{K}'')\, K' \cdot \tilde{\sigma}\ \delta(E, T) , \tag{4.1}$$

where

$$\delta(E, T) = N(E)\delta_{T'T}\, \delta_{T''T}\, \delta(\Omega' - \Omega'')\, \delta(\Gamma' - \Gamma'') . \tag{4.2}$$

Here T is a set of type variables and $N(E)$ is defined in (2.2). The CPT identity is, for $T' = T''$,[28]

$$M^N(K', -\tilde{K}'') = M^N(\tilde{K}''_T, -K'_T) , \tag{4.3}$$

where the subscript T on K indicates a transposition of the order of the variables of K. Substitution of (4.3) into (4.1) gives

$$\mathrm{Tr}\ K(E, T) = \sum_{K'K''} \int M^N(\tilde{K}''_T, -K'_T)\, K' \cdot \tilde{\sigma}^{tr}\, \delta(E, T) , \tag{4.4}$$

where the superscript tr on $\tilde{\sigma}^{tr}$ indicates the change of order of spin indices needed to compensate for the reversed order of variables on the two sides of (4.3).

This term, Tr $K(E, T)$, is to be compared to the charge-conjugate expression

$$\text{Tr } K(E, T_c) = \Sigma \int_{K'K''} M^N(K', -\tilde{K}'') \, K' \cdot \tilde{\sigma} \; \delta(E, T_c) \, , \tag{4.5}$$

in which T_c is the set of type variables obtained by changing each type index of the set T to the index specifying the conjugate antiparticle. An equivalent expression is

$$\text{Tr } K(E, T_c) = \Sigma \int_{K'K''} M^N(K''_c, -\tilde{K}'_c) \; K'' \cdot \tilde{\sigma} \; \delta(E, T) \, , \tag{4.6}$$

where K'_c and K''_c have type variable T'_c and T''_c respectively. The fact that K' and K'' are dummy variables has been used to interchange the prime and double prime.

Recalling that $\tilde{K}$ is the set obtained from K by reversing the order of variables, changing all particle-type indices to those specifying the corresponding antiparticle, and dotting the spinor indices, we see that K''_c and $\tilde{K}''_T$ differ only in that their spinor indices are relatively dotted.

The rules for changing the spinor index types on M functions have been derived in SI and SII. The "metric tensors" that effect the changes are

$$\begin{aligned}
g^{\dot{\alpha}\beta} &= (k \cdot \tilde{\sigma}^{\dot{\alpha}\beta})_{m_i}{}^{-1} \, , \\
g_{\beta\dot{\alpha}} &= (k \cdot \sigma_{\beta\dot{\alpha}})_{m_i}{}^{-1} \, , \\
g^{\beta\alpha} &= (C^{-1})^{\beta\alpha} \, , \\
g_{\alpha\beta} &= C_{\alpha\beta} \, , \\
g_{\dot{\alpha}\dot{\beta}} &= C_{\dot{\alpha}\dot{\beta}} \, , \\
g^{\dot{\beta}\dot{\alpha}} &= (C^{-1})^{\dot{\beta}\dot{\alpha}} \, .
\end{aligned} \tag{4.7}$$

When these are contracted with M functions, following the usual contraction rule that upper indices contract with lower, one obtains the M functions with altered index type. The matrices C and C^{-1} are to multiply M from the left; transposed matrices should be used if they multiply from the right. Application of these rules gives

$$M^{N}(\tilde{K}''_{T}, -K'_{T}) = \underset{(K'')}{C}\; K''\cdot\tilde{\sigma}\, M^{N}(K''_{c}, -\tilde{K}'_{c})\, K'\cdot\tilde{\sigma}\; \underset{(K')}{C^{-1}} . \tag{4.8}$$

Here it has been assumed, with no loss of generality, that the original indices of K' and K'' are all lower undotted. Using the relations

$$C^{-1}\; K'\cdot\tilde{\sigma}^{tr}\, C = K'\cdot\sigma \tag{4.9}$$

and

$$K'\cdot\tilde{\sigma}\; K'\cdot\sigma = I , \tag{4.10}$$

one finds that

$$\mathrm{Tr}\; K(E, T) = \mathrm{Tr}\; K(E, T_c) . \tag{4.11}$$

Thus the first term of the series for Δ is invariant under charge conjugation, as a consequence of CPT invariance.

The proof carries over with minor changes to the trace of any power of K(E, T). Since every term in the absolutely convergent series for Δ is a combination of traces of products of powers of K , we obtain the desired result,

$$\Delta(E, T) = \Delta(E, T_c) . \tag{4.12}$$

In the extension of the proof to traces of products of K(E, T), one problem regarding phase factors arises. The general CPT identity is

$$M(K', -\tilde{K}'') = (-1)^{N(K', -\tilde{K}'')} M(-K', \tilde{K}'')$$

$$= (-1)^{N(K', -\tilde{K}'')} \sigma(K', -\tilde{K}'') M(\tilde{K}''_T, -K'_T) , \qquad (4.13)$$

where $N(K)$ is the number of dotted indices of the set K, and $\sigma(K)$ is the phase factor associated with the reversal of order of variables of K. The normal connection between spin and statistics requires $\sigma(K', -\tilde{K}'')$ to be $(-1)^{N(K', -\tilde{K}'')}$ if $T' = T''$. This fact was used in Eq. (4.3). The unitarity relations together with analyticity and the normal connection between spin and statistics require that

$$\sigma(K', -\tilde{K}'') = \sigma^*(K'', -\tilde{K}') \qquad (4.14)$$

and

$$\sigma(K', -\tilde{K}'') = \sigma(K', -\tilde{K}) (-1)^{N(K, -\tilde{K})} \sigma(K, -\tilde{K}'') . \qquad (4.15)$$

These conditions ensure that, in the trace of a product of K's, the factors $(-1)^N$ will just cancel the σ's, as they do for the first power of K.

V. PROBLEM OF THE DISCONNECTED PARTS

Complications due to disconnected parts have been ignored in the foregoing sections. The problem is this: the analyticity postulate states that except for specified singularities the M functions are analytic on their physical sheets, boundaries included. These sheets lie in manifolds constrained by the mass conditions and conservation laws. Since it is possible that certain subsets of particles scatter independently of the rest, there may be terms in an M function, and hence also in the S function, that are restricted not only by the overall conservation law, but also by

additional conservation laws that refer to subsets of particles. The physical sheets for these terms lie in the correspondingly restricted manifold. Stated symbolically, the S function can be decomposed into a sum of terms,

$$S(K) = \sum_p S^p(K) \ , \tag{5.1}$$

where p runs over the possible partitions of the set K. The function $S^p(K)$ is nonzero only in the manifold defined by the vanishing of the sums of the momentum-energy vectors of each of the subsets K^{ip} of the pth partition. The physical sheets referred to in the analyticity postulate are sheets in the manifolds associated with the various $M^p(K)$.

A distinguished partition is the partition into a single set $K^{ip} \equiv K$. The associated $M^p(K)$, called the connected part of $M(K)$, will be denoted by $M_c(K)$. The sum of the remaining terms, called the disconnected parts, will be denoted by $M_d(K)$. In the preceding sections the contributions $M_d(K)$ were systematically ignored.

When the contributions from the disconnected parts are included the original unitarity relation,

$$M(K', -\tilde{K}'') + M^*(K'', -\tilde{K}') = -\sum_K \int M(K', -\tilde{K}) \, K\cdot\tilde{\sigma} \, M^*(K'', -\tilde{K}) \ , \tag{5.2}$$

breaks into a set of equations, the distinguished one of which is

$$M_c + M_c^\dagger = -M_c \, \rho \, M_c^\dagger - M_d \, \rho \, M_c^\dagger - M_c \, \rho \, M_d^\dagger - (M_d \, \rho \, M_d^\dagger)_c \ . \tag{5.3}$$

Here arguments E_+ should be supplied as in Eq. (2.3). The subscript c on the last term means we are to take only the connected parts, the parts whose graphical representation is a connected diagram. The contributions to (5.2) included in (5.3) are those in which there is only the single overall

conservation law; the other terms of (5.2), which correspond to different partitions p of the set K, will vanish except over restricted submanifolds.

In order to proceed, we make the ansatz that the function $S^p(K)$ has the form

$$S^p(K) = \alpha_p(K) \prod_i S_c(K^{ip}) \quad . \tag{5.4}$$

Here $\alpha_p(K)$ is a phase factor that depends on the order of variables and $S_c(K)$, the connected part of $S(K)$, is identical to $M_c(K)$ except for the case in which K contains only two variables. In this case $M_c(K)$ vanishes, whereas $S_c(K) = S_c(K', -\tilde{K}'')$ is the connected no-scattering term

$$S_c(K', -K'') = \frac{(2\pi)^4 \delta^4(k' - k'') \delta_{t't''}}{(2\pi)\, 2m\, \delta(k'^2 - m^2)}$$

$$= \frac{\omega}{m} (2\pi)^3 \delta^3(\underset{\sim}{k}' - \underset{\sim}{k}'') \delta_{t't''} \quad , \tag{5.5}$$

whose value is fixed by the normalization conventions.

The ansatz (5.4) corresponds to the physical assumption that dynamical effects of particles upon one another are associated with momentum-energy transfers. Mathematically, Eq. (5.4) effects a tremendous simplification of the equations, for the contributions to (5.2) that are not in the distinguished part (5.3) become products of the distinguished parts of other equations (5.2), which have the subsets K^{ip} in place of K. Certain consistency requirements are imposed on the α_p. Because of this redundancy of the equations it becomes sufficient to consider only the distinguished or connected part (5.3) of the general unitarity relation (5.2).

For the simple case of continuation through two-particle cuts there are no disconnected parts M_d, in the associated two-particle subspaces, and

the analysis of the previous sections is applicable.

Let us consider continuation through a three-particle cut. The initial and final configuration containing these three particles are now in the associated subspace and there can be disconnected parts of the type represented in Fig. 1.

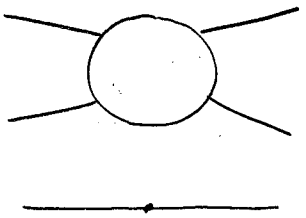

Fig. 1

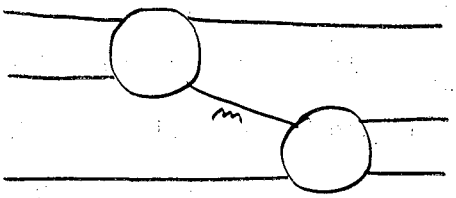

Fig. 2

Also, the term $(M_d \rho M_d^\dagger)_c$ on the right of (5.3) will have contributions of the type represented in Fig. 2. These latter give contributions on the right of (5.3) proportional to $2\pi\delta(m^2 - k^2)$, where m is the mass of the exchanged particle and k is its momentum-energy, expressed in terms of the external momentum vectors.

These mass δ-function contributions on the right can be eliminated by replacing $(M_c + M_c^\dagger)$ on the left by $(M_r + M_r^\dagger)$, where

$$M_c = M_r + M_p \quad . \tag{5.6}$$

The second term on the right, M_p, is a pole contribution, the function obtained by replacing the $2\pi\delta(k^2 - m^2)$ in $(M_d \delta M_d^\dagger)_c$ by $i(k^2 - m^2)^{-1}$. For many-particle M functions, it is only after both the disconnected parts and the pole contributions have been removed from M that the remainder, M_r, is regular in the physical region; the pole contributions are among those required by unitarity and except for the two-particle scattering functions

these pole singularities pass through physical regions.

In terms of the regular part, M_r, the unitarity relation reads

$$M_r + M_r^\dagger = -[\,(M_r + M_p)\,\rho(M_r^\dagger + M_p^\dagger) + M_d\,\rho(M_r^\dagger + M_p^\dagger) + (M_r + M_p)\,\rho\,M_d^\dagger + (M_d\,\rho\,M_d^\dagger)_r\,]\,, \tag{5.7}$$

where $(M_d\,\rho\,M_d^\dagger)_r$ is the part of $(M_d\,\rho\,M_d^\dagger)_c$ that remains after parts associated with pole contributions have been removed. For the case of three intermediate particles all contributions to $(M_d\,\rho\,M_d)_c$ are of the type shown in Fig. 2 and $(M_d\,\rho\,M_d)_r$ is zero.

To solve for $M_r^\dagger$ one first rearranges the terms of (5.7) to give

$$M_r^\dagger + (M_d + M_p + M_r)\rho M_r^\dagger = -(M_r + M_p)\,\rho(M_p^\dagger + M_d^\dagger) + M_d\,\rho\,M_p^\dagger + M_r]\,. \tag{5.8}$$

Multiplication on the right and left by $(\rho^N)^{1/2}$ and the introduction of

$$R^N = (\rho^N)^{1/2}\,M\,(\rho^N)^{1/2}$$

brings this to the form

$$(1 + R_d + R_p + R_r)R_r^\dagger = -[(R_r)(1 + R_p + R_d)^\dagger + R_pR_p^\dagger + R_pR_d^\dagger + R_dR_p^\dagger]\,, \tag{5.9}$$

where the superscript N has been dropped. As before, $R^\dagger$ can be considered a function of E^* in the first unphysical sheet associated with interval N.

On the right-hand side of (5.9) all the quantities are known. In particular, the $R_d^\dagger$ is known on the unphysical sheet as a result of the analysis for two-particle cuts. We assume now that we are considering the leading three-particle branch cut, in which case energy conservation ensures that the energy E_d of the two-particle part of $R_d^\dagger$ (see Fig. 1) is less than the three-particle threshold. Consequently, the continuation in E takes E_d through a two-particle cut. The exact details here depend on the particular way the variables Γ are chosen, but in general the two-particle energies either stay real or move through their two-particle cuts as the three-particle energy is continued through the three-particle cuts. The function $R_p^\dagger$ is known in terms of the two-particle parts R_d and $R_d^\dagger$.

The problem is on the left-hand side. The contributions R_d have, according to (5.1), (5.4), and (2.2), conservation-law δ functions, for only the single overall conservation-law δ function was factored out in passing to the reduced variables. As a consequence R_d, and in fact also R_p, is unbounded, and Fredholm theory is not immediately applicable.

The resolution of the disconnected part problem lies in closer attention to another problem, that of overlapping cuts. When one passes to processes involving more than two particles in the initial or final configurations the physical cut is generally overlaid with various cuts corresponding to subsets of particles interacting among themselves. That is, the singularities required by unitarity will include not only the thresholds corresponding to the total energy variable, indicated in Fig. 3, but also the thresholds associated with subsets of particles, as indicated in Fig. 4. Correspondingly, in the left-

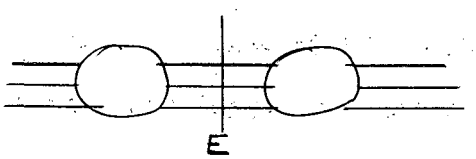

Fig. 3

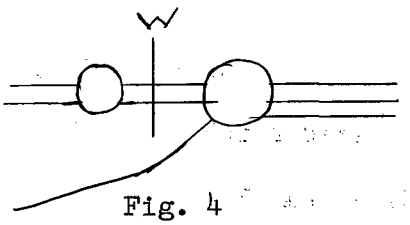

Fig. 4

hand side of the unitarity relation, $M(\alpha; E; \beta) - M(\alpha^*, E^*, \beta^*)$, it is not only the total energy variable E that has cuts separating E from E^*; there will be similar cuts in all the partial energies.

To simplify the discussion suppose we consider the continuation through the one-nucleon, two-meson cut, and consider only the disconnected parts associated with π-π scattering (see Fig. 5). Letting E be the total

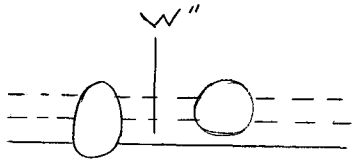

Fig. 5a

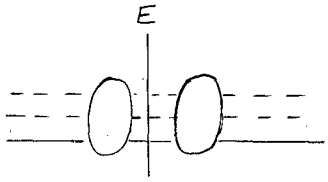

Fig. 5b

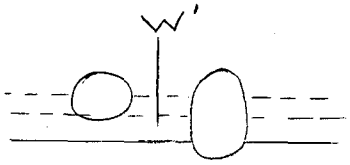

Fig. 5c

energy and W the energy of the π-π subsystem, and suppressing the other variables, one obtains the unitarity relation

$$M_c(W'_+; E_+; W''_+) - M_c(W'_-; E_-; W''_-)$$

$$= \int [M(W'_+; E_+; W_+)\rho\, M(W_-; E_-; W''_-)]_c$$

$$+ \int [M(W'_+; E_+)\rho\, M(E_-, W''_-)]_c , \qquad (5.10)$$

where W'' and W' are the energies of the initial and final meson subsystems. The last term on the right is the two-particle intermediate state contribution. The function $M(W'; E; W'')$ has independent cuts in all three variables, corresponding to the vanishing of the phase space factors in the three-particle and two-particle systems. Dividing the discontinuity into the discontinuities across the various cuts we write the left-hand side of the unitarity equation as a sum of three terms:

$$\Delta_{W''} = M_c(W'_+; E_+; W''_+) - M_c(W'_+; E_+; W''_-) \tag{5.11a}$$

$$\Delta_{E} = M_c(W_+; E_+; W''_-) - M_c(W'_+; E_-; W''_-) \tag{5.11b}$$

$$\Delta_{W'} = M_c(W'_+; E_-; W''_-) - M_c(W'_-; E_-; W''_-) \ . \tag{5.11c}$$

Referring to Fig. 5, we write the right-hand side as a sum of three corresponding terms:

$$D_{W''} = \int M_c(W'_+; E_+; W_+)\rho \ M_d(W''_-) \tag{5.12a}$$

$$D_{E} = \int M_c(W'_+; E_+; W_+)\rho \ M_c(W_-; E_-; W''_-) + \int M_c(W'_+; E_+)\rho \ M_c(E_-; W''_-) \tag{5.12b}$$

$$D_{W'} = \int M_d(W'_+) \rho \ M_c(W_-; E_-; W''_-) \tag{5.12c}$$

Let us suppose, for the moment, that the corresponding terms on the right and left are equal. The equation $\Delta_{W''} = D_{W''}$ gives

$$M_c(W'_+; E_+; W''_-) = \int M_c(W'_+; E_+; W''_+)\rho \, [\rho^{-1} - M_d(W''_-)]$$

$$= \int M_c(W'_+; E_+; W''_+)\rho \, [\rho^{-1} + M_d^{\dagger}(W''_+)] \tag{5.13}$$

and the equation $\Delta_{W'} = D_{W'}$ gives

$$M(W'_+; E_-; W''_-) = \int [\rho^{-1} + M_d(W''_+)] \rho \, M(W'_-; E_-; W''_-) \ . \tag{5.14}$$

These, in conjunction with the unitarity relation

$$[\rho^{-1} + M_d^{\dagger}(W_+)]\rho \, [\rho^{-1} + M_d(W_+)] = \rho^{-1} \ , \tag{5.15}$$

give

$$\int M(W'_+;\, E_+;\, W_+)\rho\ M(W_-;\, E_-;\, W''_-) = \int M(W'_+;\, E_+;\, W_-)\rho\, M(W_+;\, E_-;\, W''_-)\ , \tag{5.16}$$

which allows the equation $\Delta_E = D_E$ to be written in the form

$$M_c(W'_+;\, E_+;\, W''_-) - M_c(W'_+;\, E_-;\, W''_-) = \int M_c(W'_+;\, E_+;\, W_-)\rho\, M_c(W_+;\, E_-;\, W'_-)$$

$$+ \int M_c(W'_+;\, E_+)\rho\, M_c(E_-;\, W'_-)\ . \tag{5.17}$$

The same procedure works for the two-to-three, the three-to-two, and the two-to-two sectors. The result is that if one specifies that the final and initial subsystem energies are always to be taken with positive and negative imaginary parts, respectively, then one obtains the unitarity relations without the disconnected parts.

The above discussion is predicated on the assumption that the corresponding terms in (5.11) and (5.12) are equal. This is essentially the assumption made by Ball, Frazer and Nauenberg[29] in their discussion of disconnected parts and unstable particles. They consider it to be a generalized form of the unitarity condition.

The generalized unitarity relations of Cutkosky have not yet been derived from S-matrix postulates. This is the necessary first task in the development of S-matrix theory into a complete dynamical system. Pending the establishment of these relations the discussion in this section should be considered tentative. However, it may be noted that the equations $\Delta_{W'} = D_{W'}$ and $\Delta_{W''} = D_{W''}$ are just what would be obtained by a continuation of the unitarity relations from the cross channel (below threshold). One must confirm,

by a detailed examination of the singularities required by unitarity, that in this continuation no branch cuts are encountered that invalidate the relations. A preliminary analysis indicates that the continuation is indeed permitted.

So far only the π-π disconnected part has been considered. However, similar analyses can be used to remove the other disconnected parts. Contributions in which two disconnected parts contribute simultaneously on the right, as in Fig. 2, give the pole contributions discussed previously.

A sufficient condition for the applicability of Fredholm theory is the boundness of the kernel. For the many-particle case the kernel may, however, not always be bounded, because various singularities required by unitarity may enter the physical region. The simplest of these are the poles from cross channels already discussed. Also, various anomalous thresholds can enter the physical regions of cross channels. To cover such cases one can use the extension of Fredholm theory to the case of square integrable kernels.[30] For instance, anomalous thresholds associated with triangle diagrams give logarithmic singularities, which are locally square integrable. The pole contributions, however, are not square integrable. A way must be found to deal with these singularities, and with other possible singularities required by unitarity that enter the physical region, before the developments given in this paper can be applied to the poles on unphysical sheets obtained by continuation through multiparticle branch cuts.

ACKNOWLEDGMENT

The idea that the Luders-Zumino result regarding the masses and lifetimes of conjugate particles should be provable in S-matrix theory, which was the origin of this paper, is due to M. DerSarkissian, who participated in the initial stages of the work, and with whom the author had many useful discussions.

APPENDIX A: RELATIVISTIC PHASE-SPACE FACTOR

The phase-space density matrix ρ has been defined by the equation

$$\prod_{i=0}^{N} \left(\frac{d^3 k_i}{(2\pi)^3} \quad \frac{m_i}{\omega_i} \right) K \cdot \tilde{\sigma} = \frac{d^3 P}{(2\pi)^3} \frac{dE}{2\pi} \, d\Omega \, d\Gamma \, \rho(\Omega, \Gamma, E) \quad . \tag{A.1}$$

The coordinates Ω and Γ will be introduced in the following way: Let the subsystem composed of particles zero through n be called the nth subsystem. Let its rest frame and rest mass be denoted by Σ_n and M_n, respectively. Let k_{ij} and K_{ij} denote the momentum vectors of the ith particle and the ith subsystem, respectively, as measured in Σ_j. These quantities are related by the equations

$$\begin{aligned} M_n &= \sqrt{k_{nn}^2 + m_n^2} + \sqrt{K_{n-1,n}^2 + M_{n-1}^2} \\ &= \sqrt{k_{nn}^2 + m_n^2} + \sqrt{k_{nn}^2 + M_{n-1}^2} \quad . \end{aligned} \tag{A.2}$$

The total rest energy is $E = M_N$. For angle variables let us choose the angles describing the N vectors k_{nn} $(n = 1,2,\cdots,N)$, and for the Γ the $N - 1$ quantities M_n $(n = 1,\cdots,N-1)$. These, together with $E = M_N$ and $\tilde{P}$, give the required $3(N + 1)$.

The Jacobian of the transformation is calculated in two steps. First we take the variables $\tilde{k}_i$ $(i = 0,\cdots,N)$ to the variables $\tilde{k}_{ii}$ $(i = 0,\cdots,N)$ with $\tilde{k}_{00}$ defined to be the total momentum $\tilde{P}$. The Jacobian of this transformation is

$$J_1 = \left[\prod_{i=0}^{N} m_i \right] \left[\prod_{n=1}^{N} (\omega_n{}^r) \right]^{-1} M_N^{-1} \quad , \tag{A.3}$$

where the reduced energy of the nth subsystem is

$$\omega_n^r = \frac{\sqrt{k_{nn}^2 + m_n^2}\ \sqrt{k_{nn}^2 + M_{n-1}^2}}{\sqrt{k_{nn}^2 + m_n^2} + \sqrt{k_{nn}^2 + M_{n-1}^2}}$$

$$= \left(\sqrt{k_{nn}^2 + m_n^2}\ \sqrt{k_{nn}^2 + M_{n-1}^2} \right) \Big/ M_n \ . \qquad (A.4)$$

Transforming to these new variables, we have

$$\prod_{n=1}^{N} \left(\frac{k_{nn}\, dk_{nn}^2}{2(2\pi)^3} \right) K\cdot\tilde{\sigma}\ J_1 = \frac{dE}{2\pi}\, d\Gamma\, \rho \equiv \prod_{n=1}^{N} \frac{dM_n}{2\pi}\ \rho \ , \qquad (A.5)$$

where $d\Gamma$ has been interpreted as $\prod_{n=1}^{N-1} (dM_n/2\pi)$. The Jacobian $\partial(M_n)/\partial(k_{nn}^2)$, which is the determinant of a triangular matrix, is

$$J_2 = \prod_{n=1}^{N} \frac{\partial M_n}{\partial k_{nn}^2} \qquad (A.6)$$

$$= \prod_{n=1}^{N} \frac{1}{2} \left(\frac{\sqrt{k_{nn}^2 + m_n^2} + \sqrt{k_{nn}^2 + M_{n-1}^2}}{\sqrt{k_{nn}^2 + m_n^2}\ \sqrt{k_{nn}^2 + M_{n-1}^2}} \right)$$

$$= \prod_{n=1}^{N} \left(\frac{1}{2\omega_n^{red}} \right) = J_1 M_N \left[2^N \prod_{i=0}^{N} m_i \right]^{-1} .$$

Thus we have

$$\rho = \left(\frac{1}{M_N} \prod_{i=0}^{N} m_i \prod_{n=1}^{N} \frac{k_{nn}}{(2\pi)^2} \right) K\cdot\tilde{\sigma} \ . \qquad (A.7)$$

To display the behavior near threshold one may replace the M_n by

$$M'_n = M_n - M_n^{\circ} = M_n - (M_{n-1} + m_n) \simeq k_{nn}^2/2m_n^{red} \tag{A.8}$$

and introduce

$$E' = \sum_{n=1}^{N} M'_n \simeq \sum_{i=1}^{N} \frac{k_{nn}^2}{2m_n^{red}} \, . \tag{A.9}$$

In terms of the dimensionless quantities

$$X_n^2 = M'_n/E' \tag{A.10}$$

we have

$$\rho \prod_{n=1}^{N} \frac{dM'_n}{2\pi} = \frac{1}{M_N} \prod_{i=0}^{N} m_i \prod_{n=1}^{N} \frac{k_{nn}\, dM'_n}{(2\pi)^3} \, K\cdot\tilde{\sigma} \tag{A.11}$$

$$= \rho' \frac{dE'}{2\pi} \prod_{n=1}^{N-1} dX_n^2 \, ,$$

with

$$\rho' \simeq \frac{2\pi}{M_N} \prod_{i=0}^{N} m_i \prod_{n=1}^{N} \left[\frac{(2m_n^{red})^{1/2} X_n}{(2\pi)^3} \right] (E')^{(3(N+1)-5)/2} K\cdot\tilde{\sigma} \, ,$$

$$\simeq 2\pi \prod_{n=1}^{N} \frac{(2m_n^{red})^{3/2} X_n}{2(2\pi)^3} E^{(3(N+1)-5)/2} K\cdot\tilde{\sigma} \, . \tag{A.12}$$

The boundaries of the physical region are at $X_n = 0$, for any n, and at $E = 0$. The X_n are bounded by the condition that $\Sigma X_n^2 = 1$.

FOOTNOTES AND REFERENCES

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19. In the Lee model the S matrix and the propagator are closely connected. Levy has used the connection between the S matrix and a term in the propagator in his general analysis. By using the pole in the S matrix directly one avoids the need to assume the existence of a field associated with the particle.

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21. This has been particularly emphasized by M. Gell-Mann, in Proceedings of the 1958 Annual International Conference on High-Energy Nuclear Physics at CERN (CERN, Geneva, 1958) p. 143.

22. H. P. Stapp, Phys. Rev. 125, 2139 (1962); H. P. Stapp, Lectures on S-Matrix Theory (W. A. Benjamin, 1962). These will be referred to as SI and SII respectively.

23. It is the matrix $M(E)$, considered as a function of its arguments $(\Omega', \Gamma', E, \Omega'', \Gamma'')$ that is, if disconnected contributions are ignored, the finite function with postulated analyticity properties. See Ref. 22.

24. Observe that $M(E)$ in the lower half plane defined by this equation will be analytic if $M(E)$ in the upper half plane is. The choice of sign is dictated by the requirement that the difference $M(E_+) - M(E_-)$ appear on the left of Eq. (2.3a). This requirement is related to maximal analyticity; it is argued in Lecture 12 of SII that if the general requirement of maximal analyticity be that the singularities on the physical sheet (other than single-particle poles) be associated with the vanishing of phase-space factors in the unitarity relation, then Eq. (2.3a)

24. Cont.

must be true also below the various physical thresholds, provided contributions coming from cross channels and anomalous cuts, which can be considered additonal contributions, are ignored. But then below the lowest threshold, where all ρ vanish, we have $M(E_+) = M(E_-)$, and the two functions combine into a single function $M(E)$.

25. See Ref. 22. We assume that the variables Ω and Γ are chosen so that the various $k_i(\Omega, \Gamma, E)$ are analytic functions in the interior of the interval. The choice of variables is discussed later.

26. The continuation of scattering functions through the two-particle branch cut by means of the unitarity relation has been studied by many people: J. Gunson and J. G. Taylor, Phys. Rev. 119, 1121 (1961) and 121, 343 (1961); R. Oehme, Phys. Rev. 121, 1840 (1961); W. Zimmerman, Nuovo cimento 21, 249 (1961); R. Blankenbecler, N. L. Goldberger, S. W. MacDowell, and S. B. Treiman, Phys. Rev. 123, 629 (1961); P. G. O. Freund and R. Karplus, Nuovo cimento 21, 519-531 (1961).

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28. See SI. The normal connection between spin and statistics is also used in (4.3).

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