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FIBONACCI NUMBERS IN COIN TOSSING SEQUENCES

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The Fibonacci numbers and their generating function appear in a natural way in the problem of computing the expected number [2] of tosses of a fair coin until two consecutive heads appear. The problem of finding the expected number of tosses of a p-coin until k consecutive heads appear leads to classical generalizations of the Fibonacci numbers.

First consider tossing a fair coin and waiting for two consecutive heads. Let O_n be the set of all sequences of H and T of length n which terminate in HH and have no other occurrence of two consecutive heads. Let S_n be the number of sequences in O_n. Any sequence in O_n either begins with T, followed by a sequence in O_{n-1}, or begins with HT followed by a sequence in O_{n-2}. Thus,

$$S_n = S_{n-1} + S_{n-2},\; S_1 = 0,\; S_2 = 1. \tag{1}$$

Consequently, $S_{n-2} = F_n$, the nth Fibonacci number. The probability of termination in n trials is $S_n/2^n$. Letting

$$g(x) = \sum_2^\infty S_n x^n,$$

and using the generating function $(1 - x - x^2)^{-1}$ for the Fibonacci numbers, yields $g(x) = x^2/(1 - x - x^2)$. Hence, the expected number of trials is

$$\sum_{n=1}^\infty nS_n/2^n = (1/2)g'(1/2) = 6.$$

We generalize this result to the following

Theorem: Consider tossing a p-coin, $Pr(H) = p$, repeatedly until k consecutive heads appear. If P_n is the probability of terminating in exactly n trials (tosses), then the generating function

$$G(x) = \sum_k^\infty P_n x^n \text{ is given by } G(x) = \frac{(px)^k(1 - px)}{1 - x + \frac{(1-p)}{p}(px)^{k+1}} \tag{2}$$

The expected number of trials, $G'(1)$ is

$$1/p + 1/p^2 + \cdots + 1/p^k = \frac{1}{1-p}\left[\frac{1}{p^k} - 1\right]. \tag{3}$$

Proof: Let O_n be the set of all sequences of H and T of length n which terminate in k heads and have no other occurrence of k consecutive heads. Let S_n be the number of sequences in O_n and $P_n = Pr(O_n)$ be the probability of the event O_n. One possibility is that a sequence in O_n begins wi h a T, followed by a sequence in O_{n-1}; the probability of this is

$$Pr(T)Pr(O_{n-1}) = qP_{n-1}, \; q = 1 - p.$$

The next possibility to consider is that a sequence in O_n begins with HT, followed by a sequence in O_{n-2}; this has probability

$$Pr(HT)Pr(O_{n-2}) = qpP_{n-2}.$$

Continuing in this way, the last possibility to be considered is that a sequence in O_n begins with $k - 1$ H's followed by a T and then by a sequence in O_{n-k}, the probability of which is $qp^{k-1}P_{n-k}$. Hence, the recursion:

$$(4) \qquad P_n = qP_{n-1} + qpP_{n-2} + \cdots + qp^{k-1}P_{n-k},$$
$$P_1 = P_2 = \cdots = P_{k-1} = 0, \; P_k = p^k.$$

(Note that the probability of achieving k heads with k tosses is p^k, while with less than k tosses it is impossible.) The technique to find the generating function for the Fibonacci numbers applies to finding

$$G(x) = \sum_k^\infty P_n x^n.$$

Consider

$$H(x) = \sum_{n=k}^\infty P_{n+1} x^n;$$

then

$$xH(x) = \sum_k^\infty P_{n+1} x^{n+1} = \sum_k^\infty P_n x^n - P_k x^k = G(x) - (px)^k.$$

Hence,

$$H(x) = [G(x) - (px)^k]/x.$$

On the other hand,

$$H(x) = \sum_k^\infty P_{n+1} x^n = \sum_k^\infty (qP_n + qpP_{n-1} + \cdots + qp^{k-1}P_{n-k+1})x^n$$
$$= q\sum_k P_n x^n + qpx\sum_k P_{n-1}x^{n-1} + \cdots + q(px)^{k-1}\sum_k P_{n-k+1}x^{n-k+1},$$

and recalling that $P_j = 0$ for $j < k$,

$$= q\sum_k P_n x^n + qpx\sum_k P_n x^n + \cdots + q(px)^{k-1}\sum_k P_n x^n$$
$$= qG[1 + px + \cdots + (px)^{k-1}] = qG\left[\frac{1 - (px)^k}{1 - px}\right].$$

Solving for G yields (2).

In the case $p = 1/2$, the combinatorial numbers $S_n = 2^n P_n$ satisfy the recursion $S_n = S_{n-1} + S_{n-2} + \cdots + S_{n-k}$. For these numbers, the generating function $(1 - x - x^2 - \cdots - x^k)^{-1}$ was found by V. Schlegel in 1894. See [1, Chap. XVII] for this and other classical references.

An alternate solution to the problem can be obtained as follows. Consider a sequence of experiments: Toss a p-coin X_1 times, until a sequence of $k - 1$ heads occurs. Then toss the p-coin once more and if it comes up heads, set $Y = 1$. If not, toss the p-coin X_2 times until a sequence of $k - 1$ heads occurs again, and then toss the p-coin once more and if it comes up heads, set $Y = 2$. If not, continue on in this fashion until finally the value of Y is set. At this time, we have observed a sequence of k heads in a row for the first time, and we have tossed the coin $Y + X_1 + X_2 + \cdots + X_Y$ times. The X_i are independent, identically distributed random variables and Y is independent of all of the X_i. Let E_k = the expected number of tosses to observe k heads in a row. Let $Z = X_1 + \cdots + X_Y$. Then,

$$\begin{aligned} E_k &= E(Y + Z) = E(Y) + E(Z) \\ &= E(Y) + E(Z|Y = 1)Pr(Y = 1) + E(Z|Y = 2)Pr(Y = 2) + \cdots \\ &= E(Y) + \sum_{n=1}^{\infty} E(Z|Y = n)Pr(Y = n) = E(Y) + \sum_{n=1}^{\infty} nE(X_1)Pr(Y = n) \\ &= E(Y) + E(X_1)E(Y). \end{aligned}$$

But $E(Y)$ = the expected number of tosses to observe a head $= 1/p$, and $E(X_1) = E_{k-1}$. Thus $E_k = 1/p + (1/p)E_{k-1}$, which yields (3).

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STRONG DIVISIBILITY SEQUENCES WITH NONZERO INITIAL TERM

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In 1936, Marshall Hall [1] introduced the notion of a kth *order linear divisibility sequence* as a sequence of rational integers $u_0, u_1, \ldots, u_n, \ldots$ satisfying a linear recurrence relation

$$u_{n+k} = a_1 u_{n+k-1} + \cdots + a_k u_n, \tag{1}$$

where $a_1, a_2, \ldots, a_k$ are rational integers and $u_m | u_n$ whenever $m|n$. Some divisibility sequences satisfy a stronger divisibility property, expressible in terms of greatest common divisors as follows:

$$(u_m, u_n) = u_{(m,n)}$$

for all positive integers m and n. We call such a sequence a *strong divisibility sequence*. An example is the Fibonacci sequence 0, 1, 1, 2, 3, 5, 8, ...

It is well known that for any positive integer m, a linear recurrence sequence $\{u_n\}$ is periodic modulo m. That is, there exists a positive integer M depending on m and $a_1, a_2, \ldots, a_k$ such that

$$u_{n+M} \equiv u_n \pmod{m} \tag{2}$$

for all $n \geq n_0[m, a_1, a_2, \ldots, a_k]$; in particular, $n_0 = 0$ if $(a_k, m) = 1$.

Hall [1] proved that a linear divisibility sequence $\{u_n\}$ with $u_0 \neq 0$ is *degenerate* in the sense that the totality of primes dividing the terms of $\{u_n\}$ is finite. One should expect a stronger conclusion for a linear strong divisibility sequence having $u_0 \neq 0$. The purpose of this note is to prove that such a sequence must be, in the strictest sense, periodic. That is, there must exist a positive integer M depending on $a_1, a_2, \ldots, a_k$ such that

$$u_{n+M} = u_n, \qquad n = 0, 1, \ldots .$$