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TWISTOR LINES IN THE PERIOD DOMAIN OF COMPLEX TORI

NIKOLAY BUSKIN AND ELHAM IZADI

ABSTRACT. As in the case of irreducible holomorphic symplectic manifolds, the period domain $Compl$ of compact complex tori of even dimension $2n$ contains twistor lines. These are special 2-spheres parametrizing complex tori whose complex structures arise from a given quaternionic structure. In analogy with the case of irreducible holomorphic symplectic manifolds, we show that the periods of any two complex tori can be joined by a *generic* chain of twistor lines. Furthermore, we show that twistor lines are holomorphic submanifolds of $Compl$, of degree 2 in the Plücker embedding of $Compl$.

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INTRODUCTION

Let M be a complex manifold of dimension $2n \geq 2$. Then M is called *hyperkähler* with respect to a Riemannian metric g (see [6, p. 548]) if there exist covariantly constant complex structures I, J and K which are isometries of the tangent bundle TM with respect to g , satisfying the quaternionic relations

$$I^2 = J^2 = K^2 = -1, IJ = -JI = K.$$

We call the ordered triple (I, J, K) a *hyperkähler structure on M compatible with g* . A hyperkähler structure (I, J, K) gives rise to a sphere S^2 of complex structures on M :

$$S^2 = \{aI + bJ + cK \mid a^2 + b^2 + c^2 = 1\}.$$

We call the family $\mathcal{M} = \{(M, \lambda) \mid \lambda \in S^2\} \rightarrow S^2$ a *twistor family over the twistor sphere S^2* . The family \mathcal{M} can be endowed with a complex structure, so that it becomes a complex manifold and the fiber \mathcal{M}_λ is biholomorphic to the complex manifold (M, λ) , see [6, p. 554].

The well known examples of compact hyperkähler manifolds are even-dimensional complex tori and irreducible holomorphic symplectic manifolds (*IHS manifolds*). We

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recall that an IHS manifold is a simply connected compact Kähler manifold M with $H^0(M, \Omega_M^2)$ generated by an everywhere non-degenerate holomorphic 2-form σ .

Examples of IHS manifolds include $K3$ surfaces and, more generally, Hilbert schemes of points on $K3$ surfaces. For these examples there exist well-defined period domains, carrying the structure of a complex manifold, and every twistor family \mathcal{M} determines an embedding of the base S^2 into the corresponding period domain as a 1-dimensional complex submanifold. The image of such an embedding is called a *twistor line*.

It is known that in the period domain of an IHS manifold any two periods can be connected by a path of twistor lines, meaning an ordered sequence S_1, \dots, S_m of twistor spheres such that $S_i \cap S_{i+1}$ is non-empty if $1 \leq i \leq m - 1$ (see [2] or [4]). Moreover, such a path can be chosen generic, that is, the manifolds corresponding to the periods at intersections of successive lines in the path have trivial Néron Severi groups. Here we prove a similar result for the period domain $Compl$ of complex tori of dimension $2n$.

Theorem 1. *Any twistor sphere on a complex torus embeds into $Compl$ as a complex 1-dimensional submanifold. In $Compl$ any two periods can be connected by a generic path of twistor lines.*

For the case of IHS manifolds the proof of this fact relies on the realization of the period domain as the grassmanian of oriented positive real 2-planes in the second cohomology, where positivity is with respect to the Beauville-Bogomolov bilinear form, again see [2] or [4]. This bilinear form provides a very convenient tool for investigating the local topology of this period domain.

For complex tori, however, we do not have such a realization of their period domain and cannot use a similar argument. Here, instead, we need to use the (less refined) fact that the period domain of complex tori is a homogeneous space (which, certainly, the period domain of an IHS manifold is, as well). Its homogeneous nature allows us to proceed with proving the twistor path connectivity in steps that are more or less parallel to the steps of the proof of the twistor path connectivity for the period domains of IHS manifolds. Let A be a complex torus of dimension $2n$. The period domain $Compl$ can be considered as an open subset of the Grassmanian $G(2n, 4n)$, whose points are $2n$ -dimensional complex planes, realizing the real weight 1 Hodge structures on the complex $4n$ -dimensional vector space $T_{0,\mathbb{R}}A \otimes \mathbb{C}$. The open subset consists of those $2n$ -planes in $T_{0,\mathbb{R}}A \otimes \mathbb{C}$ that do not intersect the real subspace $V_{\mathbb{R}} := T_{0,\mathbb{R}}A \subset T_{0,\mathbb{R}}A \otimes \mathbb{C}$. Explicitly, a complex structure $I: V_{\mathbb{R}} \rightarrow V_{\mathbb{R}}$ corresponds to the point $(\mathbb{1} - iI)V_{\mathbb{R}} \in G(2n, 4n)$ where $\mathbb{1}$ denotes the identity map.

Remark 2. There is a relation between twistor path connectivity and rational connectedness, that is, the connectedness of points of a complex manifold by chains of rational curves (for the latter see, for example, [7]). The grassmanian $G(2n, 4n)$ being a rational variety ($G(2n, 4n) \dashrightarrow \mathbb{P}^{4n^2}$), is certainly rationally connected. However, rational connectedness is a weaker property than twistor path connectivity. Indeed, the variety of lines in \mathbb{P}^{4n^2} , passing through a fixed point, has complex dimension $4n^2 - 1$ (and the dimension of the variety of rational curves of degree $d > 1$ in \mathbb{P}^{4n^2} , passing through a fixed point, is even larger), thus its real dimension is $8n^2 - 2$. On the other hand, our dimension count in Corollary 1.4 of the space of all twistor lines, passing through a fixed point in the period domain, is $4n^2 - 1$. Thus, through a

given point, there are half as many twistor directions than general rational curves, and the problem of twistor path connectivity may be roughly considered as a “sub-Riemannian” version of rational connectedness.

The plan of the paper is as follows.

In Section 1 we describe our basic set-up and show that the twistor spheres $S^2 \subset \text{Compl}$ are complex submanifolds (Corollary 1.7). We define the union C_I of all twistor spheres passing through a given period I and show that the stabilizer group G_I of I acts transitively on the set of twistor spheres containing I . The sets C_I will serve as the main tool in the proof of twistor path connectivity.

In Section 2 we provide an argument, illustrated by a picture, that the set of periods reachable from a given one I by means of all possible triples of consecutive twistor spheres contains an open neighborhood of the initial period. Then, the connectedness of the period domain allows us to conclude that any two periods can be connected by a path of twistor lines. The three spheres argument is essentially due to the transversality formulated in its most general form in Proposition 2.5. We also show that C_I is a real analytic subset of Compl .

In Section 3 we prove the generic connectivity part of Theorem 1. The idea of the proof is to show that the space of triples of consecutive twistor spheres connecting a fixed pair of periods is not the union of its subspaces for which the first or, respectively, the second, of the two joint points belongs to the locus of tori with nontrivial Néron-Severi group in the period domain. Again, the transversality, stated in Proposition 2.5, constitutes the main tool for proving generic connectivity.

In Section 4 we prove that the degree of twistor lines in $G(2n, 4n)$ with respect to the Plücker embedding is 2. Here we use the fact that the group $G := GL^+(V_{\mathbb{R}})$ acts transitively on the set of all twistor lines together with an explicit computation on an explicit example.

In Section 5 we gather the calculations needed in the example of Section 4.

The authors are indebted to Eyal Markman for suggesting the problem of twistor path connectivity of the period domain and many useful comments that helped improve the exposition.

1. THE SPACE OF TWISTOR SPHERES

1.1. Let A be a complex torus of complex dimension $2n$. Denote by $V_{\mathbb{R}}$ the real tangent space $T_{\mathbb{R},0}A$ and by V the complex tangent space $T_{\mathbb{C},0}A \subset T_{\mathbb{R},0}A \otimes \mathbb{C}$, so that $\dim_{\mathbb{R}} V_{\mathbb{R}} = 2 \dim_{\mathbb{C}} V = 4n$. Let $I: V_{\mathbb{R}} \rightarrow V_{\mathbb{R}}$ be the operator of the complex structure on $V_{\mathbb{R}}$ induced by scalar multiplication by i on V . Let $G := GL^+(V_{\mathbb{R}}) \cong GL_{4n}^+(\mathbb{R})$ be the group of orientation-preserving automorphisms of $V_{\mathbb{R}}$, which acts via conjugation on the set of complex structures on $V_{\mathbb{R}}$: $g \cdot I = {}^g I := gIg^{-1}$ for $g \in G$. This action is clearly transitive. We consider the set Compl of all complex structures on $V_{\mathbb{R}}$ as the orbit of I : $\text{Compl} = G \cdot I$, diffeomorphic to the homogeneous space $G/GL(V) \cong GL_{4n}^+(\mathbb{R})/GL_{2n}(\mathbb{C})$. It carries the structure of a complex manifold, see [3, p. 31] and Proposition 1.6. Assume that, in addition to I , there is a complex structure J on $V_{\mathbb{R}}$ anticommuting with I . Then I and J determine a twistor sphere

$$S(I, J, K) := \{aI + bJ + cK \mid a^2 + b^2 + c^2 = 1\},$$

where $K = IJ$. In general, for two complex structures I_1, I_2 , not necessarily anticommuting, such that $I_1 \neq \pm I_2$, and such that they are contained in the same twistor sphere S , we will also denote this sphere by $S(I_1, I_2)$. Our notation is justified by the following lemma.

Lemma 1.1. *Every twistor sphere S is uniquely determined by any pair of non-proportional complex structures $I_1, I_2 \in S$.*

Proof. By definition, $S = \{aI + bJ + cK \mid a^2 + b^2 + c^2 = 1\}$ for some I, J and $K = IJ$ satisfying the quaternionic relations. Let the 3-dimensional real vector space $\langle I, J, K \rangle_{\mathbb{R}} \subset \text{End}V_{\mathbb{R}}$ be equipped with the inner product (\cdot, \cdot) defined by requiring the basis I, J, K to be orthonormal. Let $\mathbb{H} := \mathbb{H}(I, J) = \langle Id, I, J, K \rangle_{\mathbb{R}} \subset \text{End}V_{\mathbb{R}}$ be the subalgebra of quaternions generated by I and J . For arbitrary vectors $u, v \in \langle I, J, K \rangle_{\mathbb{R}} \subset \mathbb{H}$ we have the equality in \mathbb{H} , $u \cdot v = -(u, v)Id + u \times v$ where \cdot denotes the product in \mathbb{H} and $u \times v$ is the ordinary cross product of vectors in $\mathbb{R}^3 = \langle I, J, K \rangle_{\mathbb{R}}$. Note that u and v are orthogonal if and only if u and v anticommute, and the vectors of length 1 in $\langle I, J, K \rangle_{\mathbb{R}}$ are precisely those belonging to S . So our inner product does not depend on the choice of the quaternionic basis I, J, K spanning the sphere S . The complex structures I_1, I_2 determine a 2-plane $\langle I_1, I_2 \rangle_{\mathbb{R}}$ in $\text{End}V_{\mathbb{R}}$. The intersection of this 2-plane with the set of all complex structures is the set of vectors of length 1 for our inner product, hence equal to the great circle $S^1 = S \cap \langle I_1, I_2 \rangle_{\mathbb{R}}$ and independent of the choice of twistor sphere S containing I_1, I_2 . Choose $u, v \in S$ to be an orthonormal basis of our plane $\langle I_1, I_2 \rangle_{\mathbb{R}}$ or, equivalently, a pair of anticommuting complex structures in S^1 . Then $u \cdot v = u \times v$ is again a point in S whose corresponding vector is orthogonal to our 2-plane (which is equivalent to $u \cdot v$ being a complex structure anticommuting with I_1, I_2). Thus $S = \{au + bv + c u \cdot v \mid a^2 + b^2 + c^2 = 1\}$ which shows that the set S is uniquely determined by u, v (and $u \cdot v$) and hence it is uniquely determined by I_1 and I_2 . \square

1.2. By the definition of *Compl*, the group G acts transitively on the space of complex structures *Compl* on A :

$$g \in G: J \mapsto {}^gJ = gJg^{-1}.$$

In particular, G acts on the set of all twistor spheres $S(I, J)$:

$$g \cdot S(I, J) = S({}^gI, {}^gJ).$$

Consider the subgroup G_I of G fixing I via the above action: $G_I = GL(V_{\mathbb{R}}, I) \cong GL(V)$, so that *Compl* $\cong G/G_I$. For $g \in G_I$ we have $g \cdot S(I, J) = S(I, {}^gJ)$. We have

Proposition 1.2. *The group G_I acts transitively on the set N_I of complex structures anticommuting with I .*

Proof. Fix a hermitian form h on V (determining a Kähler class in $H^{1,1}(A)$) and a global holomorphic form $\sigma \in H^{2,0}(A)$. Any operator $J: V_{\mathbb{R}} \rightarrow V_{\mathbb{R}}$ defined by the equation $h(x, Jy) = \text{Re } \sigma(x, y)$ is non-degenerate, skew-symmetric with respect to h and anticommutes with I . As $h(\cdot, \cdot)$ is positive definite, for such J , its square J^2 is a diagonalizable operator with negative real eigenvalues, such that the vector space $V_{\mathbb{R}}$ is an h -orthogonal sum of the eigenspaces V_{λ} for J^2 : $V_{\mathbb{R}} = \bigoplus^{\perp} V_{\lambda}$. Moreover, as J^2 commutes with I , we have that the V_{λ} are I -invariant. It follows that every V_{λ} is an orthogonal sum of subspaces of the form $\langle v, Iv, Jv, Kv \rangle$, $K = IJ, v \neq 0$.

As $h(Jx, Jy) = \operatorname{Re} \sigma(Jx, y) = -\operatorname{Re} \sigma(y, Jx) = -h(y, J^2x) = h(-J^2x, y)$, we see that $J^2 = -Id$ if and only if J is an isometry with respect to h . Let us replace J on each eigenspace V_λ by $\frac{1}{\sqrt{-\lambda}}J$. Denoting this operator again by J we obtain $J^2 = -Id$, moreover the vectors v, Iv, Jv, Kv for $v \in V_\lambda$ all have equal h -norm and form an orthogonal basis of the subspace they span. This tells us that V_λ , and hence the whole $V_\mathbb{R}$, decomposes into an orthogonal sum of subspaces $\langle v, Iv, Jv, Kv \rangle$ such that the union of the bases of these subspaces forms an orthonormal basis of $V_\mathbb{R}$. The matrix of h on this basis is the identity matrix, and the matrix of J has the block-diagonal form with the following 4×4 blocks on the diagonal

$$\left(\begin{array}{cc|cc} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array} \right).$$

Summarizing the above observations, we see that our initial choice of a Kähler class $h \in H^{1,1}(A)$ and nondegenerate $\sigma \in H^{2,0}(A)$ defines a complex structure J which anticommutes with I , and, moreover, gives, in a non-unique way, a choice of an h -orthonormal basis where J has the canonical form above. The group $G_I \cong GL(V) < GL^+(V_\mathbb{R}) = GL_{4n}^+(\mathbb{R})$ acts transitively on the set of such bases.

Conversely, given a complex structure I , any operator $J: V_\mathbb{R} \rightarrow V_\mathbb{R}$ which anticommutes with I and whose square is $-Id$ (but which is not a priori defined as a skew-symmetric operator with respect to a form on $V_\mathbb{R}$ induced by a hermitian form on V), originates, via a procedure similar to the one described above, from an appropriate Kähler class h and a non-degenerate holomorphic 2-form σ . \square

1.3. Therefore, given a complex structure $J \in N_I$, $N_I = G_I \cdot J \cong G_I/G_{I,J}$ is the orbit of J under G_I , where $G_{I,J}$ is the stabilizer group of J in G_I . Since $G_{I,J}$ is the subgroup of elements of $G_I = GL(V)$ commuting with J , that is, preserving the quaternionic structure on $V_\mathbb{R}$ determined by I and J , we have $G_{I,J} \cong GL(V, \mathbb{H})$ which we will also denote by $G_\mathbb{H}$. So $N_I \cong GL(V)/GL(V, \mathbb{H})$ and we deduce

Corollary 1.3. *The set N_I is a real submanifold of $Compl$ of dimension $4n^2$.*

Proof. The dimension of the orbit as a complex manifold is $\dim_{\mathbb{C}} GL(V) - \dim_{\mathbb{C}} GL(V, \mathbb{H}) = (2n)^2 - 2n^2 = 2n^2$ (the complex dimension of the space of quaternionic $n \times n$ matrices is $2n^2$). The real dimension is thus equal to $4n^2$. \square

1.4. Let $S = S(I, J)$ for $J \in N_I$ be a twistor sphere. Define $G_{I,S} \subset G_I$ to be the stabilizer of S as a set, i.e., the set of elements g of G_I such that $g \cdot S \subset S$. For any $g \in G_{I,S}$, the complex structure ${}^gJ \in S$ also anticommutes with I , so gJ is of the form $aJ + bK$, $a^2 + b^2 = 1$. Setting $a = \cos t$, $b = \sin t$ we have $aJ + bK = e^{\frac{t}{2}} J e^{-\frac{t}{2}}$, where $e^{sI} = \cos s \mathbb{1} + \sin s I \in G_I$ realizes, via the adjoint action, the rotations of S around $\{\pm I\}$. Conversely, if $g \in G_I$ and ${}^gJ \in S$, then $g \in G_{I,S}$. The set of $g \in G_{I,S}$ such that ${}^gJ = J$ is the quaternionic subgroup $G_{I,J} = G_\mathbb{H} \subset G_{I,S}$. Explicitly, we have $G_{I,S} = \langle e^{tI}, t \in \mathbb{R} \rangle \times G_\mathbb{H}$, where $\langle e^{tI}, t \in \mathbb{R} \rangle \cong S^1$ (which is a subgroup of the center of G_I). This tells us, in particular, that $\dim_{\mathbb{R}} G_{I,S} = \dim_{\mathbb{R}} G_\mathbb{H} + 1 = 4n^2 + 1$.

Let M_I be the set of all twistor spheres in $Compl$ containing I . The natural map $N_I \rightarrow M_I$ identifies two complex structures J_1 and J_2 whenever they belong to the

same twistor sphere through I , i.e., $S(I, J_1) = S(I, J_2)$. More precisely, they belong to the great circle in $S := S(I, J_1)$ consisting of elements anticommuting with I . Hence, for the S^1 -action $J \in N_I \mapsto e^{tI}J = e^{\frac{tI}{2}}Je^{-\frac{tI}{2}}$ on N_I defined above, we have $N_I/S^1 = M_I$. Therefore Corollary 1.3 immediately implies

Corollary 1.4. *The set M_I is a real manifold of dimension $4n^2 - 1$.*

1.5. **The twistor cone of I .** Define the set $C_I := \bigcup_{S \in M_I} S \subset Compl$ as the union of all twistor spheres containing I . All spheres in this union contain the complex structures I and $-I$. We will sometimes refer to the set C_I as a cone. Proposition 1.2 immediately implies

Corollary 1.5. *The group G_I acts transitively on $M_I \cong G_I/G_{I,S}$ so that $C_I = \bigcup_{g \in G_I} g \cdot S(I, J)$.*

We will give an explicit local parametrization of C_I in the next section and prove that the cone C_I is a real-analytic subset of $Compl$ of dimension $4n^2 + 1$ (Proposition 2.6).

1.6. We now describe the complex structure on the tangent bundle of the orbit $Compl = G \cdot I$. Then we will see that the tangent bundle TS^2 of an arbitrary twistor sphere $S^2 \subset Compl$ is a subbundle of the restricted tangent bundle $TCompl|_{S^2}$, invariant under the complex structure of $TCompl|_{S^2}$. This will imply that S^2 is a complex submanifold in $Compl$.

Proposition 1.6. *The manifold $Compl$ is a complex manifold. Its complex structure l_I is given by left multiplication by I on $T_I Compl \subset End(V_{\mathbb{R}})$.*

Proof. In order to prove the theorem we will use the classical period matrix realizations of open charts of the period domain. The charts carry the (compatible) complex structures, induced from the complex affine space containing the period matrices, thus giving a globally defined complex structure on the period domain. We will show that the complex structure $l_I: T_I Compl \rightarrow T_I Compl$ in the statement is induced by this complex structure.

The period matrix realization of the period domain of marked complex tori uses the complex space V with a fixed basis e_1, \dots, e_{2n} so that an arbitrary full rank lattice $\Gamma \subset V_{\mathbb{R}}$, determining the complex torus $A = V_{\mathbb{R}}/\Gamma$, can be written in terms of this basis of V ,

$$\langle \gamma_1, \dots, \gamma_{2n}, \dots, \gamma_{4n} \rangle = (e_1, \dots, e_{2n})(Z_1, Z_2).$$

The complex affine space $Mat_{2n \times 4n}(\mathbb{C}) \cong \mathbb{C}^{8n^2}$ of $2n \times 4n$ complex matrices $Z = (Z_1, Z_2)$ carries a natural complex structure, given by multiplication by i . The affine chart of matrices (Z_1, Z_2) , subject to the open condition of the corresponding Γ having full rank, holomorphically embeds into $G(2n, 4n)$, where the embedding sends (Z_1, Z_2) to the $2n$ -dimensional subspace of $V_{\mathbb{R}} \otimes \mathbb{C}$ spanned by the rows of (Z_1, Z_2) . There is a mapping f from an open subspace of $Mat_{2n \times 4n}(\mathbb{C})$ to $Compl$, namely, the matrix $Z = (Z_1, Z_2)$, whose columns generate a full rank lattice Γ in $V_{\mathbb{R}}$, uniquely determines the complex structure $f(Z) = I_Z: V_{\mathbb{R}} \rightarrow V_{\mathbb{R}}$ of the torus $A \cong V_{\mathbb{R}}/\Gamma$. Such I_Z is induced by multiplication by i on V , that is, the matrix of I_Z in the basis $\gamma_1, \dots, \gamma_{2n}, \dots, \gamma_{4n}$ of Γ , denoted also by I_Z , satisfies

$$(e_1, \dots, e_{2n})ZI_Z = (e_1, \dots, e_{2n})iZ.$$

This can be written in short as the relation

$$Zf(Z) = iZ.$$

The mapping f (certainly, not bijective) is a submersion onto $Compl$. We want to use the differential $df|_Z$ to induce the complex structure on $T_{f(Z)}Compl$. If we take any small tangent vector $X \in T_Z Mat_{2n \times 4n}(\mathbb{C})$ then, writing $f(Z+X) = I_Z + df|_Z(X) + \dots$ and $f(Z+iX) = I_Z + df|_Z(iX) + \dots$, we have that the equalities

$$(Z+X)f(Z+X) = i(Z+X), \quad (Z+iX)f(Z+iX) = i(Z+iX)$$

imply

$$Zdf(X) + XI_Z = iX, \quad Zdf(iX) + iXI_Z = -X.$$

Multiplying the first of these equalities by i and comparing to the second, we obtain $iZdf(X) = Zdf(iX)$. For Z in the domain of f , this equation has a unique solution $df(iX) = I_Z df(X)$, which is precisely the statement of the proposition. \square

Corollary 1.7. *The twistor spheres $S^2 \subset Compl$ are complex submanifolds.*

Proof. The proof is based on the simple observation that the tangent space of $S^2 = S(I, J)$ at the point I , for $I, J, K = IJ$ satisfying the quaternionic identities, is the 2-plane $\langle J, K \rangle_{\mathbb{R}} \subset T_I Compl$ (this can be shown using the parametrization of S^2 by $\langle e^{sJ} e^{tK} I \mid s, t \in \mathbb{R} \rangle$ as in Paragraph 1.4) and this plane is obviously invariant under left multiplication by I . Thus, TS^2 is a complex subbundle of $TCompl|_{S^2}$ and thus $S^2 \subset Compl$ is a complex submanifold. \square

2. TWISTOR PATH CONNECTIVITY OF $Compl$

The main result of this section is Theorem 2.3. Before proving it we need to introduce a certain mapping and prove an important technical result about it (Proposition 2.1).

2.1. Let I, J, K be a triple of complex structures belonging to a twistor sphere S . Consider the smooth mapping

$$\begin{aligned} \Phi: G_J \times G_K &\longrightarrow Compl, \\ (g_1, g_2) &\longmapsto {}^{g_1 g_2} I, \end{aligned}$$

where, as before, the action on $Compl$ is by conjugation: $g \cdot I = {}^g I = g I g^{-1}$. The mapping Φ clearly sends $G_{\mathbb{H}} \times G_{\mathbb{H}}$ to I , so that its differential $d_{(e,e)} \Phi$ factors through

$$\widetilde{d_{(e,e)} \Phi} : T_e G_J / T_e G_{\mathbb{H}} \oplus T_e G_K / T_e G_{\mathbb{H}} \rightarrow T_I Compl.$$

Proposition 2.1. *Suppose I, J, K is a quaternionic triple. The mapping*

$$\widetilde{d_{(e,e)} \Phi} : T_e G_J / T_e G_{\mathbb{H}} \oplus T_e G_K / T_e G_{\mathbb{H}} \rightarrow T_I Compl$$

is injective. Hence, since the two spaces have the same dimension, it is an isomorphism.

Proof. By the definition of $\widetilde{d_{(e,e)}\Phi}$, its restrictions to the above direct summands are injective. Let us show that it is injective on the direct sum. Consider $X \in T_e G_J, Y \in T_e G_K$ and the vector $\widetilde{d_{(e,e)}\Phi}(X + T_e G_{\mathbb{H}}, Y + T_e G_{\mathbb{H}})$, which is

$$d_{(e,e)}\Phi(X + Y) = \left. \frac{d}{dt} \right|_{t=0} (e^{tX} e^{tY} \cdot I) = (X + Y)I - I(X + Y) \in T_I \text{Compl}.$$

Assume that this vector is zero, that is, $X + Y$ commutes with I :

$$(1) \quad I(X + Y) = (X + Y)I.$$

Then the conjugate $(X + Y)^J = J^{-1}(X + Y)J = X^J + Y^J = X - JYJ$ must also commute with I . Using that Y commutes with K we obtain

$$X - JYJ = X - JYKI = X - JKYI = X - IYI.$$

The commutation with I is expressed now by $I(X - IYI) = (X - IYI)I$, or

$$IX + YI = XI + IY,$$

which gives

$$I(X - Y) = (X - Y)I.$$

Adding the last equality to (1) side by side gives that $XI = IX$, hence $YI = IY$, which implies $X, Y \in T_e G_{\mathbb{H}}$. This proves the required injectivity of $\widetilde{d_{(e,e)}\Phi}$. \square

Corollary 2.2. *Suppose I, J, K is a quaternionic triple. The mapping Φ is a submersion at $(e, e) \in G_J \times G_K$, that is*

$$d_{(e,e)}\Phi(T_e G_J \oplus T_e G_K) = T_I \text{Compl} \cong \mathbb{R}^{8n^2}.$$

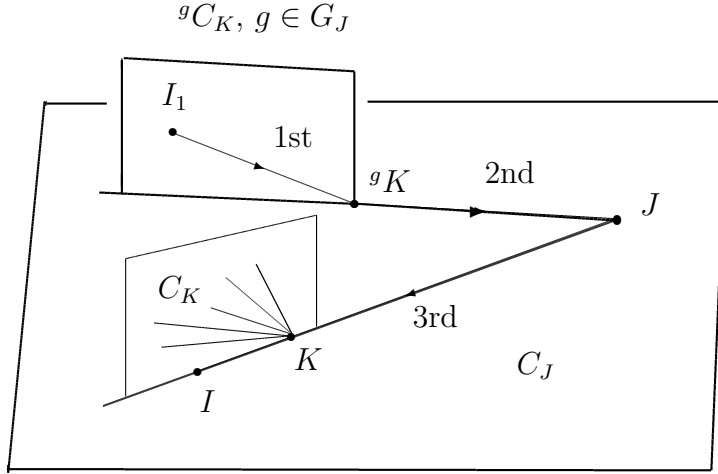
Proof. As $\dim_{\mathbb{R}} T_e G_J / T_e G_{\mathbb{H}} = \dim_{\mathbb{R}} T_e G_K / T_e G_{\mathbb{H}} = 4n^2$, the statement that Φ is a submersion follows from the fact that $d_{(e,e)}\Phi$ factors through $\widetilde{d_{(e,e)}\Phi}$ and the fact that the mapping $\widetilde{d_{(e,e)}\Phi} : T_e G_J / T_e G_{\mathbb{H}} \oplus T_e G_K / T_e G_{\mathbb{H}} \rightarrow T_I \text{Compl}$ is an isomorphism by Proposition 2.1. \square

Theorem 2.3. *Given a complex structure $I \in \text{End}(V_{\mathbb{R}})$, there is a neighborhood of I in the space of complex structures on $V_{\mathbb{R}}$ such that, for any complex structure I_1 in this neighborhood, there is a twistor path consisting of three spheres joining I to I_1 . Consequently, the space of complex structures Compl is twistor path connected.*

Proof. Choose a complex structure J , anticommuting with I , and consider the sphere $S = S(J, I)$ and the cone C_J . By Lemma 1.1, the complex structures $K = IJ$ and I span the sphere $S = S(K, I) = S(J, I)$. We can then form the cone C_K whose intersection with C_J contains S . See Picture 1 below where the cones C_J and C_K are depicted by transversal planes and the sphere S lying in their intersection is depicted by a line.

We first show that the images of C_K under the action of G_J (“rotation of C_K around J ”) sweep out an open neighborhood of I in Compl . Since Φ is a submersion by Corollary 2.2, there exist neighborhoods $U_{e,J} \subset G_J$ and $U_{e,K} \subset G_K$ of e such that the set $\Phi(U_{e,J} \times U_{e,K})$ contains an open neighborhood of I . By definition, the cone C_K contains the orbit $G_K \cdot I$. Hence the union $\bigcup_{g \in G_J} {}^g C_K$ contains the image of Φ and consequently it contains an open neighborhood of I .

Now the three twistor spheres connecting I to an arbitrary point I_1 in this neighborhood are found as illustrated in the following picture.



Picture 1.

The connectedness of the space $Compl$ allows us to conclude, as in [4, Prop. 3.7] that $Compl$ is twistor path connected. \square

2.2. Another immediate consequence of the injectivity of $\widetilde{d_{(e,e)}}\Phi$ proved in Proposition 2.1 is the following

Corollary 2.4. *For a quaternionic triple I, J, K , the triple intersection of the submanifolds $G_I/G_{\mathbb{H}}, G_J/G_{\mathbb{H}}$ and $G_K/G_{\mathbb{H}}$ of the homogeneous space $G/G_{\mathbb{H}}$ at $eG_{\mathbb{H}}$ is transversal.*

The following generalization of this transversality is one of the main ingredients of the proof of connectivity by generic twistor paths in Section 3.

Proposition 2.5. *Let I_1, I_2, I_3 be complex structures belonging to the same twistor sphere S . The submanifolds $G_{I_1}/G_{\mathbb{H}}, G_{I_2}/G_{\mathbb{H}}, G_{I_3}/G_{\mathbb{H}}$ in $G/G_{\mathbb{H}}$ intersect transversally (as a triple) if and only if I_1, I_2, I_3 are linearly independent as vectors in $End V_{\mathbb{R}}$.*

Proof. Choose anticommuting complex structures I, J in S , and set $K = IJ$. By Corollary 2.4,

$$(2) \quad T_e G / T_e G_{\mathbb{H}} = V_I \oplus V_J \oplus V_K,$$

where we set $V_I := T_e G_I / T_e G_{\mathbb{H}}, V_J := T_e G_J / T_e G_{\mathbb{H}}, V_K := T_e G_K / T_e G_{\mathbb{H}}$.

We shall prove that $T_e G / T_e G_{\mathbb{H}}$ also decomposes into the direct sum of its subspaces $V_i := T_e G_{I_i} / T_e G_{\mathbb{H}}, i = 1, 2, 3$. Put $I_i = a_i I + b_i J + c_i K, i = 1, 2, 3$. Assume, on the contrary, that for certain vectors $X \in V_1, Y \in V_2$ and $Z \in V_3$ we have $X + Y + Z = 0$. Let $X := X_I + X_J + X_K$ be the decomposition of X into the sum of its components in the respective subspaces of (2), and do similarly for Y and Z . Then for X the commutation relation $[X, I_1] = 0$ can be written as

$$a_1[X_J + X_K, I] + b_1[X_I + X_K, J] + c_1[X_I + X_J, K] = 0.$$

Note that in the above expression, the term $[X_J, I]$, for example, anticommutes with both I, J , hence commutes with $K = IJ$, and an analogous commutation relation holds for the other terms as well. Hence we can decompose the expression on the left side of the above equality with respect to (2):

$$(b_1[X_K, J] + c_1[X_J, K]) + (a_1[X_K, I] + c_1[X_I, K]) + (a_1[X_J, I] + b_1[X_I, J]) = 0.$$

From here we conclude that $b_1[X_K, J] + c_1[X_J, K] = 0$, $a_1[X_K, I] + c_1[X_I, K] = 0$ and $a_1[X_J, I] + b_1[X_I, J] = 0$. Perturbing the quaternionic triple I, J, K , we may assume that all $a_i, i = 1, 2, 3$, are nonzero. Then we can use the last two equalities to express

$$(3) \quad [X_J, I] = -\frac{b_1}{a_1}[X_I, J], \quad [X_K, I] = -\frac{c_1}{a_1}[X_I, K].$$

Note that $F_J := [\cdot, J]: V_I \rightarrow V_K$, $F_K := [\cdot, K]: V_I \rightarrow V_J$ and $F_I := [\cdot, I]: V_J \rightarrow V_K$ are isomorphisms of the respective vector spaces. Then, using (3), we can write

$$X_J = -\frac{b_1}{a_1}F_I^{-1} \circ F_J(X_I), \quad X_K = -\frac{c_1}{a_1}F_I^{-1} \circ F_K(X_I),$$

so that

$$X = X_I + \left(-\frac{b_1}{a_1}F_I^{-1} \circ F_J(X_I) \right) + \left(-\frac{c_1}{a_1}F_I^{-1} \circ F_K(X_I) \right).$$

Using $a_2, a_3 \neq 0$, we obtain similar expressions for Y and Z . Since F_I, F_J, F_K are isomorphisms, the equality $X + Y + Z = 0$ can now be written as

$$\begin{pmatrix} 1 & 1 & 1 \\ -\frac{b_1}{a_1} & -\frac{b_2}{a_2} & -\frac{b_3}{a_3} \\ -\frac{c_1}{a_1} & -\frac{c_2}{a_2} & -\frac{c_3}{a_3} \end{pmatrix} \begin{pmatrix} X_I \\ Y_I \\ Z_I \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This has a nontrivial solution if and only if the columns of the matrix, i.e., I_1, I_2, I_3 , are linearly dependent. \square

2.3. We can now prove that the cone C_I has a real analytic structure. Define the incidence correspondence

$$S_I := \{(S, J) \mid J \in S\} \subset M_I \times \text{Compl}.$$

Then S_I is an S^2 -bundle over M_I and C_I is the image of S_I by the projection to Compl :

$$\begin{array}{ccc} N_I & \hookrightarrow & S_I \xrightarrow{pr_2} C_I \subset \text{Compl} \\ & \searrow & \downarrow pr_1 \\ & & M_I. \end{array}$$

The projection $S_I \rightarrow M_I$ has two sections σ_+ and σ_- , given by $+I$ and $-I$ respectively.

Proposition 2.6. *The map $pr_2 : S_I \rightarrow C_I$ is a diffeomorphism away from the images of σ_{\pm} and contracts these images to points. Therefore the cone C_I is a real-analytic subset of Compl of dimension $4n^2 + 1$, smooth away from the points $\pm I$.*

Proof. First note that pr_2 clearly contracts the images of σ_{\pm} . Also, it is injective away from $\pm I$ by Lemma 1.1. To see that it is also an immersion away from $\pm I$, let J be in $C_I \setminus \{\pm I\}$, not necessarily anticommuting with I . Define the following mapping

$$\Phi: (T_e G_I / T_e G_{\mathbb{H}}) \times \mathbb{R} \rightarrow C_I,$$

$$(X, t) \mapsto e^X e^{tK} J e^{-tK} e^{-X},$$

where $K \in S(I, J) \setminus S^1$ for $S^1 = \langle I, J \rangle_{\mathbb{R}} \cap S(I, J)$. Then the restriction of Φ on a small enough neighborhood of $(0, 0) \in (T_e G_I / T_e G_{\mathbb{H}}) \times \mathbb{R}$ defines a parametrization of C_I around J .

Here the subgroup $e^{tK}, t \in \mathbb{R}$, rotates the sphere $S = S(I, J)$ around the axis $\{\pm K\}$ and, together with the rotation subgroup $e^{tI} \subset G_{I,S} \subset G_I$, sweeps out in S , via the above action, a neighborhood of any point of S other than $\pm I, \pm K$. Proposition 2.5 provides that K may be chosen arbitrarily in $S \setminus S^1$, which in turn gives us that C_I is a manifold, smooth away from $\pm I$, of dimension $\dim_{\mathbb{R}}(G_I / G_{I,S}) + \dim_{\mathbb{R}} S = (4n^2 - 1) + 2 = 4n^2 + 1$. The fact that the points $\pm I$ are indeed singular points of the cone C_I is easy to prove. \square

We can now use Proposition 2.1 to also prove

Corollary 2.7. *For a quaternionic triple I, J, K , the cones C_J and C_K intersect transversely at $\pm I$ in $S := S(I, J) \subset C_K \cap C_J$ and hence, by continuity, at all points in some neighborhood of $\pm I$.*

Proof. The transversality of intersection at $I \in S$ means that the intersection of the quotient subspaces $W_J = T_I C_J / T_I S$ and $W_K = T_I C_K / T_I S$ in $T_I \text{Compl} / T_I S$ is zero. The subspaces W_J and W_K are isomorphic images, under the quotient mapping $T_I \text{Compl} \rightarrow T_I \text{Compl} / T_I S$, of the subspaces $T_e G_J / T_e G_{J,S}$ and $T_e G_K / T_e G_{K,S}$ respectively, in the (non-canonical) decompositions of the transversal subspaces

$$\widetilde{d_{(e,e)}}\Phi(T_e G_J) \cong T_e G_J / T_e G_{J,S} \oplus T_e G_{J,S} / T_e G_{\mathbb{H}}$$

and

$$\widetilde{d_{(e,e)}}\Phi(T_e G_K) \cong T_e G_K / T_e G_{K,S} \oplus T_e G_{K,S} / T_e G_{\mathbb{H}}.$$

Thus, the transversality of W_J and W_K follows. \square

3. CONNECTIVITY BY GENERIC TWISTOR PATHS

Recall that a period in Compl is *generic* if the corresponding complex torus has trivial Néron Severi group. A twistor path in Compl is called *generic*, if its successive twistor spheres intersect at generic periods. In this section we prove the connectivity part of Theorem 1, i.e.,

Proposition 3.1. *Any two periods in the period domain Compl can be connected by a generic twistor path.*

In this section, with the exception of Lemma 3.4 and its proof, we do not assume that the complex structures I, J, K (with or without subscripts) anticommute.

3.1. Outline of the proof. Define \mathcal{T} to be the closure, in $\text{Compl} \times \text{Compl} \times \text{Compl}$, of the set of triples (I, J, K) that are linearly independent and belong to the same twistor sphere. Denote by

$$\begin{aligned} pr_1: \quad \text{Compl} \times \text{Compl} \times \text{Compl} &\longrightarrow \text{Compl}, \\ pr_{23}: \quad \text{Compl} \times \text{Compl} \times \text{Compl} &\longrightarrow \text{Compl} \times \text{Compl} \end{aligned}$$

the respective projections. For $(I_1, J_1, K_1) \in \mathcal{T}$, we defined, in Paragraph 2.1, the mapping Φ_{I_1, J_1, K_1} :

$$\begin{aligned} \Phi_{I_1, J_1, K_1}: G_{J_1} \times G_{K_1} &\longrightarrow \text{Compl}, \\ (g_1, g_2) &\longmapsto g_1 g_2 I_1 g_2^{-1} g_1^{-1} = {}^{g_1 g_2} I_1. \end{aligned}$$

Proposition 2.5 tells us that, when I_1, J_1, K_1 are linearly independent, Φ_{I_1, J_1, K_1} is a submersion near $(e, e) \in G_{J_1} \times G_{K_1}$. In other words, there is a neighborhood $U_{e, G} \subset G = GL^+(V_{\mathbb{R}})$ of $e \in G$ such that the map Φ_{I_1, J_1, K_1} is submersive on $U_{e, J_1} \times U_{e, K_1}$, where $U_{e, J_1} := U_{e, G} \cap G_{J_1}$ and $U_{e, K_1} := U_{e, G} \cap G_{K_1}$ (and the image is, thus, a neighborhood of I_1 in Compl).

Let I_2 be an arbitrary point in the image of Φ_{I_1, J_1, K_1} and let $(g_1, g_2) \in U_{e, J_1} \times U_{e, K_1}$ be such that $I_2 = {}^{g_1 g_2} I_1$. With this notation, the three twistor spheres connecting I_1 to I_2 are: $S_1 := S(I_1, J_1, K_1)$, $S := {}^{g_1} S_1 = S({}^{g_1} I_1, {}^{g_1} J_1 = J_1, {}^{g_1} K_1)$ and $S_2 := {}^{g_1 g_2} S_1 = S(I_2, {}^{g_1 g_2} J_1, {}^{g_1 g_2} K_1 = {}^{g_1} K_1)$, with the joint points J_1 and ${}^{g_1} K$.

We are going to show that, for a fixed I_1 , there is a neighborhood $U_{I_1} \subset \text{Compl}$ of I_1 such that for any $I_2 \in U_{I_1}$, we can choose a generic $J \in C_I$, a $K \in S(I, J)$ and find $(g_1, g_2) \in \Phi_{I_1, J, K}^{-1}(I_2)$ as above such that ${}^{g_1 g_2} K$ is also generic.

We begin by proving, in Lemma 3.2, that the set of non-generic periods in C_{I_1} is a countable union of proper analytic subsets, i.e., J can be chosen generic.

Next, for I_2 close to I_1 , and with S_1, S, S_2 as above, connecting I_1 to I_2 , the initial sphere S_1 together with the choice of $J, K \in S_1$, uniquely determines the terminal sphere S_2 together with the pair of periods ${}^{g_1 g_2} J, {}^{g_1 g_2} K$.

To justify this uniqueness we first need to control the fibers of the maps $\Phi_{I_1, J, K}$ in a neighborhood of (I_1, J_1, K_1) , which we do in Lemma 3.5. This allows us to introduce, in Paragraphs 3.4 and 3.7, two maps $\Psi^{I_1 \rightarrow I_2}$ and $\Psi^{I_2 \rightarrow I_1}$ which, roughly, switch (S_1, J, K) and $(S_2, {}^{g_1 g_2} J, {}^{g_1 g_2} K)$.

We then show in Lemma 3.9, after shrinking our various domains, that the composition of $\Psi^{I_1 \rightarrow I_2}$ and $\Psi^{I_2 \rightarrow I_1}$ is the identity. Corollary 3.10 then shows that this implies the irreducibility of the set of triples (S_1, S, S_2) joining I_1 and I_2 mentioned in the introduction, which gives that J and ${}^{g_1 g_2} K$ can both be chosen generic.

Thus the chain of three twistor spheres connecting I_1 to I_2 for every I_2 in some neighborhood of I_1 can be chosen in such a way that the periods at the intersections are generic. For arbitrary I_1 and I_2 , we connect them by a path in Compl consisting of generic triple subchains.

3.2. Let us first show that there are generic periods $J \in C_I$. Dimension-wise this is not trivial because $\dim_{\mathbb{R}} C_I = 4n^2 + 1$, whereas the real dimension of the locus of, for example, abelian varieties in Compl is $4n^2 + 2n$. For an alternating form Ω on $V_{\mathbb{R}}$ we denote by Compl_{Ω} the locus of periods in Compl at which Ω represents a class of Hodge (1,1)-type, that is

$$\text{Compl}_{\Omega} = \{I \in \text{Compl} \mid \Omega(I \cdot, I \cdot) = \Omega(\cdot, \cdot)\}.$$

If we fix a basis of $V_{\mathbb{R}}$ and switch to matrix descriptions, then the condition $\Omega(I \cdot, I \cdot) = \Omega(\cdot, \cdot)$ simply becomes ${}^t I \Omega I = \Omega$, where I and Ω also denote the matrices of the corresponding complex structure and alternating form. The locus of marked complex

tori with nontrivial Néron Severi group is

$$\mathcal{L}_{NS} = \bigcup_{0 \neq [\Omega] \in H^2(A, \mathbb{Q})} \text{Compl}_{\Omega},$$

where A is a fixed complex torus.

Lemma 3.2. *For every $I \in \text{Compl}$ the set of non-generic periods in C_I , that is $C_I \cap \mathcal{L}_{NS}$, is a countable union of closed subsets of C_I none of which contains an open neighborhood (in C_I) of any of its points.*

The proof follows from the following lemmas.

Lemma 3.3. *For any alternating form Ω and any twistor sphere S , the intersection $S \cap \text{Compl}_{\Omega}$ is either finite or all of S .*

Lemma 3.4. *For any J anti-commuting with I and any nonzero alternating form Ω on $V_{\mathbb{R}}$ there is a neighborhood $U_{\Omega} \subset C_I$ of J such that the locus Compl_{Ω} intersects U_{Ω} along a submanifold of positive codimension.*

Let us assume Lemmas 3.3 and 3.4 for a moment and prove Lemma 3.2.

Proof of Lemma 3.2. Assume that some Compl_{Ω} in \mathcal{L}_{NS} contains an open neighborhood $U \subset C_I$ of a point $I_1 \in C_I$. This implies that, for every twistor sphere $S = S(I, \cdot) \subset C_I$ intersecting U , the intersection $S \cap U$ contains a non-empty open subset of S . Hence, by Lemma 3.3, all of the periods of such S are contained in Compl_{Ω} , including the ones anticommuting with I . This holds for all spheres S intersecting U , that is, for all $g \in U_e \subset G_I$, where U_e is a neighborhood of e in G_I , we have that $S(I, {}^g J)$ is contained in Compl_{Ω} . Thus we have a contradiction with the existence of U_{Ω} as in Lemma 3.4. \square

Proof of Lemma 3.3. Follows from the fact that Compl_{Ω} and S (Corollary 1.7) are complex analytic subsets of Compl . \square

Proof of Lemma 3.4. If $J \notin \text{Compl}_{\Omega}$ there is nothing to prove. Assume $J \in \text{Compl}_{\Omega}$.

As in the proof of Proposition 1.2, choose a basis of $V_{\mathbb{R}}$ in which the matrices of J and I are bloc diagonal with 4×4 blocs

$$\left(\begin{array}{cc|cc} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array} \right), \quad \left(\begin{array}{cc|cc} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array} \right)$$

respectively. Such a basis is the union of sets of vectors of the form $\{v, Jv, Iv, JIv\}$. We denote the matrices of I, J and Ω in this basis by the same letters.

Consider the orbit of J under the conjugation action of G_I : $G_I \cdot J \cong G_I/G_{\mathbb{H}}$. Let

$$\Psi: \begin{array}{ccc} G_I & \longrightarrow & \text{Compl}, \\ g & \longmapsto & {}^g J = gJg^{-1}, \end{array}$$

be the evaluation map of the action. Put $G_{\Omega} := \Psi^{-1}(\text{Compl}_{\Omega})$, that is,

$$G_{\Omega} = \{g \in G_I \mid {}^t({}^g J)\Omega({}^g J) = \Omega\},$$

(note that G_Ω need not be a subgroup in G_I). Let $g(\tau)$ be any curve in G_Ω with tangent vector $X := g'(0) \in T_e G_\Omega$ at $e = g(0) \in G_\Omega$. Then, differentiating the constant function ${}^t(g(\tau)J)\Omega(g(\tau)J)$ at $\tau = 0$ we obtain

$$-{}^tX {}^tJ\Omega J + {}^tJ {}^tX\Omega J + {}^tJ\Omega X J - {}^tJ\Omega J X = 0.$$

The left hand side may be simplified, given that ${}^tJ\Omega J = \Omega$ and ${}^tJ = J^{-1} = -J$, to

$$\begin{aligned} & -{}^tX\Omega + {}^tJ {}^tX J {}^tJ\Omega J + {}^tJ\Omega J {}^tJ X J - \Omega X = \\ & = -{}^tX\Omega + {}^tJ {}^tX J \Omega + \Omega {}^tJ X J - \Omega X \\ & = {}^t(X^J - X)\Omega + \Omega(X^J - X), \end{aligned}$$

where

$$X^J := J^{-1}XJ = JXJ^{-1}.$$

So, denoting $Y := X^J - X$, we have the equality

$$(4) \quad {}^tY\Omega + \Omega Y = 0,$$

where Y commutes with I and anticommutes with J . Note that for any X , $X = \frac{1}{2}(X + X^J) + \frac{1}{2}(X - X^J)$, and $T_e G_{\mathbb{H}}$ is the subspace of elements of $T_e G_I$ that commute with I . Hence, the subspace of Y 's in $T_e G_I$ anticommuting with J maps isomorphically onto the quotient space $V_I := T_e G_I / T_e G_{\mathbb{H}}$ under the quotient map $T_e G_I \rightarrow V_I$. So we need to check that for a nonzero Ω the space of solutions to (4) has dimension strictly less than $\dim_{\mathbb{R}} V_I = 4n^2$ (i.e., not all of the orbit $G_I \cdot J$ lies in \mathcal{L}_{NS}).

Now conjugate equation (4) by I to obtain

$${}^tY\Omega^I + \Omega^I Y = 0.$$

Adding and subtracting this from (4) we obtain

$${}^tY(\Omega + \Omega^I) + (\Omega + \Omega^I)Y = 0 \quad \text{and} \quad {}^tY(\Omega - \Omega^I) + (\Omega - \Omega^I)Y = 0.$$

So we may assume that Ω is either I -invariant or I -anti-invariant in equation (4).

Case of I -invariant Ω . As Ω is J -invariant, it determines a skew-symmetric operator $\Omega: V_{\mathbb{R}} \rightarrow V_{\mathbb{R}}$, commuting with J . So we may choose an Ω -invariant plane $P = \langle v, Jv \rangle \subset V_{\mathbb{R}}$ corresponding to a complex eigenvector $v - iJv$ of $\Omega: V_{\mathbb{R}} \rightarrow V_{\mathbb{R}}$ such that the matrix of $\Omega|_P$ is

$$\begin{pmatrix} 0 & -\lambda \\ \lambda & 0 \end{pmatrix}.$$

The complex structure I provides another such plane $IP = \langle Iv, JIv \rangle$, which is also Ω -invariant and orthogonal to P , so that on $P \oplus IP = \langle v, Jv, Iv, JIv \rangle$ the matrices of Ω , J and I are 4×4 -block-diagonal with the following blocks on the diagonal

$$\left(\begin{array}{cc|cc} 0 & -\lambda & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \lambda \\ 0 & 0 & -\lambda & 0 \end{array} \right), \left(\begin{array}{cc|cc} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array} \right), \left(\begin{array}{cc|cc} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array} \right).$$

The condition that Y commutes with I and anticommutes with J tells us that Y has a 4×4 -block structure with blocks of the form

$$\left(\begin{array}{cc|cc} a_1 & a_2 & b_1 & b_2 \\ a_2 & -a_1 & b_2 & -b_1 \\ \hline -b_1 & b_2 & a_1 & -a_2 \\ b_2 & b_1 & -a_2 & -a_1 \end{array} \right).$$

Noting that $\Omega = JD = DJ$ for a diagonal matrix D commuting with J , we can rewrite (4) as

$$(5) \quad DY = {}^tYD, \quad Y^I = Y, \quad Y^J = -Y.$$

For notational convenience we write the matrix Y in terms of its 2×2 -blocks $Y_{k,l}$, $Y = (Y_{k,l})$, $1 \leq k, l \leq 2n$, and denote by $\mathbb{1}_{2 \times 2}$ the 2×2 identity matrix. If at least one λ_i , $1 \leq i \leq n$, in D is nonzero we get for all $1 \leq j \leq n$ the equalities of 4×4 -blocks

$$\begin{aligned} & \begin{pmatrix} \lambda_i \mathbb{1}_{2 \times 2} & 0 \\ 0 & -\lambda_i \mathbb{1}_{2 \times 2} \end{pmatrix} \cdot \begin{pmatrix} Y_{2i-1,2j-1} & Y_{2i-1,2j} \\ Y_{2i,2j-1} & Y_{2i,2j} \end{pmatrix} = \\ & = \begin{pmatrix} {}^tY_{2j-1,2i-1} & {}^tY_{2j,2i-1} \\ {}^tY_{2j-1,2i} & {}^tY_{2j,2i} \end{pmatrix} \cdot \begin{pmatrix} \lambda_j \mathbb{1}_{2 \times 2} & 0 \\ 0 & -\lambda_j \mathbb{1}_{2 \times 2} \end{pmatrix}. \end{aligned}$$

These matrix equalities completely determine all $n - 1$ off-diagonal 4×4 -entries of Y in the i -th ‘‘fat’’ row of 4×4 -blocks in terms of the off-diagonal 4×4 -entries of the i -th ‘‘fat’’ column, $1 \leq i \leq n$. So the codimension of the space of solutions of (5) is at least $4(n - 1)$ (precise lower bound that is reached in the least restrictive case $\lambda_j = \lambda_i$ for all j). For the diagonal 4×4 -entry, $i = j$, we obtain $b_2 = 0$ in $Y_{2i-1,2i}$, so that the codimension is at least $4n - 3$.

Case of I -anti-invariant Ω . The only difference with the previous case is that in the basis $\langle v, Jv, Iv, JIv \rangle$ as above the matrix of Ω is

$$\left(\begin{array}{cc|cc} 0 & -\lambda & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -\lambda \\ 0 & 0 & \lambda & 0 \end{array} \right).$$

So equation (5), written block-wise, gives that in addition to the $4(n - 1)$ conditions for the off-diagonal entries, for the diagonal 4×4 -entry we have $b_1 = 0$, which still results in the lower bound $4(n - 1) + 1 = 4n - 3$ for the codimension of the space of solutions of (5).

Now, by Lemma 3.3, either a twistor sphere in C_I entirely lies in some $Compl_\Omega$ or its intersection with \mathcal{L}_{NS} contains only finitely many points of each $Compl_\Omega$. If $I \notin \mathcal{L}_{NS}$ then no twistor sphere in C_I is contained in \mathcal{L}_{NS} . The codimension estimate above then allows us to conclude that, for every nonzero Ω , the subset $C_I \cap Compl_\Omega$ is of codimension at least $(4n - 3) + 2 = 4n - 1 > 0$ in C_I . If $I \in \mathcal{L}_{NS}$, the lower bound for the codimension is still at least $4n - 3 > 0$. The proof is now complete. \square

3.3. The transversality of the triple intersection of $G_{I_1}/G_{\mathbb{H}}$, $G_{J_1}/G_{\mathbb{H}}$, $G_{K_1}/G_{\mathbb{H}}$ at $eG_{\mathbb{H}}$, which is equivalent to the direct sum decomposition $T_eG/T_eG_{\mathbb{H}} = T_eG_{I_1}/T_eG_{\mathbb{H}} \oplus T_eG_{J_1}/T_eG_{\mathbb{H}} \oplus T_eG_{K_1}/T_eG_{\mathbb{H}}$, is preserved if we perturb $(I_1, J_1, K_1) \in \mathcal{T}$ a little. In other words, there is a compact neighborhood $U_{I_1, J_1, K_1} \subset \mathcal{T}$ of (I_1, J_1, K_1) and a compact neighborhood $U_{e,G} \subset G$ such that $\Phi_{I,J,K}: U_{e,J} \times U_{e,K} \rightarrow Compl$ is a submersion

onto its image for all $(I, J, K) \in U_{I_1, J_1, K_1}$. Moreover, there is a compact neighborhood $U_{I_1} \subset \text{Compl}$ of I_1 which is contained in the image $\Phi_{I, J, K}(U_{e, J} \times U_{e, K})$ for all $(I, J, K) \in U_{I_1, J_1, K_1}$. We will always assume that for each neighborhood $U_{e, G}$ we made a choice of such $U_{I_1} = U_{I_1}(U_{e, G})$. Note that every $I_2 \in U_{I_1}$ is a regular value of $\Phi_{I, J, K}$ for all $(I, J, K) \in U_{I_1, J_1, K_1}$.

Lemma 3.5. *There exists a neighborhood $U_{e, G}$ such that for all $I_2 \in U_{I_1}$ and for all $(I, J, K) \in U_{I_1, J_1, K_1}$, the full preimage $\Phi_{I, J, K}^{-1}(I_2)$ is an $8n^2$ -dimensional submanifold in $U_{e, J} \times U_{e, K}$ of the form*

$$(6) \quad \{(f_1 h_1, h_1^{-1} f_2 h_2) \mid h_1, h_2 \in G_{\mathbb{H}}\} \cap (U_{e, J} \times U_{e, K}),$$

where (f_1, f_2) is a pair in $U_{e, J} \times U_{e, K}$ such that $\Phi_{I, J, K}(f_1, f_2) = I_2$.

Proof of Lemma 3.5. The fact that $\Phi_{I, J, K}^{-1}(I_2) \cap (U_{e, J} \times U_{e, K})$ consists of a finite number of $8n^2$ -dimensional manifolds follows from the regularity of I_2 .

While the part of the fiber in (6) may have been easily guessed, the fact that for a small enough $U_{e, G}$ this is the whole fiber follows from Proposition 2.5. Indeed, assuming that we have $(f_1, f_2), (g_1, g_2) \in \Phi_{I, J, K}^{-1}(I_2) \subset G_J \times G_K$, we see that $f_2^{-1} f_1^{-1} g_1 g_2 \in G_I$. Setting $g_I = f_2^{-1} f_1^{-1} g_1 g_2$ and $g_J = f_1^{-1} g_1 \in G_J$ we have the equality

$$f_2 g_I = g_J g_2.$$

The left side of the equality lies in $G_K G_I$ and the right side lies in $G_J G_K$. If we restrict ourselves to $\Phi_{I, J, K}^{-1}(I_2) \cap (U_{e, G} \times U_{e, G})$ for a small enough neighborhood $U_{e, G} \subset G$ then Proposition 2.5 tells us that, for every element in the product $U_{e, J} U_{e, K} U_{e, I}$, each of its three factors is uniquely determined up to a $G_{\mathbb{H}}$ -correction.

So from our equality $f_2 g_I = g_J g_2$ we obtain $g_I, g_J \in G_{\mathbb{H}}$, which, after setting $h_1 := g_J = f_1^{-1} g_1$ and $h_2 := g_I$, implies that $g_1 = f_1 h_1$ and $g_2 = g_J^{-1} f_2 g_I = h_1^{-1} f_2 h_2$.

Since $U_{I_1}, U_{I_1, J_1, K_1}, U_{e, G}$ are compact and $U_{e, G}$ is independent of the choice of $(I, J, K) \in U_{I_1, J_1, K_1}$, there is a universal upper bound for the number of connected components of $\Phi_{I, J, K}^{-1}(I_2) \cap (U_{e, J} \times U_{e, K})$, for all $I_2 \in U_{I_1}$ and all $(I, J, K) \in U_{I_1, J_1, K_1}$. Therefore we can shrink the compact neighborhood $U_{e, G}$ so that the fibers $\Phi_{I, J, K}^{-1}(I_2) \cap (U_{e, J} \times U_{e, K})$ for all $I_2 \in U_{I_1}$ and all $(I, J, K) \in U_{I_1, J_1, K_1}$ contain only the component specified in (6). \square

Regarding the proof of Lemma 3.5, we note the following.

Remark 3.6. It is not hard to see that the fiber $\Phi_{I, J, K}^{-1}(I_2)$ in Lemma 3.5, as a topological subspace of $G \times G$, depends continuously on $I_2 \in U_{I_1}$ and $(I, J, K) \in U_{I_1, J_1, K_1}$.

Remark 3.7. In general, it is possible that $g \in U_{e, G}$ is not uniquely representable as a triple product of elements in the larger sets G_J, G_K, G_I and thus we cannot say if the whole fiber $\Phi_{I, J, K}^{-1}(I_2) \subset G_J \times G_K$ consists of just one $G_{\mathbb{H}} \times G_{\mathbb{H}}$ -orbit as in Lemma 3.5. This is why we possibly need to shrink $U_{e, G}$.

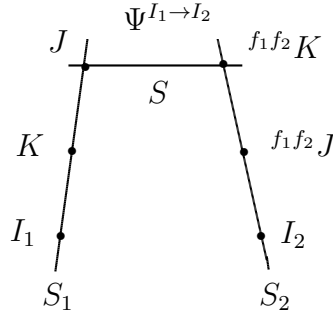
3.4. Recall that, for any I , $M_I = G_I/G_{I, S}$ parametrizes the twistor lines through I (see Paragraph 1.4). For all I , put $U_{I_1, J_1, K_1}(I) := \text{pr}_{23}(\text{pr}_1^{-1}(I) \cap U_{I_1, J_1, K_1})$. Then $U_{I_1, J_1, K_1}(I_1)$ is a neighborhood of (J_1, K_1) in $C_{I_1} \times_{M_{I_1}} C_{I_1} = \text{pr}_{23}(\text{pr}_1^{-1}(I_1) \cap \mathcal{T})$.

Consider the map

$$\begin{aligned} \Psi^{I_1 \rightarrow I_2}: \quad U_{I_1, J_1, K_1}(I_1) &\longrightarrow C_{I_2} \times_{M_{I_2}} C_{I_2} = pr_{23}(pr_1^{-1}(I_2) \cap \mathcal{T}), \\ (S(J, K), J, K) &\longmapsto (S(f_1 f_2 J, f_1 f_2 K), f_1 f_2 K, f_1 f_2 J), \end{aligned}$$

where $(f_1, f_2) \in \Phi_{I_1, J, K}^{-1}(I_2) \cap (U_{e, J} \times U_{e, K})$, and we use, in an obvious way, the triple notation of the kind $(S(J, K), J, K)$ for the elements of the fiber products above. Note the switched order of $f_1 f_2 K, f_1 f_2 J$. The role of this change of order will be clarified later.

Lemma 3.5 guarantees that the mapping $\Psi^{I_1 \rightarrow I_2}$ is well-defined, as its value at $(S(J, K), J, K)$ is uniquely determined by the fiber $\Phi_{I_1, J, K}^{-1}(I_2) \cap (U_{e, J} \times U_{e, K})$, so it does not depend on the choice of a particular point in the fiber.



Picture 2: For fixed I_1 and I_2 any pair $(J, K) \in C_{I_1} \times_{M_{I_1}} C_{I_1}$ near (J_1, K_1) determines a unique pair $(f_1 f_2 K, f_1 f_2 J) \in C_{I_2} \times_{M_{I_2}} C_{I_2}$.

3.5. Next, for each $(I, J, K) \in U_{I_1, J_1, K_1}$ consider the mapping $\Phi_{I, K, J}$ (note that we switched J and K in the subscript). By shrinking the original U_{I_1, J_1, K_1} and $U_{e, G}$ if needed, we can find a compact neighborhood $V_{e, G} \subset G$ such that for each $(I, J, K) \in U_{I_1, J_1, K_1}$ we have

- (a) $\Phi_{I, K, J}: V_{e, K} \times V_{e, J} \rightarrow Compl$ is a submersion onto its image;
- (b) every fiber of this mapping is of the form described in Lemma 3.5;

and

- (c) the image $\Phi_{I, K, J}(V_{e, K} \times V_{e, J})$ contains $U_{I_1} \subset \bigcap_{(I, J, K) \in U_{I_1, J_1, K_1}} \Phi_{I, J, K}(U_{e, J} \times U_{e, K})$

(see Paragraph 3.3).

By Lemma 3.5, conditions (a) and (b) are satisfied. We need only to comment on (c). By Lemma 3.5, for the original triple $(I_1, J_1, K_1) \in U_{I_1, J_1, K_1}$, we can find $V_{e, G}$ such that $\Phi_{I_1, K_1, J_1}: V_{e, K_1} \times V_{e, J_1} \rightarrow Compl$, where $V_{e, K_1} := V_{e, G} \cap G_{K_1}, V_{e, J_1} := V_{e, G} \cap G_{J_1}$, satisfies (a) and (b). Shrinking $U_{e, G}$ and, thus, U_{I_1} , if needed, we can satisfy (c) for Φ_{I_1, K_1, J_1} . Now shrinking U_{I_1, J_1, K_1} and again $U_{e, G}$, if needed, we can satisfy conditions (a), (b) and (c) for all $(I, J, K) \in U_{I_1, J_1, K_1}$.

3.6. Now introduce $V_{I_1, K_1, J_1} := \{(I, K, J) \mid (I, J, K) \in U_{I_1, J_1, K_1}\}$ and $V_{I_1, K_1, J_1}(I) := pr_{23}(pr_1^{-1}(I) \cap V_{I_1, K_1, J_1})$.

Then, for all (I, K, J) in the interior of V_{I_1, K_1, J_1} , the set $pr_1(V_{I_1, K_1, J_1})$ is a neighborhood of I in $Compl$ and $V_{I_1, K_1, J_1}(I)$ is a neighborhood of $(K, J) \in C_I \times_{M_I} C_I$. Note

that due to Condition (c) in Paragraph 3.5, for all $I \in U_{I_1} \cap pr_1(V_{I_1, K_1, J_1})$ and for all $(K, J) \in V_{I_1, K_1, J_1}(I)$, the image $\Phi_{I, K, J}(U_{e, K} \times U_{e, J})$ contains the neighborhood U_{I_1} .

3.7. Choose $I_2 \in U_{I_1} \cap pr_1(V_{I_1, K_1, J_1})$ and K, J such that $(I_2, K, J) \in V_{I_1, K_1, J_1}$. Conditions (a), (b) and (c) in Paragraph 3.5 allow us to define, analogously to $\Psi^{I_1 \rightarrow I_2}$, the map

$$\begin{aligned} \Psi^{I_2 \rightarrow I_1}: \quad V_{I_1, K_1, J_1}(I_2) &\longrightarrow C_{I_1} \times_{M_{I_1}} C_{I_1}, \\ (S(J, K), K, J) &\longmapsto (S(d_1 d_2 J, d_1 d_2 K), d_1 d_2 J, d_1 d_2 K), \end{aligned}$$

for $(d_1, d_2) \in \Phi_{I_2, K, J}^{-1}(I_1)$ (again, note the reversed order of J and K in the subscript).

The period $f_1 f_2 K$ in Picture 2 above will play the role of the ‘‘rotation center’’ for $\Phi_{I_2, f_1 f_2 K, f_1 f_2 J}$ (here $J, K \in C_{I_1}$), similar to the role that J plays for $\Phi_{I_1, J, K}$. This explains why we switched J and K .

Below we will impose restrictions on the domain of $\Psi^{I_1 \rightarrow I_2}$ in order for the image of this map to be contained in the domain of $\Psi^{I_2 \rightarrow I_1}$, so that we can compose them.

We begin by choosing a compact neighborhood U_{J_1, K_1} of (J_1, K_1) in $U_{I_1, J_1, K_1}(I_1)$, which can at first be all of $U_{I_1, J_1, K_1}(I_1)$. We will later modify U_{J_1, K_1} , without changing the original U_{I_1, J_1, K_1} .

Lemma 3.8. *For fixed V_{I_1, K_1, J_1} , we can shrink $U_{e, G}$ and U_{J_1, K_1} so that for arbitrary $I_2 \in U_{I_1}$,*

$$\Psi^{I_1 \rightarrow I_2}(U_{J_1, K_1}) \subset V_{I_1, K_1, J_1}(I_2).$$

Proof. As in Paragraph 3.5, this follows from the fact that the mapping $\Psi^{I_2 \rightarrow I_1}$ depends continuously on I_2 (see Remark 3.6), and that

$$\begin{aligned} \lim_{I_2 \rightarrow I_1} \Psi^{I_2 \rightarrow I_1} &= (12): U_{J_1, K_1} \rightarrow V_{I_1, K_1, J_1}(I_1), \\ (S(J, K), J, K) &\mapsto (S(J, K), K, J), \end{aligned}$$

the latter mapping is trivially defined on the whole U_{J_1, K_1} , so that the sizes of the domains $V_{I_1, K_1, J_1}(I_2)$ of $\Psi^{I_2 \rightarrow I_1}$'s are bounded away from zero, when I_2 is close to I_1 .

As before, we can further shrink $U_{e, G}$ (and hence U_{I_1}), if needed, so that properties (a), (b), (c) in Paragraph 3.5 hold independently of the point $I_2 \in U_{I_1}$. \square

3.8. Possibly shrinking $U_{e, G}$, we can and will assume that it is invariant under taking inverses, $g \mapsto g^{-1}$.

Lemma 3.9. *Possibly further shrinking $U_{e, G}$ and U_{J_1, K_1} , satisfying the conclusion of Lemma 3.8, we have for all $I_2 \in U_{I_1}$*

$$\Psi^{I_2 \rightarrow I_1} \circ \Psi^{I_1 \rightarrow I_2} = Id|_{U_{J_1, K_1}}.$$

Proof. For all $(f_1, f_2) \in \Phi_{I_1, J, K}^{-1}(I_2) \cap (U_{e, G} \times U_{e, G})$ and all $(S(J, K), J, K) \in U_{J_1, K_1}$, we want the neighborhoods $V_{e, f_1 f_2 K} = V_{e, G} \cap G_{f_1 f_2 K}$, $V_{e, f_1 f_2 J} = V_{e, G} \cap G_{f_1 f_2 J}$ to contain, respectively, the neighborhoods $f_1 f_2 U_{e, K} = f_1 f_2 U_{e, K} f_2^{-1} f_1^{-1}$ and $f_1 f_2 U_{e, J} = f_1 f_2 U_{e, J} f_2^{-1} f_1^{-1}$, so that, in particular, $V_{e, f_1 f_2 K} \times V_{e, f_1 f_2 J}$ contains the pair

$$(d_1, d_2) = (f_1 f_2 \cdot f_2^{-1} \cdot f_2^{-1} f_1^{-1}, f_1 f_2 \cdot f_1^{-1} \cdot f_2^{-1} f_1^{-1}) = (f_1 f_2^{-1} f_1^{-1}, f_1 f_2 \cdot f_1^{-1} \cdot f_2^{-1} f_1^{-1}).$$

Here the invariance of $U_{e,G}$ under taking inverses is used. The pair (d_1, d_2) certainly belongs to the preimage $\Phi_{I_2, f_1 f_2 K, f_1 f_2 J}^{-1}(I_1)$ as the product of its entries is $f_1 f_2^{-1} f_1^{-1} \cdot f_1 f_2 \cdot f_1^{-1} \cdot f_2^{-1} f_1^{-1} = f_2^{-1} f_1^{-1}$.

Note that, for $U_{e,G}$ small enough, the neighborhoods $f_1 f_2 U_{e,K} \times f_1 f_2 U_{e,J}$ will also be uniformly small for all $(f_1, f_2) \in \Phi_{I_1, J, K}^{-1}(I_2) \cap (U_{e,G} \times U_{e,G})$, so that the fiber of $\Phi_{I_2, f_1 f_2 K, f_1 f_2 J}^{-1}(I_1)$ in $f_1 f_2 U_{e,K} \times f_1 f_2 U_{e,J}$ consists of a unique connected component of the form described in Lemma 3.5. Then the pair (d_1, d_2) is contained in this ‘‘good’’ part of the fiber $\Phi_{I_2, f_1 f_2 K, f_1 f_2 J}^{-1}(I_1)$ and we can use (d_1, d_2) to evaluate $\Psi^{I_2 \rightarrow I_1}$ at $(S(f_1 f_2 J, f_1 f_2 K), f_1 f_2 J, f_1 f_2 K)$. Thus

$$\Phi_{I_2, f_1 f_2 K, f_1 f_2 J}(d_1, d_2) = I_1$$

and

$$d_1 d_2 f_1 f_2 J = J, \quad d_1 d_2 f_1 f_2 K = K,$$

so that

$$\Psi^{I_2 \rightarrow I_1}(S(f_1 f_2 J, f_1 f_2 K), f_1 f_2 K, f_1 f_2 J) = (S(J, K), J, K),$$

where, certainly, $S(J, K) = S(I_1, J, K)$, proving that the composition $\Psi^{I_2 \rightarrow I_1} \circ \Psi^{I_1 \rightarrow I_2}$ is the identity on U_{J_1, K_1} .

In order to ensure that $V_{e, f_1 f_2 K} \times V_{e, f_1 f_2 J}$ contains (d_1, d_2) , we assume, shrinking $U_{e,G}$ and U_{J_1, K_1} if necessary, but not changing $V_{e,G}$ and the previously fixed V_{I_1, K_1, J_1} , that for all $(S(J, K), J, K) \in U_{J_1, K_1}$ and for all points $(f_1, f_2) \in \Phi_{I_1, J, K}^{-1}(I_2) \cap (U_{e,G} \times U_{e,G})$, the neighborhoods $V_{e, f_1 f_2 K}, V_{e, f_1 f_2 J}$ contain, respectively, the neighborhoods $f_1 f_2 U_{e,K}$ and $f_1 f_2 U_{e,J}$. \square

Corollary 3.10. *Let U_{I_1} be defined by $U_{e,G}$ ($U_{e,G}$ satisfying Lemma 3.9). For arbitrary $I_2 \in U_{I_1}$, both joint points $J \in C_{I_1}$ and $f_1 f_2 K \in C_{I_2}$ of a triple of twistor spheres connecting I_1 and I_2 , can be chosen generic.*

Proof. Define

$$\begin{aligned} pr_K: \quad V_{I_1, K_1, J_1}(I_2) &\longrightarrow C_{I_2} \subset Compl, \\ (S(J, K), K, J) &\longmapsto K. \end{aligned}$$

This projection is a submersion onto its image. By Lemma 3.2, the locus \mathcal{L}_{NS} intersects C_{I_2} in a countable union of closed submanifolds of positive codimension in C_{I_2} . As the mapping pr_K is a submersion onto its image, the preimage $pr_K^{-1}(\mathcal{L}_{NS} \cap C_{I_2})$ is also a countable union of closed submanifolds of positive codimension in $V_{I_1, K_1, J_1}(I_2)$. Similarly, for

$$\begin{aligned} pr_J: \quad U_{J_1, K_1} &\longrightarrow C_{I_1} \subset Compl, \\ (S(J, K), J, K) &\longmapsto J, \end{aligned}$$

$pr_J^{-1}(\mathcal{L}_{NS} \cap C_{I_1}) \subset U_{J_1, K_1}$ is a countable union of closed submanifolds of positive codimension. The mapping $\Psi^{I_2 \rightarrow I_1}$ is real-analytic, so the closure of $\Psi^{I_2 \rightarrow I_1}(pr_J^{-1}(\mathcal{L}_{NS} \cap C_{I_1}))$ in U_{J_1, K_1} does not contain interior points. Therefore

$$(7) \quad U_{J_1, K_1} \neq pr_J^{-1}(\mathcal{L}_{NS} \cap C_{I_1}) \cup \Psi^{I_2 \rightarrow I_1}(pr_K^{-1}(\mathcal{L}_{NS} \cap C_{I_2})).$$

Since, by Lemma 3.9, $\Psi^{I_2 \rightarrow I_1} \circ \Psi^{I_1 \rightarrow I_2} = Id|_{U_{J_1, K_1}}$, the inequality (7) tells us that the image of the mapping $\Psi^{I_1 \rightarrow I_2}$ is not contained in $pr_K^{-1}(\mathcal{L}_{NS} \cap C_{I_2})$. Thus we may find a pair $(J, K) \in U_{J_1, K_1}$ such that $J = pr_J(S(J, K), J, K) \notin \mathcal{L}_{NS} \cap C_{I_1}$ and $f_1 f_2 K = pr_K(\Psi^{I_1 \rightarrow I_2}(S(J, K), J, K)) \notin \mathcal{L}_{NS} \cap C_{I_2}$, that is, both periods are generic. \square

4. THE DEGREE OF TWISTOR LINES

In this section we show that, as in the case of K3 surfaces, twistor lines in $Compl$ have degree 2 in the Plücker embedding. We first show that the group $G = GL^+(V_{\mathbb{R}})$ acts transitively on the set of twistor lines in $Compl$ and then compute the degree of an explicit twistor line.

Lemma 4.1. *The group $G = GL^+(V_{\mathbb{R}})$ acts transitively on the set of twistor lines in $Compl$.*

Proof. Given two twistor spheres $S_1 = S(I_1, J_1)$ and $S_2 = S(I_2, J_2)$, there is an element $g \in G$ sending I_1 to I_2 , hence sending S_1 to a twistor sphere through I_2 . The lemma now follows from Corollary 1.5. \square

4.1. To construct our example, consider the affine chart in the Grassmannian $G(2n, 4n)$ of normalized period matrices $(\mathbb{1}|Z)$, where $\mathbb{1}$ is, in general, the $2n \times 2n$ identity matrix and Z now denotes a non-degenerate $2n \times 2n$ complex matrix. Let us fix a basis of $V_{\mathbb{R}}$ and write the matrix of an arbitrary complex structure $I: V_{\mathbb{R}} \rightarrow V_{\mathbb{R}}$ in the following block form

$$I = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

for $2n \times 2n$ real matrices A, B, C, D . Then the relation

$$(\mathbb{1}|Z)I = (i\mathbb{1}|iZ),$$

gives the matrix equations

$$A + ZC = i\mathbb{1}, \quad B + ZD = iZ.$$

Assume that C is invertible so that the first equation allows us to write $Z = (i\mathbb{1} - A)C^{-1}$. The condition that I is a complex structure will then guarantee that the second equation is automatically satisfied.

4.2. **The case $n = 1$.** Momentarily assume $n = 1$ and consider the twistor sphere $S = S(I, J)$ where I and J have the respective matrices

$$\begin{pmatrix} 0 & -1 & | & 0 & 0 \\ 1 & 0 & | & 0 & 0 \\ \hline 0 & 0 & | & 0 & -1 \\ 0 & 0 & | & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & | & -1 & 0 \\ 0 & 0 & | & 0 & 1 \\ \hline 1 & 0 & | & 0 & 0 \\ 0 & -1 & | & 0 & 0 \end{pmatrix}$$

and put $K = IJ$. So for $\lambda \in S$,

$$\lambda = aI + bJ + cK = \begin{pmatrix} 0 & -a & | & -b & -c \\ a & 0 & | & -c & b \\ \hline b & c & | & 0 & -a \\ c & -b & | & a & 0 \end{pmatrix}.$$

Assume additionally that $b^2 + c^2 \neq 0$, that is, $\lambda \in S \setminus \{\pm I\}$. Here

$$A = \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}, C = \begin{pmatrix} b & c \\ c & -b \end{pmatrix}, C^{-1} = \frac{1}{b^2 + c^2} \begin{pmatrix} b & c \\ c & -b \end{pmatrix}.$$

Then

$$Z = (i\mathbb{1} - A)C^{-1} = \frac{1}{b^2 + c^2} \begin{pmatrix} ac + ib & -ab + ic \\ -ab + ic & -ac - ib \end{pmatrix} = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix},$$

where Z clearly satisfies the equations

$$z_1 + z_4 = 0, \quad z_2 - z_3 = 0, \quad \det Z = z_1 z_4 - z_2 z_3 = 1.$$

4.3. Now, for a general n , we can construct a twistor line in the period domain of complex $2n$ -dimensional tori, which, in the affine chart of $G(2n, 4n)$ above, corresponds to the locus of matrices $(\mathbb{1}|Z)$ where Z is the bloc diagonal matrix with the same 2×4 -block

$$\left(\begin{array}{cc|cc} 1 & 0 & u & v \\ 0 & 1 & v & -u \end{array} \right), \quad u^2 + v^2 = -1,$$

on the diagonal.

4.4. The degree of the curve in the example is 2 in the Plücker embedding $G(2n, 4n) \hookrightarrow \mathbb{P}^{\binom{4n}{2n}-1}$. Indeed, the Plücker coordinates in the above affine chart are given by the maximal minors of the matrix $(\mathbb{1}|Z)$. The twistor line S in our example is contained in the plane $P(S)$ with parameters u, v . The coordinates of the restriction of the Plücker embedding are $0, 1, \pm u, \pm v, u^2 + v^2$, or products of such, with the highest degree coordinate equal to $(u^2 + v^2)^n$. The image of $P(S)$ by the Plücker embedding is therefore the intersection of the image of the Grassmannian with a linear space and has degree $2n$ in that linear space since its highest degree coordinate is $(u^2 + v^2)^n$. It is now immediate that the image of S is the intersection of the image of $P(S)$ with a linear space given by setting all the coordinates that are powers of $u^2 + v^2$ equal to 1 (and some of the other coordinates equal to each other). This shows that the image of S by the Plücker embedding is contained in a plane and is a conic is that plane.

Corollary 4.2. *Twistor lines have degree 2 in the Plücker embedding of Compl .*

Proof. Follows from Lemma 4.1 and Paragraph 4.4. □

5. APPENDIX

Let A be a 2-dimensional complex torus with period belonging to the twistor line S constructed in Paragraph 4.2.

For any $\Omega \in \text{Hom}(\wedge^2 \Gamma, \mathbb{Q})$ such that $S \cap \text{Compl}_\Omega$ is infinite, by Lemma 3.3, the whole twistor line S is contained in Compl_Ω . Below we determine all Ω such that $S \subset \text{Compl}_\Omega$: these are specified by the invariance conditions $\Omega(I \cdot, I \cdot) = \Omega(J \cdot, J \cdot) = \Omega(\cdot, \cdot)$ and form a 3-dimensional subspace in $\text{Hom}(\wedge^2 \Gamma, \mathbb{Q})$. The invariance conditions mean that the first Riemann bilinear relation is satisfied.

On the other hand, the second Riemann bilinear relation does not hold: these Ω determine $(1, 1)$ -classes in the cohomology of tori in this twistor line whose hermitian forms are always indefinite. Thus, none of the classes determined by these Ω is Kähler. For the formulation of the Riemann bilinear relations see, for example, [5, Ch. 2].

5.1. The first bilinear relation. Let Q be the matrix of the alternating form corresponding to an I, J -invariant cohomology class in $H^2(A, \mathbb{Q})$, written in the basis in which the matrices of I, J are as in the previous section. The I, J -invariance then translates into the commutation relations $QI = IQ$ and $QJ = JQ$. A general skew-symmetric such Q has the form

$$Q = \begin{pmatrix} 0 & -b & c & -d \\ b & 0 & d & c \\ -c & -d & 0 & b \\ d & -c & -b & 0 \end{pmatrix}, \quad b, c, d \in \mathbb{Q}.$$

Such Q , by definition, determines a rational class of Hodge type $(1, 1)$ for all tori with periods in $S(I, J)$, so that the first bilinear relation $\Omega Q^{-1} {}^t \Omega = 0$ is automatically guaranteed by the choice of Q .

5.2. The second bilinear relation $-i\Omega Q^{-1} {}^t \bar{\Omega} > 0$. For Q as above it is easy to find Q^{-1} . Indeed, note that

$$Q^2 = -(b^2 + c^2 + d^2) \mathbb{1}_{4 \times 4},$$

where $\mathbb{1}_{4 \times 4}$ is the 4×4 identity matrix, so that $Q^{-1} = -\frac{1}{(b^2 + c^2 + d^2)} Q$ and $-i\Omega Q^{-1} {}^t \bar{\Omega} > 0$ is equivalent to $i\Omega Q {}^t \bar{\Omega} > 0$. We have

$$\Omega Q = \begin{pmatrix} -uc + vd & -b - ud - vc & c - vb & -d + ub \\ b - vc - ud & -vd + uc & d + ub & c + vb \end{pmatrix},$$

so that $\Omega Q {}^t \bar{\Omega}$ is equal to

$$\begin{pmatrix} (\bar{u} - u)c + (v - \bar{v})d + (u\bar{v} - \bar{u}v)b & (\bar{u} - u)d + (\bar{v} - v)c - (1 + |u|^2 + |v|^2)b \\ (\bar{u} - u)d + (\bar{v} - v)c + (1 + |u|^2 + |v|^2)b & (u - \bar{u})c + (\bar{v} - v)d + (u\bar{v} - \bar{u}v)b \end{pmatrix}.$$

Setting $u = u_1 + iu_2$ and $v = v_1 + iv_2$, we compute

$$i\Omega Q {}^t \bar{\Omega} = \begin{pmatrix} 2(u_2c - v_2d) + 2(u_1v_2 - u_2v_1)b & 2(u_2d + v_2c) - i(1 + |u|^2 + |v|^2)b \\ 2(u_2d + v_2c) + i(1 + |u|^2 + |v|^2)b & -2(u_2c - v_2d) + 2(u_1v_2 - u_2v_1)b \end{pmatrix}.$$

Now the determinant $\det i\Omega Q {}^t \bar{\Omega}$ is

$$\begin{aligned} \det i\Omega Q {}^t \bar{\Omega} &= (4b^2(u_1v_2 - u_2v_1)^2 - 4(u_2c - v_2d)^2) - (b^2(1 + |u|^2 + |v|^2)^2 + 4(v_2c + u_2d)^2) \\ &= b^2(4(u_1v_2 - u_2v_1)^2 - (1 + |u|^2 + |v|^2)^2) - 4(u_2c - v_2d)^2 - 4(v_2c + u_2d)^2. \end{aligned}$$

Let us show that indeed $4(u_1v_2 - u_2v_1)^2 - (1 + |u|^2 + |v|^2)^2 \leq 0$ for all $b, c, d \in \mathbb{R}$ and $u, v \in \mathbb{C}$ such that $u^2 + v^2 = -1$. This would prove that $i\Omega Q {}^t \bar{\Omega} > 0$ never holds for the periods in our twistor line.

The complex equation $u^2 + v^2 = -1$ is equivalent to the two real equations $u_2^2 + v_2^2 = u_1^2 + v_1^2 + 1$ and $u_1u_2 + v_1v_2 = 0$. Introducing the vectors

$$X = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, Y = \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}.$$

these equalities can be written as $|Y|^2 = |X|^2 + 1$ and $X \perp Y$. The term $(u_1v_2 - u_2v_1)^2$ above is the square of the dot product of X and the result of rotation of Y by $\frac{\pi}{2}$, so

that $X \perp Y$ implies that $(u_1v_2 - u_2v_1)^2 = |X|^2|Y|^2$. Now the equality $|Y|^2 = |X|^2 + 1$ allows us to write $1 + |u|^2 + |v|^2 = |X|^2 + |Y|^2$ and we have

$$4|X|^2|Y|^2 - (|X|^2 + |Y|^2)^2 = -(|X|^2 - |Y|^2)^2 = -1 < 0.$$

So, finally we obtain $\det i\Omega Q {}^t\bar{\Omega} < 0$ and none of the Q above determines a Kähler class.

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