Title
FIBONACCI NUMBERS IN COIN TOSSING SEQUENCES

Permalink
https://escholarship.org/uc/item/9nx172g1

Journal
FIBONACCI QUARTERLY, 16(6)

ISSN
0015-0517

Authors
FINKELSTEIN, M
WHITLEY, R

Publication Date
1978

License
https://creativecommons.org/licenses/by/4.0/ 4.0

Peer reviewed
The Fibonacci numbers and their generating function appear in a natural way in the problem of computing the expected number [2] of tosses of a fair coin until two consecutive heads appear. The problem of finding the expected number of tosses of a $p$-coin until $k$ consecutive heads appear leads to classical generalizations of the Fibonacci numbers.

First consider tossing a fair coin and waiting for two consecutive heads. Let $O_n$ be the set of all sequences of $H$ and $T$ of length $n$ which terminate in $HH$ and have no other occurrence of two consecutive heads. Let $S_n$ be the number of sequences in $O_n$. Any sequence in $O_n$ either begins with $T$, followed by a sequence in $O_{n-1}$, or begins with $HT$ followed by a sequence in $O_{n-2}$. Thus,

$$S_n = S_{n-1} + S_{n-2}, \quad S_1 = 0, \quad S_2 = 1.$$  

Consequently, $S_{n-2} = F_n$, the $n$th Fibonacci number. The probability of termination in $n$ trials is $S_n/2^n$. Letting

$$g(x) = \sum_{n=0}^\infty S_n x^n,$$

and using the generating function $(1 - x - x^2)^{-1}$ for the Fibonacci numbers, yields $g(x) = x^2/(1 - x - x^2)$. Hence, the expected number of trials is

$$\sum_{n=1}^\infty n S_n/2^n = (1/2)g'(1/2) = 6.$$  

We generalize this result to the following

**Theorem:** Consider tossing a $p$-coin, $Pr(H) = p$, repeatedly until $k$ consecutive heads appear. If $P_n$ is the probability of terminating in exactly $n$ trials (tosses), then the generating function

$$G(x) = \sum_{n=0}^\infty P_n x^n$$

is given by $G(x) = (px)^k(1 - px)/[1 - x + (1 - p)(px)^{k+1}]$.

The expected number of trials, $G'(1)$, is

$$1/p + 1/p^2 + \cdots + 1/p^k = \frac{1}{1 - p} \left[ \frac{1}{p^k} - 1 \right].$$

**Proof:** Let $O_n$ be the set of all sequences of $H$ and $T$ of length $n$ which terminate in $k$ heads and have no other occurrence of $k$ consecutive heads. Let $S_n$ be the number of sequences in $O_n$ and $P_n = Pr(O_n)$ be the probability of the event $O_n$. One possibility is that a sequence in $O_n$ begins with a $T$, followed by a sequence in $O_{n-1}$; the probability of this is
FIBONACCI NUMBERS IN COIN TOSsing SEQUENCES

The next possibility to consider is that a sequence in \( O_n \) begins with \( HT \), followed by a sequence in \( O_{n-2} \); this has probability

\[
Pr(HT)Pr(O_{n-2}) = qP_{n-2}.
\]

Continuing in this way, the last possibility to be considered is that a sequence in \( O_n \) begins with \( k-1 \) 's followed by a \( T \) and then by a sequence in \( O_{n-k} \), the probability of which is \( qP^{k-1}P_{n-k} \). Hence, the recursion:

\[
(4) \quad P_n = qP_n + qP_{n-2} + \cdots + qP^{k-1}P_{n-k},
\]

\[
P_1 = P_2 = \cdots = P_{k-1} = 0, \quad P_k = p^k.
\]

(\text{Note that the probability of achieving } k \text{ heads with } k \text{ tosses is } p^k, \text{ while with less than } k \text{ tosses it is impossible.})\) The technique to find the generating function for the Fibonacci numbers applies to finding

\[
G(x) = \sum_{k} P_n x^n.
\]

Consider

\[
H(x) = \sum_{n=k} P_n x^n;
\]

then

\[
x^n H(x) = \sum_{n=k} P_n x^{n+1} = \sum_{k} P_n x^n - P_k x^k = G(x) - (px)^k.
\]

Hence,

\[
H(x) = [G(x) - (px)^k]/x.
\]

On the other hand,

\[
H(x) = \sum_{k} P_n x^n = \sum_{k} (qP_n + qP_{n-1} + \cdots + qP^{k-1}P_{n-k+1})x^n
\]

\[
= q \sum_{k} P_n x^n + qpx \sum_{k} P_{n-1} x^{n-1} + \cdots + q(px)^{k-1} \sum_{k} P_{n-k+1} x^{n-k+1},
\]

and recalling that \( P_j = 0 \) for \( j < k \),

\[
= q \sum_{k} P_n x^n + qpx \sum_{k} P_n x^n + \cdots + q(px)^{k-1} \sum_{k} P_n x^n
\]

\[
= qG[1 + px + \cdots + (px)^{k-1}] = qG\left[\frac{1 - (px)^k}{1 - px}\right].
\]

Solving for \( G \) yields (2).

In the case \( p = 1/2 \), the combinatorial numbers \( S_n = 2^n P_n \) satisfy the recursion \( S_n = S_{n-1} + S_{n-2} + \cdots + S_{n-k} \). For these numbers, the generating function \((1 - x - x^2 - \cdots - x^k)^{-1}\) was found by V. Schlegel in 1894. See [1, Chap. XVII] for this and other classical references.

An alternate solution to the problem can be obtained as follows. Consider a sequence of experiments: Toss a \( p \)-coin \( X_1 \) times, until a sequence of \( k-1 \) heads occurs. Then toss the \( p \)-coin once more and if it comes up heads, set \( Y = 1 \). If not, toss the \( p \)-coin \( X_2 \) times until a sequence of \( k-1 \) heads occurs again, and then toss the \( p \)-coin once more and if it comes up heads, set \( Y = 2 \). If not, continue on in this fashion until finally the value of \( Y \) is set. At this time, we have observed a sequence of \( k \) heads in a row for the first time, and we have tossed the coin \( Y + X_1 + X_2 + \cdots + X_{Y} \) times. The \( X_i \) are independent, identically distributed random variables and \( Y \) is independent
of all of the \(X_i\). Let \(E_k\) denote the expected number of tosses to observe \(k\) heads in a row. Let \(Z = X_1 + \cdots + X_p\). Then,

\[
E_k = E(Y + Z) = E(Y) + E(Z)
\]

\[
= E(Y) + E(Z|Y = 1)Pr(Y = 1) + E(Z|Y = 2)Pr(Y = 2) + \cdots
\]

\[
= E(Y) + \sum_{n=1}^{\infty} E(Z|Y = n)Pr(Y = n) = E(Y) + \sum_{n=1}^{\infty} nE(X_1)Pr(Y = n)
\]

\[
= E(Y) + E(X_1)E(Y).
\]

But \(E(Y)\) is the expected number of tosses to observe a head = \(1/p\), and \(E(X_1) = E_{k-1}\). Thus \(E_k = 1/p + (1/p)E_{k-1}\), which yields (3).

**REFERENCE**


*****

**STRONG DIVISIBILITY SEQUENCES WITH NONZERO INITIAL TERM**

CLARK KIMBERLING

*University of Evansville, Evansville, IN 47702*

In 1936, Marshall Hall [1] introduced the notion of a \(k\)th order linear divisibility sequence as a sequence of rational integers \(u_0, u_1, \ldots, u_n, \ldots\) satisfying a linear recurrence relation

\[(1) \quad u_{n+k} = a_1u_{n+k-1} + \cdots + a_ku_n,\]

where \(a_1, a_2, \ldots, a_k\) are rational integers and \(u_m|u_n\) whenever \(m|n\). Some divisibility sequences satisfy a stronger divisibility property, expressible in terms of greatest common divisors as follows:

\[(u_m, u_n) = u_{(m,n)}\]

for all positive integers \(m\) and \(n\). We call such a sequence a **strong divisibility sequence**. An example is the Fibonacci sequence \(0, 1, 1, 2, 3, 5, 8, \ldots\).

It is well known that for any positive integer \(m\), a linear recurrence sequence \(\{u_n\}\) is periodic modulo \(m\). That is, there exists a positive integer \(M\) depending on \(m\) and \(a_1, a_2, \ldots, a_k\) such that

\[(2) \quad u_{n+M} \equiv u_n \pmod{m}\]

for all \(n \geq n_0[m, a_1, a_2, \ldots, a_k]\); in particular, \(n_0 = 0\) if \((a_k, m) = 1\).

Hall [1] proved that a linear divisibility sequence \(\{u_n\}\) with \(u_0 \neq 0\) is degenerate in the sense that the totality of primes dividing the terms of \(\{u_n\}\) is finite. One should expect a stronger conclusion for a linear strong divisibility sequence having \(u_0 \neq 0\). The purpose of this note is to prove that such a sequence must be, in the strictest sense, periodic. That is, there must exist a positive integer \(M\) depending on \(a_1, a_2, \ldots, a_k\) such that

\[u_{n+M} = u_n, \quad n = 0, 1, \ldots.\]