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Error Analysis and Stochastic Differentiability in Subsurface Flow Modeling

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In the stochastic analysis of steady aquifer flow, the log hydraulic conductivity is the random "input" and the piezometric potential is the random "output." Their joint behavior is governed by a differential equation that, in view of the random nature of its dependent (piezometric potential) and independent (log hydraulic conductivity) variables, represents a stochastic differential equation. The analysis of the distributional properties of the piezometric potential field involves the product of the gradients of log hydraulic conductivity and of the piezometric potential. Previous research in this field assumes that such product of gradients is small in some sense. This paper derives a closed-form expression for the standard deviation, and hence the order of magnitude, of the product of the random gradients of log hydraulic conductivity and of piezometric potential. It was found in this research that for statistically homogeneous log hydraulic conductivity fields (1) the product of the random gradients may or may not have a zero mean, depending on whether the specific discharge is a constant or a random quantity, respectively; (2) under joint normality of the log hydraulic conductivity and the piezometric potential fields, their random gradients are statistically independent if the specific discharge is constant but are dependent when the specific discharge is random; (3) the standard deviation of the product of random gradients is proportional to the variance of log-hydraulic conductivity times a term involving three quantities: the covariance of the piezometric potential, the covariance of the log hydraulic conductivity, and the cross covariance of the latter two fields; (4) a necessary and sufficient condition for the smallness of the product of random gradients is that the second derivatives of the covariance of the log hydraulic conductivity, the covariance of the piezometric potential, and the cross covariance of the latter two random fields be finite and that the variance of log hydraulic conductivity be much less than one. This paper also reviews some fundamental principles on the stochastic analysis of random fields and their importance to the modeling of log hydraulic conductivity fields and to the analysis of subsurface flow. Specifically, the paper highlights the role of Gaussian distributional assumptions in deriving key results of stochastic groundwater flow via perturbation analysis.

INTRODUCTION

Consider steady state flow in an aquifer (tensorial notation is used)

$$\frac{\partial}{\partial x_i}\left(K\frac{\partial \phi}{\partial x_i}\right)=0 \qquad (1)$$

where K is the hydraulic conductivity and ϕ is the piezometric potential. The piezometric potential and the logarithm of hydraulic conductivity (i.e., the log hydraulic conductivity) are expressed as the sum of a mean and a (zero mean) perturbation,

$$\phi = H + h \qquad (2)$$

$$\ln K = F + f \qquad (3)$$

where the lowercase variables represent the perturbations. In a statistically homogeneous aquifer, the log hydraulic conductivity has a constant mean (F) and its perturbation f has an isotropic covariance. Interestingly, even if the log hydraulic conductivity field is statistically homogeneous the piezometric potential is not [*Bakr et al.*, 1978; *Dagan*, 1985].

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In the developments of this paper it will be assumed that the perturbation can be approximated as a statistically homogeneous random field for the purpose of the error analysis (this assumption is consistent with previous studies of stochastic subsurface flow based on the spectral representation of statistically homogeneous fields (see *Gelhar* [1986] for a review) for the piezometric potential). This assumption turns out to be viable as shown by *Dagan* [1985] in an error analysis study based on Fourier transforms of generalized functions. Substitution of the right-hand sides of (2) and (3) into (1), taking the expected value (E) of the resulting expression and subtracting it from (1), yields the stochastic differential equation relating piezometric potential perturbations to log hydraulic conductivity perturbations (F in (3) is assumed constant),

$$\frac{\partial^2 h}{\partial x_i\,\partial x_i}+\frac{\partial f}{\partial x_i}\frac{\partial H}{\partial x_i}+\left\{\frac{\partial f}{\partial x_i}\frac{\partial h}{\partial x_i}-E\left(\frac{\partial f}{\partial x_i}\frac{\partial h}{\partial x_i}\right)\right\}=0 \qquad (4)$$

Equation (4) is the basis for the analysis in this paper. Previous stochastic analyses of aquifer flow have neglected the term within braces in (4) that involves the product of random gradients of f and h (see, for example, the review by *Gelhar* [1986]). It should be indicated that there are alternative approaches to modeling groundwater flow within a stochastic framework besides the perturbation analysis. One such alternative approach is based on functional analysis

[*Serrano et al.*, 1985; *Unny*, 1989]. Another promising approach to stochastic groundwater flow modeling is based on numerical simulation [*Ababou*, 1987]. The latter two approaches have appeared only recently in the hydrologic literature whereas the perturbation analysis has received more attention in groundwater flow analysis.

The purpose of this work is (1) to derive a general, closed-form, expression for the order of magnitude of the product of random gradients of f and h; and (2) to derive necessary and sufficient conditions for the smallness of the product of random gradients. The approach to be followed is first to compute the standard deviation of the product of random perturbations in terms of the covariances and cross covariances of the log hydraulic conductivity and piezometric potential fields (the standard deviation is a measure of the order of magnitude of the product of gradients of f and h). Subsequently, the necessary and sufficient conditions for the smallness of the product of random gradients are established. First, however, a few foundational concepts concerning random gradients and stochastic differentiability and continuity are reviewed.

Some Important Results on Stochastic Differentiability

The random gradients in (4) are meaningful in a mean square sense only, since the formal, deterministic, limit for partial differentiation,

$$\lim_{\delta \to 0} \frac{f(x_1, \cdots, x_j + \delta, \cdots, x_n) - f(x_1, \cdots, x_j, \cdots, x_n)}{\delta}$$

is not defined for a random variable f. Instead, the differentiability of a random field (with finite second moments) requires the concept of mean square convergence. For example, the operation,

$$f_j' \equiv \frac{\partial f}{\partial x_j} \quad (5)$$

$$\lim_{\delta \to 0} E\left\{\frac{f(x_1, \cdots, x_j + \delta, \cdots, x_n) - f(x_1, \cdots, x_j, \cdots, x_n)}{\delta} - f_j'\right\}^2 = 0 \quad (6)$$

means that [*Papoulis*, 1965]

If the limit in (6) exists, then the random field f is said to be stochastically differentiable or differentiable in mean square. It is in the true sense of (6) that random gradients are defined in this work. Based on (6) it is easily shown that a necessary and sufficient condition for stochastic differentiability of a statistically homogeneous random field (f) is that its covariance (σ_{ff}) have the following property [*Priestley*, 1981]

$$\left.\frac{\partial \sigma_{ff}(\tau)}{\partial \tau_j}\right|_{\tau = 0} = 0 \qquad \text{for all } j \quad (7)$$

where

$$\tau = \left(\sum_{j=1}^{n} \tau_j^2\right)^{1/2} \quad (8)$$

$$\tau_j = (x_1 - x_2)_j, \qquad j = 1, 2, \cdots, n \quad (9)$$

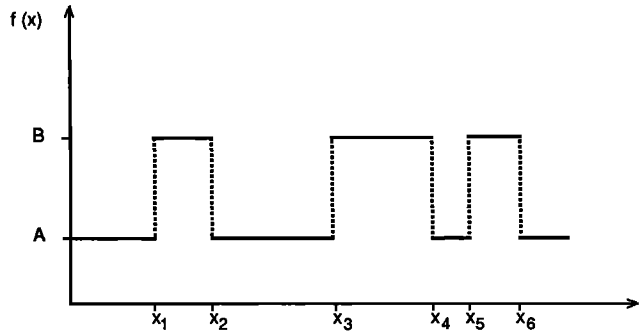

Fig. 1. A two-valued stochastic process.

(τ is the separation distance between vector locations x_1 and x_2 and τ_j is the jth component of the separation vector). For example, the exponential, three-dimensional, and isotropic covariance model for log hydraulic conductivity,

$$\sigma_{ff}(\tau) = \sigma_f^2 \exp(-\tau/\lambda) \quad (10)$$

(λ is the correlation scale and σ_f^2 is the variance of log hydraulic conductivity) satisfies condition (7). On the other hand, the one-dimensional ($n = 1$) case of (10) does not satisfy (7), meaning that the gradient of a (point) one-dimensional random field with exponential covariance is not defined. Because of (7), the covariance models for random fields appearing in stochastic differential equations (such as in (1)) must be properly chosen to ensure compatibility of the random gradients with the definition of mean square differentiability (see (6)). In order to satisfy (7) it usually suffices to average the random field variable over a spatial domain [*Vanmarcke*, 1983]. Otherwise, the choice of admissible covariance models for point random fields may be problematic when dealing with spatial processes governed by differential equations such as (1) that involve derivatives of field variables.

To illustrate the rather abstract nature of the concepts of continuity and differentiability for statistically homogeneous fields, relative to their counterparts for deterministic functions, it is worthwhile to examine the line process in Figure 1. Shown there is a process that takes only two possible values, say A or B, and the distances $\{x_j\}$ at which the transitions occur from a Poisson process (with parameter λ). The process of Figure 1 could be thought of as representing a complex stratified formation with two predominant permeability values, A or B [*Davis*, 1986]. It may be shown (see the appendix) that the process in Figure 1 is statistically homogeneous with an exponential covariance function,

$$\sigma_{ff}(\tau) = \sigma_f^2 \exp(-|\tau|/\nu) \quad (11)$$

where $\nu = 1/2\lambda$ and σ_f^2 is the variance of the two-valued process. The process of Figure 1 is stochastically continuous [*Priestley*, 1981] in a mean square sense, i.e.,

$$\lim_{x \to x_0} E\{f(x) - f(x_0)\}^2 = 0 \quad (12)$$

at any $x = x_0$; yet, f is not stochastically differentiable, since (11) does not satisfy (7). What one sees in Figure 1 is a realization (of the random process f) which, if taken as a deterministic function, would clearly be discontinuous and nondifferentiable in a classical sense. The mean square properties of a random field (continuity and differentiability) do not necessarily coincide with those suggested by the geometric structure of its individual realizations as shown in the example of Figure 1.

Several important issues have been brought up in this section that play a central role in the stochastic analysis of subsurface flow: (1) the role of spatial averaging vis à vis point random processes in relation to the choice of covariance model and the existence of random gradients; and (2) the importance of statistical homogeneity in the analysis of stochastic differential equations, in particular, in the study of gradients of random fields. These two issues are key to the error analysis that follows.

Order of Magnitude of the Product of Random Gradients

Two questions are addressed in this section: (1) what is the expected value of the product $f_i'h_i'$ $(=(\partial f/\partial x_i)(\partial h/\partial x_i))$ in (4)?, and (2) what is the standard deviation of this product of random gradients? The expected value is a measure of central tendency or location, whereas the standard deviation measures dispersion about the mean and, hence, indicates the order of magnitude of the product of random gradients. From the analysis of the second moment of the product of random gradients it is possible to determine conditions for the smallness of this term and, thus, its relative importance in stochastic differential equation (4). It should be noted that *Cushman* [1983] argued that the product of random gradients can be the dominant term in the stochastic differential equation (4).

Expected Value of the Product of Random Gradients

The analysis begins with Darcy's law,

$$K \frac{\partial H}{\partial x_i} = -q_i \tag{13}$$

in which q_i is the specific discharge. Substitution of (2) into (13) and taking expected value yields (observing that expectation and differentiation operations commute within a mean square context)

$$\frac{\partial H}{\partial x_i} = -E(q_i/K) \tag{14}$$

The right-hand side of (14) is worth a close examination. If q_i is a constant, then the right-hand side of (14) can be expressed in terms of the mean and variance of log hydraulic conductivity (see (3), and assuming that hydraulic conductivity is lognormally distributed) so that (14) becomes

$$\frac{\partial H}{\partial x_i} = -q_i/\exp\,(F + \sigma_f^2/2) \tag{15}$$

and it follows at once that

$$\frac{\partial^2 H}{\partial x_i\, \partial x_i} = 0 \tag{16}$$

since the right-hand side of (15) is a constant also. It is straightforward to show that the mean equation corresponding to (1) (and obtained by substituting (2) and (3) into (1) and then taking the expected value of the resulting expression) is given by

$$\frac{\partial^2 H}{\partial x_i\, \partial x_i} + E\left[\frac{\partial f}{\partial x_i}\frac{\partial h}{\partial x_i}\right] = 0 \tag{17}$$

From (16) and (17) it follows that

$$E\left[\frac{\partial f}{\partial x_i}\frac{\partial h}{\partial x_i}\right] = 0 \qquad q_i \text{ constant} \tag{18}$$

and the expected value of the product of random gradients is zero (a result similar to (18) was obtained by *Gelhar and Axness* [1983] using spectral analysis). Notice that since the perturbations f and h have zero mean, the left-hand side of (18) is equal to the cross covariance of the random gradients $(\sigma_{f_i'h_i'})$ evaluated at the origin

$$\sigma_{f_i'h_i'}\big|_{\tau=0} = \sigma_{f_i'h_i'}(0) = 0 \qquad q_i \text{ constant} \tag{19}$$

If the perturbations f and h are jointly normal or Gaussian (in fact f is by assumption normal and h is approximately normal [see *Dagan*, 1985]) then (19) indicates that the random gradients $f_i'(=\partial f/\partial x_i)$ and $h_i'(=\partial h/\partial x_i)$, at any particular point, are independent when the specific discharge is assumed constant.

If the specific discharge q_i in (13) is not a constant but, rather, a random variable with spatial variability, then the right-hand side of (14) depends on the spatial coordinates. In this case it is easily seen from (14) and (17) that

$$E\left[\frac{\partial f}{\partial x_i}\frac{\partial h}{\partial x_i}\right] = E\left\{\frac{\partial}{\partial x_i}(q_i/K)\right\} \qquad q_i \text{ random} \tag{20}$$

and the product of random gradients of the perturbations has nonzero mean. Notice, however, that substitution of (13) for q_i in (20) and using the fact that $\phi = H + h$, where $E[\partial^2 h/\partial x_i^2]$ equals zero, yields

$$E\left\{\frac{\partial f}{\partial x_i}\frac{\partial h}{\partial x_i}\right\} = -E\left\{\frac{\partial^2 H}{\partial x_i\, \partial x_i}\right\} \qquad q_i \text{ random} \tag{21}$$

Equation (21) shows that whenever the gradient of H, $\nabla \cdot H$, is a constant, then the cross covariance of the perturbations f and h at zero would vanish even if the specific discharge q_i is random. Incidentally, under the assumption of jointly normal perturbations, (21) shows that the random gradients of the perturbations f and h are not in general independent of each other when the specific discharge is random. This is a result of a much more intuitive appeal from a physical standpoint than that derived in (18) under the assumption of a constant specific discharge.

The previous analysis has shown that the nature of the specific discharge (i.e., random or constant) is an important determinant of the distributional properties of the product of gradients of the log hydraulic conductivity and piezometric potential. The expected value of the product of random gradients is not in general equal to zero, but rather, its value depends on the nature of the mean gradient of the potentiometric potential. The reader is referred to *Gutjahr and Gelhar* [1981] for a relevant discussion on the role of the nature of the specific discharge in one-dimensional subsurface flow problems.

Standard Deviation of the Product of Random Gradients

The mean or expected value of a random variable is a measure of central tendency: it tells of the likely average value of the random variable. However, the standard deviation is the true measure of the order of magnitude of a random variable since it describes the dispersion or variability of the random variable about the mean. An analysis of the standard deviation of the product of random gradients follows next.

Let the random variable z be defined as follows (the subindex i is now fixed):

$$z = f_i'h_i' \tag{22}$$

The expected value of z is given by either (19) or (21), depending on whether or not the specific discharge is constant. The covariance of the field $z(\sigma_{zz})$ is, by definition (where the subindex i is now fixed),

$$\sigma_{zz}(\tau) = E\{f_i'(x+\tau)h_i'(x+\tau)f_i'(x)h_i'(x)\} - E\{f_i'(x)h_i'(x)\}^2 \tag{23}$$

in which x and τ are position and shift vectors, respectively. We seek to determine the covariance in (23) and then evaluate it at the origin ($\tau = 0$) to obtain the variance (and, therefore, the standard deviation) of z. Unfortunately, the probability distribution of z is hopelessly complicated. The only viable possibility to evaluate the right-hand side of (23) is by assuming that f_i' and h_i' are jointly Gaussian. The Gaussian assumption on the product of gradients, z, will provide us with an approximation to the order of magnitude of $f_i'h_i'$ and the results to be obtained should be interpreted accordingly.

Using a result on the fourth moment about the mean of a bivariate normal distribution [*Anderson*, 1985], it is readily shown that the first term within braces in the right-hand side of (23) (call it T) is given by the following expression,

$$T = \sigma_{f_i'h_i'}(0)^2 + \sigma_{f_i'f_i'}(\tau)\sigma_{h_i'h_i'}(\tau) + \sigma_{f_i'h_i'}(\tau)\sigma_{h_i'f_i'}(\tau) \tag{24}$$

(notice that $\sigma_{f_i'h_i'}(\tau) = \sigma_{h_i'f_i'}(-\tau)$), where $\sigma_{f_i'h_i'}(\tau)$ represents the cross covariance of the gradients f_i' and h_i' separated by a distance, τ,

$$\sigma_{f_i'h_i'}(\tau) = E\left[\left(\frac{\partial f}{\partial x_i}\bigg|_{x+\tau}\right)\left(\frac{\partial h}{\partial x_i}\bigg|_{x}\right)\right] \tag{25}$$

and similar definitions hold for the other covariance terms in the right-hand side of (24). We seek to express the covariances of the random gradients in the right-hand side of (23) and (24) in terms of the second derivatives of the cross covariances of f and h to obtain results more amenable to analysis. For the purpose of illustration, consider the derivation of the cross covariance of the product of gradients. Because of the statistical homogeneity and zero mean assumptions of f and h one can write

$$\sigma_{fh}(\tau) = E[f(x+\tau)h(x)]$$
$$= E[f(x_1+\tau_1, \cdots, x_i+\tau_i, \cdots, x_n+\tau_n)h(x_i, \cdots, x_n)] \tag{26}$$

(where τ is the separation distance defined in (8)). Taking the partial derivative with respect to τ_i in (26) yields

$$\frac{\partial\sigma_{fh}}{\partial\tau_i}(\tau) = E\left[\frac{\partial f}{\partial\tau_i}(x+\tau)h(x)\right] \tag{27}$$

By letting

$$x_i' = x_i + \tau_i \qquad \text{for all } i\ (i = 1, 2, 3) \tag{28}$$

Equation (27) can be rewritten as follows,

$$\frac{\partial\sigma_{fh}(\tau)}{\partial\tau_i} = E\left[\frac{\partial f}{\partial x_i'}(x_1', \cdots, x_i', \cdots, x_n') \cdot h(x_1'-\tau_1, \cdots, x_i'-\tau_i, \cdots, x_n'-\tau_n)\right] \tag{29}$$

which implies, after differentiation with respect to τ_i (and using the chain rule of differentiation), that

$$\frac{\partial\sigma_{fh}(\tau)}{\partial\tau_i\,\partial\tau_i} = -E\left[\left(\frac{\partial f}{\partial x_i'}\bigg|_{x'}\right)\left(\frac{\partial h}{\partial x_i}\bigg|_{x'-\tau}\right)\right] \tag{30}$$

and letting $\tau = 0$ in (30) results in

$$E[f_i'h_i'] = -\sigma''_{f_ih_i}(0) \tag{31}$$

where

$$\sigma''_{f_ih_i}(0) = \frac{\partial\sigma_{fh}(\tau)}{\partial\tau_i\partial\tau_i}\bigg|_{\tau=0} \tag{32}$$

Equation (31) expresses the expected value of the product of random gradients. It is equal to minus the second derivative of the cross covariance of log hydraulic conductivity and piezometric potential. The derivation of (31) has led us, following a completely independent route, to expressions alternative to those presented in (18) and (21). While (31) was obtained from statistical considerations, (18) and (21) were derived from the governing flow equation. A corollary to (18) and (31) is that

$$\sigma''_{f_ih_i}(0) = 0 \tag{33}$$

when the specific discharge (q_i) is constant, and from (21) and (31)

$$\sigma''_{f_ih_i}(0) = E\left\{\frac{\partial^2 H}{\partial x_i\,\partial x_i}\right\} \tag{34}$$

when q_i is random. Equations (33) and (34) provide information about the behavior of the cross covariance of the log hydraulic conductivity and piezometric potential perturbations at the origin ($\tau = 0$).

Operating on each of the terms on the right-hand side of (24) (in a manner similar to that leading to (31)) permits expressing them in terms of the second derivatives of the corresponding cross covariances of the log hydraulic conductivity and piezometric potential perturbations. Evaluating the resulting expression for (24) at $\tau = 0$, substituting it into the right-hand side of (23), and taking the square root yields the standard deviation of the product of random gradients (σ_z),

$$\sigma_z = \{\sigma''_{f_i h_i}(0)^2 + \sigma''_{f_i f_i}(0)\sigma''_{h_i h_i}(0)\}^{1/2} \quad (35)$$

Using a result by *Dagan* [1985] stating that the cross covariances $\sigma_{f_i h_i}$ and $\sigma_{h_i h_i}$ are proportional to the variance of log hydraulic conductivity (σ_f^2), (35) simplifies to

$$\sigma_z = \sigma_f^2\{R''_{f_i h_i}(0)^2 + R''_{f_i f_i}(0)R''_{h_i h_i}(0)\}^{1/2} \quad (36)$$

in which $R_{f_i h_i}$ denotes the cross correlation of f and h and similar definitions hold for $R_{f_i f_i}$ and $R_{h_i h_i}$. *Dagan* [1985] has provided closed-form expressions for the cross correlations appearing in (36).

DISCUSSION AND CONCLUSIONS

Equation (36) constitutes a key result of this work. It follows from that result that the order of magnitude of the product of random gradients is directly proportional to (1) the variance of log hydraulic conductivity and (2) the cross covariance of log hydraulic conductivity and piezometric potential as well as the (auto) covariances of these two random fields. One of the most interesting implications of the error analysis conducted in this work is the parallelism with the findings obtained by *Dagan* [1985] on the behavior of first-order piezometric potential approximations. In our work, the goal was to obtain criteria under which the stochastic differential equation governing the distribution of piezometric potential (see (4)) could be linearized by neglecting the nonlinear terms associated with the product of random gradients. *Dagan* [1985] examined the error involved in using a first-order approximation for the piezometric potential in an asymptotic expansion via infinite power series. *Dagan* [1985] wrote

> In a recent paper, *Gutjahr* [1984] has shown that if the log-conductivity and head fields are jointly normal, the first-order approximation becomes exact and is valid for arbitrary σ_f^2. The present study indicates that this is not the case, in the sense that the first-order approximation has to be supplemented by additional terms. Nevertheless, for small to moderate values of σ_f^2, it is seen that these additional terms may be neglected.

Our result (36) reinforces those statements by *Dagan* [1985] and it unambiguously indicates that (1) the linearization of the stochastic differential equation for steady state flow (see (4)) requires small values of σ_f^2 and small values of the square root term involving the products of log hydraulic conductivity and piezometric potential cross covariance and covariances in (36); and (2) the often quoted condition [*Gutjahr*, 1984]

$$\sigma_f^2 \ll 1 \quad (37)$$

for the smallness of nonlinear terms in (4) is only valid provided that the second derivatives of the cross covariances and covariances of the log hydraulic conductivity and piezometric potential (see (36)) are finite and small also.

A second result of this work concerns the expected value of the product of random gradients. Specifically, it is always zero when the specific discharge is constant (see (18)). Otherwise, the mean of the product of random gradients is nonzero (see (21)) unless the mean piezometric gradient is constant. This result concerning the expected value of the product of random gradients and the role of the specific discharge in a three-dimensional context seems to be novel, although *Gelhar and Axness* [1983] provided a solution to this problem for the case of constant specific discharge in one-dimensional flow. *Gutjahr and Gelhar* [1981] examined the role of the specific discharge (constant or random) in one-dimensional flow. The importance of the expected value of random gradients in the error analysis of stochastic subsurface flow becomes evident when an alternate form of the stochastic differential equation (4) is used, i.e.,

$$\frac{\partial^2 h}{\partial x_i\,\partial x_i} + \frac{\partial^2 H}{\partial x_i\,\partial x_i} + \frac{\partial f}{\partial x_i}\frac{\partial H}{\partial x_i} + \frac{\partial f}{\partial x_i}\frac{\partial h}{\partial x_i} = 0 \quad (38)$$

((38) is obtained by adding the expected value of (1) to (4)). From (38) it is clear that both the expected value and the standard deviation of the product of random gradients determine its magnitude relative to other terms in (38).

A third contribution of this research has been in analyzing the meaning of mean square concepts (in particular, continuity and differentiability) in the context of subsurface flow analysis. Specifically, we have analyzed: (1) the conditions for the existence of a random gradient (7); (2) the role of spatial averaging in the proper selection of covariance models vis à vis the restrictions on the covariance structure of point random fields; and (3) the profound difference between the deterministic continuity and differentiability of a realization of a random field (Figure 1) and the properties of the random field itself. The paper has also shown the role of the Gaussian distributional assumptions in the stochastic analysis of groundwater flow. This is a sine qua non condition to derive closed-form results for the moments of the piezometric potential. The same type of error analysis can also be carried out for transient systems, although at considerably more difficulty and leading to significantly more complex results that those derived from the steady state case.

As a final conclusion, this research points to the need for opening new avenues of nonlinear analysis of stochastic differential equations in subsurface flow studies. The rationale for such a type of analysis arises from two facts: (1) the closed-form results obtained from linear analysis are somewhat limited in revealing the behavior of piezometric potential fields (i.e., limited to covariance characteristic Gaussian fields); and (2) empirical evidence [*Bakr*, 1976; *Delhomme*, 1979; *Hufschmied*, 1985] of sampling studies revealing large conductivity variability ($\sigma_f > 1$) is in little agreement with one of the basic conditions for the validity of first-order or linear analysis of stochastic subsurface flow. The limitations of classical time domain and spectral methods to cope with nonlinear stochastic differential equations suggest future intensive activity in the application of numerical and simulation methods to tackle these kinds of problems.

APPENDIX

In proving (11), the covariance function of the two-valued process represented in Figure 1, let

$$f(x) = \frac{A+B}{2} + \left(\frac{B-A}{2}\right) w_0(-1)^{N(x)} \quad (A1)$$

where w_0 is a binary process such that

$$\begin{aligned} w_0 &= 1 \text{ with probability } 1/2 \\ w_0 &= -1 \text{ with probability } 1/2 \end{aligned} \quad (A2)$$

and $N(x)$ is a Poisson process with parameter λ. Moreover, w_0 and $N(x)$ are independent processes. The expected value of $f(x)$ is easily derived,

$$E[f(x)] = \frac{A + B}{2} \quad \text{(A3)}$$

result (A3) follows from the independence of w_0 and $N(x)$ and the fact that the expected value of w_0 is equal to zero. The covariance of $f(x)$ is defined by

$$\sigma_{ff}(x - s) = E[f(x)f(s)] - E[f(x)]E[f(s)] \qquad x > s \quad \text{(A4)}$$

Substitution of (A1) into (A4), and using the fact that w_0 is zero mean and independent of $N(x)$ yields

$$\sigma_{ff}(x - s) = \left(\frac{B - A}{2}\right)^2 E(w_0)^2 E[(-1)^{N(x)}(-1)^{N(s)}] \quad \text{(A5)}$$

The term in (A5) involving the Poisson process can be conveniently rewritten to use the independence of nonoverlapping increments of a Poisson process [*Parzen*, 1962],

$$E[(-1)^{N(x)}(-1)^{N(s)}] = E[(-1)^{N(x) + N(s) - N(s)}(-1)^{N(s)}]$$

$$= E[(-1)^{N(x) - N(s)}]E[(-1)^{N(s)}]^2 = e^{-2\lambda(x - s)}1 \quad \text{(A6)}$$

Finally, $E(w_0)^2$ is easily verified to be equal to one, which along with result (A6) yields the covariance of f in (A5) (letting $\tau = x - s$ and $\nu = 1/2\lambda$),

$$\sigma_{ff}(\tau) = \left(\frac{B - A}{2}\right)^2 e^{-\tau/\nu} \quad \text{(A7)}$$

which is precisely equal to (11) since

$$\sigma_f^2 = \left(\frac{B - A}{2}\right)^2 E[w_0^2((-1)^{N(t)})^2] = \left(\frac{B - A}{2}\right)^2 E(w_0)^2$$

$$\cdot E[(-1)^{N(t)}]^2 = \left(\frac{B - A}{2}\right)(1)(1) = \left(\frac{B - A}{2}\right)^2 \quad \text{(A8)}$$

completing the proof.

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