Submitted to Journal of
Fluid Mechanics

LBL-4246
Preprint c.1

GRAVITY WAVES ON STEADY WAVE-LIKE AND OTHER NON-UNIFORM FLOWS

David L. Judd

August 29, 1975

Prepared for the U.S. Energy Research and
Development Administration under Contract W-7405-ENG-48

DISCLAIMER

GRAVITY WAVES ON STEADY WAVE-LIKE AND OTHER NON-UNIFORM FLOWS

David L. Judd

Department of Physics and Lawrence Berkeley Laboratory
University of California, Berkeley, California 94720

August 29, 1975

ABSTRACT

A new theory of small-amplitude gravity waves on any given steady two-dimensional deep flow is developed. Rigorous solutions of linearized equations in curvilinear coordinates for quasi-sinusoidal traveling waves are indicated for the dispersion relation, wave amplitude, wave-number, phase and group velocities, wave energy, energy flux, and its spatial change; all are functions of known surface flow speed and equivalent "gravity" reduced from g by surface tilt and modified by centrifugal acceleration. A WKB approximation is justified and employed for all but special classes of small waves traveling on all given Stokes wave-trains, including maximum and near-maximum trains (except at their crests); for these flows analytical models are developed. Spatial change of small-wave energy flux differs greatly, for all wave-like and most other steady flows, from that in the widely-used "radiation stress tensor" theory of Longuet-Higgins and Stewart. It is argued that the linearized results retain quasi-quantitative validity for finite amplitudes; they are used to analyze particle acceleration at a confluence of crests, and to describe in detail a mechanism for triggering whitecapping of large waves by small shorter ones moving up their downwind sides.

TABLE OF CONTENTS

I. INTRODUCTION

The behavior of gravity waves on water has always fascinated the eyes and challenged the reasoning powers of men. Scientific observations and mathematical analyses of their properties have proceeded hand in hand with the general advance of experimental and theoretical science from its beginnings to the present. An impressive body of observational data and theoretical calculations has been accumulated on this aspect of what is now often termed the air-sea interface, but it appears that we have only scratched the surface of many difficult problems which are of practical importance to physical oceanography, meteorology, and other areas of study and application.

An illustration of the incompleteness of our knowledge is provided by the problem of the interaction of two trains of waves having different wavelengths and amplitudes. This problem may be considered an idealization of the interaction problem in a real wind-generated wave field characterized by a statistical distribution of amplitudes, wavelengths, phases, and directions of propagation. Many authors have carried out elaborate calculations to explore these interactions. These include studies linearized with respect to various small quantities; sophisticated higher-order perturbation analyses, sometimes based on techniques, and using notations, borrowed from quantum theory; and interesting new methods appropriate to the general study of nonlinear waves in dispersive media. There is much heavy weather to be encountered in trying to make a safe passage through this sea of papers, yet no one seems to have put forward in simple but elegant form a fully adequate treatment, in linear approximation, of the behavior of a wave train superposed on an arbitrary steady wave-like or other nonuniform flow in deep water. The methods developed here lead to such a treatment and shed new light on this behavior.

The evaluation of energy associated with these wave trains, its propagation, and the exchange of energy between a wave train and a given steady flow (whether or not it represents another train of waves) has occupied many investigators, with frequent reference being made in recent years to the concept of a radiation stress tensor introduced in 1960 by Longuet-Higgins and Stewart. The methods we develop here for two-dimensional flows lead directly (for linearized waves) to expressions for wave energy, energy flux, and energy interchange. We find that the L-H&S radiation stress must be multiplied by a correction factor which tends to zero for small-amplitude wave-wave interactions.

Another example of our incomplete knowledge is provided by the phenomenon of whitecapping or breaking over of waves in deep water. This is obviously the mechanism preventing unlimited growth of wave energy from the wind, and is of fundamental importance in determining the character of wind-generated wave spectra, but its details still do not appear to be completely understood. Dr. William van Dorn of the Scripps Institution of Oceanography told me that he attached importance to the role of small waves of short wavelength traveling, relative to longer waves of near-maximum height, up their downwind sides, in providing a mechanism* for triggering the breaking of the large waves.

* I later became aware of the papers of Phillips (1963), Longuet-Higgins (1969b), and Hasselmann (1971) relating to this mechanism of whitecapping. The contribution of the present work to this problem is principally to provide and justify quantitative expressions for the growth of small short waves near the crests of long high ones, and for their exchange of energy.

An experimental program to observe this type of interaction was proceeding in his laboratory. There did not seem to be an appropriate analytical treatment with which to compare such observations, so the work described herein was begun with the aim of filling this gap, but has led beyond it to more general results.

This work was originally designed to extend and generalize the calculation by Longuet-Higgins and Stewart (1960), henceforth called LS I, of variations of amplitude and wave-number of a train of short waves passing through a train of longer waves. They used expansions in Cartesian coordinates, neglecting terms of second and higher orders in each amplitude and keeping only quadratic interaction terms of first order in each. They also assumed that the ratio of short to long wavelengths was very small. In addition, the generally used assumptions of constant pressure at the interface and of incompressible, irrotational, inviscid flow were made, surface tension was neglected, and the problem solved was in two space dimensions. Their results for these variations in deep water were obtained as a prelude to the case of finite depth and the introduction of their radiation stress tensor.

Our calculations retain their general assumptions, are confined to the deep-water limit, and also treat flow in two space dimensions. However, the restriction to terms of first order in the long-wave amplitude is fully removed; these waves may be of any height up to the instability limit. Only the train of short wavelength is treated in linear approximation. In addition, the ratio of short to long wavelengths is arbitrary; some implications of near-equality are discussed.

We may choose the long-wave flow pattern to be a given steady train of "Stokes waves" of finite amplitude. However, the periodic character of such a flow, which is stationary in our coordinates, is not of importance; our work constitutes a linearized treatment of a wave train superposed on an arbitrary steady nonuniform flow. A special case of this problem was treated by Longuet-Higgins and Stewart (1961), denoted by LS II below. Because our results include as special cases some of the problems attacked in both LS I and LS II we are able to draw an important distinction between wave-like steady flows and others having negligible surface profile inclination and curvature. Within the domain limited by our linearization of one wave-train, we find the form of energy exchange to be significantly different from that predicted from the radiation stress theory of LS I and LS II.

Treating the small waves as linear will limit the rigor of applying the results to the triggering of whitecapping, but it seems a necessary first step and provides a framework for comparison with wave-tank experiments. One may remark that nature has frequently been kind enough to allow a greater domain of usefulness to linearized analyses than the linearizers have had any right to expect. This appears to be the case with respect to many properties of gravity waves of large amplitude, in spite of their strikingly nonlinear features.

II. SUMMARY OF RESULTS

A major part of the present work may be described as the development of a small-signal theory of the propagation of gravity waves on a steady but otherwise arbitrary two-dimensional flow of an ideal liquid. After defining the given steady flow we introduce curvilinear coordinates appropriate to its free surface, and derive linear partial differential equations for the velocity potential and surface displacement of a small disturbance on it. We obtain a general quasi-sinusoidal traveling-wave solution of these equations, and from it obtain the dispersion relation and expressions for wave-number, amplitude, phase and group velocities, wave energy, energy flux, and its spatial rate of change at any point on the given free surface. These expressions are rigorous solutions of the linearized problem. Their interpretation is straightforward and exhibits the physical concepts clearly; it is especially simple if a WKB approximation is well justified. We discuss this approximation mainly in connection with wave-like steady flows. We prove WKB validity for a large and important portion of the entire small-wave spectrum on all steady long-wave trains except those of nearly-maximum height; for these its validity is proved for all but a well-defined fraction of this portion. With this exception, the approximation is valid for all small waves propagating in bottom-fixed coordinates in the same direction but with lesser phase velocities; these are the waves traveling up the downwind sides of the longer waves. Some of these waves are not short with respect to the waves of the given flow. The properties of all other linearized waves are contained in our general expressions and can be worked out as desired. The WKB approximation becomes invalid on some wave-like flows only for small waves of great length and for "anomalous" waves with group velocity nearly the same as the phase velocity of the given wave train; their existence and properties have been noted by others.

The results of LS I for amplitude and wave-number changes emerge as special cases of ours in the limit of small-amplitude long waves of great length as expected, but our more general results are obtained by a simpler calculation once our foundation has been laid. They depart significantly from those of LS I for long waves of large height. In particular, for rather short small waves the wave-number increases much more rapidly in a small region near the crest of a long wave of nearly-maximum height. This region is larger for longer short waves and includes the entire long-wave profile if the "short" and "long" wavelengths are comparable. Variations of amplitude, wave-number, energy, and energy flux in this region, and in the more familiar one, are described by limiting forms of the more general results.

In order to generate analytically tractable expressions for these results, and to test the validity of the WKB approximation on which their simple form depends, we have developed an extremely simple and fairly accurate mathematical model of the profile of a wave of maximum height. This model may have uses in other contexts. In addition, we have made an approximate analysis of flow properties near the crests of waves of nearly maximum height, which enables us to estimate the particular waves for which the WKB approximation becomes invalid in this region, as well as to establish the forms of upper limits on the growth of wave-number and amplitude. These mathematical models are described in a companion paper, in which comparisons are also made with the model profile of Longuet-Higgins (1973).

The comparison of our results for waves on an arbitrary steady nonuniform flow with those of LS II is of particular interest. Because our results are two-dimensional the comparison is with the flow they describe as vertically upwelling. Their results for wave-number and amplitude changes are obtained in differential form from a linearized calculation for a specific shape of flow; they are expressed in terms of the co-moving phase velocity which itself depends on position along the flow through the wave-number. In contrast, our results are obtained directly in integrated form and depend only on known properties of the arbitrarily given flow; they must be differentiated to make comparisons. Their differential forms agree with our more general ones only for the very special flow shape they selected, and will differ for any flow with appreciable surface tilt or curvature; however, their integrated results for wave-number and amplitude agree with ours.

They employ their amplitude result as a criterion for deciding among several alternative general expressions for interchange of energy between waves and a given flow. However, their expressions for wave energy, energy flux, and its spatial rate of change are all in disagreement with ours except on their particular flow shape. We show that the general energy relation they select, which depends on their radiation stress, must be greatly modified* for small waves riding on all wave-like steady flows. This result is important because their radiation stress has been used by many workers in the study of wave-wave interactions. We introduce a "wave-interaction function" which appears as a factor multiplying their radiation stress, and show that it is very small on wave-like steady flows unless they are of nearly-maximum height, and even there is small except near their crests.

The various differences mentioned above and described in detail below are all related to a local parameter G of the given flow. This

* Whitham (1962) expressed dissatisfaction with their radiation stress and discussed energy transport for finite depth but his analysis does not appear to be directly extendable to the deep water limit. See also our comments in Section XIII on the "previously overlooked mechanism" described by Hasselmann (1971).

is the component of the equivalent or effective gravity-like acceleration normal to the given free surface experienced by the small waves; G is reduced from the acceleration of gravity g by surface tilt and altered by centrifugal acceleration. We show that G replaces g both in the small-wave dispersion relation and in the expression for small-wave energy; we exhibit the interesting property of most wave-like steady flows that G is closely proportional to the surface flow speed. If this proportionality is exact there is, in our linearized treatment of the small waves, no exchange of energy between the two wave trains; however, this is not true on nearly-maximum waves (and especially near their crests) or for flows of constant small slope such as the one selected in LS-II, leading to significant rates of energy interchange in these regimes.

The function G has another interesting property. The only effect on our dispersion relation of a moderate departure from validity of the WKB approximation is to introduce an additional variation in G.

The smallness of the product ak of small-wave amplitude and wave-number is a measure of the validity of linearization. In addition, the growth of this product measures the approach toward maximum height. We hope that our results retain an element of qualitative validity beyond the domain of quasi-linearity up toward the stability limit. Within the limitations of this plausible extension, we have investigated the acceleration of surface particles of fluid at a confluence of wave crests. The results support our conjecture that this acceleration may range between $\frac{1}{2}$ g and g for limiting situations on the border of instability; we exhibit examples in which it ranges from $\frac{1}{2}$ g to 3g/4. In the concluding section we also give a qualitative picture of the relation of our results to the triggering of whitecapping.

This description indicates that our methods may be useful in other hydrodynamical free-surface problems. Therefore we have tried to describe them clearly and completely in the sections that follow.

III. THE LONG-WAVE TRAIN OR OTHER STEADY FLOW

In coordinates fixed with respect to deep water at rest, our long-wave train moves to the left with phase velocity U_o . The shorter-wave trains to be emphasized below also move to the left in these coordinates with a smaller speed. We adopt a coordinate system which moves with the long-wave crests; their flow pattern is then steady and is assumed known in full detail. For wave-like flows such knowledge could in principle be gained (1) by keeping many terms of the Stokes expansion, (2) from more sophisticated techniques, such as those reviewed by Wehausen (1960), or (3) from the output of a digital calculation on a large computer. Alternatively, any suitable nonwave-like steady flow may be postulated. The velocity potential of the steady flow is $\bar{\phi}(x,y)$; its velocity is $\vec{U} = \vec{\nabla}\bar{\phi}$, and $\nabla^2\bar{\phi} = 0$. With y increasing upward and x to the right, we have $\bar{\phi} \to U_o x$ for wave-like flows as $y \to -\infty$. Because of our choice of leftward propagation in bottom-fixed coordinates, both U_o and the velocity of propagation of the shorter waves to be emphasized are positive in our coordinate system.

The free surface is defined by $y = y_f(x)$. On this surface there are two boundary conditions: $(\partial\bar{\phi}/\partial x)(dy_f/dx) = \partial\bar{\phi}/\partial y$, expressing the kinematic condition that flow at the surface is parallel to it, and $-p/\rho = gy_f(x) + \frac{1}{2} \,|\vec{\nabla}\Phi|^2$ = constant, expressing conservation of potential plus kinetic energy, through the Bernoulli relation, for particles flowing along the surface streamline. The free surface departs from the horizontal by an angle α ; its slope is given by $\tan \alpha = dy_f/dx$. We reserve the scalar symbol U for flow speed at the surface; $U^2 = (\partial\Phi/\partial x)^2 + (\partial\Phi/\partial y)^2$ at $y = y_f(x)$. In what follows we assume that $y_f(x)$ and $U(x)$ are known; they contain all we need to know about the steady flow.

IV. CURVILINEAR COORDINATE SYSTEMS

We introduce curvilinear coordinates (s,n) in the neighborhood of the unperturbed free surface, whose equation becomes $n = 0$. Distance normal to this surface, measured positive away from the fluid, is n ; distance measured along the surface in the direction of increasing x from some fixed reference point is s . A point in the neighborhood of the surface has the value of s associated with the base of the perpendicular of length n from the point to the surface. These locally defined (s,n) cordinates, shown in Figure 1a, are useful in the differential neighborhood of $n = 0$ because they have metrical simplicity (they measure lengths directly) and are orthogonal there but are inappropriate for investigations at finite values of n where their metric properties are less simple; the system even becomes singular where the perpendiculars intersect.

An orthogonal set of curvilinear coordinates shown in Figure 1b and having useful properties throughout the flow is provided by the equipotentials Φ = constant and streamlines Ψ = constant of the complex velocity potential $W(z) = W(x + iy) = \Phi(x,y) + i\Psi(x,y)$ of the given flow. This function may be extended as far as needed by analytic continuation outside the unperturbed free surface with due regard, in their differential neighborhoods, for branch points which appear at the crests of a wave of maximum height; it is therefore valid to extrapolate properties of this unperturbed flow beyond its boundary by a Taylor series in the displacement of a small added wave.

We will employ the (Φ, Ψ) coordinates in Section VI, because of the useful property that under a transformation of coordinates from (x,y) to (Φ, Ψ) by a conformal mapping $W(z) = \Phi + i\Psi$ the Laplacian operator $\partial^2/\partial x^2 + \partial^2/\partial y^2$ becomes proportional to $\partial^2/\partial\Phi^2 + \partial^2/\partial\Psi^2$ (Morse and Feshbach (1953), sections 5.1 and 10.2).

The surface boundary conditions at $n = 0$ for steady flow are $\partial\Phi/\partial n = 0$ and $gy_f(s) + \frac{1}{2} U^2(s) =$ constant in these coordinates. The tangential component a_s of acceleration $\vec{a}$ of a surface particle is given* by $dU/dt = (dU/ds)(ds/dt) = UU'$; differentiating the Bernoulli boundary condition with respect to s, and noting that $dx/ds = \cos\alpha$,

$$a_s = \vec{a} \cdot \hat{e}_s = UU' = -g \sin\alpha ,$$

the tangential component of $\vec{g}$, as on any inclined surface without friction. The normal component a_n of $\vec{a}$ is centripetal and has magnitude U^2/R , with R the radius of curvature of the surface. We take R positive where the surface is concave upward; in these coordinates $R^{-1} = d\alpha/ds = \alpha'$, and

$$a_n = \vec{a} \cdot \hat{e}_n = U^2/R = U^2\alpha' .$$

---

* A prime denotes a derivative with respect to s of a quantity depending only on s ; $\hat{e}_s$ and $\hat{e}_n$ are unit vectors.

For a wave of maximum height $\alpha' \to 0$ and $\alpha \to \pm 30^\circ$ near the crests, so that $a_n \to 0$ and $a_s \to \pm \frac{1}{2} g$.

The complete Bernoulli equation holds throughout the fluid in the form $(p/\rho) + \frac{1}{2} |\vec{\nabla} \not{\Phi}|^2 + gy$ = constant for steady flow. Taking the normal component of the gradient of this equation, evaluated at the surface $n = 0$, we have

$$-\rho^{-1}(\partial p/\partial n) = g(\partial y/\partial n) + \frac{1}{2} \hat{e}_n \cdot \vec{\nabla} \, | \vec{\nabla} \not{\Phi} |^2 .$$

On the surface, $\partial y/\partial n = \cos \alpha$; the second term is

$$\hat{e}_n \cdot (\vec{\nabla} \not{\Phi} \cdot \vec{\nabla}) \vec{\nabla} \not{\Phi} = \hat{e}_n \cdot [U \partial(U\hat{e}_s)/\partial s] = \hat{e}_n \cdot [UU'\hat{e}_s + U^2 \partial \hat{e}_s/\partial s] = U^2 \alpha' ,$$

where we have used $\partial \hat{e}_s/\partial s = \alpha' \hat{e}_n$. Therefore on $n = 0$

$$-\rho^{-1}(\partial p/\partial n) = g \cos \alpha + U^2 \alpha' .$$

The first term in the pressure gradient supports the fluid against the normal component of gravity while the second provides the force to produce centripetal acceleration. We shall need this quantity in the following section.

The curvature of the surface streamline is given by $\hat{e}_n \cdot \vec{\nabla} \ln |\vec{\nabla} \not{\Phi}|$; and that of an equipotential, at the surface, by $\hat{e}_s \cdot \vec{\nabla} \ln |\vec{\nabla} \not{\Phi}|$. These curvatures are, respectively, R^{-1} and U'/U in magnitude. We have already defined the sign of R; the equipotentials are concave in the direction of increasing s if U' is positive.

V. DERIVATION OF THE WAVE EQUATION

We now add a small time-dependent perturbation $\phi(s,n,t)$ to the velocity potential. The displaced surface will be located at $n = \xi(s,t)$. The gravitational potential associated with ξ is

$g\xi \cos\alpha$ (see Figure 2). The quantities ϕ, ξ, and their derivatives are taken as small of first order, and higher order terms are dropped. To develop the kinematic boundary condition on the displaced free surface we note that the component of flow velocity at $n = \xi(s,t)$ which is normal to the direction of the unperturbed surface at the same value of s is given by

$$\hat{e}_n\Big|_{n=0} \cdot \vec{\nabla}(\Phi + \phi)\Big|_{s,n} = \hat{e}_n \cdot \vec{\nabla}(\Phi + \phi)\Big|_{n=0}$$

$$+ \hat{e}_n \cdot \left[\xi \frac{\partial}{\partial n} \vec{\nabla}(\Phi + \phi)\right]\Big|_{n=0} + \cdots$$

$$= (\partial\phi/\partial n)\Big|_{n=0} + \xi(\partial^2\Phi/\partial n^2)\Big|_{n=0} + \text{second order.}$$

The second term may be evaluated by considering s and n to form a locally Cartesian system, whence $\xi\partial^2\Phi/\partial n^2 = -\xi\partial^2\Phi/\partial s^2 = -\xi U'$ on $n = 0$ since $\nabla^2\Phi = 0$. Alternatively, we may notice that the flow velocity $\vec{\nabla}\Phi$ is rotated as we move outward along an equipotential, so as to remain orthogonal to it; this develops an outward component equal to its magnitude U times the rotation angle. For small ξ this angle has magnitude $\xi U'/U$ (where U'/U is the equipotential's curvature) and is a rotation <u>toward</u> $n = 0$, generating an inward normal component, if U' is positive. Thus the second term is $-U(\xi U'/U) = -\xi U'$, in agreement with the first evaluation.

The boundary itself moves normal to the unperturbed boundary with velocity $\partial\xi/\partial t$, so the net normal velocity is $(\partial\phi/\partial n)\Big|_{n=0} - U'\xi - \partial\xi/\partial t$. The tangential component of velocity, $\partial\Phi/\partial s = U$, is needed only to zero order. The ratio of these velocity components is the slope $\partial\xi/\partial s$, to first order. Thus the linearized kinematic boundary condition is

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial s}\right)\xi + U'\xi = \frac{\partial\phi}{\partial n} \quad \text{on} \quad n = 0 .$$

Since $\partial U/\partial t = 0$, we may write this as

$$\boxed{\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial s}\right)(U\xi) = U\frac{\partial\phi}{\partial n} \quad \text{on} \quad n = 0 .}$$

The Bernoulli surface condition for time-dependent flow is

$$\left[\frac{\partial}{\partial t}(\bar{\phi} + \phi) + \frac{1}{2}\left|\vec{\nabla}(\bar{\phi} + \phi)\right|^2 + g(y_f + \xi\cos\alpha)\right]\Bigg|_{n=\xi} = 0 ;$$

cancelling the terms pertaining to the steady flow at $n = 0$, we have to first order in ξ and ϕ

$$\left[\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial s}\right)\phi + \frac{1}{2}\xi\,\hat{e}_n\cdot\vec{\nabla}\left|\vec{\nabla}\bar{\phi}\right|^2 + g\xi\cos\alpha\right]\Bigg|_{n=0} = 0 .$$

The gradient term was evaluated in the preceding section; the coefficient of ξ is $g\cos\alpha + U^2\alpha'$, equal to $-\rho^{-1}(\partial p/\partial n)$ on $n = 0$. It represents the equivalent or effective gravity-like acceleration experienced by the small waves; it is reduced by tilt ($g \rightarrow g\cos\alpha$) and altered by centrifugal acceleration ($U^2\alpha' = U^2/R$). We call this important known quantity

$$\boxed{G(s) \equiv g\cos\alpha + U^2/R .}$$

Thus our Bernoulli condition is

$$G\xi = -\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial s}\right)\phi \qquad \text{on} \quad n = 0\ ,$$

which may be written as

$$\boxed{(U\xi) = -\frac{U}{G}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial s}\right)\phi \qquad \text{on} \quad n = 0\ .}$$

The linearized kinematic and Bernoulli boundary conditions above are simpler, and display the physical considerations more clearly, than those resulting from their expression in Cartesian coordinates, utilizing a displaced free surface located at $y = y_f(x) + \zeta(x,t)$ (see Figure 2b):

$$\left[\frac{\partial\zeta}{\partial t} + \frac{\partial\bar{\phi}}{\partial x}\frac{\partial\zeta}{\partial x} + \frac{dy_f}{dx}\frac{\partial\phi}{\partial x} - \frac{\partial\phi}{\partial y} + \zeta\left(\frac{dy_f}{dx}\frac{\partial^2\bar{\phi}}{\partial x\partial y} - \frac{\partial^2\bar{\phi}}{\partial y^2}\right)\right]\Bigg|_{y=y_f(x)} = 0,$$

and

$$\left[\left(g + \frac{\partial\bar{\phi}}{\partial x}\frac{\partial^2\bar{\phi}}{\partial x\partial y} + \frac{\partial\bar{\phi}}{\partial y}\frac{\partial^2\bar{\phi}}{\partial y^2}\right)\zeta + \frac{\partial\phi}{\partial t} + \frac{\partial\bar{\phi}}{\partial x}\frac{\partial\phi}{\partial x} + \frac{\partial\bar{\phi}}{\partial y}\frac{\partial\phi}{\partial y}\right]\Bigg|_{y=y_f(x)} = 0.$$

It was a tedious and nontrivial exercise to demonstrate in detail the equivalence of these pairs of equations in the different coordinate systems; this will not be done here. In addition to the inherent complexity of the second pair, the two sets differ intrinsically due to the angular displacement of the (s,n) axes which varies as a function of s or x. The Cartesian set acquires several additional terms when altered (as in LS I) through Taylor expansions in y_f (implying that the long-wave amplitude is also small of first order), so as to be evaluated at $y = 0$. Comparison of the resulting equations with ours then becomes even more tedious, although the Cartesian second-order partial differential equations for ϕ obtained before and after this additional expansion (in the limit of small y_f) can be shown, again with difficulty, to be identical.

Inserting $U\xi$ from the second boundary condition into the first and multiplying by U/G we obtain the linearized wave equation for ϕ:

$$\left(\frac{U}{G}\frac{\partial}{\partial t} + \frac{U^2}{G}\frac{\partial}{\partial s}\right)^2 \phi + \frac{U^2}{G}\frac{\partial\phi}{\partial n} = 0 \qquad \text{on} \quad n = 0.$$

From the form of this equation it is evident that the quantities U/G and U^2/G are the appropriate local (space-varying) units of time and length, respectively; both are known functions of s. We may define a dimensionless distance σ along the unperturbed surface by $d\sigma = (G/U^2)ds = (G/U^3)d\Phi$; σ is a known function of s. The corresponding dimensionless normal distance η may be defined near $n = 0$ by $d\eta = (G/U^2)dn = (G/U^3)d\Psi$, and a local time τ by $d\tau = (G/U)dt$; then

$$\boxed{\left(\frac{\partial}{\partial\tau} + \frac{\partial}{\partial\sigma}\right)^2 \phi + \frac{\partial\phi}{\partial\eta} = 0 \qquad \text{on} \quad n = 0.}$$

We emphasize that the linearized partial differential equations developed here constitute one of our most important results. The use we have made of them in this paper suggests that their generalization to other hydrodynamical and oceanographic free-surface and other interface problems can be expected to yield results of greater physical clarity in simpler form and with less effort than by Cartesian expansions. We will not undertake here to extend them to the cases of finite depth, three space dimensions, or internal waves at a curved interface, although these extensions should be practical and equally effective.

VI. SIMULTANEOUS SOLUTION OF LAPLACIAN AND WAVE EQUATIONS

The simplest limiting situation is that in which the long-wave train has zero amplitude and the fluid is at rest, with a horizontal surface, in bottom-fixed coordinates. Then $U = U_o$, $G = g$, $\alpha = 0$, $s = x$, $n = y$, and $\bar{\phi} = U_o s$. The Laplacian becomes $[(\partial^2/\partial s^2) + (\partial^2/\partial n^2)]\phi = 0$ and the wave equation becomes $[(\partial/\partial t) + U_o(\partial/\partial s)]^2\phi + g(\partial\phi/\partial n) = 0$. The traveling wave solution is $\phi = A_o e^{k_o n} \cos(k_o s - \omega t)$, the dispersion relation is $(U_o k_o - \omega)^2 = gk_o$, the phase velocity is $v_p = \omega/k_o = U_o \pm (g/k_o)^{\frac{1}{2}}$; we have the expected linearized gravity waves traveling in both directions with phase velocity $(g/k_o)^{\frac{1}{2}}$ relative to the uniformly moving fluid.

We therefore seek a solution reducing to this form in this limit, which will be particularly appropriate for waves sufficiently short that the fractional changes in U, G, and α over one wave-length are small. The parameters of this wave train are shown in Figure 2a. Specifically, we require that

$$\left[\phi(s,n,t)\Big|_{n=0} = A(s) \cos\left[\int^{s} k(s)ds - \omega t\right] \right] .$$

We must now determine the n-dependence of this function near $n = 0$ which satisfies Laplace's equation and use it to evaluate $\partial\phi/\partial n$ in the wave equation. This can be accomplished with the aid of the $\bar{\phi}$ - $\bar{\Psi}$ coordinate system described earlier, in which the Laplacian operator has the form $(\partial^2/\partial\bar{\phi}^2) + (\partial^2/\partial\bar{\Psi}^2)$. We note that ϕ is the real part of a complex perturbation velocity potential

$$w(Z,t) = w(\bar{\phi} + i\bar{\Psi},t) = \phi(\bar{\phi},\bar{\Psi},t) + i\psi(\bar{\phi},\bar{\Psi},t).$$

Choosing w to be an analytic function of Z of the form $\exp\{-i[F(Z) - \omega t]\}$, with F an analytic function having real and imaginary parts $R(\Phi, \Psi)$ and $I(\Phi, \Psi)$, we have

$$\phi = e^I \cos \chi, \qquad \psi = -e^I \sin \chi,$$

where $\chi \equiv R - \omega t$. It is a general property of analytic functions that $\nabla^2\phi = \nabla^2\psi = 0$.

The Cauchy-Riemann relation is

$$\frac{\partial\phi}{\partial\Psi} = -\frac{\partial\psi}{\partial\Phi}.$$

Because $d\Phi/d\Psi = d\sigma/d\eta = ds/dn$ near $n = 0$,

$$\left.\frac{\partial\phi}{\partial n}\right|_{n=0} = -\left.\frac{\partial\psi}{\partial s}\right|_{n=0} = e^I \left[\frac{dI}{ds} \sin \chi + \frac{dR}{ds} \cos \chi\right]\Bigg|_{n=0};$$

henceforth I and R are regarded as functions of Φ only. To complete the identification we set $I(\Phi) \equiv \ln A(s)$ and $R(\Phi) \equiv \int^s k(s')ds'$. With $d\Phi = U\,ds$ on $n = 0$, $dR/d\Phi = k/U$ and $dI/d\Phi = (AU)^{-1}\,dA/ds$.

Applying the operators $\left(\frac{U}{G}\frac{\partial}{\partial t} + \frac{\partial}{\partial\sigma}\right)^2$ and $\partial/\partial\eta$ to ϕ, inserting in the wave equation, and dividing by A, we obtain

$$\left[-\left\{\frac{U}{G}(Uk - \omega)\right\}^2 + \frac{U^2k}{G} + \frac{1}{A}\,\frac{d^2A}{d\sigma^2}\right] \cos \chi$$
$$- \left[\frac{d}{d\sigma}\left\{\frac{U}{G}(Uk - \omega)\right\} + 2\left\{\frac{U}{G}(Uk - \omega) - \frac{1}{2}\right\}\frac{1}{A}\frac{dA}{d\sigma}\right] \sin \chi = 0.$$

Both square brackets must vanish,* since the equation must hold for all t. The second is a perfect differential which, when integrated, yields the relation between $A(s)$ and $k(s)$:

$$A^2 = C \left| \frac{1}{2} + (U/G)(\omega - Uk) \right|^{-1}.$$

Here, C is a constant of integration whose value is expected to be unspecified in the solution of a set of linear homogeneous equations. Its value is irrelevant and we set it equal to unity. [This form of analysis becomes inapplicable near the singularity at $(U/G)(Uk - \omega) = \frac{1}{2}$, corresponding to vanishing group velocity in our coordinates. This anomalous wave is outside the spectrum of interest to us, but we discuss it briefly in a later section.]

We use this relation between A and k to eliminate $k(s)$ from the first square bracket, which then determines the dependence of A on σ. Inserting $U^2k/G = (U\omega/G) + \frac{1}{2} \pm A^{-2}$ and combining terms, we obtain

$$d^2A/d\sigma^2 + [\tfrac{1}{4} + (U\omega/G)]A - A^{-3} = 0.$$

This apparently unpleasant nonlinear differential equation arises in the development of the phase-amplitude form of solution of equations of the Mathieu-Hill type (Courant and Snyder (1958), pp. 9-11) and more generally of the type to which the WKB approximation is commonly applied. If the solution of the differential equation

---

* This does not apply to the static case $\omega = 0$, in which the arguments of the trigonometric functions do not depend on time. The two conditions then reduce to a single one, preventing the separate determination of A and k, but one may take the limit as $\omega \to 0$.

$$d^2F/d\sigma^2 \quad + \quad P(\sigma)F \quad = \quad 0$$

is expressed as the real part of $A(\sigma)\ \exp[i\Gamma(\sigma)]$, with A and Γ real, the differential equation is equivalent to

$$A^2\ d\Gamma/d\sigma = C \qquad \text{and} \qquad d^2A/d\sigma^2 + P(\sigma)A - C^2A^{-3} = 0,$$

with C a constant. [The form of this pair of equations depends only on A^2/C, verifying that C is arbitrary.] In this way we see that the dependence of A (and thence of k) on σ (and thence on s) is related to the linear ordinary differential equation

$$d^2F/d\sigma^2 \quad + \quad [\tfrac{1}{4} + (U\omega/G)]F \quad = \quad 0;$$

$A(\sigma)$ is the amplitude of the envelope surrounding the quasi-sinusoidal oscillations of its solutions.

Application of the conventional WKB approximation to this equation yields the approximate solution

$$F(\sigma) \quad \propto \quad [\tfrac{1}{4} + (U\omega/G)]^{-\frac{1}{4}} \exp\left\{\pm\ i\int^{\sigma} [\tfrac{1}{4} + (U\omega/G)]^{\frac{1}{2}}\ d\sigma\right\},$$

$$\boxed{A \quad \propto \quad [\tfrac{1}{4} + (U\omega/G)]^{-\frac{1}{4}} ,}$$

corresponding exactly to neglect of $d^2A/d\sigma^2$ in the nonlinear equation for A. [This form becomes inapplicable in the limit $U\omega/G \to -\frac{1}{4}$, the anomalous wave mentioned above.] Subject only to the validity of this approximation, we have thus obtained a complete solution in closed form for the velocity potential of the linearized waves propagating on an arbitrary steady flow defined by the functions $U(s)$ and $G(s)$.

The displacement $\xi(s,t)$, shown in Figure 2a, is given by

$$\xi = -G^{-1}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial s}\right)\phi\Big|_{n=0} = \frac{A}{G}\left[(Uk - \omega)\sin\chi - \frac{U}{A}\frac{dA}{ds}\cos\chi\right]$$

$$= \frac{(Uk - \omega)A}{G}\left[1 + \left\{\frac{U}{Uk - \omega}\frac{1}{A}\frac{dA}{ds}\right\}^2\right]^{\frac{1}{2}} \sin\left[\int^s k\,ds - \omega t - \tan^{-1}\left\{\frac{UA'}{(Uk - \omega)A}\right\}\right];$$

this may be written as

$$\xi = a(s)\sin\left[\int^s k\,ds - \omega t - \delta(s)\right],$$

which defines the small-wave amplitude $a(s)$ and phase $\delta(s)$.

VII. THE RIGOROUS DISPERSION RELATION: AMPLITUDE, WAVE-NUMBER, WAVE-ENERGY, AND ENERGY FLUX

The dispersion relation in familiar terms may be obtained from the first of the vanishing square brackets in the preceding section:

$$(Uk - \omega)^2 = Gk[1 + (U^2 kA/G)^{-1}\, d^2A/d\sigma^2] \equiv Gk(1 + \epsilon),$$

in which the smallness of ϵ thus defined is associated with the smallness of $d^2A/d\sigma^2$ and therefore with the degree of validity of the WKB approximation for $A(\sigma)$. Specifically, $d^2A/d\sigma^2$ may be neglected if it is small with respect to either of the two remaining terms in the nonlinear differential equation for A; this is equivalent to the condition $|A^3 \cdot d^2A/d\sigma^2| \ll 1$ with our normalization of A. We will relate ϵ to this quantity below. For small ϵ the familiar dispersion relation $(Uk - \omega)^2 = Gk$ is then valid locally, using local values of U and G.

It is convenient to introduce dimensionless frequency and wave-number variables appropriate to the local units of time and length described in an earlier section:

$$p \equiv U\omega/G^*, \qquad q \equiv U^2k/G^* ,$$

with $G^* \equiv G(1 + \epsilon)$ so that $G^* \approx G$ when the WKB approximation is justified. The rigorous dispersion relation is

$$\boxed{(q - p)^2 = q, \qquad \text{or} \qquad p = q \pm q^{\frac{1}{2}} .}$$

The two branches are shown in Figure 3. It is simplest to describe them with reference to a given flow with nearly constant G and U, a condition corresponding to a long-wave train of very small amplitude. The phase velocity $v_p = \omega/k$, measured in units of U, is the slope of a line from the origin 0 to any point E in question; the group velocity $v_g = d\omega/dk$, in the same units, is the slope of the tangent at E. On the upper branch OA both are positive and both exceed U. In bottom-fixed coordinates these waves propagate backward with respect to the given wave train; we shall not emphasize them in our treatment here, although we justify the WKB approximation on a large part of this branch. On that part CD of the lower branch for which $\omega > 0$ both v_p and v_g are positive but less than U; these waves are of greatest interest to us. On BC v_p is negative and v_g positive, both being less than U in magnitude; in bottom-fixed coordinates these waves move in the same direction as the given wave but with greater phase velocity and lesser group velocity than U. Finally, on OB both v_p and v_g are negative and greater than U in magnitude. These waves move in the same direction as the given wave in bottom-fixed coordinates with a phase velocity more than twice as great as U. To keep signs straight we assume that U and k are always positive. We refer repeatedly to these branches in several sections below. [The corresponding relationships in bottom-fixed coordinates, in which $p^2 = q$, are obtained by rotating the parabola clockwise by 45°. The given wave is then represented by the point $q = -p = 1$ if its amplitude is small.]

For long-wave trains of finite amplitude and other steady non-uniform flows U and G vary, causing the representative point to move along the curve. In our coordinates a steady progression of small waves moves along a flow of fixed profile; the same number of crests must pass each point per unit time, so ω is a constant. Its sign determines the direction of the phase velocity, which is positive for the waves we emphasize here. Its magnitude defines the short-wave train of interest by selecting the short-wave number $k_o = k(s_o)$ at a reference point s_o (for example, in a long-wave trough) and solving the dispersion relation for ω using $U(s_o)$ and $G(s_o)$ for the steady flow under study.

The local phase and group velocities are given by

$$v_p = U \mp (G^*/k)^{\frac{1}{2}} = U + c = U[1 - (q - p)^{-1}]$$

and

$$v_g = U \mp \frac{1}{2} (G^*/k)^{\frac{1}{2}} = U + c_g,$$

with c and c_g denoting the local <u>co-moving</u> small-wave phase and group velocities, respectively; $c_g = \frac{1}{2} c$ as expected. The upper signs apply to branch OBCD and the lower to OA. The quantity $(G^*/k)^{\frac{1}{2}}$ is the magnitude of c, and $(U/G^*)(\omega - UK) = p - q$ is the ratio U/c; $c > 0$ on OA and $c < 0$ on OBCD. The quantity $|\frac{1}{2} + (U/G)(\omega - Uk)| = |\frac{1}{2} + (1 + \epsilon)(p - q)| = A^{-2}$ may be identified, for small $|\epsilon|$, with $|v_g/c| = |(U + c_g)/c|$; v_g/c is positive on AOB and negative on BCD. Although the only effect on the dispersion relation of a failure of $|\epsilon|$ to be negligible is to provide a varying correction to G^*, the interpretation of A given above will also be altered.

We introduce the symbol Δ for the ratio U/c;

$$\Delta \equiv U/c \equiv (U/G)(\omega - Uk) \equiv p - q.$$

The solution of the rigorous dispersion relation is given by any of the following:

$$p = q \mp q^{\frac{1}{2}}; \qquad q = \frac{1}{2} + p \pm (\frac{1}{4} + p)^{\frac{1}{2}};$$

$$\Delta = \mp (\frac{1}{4} + p)^{\frac{1}{2}} - \frac{1}{2}; \qquad \Delta^{-1} = p^{-1}[\frac{1}{2} \mp (\frac{1}{4} + p)^{\frac{1}{2}}].$$

In the first equation the upper sign applies to segments OBCD and the lower to OA; in the others, the upper sign applies to BCD and the lower to AOB. On the segment CD of special interest to us, $p > 0$, $q > 1$, and $\Delta < -1$. In this notation $\frac{1}{2} + (p - q) = \mp (\frac{1}{4} + p)^{\frac{1}{2}}$, the upper sign applying to BCD and the lower to AOB. Thus A^{-2} is equal to $(\frac{1}{4} + p)^{\frac{1}{2}} = A_{WKB}^{-2}$ for small $|\epsilon|$; then

$$a \propto |A\Delta/U| \sec \delta,$$

$$\tan \delta = -\Delta^{-1} d(\ln A)/d\sigma .$$

The quantity ϵ is equal to $(qA)^{-1} d^2A/d\sigma^2$; if we use the WKB value for A, we have

$$\epsilon \approx [(\tfrac{1}{4} + p)/q]A^3 \, d^2A/d\sigma^2.$$

Thus the smallness of ϵ is connected to the validity of the WKB approximation for A. The square bracket is less than unity everywhere on segments BCD, so that the smallness of ϵ follows directly from WKB validity for the small waves of special interest here. Additional examination is required for the other branches. This will be deferred to the section on WKB validity; henceforth we neglect ϵ.

It is well known that the potential and kinetic energies per unit surface area associated with a train of small-amplitude gravity waves, when averaged over a wavelength, are each equal to $\frac{1}{4}\rho g a^2$, so the total energy E per unit surface area is $\frac{1}{2}\rho g a^2$. However, our waves are propagating on a surface with local effective gravity G, so we have

$$E = \tfrac{1}{2}\rho G a^2 \propto G(A\Delta/U)^2 = (\tfrac{1}{4} + p)^{-\frac{1}{2}} G/c^2 = (\tfrac{1}{4} + p)^{-\frac{1}{2}} Gq/U^2;$$

from this we have $E \propto k(\frac{1}{4} + p)^{-\frac{1}{2}}$, or as a function of p and U

$$E \propto (\tfrac{1}{4} + p)^{-\frac{1}{2}} \left[\tfrac{1}{2} + p \pm (\tfrac{1}{4} + p)^{\frac{1}{2}}\right] / [|p| \, U].$$

From a physical point of view it is more illuminating to write this in the equivalent form

$$\boxed{E \propto G/|c(U + c_g)| = G/|cv_g| \propto |c/(v_p v_g)|,}$$

obtained by use of the relations above. In the same notation

$$\boxed{k = |\omega/(U + c)|}$$

and

$$a \propto |c(U + c_g)|^{-\frac{1}{2}} = |cv_g|^{-\frac{1}{2}} .$$

Because the energy of linearized waves is propagated with velocity c_g in fluid at rest, and with velocity $v_g = U + c_g$ in general, the flux of wave energy J (in units of energy crossing a unit length parallel to the wave front per unit time) is given in magnitude by

$$|J| = E|v_g| \propto G/|c| \propto |c/v_p| .$$

However, the signs of v_g and c are the same on AOB and opposite on BCD, while v_g is negative on OB and positive elsewhere, giving rise to the following scheme:

Region	Wave Energy Flux $J \propto$:	J and v_g	$\omega, p,$ and v_p	c and c_g
OA	$+(G/U)[(\frac{1}{4} + p)^{\frac{1}{2}} - \frac{1}{2}]$	> 0	> 0	> 0
OB	$-(G/U)[(\frac{1}{4} + p)^{\frac{1}{2}} + \frac{1}{2}]$	< 0	< 0	< 0
BC	$+(G/U)[(\frac{1}{4} + p)^{\frac{1}{2}} + \frac{1}{2}]$	> 0	< 0	< 0
CD	$+(G/U)[(\frac{1}{4} + p)^{\frac{1}{2}} + \frac{1}{2}]$	> 0	> 0	< 0

If J is not conserved in passing along the profile of the given flow, energy is being exchanged between the small-wave train and the steady flow. Our interpretation of these expressions for wave energy and energy flux is given in a later section.

It is implicit in our linearized treatment that small-wave energy and energy flux are small of second order and that changes of wave-energy flux produce negligible perturbations on the given steady flow. The back-effect of these perturbations on the small waves is neglected in this paper but we note the resulting lack of overall self-consistency and the related lack of accuracy for small waves of finite amplitude where the given flow is very slow.

VIII. APPROXIMATE EVALUATIONS FOR LARGE AND SMALL p

On the branch CD of greatest interest to us the dispersion relation displays two distinct and basically different limiting regimes of behavior, characterized by $U\omega/G \gg 1$ and $U\omega/G \ll 1$, and separated by an intermediate transition range. For $p \gg 1$, corresponding to very short small waves, we expand in powers of p^{-1}, obtaining

$$q = p \pm p^{\frac{1}{2}} - \cdots$$

$$\Delta = \mp p^{\frac{1}{2}} - \frac{1}{2} + \cdots$$

$$\Delta^{-1} = \mp p^{-\frac{1}{2}} + (2p)^{-1} - \cdots$$

$$A = |p|^{-\frac{1}{4}} [1 - p^{-1}/16 + \cdots]$$

$$a \propto U^{-1} |p|^{\frac{1}{4}} [1 \pm \frac{1}{2} p^{-\frac{1}{2}} - \cdots]$$

$$|\tan \delta| = |p^{-\frac{1}{2}} d(\ell n A)/d\sigma| \ll 1$$

$$E \propto U^{-1} p^{-\frac{1}{2}}(1 \pm p^{-\frac{1}{2}} + \cdots)$$

$$J \propto p^{-\frac{1}{2}} \pm \frac{1}{2} p^{-1} + \cdots ,$$

with upper signs for CD and lower for OA. This is the regime appropriate to the description of short waves riding on long waves of small or moderate amplitude, and also on long waves of maximum or nearly-maximum height but sufficiently far away from their crests. In both cases $|\Delta| \gg 1$; the local given flow velocity U is large with respect to the small-wave co-moving phase velocity c, so p is large of order λ_L/λ with λ the local short-wave length. In such regions we have

$$Uk \approx \text{constant}, \qquad A^4U/G \approx \text{constant}, \qquad a^4U^3G \approx \text{constant},$$
$$E^2U^3/G \approx \text{constant}, \quad \text{and} \quad J^2U/G \approx \text{constant}.$$

In order to compare these results with those of LS I it is necessary to pass to the limit of small amplitude for the long waves. To first order in a_Lk_L,

$$\phi \to (g/k_L)^{\frac{1}{2}}(x + a_L e^{k_L y} \cos k_L x), \qquad x \to s;$$

$$y_f(x) \to a_L \sin k_L x, \qquad \alpha \approx dy_f/dx \to a_Lk_L \cos k_L x,$$

$$U \to (g/k_L)^{\frac{1}{2}} (1 - a_Lk_L \sin k_Lx), \qquad U^2\alpha' \to -ga_Lk_L \sin k_Lx,$$

$$G \to g(1 - a_Lk_L \sin k_Lx), \qquad G \propto U, \qquad A \to \text{constant}.$$

Thus in this limit $k'/k \to -U'/U$, $a'/a \to -U'/U$, so that $a/a_o = k/k_o = U_o/U = 1 + a_Lk_L \sin k_Lx = 1 + k_Ly_f$, in agreement with their results. However, we will show later a departure, in terms of higher order in a_Lk_L, from exact proportionality between U and G. The numerical effects of this departure on small-wave amplitude and wave-number are small for small values of a_Lk_L but will prove to be significant if the long waves are of nearly-maximum height.

These results for large p describe the variations of short small waves as they pass along the profile of any long wave of appreciably less than maximum height. The variations may be of interest for other purposes, but it would not be expected that small-amplitude short waves would influence the breaking of a long wave whose height is well below its own instability limit. Therefore we do not present numerical evaluations for intermediate-height long waves, but focus our attention in later sections on long waves of maximum or nearly-maximum height. Again we defer discussion of E and J to a later section.

In the opposite limiting case of small p we discard the branches near point 0 which correspond to small waves of very great length, and work near point C, expanding in powers of p. We first remark that for long waves of small amplitude this situation implies a denial of the roles we have consistently ascribed to the long and short wave trains. For small-amplitude long waves $U^2/G \approx k_L^{-1}$, so that $q \approx k/k_L = \lambda_L/\lambda$. Thus $p \approx (\lambda_L/\lambda)^{\frac{1}{2}}[(\lambda_L/\lambda)^{\frac{1}{2}} - 1]$ and can only be small relative to unity if λ is very close to λ_L.

The expansions in powers of p, for either sign of p, are:

$$q = 1 + 2p - p^2 + \cdots$$

$$\Delta = -1 - p + p^2 - \cdots$$

$$\Delta^{-1} = -1 + p - \cdots$$

$$A = 2^{\frac{1}{2}}(1 - p + 5p^2/2 + \cdots)$$

$$a \propto U^{-1}(1 + \tfrac{1}{2} p^2 + \cdots)$$

$$|\tan \delta| = |d(\ln A)/d\sigma| \ll 1 \quad \text{if WKB is valid}$$

$$E \propto |p|^{-1} U^{-1} [1 + \mathcal{O}(p^2)]$$

$$J \propto p^{-1}(1 + p + \cdots).$$

In this less familiar domain we have

$$U^2k/G \approx \text{constant}, \qquad A \approx \text{constant}, \qquad aU \approx \text{constant},$$
$$EU^2/G \approx \text{constant}, \qquad \text{and} \qquad JU/G \approx \text{constant}.$$

Unless p is already small in the long-wave trough (in which case the "short" waves are not short) this regime can only be entered by moving up away from the trough far enough so that U has decreased enough to approach c. However, this phase velocity also decreases with decreasing U, but less rapidly than as U^{-1}. Therefore for all but rather long short waves the required decrease in U will only be available near the crests of long waves of nearly-maximum height. In order-of-magnitude terms, $\omega \sim (\lambda_L/\lambda_o)(U_o/G_o)$, where o denotes a value in a trough. If $p \ll 1$ at s, we must have $U_s/G_s \ll (U_o/G_o)(\lambda_o/\lambda_L)$. We shall show later that $1/2 \lesssim G/g \lesssim 4/3$ at any point on any long-wave train. Therefore the condition, in order of magnitude, is $U_s/U_o \ll \lambda_o/\lambda_L$.

From a mathematical point of view we can always find, for any trough-value of p, a long-wave train for which this will occur, since the crest-value of U approaches arbitrarily close to zero as the long-wave height approaches its maximum. Physically, we must remember that for any value of p the short-wave amplitude and wave-number continually increase on their way up the long waves. Therefore unless the trough-value a_o is very small this growth may at some point invalidate our linearized analysis, and a little farther up may bring the short waves to their own maximum height. Depending on a_o and k_o, this may occur before entering the regime $p \ll 1$ if the long wave is high enough to include it, or before reaching the long-wave crest if it is not. Alternatively, the small waves may pass over the crest and descend without attaining their maximum height.

Thus we see that for short small waves the interesting behavior (specifically, $ak \propto U^{-3}$ rather than U^{-2}) associated with this regime is confined to small regions almost at the crests of long waves of almost-maximum height. Therefore it may be difficult to observe in laboratory wave-tanks; the difficulty will be least for the longest "short waves" that can be considered.

IX. MODELS OF MAXIMUM AND NEAR-MAXIMUM WAVE TRAINS

To proceed further, and in particular to examine the validity of the WKB approximation used in the preceding section, we must have values of $U(s)$ and $G(s)$ for long-wave-trains of interest. In a companion paper I have developed a model of the profile of the maximum Stokes wave, tested it by several criteria, and compared it with a good numerical solution as well as with the model discovered by Longuet-Higgins (1973) after my tests had been made. My model is simpler than his and represents most properties slightly better, except that in the trough it yields a small underestimate of my wave-interaction function Ω to be introduced below. It will be used in what follows, with the effects of this discrepancy being noted where appropriate.

The wave-train of maximum height is a mathematical abstraction or limiting case. It is therefore important to compare the results obtained for it with those for more realistic waves which fall barely short of attaining this configuration. To estimate $U(s)$ and $G(s)$ in the neighborhood of the crests of such waves I have developed a model based on the method of Havelock (1919), by expanding in powers of a parameter of smallness $\mathcal{E}$ which goes to zero for the maximum wave. This model is also described in the companion paper. After this work was completed the papers of Grant (1973) and Schwartz (1974) appeared, in which the complicated nature of the analytic behavior of the complex potential near the crests of maximum and nearly-maximum waves is discussed. Although their work implies that Havelock's computational scheme lacks mathematical rigor, it seems probable that these complexities do not vitiate the qualitative validity of the behavior exhibited by the model, whose predictions seem too reasonable to be wrong in their general implications.

1. The Maximum-Wave Model

My maximum-wave model profile is a simple parabola adjusted to have a 30° slope angle at the crest; it is given, with origin at the surface in a trough, by

$$y(x) = 3^{-\frac{1}{2}} x^2/\lambda_L$$

with $|x| \leq \frac{1}{2}\lambda_L$ and λ_L the wavelength. The surface flow speed is postulated to be given by

$$U^2 = 2g(y_{max} - y)$$

although this would require a surface pressure distribution varying from a constant one by perhaps $\pm$ 1% of the hydrostatic difference $\rho g y_{max}$ (because the model is not an exact solution) rather than the constant pressure normally assumed. We define $\beta \equiv 2x/\lambda_L$; $0 \leq |\beta| \leq 1$ between adjacent crests. Then we find

$$U^2(\beta) = 2^{-1}\, 3^{-\frac{1}{2}}\, g\lambda_L(1 - \beta^2)\,,$$

$$G(\beta) = (4g/3)\,[1 + (\beta^2/3)]^{-3/2}\,.$$

In this notation

$$p = (U\omega/G) = (3/4)(\omega^2\lambda_L/3^{\frac{1}{2}}2g)^{\frac{1}{2}}[1 + (\beta^2/3)]^{3/2}(1 - \beta^2)^{\frac{1}{2}}\,;$$

in a later section we will need

$$dp/d\sigma = (2U^2/\lambda_L g)(dx/ds)(dp/d\beta) = 4^{-1}\, 3^{\frac{1}{2}}[1 + (\beta^2/3)](1 - \beta^2)(dp/d\beta).$$

The dimensionless parameter $(\omega^2\lambda_L / 3^{\frac{1}{2}}\, 2g)$ is determined by specifying the wave number k_o, or wavelength λ_o, of the short waves of interest at the trough of the long wave of length λ_L. This parameter is equal to $4p_o/3$, and on branch CD $p_o = q_o - q_o^{\frac{1}{2}}$, with $q_o = U_o^2 k_o/G_o = (3^{\frac{1}{2}}\pi/4)(\lambda_L/\lambda_o)$. Since the smallest value of q_o on this branch is unity $(\omega \to 0)$ we see that $\lambda_o/\lambda_L < 3^{\frac{1}{2}}\pi/4 \simeq 1.36\cdots$ for $\omega > 0$. This trough value is greater than unity because the "local" wave number, differentially defined, increases away from the trough; the total distance between short-wave crests becomes comparable to the long-wave length as $\omega \to 0$. This limiting behavior is discussed in a later section.

We employ the model to obtain expressions on branch CD for the amplification factors a/a_o and k/k_o (where a_o and k_o are trough values) as functions of β. These factors depend on the choice of ω, or equivalently of λ_o/λ_L or of p_o. They are most compactly written in terms of $H \equiv 4p_o$ and $H\gamma(\beta) \equiv 4p(\beta)$ $= H(1 - \beta^2)^{\frac{1}{2}}\,(1 + \beta^2/3)^{3/2}$:

$$a/a_o = [(1 + H)/(1 + H\gamma)]^{\frac{1}{4}} \frac{1 + (1 + H\gamma)^{\frac{1}{2}}}{1 + (1 + H)^{\frac{1}{2}}} (1 - \beta^2)^{-\frac{1}{2}} ,$$

$$k/k_o = \frac{\frac{1}{2} H + [1 + (1 + H\gamma)^{\frac{1}{2}}]\gamma^{-1}}{\frac{1}{2} H + 1 + (1 + H)^{\frac{1}{2}}} (1 - \beta^2)^{-\frac{1}{2}} .$$

These expressions are complicated, and depend on the parameter H which is not simply related to λ_o/λ_L. However, they contain some of the principal results of this study and provide a theoretical framework against which to compare the results of experiments. It has not been easy to guess the most useful graphical forms in which to present numerical evaluations. For large values of H the behavior is simple (as predicted for $p \gg 1$) for values of β such that $(1 - \beta^2)^{-\frac{1}{2}}$ is considerably smaller than H, but the transition zone then occurs very near to $\beta = 1$. For small values of H the transition occurs at smaller values of β which approach zero as $H \to 0$ and λ_o/λ_L approaches the upper limit derived above.

From the forms of these expressions one can see that a/a_o will always be slightly less than $(1 - \beta^2)^{-\frac{1}{2}}$; also, k/k_o will always be greater than this quantity, the difference increasing rapidly well past the transition. In Figure 4 we present the relation between H and λ_o/λ_L . Figure 5 shows $(1 - \beta^2)^{\frac{1}{2}}(a/a_o)$ and $(1 - \beta^2)^{\frac{1}{2}}(k/k_o)$ vs. $(1 - \beta^2)^{-\frac{1}{2}}$ for selected values of λ_o/λ_L . In addition, in Figure 6 we have plotted as functions of λ_o/λ_L the values of β at which selected values of ak/a_ok_o, the product of these amplification factors, are attained.

Of particular interest is the behavior of the product ak, which must be small to justify linearization and whose growth measures the approach to maximum height. In a later section we make some comparisons relating to the belief that our linearized analysis retains qualitative validity up to this maximum height, corresponding to $ak \simeq 0.44$ if we identify 2a with the trough-to-crest height of finite-amplitude waves. We have already shown that ak is proportional to U^{-2} for large p and to U^{-3} for small p; the transition behavior on a maximum wave is given by the formulas above. Lest it be thought that the ratio $ak/a_o k_o$ becomes appreciable only at infinitesimally small distances from the crest, we point out that it attains the value seven between $\beta = 0.885$ and $\beta = 0.94$ for all values of λ_o in the relevant range, smaller values of β corresponding to larger values of λ_o. Amplification of ak by a factor of seven is sufficient to bring a small wave whose trough amplitude is only 1% of its wavelength up to the value 0.44 (an estimate of its instability limit) at the appropriate value of β in this range.

2. The Nearly-Maximum Wave Model

The profile of a nearly-maximum wave train is characterized by very small sharply curved caps; long slowly changing regions between crests, essentially the same as for a maximum wave; and small transition regions connecting these, containing points of inflection of the profile. We have replaced Havelock's parameter α characterizing wave amplitude by $\mathcal{E}$ to emphasize its role as a parameter of smallness here; $\mathcal{E} = 0$ gives the maximum wave and $\mathcal{E} \to \infty$ for infinitesimal waves, the amplitude being related to $e^{-2\mathcal{E}}$. His dimensionless units are denoted here by a tilde. The parameter replacing s along the surface is his dimensionless velocity potential $\tilde{\phi}$ (not to be confused with our perturbation potential ϕ for small waves); it is zero at the origin and is equal to $\pm n\pi$ (n integral) at the other crests. In the cap region we show that another parameter θ is more appropriate; it is defined by $\tan\theta \equiv \tan\tilde{\phi}/\tanh\mathcal{E}$, but we approximate it here by $\tan\theta \simeq \tilde{\phi}/\mathcal{E}$ since $|\tilde{\phi}|$ is of order $\mathcal{E}^{\frac{1}{2}}$ or less within the cap region near the origin. This cap region is defined by $0 \lesssim |\theta| \lesssim \frac{1}{2}\pi - (D\mathcal{E})^{\frac{1}{2}}$, and the main part of the wave from the inflection point down to the first trough by $(\mathcal{E}/D)^{\frac{1}{2}} \lesssim \tilde{\phi} \lesssim \frac{1}{2}\pi$, with $D \simeq 3/2$.

In the cap (but excluding the transition) the surface slope angle $\alpha(\theta)$ and the functions $\tilde{U}(\theta)$ and $\tilde{G}(\theta)$ are shown in the companion paper to be given to the lowest relevant order in $\mathcal{E}$ by

$$\alpha \simeq \theta/3 , \qquad \tilde{U} \simeq [\mathcal{E}\tilde{g}P(\theta)]^{1/3}, \qquad \tilde{G} \simeq \tilde{g}Q(\theta) ,$$

with α positive downward for positive θ ; here

$$P(\theta) \equiv (3/2) + 3\int_0^\theta \sin(\theta/3)\sec^2\theta\, d\theta$$

$$= 3\sec\theta\cos(2\theta/3) - (3/2) - C\,\ln\left[\frac{1-C}{1+C}\ \frac{\cos(\theta/3)+C}{\cos(\theta/3)-C}\right] ,$$

with $C \equiv \cos(\pi/6) = 3^{\frac{1}{2}}/2$, and

$$Q(\theta) = \cos(\theta/3) - 3^{-1}P(\theta)\cos^2\theta .$$

The function $P(\theta)$ diverges as $\theta \to \frac{1}{2}\pi$ but this value is not attained before entering the transition region; $P_{max} \simeq (3/2)(D\mathcal{E})^{-\frac{1}{2}}$.

In Figure 7 we display $P\cos\theta$, U, G, and U/G in suitable units as functions of θ. We see that in passing up the wave from transition to crest G decreases from almost $g\cos(\pi/6) = 0.866\,g$ to almost $\frac{1}{2}g$; this large change takes place over a very small distance of order $\mathcal{E}^{2/3}$. The velocity U also decreases rapidly from a value of order $\mathcal{E}^{1/6}$ to a value of order $\mathcal{E}^{1/3}$. We may expect, and will show later, that these rapid changes impair the validity of the WKB approximation for a portion of the small-wave spectrum in the cap regions of nearly-maximum waves. It is noteworthy that although both U and G decrease on approaching the crest the ratio U/G increases for θ less than about 55°. This fact becomes significant in connection with our discussion of energy exchange to follow.

The determination of the wave profile in the cap region is described in the companion paper; it is plotted in appropriate units in Figure 8, in which are also shown values of θ.

These expressions are all valid for $0 < \mathcal{E} \ll 1$ and for θ in the indicated range within the cap. The second term in $\tilde{G}$ is $\tilde{U}^2\, d\alpha/ds$, with sign (α is positive downward here) corresponding to a reduction in G arising from centrifugal acceleration, since the surface is convex upward in the cap. We restore physical dimensions to G by replacing $\tilde{g} = 0.833\cdots$ by g; velocities are obtained from $U^2 = \tilde{U}^2 \lambda_L g/(2^{1/3}\pi\tilde{g})$ and lengths from $x = \tilde{x}\lambda_L/(2^{1/3}\pi)$, etc., with λ_L the physical distance between crests.

In Havelock's units $p = U\omega/G = \mathcal{E}^{1/3}\,\omega\,\tilde{g}^{-2/3}\,P^{1/3}\,Q^{-1}$; in ordinary physical units

$$p = \left(\frac{\lambda_L g}{2^{1/3}\pi\tilde{g}}\right)^{1/2}\frac{(\mathcal{E}\tilde{g}P)^{1/3}}{gQ}\,\omega = \left(\frac{3^{\frac{1}{2}}\,12}{\pi^3\,\tilde{g}}\right)^{1/6}\left(\frac{\omega^2\lambda_L}{3^{\frac{1}{2}}\,2g}\right)^{1/2}\frac{(\mathcal{E}\,P)^{1/3}}{Q}\,.$$

The numerical factor $[(3^{\frac{1}{2}}\,12)/(\pi^3\,\tilde{g})]^{1/6}$ is equal to $0.964\cdots$; we approximate it by unity. The dimensionless parameter $(\omega^2\,\lambda_L/3^{\frac{1}{2}}\,2g)^{\frac{1}{2}} = 4p_o/3$ is the same one introduced earlier in this section to characterize a small wave-train riding on our model maximum wave and evaluated there in terms of the ratio of λ_L to the small-wave length λ_o in a trough. In a later section we will need for a nearly-maximum wave the quantity

$$dp/d\sigma = (U^2/G)(dp/ds) = (U^2/G)(d\phi/ds)(dp/d\phi) = (U^3/G)(dp/d\phi) .$$

The dimensionless operator $(U^3/G)(d/d\phi)$ is equal to $(\tilde{U}^3/\tilde{G})(d/d\tilde{\phi})$, and $d/d\tilde{\phi} \simeq \mathcal{E}^{-1}\cos^2\theta(d/d\theta)$, so that

$$dp/d\sigma \simeq (P/Q)\cos^2\theta \;\; dp/d\theta .$$

No special interest attaches to the transition region; all quantities merely connect smoothly across it.

We are now in a position to evaluate the components of acceleration of surface particles of fluid in the crest region of a nearly-maximum wave. The tangential component $a_s = \vec{a}\cdot\hat{e}_s$ is given by $g\sin\alpha$ if s increases away from the crest; the normal component $a_n = \vec{a}\cdot\hat{e}_n$ is inwardly directed here, the surface being convex upward, and is given in magnitude by $U^2\alpha' = U^3\,d\alpha/d\phi \simeq (gP\cos^2\theta)/3$. At the crest $a_s = 0$ and $a_n \simeq -\frac{1}{2}g$, while at the point of inflection $a_s \simeq \frac{1}{2}g$ and $a_n = 0$. The magnitude $|\vec{a}|$ is nearly constant in crossing the cap while its direction rotates steadily through a total range of almost 120°, being tangential to the surface at the inflection points and downward at the crest. These predictions of the model agree with known results (Longuet-Higgins, 1963, 1969a).

Finally the results of this section enable us, in principle, to place bounds on the growth factors a/a_o and k/k_o on branch CD as functions of $\mathcal{E}$, replacing the results of the preceding sub-section in which they were unbounded on a maximum wave as $\beta \to 1$. As mentioned above, we have obtained the properties of nearly-maximum waves only in the limit $\mathcal{E} \to 0$. At the crest, $4p$ approaches a value close to $12 p_o \mathcal{E}^{1/3} = 3H\mathcal{E}^{1/3}$ and U_o/U approaches $\sim 0.9\,\mathcal{E}^{-1/3}$. Therefore the crest values of the growth factors are obtained by replacing $\gamma(\beta)$ by $3\mathcal{E}^{1/3}$ and $(1-\beta^2)^{-\frac{1}{2}}$ by $0.9\mathcal{E}^{-1/3}$ in the earlier formulas. The forms taken by these bounds depend on whether $3H\mathcal{E}^{1/3}$ is very large, very small, or of order unity, as well as on the value of H itself. In the first case

$$\left.\begin{array}{l} a/a_o \to 1.2\,\mathcal{E}^{-1/4}, \qquad k/k_o \to 0.9\,\mathcal{E}^{-1/3}, \\ \qquad ak/a_o k_o \to 1.1\,\mathcal{E}^{-7/12} \end{array}\right\} \text{ for } \; 3H\,\mathcal{E}^{1/3} \gg 1.$$

In the second

$$\left.\begin{array}{l} a/a_o \to 1.8(1+H)^{\frac{1}{4}}\left[1+(1+H)^{\frac{1}{2}}\right]^{-1}\mathcal{E}^{-1/3} \\ k/k_o \to 0.6\left[1+\frac{1}{2}H+(1+H)^{\frac{1}{2}}\right]^{-1}\mathcal{E}^{-2/3} \end{array}\right\} \text{ for } \; 3H\,\mathcal{E}^{1/3} \ll 1.$$

We shall see in the next section that for $3H\,\mathcal{E}^{1/3}$ neither large nor small the WKB approximation becomes invalid very near the crests of nearly-maximum waves.

As a numerical example consider the nearly-maximum wave for which $\mathcal{E} = 10^{-3}$. The cap region includes $0 \leqslant |\theta| \lesssim 88^{\circ}$. At $\theta = 88^{\circ}$, $P \simeq 39$, $\tilde{U} \simeq 0.319$, $\tilde{x}_f \simeq 0.114$, and

$|\tilde{y}_{crest} - \tilde{y}_f| \simeq 0.055$; at the crest $\tilde{U} \simeq 0.108$. Converting to physical units, the cap includes $|x_f| \lesssim 0.029\ \lambda_L$, $|y_{crest} - y_f| \lesssim 0.014\ \lambda_L$; the latter is to be compared with the total wave height $h \sim 0.14\ \lambda_L$. Within the cap $U^2/(2\lambda_L g)$ increases from 0.0018 at the crest to 0.015 at its edge, to be compared with its value of ~ 0.14 in the trough. Now take a small wave whose trough value of λ_o is $\lambda_L/20$; for this wave $q_o = 27.2$, $p_o = 22$, $H = 88$, and $3H\mathcal{E}^{1/3} = 26.4$. Using the approximate forms for $3H\mathcal{E}^{1/3} >> 1$, we find $a/a_o \sim 6.8$, $k/k_o \sim 9$, and $ak/a_ok_o \sim 61$ at the crest; the complete forms give 7.3, 10.8, and 78, respectively. This small wave will attain its maximum height before reaching the crest unless $a_ok_o \gtrsim 6 \times 10^{-3}$. The value of β on entering the cap is $1 - (2 \times 0.029) = 0.942$; from the maximum-wave model $a/a_o = 2.6$, $k/k_o = 3.2$, and $ak/a_ok_o = 6.7$ at that point.

X. VALIDITY OF THE WKB APPROXIMATION

The validity of the WKB approximation for A depends on the smallness of the neglected term in the nonlinear differential equation for A with respect to either of the other two; if the product $|A^3\ d^2A/d\sigma^2|$ is small with respect to unity this approximation is valid. The smallness of the dispersion relation correction $\epsilon \equiv (qA)^{-1}\ d^2A/d\sigma$, the wave phase-shift $\delta \equiv \tan^{-1}[(q-p)^{-1}\ d(\ell n\ A)/d\sigma]$, and the correction of order δ^2 to the simple expression for wave amplitude a, are also related to its validity. In this section we examine these questions on branches CD and OA (the regions with $\omega > 0$) for the wave train of maximum height, and on branch CD in the cap region of a nearly-maximum wave train, using the models described in the preceding section.

Because $A \approx (\frac{1}{4} + p)^{-\frac{1}{4}}$, one can show that

$$A^3\, d^2A/d\sigma^2 \;\equiv\; W(p,\sigma) \;\simeq\; \frac{4p}{(1+4p)^2}\left[\frac{4p}{1+4p}\,a_1 - a_2\right]$$

with $a_1 \equiv 5M^2/4$, $a_2 \equiv M^2 + dM/d\sigma$, and $M \equiv d(\ln p)/d\sigma$. The quantitites M, a_1, and a_2 thus defined are independent of p and depend only on σ or s through the functions $U(s)$ and $G(s)$ which are assumed known and are given approximately by the models. The quantity $p \equiv U\omega/G$ depends on s through these functions, and also on the parameter ω which determines the trough-value of the short-wave length of interest. In this notation,

$$\epsilon \;\approx\; (\tfrac{1}{4} + p)\left[\tfrac{1}{2} + p \;\pm\; (\tfrac{1}{4} + p)^{\frac{1}{2}}\right]^{-1} W\,,$$

$$\tan\delta \;\approx\; \tfrac{1}{4}\,(\tfrac{1}{4} + p)^{-1}\left[\tfrac{1}{2} \mp (\tfrac{1}{4} + p)^{\frac{1}{2}}\right] M\,,$$

with the upper sign on BCD and lower on AOB.

1. The Maximum Wave

Using U, G, and $dp/d\sigma$ of the maximum-wave model, we find

$$M = -3^{-\frac{1}{2}}\,\beta^3, \qquad a_1 = 5\beta^6/12, \qquad a_2 = (\beta^2/12)(7\beta^4 + 6\beta^2 - 9).$$

The ranges of interest in p and β on branches CD and OA are $0 < p < \infty$ and $0 \leqslant |\beta| \leqslant 1$, but p contains the factor $(1 - \beta^2)^{\frac{1}{2}}$ so that $p \to 0$ as $\beta \to 1$ for any value of ω. We may therefore determine an upper bound on $|W|$ by examining this entire $p - \beta$ domain.

It is obvious that $|W| \propto p$ as $p \to 0$ and $|W| \propto p^{-1}$ as $p \to \infty$. For fixed β, W has extrema at

$4p = \left\{ a_1 \pm [(a_1 - a_2)^2 + a_1 a_2]^{\frac{1}{2}} \right\} / (a_1 - a_2)$ and for fixed p it has an extremum at $4p = (\partial a_2/\partial\beta)/[\partial(a_1 - a_2)/\partial\beta]$. These equations have only a single simultaneous solution for $p > 0$, $0 < \beta < 1$, at $\beta^2 \simeq 0.489$, $p \simeq 0.2880$, at which the value of W is $\simeq 0.051$. If p were independent of β there would also be a negative extremum at the boundary $\beta = 1$, whose magnitude is greatest at $p \simeq 0.104$; here $W \simeq -0.044$. Therefore we have shown that $|W| \lesssim 0.05$ for all ω and β in this domain and that the WKB approximation for A is valid everywhere for small waves on branches CD and OA traveling on a long wave of maximum height. These conclusions do not apply to waves with reverse propagation $(\omega < 0)$ which we do not analyze here. They obviously fail to apply near point B where $p = -\frac{1}{4}$, the anomalous wave mentioned earlier, and at singular points at wave crests.

We have already shown that the smallness of $|\epsilon|$ can be inferred directly from the smallness of $|W|$ on branch CD; $|\epsilon| = (\frac{1}{4} + p)q^{-1}|W| < |W|$ there. However, on OA the conclusion is the same for large p but ϵ diverges for $a_2 \neq 0$ as $p \to 0$, showing that neglect of ϵ is invalid for very long small waves on this branch. Quantitatively, the largest value attained by $|\epsilon|$ in $0 \leqslant |\beta| \leqslant 1$ for $p = 2$, $q = \frac{1}{2}$ is 0.04, and even as near to point O as $p = 3/4$, $q = 1/4$ this largest value is ~ 0.16. The "local wavelength" $2\pi/k$ of this wave is more than five times λ_L in the trough, and even as near the crest as $\beta \approx 0.85$ it is twice λ_L. Thus we may neglect ϵ everywhere on OA except for very long small waves.

We find from the model that $|\tan\delta| = (3^{-\frac{1}{2}}\beta^3/4\ [(\frac{1}{4} + p)^{\frac{1}{2}} - \frac{1}{2}]/(\frac{1}{4} + p)$ on branch CD. Examining this function as though p and β were independent, we see that $|\tan \delta| \to 0$ as $p \to 0$, as $p \to \infty$, and as $\beta \to 0$. For fixed β it has a single maximum at $p = 3/4$, where its value is $3^{-\frac{1}{2}} \beta^3/8$. We cannot realize its largest value by letting $\beta \to 1$ since $p = 0$ for all ω there, but we obtain an upper bound; $|\tan \delta| < 0.072$. Thus the phase shift is always less than about 4°; since the correction to the amplitude $a(s)$ is of order δ^2, both may be neglected. Using the lower sign on branch OA, $|\tan \delta|$ increases as p decreases, approaching a maximum value of $3^{-\frac{1}{2}} \beta^3$ as $p \to 0$. Thus $|\delta| < 30^{\circ}$ everywhere on OA; if we stay above $p = 3/4$, $q = 1/4$, $|\tan \delta| < 3^{\frac{1}{2}} \beta^3/8$ and $|\delta| < 12^{\circ}$.

In this way we have justified use of the WKB approximation for A, and neglect of ϵ, δ, and the correction of order δ^2 to a, for all small waves on CD and for $q > \frac{1}{4}$ on OA, when riding on our model of a wave of maximum height. In the companion paper we note that the model slightly underestimates our wave-interaction function in the trough of the maximum wave. Through a numerical study not detailed here, using the best available digital representation of the maximum wave, we have found that correcting for this discrepancy has no effect on the validity of these conclusions. The reason is that it exists only in the trough where U and G are varying most slowly; a_1 is unaltered, and the shape and magnitude of a_2 are changed but little by the correction. In Figure 9 we show M, a_1, and a_2 as functions of β.

For given long waves of height substantially less than maximum the variations of U and G are much less than in this limiting case. It is not difficult to show that for a Stokes wave represented by a power series expansion in $a_L k_L$ the first nonvanishing term in W is $-8p(1 + 4p)^{-2}(a_L k_L)^3 \cos k_L x$ with $x = 0$ in the trough; its magnitude is greatest for $p = \frac{1}{4}$ and is less than 0.05 there even if the value $a_L k_L = 0.44$ associated with a wave of maximum height is inserted. Therefore our approximations are valid, for all the small waves mentioned just above, everywhere on every long-wave train except near the sharply curved crests of waves of nearly-maximum height. It is not clear that this rather remarkable property could have been anticipated by physical intuition.

2. The Nearly-Maximum Wave

Here we examine the validity of the WKB approximation near the crests of nearly-maximum waves, confining our attention to branch CD. Using U, G, and $dp/d\sigma$ for our model of the cap region excluding the transition, we have as $\varepsilon \to 0$

$$M = d(\ln p)/d\sigma = (P/Q)\cos^2\theta \, d(\ln p)/d\theta$$

$$= (P/Q)\cos^2\theta \, [(3P)^{-1}dP/d\theta - Q^{-1}dQ/d\theta] \; ;$$

$$dP/d\theta = 3 \sin(\theta/3)\sec^2\theta \, , \qquad dQ/d\theta = -[4 \sin(\theta/3) - P \sin 2\theta]/3 \; ;$$

$$M = Q^{-2}\left\{\frac{1}{2}\sin(2\theta/3) + P\cos^2\theta[\sin(\theta/3) - 3^{-1} P \sin 2\theta]\right\} .$$

As before, $a_1 \equiv 5M^2/4$ and $a_2 \equiv M^2 + dM/d\sigma = M^2 + (P/Q)\cos^2\theta \, dM/d\theta$. In Figure 10 we plot M, a_1, and $\frac{1}{4} a_2$ as functions of θ. For each value of θ from zero at the crest to nearly 90° in the transition region we may find the value of p at which $|W|$ attains a maximum, and the value of this maximum. As expected, the results at $\theta = 90^\circ$ agree with and join those for the maximum wave as $\beta \to 1$; that is, W has a negative extremum of $\simeq -0.04$ at $p \simeq 0.1$. As we move toward the crest the magnitude of this extremum increases, reaching -0.1 for $p \sim 0.2$ at $\theta = 75^\circ$ and -0.3 for $p \sim \frac{1}{4}$ at $\theta = 60^\circ$. The variation of W_{ext} with θ is also shown in Figure 10; W_{ext} is always close to $-\frac{1}{4}a_2$. The value of p at which it is attained remains near to $\frac{1}{4}$ for $0 < \theta < 60^\circ$, except for a very small region centered at $\theta \sim 23.6^\circ$ of extent $\sim 1.5^\circ$ in which the equation for the extremum yields two positive solutions for p; this feature may be an artifact of the model.

As indicated in the preceding section, $4p \to 3H\mathcal{E}^{1/3}$ at the crests, where $a_1 \to 0$ and $a_2 \to -8$. Therefore at this point W will have the value $24H\mathcal{E}^{1/3}/(1 + 3H\mathcal{E}^{1/3})^2$. This is only small (say $< \frac{1}{2}$ as an extreme) for $3H\mathcal{E}^{1/3} > 14$ or $< (14)^{-1}$; W attains the maximum value 2 at $3H\mathcal{E}^{1/3} = 1$. Because we have not established the largest value of the infinitesimal quantity $\mathcal{E}$ for which our limiting forms for $\mathcal{E} \to 0$ retain qualitative validity, we cannot be quantitative about this very local and not unexpected failure of the WKB approximation. It is clear that for any given $\mathcal{E}$ within the acceptable range there will be some portion of the small-wave spectrum for which the approximation loses validity.

As one might expect, it can be shown that these "critical waves" are ones whose wavelength has shrunk in traveling up from the trough so as to be comparable to the physical extent s_c of the cap, which is small of order $\mathcal{E}^{2/3}$, when they arrive. Using $(\tilde{g}\mathcal{E}^2)^{1/3}\,\tilde{s}_c \sim 2\pi$ from Figure 8 as an estimate of the cap's extent, we find $s_c \sim 1.7\,\mathcal{E}^{2/3}\lambda_L$, and $0.4 < \lambda_{crest}/s_c < 2.5$ for the critical domain as defined above. The shortest critical waves are those whose wavelengths in the trough are small of order $\mathcal{E}^{1/3}$. For the numerical example of the preceding section with $\mathcal{E} = 10^{-3}$, we find $0.09 < \lambda_o/\lambda_L < 1.2$ in the critical region. The WKB approximation retains its validity throughout the crest region for waves either longer or shorter than these critical waves.

Except for these critical waves the parameter ϵ which modifies G, the phase shift δ, and corrections of order δ^2 to the wave amplitude a are also negligible on branch CD throughout the cap regions of nearly-maximum waves.

The failure of the WKB parameter W to remain small with respect to unity does not necessarily have drastic consequences for these critical waves. Rather, it means that their wavelengths, amplitudes, and phases will undergo variations somewhat different from those predicted by the approximation. For most of the critical waves these differences will be only quantitative and not qualitative, since $|W|_{crest} < 1$ for all but 60% of the domain in p_{crest} within which it is $< \frac{1}{2}$. This critical part is in turn only a part of the total spectrum of the small waves of interest. Although one could study the departures from adiabaticity in detail, or examine branch OA, we will content ourselves in what follows with use of the WKB results throughout.

XI. WAVES OF NEARLY EQUAL LENGTH; THE ANOMALOUS WAVE

We have taken the spectrum of small waves of special interest to be those on branch CD of Figure 3, whose crests move up the down-wind sides of the long waves toward their crests, and have pointed out that the lower end of this branch near C, where $\omega \to 0$, corresponds to small waves whose trough-values of wavelength are not short compared to the given long-wave length. In fact, for long waves of less than maximum height the wave at C with $\omega = 0$ merely represents a very small addition to the given wave, since the total wave profile remains static. Points near C with very small positive or negative ω represent small-amplitude waves moving backward or forward, respectively, very slowly with respect to the given waves. Our analysis remains valid for them, but they have the property of being in the domain $p \ll 1$ everywhere since they already have this property in the long-wave troughs. As we remarked earlier for the shorter small waves, our results for their changes in amplitude and wave number in passing along a given wave-train of small or moderate amplitude may be of interest for other purposes, but our attention here is focused on their influence on long waves of maximum or near-maximum height.

As $\omega \to 0$ near point C our results approach a particularly simple form; $A \to$ constant, $U^2k/G \to 1$, $\int kds \to \int (G/U^2)ds = \sigma - \sigma_o$, $\phi \to \text{const.} \cos(\sigma - \sigma_o)$, and $\xi \to \text{const.}\ U^{-1} \sin(\sigma - \sigma_o)$. Using our model for the wave of maximum height

$$\sigma \simeq \int kds \to 3^{-\frac{1}{2}} \cdot 4 \cdot \int \left[(1 + \beta^2/3)(1 - \beta^2)\right]^{-1} d\beta = \tan^{-1}(3^{-\frac{1}{2}}\beta) + 3 \tanh^{-1}\beta.$$

Thus we see directly the previously discussed rapid growth in amplitude and wave-number. Plots of ξ vs. β in this limit are shown in Figure 11 for $\sigma_o = 0$ and $\pi/2$.

Our analysis degenerates at point B, a small wave with reversed phase velocity for which $p = -\frac{1}{4}$. At this point the group velocity in our coordinates vanishes; in bottom-fixed coordinates the small wave crests move down-wind with speed 2U while their group velocity is U. This phenomenon has been noted by other workers, some of whom have characterized it as a barrier. If the given steady flow is that of a wave of small amplitude, this anomalous wave is one with wave length four times as great. Different methods are needed to analyze the behavior of a and k on passing along branch OB or CB toward B. Our rigorous equation for A is $d^2A/d\sigma^2 + \left[\frac{1}{4} + p(\sigma)\right] A - A^{-3} = 0$; if we replace G^* by G in our definitions of p and q, $q = p + \frac{1}{2} \pm A^{-2}$ exactly, with upper sign on BC and lower on OB. This relation is equivalent to $(p - q)^2 = q + A^{-1} d^2A/d\sigma^2$. We may now assign a suitable form to $p(\sigma)$ and integrate the nonlinear equation for $A(\sigma)$ (numerically, if necessary), working back from a region of WKB validity; q and a may then be found. For example, if $p(\sigma) = -\frac{1}{4} + K^2\sigma$ (with K a constant) the equation for F becomes $d^2F/d\sigma^2 + K^2\sigma F = 0$, whose solution for $\sigma > 0$ is $F = v^{1/3}\left[aJ_{1/3}(v) + bJ_{-1/3}(v)\right]$, with $v \equiv 2K\sigma^{3/2}/3$; A is the envelope surrounding this oscillation, nearly $(K^2\sigma)^{-\frac{1}{4}}$ for large σ. We shall not pursue this problem further here.

XII. WAVE ENERGY, ENERGY FLUX, AND ITS SPATIAL CHANGE; THE RELATION BETWEEN G AND U, AND THE WAVE-INTERACTION FUNCTION Ω

We introduce our discussion of energy by making a detailed comparison of our results for wave-number k, amplitude a, wave energy E, its flux J, and its spatial rate of change with those of LS II for the two-dimensional steady flow characterized there as upwelling. These authors found the relations

$$\frac{1}{k}\frac{dk}{dx} = -\frac{2}{c + 2U}\frac{dU}{dx} \quad \text{and} \quad \frac{1}{a}\frac{da}{dx} = -\frac{2c + 3U}{(c + 2U)^2}\frac{dU}{dx} \qquad \text{(LS II)}$$

by solving a specific flow problem in which the interaction term was linearized. They integrated these differential relations by using $c^2 = g/k$ with g constant, finding $k(U + c) = \omega = $ constant and

$a^2 \propto |c(c + 2U)|^{-1}$. The first result agrees with the physical postulate that the distance between crests varies only through surface stretching, which leads directly to (frequency) × (wavelength) = wave-velocity, or $k(U + c) = \omega$. Their application of the second result is not so simple. They considered the transport of wave energy and examined several alternative expressions for energy conservation. A selection among these options was made by requiring its differential form to be consistent with their result above for amplitude change. The one thus selected, under the assumption that $E = \frac{1}{2} \rho g a^2$ with g constant, was

$$(d/dx)[E(U + c_g)] + S_x \, dU/dx = 0 \, , \qquad \text{(LS II)}$$

with S_x, the only component of their radiation stress tensor relevant in a two-dimensional problem, equal to $\frac{1}{2} E$ in deep water. They were gratified to find this in complete agreement with their result from LS I for small short waves moving on small long ones; the general validity of this relation seemed to have been established.

Our more general expressions for wave-number and amplitude are already in integrated form but must be manipulated to put them in this notation. We find

$$k \equiv Gq/U^2 = (\omega q/U)(G/U\omega) = (\omega/U)(q/p) = (\omega/U)(U/v_p) = \omega/(U + c) \, ,$$

$$a^2 \propto (A\Delta/U)^2 = [(p - q)/(\tfrac{1}{4} + p)^{\frac{1}{4}}U]^2 = [(U/c)/(v_g/c)^{\frac{1}{2}}U]^2$$

$$= |cv_g|^{-1} \propto |c(c + 2U)|^{-1} \, ,$$

displaying agreement with theirs. However, our differential forms will not agree in general;

$$dk/k = -dU/U + dq/q = -dU/U + [(p/q)(dq/dp) - 1]dp/p$$

$$= -dU/U + [(v_p/v_g) - 1]dp/p = -dU/U + [c/(c+2U)]dp/p$$

$$= -du/U + [c/(c+2U)](dU/U - dG/G)$$

$$= -\frac{2U}{c+2U}\frac{dU}{U} - \frac{c}{c+2U}\frac{dG}{G},$$

and by a similar but more lengthy manipulation

$$da/a = -dU/U + (v_p/2v_g)^2 dp/p = -\frac{2c+3U}{(c+2U)^2}dU - \frac{(c+U)^2}{(c+2U)^2}\frac{dG}{G}.$$

There is complete agreement for the flow selected in LS II, in which the free surface is a straight line whose slope is small of first order so that centripetal acceleration is absent and $G \simeq g =$ constant. However, in addition to our added terms in the differentials of k and a, important differences involving the energy and its transport and exchange with the given flow will exist for any steady flow having varying G. It is significant that our expression for the energy, $E = \frac{1}{2}\rho G a^2$, differs from theirs in that g is replaced by G. Their energy is proportional to $|c(c+2U)|^{-1}$, while ours, in this notation, is

$$E \propto (p-q)^2(\tfrac{1}{4}+p)^{-\frac{1}{2}}(G/U)U^{-1} \propto |(p-q)(\tfrac{1}{4}+p)^{-\frac{1}{2}}[1-(q/p)]U^{-1}|$$

$$= |[1-(U/v_p)]/v_g| = |c/[(c+U)(c+2U)]|.$$

Their energy flux $J = Ev_g$ is proportional to $|c|^{-1}$, while ours varies as $|c/(c+U)|$.

The importance of these differences is particularly great on the given small-amplitude long-wave flow studied in LS I which originally led to their concept of radiation stress. We have shown that on this flow G is closely proportional to U. If we adopt the assumption that G/U is a constant, then $p = U\omega/G$ is constant, so that kU, aU, EU, and all velocity ratios are constants, and $\epsilon = \delta = 0$, for all values of p. Finally, and of greatest significance, J is constant, so that

$$dJ/dx = (d/dx)[E(U + c_g)] = 0 , \quad \text{(for } G/U \text{ constant)}$$

completely removing the necessity and the justification for the radiation stress!

We defer to the next section an examination of the difficulties associated with their radiation stress tensor, turning instead to evaluation of the spatial rate of change of the small-wave energy flux in the general case. One must be careful with signs. Since E is proportional to $G/|cv_g|$, we set the conserved quantity $E|cv_g|/G$ equal to a positive constant K. Then

$$J = Ev_g = K(\operatorname{sgn} v_g)G/|c|, \quad \text{and} \quad G/|c| = \omega U/p|c| = \omega[1 - (q/p)](\operatorname{sgn} c).$$

Thus $J = K\omega \operatorname{sgn}(cv_g)[1 - (q/p)]$ and

$dJ/ds = - Ecv_g\, \omega G^{-1}\, d(q/p)/ds = - Ecv_g(p/U)d(q/p)/ds$. Now

$$\frac{d}{ds}\left(\frac{q}{p}\right) = \left(\frac{dq}{dp} - \frac{q}{p}\right)\frac{1}{p}\frac{dp}{ds} = \left(\frac{1}{v_g} - \frac{1}{v_p}\right)\frac{U}{p}\frac{dp}{ds} = \frac{c}{2v_g v_p}\frac{U}{p}\frac{dp}{ds} ,$$

$$\frac{dJ}{ds} = -\frac{E}{2}\left(\frac{pc^2}{Uv_g}\right)\left(\frac{U}{p}\frac{dp}{ds}\right) ,$$

and $(pc^2/Uv_g) = p(c/U)^2(U/v_p) = p(p - q)^{-2}(q/p) = 1$ from the dispersion relation. We express $(U/p)(dp/ds)$ as $U[U^{-1}(dU/ds) - G^{-1}(dG/ds)] = [1 - (U/G)(dG/dU)](dU/ds)$, which motivates the definition of our last important parameter, the wave-interaction function Ω:

$$\Omega \equiv 1 - \frac{U}{G}\frac{dG}{dU} .$$

The exchange of energy between the small wave and the given flow is thus governed by

$$dJ/ds = -\frac{1}{2} E\, \Omega(dU/ds) ;$$

we see that the "radiation stress" must be multiplied by Ω, whose value is unity for the flow of LS II and nearly zero for that of LS I.

We next analyze the relation between G and U, and the behavior of Ω, for four flows: our maximum-wave model, our model of the cap region of a nearly-maximum wave, a local region of an arbitrary flow, and a moderate-amplitude wave train represented by the classical Stokes expansion. For any steady flow with constant $\vec{g}$, G(s) is determined by the shape of the wave profile and the surface flow speed U(s), which in turn is also determined by the profile (from the Bernoulli surface condition) and an integration constant specifying its value at one point. Therefore Ω is a purely geometrical property of the profile and that constant. From our parabolic maximum-wave model

$$U/G \propto (1 - \beta^2)^{\frac{1}{2}}(1 + \beta^2/3)^{3/2} ;$$

U/G is constant to within about 10% over the lower 70% of the surface. For this simple model

$$\Omega = (4\beta^2/3)/(1 + \beta^2/3) ,$$

which rises quasi-parabolically from zero in the trough and approaches unity at the crest. Correcting for the model's under-estimate of Ω in the trough leads to the plausible fit

$$\Omega \simeq (1 + 4\beta^2)/5 ,$$

as shown in the companion paper.

From our model of the cap region of a nearly-maximum wave,

$$\Omega = 1 - (3P/Q)(dQ/d\theta)/(dP/d\theta) ,$$

which can readily be evaluated from equations given in an earlier section. This function is plotted in Figure 12. At the inflection point near $\theta = 90^\circ$ it approaches unity, joining smoothly the maximum-wave value just obtained. As $\theta \to 0$, $\Omega \to -4$; it crosses zero near $\theta = 56^\circ$. For the entire profile of a nearly-maximum wave we then have a nearly-parabolic rise from a value near 0.2 in the trough to nearly unity at the inflection point, followed by a rapid decline to zero well up on the cap and a precipitous drop to -4 at the crest.

Expressions for Ω on a general steady-flow profile $y(x)$ on which $U^2 = U_o{}^2 - 2g(y - y_o)$ may be written in several forms. Using primes to denote derivatives with respect to x, we have $y' = \tan\alpha$ and $y'' \cos^3\alpha = d\alpha/ds = R^{-1}$, the curvature. Then

$$G/g = \cos\alpha[1 + (U^2/g)y'' \cos^3\alpha] ,$$

$$d(G/g)/dx = -3(G/g)y'' \sin\alpha\cos\alpha + (U^2/g)y''' \cos^3\alpha,$$

$$d(\ln G)/dx = -3y'' \sin\alpha\cos\alpha + [(U^2/g)y''' \cos^2\alpha]/[1 + (U^2/g)y'' \cos^2\alpha] ,$$

$$d(\ln U)/dx = -(U^2/g)^{-1} \tan\alpha ,$$

so that

$$\Omega = 1 - 3F + (1 + F)^{-1} F^2 \sec^2\alpha (y')^{-1}(y'')^{-2} y''' \quad \text{with}$$

$F \equiv (U^2 y'' \cos^2\alpha)/g$. [An alternative form, independent of explicit reference to coordinates, is

$$\Omega = [1 - 2NC + N^2(dC/ds)/\tan\alpha]/(1 + NC)$$

with $N \equiv U^2/(g \cos\alpha)$ and $C \equiv R^{-1}$ = curvature.] From this we first see that $\Omega \to 1$ at the crest of a maximum wave because $U \to 0$ there. At the inflection point near the cap of a nearly-maximum wave $y'' = 0$, $\alpha \simeq 30^\circ$, and $\Omega = 1 + 1.3(U^2/g)^2\, y'''$. Because U^4 is of order $\mathcal{E}^{4/3}$ there, $\Omega \simeq 1$ as we have found. At the bottom of a symmetrical trough we must be more careful; both y''' and y' vanish there, but their ratio approaches that of y'''' to y''. For our parabolic model $y'''' = 0$, and the value of $3(U^2/g)y'' \cos^2\alpha$ (the second term in Ω) at the trough is exactly unity, confirming that $\Omega = 0$ there. However, for a maximum-wave profile of slightly lesser wave height or trough curvature the second term will be slightly less than unity, and Ω will be small and positive in the trough if $y'''' \sim 0$ there; this seems to be the case for the correct profile. At the bottom of an unsymmetrical trough with $y''' \neq 0$, a symmetrical quartic trough with $y'' = y''' = 0$ but $y'''' \neq 0$, and in general at any point where $U' = 0$ but $G' \neq 0$ we find $|\Omega| \to \infty$, but in the equation for dJ/ds it is multiplied by dU/ds, which vanishes there, so that no physical divergence occurs; this situation is discussed in the following section.

We now verify our earlier result that in linear approximation G/U is constant and $\Omega = 0$ on a small-amplitude wave train. To first order in ak (we drop subscripts L here), $y = a \cos kx$, $U_o^2/g = k^{-1}$, $\alpha = -ak \sin kx$, $(U_o^2/g)y'' = -ak \cos kx$, and $y''' = ak^3 \sin kx$, so the second term in Ω is small of first order and the third is $\simeq -1$, giving $\Omega = 0$, plus small corrections of order ak, for all x.

It is not difficult, by use of Stokes-type expansions, to show that these first-order corrections cancel. We have investigated the second-order corrections to Ω, which depend on expansions of G and U to third order in $a_L k_L$. High-order expansions for the wave velocity and profile were carried out a long time ago by Rayleigh and many others, but expansions of G and U were not previously of interest; therefore some details of our third-order calculations are included in the Appendix, where a subtle point that could lead to an error is noted.

The result is that Ω contains a constant term of second order equal to $2(a_L k_L)^2$ but no oscillating terms to this order, although the expansions for U and G contain terms differing in both second and third orders. From the structure of the calculation one can see that to the next order

$$\boxed{\Omega = 2(a_L k_L)^2 + (d_1 \cos k_L x + d_3 \cos 3k_L x)(a_L k_L)^3 + O(a_L k_L)^4,}$$

where d_1 and d_3 are dimensionless constants of order unity not evaluated here. This result is consistent with the fact that Phillips (1960) and Hasselmann (1962, 1963), who have examined wave-wave interactions by successive approximation or perturbation methods, found no interactions to second order, the first nonvanishing contributions being for specific wave-number combinations of third order.

As we will see shortly, the constant term in Ω yields an oscillatory interchange of energy between the two wave trains which is not related to specific wave-number combinations but is proportional to the product of the squares of both wave amplitudes; $\frac{1}{2} E\Omega = \text{const. } a^2 a_L^2$.* In addition the unevaluated oscillatory terms in Ω will produce energy exchanges involving specific harmonics of the given wave and additional powers of its amplitude. Although the connection between our result and their work requires further study, it seems likely that our general expression for the exchange of energy between a small wave and one of arbitrary height should be useful to workers trying to extend or interpret analyses of this kind.

By an argument invoking continuity between waves of small and large amplitude we may expect for intermediate amplitudes a positive average for Ω, with a decrease at crests and an increase where the wave is steepest. As the amplitude increases we may expect a moderate decrease below the average to develop in the troughs. The minimum value of Ω will occur at the crests, and will become negative for waves above a certain height. As the height increases further the region of negative Ω will become more negative, and will be increasingly confined near the crest, approaching the very narrow zone with width of order $\varepsilon^{2/3}$ and depth ~ -4 that we have found for nearly-maximum waves.

---

* We take this symmetry as a confirmation of the validity of our results; if both wave trains are treated as small of the same order, an interchange of their roles in the analysis should not affect the amplitude-dependence of this interaction.

For waves on branch CD, v_g and J are positive. The energy E is intrinsically positive, so the effect of a small positive constant Ω is to produce an increase in the energy flux of a small short wave as it travels from trough to crest up the down-wind side of a long wave, where dU/ds is negative, and a symmetrical decrease on descending. This oscillating energy interchange between long and short waves is very feeble for long waves of rather small amplitude, in marked contrast with the result of the radiation-stress theory. For long waves of large height the transfer of energy from long to short waves becomes significant as the small waves climb up near to the inflection point. It seems unlikely that rapid reversal within the narrow cap region of this energy transfer can be complete with those small waves which have not broken over themselves but for which the WKB approximation is failing there.

XIII. DISCUSSION OF THE SURFACE STRESS TENSOR CONCEPT

Following publication of the papers referred to as LS I and II and a third paper on shallow-water phenomena, Longuet-Higgins and Stewart (1964) presented a review paper called LS IV here* devoted entirely to their radiation stress tensor. In it they reviewed their physical reasoning, recounted previous applications, and described some new ones. Because our work has led to different conclusions it is a part of our task to discuss in some detail the basis of the differences. We confine our attention to two-dimensional deep-water flows of ideal fluids, related to pp. 530-535, 551-553, and 556-558 of LS IV.

---

* This notation is chosen because in LS IV they designated their third paper as III.

The separation of power per unit volume into two portions, one due to stress-strain rate of work and the other to the divergence of energy flux, is somewhat artificial; the terms which are isolated, integrated vertically, and discussed on pp. 530-535 of LS IV may be viewed as coming partly from each portion. For irrotational incompressible flow (using Cartesian tensors) the first portion depends only on the terms $\rho v_i v_j$ in the stress tensor $\Pi_{ij} = (p + \rho gy)\delta_{ij} + \rho v_i v_j$ and is given by $\underset{\sim}{\Pi}:\underset{\sim}{\epsilon} = \vec{\nabla} \cdot \left[(\frac{1}{2}\rho v^2)\vec{v}\right]$, with $\epsilon_{ij} = \partial v_i/\partial x_j$ the traceless symmetric rate-of-strain tensor. The second portion comes from $\vec{\nabla}\cdot\vec{j}$, with the energy flux $\vec{j} = (p + \rho gy + \rho v^2)\vec{v}$ given by $j_i = \Pi_{ij}v_j$. The time rate of energy loss per unit volume is

$$\vec{\nabla}\cdot\vec{j} - \underset{\sim}{\Pi}:\underset{\sim}{\epsilon} = \vec{\nabla}\cdot\left\{(p + \rho gy + \rho v^2)\vec{v} - (\frac{1}{2}\rho v^2)\vec{v}\right\},$$

displaying the mixed origins of contributions from $p\delta_{ij} + \rho v_i v_j$.

For flows with $G \neq g$ the use of Cartesian coordinates tends to obscure useful physical properties. For example, in the $(\bar{\phi}, \bar{\Psi})$ coordinates of Figure 1b we find $\epsilon^{\mu}_{\nu} = V^{\mu}_{,\nu} = V^{\nu}_{,\mu} = \begin{pmatrix} \partial V/\partial s & V/R \\ V/R & -\partial V/\partial s \end{pmatrix}$ for the steady flow; each term arises from Christoffel symbols needed for covariant differentiation, and $\vec{V} = \vec{\nabla}\phi$ is the flow speed at any point. Further, $f_\nu = \Pi^{\mu}_{\nu,\mu} = 0$ for this flow, yielding

$$-\rho^{-1}\,\partial p/\partial s = g \sin\alpha + V\,\partial V/\partial s, \qquad -\rho^{-1}\,\partial p/\partial n = g\cos\alpha + V^2/R = G$$

by obvious extensions of our definitions of G, α, s, n, and R at the surface. The natural way in which G replaces g is noteworthy.

In applications a knowledge of their radiation stress tensor is of little interest in itself; its utility arises from its contribution, together with the corresponding rate-of-strain tensor, to the spatial rate of change dJ/ds of wave energy flux. One cannot deduce the form of dJ/ds on any surface from general tensor arguments without resort to detailed calculations. A four-element Cartesian surface tensor obtained by vertical integration of four of the nine elements of a stress tensor cannot represent the integrated vertical component of force, or any component exerted across a horizontal surface; but the essence of gravity-wave motion is variation in space and time of vertical components of force, acceleration, and velocity, producing surface slope and curvature. These quantities enter the surface tensor only through its dependence on parameters of the surface waveform through the detailed calculations just mentioned. Therefore the value of separating certain terms from the total stress, integrating them, and identifying the resulting array as a radiation stress tensor will differ for various flows on which the detailed calculations give differing results.

We do not find their radiation stress tensor useful in this sense except on flows for which $G \simeq \text{constant} = g$, the only ones for which their results and ours agree. This special property of the two-dimensional "upwelling" flow selected in LS II, and reviewed on pp. 551-553 of LS IV, renders their result for dJ/dx inapplicable on flows with tilted or curved profiles in general and on wave-like flows in particular. However, they have taken it to be a general one; for the long-wave flow of LS I, reviewed on pp. 556-558 of LS IV, this error is exactly compensated by another one in their detailed calculation of dJ/dx for this flow, as will next be shown.

Consider the equation (with $S_x = \frac{1}{2} E$)

$$(d/dx)\left[E(U + c_g)\right] + QS_x \, dU/dx = 0 ,$$

the form of their general relation in our moving cordinates if $Q = 1$. Here U contains not only the horizontal part of the long-wave orbital velocity but also the large constant term (equal to $-c_2$ for their long-wave propagation to the right) needed to transform to our coordinates. They have adopted for wave energy the expression $E = \frac{1}{2}\rho g a^2(1 + P\Delta g/g)$ with $P = \frac{1}{2}$ and $\Delta g = g' - g \simeq G - g$, but we regard P and Q as constants to be determined. Our results and theirs agree that on this flow $da/a = -\frac{3}{4}\, dU/U - \frac{1}{4}\, dG/G = -\, dU/U$ in deep water. In lowest approximation [neglecting c_g/U and $(dc_g/dx)/(dU/dx)$, which are both small of order $(k_2/k_1)^{\frac{1}{2}}$] the equation yields

$$(P - 1 + \tfrac{1}{2}\,Q)E\, dU/dx = 0.$$

Because they are committed to $Q = 1$ they must find $P = \frac{1}{2}$; this corresponds to their replacement of g by G only in the kinetic but not the potential contribution to wave energy. Correcting this error gives $P = 1$ so that Q must be zero, our rigorous result for $G \propto U$.

With hindsight we can identify the source of this error. It is connected with their choice of coordinates, in which their U is small of first order in long-wave amplitude, being only the orbital part. Therefore contributions to $E(U + c_g)$ and $S_x U$ arising from $\partial W/\partial t \simeq G - g$ are small of second order in each amplitude, or fourth order overall. Had they worked in our coordinates, terms in $U\, dG/dx$ and $G\, dU/dx$ would have been of the same order because of the large constant term in U. Furthermore, in our coordinates the long-wave flow is steady and $\partial W/\partial t = 0$; its role is replaced by our time-independent centrifugal term in G, whose equal contributions to kinetic and potential energies are more evident. The local relationships among amplitude, wave-number, phase and group velocities, and kinetic and potential energies are those of a wave moving on a space-fixed surface with locally constant parameters. We have derived equations showing that the surface properties are determined by U and G. Because of U we must replace the dispersion relation $\omega^2 = gk$ by $(\omega - Uk)^2 = gk$. We must replace g by G not only in the expression for potential energy but also in the dispersion relation and, through its use, in the kinetic energy as well. These replacements are required by the principle of equivalence; the <u>only</u> effect of the acceleration of a system is equivalent to that of a gravitational field.

The following physical argument displays the inadequancy of the expression $dJ/dx = -\frac{1}{2} E\, dU/dx$ and the need in general to include variations of G even for a flow without surface slope or curvature. Consider the following hypothetical but theoretically achievable situation. A body of fluid, deep with respect to the wavelengths λ to be considered, moves with uniform constant horizontal velocity U_o in the x direction on the surface of a planet having an anomalous mass distribution producing a gravitational field $\vec{g} = -G(x)\,\hat{e}_y$ at the horizontal free surface $y = 0$ of the fluid. Of course $\vec{g}$ cannot have the same direction everywhere, but if $|(\lambda/G)\, dG/dx| \ll 1$ we may ignore g_x down to depths of order λ. Here there is no acceleration of coordinates tied to surface particles of the unperturbed flow, and it is clear that the local value of G must be used to calculate potential energy. Because $dU/dx = 0$ the LS equation gives $dJ/dx = 0$, but there is no reason to assume a priori that waves on this flow may not exchange energy with it. Our equation, which may be written $dJ/ds = -\frac{1}{2} EG\, d(U/G)/ds$, applies here; for $U = U_o$ it becomes $dJ/dx = +\frac{1}{2} (EU_o/G)\, dG/dx$, which vanishes only for $U_o = 0$.

It is indeed remarkable that the rather subtle effects uncovered by our approach should have conspired to lead to the conclusions reviewed in LS IV from their several correct and detailed calculations on specific flows. Very few workers appear to have questioned their concept of the radiation stress tensor. The misgivings of Whitham have already been mentioned. The only other work I have found in which doubt has been expressed is that of Hasselmann (1971), who identified a "previously overlooked mechanism" he describes as a "loss of potential energy arising from mass transfer" whose effect is stated to be a nearly-total cancellation of work done by his interaction stress which is related to their radiation stress.

His more elaborate derivations connected with this matter are also conducted by means of Cartesian expansions and integrations with respect to the vertical coordinate over the entire fluid depth, which makes direct comparisons with our results of simpler origin, structure, and interpretation nearly impossible. We have given a quantitative evaluation of his nearly-total cancellation in terms of the smallness of our wave-interaction function Ω ; it seems probable that he has found an arcane version of this effect. Our comments on the limitations of a vertically-averaged Cartesian surface stress tensor do not, of course, imply that vertical integrations and Cartesian expansions of other quantities, such as those of Whitham and Hasselmann, contain flaws. Nevertheless, we have become convinced that these techniques are at least inelegant, and at worst may be misleading, in studying some aspects of these processes.

XIV. PARTICLE ACCELERATION AT A CONFLUENCE OF CRESTS; DISCUSSION OF WHITECAPPING

Several results of linearized gravity-wave analyses retain qualitative validity in the nonlinear finite-amplitude region all the way up to waves of maximum height. In making numerical comparisons one is allowed a single free choice; he may fix the numerical value of the dimensionless parameter ak of the linear wave to agree with one property of the maximum wave. Then all other properties may be compared with linear predictions, with the exception of wave velocity which exceeds the linear value $(g/k)^{\frac{1}{2}}$ by 9.5%. Longuet-Higgins (1969a) chose $ak = \frac{1}{2}$ to agree with the downward acceleration of a surface particle at the crest of a maximum wave, and then showed agreement of order 10% for the linearized predictions of wave height and maximum slope. Elsewhere in this paper we have identified $2a$ with the maximum-wave height, corresponding to $ak = 0.44$. One can give an argument that even this value is too high for the fairest overall comparison; the first nonvanishing correction term in the Stokes-Rayleigh expansion for wave height is positive and gives a good fit, when included, to the maximum wave for $ak \sim 0.4$, at which value the kinetic and potential energies per unit area (which are quadratic in ak from the linearized theory), and the maximum slope and crest-acceleration with first nonvanishing corrections, all agree with their maximum-wave values to within $\pm$ 15%. Without <u>any</u> use of nonlinear corrections the largest discrepancy among all these properties is +40% for the potential energy with $ak = 0.44$, and -30% for the maximum slope with $ak = 0.4$.

From all this we are encouraged to believe that our results obtained by linearization also retain semi-quantitative validity beyond the domain of linearity up to the instability limit of the small waves.* We are therefore emboldened to apply them to the study of particle acceleration at a confluence of crests on the verge of instability, and the growth toward instability of small waves riding up large ones as a mechanism causing whitecapping.

The value $\frac{1}{2}g$ for the downward acceleration of a surface particle has been taken by Longuet-Higgins (1969a) as an empirical criterion for the onset of whitecapping in a statistically random wave field (although he has pointed out (1963) that vertical acceleration is not a sufficient condition for it). As we have just seen, this value, which is exact for a single traveling wave-train of maximum height, is reasonably well reproduced by a linear analysis with $ak \sim 0.45$ which also reproduces other properties qualitatively. If two wave-trains are superposed and treated in linear approximation each behaves as if the other were absent, and the downward particle acceleration at a confluence of crests is estimated as $(a_1k_1 + a_2k_2)g$.

---

* Some effects of nonlinearity have been indicated by Crapper (1972). It is not clear whether his results for energy, which contain a radiation stress tensor, depend on properties of the tensor we have questioned here.

To assess the limit of validity of this approximation, and to evaluate the limit of instability as a function of the two parameters $a_1 k_1$ and $a_2 k_2$, would seem to require complicated digital computer calculations of many nonlinear time-dependent flows. However, there is one situation in which the answer is known. Penney and Price (1952) and Taylor (1953) have studied standing waves of maximum height theoretically and experimentally, respectively. They found the downward opening angle of the sharp-crested peaks created at the instants of maximum displacement to be 90°, and the acceleration of fluid particles at the peaks to be g, not $\frac{1}{2} g$. Here we may picture two identical wave-trains of finite amplitude moving in opposite directions. The curious superficial validity of "superposition" of particle accelerations at crests in this very nonlinear situation is an accident, since neither of the two constituent traveling waves would be a maximum one in the absence of its oppositely traveling partner. Of course g is an obvious maximum; Taylor observed actual detachment of fluid droplets at the crests of slightly higher standing waves.

We put forward here our conclusion that between the limiting cases of a single traveling wave and two equal and opposite waves there exists a continuum of cases with two wave-trains of different amplitudes and wave-numbers which combine to produce limiting flows on the border of instability, and that surface particles at a confluence of crests experience accelerations in the range between $\frac{1}{2} g$ and g. The results of this paper allow us to exhibit the lower half of this range explicitly. If the given flow is that of a long wave of small amplitude and the added "small wave" is really small, the particle acceleration a_c at a confluence of crests is given by $akG + a_L k_L g$, and $G \simeq g$. We may consider larger and larger small waves up to the value $ak \simeq \frac{1}{2}$; this wave will break, with $a_c \simeq \frac{1}{2} G \simeq \frac{1}{2} g$, the known result. If the given wave is larger we may expand its properties in powers of $a_L k_L$; at its crests $G = g(1 - a_L k_L)$ and U^2/R (particle acceleration at its crests) = $a_L k_L g$, plus terms of third and higher orders. Therefore $a_c \simeq [ak(1 - a_L k_L) + a_L k_L]g$. In the limit $ak \ll 1$ and $a_L k_L \to \frac{1}{2}$ we find again $a_c \simeq \frac{1}{2} g$ at the stability limit, but we may impose on the small wave the role of instability $(ak \simeq \frac{1}{2})$ on any given wave, obtaining $a_c \simeq \frac{1}{2}(1 + a_L k_L)g$. As the given wave approaches its maximum, $a_c \to 3g/4$. The physical picture and the mathematical justification of these results in the range $\frac{1}{2} g < a_c < 3g/4$ are particularly convincing in the domain $\lambda/\lambda_L \ll 1$. To explore the range $3g/4 < a_c < g$ would require treating the two flows on an equal footing (and with $\lambda \sim \lambda_L$), which is precluded by the unsymmetrical roles assigned them in our analysis.

Finally, we discuss the application of our results to the triggering of whitecapping. The fate of a small-amplitude short-wave train moving from the trough of a long wave up its downwind side depends on three dimensionless parameters. These are (1) the wave-length ratio λ_o/λ_L in the trough, which determines p_o and H, and thence the functional dependence of the growth factors a/a_o and k/k_o on properties of the long-wave flow; (2) the long-wave height-to-length ratio, which defines the given flow and allows evaluation of the growth factors as functions of position along it, including their crest-values; and (3) the short-wave height-to-length ratio

$\propto a_o k_o$ in the trough, which then fixes the position at which they break, or how close they are to breaking at the long-wave crest. We have obtained approximate analytical expressions for these relations.

If the critical value of ak is reached before attaining the crest the small-wave train will become unstable and dump some amount of turbulent fluid onto the downwind side of the long wave. Its effect on the long wave will be greater the lower down on its surface the breakup occurs, since the effect tends toward zero in the opposite limit. This turbulent water mass will be dragged up the long-wave slope by the flow toward its crest. Its dead weight will create a pressure asymmetry about the crest and the flow speed will be reduced, tending to make the long-wave crest "stub its toe" on the obstacle and fall over. The effect is obviously greatest for the most nearly maximum long waves.

Even if the critical value of ak is not attained, the small waves take energy away from the downwind sides of nearly-maximum long ones at a rapidly increasing rate near their crests, for which we have also given an approximate analytic expression. Loss of potential energy from the main flow corresponds to a lowering of the free surface above the region of loss and a rise below. Loss of kinetic energy reduces the main flow's speed, and the ensuing surface displacement is of the same form. Any attempt to sketch the influence of these losses on the long-wave profile just downwind of the crest yields a shape strongly suggestive of breaking over.

In a real wind-generated wave field conditions are very different from the idealized ones hypothesized in this study. In the presence of a continuous random wave spectrum short sequences of waves of varying lengths and amplitudes will continually be generated and dissipated by constructive and destructive interference. These processes occur fairly rapidly and over small numbers of wavelengths because of the strongly dispersive nature of gravity waves. The crests of the longer waves in the spectrum, whose mechanism of breaking over is being considered here, will also have ephemeral existence and varying properties. All of the waves have finite extent in the third space dimension neglected here, and their properties will vary in this direction. The whitecapping mechanism described will operate in spite of these complications, and is in accord with visual impressions of the phenomenon under suitable conditions. The concentration of its effect in a very small region near the crest is an important prediction, in quantitative detail, of our analysis. It results from existence of the special regime very near the crests of nearly-maximum waves in which $ak \propto U^{-3}$ rather than U^{-2}, and from growth of the wave-interaction function Ω from a small value in the trough to nearly unity at the inflection point near the crest. Because of this concentration the mechanism will be difficult to observe and study in wave-tank experiments.

It is not claimed that this qualitative picture is complete. A single wave-train will become unstable on its own if energy is added steadily to it without creating other waves as in a slowly converging channel, by the wind, or by variations of the flow on which it rides which we have studied in this paper. Other authors have described interchanges of momentum and energy in crest regions and a variety of other effects. Surface tension may become important for the shrinking small waves near long-wave crests, as may capillary waves on the upwind side. In a whole gale the general presence of foam and the bodily detachment of water from nearly-discontinuous wave crests by local wind stress may dominate the picture. Many interesting questions remain to be answered.

ACKNOWLEDGMENTS

This work was supported by the United States Atomic Energy Commission through the Lawrence Berkeley Laboratory of the University of California. The Laboratory Director, Professor Edwin M. McMillan, arranged to relieve me of administrative duties and approved support of an extended leave without which it could not have been undertaken. I am grateful to Professor Walter H. Munk and to the Scripps Institute of Oceanography and the Institute of Geophysics and Planetary Physics for providing office space at the San Diego campus of the University in 1971-72.

APPENDIX: Stokes-Rayleigh Expansions of U, G, and Ω.

Rayleigh (see Lamb (1932), p. 417) has shown that to third order in $ak \equiv \epsilon$ the velocity potential of a traveling wave train of finite amplitude is given by $\phi = c(x - a\, e^{ky} \sin kx)$ and the stream function by $\psi = c(y - a\, e^{ky} \cos kx)$ in co-moving coordinates; we have changed signs to use $\vec{v} = + \vec{\nabla} \phi$. To this order the phase velocity c is given by

$$c^2 = (g/k)(1 + \epsilon^2)$$

and the free surface $y = y_f(x)$ at the streamline $\psi = 0$ by

$$ky_f = \epsilon[\cos X + \frac{1}{2}\epsilon(1 + \cos 2X) + (3\epsilon^2/8)(3 \cos X + \cos 3X)]$$

with $X \equiv kx$. The surface flow speed $U(x)$ on $\psi = 0$ is given to the same order by

$$U^2 = |\nabla \phi|^2 \Big|_{\psi=0} = c^2[1 - 2ky_f + \epsilon^2(1 + 2ky_f)] \ .$$

In making our expansions through third order it is convenient to use powers rather than multiple angles. We find

$$ky_f = \epsilon[\mathrm{cox}\ X + \epsilon \cos^2 X + (3\epsilon^2/2)\cos^3 X],$$

$$U^2 = c^2[1 - 2\epsilon \cos X + \epsilon^2(1 - 2\cos^2 X) + \epsilon^3(2\cos X - 3\cos^3 X)],$$

$$U = c[1 - \epsilon \cos X + \frac{1}{2}\epsilon^2(1 - 3\cos^2 X)$$
$$+ \frac{1}{2}\epsilon^3(3\cos X - 6\cos^3 X)].$$

The tangent of the surface slope angle α, given by dy_f/dx, is

$$\tan \alpha = -(\epsilon \sin X)[1 + 2\epsilon \cos X + (9\epsilon^2/2)\cos^2 X];$$

we also need $\cos\alpha = 1 - \frac{1}{2}\tan^2\alpha$ and $\cos^3\alpha = 1 - (3/2)\tan^2\alpha$ to this order;

$$\cos\alpha = 1 - \frac{1}{2}\epsilon^2\sin^2 X - 2\epsilon^3\sin^2 X\cos X\,,$$

$$\cos^3\alpha = 1 - (3\epsilon^2/2)\sin^2 X - 6\epsilon^3\sin^2 X\cos X\,.$$

The radius of curvature R of the surface is given by $R^{-1} = (d^2y_f/dx^2)\cos^3\alpha$;

$$R^{-1} = -k\epsilon[\cos X + 2\epsilon(2\cos^2 X - 1) - (3\epsilon^2/2)(7\cos X - 10\cos^3 X)].$$

To find the term U^2/R in G we must multiply through the factor $(1+\epsilon^2)$ in c^2, obtaining

$$U^2/R = -g\epsilon[\cos X + 2\epsilon(\cos^2 X - 1) - \frac{1}{2}\epsilon^2(9\cos X - 10\cos^3 X)]\;;$$

then $G(x) \equiv g\cos\alpha + U^2/R$ is found to be

$$G = g[1 - \epsilon\cos X + (3\epsilon^2/2)(1 - \cos^2 X) + \frac{1}{2}\epsilon^3(5\cos X - 6\cos^3 X)].$$

To find $\Omega \equiv 1 - (U/G)(dG/dU)$ to second order we need

$$dG/dx = (gk\epsilon\sin X)[1 + 3\epsilon\cos X - \frac{1}{2}\epsilon^2(5 - 18\cos^2 X)],$$

$$dU/dx = (ck\epsilon\sin X)[1 + 3\epsilon\cos X - \frac{1}{2}\epsilon^2(3 - 18\cos^2 X)];$$

$$\Omega = 1 - \frac{1 - \epsilon C' + \frac{1}{2}\epsilon^2(1 - 3C^2)}{1 - \epsilon C + \frac{1}{2}\epsilon^2(3 - 3C^2)} \cdot \frac{1 + 3\epsilon C - \frac{1}{2}\epsilon^2(5 - 18C^2)}{1 + 3\epsilon C - \frac{1}{2}\epsilon^2(3 - 18C^2)}\,,$$

with $C \equiv \cos X$. The terms of first order cancel, as do those of second order which are proportional to C^2, giving

$$\Omega = 2(ak)^2$$

plus unevaluated terms of third order in C and C^3, and higher order, depending on fourth and higher order terms in U and G.

A number of authors (e.g. Wehausen (1960), p.658, Eqs. 27.25-27.27) give the same equations with which we have commenced except that the origin of y has been shifted to the mean surface level;

$$ky_f = \epsilon[\cos X + \frac{1}{2}\epsilon \cos 2X + (3\epsilon^2/8)(3 \cos X + \cos 3X)].$$

Although this equation is also accurate to third order, its use in the Rayleigh equation $U^2 = c^2[1 - 2ky_f + \epsilon^2(1 + 2ky_f)]$ is incorrect. The curious result of using it is that Ω is wrongly found to vanish identically through second order!

REFERENCES

Courant, E. D. & Snyder, H. S. 1958 Theory of the alternating-gradient synchrotron. Ann. Phys. 3, 1-48.

Crapper, G. D. 1972 Nonlinear gravity waves on steady non-uniform currents. J. Fluid Mech. 52, 713-724.

Grant, M. A. 1973 The singularity at the crest of a finite amplitude progressive Stokes wave. J. Fluid Mech. 59, 257-262.

Hasselmann, K. 1962, 1963 On the non-linear energy transfer in a gravity wave spectrum. J. Fluid Mech.; Part 1, 12, 481-500; Part 2, 15, 273-281; Part 3, 15, 385-398.

Hasselmann, K. 1971 On the mass and momentum transfer between short gravity waves and larger-scale motions. J. Fluid Mech. 50, 189-205.

Havelock, T. H. 1919 Periodic irrotational waves of finite height. Proc. Roy. Soc. A, 95, 38-51.

Lamb, H. 1932 Hydrodynamics (6th ed.) Cambridge Univ. Press.

Longuet-Higgins, M. S. 1963 The generation of capillary waves by steep gravity waves. J. Fluid Mech. 16, 138-159.

Longuet-Higgins, M. S. 1969a On wave-breaking and the equilibrium spectrum of wind-generated waves. Proc. Roy. Soc. A, 310, 151-296.

Longuet-Higgins, M. S. 1969b A nonlinear mechanism for the generation of sea waves. Proc. Roy Soc. A, 311, 371-389.

Longuet-Higgins, M. S. 1973 On the form of the highest progressive and standing waves in deep water. Proc. Roy. Soc. A, 331, 445-456.

Longuet-Higgins, M. S. & Stewart, R. W. 1960 Changes in the form of short gravity waves on long waves and tidal currents. J. Fluid Mech. 8, 565-583.

Longuet-Higgins, M. S. & Stewart, R. W. 1961 The changes in amplitude of short gravity waves on steady non-uniform currents. J. Fluid Mech. 10, 529-549.

Longuet-Higgins, M. S. & Stewart, R. W. 1964 Radiation stresses in water waves; a physical discussion, with applications. Deep Sea Res. 11, 529-562.

Morse, P. M. & Feshbach, H. 1953 Methods of Theoretical Physics McGraw-Hill, N. Y.

Penney, W. G. & Price, A. T. 1952 Finite periodic stationary gravity waves in a perfect liquid. Phil. Trans. A, 244, 254-284.

Phillips, O. M. 1960 On the dynamics of unsteady gravity waves of finite amplitude. J. Fluid Mech. 9, 193-217.

Phillips, O. M. 1963 On the attenuation of long gravity waves by short breaking waves. J. Fluid Mech. 16, 321-332.

Schwartz, L. W. 1974 Computer extension and analytic continuation of Stokes' expansion for gravity waves. J. Fluid Mech. 62, 553-578.

Taylor, G. I. 1953 An experimental study of standing waves. Proc. Roy. Soc. A, 218, 44-59.

Wehausen, J. W. 1960 Chapters D and F of "Surface Waves" by Wehausen, J. W. and Laitone, E. V., Handbuch der Physik, vol 9, Fluid Mechanics, Springer.

Whitham, G. B. 1962 Mass, momentum, and energy flux in water waves. J. Fluid Mech. 12, 135-147.

FIGURE CAPTIONS

Figure 1. Curvilinear coordinate systems. (a) s-n coordinates; (b) Φ-Ψ coordinates.

Figure 2. (2) Parameters of the short-wave train. (b) Relations between curvilinear and Cartesian descriptions of small-wave parameters.

Figure 3. The dispersion relation for ω vs. k relative to flow of speed U with equivalent gravitational acceleration G; $U\omega/G = p$ is plotted vs. $U^2k/G = q$. In units of U, phase and group velocities at any point E are the slopes of the chord from the origin and the tangent to the curve at E, respectively.

Figure 4. The relation between H and λ_o/λ_L (logarithmic scales).

Figure 5. Growth factors k/k_o (solid) and a/a_o (dashed), each multiplied by $(1 - \beta^2)^{\frac{1}{2}}$, vs. $(1 - \beta^2)^{-\frac{1}{2}}$, for selected values of λ_o/λ_L as indicated, on a wave of maximum height. A scale of β is also shown.

Figure 6. Values of β at which selected values of ak/a_ok_o are attained vs. λ_o/λ_L on a maximum wave.

Figure 7. Values of the functions $P \cos \theta$, $\tilde{U}$, G, and $\tilde{U}/\tilde{G}$, in suitable units as indicated, vs. θ in the cap region of a nearly-maximum wave. P is not plotted because it diverges as $\sec \theta$ near $\theta = 90^\circ$.

Figure 8. The surface profile $|y|$ vs. $|x|$, with origin at crest, in the cap region of a nearly-maximum wave. The units of length are $(\frac{1}{2}\,\mathcal{E}^2/\tilde{g})^{1/3}\,\pi^{-1}\lambda$ (see text); values of the parameter θ are also shown.

Figure 9. The parameters M, a_1, and a_2 vs. β on a maximum wave. Dashed curves for M and a_2 are from the parabolic model; solid curves are best estimates.

Figure 10: The parameters M, a_1, and $\frac{1}{4}a_2$ vs. θ in the cap region of a nearly-maximum wave. The negative of the extremum of the WKB test function W is also shown, as a dashed curve; $W_{ext} \simeq -\frac{1}{4}a_2$.

Figure 11. Small-wave displacement ξ vs. $\beta \equiv 2x/\lambda_L$ in the limit $p \to 0$ on a maximum wave for $\sigma_o = 0$ and $\frac{1}{2}\pi$.

Figure 12. The wave-interaction function $\Omega \equiv 1 - (U/G)(dG/dU)$ vs. θ in the cap region of a nearly-maximum wave.

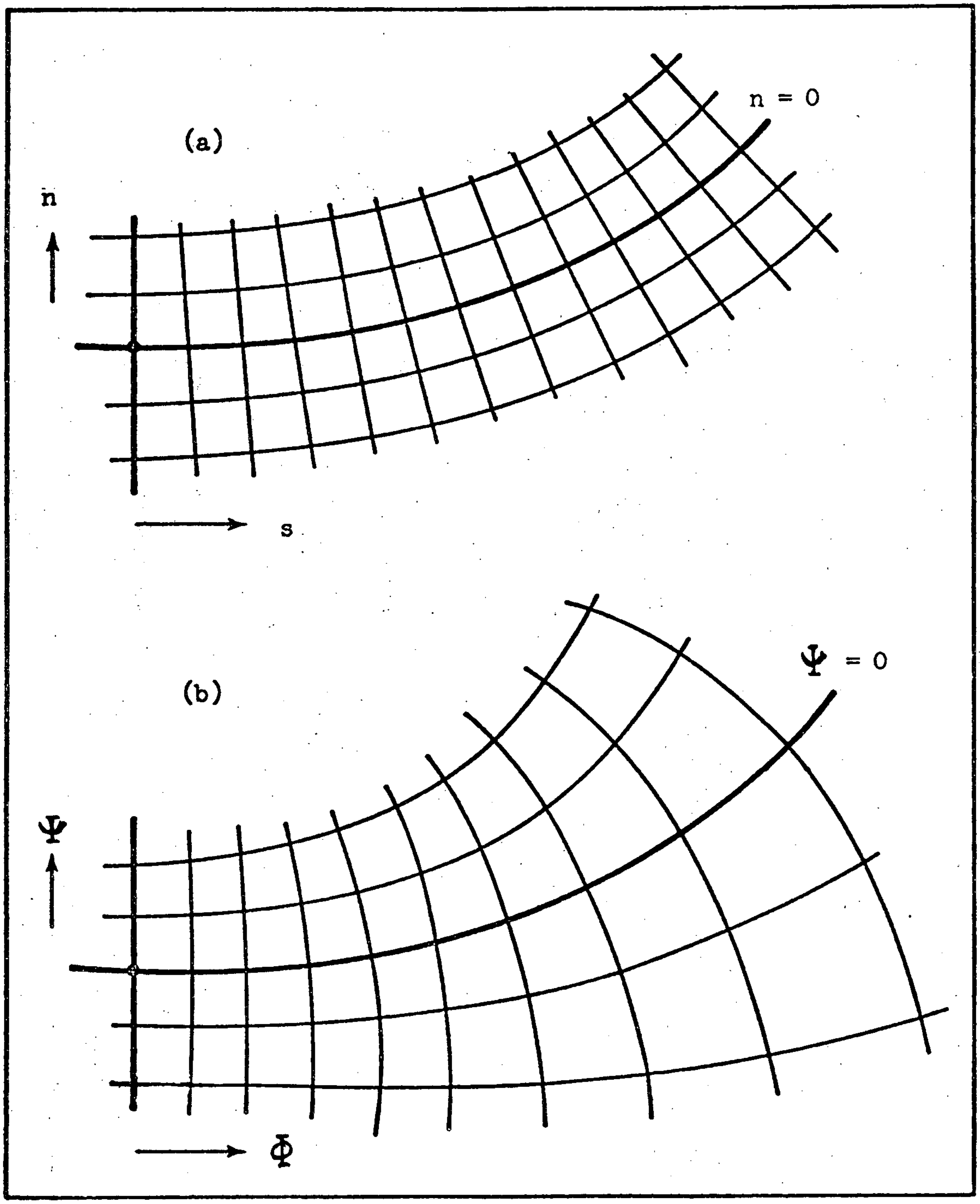

FIGURE 1

XBL 759-4004

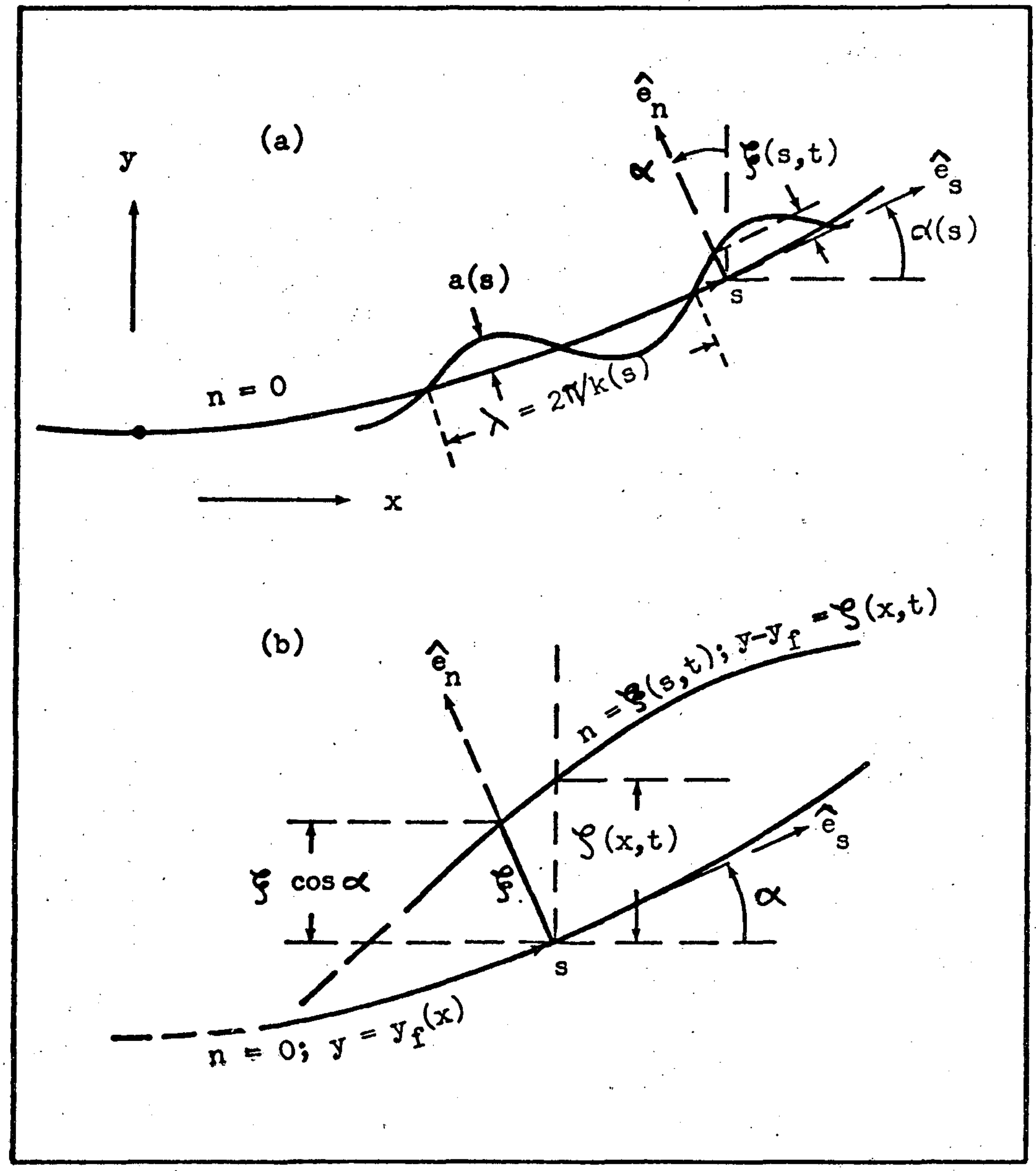

FIGURE 2

XBL 759-4005

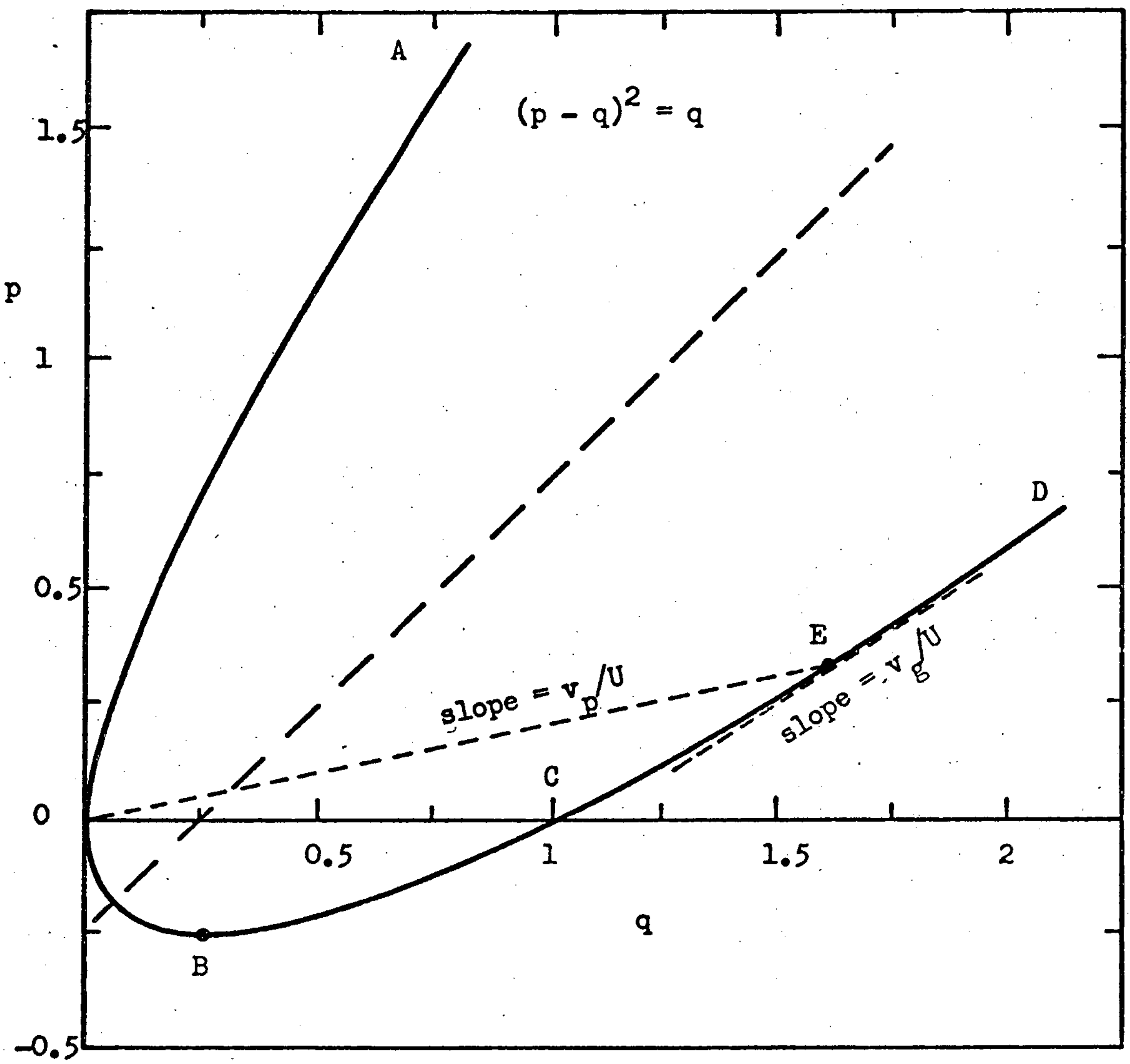

FIGURE 3

XBL 759-4006

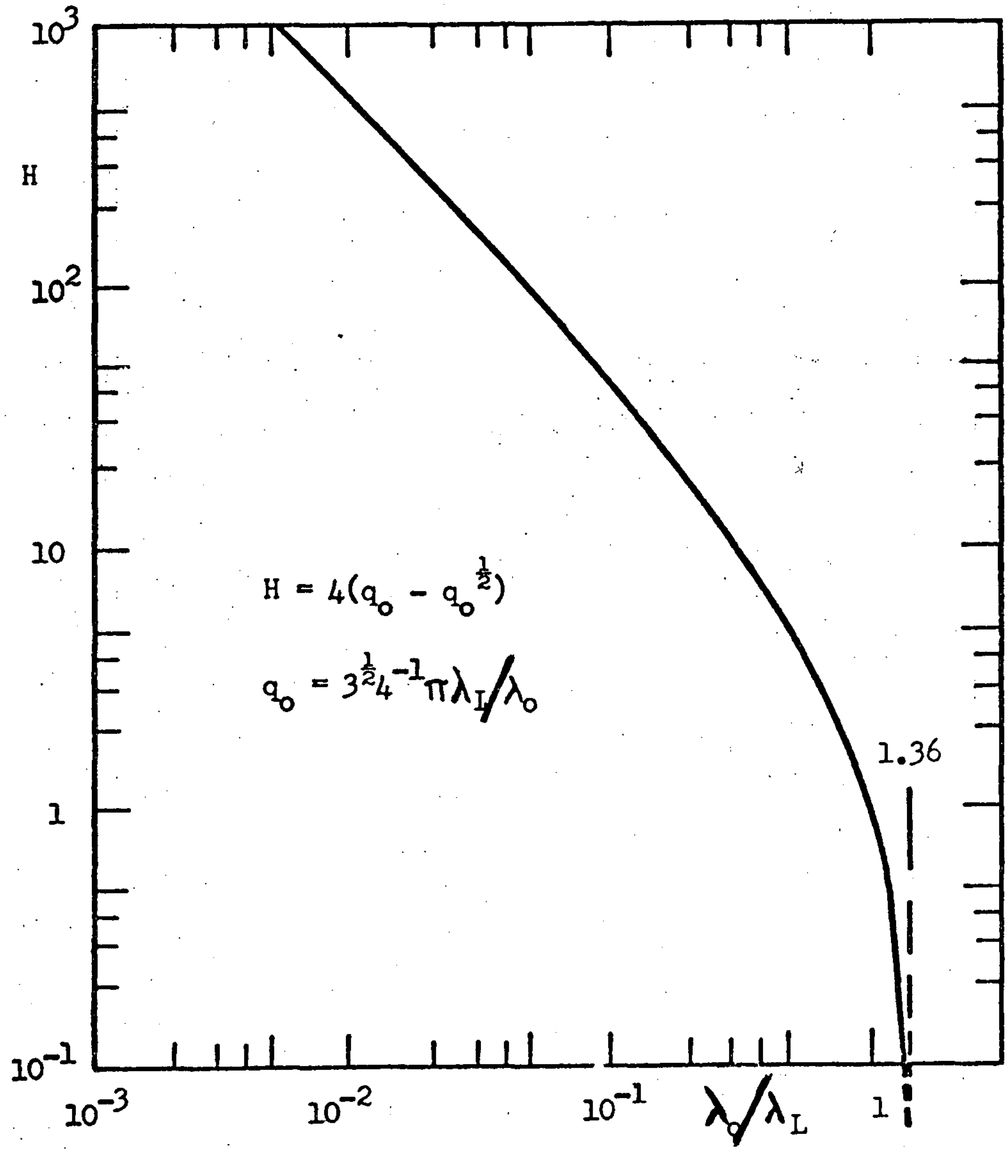

FIGURE 4

XBL 759-4007

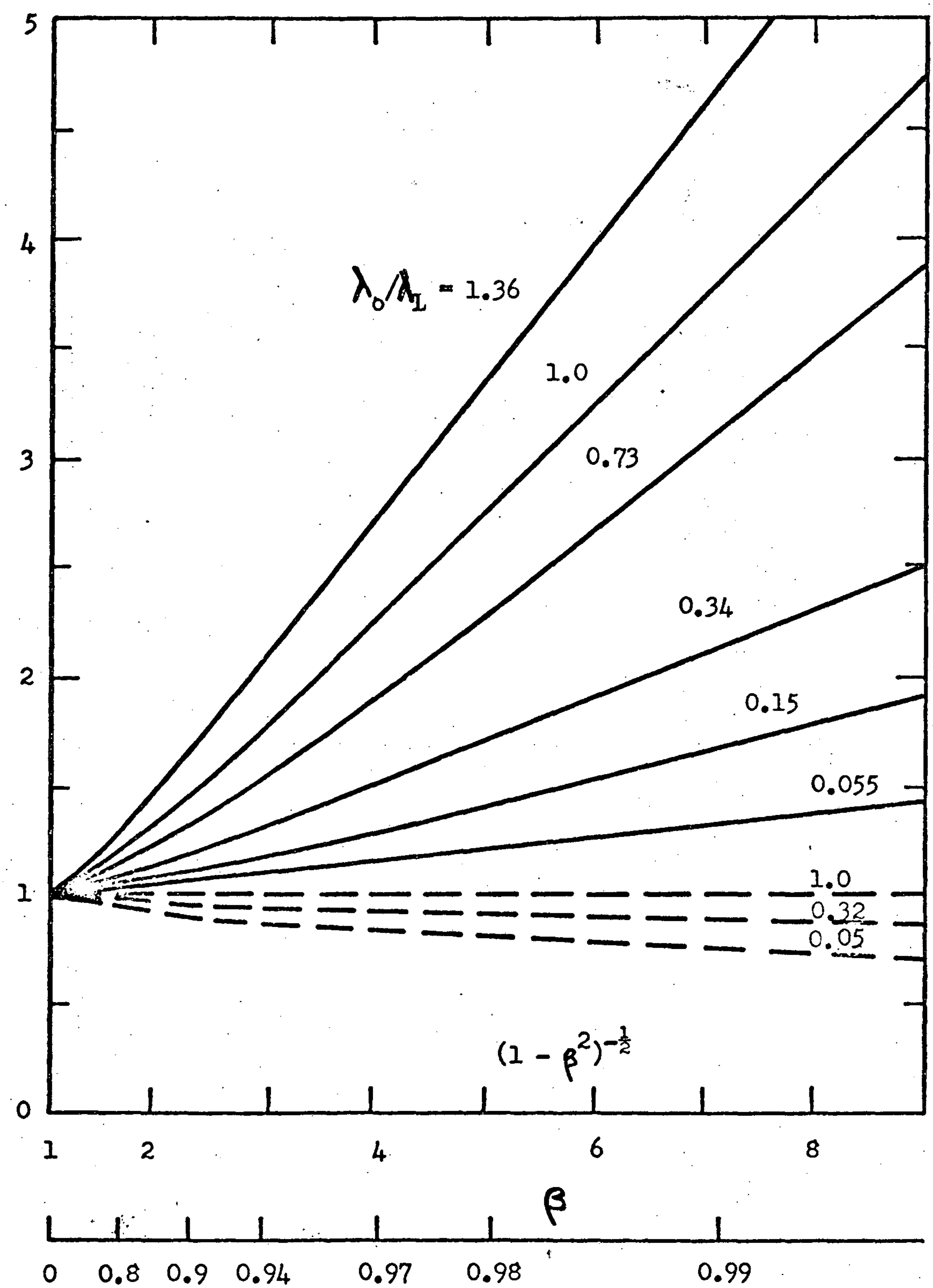

FIGURE 5

XBL 759-4008

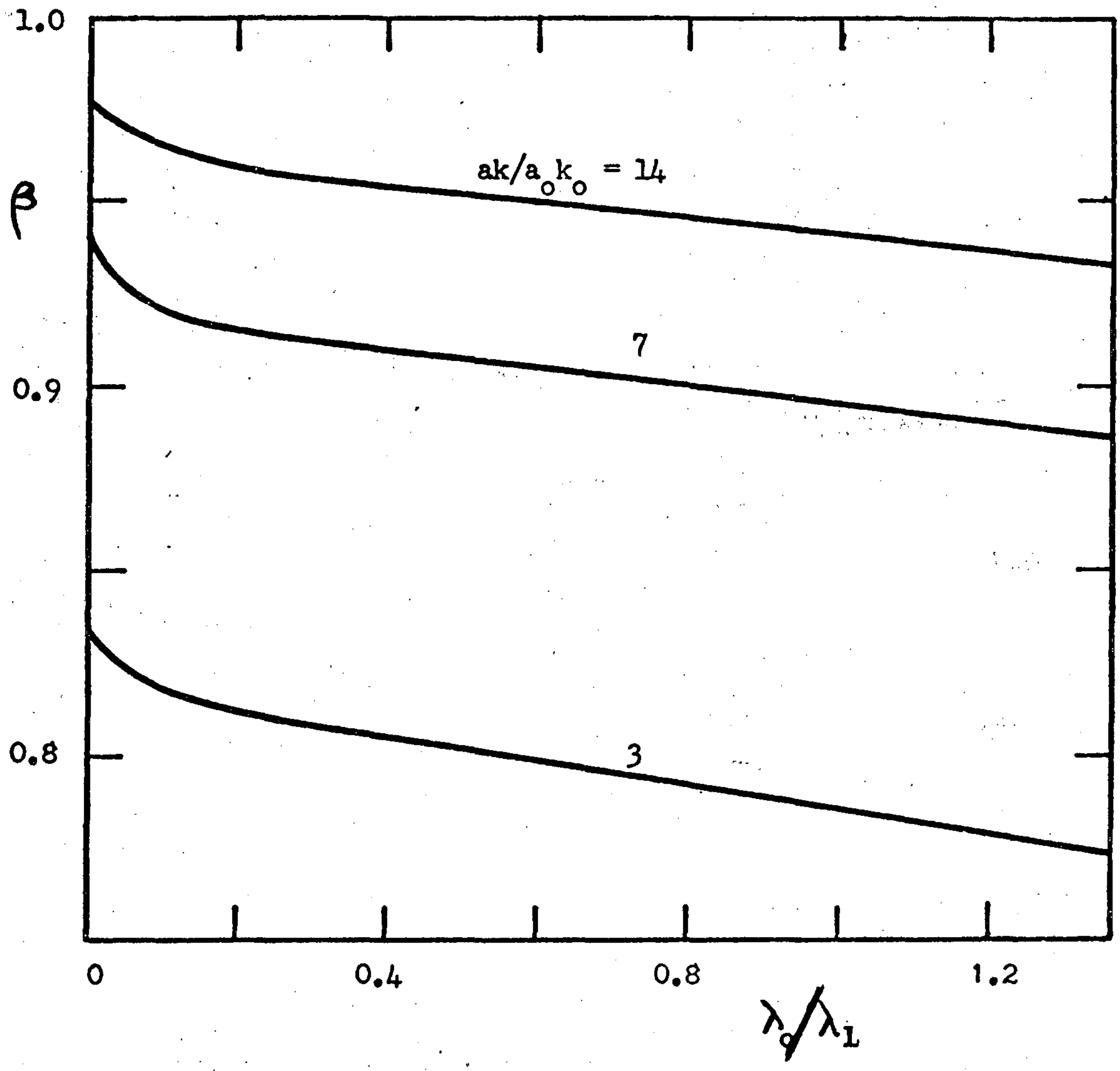

FIGURE 6

XBL 759-4009

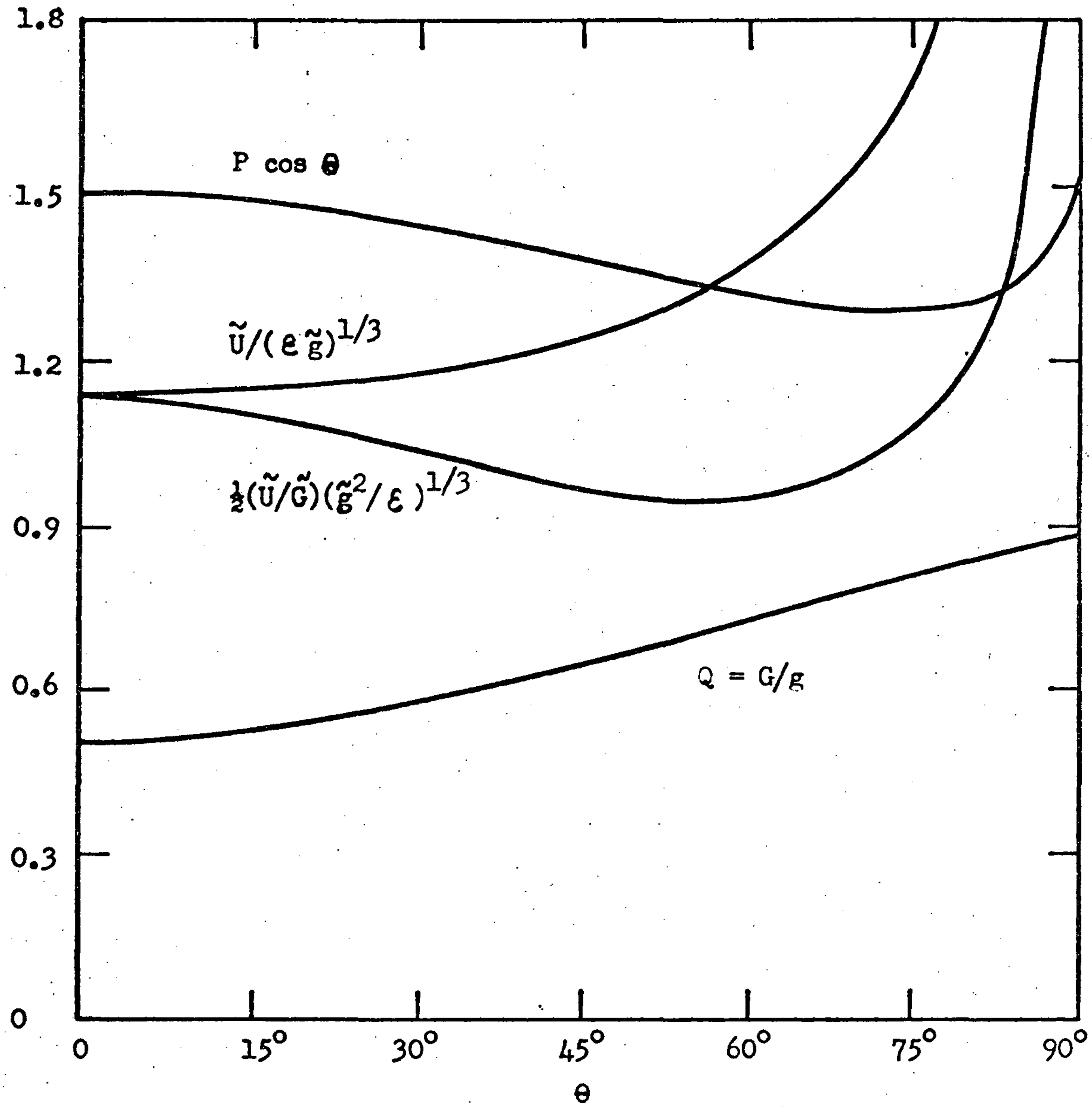

FIGURE 7

XBL 759-4010

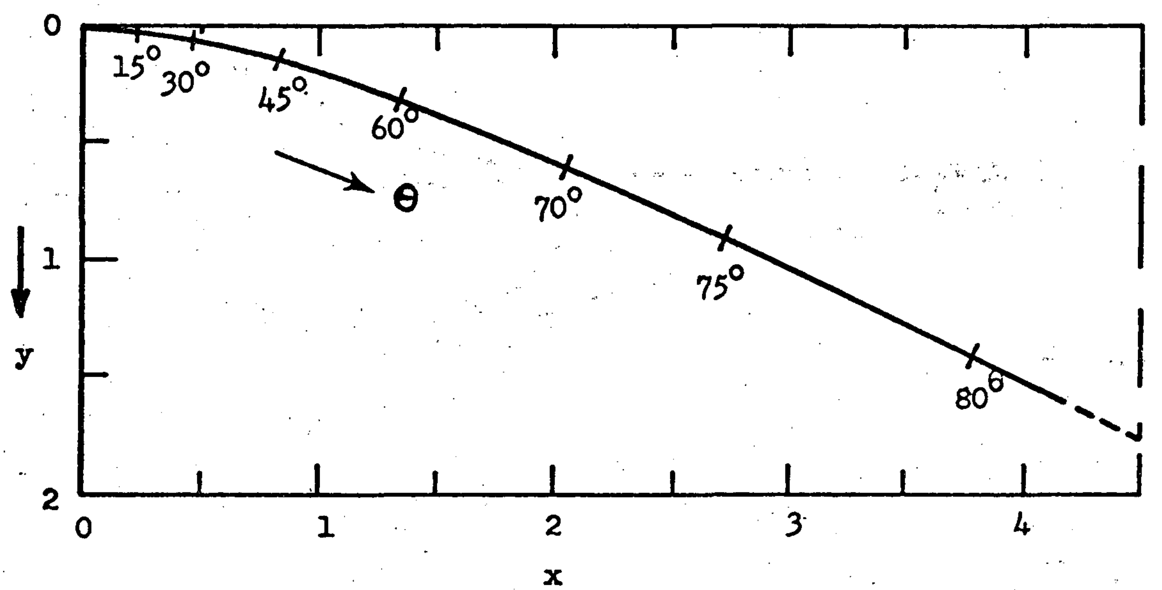

FIGURE 8

XBL 759-4011

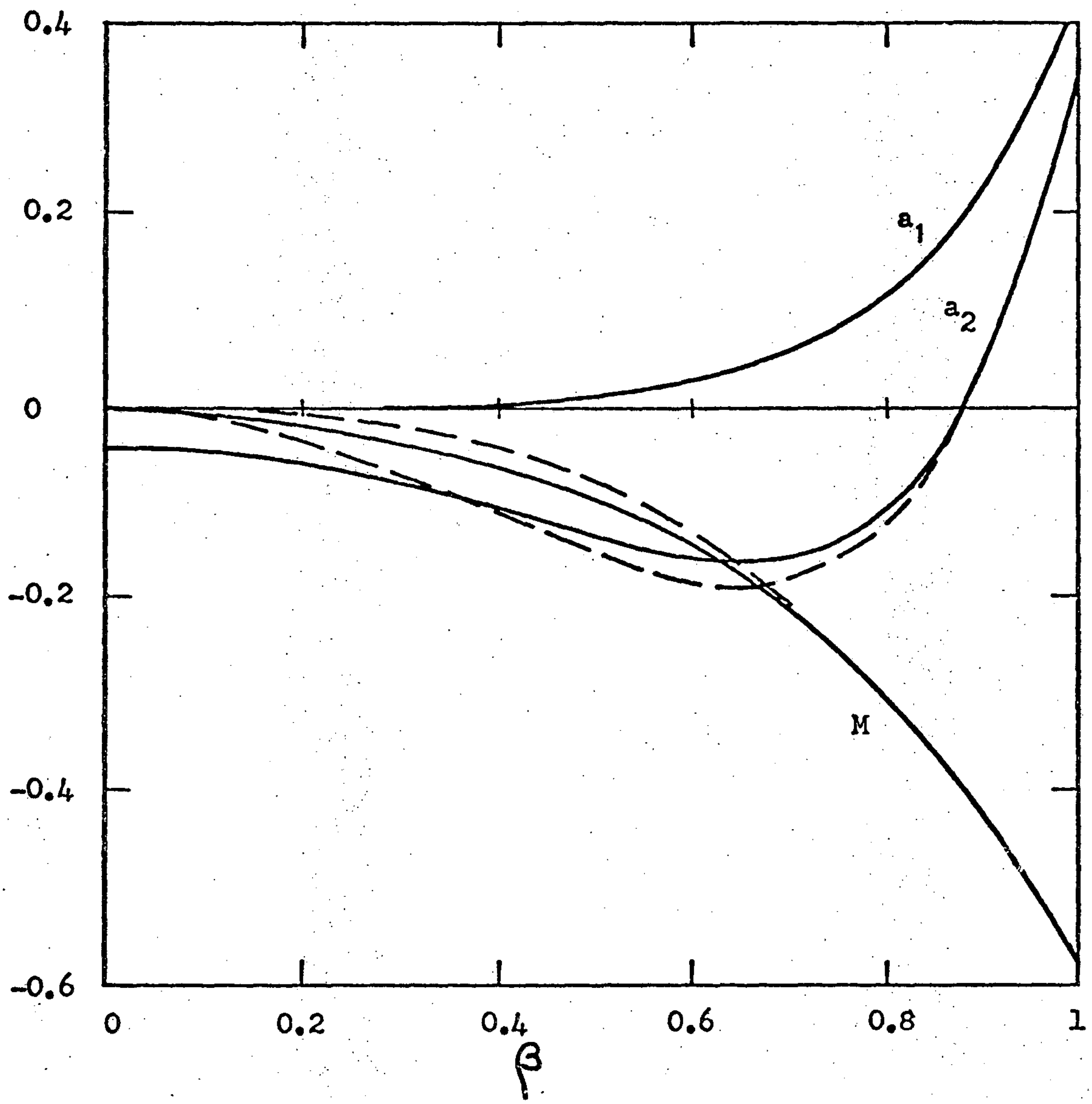

FIGURE 9

XBL 759-4012

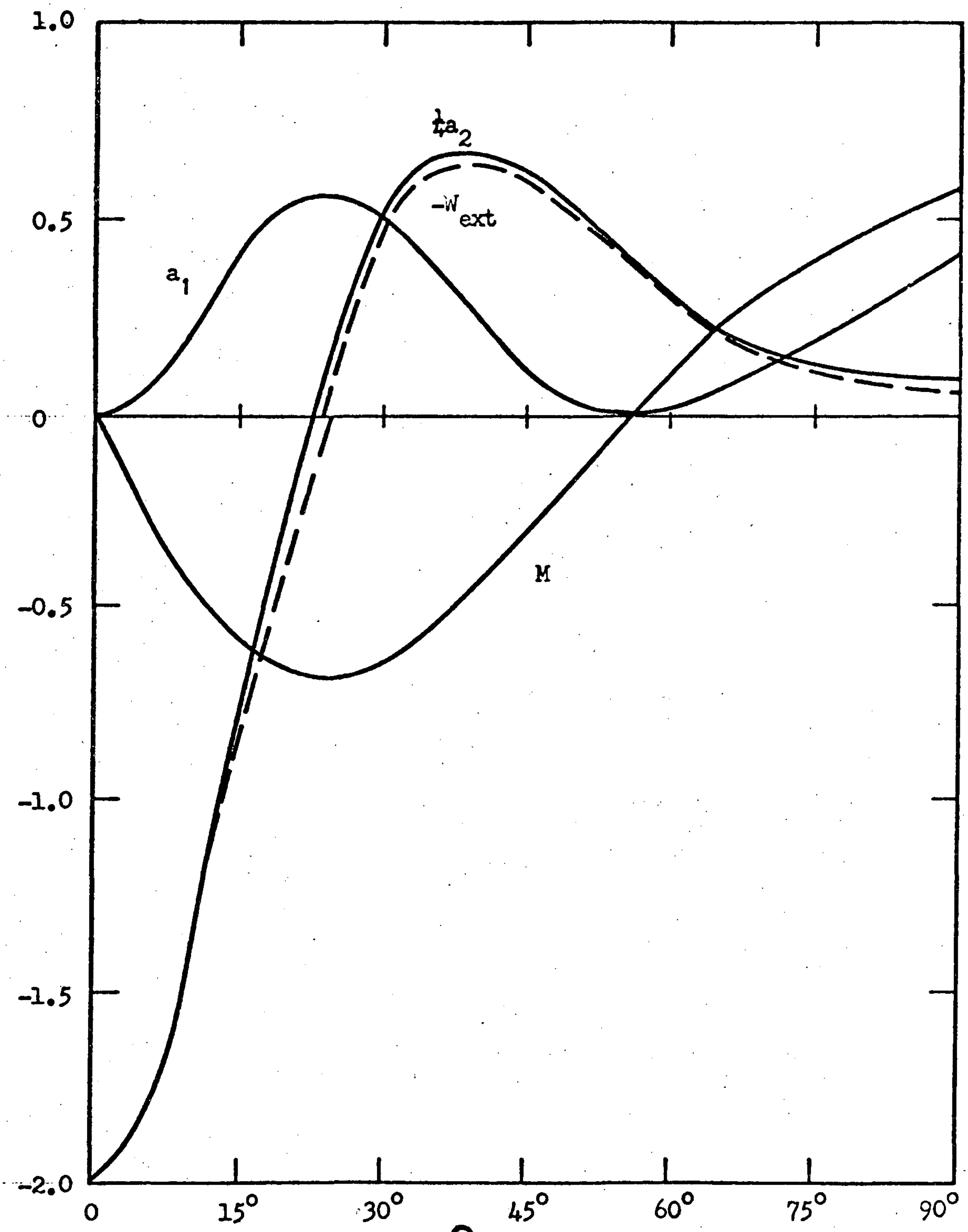

FIGURE 10

XBL 759-4013

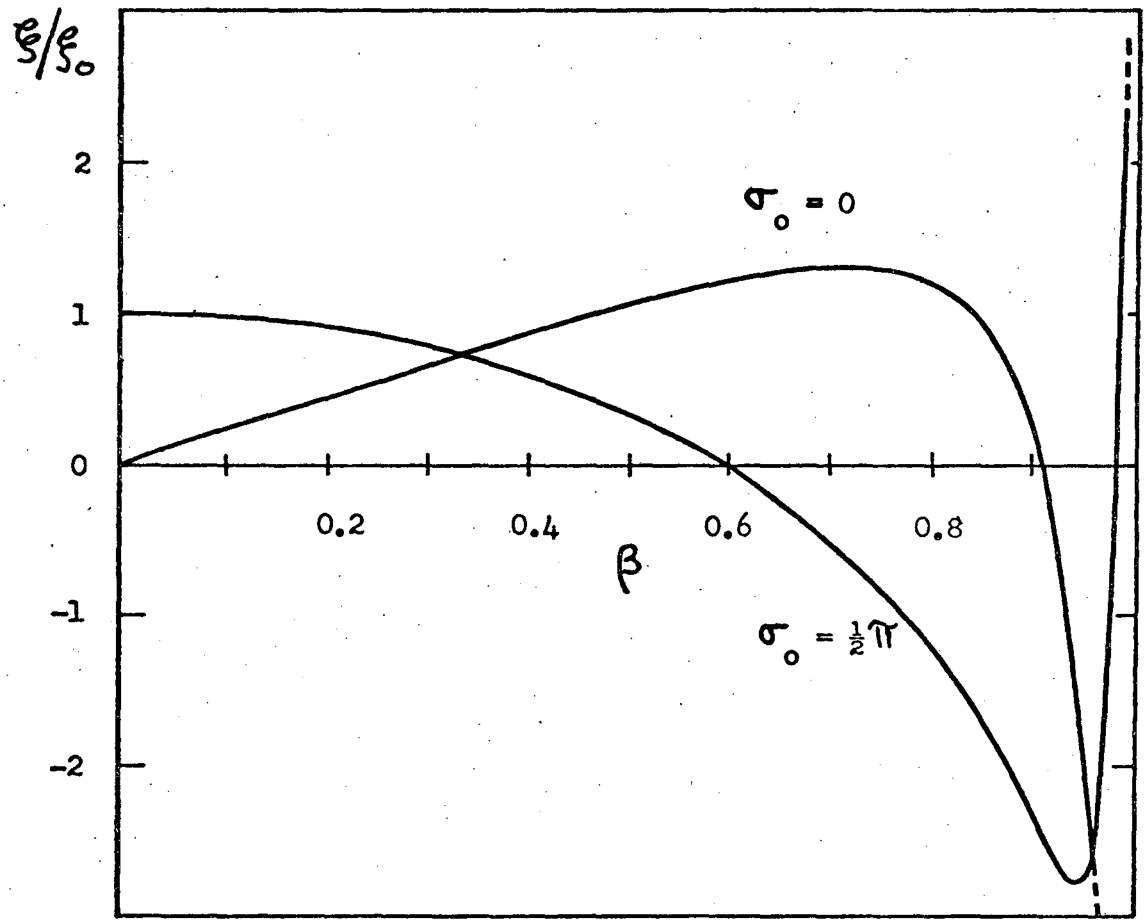

FIGURE 11

XBL 759-4014

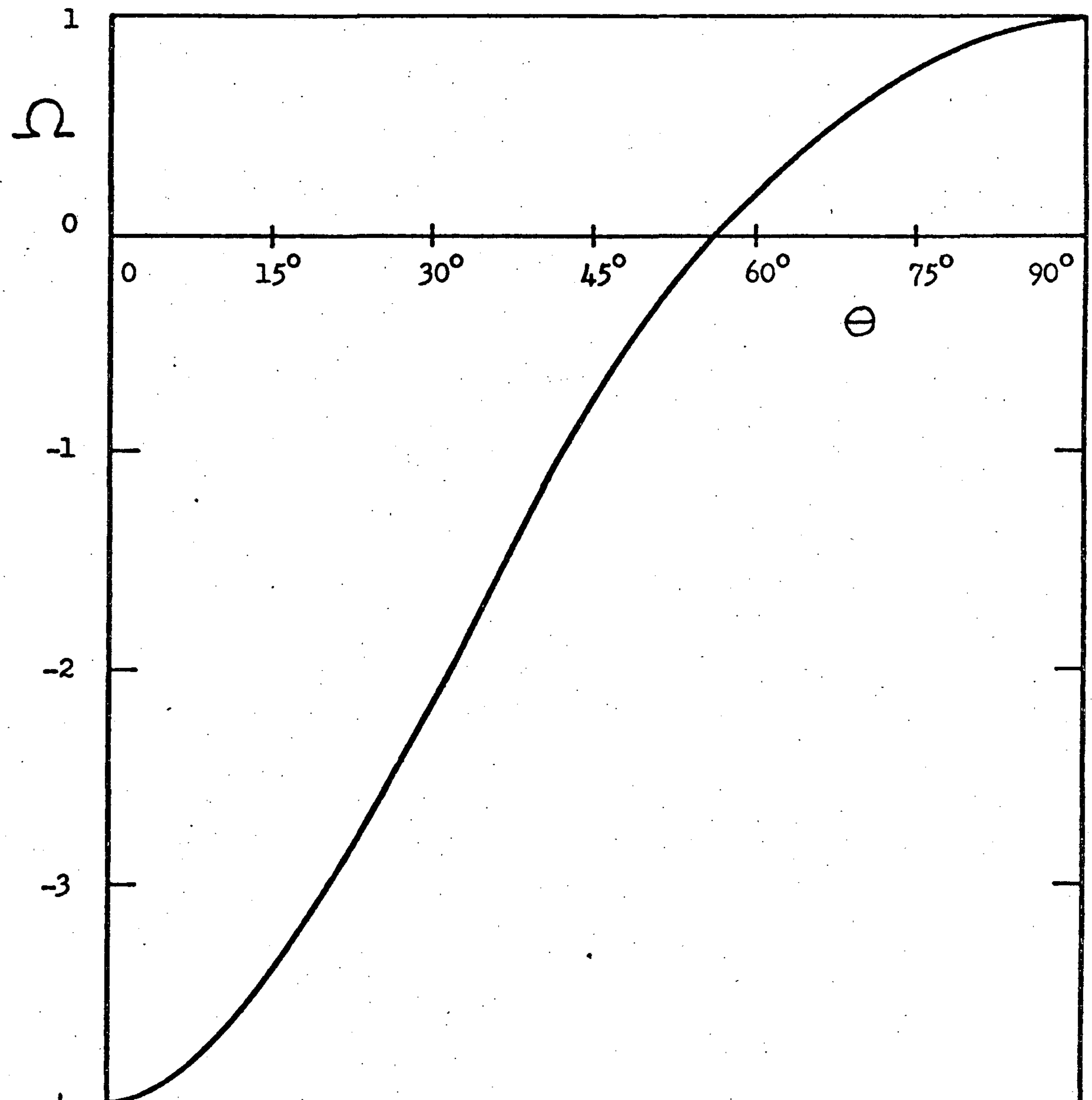

FIGURE 12

XBL 759-4015

TECHNICAL INFORMATION DIVISION
LAWRENCE BERKELEY LABORATORY
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA 94720