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ABSTRACT

We study the stability, in terms of minimal total potential energy, of liquid configurations in axisymmetric containers, such that the liquid-vapor interface is annular and meets the container walls at zero contact angle. The proper limits of sufficient and necessary conditions for stability, respectively, as the contact angle tends to zero, are formulated in terms of the Jacobi accessory differential equations. The stability is shown to depend crucially on whether the equilibrium liquid-vapor interface stays inside the container or not when continued analytically past the three-phase contact lines.

1. INTRODUCTION

We shall study in this paper the stability of certain configurations of liquid partially filling an axially symmetric tank in a gravitational field directed along the axis of symmetry. We require, that the tank shape and the liquid volume are such that the liquid-vapor interface is annular, i.e. it does not intersect the axis of symmetry, cf. figure 1:

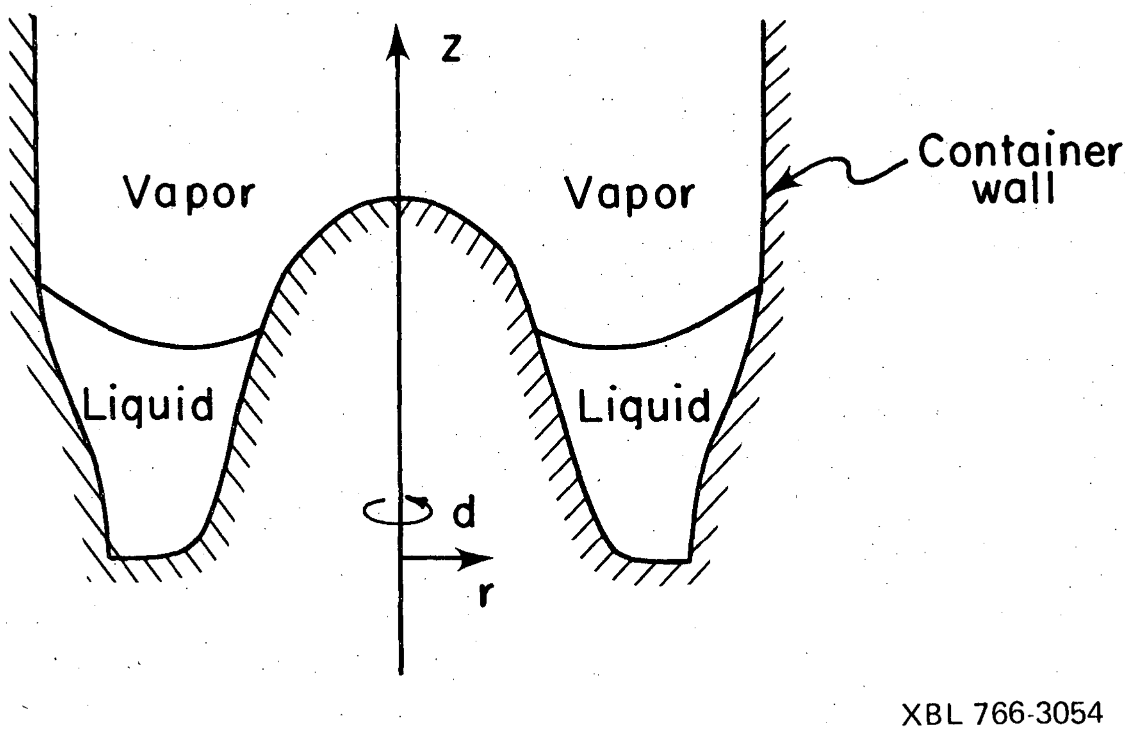

Figure 1: Example of permissible liquid-tank configuration and associated coordinate system.

A configuration is in stable equilibrium if and only if it strictly minimizes the total static potential energy of the system,

$$E = \sigma \cdot (A_f - \cos\gamma \cdot A_w) + E_g \tag{1.1}$$

among all nearby configurations with the same liquid volume V. Here $\sigma > 0$ (the liquid-vapor surface tension) and $0 \leq \gamma \leq \pi$ (the contact angle between the liquid-vapor surface and the container wall) are constants, A_f and A_w are the areas of the liquid-vapor and the liquid-wall interfaces, respectively, and E_g is the gravitational potential energy of the liquid. This constrained minimization problem has received much attention in the literature, see e.g. Huh [5] and Gillette [4] for extensive lists of references. By a suitable choice of variables, it may be viewed as a variable-endpoint problem of variational calculus ([4] p. 21 and p. 145). When $\gamma > 0$, this approach results in conditions which distinguish between stable and unstable cases in a rather satisfactory way. There appear to be fewer rigorous results, however, concerning to what extent these stability conditions also apply to the limiting case $\gamma = 0$ (cf. [4], p. 23). The purpose of this paper is to analyze this limiting case for axially symmetric liquid configurations of the above kind. More specifically, we shall look at necessary and sufficient conditions, respectively, for minimum of E based on the Jacobi accessory minimization problem for the second variation of E (see e.g. Akhiezer [1], p. 69), as $\gamma \to 0$. The formal limits, as $\gamma \to 0$, of the boundary conditions associated with the Jacobi accessory differential equations depend crucially on whether the curvatures of the equilibrium liquid-vapor interface and the container wall coincide or not at the three-phase contact lines. In the latter case these limiting boundary conditions will be of the fixed end-point type (when using a parametric representation of the surfaces, see further Section 2.1 below).

We will show, firstly (Theorems 3.1 and 3.2) that the stability conditions (sufficient and necessary, respectively) based on the fixed end-point boundary conditions in fact apply to (1.1) with $\gamma = 0$ if only the analytic continuation of the equilibrium liquid-vapor interface does not penetrate the container walls at the three-phase contact lines. Secondly (Theorem 3.3), we show that if the analytic continuation of the equilibrium liquid-vapor interface does penetrate the wall at either of the contact lines, the configuration will be unstable regardless of the conditions on the second variation of E.

2. NOTATION AND SOME PRELIMINARY RESULTS

2.1 The Euler-Lagrange and Jacobi Conditions

It will suffice to consider (1.1) at small perturbations from axially symmetric configurations. We will use a parametric arc-length, normal displacement representation of the surfaces (see e.g. Reynolds, Saad, Satterlee [8]). Thus the unperturbed liquid-vapor interface is described by

$$\begin{cases} r = R(s) \\ z = Z(s) \end{cases} ; \quad \begin{matrix} s_o \le s \le s_1 \\ 0 \le \phi \le 2\pi \end{matrix} \tag{2.1}$$

in the polar co-ordinate system of figure 1, where s is the arc-length along the curve of intersection between the interface and any plane ϕ = constant. Then the equations

$$\begin{cases} r = R(s) - \eta(s,\phi)Z'(s) \\ z = Z(s) + \eta(s,\phi)R'(s) \end{cases} ; \quad \begin{matrix} s_o(\phi) \le s \le s_1(\phi) \\ 0 \le \phi \le 2\pi \end{matrix} \tag{2.2}$$

describe a surface obtained by moving each point of the surface (2.1) the distance $\eta(s,\phi)$ in the direction of the normal at (s,ϕ). (In general, since we want the perturbed surface (2.2) to intersect the container walls, the functions R and Z of (2.1) must be continued to some open interval containing $[s_o,s_1]$. A convenient way of doing this, which we will use in the sequel, is provided by the differential equations (2.9) below). Similarly, in some neighborhood of the unperturbed contact lines ($s = s_o$ and $s = s_1$ in (2.1)), the container wall will be given by

$$\begin{cases} r = R(s) - w(s)Z'(s) \\ \\ z = Z(s) + w(s)R'(s) \end{cases} \quad ; \quad 0 \leq \phi \leq 2\pi. \tag{2.3}$$

Then, denoting by $\delta E(\eta)$ and $\delta V(\eta)$ the increments of the energy (1.1) and the liquid volume at the perturbation (2.2), we obtain in a straightforward way

$$\delta E(\eta) = \int_0^{2\pi} \int_{s_o(\phi)}^{s_1(\phi)} \left\{ f_A(\underline{\eta},s) - f_A(\underline{0},s) + f_g(\underline{\eta},s) \right\} ds d\phi$$

$$- \int_0^{2\pi} \int_{\Delta\phi} \left\{ \cos\gamma \cdot f_A(\underline{w},s) - f_A(\underline{0},s) + f_g(\underline{w},s) \right\} ds d\phi \quad , \tag{2.4}$$

$$\delta V(\eta) = \int_0^{2\pi} \int_{s_o(\phi)}^{s_1(\phi)} f_V(\underline{\eta},s) ds d\phi - \int_0^{2\pi} \int_{\Delta\phi} f_V(\underline{w},s) ds d\phi \quad , \tag{2.5}$$

where we have put

$$\begin{aligned} \underline{\eta} &= \underline{\eta}(s,\phi) = (\eta(s,\phi), \eta_s(s,\phi), \eta_\phi(s,\phi))^T \\ \underline{w} &= \underline{w}(s) = (w(s), w'(s), 0)^T \\ \Delta_\phi &= \text{the interval } (s_o(\phi), s_o) \cup (s_1, s_1(\phi)) \end{aligned} \tag{2.6}$$

and, denoting $R = R(s)$, $Z = Z(s)$,

$$f_A(\underline{\eta},s) = \sigma \cdot \{(R - \eta Z')^2(\eta_s^2 + (1 + \eta(R''Z' - Z''R'))^2)$$

$$+ \eta_\phi^2(1 + \eta(R''Z' - Z''R'))^2\}^{1/2} \quad ,$$

$$f_V(\underline{\eta},s) = \eta\{1 + \frac{\eta}{2}(R''Z' - Z''R')\}(R - \frac{\eta}{2}Z') \quad ,$$

$$f_g(\underline{\eta},s) = \rho g \cdot (Z + \frac{\eta}{2}R') \cdot f_V(\underline{\eta},s) \ . \tag{2.7}$$

Here ρ is the constant liquid density and g is the gravitation constant with $g > 0$ if the gravitation force acts towards the negative z-axis in figure 1.

The condition, that all first order η-terms in $\delta E(\eta)$ should vanish for all η such that $\delta V(\eta) = 0$ then leads to

$$\left.\begin{array}{l} \frac{\partial f_A}{\partial \eta}(\underline{0},s) + \frac{\partial f_g}{\partial \eta}(\underline{0},s) - \lambda\frac{\partial f_V}{\partial \eta}(\underline{0},s) = 0 \\ \text{in } s_o \leq s \leq s_1, \text{ with the boundary conditions} \\ \cos\gamma\ f_A(\underline{w}(s_i),s_i) - f_A(\underline{0},s_i) = 0 \quad ; \quad i=0,1 \ , \end{array}\right\} \tag{2.8}$$

where λ is a constant (the Lagrange multiplier). Putting $B_o = \rho g/\sigma$, $H_o = \lambda/\sigma$, using (2.6), (2.7) and the identity $R'(s)^2 + Z'(s)^2 \equiv 1$,(2.8) becomes the Euler-Lagrange boundary value problem

$$\left.\begin{array}{l} R'' = -Z'(B_oZ - H_o - Z'/R) \\ Z'' = R'(B_oZ - H_o - Z'/R) \\ \\ \left.\begin{array}{l} w(s_i) = 0 \\ w'(s_i) = (-1)^{i+1}\tan\gamma \end{array}\right. \quad ; \quad i=0,1. \end{array}\right\} \tag{2.9}$$

(Without loss of generality we have excluded the contact angles $\pi - \gamma$ allowed by (2.8)).

Assuming now that (2.9) is satisfied, the condition, that all second order η-terms should give a non-negative contribution to $\delta E(\eta)$ for all η such that $\delta V(\eta) = 0$, takes the form

$$Q_o(\underline{\mu}) = \int_0^{2\pi}\int_{s_o}^{s_1} \left\{A(s)\mu_s^2 + B(s)\mu_\phi^2 + C(s)\mu^2\right\} ds\,d\phi + \int_0^{2\pi} \left\{\alpha_o\mu(s_o,\phi)^2 + \alpha_1\mu(s_1,\phi)^2\right\} d\phi \geq 0 \qquad (2.10)$$

for all $\mu(s,\phi)$ such that

$$\int_0^{2\pi}\int_{s_o}^{s_1} R(s)\mu(s,\phi)ds\,d\phi = 0\ . \qquad (2.11)$$

Here we have put

$$f_{A\underline{\eta\eta}}(\underline{0},s) + f_{g\underline{\eta\eta}}(\underline{0},s) - f_{V\underline{\eta\eta}}(\underline{0},s) = \text{diag}\left\{C(s),A(s),B(s)\right\}\ , \qquad (2.12)$$

$$\frac{(-1)^i}{\tan^2\gamma}\cdot\frac{d}{ds}\left\{\cos\gamma\ f_A(\underline{w}(s),s) - f_A(\underline{0},s) + f_g(\underline{w}(s),s) - \lambda f_V(\underline{w}(s),s)\right\}_{s=s_i} = \alpha_i\ , \quad i=0,1. \qquad (2.13)$$

By (2.12) and (2.7), the A, B and C of (2.10) are

$$\begin{aligned} A(s) &= \sigma R(s)\ , \\ B(s) &= \sigma/R(s)\ , \\ C(s) &= -2\sigma R'' + \rho g\left\{RR' - ZZ' + ZR(R''Z' - Z''R')\right\} \\ &\quad + \lambda\left\{Z' - R(R''Z' - Z''R')\right\}\ . \end{aligned} \qquad (2.14)$$

By (2.9), since $R(s) \geq R_{min} > 0$ in $s_o \leq s \leq s_1$, A, B and C will be smooth (in fact analytic) in some open interval containing $[s_o, s_1]$ and $A(s) \geq A_{min} > 0$, $B(s) \geq B_{min} > 0$ will hold there. By standard results for symmetric, semibounded quadratic forms in Hilbert space (see e.g. Kato [7], p. 322 and 352-353), (2.10) may be analyzed in terms of the eigenvalues and eigenfunctions of an associated selfadjoint differential operator:

$$T\mu = -\frac{\partial}{\partial s}(A(s)\mu_s) - \frac{\partial}{\partial \phi}(B(s)\mu_\phi) + C(s)\mu$$

in $s_o \leq s \leq s_1$, $0 \leq \phi \leq 2\pi$, with the boundary conditions, that μ should be periodic in ϕ with period 2π and

$$A(s_i)\mu_s(s_i,\phi) = (-1)^i \alpha_i \mu(s_i,\phi) \quad , \quad 0 \leq \phi \leq 2\pi \qquad i = 0,1 .$$

T has a complete, orthogonal system of eigenfunctions of the form

$$\{\mu_{ik}(s) \cos k\phi\}_{i=1,k=0}^{\infty} \quad ; \quad \{\mu_{ik}(s) \sin k\phi\}_{i=1,k=1}^{\infty}$$

with associated eigenvalues $\{\kappa_{ik}\}_{i=1,k=0}^{\infty}$ (ordered increasingly in the index i), determined from the boundary value problems

$$\begin{cases} -\frac{d}{ds}(A(s)\mu_{ik}(s)) + \{k^2 \cdot B(s) + C(s)\}\mu_{ik}(s) = \kappa_{ik}\mu_{ik}(s) \\ A(s_j)\mu'_{ik}(s_j) = (-1)^j \alpha_j \mu_{ik}(s_j) \quad ; \quad j=0,1 \; ; \; i=1,2,3,\ldots \; ; \; k=0,1,2,\ldots . \end{cases} \tag{2.15}$$

We notice, that all eigenfunctions but those with $k = 0$ satisfy the constraint (2.11) and that, since $B(s) > 0$ in $s_o \leq s \leq s_1$, the eigenvalues κ_{ik} are increasing functions of k. It then follows that (2.10) with the side-condition (2.11) holds for all μ in the class of continuous functions in $s_o \leq s \leq s_1$, $0 \leq \phi \leq 2\pi$, which are periodic in ϕ with period 2π and have square integrable first derivatives (see e.g. Kato [7], p. 322-323, Cor. 2.3) if and only if

$$\kappa_o = \min\{\beta_1^2\kappa_{10} + \beta_2^2\kappa_{20}, \kappa_{11}\} \geq 0 \tag{2.16}$$

where, denoting $(f,g)_o = \int_0^{2\pi}\int_{s_o}^{s_1} f(s,\phi)g(s,\phi)ds d\phi$, β_1^2 and β_2^2 are the solutions to

$$\begin{cases} \beta_1^2(\mu_{10},R)_o^2 - \beta_2^2(\mu_{20},R)_o^2 = 0 \\ \beta_1^2(\mu_{10},\mu_{10})_o + \beta_2^2(\mu_{20},\mu_{20})_o = 1 \end{cases} \tag{2.17}$$

with $\beta_2 = 0$ if the solutions are non-unique.

(2.15) - (2.17) are(the equivalent of)the Jacobi accessory boundary value problems for our constrained minimization problem. By (2.13), (2.6) and (2.7), the boundary conditions are

$$(-1)^i \tan\gamma\, \mu'_{ik}(s_j) = \{\sin^2\gamma\,(R''Z' - Z''R') - \cos^2\gamma\, w''\}_{s=s_j} \cdot \mu_{ik}(s_j) \qquad j=0,1. \tag{2.18}$$

We notice, that <u>if $w''(s_j) \neq 0$, j=0,1, (2.18) converges formally to</u>

$$\mu_{ik}(s_j) = 0 \quad , \quad j=0,1 \quad , \tag{2.19}$$

as $\gamma \to 0$.

2.2 Permissible Perturbations. Two Lemmas.

By the contact lines of the liquid-vapor interface we mean the two closed curves within the container wall, any open neighborhoods of which intersects the interiors of both the liquid and the vapor inside the container (cf. fig. 1). The contact lines determine by (2.2) a closed region $s_o(\phi) \leq s \leq s_1(\phi)$, $0 \leq \phi \leq 2\pi$ in the (s,ϕ)-plane. We denote this region with Σ and require the following regularity properites from Σ and the associated function η:

a) η is continuous in Σ and periodic in ϕ with period 2π.

b) η_s and η_ϕ are continuous in Σ except possibly at finitely many isolated points or finitely many piecewise smooth curves with finite length. In particular, η_s is piecewise continuous as function of s for all $0 \leq \phi \leq 2\pi$.

c) $$d(\eta) = \sup|\eta(s,\phi)| + \sup|\eta_s(s,\phi)| + \sup|\eta_\phi(s,\phi)| < \infty, \tag{2.20}$$

where the supremum is taken over all points of Σ where η_s and η_ϕ are continuous.

d) $s_i(\phi)$, i=0,1 are continuous and such that $s_i(\phi) - s_i$, i=0,1, change sign at most finitely many times in $0 \leq \phi \leq 2\pi$. $s_i'(\phi)$, i=0,1, are continuous except possibly at finitely many points in $0 \leq \phi \leq 2\pi$ and

$$d(\Sigma) = \sum_{i=0}^{1} (\sup|s_i(\phi) - s_i| + \sup|s_i'(\phi)|) < \infty \qquad (2.21)$$

where the supremum is taken over all points of $0 \leq \phi \leq 2\pi$ where $s_o'(\phi)$ and $s_1'(\phi)$ are continuous.

Remark: The sufficient conditions to be considered below will ensure the stability of the surface (2.1) with respect to all perturbations (2.2) which satisfy a) - d) above and for which $d(\eta) + d(\Sigma)$ is sufficiently small. Thus in terms of variational calculus (see e.g. Bolza [2], p. 68-70) the extremum will be "weak". The detailed assumptions under b) and d) are introduced for simplicity in what follows, and could be relaxed slightly by introducing more advanced concepts from the theory of Lebesque integrals. With regard to the physical background, however, nothing essential is lost by the above.

We will denote the closed rectangle $s_o \leq s \leq s_1$, $0 \leq \phi \leq 2\pi$ by Σ_o and use, for any $f(s,\phi)$ which is square integrable on Σ_o,

$$\|f\|_o^2 = (f,f)_o \qquad (2.22)$$

where $(\ , \ )_o$ is defined as in (2.17). Then the following result will be useful:

Lemma 2.1: *Let $\mu(s,\phi)$ satisfy the requirements a)-c) on Σ_o and the condition*

$$\mu(s_o,\phi) = \mu(s_1,\phi) = 0 \quad , \quad 0 \leq \phi \leq 2\pi. \tag{2.23}$$

Let $Q_o(\underline{\mu})$ be defined as in (2.10) and assume that $A_{max} \geq A(s) \geq A_{min} > 0$, $B_{max} \geq B(s) \geq B_{min} > 0$ and $|C(s)| \leq C_{max}$ hold in $s_o \leq s \leq s_1$. Then there exist positive constants K_o, K_1, L_o and L_1, depending on s_o, s_1, A_{min}, A_{max}, B_{min}, B_{max} and C_{max} but not on μ, such that

$$K_oQ_o(\underline{\mu}) - L_o\|\mu\|_o^2 \leq \|\mu_s\|_o^2 + \|\mu_\phi\|_o^2 \leq K_1Q_o(\underline{\mu}) + L_1\|\mu\|_o^2 . \tag{2.24}$$

(2.24) follows from the mean value theorem in a straightforward way and the proof is omitted. (By use of a Sobolev-type inequality (see e.g. Kato [7], p. 193), condition (2.23) could in fact be omitted, and this stronger result could be used for a similar treatment of the case $\gamma > 0$, Karasalo [6]).

Before stating our second Lemma we need some further notation. For any $\mu(s,\phi)$ satisfying a) - c) on Σ_o and (2.23) we put, for clarity

$$Q_o(A,B,C,\underline{\mu}) = Q_o(\underline{\mu}) \tag{2.25}$$

where $Q_o(\underline{\mu})$ is defined in (2.10). Further, if $\delta A(s,\phi)$, $\delta B(s,\phi)$ and $\delta C(s,\phi)$ are bounded and integrable on Σ_o and ε is a positive constant, we put

$$\Phi = \Phi(\delta A,\delta B,\delta C,\varepsilon) = \inf Q_o(A + \delta A,B + \delta B,C + \delta B,\underline{\mu}) \tag{2.26}$$

over all μ satisfying a) - c) on Σ_o, (2.23) and the conditions

$$\|\mu\|_o = 1 \ , \tag{2.27}$$

$$|(R,\mu)_o| \leq \varepsilon(\|\mu_s\|_o + \|\mu_\phi\|_o + \|\mu\|_o). \tag{2.28}$$

If $f(s,\phi)$ is bounded on Σ_o, we will let, as usual, $\|f\|_\infty$ denote the supremum of $|f(s,\phi)|$ over (s,ϕ) in Σ_o. Then we have

Lemma 2.2: *Let $A(s)$, $B(s)$ and $C(s)$ satisfy the requirements in Lemma 2.1 and let $\delta A(s,\phi)$, $\delta B(s,\phi)$ and $\delta C(s,\phi)$ be bounded and integrable on Σ_o. Let $\varepsilon > 0$, define Φ as in (2.25) - (2.28), denote $\Phi_o = \Phi(0,0,0,0)$ and put*

$$\delta = \|\delta A\|_\infty + \|\delta B\|_\infty + \|\delta C\|_\infty + \varepsilon \; . \tag{2.29}$$

Then there exist positive constants C and δ_o, independent of δA, δB, δC and ε, such that

$$|\Phi - \Phi_o| \leq C \cdot \delta \tag{2.30}$$

holds true, if only $\delta \leq \delta_o$.

Proof: Throughout this proof, M_i, N_i, δ_i, $i=1,2,3,\ldots$ will denote positive constants, independent of δA, δB, δC, ε and μ. Let μ satisfy conditions a) - c) on Σ_o, (2.23) and (2.27). With the notation of (2.25), put for brevity, $\delta Q_o(\underline{\mu}) = Q_o(\delta A,\delta B,\delta C,\underline{\mu})$. ($\delta Q_o$ will be well defined because of the assumptions.) Then there exist M_1, M_2, N_1 and N_2, such that

$$\begin{aligned}(1 - M_1\delta)Q_o(\underline{\mu}) - N_1\delta &\leq Q_o(\underline{\mu}) + \delta Q_o(\underline{\mu}) \\ &\leq (1 + M_1\delta)Q_o(\underline{\mu}) + N_1\delta \quad ,\end{aligned} \tag{2.31}$$

because of Lemma 2.1 and the mean value theorem. Noting by (2.26) and (2.28), that Φ is a non-increasing function of ε and that, by (2.10) and (2.25), $Q_o(A,B,C,\underline{\mu})$ is a linear function of A, B and C we obtain from (2.31)

$$\Phi \leq (1 + M_1\delta)\Phi_o + N_1\delta \ . \tag{2.32}$$

It follows, that we need only consider those μ which satisfy, e.g., the additional condition $Q_o(\underline{\mu}) + \delta Q_o(\underline{\mu}) \leq (1 + 2M_1\delta)|\Phi_o| + 2N_1\delta$ when forming the infimum in (2.26). By (2.31) and Lemma 2.1, however, for all such μ

$$\|\mu_s\|_o^2 + \|\mu_\phi\|_o^2 \leq M_2|\Phi_o| + N_2 = M_3 \tag{2.33}$$

if only, e.g., $\delta \leq \delta_1 = \frac{1}{2M_1}$. Let μ be any function satisfying a) - c) on Σ_o, (2.23), (2.27), (2.28) and (2.33). Put

$$f^*(s,\phi) = (s - s_o)(s_1 - s) \quad ,$$

$$\tilde{\mu} = C_1(\mu - C_2 f^*)$$

where C_1 and C_2 are chosen so as to make $\tilde{\mu}$ satisfy (2.27) and (2.28) with $\varepsilon = 0$, i.e. $C_2 = (R,\mu)_o/(R,f^*)_o$, $C_1 = 1/\|\mu - C_2 f^*\|_o$. Noting that $(R,f^*)_o$ is a positive constant, dependent only on s_o, s_1 and R(s), it follows from (2.27) - (2.29) and (2.33) that

$$|C_2| \leq M_4\delta \quad ; \quad \frac{1}{C_1} \geq 1 - M_5\delta \ . \tag{2.34}$$

Furthermore, since f^* vanishes on $s = s_o$ and $s = s_1$, $\tilde{\mu}$ satisfies (2.23).

Then form

$$Q_o(\tilde{\underline{\mu}}) - \{Q_o(\underline{\mu}) + \delta Q_o(\underline{\mu})\}$$

$$= (C_1^2 - 1)Q_o(\underline{\mu}) + C_1^2C_2^2Q_o(\underline{f}^*) - \delta Q_o(\underline{\mu})$$

$$- 2C_1^2C_2\int_0^{2\pi}\int_{s_o}^{s_1}\{A(s)\mu_s f_s^* + C(s)\mu f^*\}\, ds d\phi \quad .$$

Here we use Lemma 2.1, (2.31), (2.33), (2.34), the mean value theorem and the Schwartz inequality to find upper bounds for the terms to the right. We obtain, that for some M_6, δ_2

$$Q_o(\tilde{\underline{\mu}}) - \{Q_o(\underline{\mu}) + \delta Q_o(\underline{\mu})\} \leq M_6 \cdot \delta \qquad (2.35)$$

holds true, if only $\delta \leq \delta_2$. Hence

$$\Phi_o - \Phi \leq M_6 \cdot \delta \qquad (2.36)$$

if $\delta \leq \delta_2$. The statement of the lemma follows by combining (2.32) and (2.36).

3. STABILITY RESULTS AT $\gamma = 0$.

Our first statement concerns sufficient conditions for stability at zero contact angle:

Theorem 3.1: *Let $\gamma = 0$ in (1.1) and assume that the unperturbed surface satisfies the Euler-Lagrange equations with the associated boundary conditions (2.9), and does not intersect the z-axis. Assume further that the function $w(s)$ of (2.3) is twice continuously differentiable and that*

$$w(s) \leq 0 \text{ in some open neighborhoods of } s = s_o \text{ and } s = s_1 . \quad (3.1)$$

Let κ_o be defined as in (2.15) - (2.17) but with the boundary conditions in (2.15) replaced by the fixed end-point conditions (2.19), and assume that $\kappa_o > 0$. Let $d(\eta)$ and $d(\Sigma)$ be defined as in (2.20) and (2.21). Then there exists a constant $d_o > 0$, such that in (2.4)

$$\delta E(\eta) \geq 0 \text{ with equality iff } \eta = 0 \text{ in } \Sigma \quad (3.2)$$

holds for all η satisfying the volume constraint $\delta V(\eta) = 0$ in (2.5), the conditions a) - d) of Section 2.2 and the condition

$$d(\eta) + d(\Sigma) \leq d_o . \quad (3.3)$$

Remark: When $\gamma = 0$, both the energy (1.1) and the liquid volume remain unchanged if the liquid-vapor interface is continued past the contact lines by "wetting" dry parts of the container walls. Thus, when $\gamma = 0$, any configuration is neutrally unstable with respect to such "wetting" perturbations. With our notation, however, the region Σ is unchanged at "wetting" (see the beginning of Section 2.2), and there is no ambiguity in (3.2) in this respect.

Proof of Theorem 3.1: Let Q_o be defined as in (2.10). Then, by (2.15) - (2.17) and the representation theorem for quadratic forms in Hilbert space (see e.g. Kato [7], p. 322-323):

$$\inf Q_o(\underline{\mu}) = \kappa_o > 0 \tag{3.4}$$

where the infimum is taken over all μ satisfying in Σ_o the conditions a) - c) of Section 2.2, (2.11), (2.23) and (2.27).

Using the notation of (2.7) we put

$$f_E(\underline{\eta},s) = f_A(\underline{\eta},s) - f_A(\underline{0},s) + f_g(\underline{\eta},s) - \lambda f_V(\underline{\eta},s) \tag{3.5}$$

where λ is the constant in (2.8). Then, by (2.4) and (2.5),

$$\delta E(\eta) = \int_0^{2\pi}\int_{s_o(\phi)}^{s_1(\phi)} f_E(\underline{\eta},s)\, ds d\phi$$

$$- \int_0^{2\pi}\int_{\Delta\phi} f_E(\underline{w},s)\, ds d\phi \tag{3.6}$$

for all η satisfying the volume constraint $\delta V(\eta) = 0$. For convenience in the following we denote

$$\Sigma_+ = \text{complement of } (\Sigma\cap\Sigma_o) \text{ w.r.t. } \Sigma,$$

$$\Sigma_- = \text{complement of } (\Sigma\cap\Sigma_o) \text{ w.r.t. } \Sigma_o \text{ and}$$

$$\tilde{\Sigma} = \Sigma\cup\Sigma_o = \Sigma_o\cup\Sigma_+ = \Sigma\cup\Sigma_- \tag{3.7}$$

and define a function $\tilde{\eta}(s,\phi)$ on $\tilde{\Sigma}$ by

$$\tilde{\eta}(s,\phi) = \begin{cases} \eta(s,\phi) & , \quad (s,\phi) \in \Sigma \\ w(s) & , \quad (s,\phi) \in \Sigma_- \end{cases} , \tag{3.8}$$

(i.e. $\tilde{\eta}$ is obtained by extending η by wetting those parts of the wall which dried because of the perturbation η). By (2.6) and (3.7), $\tilde{\Sigma}$ has the boundaries

$$\begin{cases} \tilde{s}_o(\phi) = \min\{s_o(\phi), s_o\} \\ \tilde{s}_1(\phi) = \max\{s_1(\phi), s_1\} \end{cases} ; \quad 0 \le \phi \le 2\pi . \tag{3.9}$$

Putting further

$$\eta^*(s,\phi) = \begin{cases} 0 & , \quad (s,\phi) \in \Sigma_o \\ w(s) & , \quad (s,\phi) \in \Sigma_+ \end{cases} , \tag{3.10}$$

$$\nu(s,\phi) = \tilde{\eta}(s,\phi) - \eta^*(s,\phi) \quad , \quad (s,\phi) \in \tilde{\Sigma} , \tag{3.11}$$

we obtain by (3.6) and (2.5), noting that $f_E(\underline{0},s) = 0$ and $f_V(\underline{0},s) = 0$,

$$\delta E(\eta) = \int_0^{2\pi}\int_{\tilde{s}_o(\phi)}^{\tilde{s}_1(\phi)} f_E(\underline{\tilde{\eta}},s)\,dsd\phi - \int_0^{2\pi}\int_{\tilde{\Delta}\phi} f_E(\underline{w},s)\,dsd\phi$$

$$= \int_0^{2\pi}\int_{\tilde{s}_o(\phi)}^{\tilde{s}_1(\phi)} \{f_E(\underline{\eta}^* + \underline{\nu},\ s) - f_E(\underline{\eta}^*,s)\}\,dsd\phi \quad , \tag{3.12}$$

for all η satisfying the volume constraint

$$\delta V(\eta) = \int_0^{2\pi}\int_{\tilde{s}_o(\phi)}^{\tilde{s}_1(\phi)} \{f_V(\underline{\eta}^* + \underline{\nu},s) - f_V(\underline{\eta}^*,s)\}\,dsd\phi = 0. \tag{3.13}$$

In the sequel M_i, N_i and d_i, i=1,2,3,..., will denote positive constants, independent of μ, η, Σ, s and ϕ. By (3.8) - (3.11), $\tilde{\eta}$, η^*, ν and $\tilde{s}_i(\phi)$ will satisfy the requirements a) - d) of Section 2.2. Furthermore, since $w(s) = 0(s - s_i)^2$ in the neighborhood of $s = s_i$, we may find some M_1 and d_1, such that

$$d(\tilde{\Sigma}) + d(\eta^*) + d(\tilde{\eta}) + d(\nu) \leq M_1\{d(\Sigma) + d(\eta)\} \tag{3.14}$$

if only $d(\Sigma) + d(\eta) \leq d_1$.

By (2.9) and since $R(s) > 0$ in $s_o \leq s \leq s_1$, the functions R and Z can be continued analytically to some open interval containing $[s_o,s_1]$ and it will hold $R(s) \geq R_{min} > 0$ there. It then follows from (2.7), (3.5) and (3.9) that there exist some d_2 and d_3, such that $f_E(\underline{\eta},s)$ and $f_V(\underline{\eta},s)$ are analytic functions of the arguments η, η_s, and η_ϕ in the region $|\eta| + |\eta_s| + |\eta_\phi| \leq d_2$ at all points of $\tilde{\Sigma}$, if only $d(\Sigma) \leq d_3$. Hence, putting in (3.12) and (3.13)

$$f_E(\underline{\eta}^* + \underline{\nu}, s) - f_E(\underline{\eta}^*, s)$$

$$= f_{E\underline{\eta}}(\underline{\eta}^*, s)^T \underline{\nu} + \frac{1}{2} \underline{\nu}^T f_{E\underline{\eta\eta}}(\underline{0}, s)\underline{\nu} + h_E(\underline{\eta}^*, \underline{\nu}, s) \tag{3.15}$$

and

$$f_V(\underline{\eta}^* + \underline{\nu}, s) - f_V(\underline{\eta}^*, s) = \frac{\partial f_V}{\partial \eta} (\underline{0}, s)\nu + h_V(\underline{\eta}^*, \underline{\nu}, s) \tag{3.16}$$

there will exist some M_2 and d_4, such that

$$|h_E(\underline{\eta}^*, \underline{\nu}, s)| \leq M_2\{d(\eta) + d(\Sigma)\}(\nu_s^2 + \nu_\phi^2 + \nu^2)$$

$$|h_V(\underline{\eta}^*, \underline{\nu}, s)| \leq M_2\{d(\eta) + d(\Sigma)\}(|\nu_s| + |\nu_\phi| + |\nu|) \tag{3.17}$$

holds at all points of $\tilde{\Sigma}$ where $\underline{\eta}^*$ and $\underline{\nu}$ are continuous, if only $d(\eta) + d(\Sigma) \leq d_4$.

We now observe, <u>that since $w(s) \leq 0$ in some open neighborhoods of $s = s_o$ and $s = s_1$, the first term to the right in (3.15) gives a non-negative contribution to $\delta E(\eta)$</u>. To see this, first note that $f_{E\underline{\eta}}(\underline{\eta}^*, s) = 0$ in Σ_o, by (3.10), (3.5), (2.7) and (2.8). Second, when $(s,\phi) \in \Sigma_+$ we have by (3.10), (2.12) and (2.14)

$$f_{E\underline{\eta}}(\underline{\eta}^*, s)^T \underline{\nu} = \{\sigma R(s) + D_R(s)\} w'(s)\nu_s + \{C(s) + D_C(s)\} w(s)\nu \tag{3.18}$$

where, for some M_3 and d_5,

$$|D_R(s)| + |D_C(s)| \leq M_3\{|w(s)| + w'(s)^2\}$$

if only $d(\Sigma) \leq d_5$. Furthermore, since $w''(s)$ is continuous in Σ_+, D_R will be continuously differentiable there. Hence, after partial integration of the first term in (3.18) noting that $\nu(s,\phi)w'(s) = 0$ on the boundaries of Σ_+, we obtain

$$\int_0^{2\pi}\int_{\tilde{s}_o(\phi)}^{\tilde{s}_1(\phi)} f_{E\underline{\eta}}(\underline{\eta}^*,s)^T \cdot \underline{\nu}\, ds d\phi$$

$$= \int_0^{2\pi}\int_{\tilde{\Delta}\phi} \left\{-\sigma R(s)w''(s) + D(s)\right\}\nu\, ds d\phi \tag{3.19}$$

where, for some d_6, M_4, $|D(s)| \leq M_4|w'(s)|$ if only $d(\Sigma) \leq d_6$. Now by (3.1), since $w(s_i) = w'(s_i) = 0$, $i=0,1$, $w''(s) \leq 0$ in some open neighborhoods of $s = s_o$ and $s = s_1$, whence the first factor of the last integrand will be non-negative, if only $d(\Sigma)$ is sufficiently small. Since further, by (3.8) - (3.11), $\nu(s,\phi) > 0$ at interior points of Σ_+, we can then find some d_7, such that for all permissible η for which $\delta V(\eta) = 0$

$$\delta E(\eta) \geq \frac{1}{2}\int_0^{2\pi}\int_{\tilde{s}_o(\phi)}^{\tilde{s}_1(\phi)} \left\{A(s)\nu_s^2 + B(s)\nu_\phi^2 + C(s)\nu^2 + 2h_E(\underline{\eta}^*,\underline{\nu},s)\right\} ds d\phi\ , \tag{3.20}$$

if only $d(\eta) + d(\Sigma) \leq d_7$. Furthermore, (3.20) holds with equality if Σ_+ is empty or if $w(s) \equiv 0$ in Σ_+.

In (3.13) and (3.20) we introduce a change of variable by putting

$$\begin{cases} s = s(s',\phi) = \tilde{s}_o(\phi) + \dfrac{s' - s_o}{s_1 - s_o}(\tilde{s}_1(\phi) - \tilde{s}_o(\phi)) & (3.21) \\ \mu(s',\phi) = \nu(s(s',\phi),\phi) \; . & (3.22) \end{cases}$$

(3.21) takes $\tilde{\Sigma}$ onto the rectangle Σ_o in the (s',ϕ)-plane. It follows by (3.17), (3.21), (3.22) and the smoothness properties of A, B and C that in the notation of (2.25)

$$\delta E(\eta) \geq \frac{1}{2} Q_o \, (A + \delta A, B + \delta B, C + \delta C, \underline{\mu})$$

where $\delta A(s',\phi)$, $\delta B(s',\phi)$ and $\delta C(s',\phi)$ are bounded and integrable on Σ_o and such that for some M_5 and d_8

$$\|\delta A\|_\infty + \|\delta B\|_\infty + \|\delta C\|_\infty \leq M_5 \{d(\eta) + d(\Sigma)\}$$

if $d(\eta) + d(\Sigma) \leq d_8$. Similarly, by (3.13), (3.16), (3.17) and (2.7), there exist M_6 and d_9, such that

$$|(R,\mu)_o| \leq M_6\{d(\eta) + d(\Sigma)\}\{\|\mu_s\|_o + \|\mu_\phi\|_o + \|\mu\|_o\}$$

if $d(\eta) + d(\Sigma) \leq d_9$. Furthermore, $\mu(s',\phi)$ satisfies (2.23) and requirements a) - c) of Section 2.2 on Σ_o. We may then use (3.4) and Lemma 2.2 to conclude that there exists some d_o such that, e.g.,

$$\delta E(\eta) \geq \frac{\kappa_o}{3} \|\mu\|_o^2$$

for all η satisfying the volume constraint, if only $d(\eta) + d(\Sigma) \leq d_o$. Since, by (3.7) - (3.11) and (3.21) - (3.22) $\|\mu\|_o = 0$ if and only if

$\eta \equiv 0$ in Σ, this completes the proof of Theorem 3.1.

The second statement of this section is concerned with necessary conditions for stability at zero contact angle, based on the fixed end-point conditions (2.19). As may be expected, these will apply regardless of the additional condition (3.1):

Theorem 3.2: *Let $\gamma = 0$ and assume that the unperturbed surface satisfies (2.9) and does not intersect the z-axis. Let κ_o be defined as in (2.15) - (2.17) but with the boundary conditions (2.19) and assume that $\kappa_o < 0$. Then, for any $d_o > 0$ we may find a function η satisfying a) - d) of Section 2.2, the volume constraint $\delta V(\eta) = 0$ in (2.5) and the condition $d(\eta) + d(\Sigma) \leq d_o$, such that in (2.4)*

$$\delta E(\eta) < 0 \ . \tag{3.23}$$

Proof: We note that the infimum κ_o in (3.4) under the conditions stated there is attained for $\mu = \hat{\mu}$ where $\hat{\mu}$ is either μ_{11} or $\beta_1\mu_{10} + \beta_2\mu_{20}$ in the notation of (2.15) - (2.17). We can use $\hat{\mu}$ to construct a function η with the properties required in the theorem as follows: Let e.g. $s_{00} = 2s_o/3 + s_1/3$, $s_{10} = s_o/3 + 2s_1/3$ and put

$$g(s) = \begin{cases} (s - s_{00})(s_{10} - s) \ , & s_{00} \leq s \leq s_{10} \\ 0 & \text{otherwise} \ . \end{cases} \tag{3.24}$$

Then put

$$\eta(s,\phi) = \varepsilon\hat{\mu}(s,\phi) + a(\varepsilon)g(s) \tag{3.25}$$

where $a(\varepsilon)$ is chosen so as to make $\eta(s,\phi)$ satisfy the volume constraint. Using (2.5), the assumption $(R,\hat{\mu})_o = 0$ and the contraction mapping theorem it can be shown that $a(\varepsilon)$ is well defined when $|\varepsilon|$ is small and that $a(\varepsilon) = o(\varepsilon)$, $\varepsilon \to 0$. Since $\hat{\mu}$ and g are zero on $s = s_o$ and $s = s_1$, Σ_+ as defined in (3.7) will be empty. Hence, by putting

$$\eta^*(s,\phi) = \begin{cases} 0 & ; \quad (s,\phi) \in \Sigma \\ w(s) - \varepsilon\hat{\mu}(s,\phi) & ; \quad (s,\phi) \in \Sigma_- \end{cases} \tag{3.26}$$

we obtain by (3.6)

$$\delta E(\eta) = \int_0^{2\pi}\int_{s_o}^{s_1} f_E(\varepsilon\underline{\hat{\mu}} + \underline{\eta}^* + a(\varepsilon)\underline{g},s)\, ds d\phi \tag{3.27}$$

if only ε is small enough for $g(s)$ to be zero within Σ_-, cf. (3.24), (3.25). Noting that $|\eta^*(s,\phi)| \leq |\varepsilon\hat{\mu}(s,\phi)|$ in Σ_- and that the area of Σ_- tends to zero as $\varepsilon \to 0$, we get from (3.27)

$$\delta E(\eta) = \varepsilon^2 \int_0^{2\pi}\int_{s_o}^{s_1} \frac{1}{2}\underline{\hat{\mu}}^T f_{E\underline{\eta}\underline{\eta}}(\underline{0},s)\underline{\hat{\mu}}\, ds d\phi + o(\varepsilon^2)$$

$$= \frac{\varepsilon^2}{2}(\kappa_o + o(1)) \quad , \quad \varepsilon \to 0.$$

Since $\kappa_o < 0$, the statement of Theorem 3.2 follows.

Finally, the following theorem states that if the analytic continuation of the equilibrium liquid-vapor interface penetrates the container walls at either of the three-phase contact lines, the configuration is unstable:

Theorem 3.3: *Let $\gamma = 0$ and assume that the unperturbed surface satisfies (2.9) and does not intersect the z-axis. Assume further, that for $i=0$ or $i=1$ the function $w(s)$ is twice continuously differentiable in some open neighborhood of $s = s_i$ and changes sign at $s = s_i$. Then, for any $d_o > 0$, we may find a function η satisfying a) - d) of Section 2.2, the volume constraint (2.5) and the condition $d(\eta) + d(\Sigma) \leq d_o$, such that (3.23) holds.*

Proof: We may assume, e.g. $i=1$. Then $w(s) > 0$ for $s > s_1$, and since $w(s_1) = w'(s_1) = 0$, $w''(s) > 0$ holds in some open interval $s_1 < s < s_2$. Hence, the integral (3.19) can be made negative by a suitable choice of $\tilde{\Delta}\phi$. Furthermore, the integral is linear in ν while by (3.12), (3.15) and (3.17) the other contributions to $\delta E(\eta)$ are of higher order in ν. The proof may then be completed by choosing some appropriate $\nu(s,\phi)$ and proceeding as in the proof of Theorem 3.2 to satisfy the volume constraint.

Remark: We note, that the instability stated in Theorem 3.3 holds regardless of the stability conditions based on the second variation. The method of proof suggests that the instability should show by liquid building up towards the container wall at the contact line $s = s_i$ in (3.28). This kind of liquid behavior has some support in experimental evidence [3].

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