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Affine structure of facially symmetric spaces

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In [**7**], the authors proposed the problem of giving a geometric characterization of those Banach spaces which admit an algebraic structure. Motivated by the geometry imposed by measuring processes on the set of observables of a quantum mechanical system, they introduced the category of *facially symmetric spaces*. A discrete spectral theorem for an arbitrary element in the dual of a *reflexive* facially symmetric space was obtained by using the basic notions of *orthogonality*, *projective unit*, *norm exposed face*, *symmetric face*, *generalized tripotent* and *generalized Peirce projection*, which were introduced and developed in this purely geometric setting.

In this paper, we investigate geometric properties of these spaces. We analyze the generalized Peirce decomposition associated with a face and give a useful condition for two such decompositions to commute. A polar decomposition is proved for an arbitrary element and a characterization is given of semi-exposed faces in these spaces and their duals.

The primary example of a facially symmetric space is a Banach space whose dual is a JB^*-triple. In particular, this includes the preduals of von Neumann algebras, the duals of C^*-algebras and JB^*-algebras (= Jordan C^*-algebras) as well as those of J^*-algebras. The latter includes Hilbert spaces and Cartan factors as special cases. For an introduction to JB^*-triples and JBW^*-triples, see [**5**, **6**, **10**].

Characterizations of the state spaces of C^*-algebras and of JB^*-algebras, based on physically significant axioms, are known and were constructed in a framework of ordered Banach spaces (cf. [**1**, **2**]). On the other hand, the predual of a JBW^*-triple enjoys analogues of essentially all the properties which are needed in these characterizations (cf. [**5**]). We expect that these properties, formulated in a facially symmetric space, will lead to an algebraic structure on its dual. Because of the lack of a global order structure however, this will be a triple product rather than a binary one (cf. the introduction in [**7**]). This will solve (and give precise meaning to) the problem stated in the first paragraph.

One approach to solving this problem is via a spectral theorem, functional calculus, and polarization. However, obtaining a satisfactory *continuous* spectral theorem seems to be a difficult task, requiring a new version of Choquet theory. Moreover, even though spectral theory and functional calculus can be used to define cubes, it is non-trivial to show that the triple product defined via polarization is additive in each argument, since this requires a treatment of non-compatible elements, discussed below.

Fortunately, there is another approach which we believe is tractable for our problem. It is based on the recent work of Dang and Friedman [3], which gives an elementary constructive proof of the classification of JBW^*-triple factors of type I (cf. also [9]). In the work of Dang and Friedman, a basis is constructed which consists of tripotents which are related in certain basic ways (orthogonal, collinear, or governing). All of these notions have meaning in a facially symmetric space.

In order to facilitate the carrying out of this approach, a penetrating analysis is needed of the facial structure of the unit ball of a facially symmetric space and its dual. This is the purpose of the present paper. The corresponding analysis in the predual of a JBW^*-triple appears in [5] and the recent work of Edwards and Rüttiman [4].

An important feature of a JBW^*-triple is the existence of a Peirce decomposition associated with a given tripotent (or equivalently, a norm exposed face) in which each summand is a JBW^*-subtriple. A similar decomposition occurs in the context of [7], and the stability of the category under generalized Peirce projections will be of equal importance in our study.

We now discuss the notion of compatibility. A useful general technique is to decompose a space simultaneously with respect to a family of individual decompositions. It is therefore important to know conditions under which this joint decomposition does not depend on the order, i.e. when the corresponding projections all commute. In this case we say that the members of the family are *compatible*. For a comprehensive study of this notion in the context of Jordan triple systems, see [8].

In the globally ordered case, faces correspond to idempotents, and compatibility implies commutativity of the idempotents. Therefore it is not possible to study non-commutative phenomena in this framework without abandoning compatibility, and thus increasing significantly the complexity of the problem. On the other hand, by dropping the requirement of a global order structure, it is possible to describe non-commutative objects by using compatible families. This method, which we intend to employ toward this end for facially symmetric spaces, was illustrated in the factor classification of JBW^*-triples in [3].

The contents of this paper are as follows. In §1 we begin by establishing properties of orthogonality in a normed space Z which depend only on the assumption that the orthogonal complement of any norm exposed face is a linear space. With this minimal assumption we are able to prove, for example, the existence part of a polar decomposition of an arbitrary element. Then, under the basic assumption that every norm exposed face is symmetric (we shall call such spaces *weakly facially symmetric*, WFS), we discuss the one-to-one correspondence between generalized tripotents and symmetric faces, and related matters.

In §2 we introduce the notion of neutrality and use it to prove the uniqueness of the symmetry occurring in the definition of symmetric face. We also explore the duality between the generalized Peirce spaces in Z and its dual Z^* and use it to show that the latter are spanned in the weak*-topology by certain generalized tripotents. This is used in §3 to prove the fundamental result which states that two generalized tripotents are compatible if one of them belongs to a generalized Peirce space of the other.

Also in §3 we study the behaviour of our spaces under generalized Peirce projections. We show that the properties neutrality and WFS are inherited by the range of any generalized Peirce projection. In the final §4, we restrict our attention to the subclass of (strongly) facially symmetric spaces. We study the order structure on the set of generalized tripotents and use it to prove the uniqueness, minimality, and faithfulness of the polar decomposition. We also study semi-exposed faces in Z and Z^*. We show that every norm semi-exposed face in Z_1 is exposed and give a complete description of weak*-semi-exposed faces in the dual ball.

This paper builds on its predecessor [7], but is otherwise self-contained, using only elementary functional analysis. Although the main result of [7] is not needed here, we do refer to most of the machinery used in its proof, up to and including lemma 2·8 of [7].

The theory in this paper is developed for real or complex spaces. The proof that the predual of a JBW^*-triple is a neutral facially symmetric space can be seen from proposition 8, proposition 1, lemma 1·6 and (1·9) of [5] and the Jordan decomposition for normal functionals on a JBW-algebra. It follows from this fact and the results of this paper that purely geometric proofs can be given for the commutation formulas (lemma 1·10 and proposition 3 in [5]) and for the polar decomposition of a normal functional (proposition 2 in [5]).

It seems doubtful whether the predual of a real JBW^*-triple will be a facially symmetric space.

1. *Symmetric faces and orthogonality*

In this section we review the fundamental concepts which were introduced in [7] and prove some new properties which are needed in the present paper.

Let Z be a real or complex normed space. Elements $f, g \in Z$ are *orthogonal*, notation $f \diamond g$, if

$$\|f+g\| = \|f-g\| = \|f\|+\|g\|. \tag{1}$$

This condition is equivalent to the existence of elements $u, v \in Z^*$ satisfying

$$\|u\| = \|v\| = 1 = \|u \pm v\|, \quad f(u) = \|f\|, \quad g(v) = \|g\|, \quad f(v) = g(u) = 0 \tag{2}$$

(cf. [7], proposition 1·1).

Recall that a *norm exposed face* of the unit ball Z_1 of Z is a non-empty set (necessarily $\neq Z_1$) of the form $F_x = \{f \in Z_1 : f(x) = 1\}$, where $x \in Z^*$, $\|x\| = 1$. Note that $F_x \cap F_y = F_z$, where $z = \frac{1}{2}(x+y)$.

For subsets S, T of Z, $S \diamond T$ means $f \diamond g$ for all $f \in S, g \in T$. From (2) it follows that

$$\|x\| = \|y\| = \|x \pm y\| = 1 \text{ implies } F_x \diamond F_y. \tag{3}$$

For a subset S of Z, $S^\diamond$ denotes $\{f \in Z : f \diamond g \text{ for all } g \in S\}$. It is easy to see that $S \diamond T$ if and only if $T \subset S^\diamond$. Moreover $S^\diamond = S^{\diamond\diamond\diamond}$ and $S^\diamond$ is closed, but in general $S \neq S^{\diamond\diamond}$ and $S^\diamond$ is not additive or complex homogeneous. However, under a mild assumption, $S^\diamond$ will always be a linear subspace. Namely, we say that a space is *facially linearly complemented* if it is a real or complex normed space in which the orthogonal complement $F^\diamond$ of every norm exposed face F (of the unit ball) is a linear subspace.

PROPOSITION 1·1. *Let Z be a facially linearly complemented space. Then $S^\diamond$ is a linear subspace for every subset S of Z. Moreover $\overline{\text{sp}}\, S$ is orthogonal to $S^\diamond$.*

Proof. Since $S^\diamond = \bigcap\{\{f\}^\diamond : f\in S\}$, it suffices to prove that $\{f\}^\diamond$ is a linear space. Let $g,h\in\{f\}^\diamond$ and assume without loss of generality that $\|f\| = \|g\| = \|h\| = 1$. By [7], corollary 1·3(*b*) there exist pairs of orthogonal norm exposed faces (F,G) and (H,K) with $f\in F\cap H$, $g\in G$, $h\in K$. Then $J:=F\cap H$ is a norm exposed face and $g,h\in J^\diamond$. By assumption, for any scalar α we have $g+\alpha h\in J^\diamond$, implying $g+\alpha h\in\{f\}^\diamond$.

Since $S^{\diamond\diamond}$ is a closed linear subspace containing S, it contains $\overline{\mathrm{sp}}\,S$, i.e. $(\overline{\mathrm{sp}}\,S)\diamond S^\diamond$. ∎

From this proposition, by using induction, we have

COROLLARY 1·2. *Let Z be a facially linearly complemented space. Then for any mutually orthogonal family $\psi_1,\ldots,\psi_n$ we have $\|\Sigma\,\psi_i\| = \Sigma\,\|\psi_i\|$.*

The following is a converse of Proposition 1·1, and does not require any hypothesis on norm exposed faces.

Remark 1·3. Suppose $g\diamond h$ and $(g+h)\diamond f$. Then $g\diamond f$ and $h\diamond f$.

Proof. We have

$$\|g\|+\|h\|+\|f\| = \|g+h\|+\|f\| = \|g+h\pm f\|$$
$$\leqslant \|g\pm f\|+\|h\| \leqslant \|g\|+\|f\|+\|h\|.$$

Therefore $\|g\pm f\| = \|g\|+\|f\|$, i.e. $g\diamond f$. By symmetry $h\diamond f$. ∎

An element $u\in Z^*$ is called a *projective unit* if $\|u\| = 1$ and $\langle u, F_u^\diamond\rangle = 0$. This means that the norm exposed face F_u is 'parallel' to $F_u^\diamond$.

In the following proposition, this notion is used to prove the existence of a 'polar decomposition' for elements of Z.

PROPOSITION 1·4. *For each non-zero f in a facially linearly complemented space Z, there is a projective unit u with $f(u) = \|f\|$ and $\langle u,\{f\}^\diamond\rangle = 0$.*

Proof. Assume $\|f\| = 1$. Define ϕ on $\mathrm{sp}\{f\}+\{f\}^\diamond$ by $\phi(\alpha f+g) = \alpha$ for $g\in\{f\}^\diamond$ and scalar α. Then ϕ is a linear functional. Since $f\diamond g$, by Proposition 1·1 we have $\alpha f\diamond g$ and

$$|\phi(\alpha f+g)| = |\alpha| \leqslant |\alpha|+\|g\| = \|\alpha f+g\|.$$

By the Hahn–Banach theorem, there exists $u\in Z^*$ with $\|u\| = 1$, $f(u) = 1$, and $\langle u,\{f\}^\diamond\rangle = 0$. Since $\{f\}\subset F_u$, we have $\{f\}^\diamond\supset F_u^\diamond$. Thus $\langle u,F_u^\diamond\rangle = 0$, i.e. u is a projective unit. ∎

We shall improve this result and prove the uniqueness of u under appropriate hypothesis in §4 (cf. Theorem 4·3).

Let $\mathscr{F}$ and $\mathscr{U}$ denote the collections of norm exposed faces of Z_1 and projective units in Z^*, respectively. The map $\mathscr{U}\ni u\mapsto F_u\in\mathscr{F}$ is not onto in general (cf. [7], example 4).

In order to get a bijection between distinguished subsets of projective units and norm exposed faces, we need to recall the definitions of symmetric face and generalized tripotent from [7].

Motivated by measuring processes in quantum mechanics, we define a *symmetric face* to be a norm exposed face F in Z_1 with the following property: there is a linear isometry S_F of Z onto Z, with $S_F^2 = I$, such that the fixed point set of S_F is $(\overline{\mathrm{sp}}\,F)\oplus F^\diamond$ (topological direct sum). In particular, $F^\diamond$ is a closed linear space. We shall show in §2 that under a mild assumption, S_F is unique. Hence the assumption of uniqueness made in [7] is not needed.

The rest of this paper will be concerned with the following spaces. In §4 we will restrict attention to a subclass of this class.

Definition 1. A real or complex normed space Z is said to be *weakly facially symmetric* (*WFS*) if every norm exposed face in ∂Z_1 is symmetric.

Since a WFS space is facially linearly complemented, the above results are valid for it.

For each symmetric face F we define contractive projections $P_k(F)$ for $k = 0, 1, 2$ on Z as follows. First $P_1(F) = \frac{1}{2}(I - S_F)$ is the projection on the -1 eigenspace of S_F. Next we define $P_2(F)$ and $P_0(F)$ as the projections of Z onto $\overline{\text{sp}}\, F$ and $F^\diamond$ respectively, so that $P_2(F) + P_0(F) = \frac{1}{2}(I + S_F)$. These projections are called *generalized Peirce projections*.

The projections $P_k(F)$ depend, *a priori*, on the choice of S_F. It is immediate from the definition that

$$P_2(F) + P_1(F) + P_0(F) = I$$

and

$$P_2(F) - P_1(F) + P_0(F) = S_F.$$

The following, concerning the generalized Peirce projections, is an immediate consequence of Proposition 1·1.

PROPOSITION 1·5. *Let Z be a WFS space. Then $P_2(F)\,Z \diamond P_0(F)\,Z$ for every norm exposed face F. In particular,*

$$\|P_2(F)\,\rho\| + \|P_0(F)\,\rho\| = \|(P_2(F) + P_0(F))\,\rho\| \quad (\rho \in Z). \tag{4}$$

A *generalized tripotent* is a projective unit $u \in \mathscr{U}$ with the property that $F := F_u$ is a symmetric face and $S_F^* u = u$ for some choice of symmetry S_F corresponding to F.

Denote by $\mathscr{GT}$ and $\mathscr{SF}$ the collections of generalized tripotents and symmetric faces respectively. By using Proposition 1·5 and the proof of [7], proposition 1·4, we now have

PROPOSITION 1·6. *Let Z be a WFS space. Then the map $\mathscr{GT} \ni u \mapsto F_u \in \mathscr{SF}$ is a bijection of the set of generalized tripotents and the set of symmetric faces.*

Propositions 1·5 and 1·6 were stated in [7] without the assumption that every norm exposed face is symmetric. The authors wish to thank Professor Kevin McCrimmon for pointing out that (4) does not follow directly from the definition of symmetric face.

For each generalized tripotent u in the dual of a WFS space Z, we shall denote the generalized Peirce projections by $P_k(u) = P_k(F_u)$, for $k = 0, 1, 2$. Also we let

$$U := Z^*, \quad Z_k(u) = Z_k(F_u) := P_k(u)\,Z \quad \text{and} \quad U_k(u) = U_k(F_u) := P_k(u)^*(U),$$

so that

$$Z = Z_2(u) + Z_1(u) + Z_0(u) \quad \text{and} \quad U = U_2(u) + U_1(u) + U_0(u).$$

The inverse of the map $u \mapsto F_u$ of Proposition 1·8 will be denoted by $F \mapsto v_F$. A symmetry corresponding to the symmetric face F_u will sometimes be denoted by S_u.

Example 1·7. Let $Z = \mathbb{R}^2$ with unit ball given by a regular hexagon with vertices $(\pm 1, 0), (\frac{1}{2}, \pm\frac{1}{2}\sqrt{3}), (-\frac{1}{2}, \pm\frac{1}{2}\sqrt{3})$. Then

(1) ∂Z_1 has 12 norm exposed faces, each of which is symmetric; thus Z is a WFS space;

(2) $\mathscr{U} = \partial U_1$ (since $S^\diamond = \{0\}$ for all $\varnothing \neq S \subset Z$);

(3) the generalized tripotents are $(\pm 1, 0)$, $(\frac{1}{2}, \pm\frac{1}{2}\sqrt{3})$, $(-\frac{1}{2}, \pm\frac{1}{2}\sqrt{3})$, $(1, \pm\frac{1}{3}\sqrt{3})$, $(0, \pm\frac{2}{3}\sqrt{3}), (-1, \pm\frac{1}{3}\sqrt{3})$;

(4) if $f = (1,0) \in Z$, then any $u \in \{(1,0), (1, \pm\frac{1}{3}\sqrt{3})\}$ satisfies $f(u) = \|f\|$ and $\langle u, \{f\}^\diamond \rangle = 0$.

Properties (1), (2), (3) show that there exist projective units u which are not generalized tripotents but for which F_u is a symmetric face. Hence the condition $S_u^* u = u$ in the definition of generalized tripotent does not follow from the other part of the definition. Property (4) shows the non-uniqueness of a projective unit occurring in the polar decomposition of Proposition 1·4 (uniqueness will be shown in §4 under additional assumption).

In the rest of this paper we shall need the basic orthogonality properties of generalized tripotents and other elements of the dual space Z^* of a WFS space Z, as developed in [7], §2. These results were stated in [7], for convenience, under an additional assumption. An examination of their proofs shows that lemma 2·1, lemma 2·4 and lemma 2·5 of [7] are valid in WFS spaces, and therefore they can be used here in §2 and §3.

The following lemma will be used in §4 to study the properties of a partial order on the set of generalized tripotents.

LEMMA 1·8. *Let Z be a WFS space. For $v \in \mathscr{GT}$ and $b \in U_0(v)$ with $\|b\| = 1$ and $F_b \neq \varnothing$, we have*

$$(F_{v+b})^\diamond = F_v^\diamond \cap F_b^\diamond. \tag{5}$$

In particular, if $u, v \in \mathscr{GT}$ are orthogonal, then $Z_0(u+v) = Z_0(u) \cap Z_0(v)$.

Proof. By [7], lemma 2·1, we have $F_b \subset F_{b+v}$. Therefore $(F_{v+b})^\diamond \subset F_v^\diamond \cap F_b^\diamond$.

For any $\phi \in F_{b+v}$, we have

$$\begin{aligned} 1 = \langle \phi, b+v \rangle &= \langle \phi, (P_2(v)^* + P_0(v)^*)(b+v) \rangle \\ &= \langle P_2(v)\phi, v \rangle + \langle P_0(v)\phi, b \rangle \\ &\leqslant \|P_2(v)\phi\| + \|P_0(v)\phi\| \leqslant \|\phi\| = 1, \end{aligned}$$

which implies that

$$\|P_2(v)\phi\|^{-1} P_2(v)\phi \in F_v \quad \text{if} \quad P_2(v)\phi \neq 0,$$

and

$$\|P_0(v)\phi\|^{-1} P_0(v)\phi \in F_b \quad \text{if} \quad P_0(v)\phi \neq 0.$$

Let $\psi \in F_v^\diamond \cap F_b^\diamond$. Then $\psi \diamond P_2(v)\phi$ and $\psi \diamond P_0(v)\phi$, so by Corollary 1·2 we have

$$1 + \|\psi\| \geqslant \|\phi \pm \psi\| \geqslant \|(P_2(v) + P_0(v))(\phi \pm \psi)\| = \|P_2(v)\phi\| + \|P_0(v)\phi\| + \|\psi\| = 1 + \|\psi\|,$$

implying $\psi \diamond \phi$, and (5). ∎

2. *Neutrality and uniqueness of symmetries*

In this section we introduce the notion of neutrality and use it to prove that the generalized Peirce projections are uniquely determined by the symmetric face F which defines them.

We shall also characterize the generalized tripotents which belong to $U_k(F)$ for $k \in \{0, 1, 2\}$, and show that they span this space in the weak*-topology.

A contractive projection Q on a normed space X is said to be *neutral* if for each $\xi \in X$, $\|Q\xi\| = \|\xi\|$ implies $Q\xi = \xi$. A normed space Z is *neutral* if for every symmetric face F, the projection $P_2(F)$ corresponding to some choice of symmetry S_F, is neutral.

If Z is the predual of a JBW^*-triple then Z is neutral (see [**5**], proposition 1). Moreover, in a JBW^*-triple, $P_0(F)$ is always neutral and $P_1(F)$ is not neutral in general. This situation prevails in general according to the following result.

LEMMA 2·1. *Let Z be a neutral WFS space. Then $P_0(F)$ is neutral for every symmetric face F.*

Proof. Let $g \in Z$ and $\|P_0(F)g\| = \|g\| = 1$ for some symmetric face F. There exist $x \in Z^*$ with $\|x\| = 1$ and with $\langle P_0(F)g, x\rangle = 1$. We may therefore assume that $x \in U_0(F)$. Let w and v be the unique elements in $\mathcal{GT}$ corresponding to the symmetric faces F_x and F respectively. By lemma 2·1 of [**7**]

$$x \diamond v \Rightarrow F_x \diamond F_v \Rightarrow F_w = F_x \subset F_v^{\diamond} \Rightarrow Z_2(w) \subset Z_0(v) \Rightarrow P_0(v)\, P_2(w) = P_2(w).$$

Also, since $g \in F_x = F_w$, we have $\|P_2(w)\, g\| = \|g\|$, and by neutrality we have $g = P_2(w)\, g = P_0(v)\, P_2(w)\, g = P_0(v)\, g$. ∎

The following lemma will be used to prove the uniqueness of a symmetry associated with a symmetric face. It will also play an important role in §3 for proving the compatibility of two generalized tripotents u, v with $u \in U_1(v)$. It is based on a simple relation between neutrality of a contractive projection Q and a unique Hahn–Banach extension property of the range of Q^*.

LEMMA 2·2. *Let Q be a neutral contractive projection on a normed space X. Then Q^* is the unique weak*-continuous contractive projection on X^* with range $Q^*(X^*)$.*

Proof. We show first that $V := Q^*(X^*)$ has the following unique Hahn–Banach extension property: if $\phi_1, \phi_2 \in X \subset X^{**}$ and $\phi_1|_V = \phi_2|_V = \phi$ say, with $\|\phi_1\| = \|\phi_2\| = \|\phi\|_{V^*}$, then $\phi_1 = \phi_2$.

To see this, note that $\| Q\phi_i \|_X = \| \phi \|_{V^*} = \|\phi_i\|_X$. Hence, by neutrality, $\phi_i \in Q(X)$. Thus for all $a \in X^*$, we have $\langle a, \phi_i\rangle = \langle Q^*a, \phi_i\rangle = \langle Q^*a, \phi\rangle$, and $\phi_1 = \phi_2$.

Now let $R : X^* \to X^*$ be a weak*-continuous contractive projection with range V and let $\psi \in X$. Then $\psi|_V \in V^*$ and, with $\psi_1 = \psi \circ R$, $\psi_2 = \psi \circ Q^*$, we have $\psi_i \in X$, $\|\psi_i\| = \|\psi|_V\|$ and $\psi_i|_V = \psi|_V$. By the previous paragraph, $\psi_1 = \psi_2$ so that $R = Q^*$. ∎

For an arbitrary projection Q on X, the kernel of Q^* is determined by the range of Q in the sense that $R(X) = Q(X)$ implies $\ker R^* = \ker Q^*$. The following theorem shows that the range of Q^* is also determined by the range of Q in the case that $Q = P_2(F)$ or $Q = P_0(F)$. This fact will be used in the next section to prove compatibility of generalized tripotents.

THEOREM 2·3. *Let Z be a neutral WFS space, and let F be any norm exposed face. For $k = 0$ or 2, any $x \in U_k(F)$ with $\|x\| = 1$ satisfies $F_x \subset Z_k(F)$; conversely, if x is a generalized tripotent with $F_x \subset Z_k(F)$, then $x \in U_k(F)$. Moreover, all such generalized tripotents generate $U_k(F)$ in the following sense:*

$$U_k(F) = \overline{\mathrm{sp}}^{w^*}\{v_G : G \in \mathcal{SF}, G \subset Z_k(F)\} \quad (k = 0, 2).$$

Proof. Throughout let $k = 0$ or 2.

If $x \in U_k(F)$ and $\|x\| = 1$, then

$$\rho \in F_x \quad \text{implies} \quad \langle P_k(F)\rho, x\rangle = \langle \rho, P_k(F)^* x\rangle = \langle \rho, x\rangle = 1.$$

Hence, by neutrality of $P_k(F)$, we have $\rho \in Z_k(F)$ and $F_x \subset Z_k(F)$.

If $G \in \mathcal{SF}$ and $G \subset Z_k(F)$, then $S_F(G) = G$. Hence, by [7], lemma 2·4, $S_F^* v_G = v_G$, i.e. $v_G \in U_2(F) + U_0(F)$.

Suppose $G \subset Z_0(F) = F^\diamond$. Then $F^{\diamond\diamond} \subset G^\diamond$. By Proposition 1·5, $(\overline{\mathrm{sp}}\, F) \diamond F^\diamond$, which implies $\overline{\mathrm{sp}}\, F \subset F^{\diamond\diamond} \subset G^\diamond$. Since $Z_2(F) = \overline{\mathrm{sp}}\, F$ and $\langle v_G, G^\diamond\rangle = 0$, we have for arbitrary $\sigma \in Z$

$$\langle P_2(F)^* v_G, \sigma\rangle = \langle v_G, P_2(F)\sigma\rangle = 0.$$

Thus $P_2(F)^* v_G = 0$ and $v_G \in U_0(F)$.

Now let $G \subset Z_2(F)$. Then once more by Proposition 1·5 we have $G \diamond F^\diamond$, implying $F^\diamond \subset G^\diamond$. For arbitrary $\sigma \in Z$ we have

$$\langle P_0(F)^* v_G, \sigma\rangle = \langle v_G, P_0(F)\sigma\rangle = 0.$$

Thus $P_0(F)^* v_G = 0$ and $v_G \in U_2(F)$.

We have shown that $\overline{\mathrm{sp}}^{w^*}\{v_G : G \in \mathcal{SF}, G \subset Z_k(F)\} \subset U_k(F)$. To show equality, suppose there exists $\phi \in Z$ with $\|\phi\| = 1$ and $\phi(v_G) = 0$ for all norm exposed faces $G \subset Z_k(F)$, and suppose that $\phi(U_k(F)) \neq \{0\}$. We may assume that $\phi \in Z_k(F)$ and that there is an $x \in U_k(F)$ with $\|x\| = 1$ and with $\phi \in F_x$. From above, $F_x \subset Z_k(F)$ and thus $\phi(v_H) = 0$ where $H = F_x$. This contradicts the fact that $\phi \in F_x$. ∎

We are now able to prove the uniqueness of a symmetry corresponding to a given symmetric face.

THEOREM 2·4. *Let Z be a neutral WFS space, and let F be any norm exposed face. If S_F and $\tilde{S}_F$ are isometric symmetries with the same fixed point set $\overline{\mathrm{sp}}\, F \oplus F^\diamond$, then $S_F = \tilde{S}_F$. Moreover the generalized Peirce projections $P_k(F), k = 0, 1, 2$, are uniquely determined by F.*

Proof. Let $P_k(F)$ and $\tilde{P}_k(F)$ be the generalized Peirce projections corresponding to the symmetries S_F and $\tilde{S}_F$ respectively. Since

$$P_2(F)\,Z = \overline{\mathrm{sp}}\, F = \tilde{P}_2(F)\,Z = Z_2(F) \quad \text{and} \quad P_0(F)\,Z = F^\diamond = \tilde{P}_0(F)\,Z = Z_0(F),$$

Theorem 2·3 implies that $P_k(F)^*$ and $\tilde{P}_k(F)^*$ have the same range $(k = 0, 2)$. By Lemma 2·2, these projections are equal, so that

$$\tfrac{1}{2}(I + S_F^*) = P_2(F)^* + P_0(F)^* = \tilde{P}_2(F)^* + \tilde{P}_0(F)^* = \tfrac{1}{2}(I + \tilde{S}_F^*).$$

Hence $S_F^* = \tilde{S}_F^*$ and $S_F = \tilde{S}_F$. ∎

Our final theorem in this section describes the generalized tripotents in the space $U_1(F)$. It is similar to Theorem 2·3, but since $P_1(F)$ is not neutral and $x \in U_1(F)$ does

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not imply $F_x \subset Z_1(F)$, the condition on the face F_x involves the action of the symmetry rather than the range of $P_1(F)$.

THEOREM 2·5. *Let Z be a WFS space, let $F \in \mathcal{SF}, x \in U_1(F)$, $\|x\| = 1$. Then $S_F(F_x) = -F_x$ and $v_H \in U_1(F)$, where $H = F_x$; in particular, if x is a generalized tripotent and $S_F(F_x) = -F_x$, then $x \in U_1(F)$. Moreover the generalized tripotents in $U_1(F)$ generate $U_1(F)$ in the following sense:*

$$U_1(F) = \overline{\mathrm{sp}}^{w^*}\{v_G : G \in \mathcal{SF}, S_F(G) = -G\}.$$

Proof. Let $\phi \in Z$, where $\|\phi\| = 1$. Then

$$S_F\phi \in -F_x \Leftrightarrow \langle -S_F\phi, x\rangle = 1 \Leftrightarrow \langle \phi, S_F^* x\rangle = -1 \Leftrightarrow \langle \phi, x\rangle = 1 \Leftrightarrow \phi \in F_x.$$

Hence $S_F(F_x) = -F_x$. By [7], lemma 2·4, with $H = F_x$, we have $S_F^*(v_H) = v_{-H} = -v_H$, i.e. $v_H \in U_1(F)$.

Let $G \in \mathcal{SF}$ satisfy $S_F(G) = -G$. Then

$$P_1(F)^* v_G = \tfrac{1}{2}(I - S_F)^* v_G = v_G.$$

Thus $\overline{\mathrm{sp}}^{w^*}\{v_G : G \in \mathcal{SF}, S_F(G) = -G\} \subset U_1(F)$. To show equality, let ψ belong to Z, $\psi(v_G) = 0$ for all $G \in \mathcal{SF}$ with $S_F(G) = -G$ and suppose $\psi(U_1(F)) \neq 0$. We may assume $\psi \in Z_1(F)$ and $\|\psi\| = 1$. Thus there exists $x \in U_1(F)$ with $\|x\| = 1 = \langle \psi, x\rangle$, and by the first statement $S_F(F_x) = -F_x$. We now have $\psi(v_H) = 1$, a contradiction, where $H = F_x$. ∎

3. *Compatible generalized tripotents*

Two generalized tripotents u and v are said to be *compatible* if their generalized Peirce projections commute, i.e. $[P_k(u), P_j(v)] = 0$ for $k, j \in \{0, 1, 2\}$. Theorem 3·3, which is of fundamental importance in the sequel, gives sufficient conditions for compatibility.

In the next lemma and theorem, the following idea occurs a few times. Once it is proved that $P_k(u)T = T$ for a suitable operator T, then a commutativity formula follows by duality using Theorem 2·3 to reverse the order of the product. In Lemma 3·1, $T = S_v P_k(u)$ and $[S_v, P_k(u)] = 0$ follows. In Theorem 3·3, $T = P_j(v)$ and $[P_k(u), P_j(v)] = 0$ follows.

LEMMA 3·1. *Let Z be a neutral WFS space. Let $u, v \in \mathcal{GT}$ and suppose $S_v(F_u) = \pm F_u$. Then*

$$[S_v, P_j(u)] = 0 = [P_1(v), P_j(u)] \quad \textit{for } j \in \{0, 1, 2\}. \tag{6}$$

In particular, if $u \in U_k(v)$ for some $k \in \{0, 1, 2\}$, then (6) *holds.*

Proof. Suppose that $S_v(F_u) = \pm F_u$. Then $P_2(u)S_v(F_u) = S_v(F_u)$ so that

$$P_2(u)S_v P_2(u) = S_v P_2(u). \tag{7}$$

Since
$$S_v(F_u^\diamond) = (S_v(F_u))^\diamond = (\pm F_u)^\diamond = F_u^\diamond$$

we have $S_v(Z_0(u)) = Z_0(u)$ so that

$$P_0(u)S_v P_0(u) = S_v P_0(u). \tag{8}$$

For S_v^*, by Theorem 2·3 and [7], lemma 2·4, we obtain

$$S_v^*(U_k(u)) = \overline{\mathrm{sp}}^{w^*}\{S_v^*(v_H): H\in\mathscr{SF}, H\subset Z_k(u)\}$$
$$= \overline{\mathrm{sp}}^{w^*}\{v_K: K = S_v(H), H\in\mathscr{SF}, H\subset Z_k(u)\}$$

for $k = 0$ or 2. Since (7) and (8) imply that $S_v(H)\subset Z_k(u)$, we have

$$P_2(u)^* S_v^* P_2(u)^* = S_v^* P_2(u)^*, \tag{9}$$

and

$$P_0(u)^* S_v^* P_0(u)^* = S_v^* P_0(u)^*. \tag{10}$$

Using (9) and (7), we have

$$P_2(u)^{**}S_v^{**} = P_2(u)^{**}S_v^{**}P_2(u)^{**} = (P_2(u)\,S_v\,P_2(u))^{**}$$
$$= (S_v\,P_2(u))^{**} = S_v^{**}P_2(u)^{**}.$$

Thus $[P_2(u), S_v] = 0$.

Similarly, (8) and (10) imply $[P_0(u), S_v] = 0$. Since $P_1(u) = I - P_2(u) - P_0(u)$ and $P_1(v) = \frac{1}{2}(I - S_v)$, assertion (6) follows.

Finally, if $u\in U_k(v)$, then by Theorem 2·3, $S_v(F_u) = F_u$ if $k = 0$ or 2, and by Theorem 2·5, $S_v(F_u) = -F_u$ if $k = 1$. ∎

The following remark, similar to Remark 1·3, is needed in the proof of the fundamental Theorem 3·3.

Remark 3·2. Let F be a symmetric face in a neutral normed space Z. If $\phi,\psi\in Z$, $\phi\diamond\psi$ and $\phi+\psi\in Z_2(F)$, then $\phi\in Z_2(F)$ and $\psi\in Z_2(F)$.

Proof.

$$\|\phi\|+\|\psi\| = \|\phi+\psi\| = \|P_2(F)(\phi+\psi)\| \leqslant \|P_2(F)\,\phi\| + \|P_2(F)\,\psi\| \leqslant \|\phi\|+\|\psi\|,$$

so by neutrality, $\phi,\psi\in Z_2(F)$. ∎

THEOREM 3·3. *Let Z be a neutral WFS space. If u and v are two generalized tripotents such that $u\in U_k(v)$ for some $k\in\{0,1,2\}$, then u and v are compatible.*

Proof. In each of the following three cases, by Lemma 3·1 we have

$$[P_1(v), P_j(u)] = 0 \quad \text{for} \quad j\in\{0,1,2\}. \tag{11}$$

Case 1. $k = 0$. By Theorem 2·3, $F_u\subset Z_0(v)$ and therefore $Z_2(u) = \overline{\mathrm{sp}}\,F_u\subset Z_0(v)$. This means that $P_0(v)\,P_2(u) = P_2(u)$. Also by Theorem 2·3,

$$U_2(u) = \overline{\mathrm{sp}}^{w^*}\{v_K: K\subset Z_2(u)\}\subset\overline{\mathrm{sp}}^{w^*}\{v_K: K\subset Z_0(v)\} = U_0(v)$$

so that $P_0(v)^*P_2(u)^* = P_2(u)^*$. Therefore

$$P_2(u)^{**}P_0(v)^{**} = P_2(u)^{**} = (P_0(v)\,P_2(u))^{**} = P_0(v)^{**}P_2(u)^{**},$$

i.e. $[P_2(u), P_0(v)] = 0$.

Since $v\in U_0(u)$ by [7], lemma 2·5, by symmetry $[P_2(v), P_0(u)] = 0$. Finally

$$[P_2(u), P_2(v)] = [P_2(u), I - P_1(v) - P_0(v)] = 0$$

and similarly

$$[P_2(v), P_1(u)] = [P_1(u), P_0(v)] = [P_0(u), P_0(v)] = 0.$$

Thus u and v are compatible.

Case 2. $k = 2$. Once more, by Theorem 2·3, $F_u \subset Z_2(v)$. Thus $P_2(v)P_2(u) = P_2(u)$, and as above, by the same theorem, we obtain

$$P_2(v)^*P_2(u)^* = P_2(u)^*, \quad \text{and} \quad [P_2(v), P_2(u)] = 0.$$

Since $F_u \subset Z_2(v) \subset F_v^{\diamond\diamond}$ by Proposition 1·7, we have $F_u^{\diamond} \supset F_v^{\diamond\diamond\diamond} = F_v^{\diamond}$. Therefore $Z_0(u) \supset Z_0(v)$ and so $P_0(u)P_0(v) = P_0(v)$.

By Theorem 2·3 again, $P_0(u)^*P_0(v)^* = P_0(v)^*$, and $[P_0(u), P_0(v)] = 0$. It now follows, as in the previous case, that $[P_i(v), P_j(u)] = 0$ for all $i, j \in \{0, 1, 2\}$.

Case 3. $k = 1$. Since $P_2(v) + P_0(v) = \frac{1}{2}(I + S_v)$, Lemma 3·1 implies that

$$(P_2(v) + P_0(v))\, Z_j(u) \subset Z_j(u) \quad \text{for } j \in \{0, 1, 2\}. \tag{12}$$

Using the fact that $Z_2(v) \diamond Z_0(v)$, from Remark 1·3 (for $j = 0$) and Remark 3·2 (for $j = 2$), we have

$$P_i(v)\, P_j(u) = P_j(u)\, P_i(v)\, P_j(u) \quad \text{for } i, j \in \{0, 2\}. \tag{13}$$

Moreover $Q := P_i(v)\, P_j(u)$ is a neutral contractive projection with range $Z_i(v) \cap Z_j(u)$, for $i, j \in \{0, 2\}$.

Obviously the subspace $Q^*(U)$ contains $U_i(v) \cap U_j(u)$. If these subspaces are not equal, we may choose $\phi \in Q(Z)$ such that $\phi(U_i(v) \cap U_j(u)) = \{0\}$ and $\|\phi\| = 1$. Then there exists $x \in Q^*(U)$ with $\|x\| = 1$ and with $\phi \in F_x$. By neutrality of Q we have $F_x \subset Q(Z)$ and by Theorem 2·3, $v_H \in U_i(v) \cap U_j(u)$, where $H = F_x$. Hence $\phi(v_H) = 0$, contradicting $\phi \in F_x$. Thus we have $Q^*(U) = U_i(v) \cap U_j(u)$.

Now let $R := P_i(v)\, P_j(u)\, P_i(v)$. From (13) it follows that $R^2 = R$, so that R is a neutral contractive projection. For any $\phi \in Z$ we have

$$\|R\phi\| = \|R^2\phi\| = \|P_i(v)\, P_j(u)\, R\phi\| \leqslant \|P_j(u)\, R\phi\| \leqslant \|R\phi\|.$$

Since $P_j(u)$ is neutral for $j = 0$ or 2, we have $R\phi = P_j(u)\, R\phi$. Thus we have $R(Z) = Z_j(u) \cap Z_i(v)$. This implies, as above, $R^*(U) = U_i(v) \cap U_j(u)$ and thus by Lemma 2·2, $Q = R$, i.e.

$$P_i(v)\, P_j(u) = P_i(v)\, P_j(u)\, P_i(v) \quad \text{for } i, j \in \{0, 2\}. \tag{14}$$

From this it follows that

$$P_i(v)\, P_1(u) = P_i(v)\,(P_2(u) + P_1(u) + P_0(u))\, P_i(v)\, P_1(u) = (P_i(v)\, P_1(u))^2,$$

implying, by neutrality of $P_i(v)$,

$$P_i(v)\, P_1(u) = P_1(u)\, P_i(v)\, P_1(u) \quad \text{for } i \in \{0, 2\}. \tag{15}$$

Finally, using (13) and (15), we have, for $i, j \in \{0, 2\}$,

$$P_j(u)\, P_i(v) = P_j(u)\, P_i(v)\,(P_2(u) + P_1(u) + P_0(u))$$
$$= P_j(u)\, P_i(v)\, P_j(u) = P_i(v)\, P_j(u),$$

i.e. $[P_i(v), P_j(u)] = 0$ for $i, j \in \{0, 2\}$. This fact, together with (11) shows that u and v are compatible. ∎

COROLLARY 3·4. *Let Z be a neutral WFS space. If u and v are generalized tripotents such that $u \in U_k(v)$ for some $k \in \{0,1,2\}$, then $P_i(v)P_j(u)$ is a projection with range $Z_i(v) \cap Z_j(u)$ for all $i,j \in \{0,1,2\}$. Moreover*

(*a*) *if $u \in U_0(v)$, then*	(*b*) *if $u \in U_2(v)$, then*
(i) $P_0(v)P_2(u) = P_2(u)$	(i) $P_2(v)P_2(u) = P_2(u)$
(ii) $P_0(u)P_2(v) = P_2(v)$	(ii) $P_0(u)P_0(v) = P_0(v)$
(iii) $P_2(u)P_2(v) = 0$	(iii) $P_1(u)P_0(v) = 0$
(iv) $P_1(u)P_2(v) = 0$	(iv) $P_2(u)P_0(v) = 0$
(v) $P_1(v)P_2(u) = 0$	(v) $P_1(v)P_2(u) = 0$.

Proof. The proofs of (*a*)(i–ii) and (*b*)(i–ii) are included in the proof of Theorem 3·3.

If $u \in U_0(v)$, then

$$P_2(v)P_2(u) = P_2(v)(P_0(v)P_2(u)) = 0,$$

and
$$P_1(u)P_2(v) = (I - P_2(u) - P_0(u))P_2(v) = P_2(v) - 0 - P_2(v) = 0.$$

By symmetry $P_1(v)P_2(u) = 0$, proving (*a*).

If $u \in U_2(v)$, the proof of Theorem 3·3 showed that $Z_0(u) \supset Z_0(v)$ and $Z_2(v) \supset Z_2(u)$. Therefore

$$P_1(u)P_0(v) = P_2(u)P_0(v) = 0 \quad \text{and} \quad P_1(v)P_2(u) = 0,$$

proving (*b*). ∎

We close this section with an important consequence of Theorem 3·3. Namely, we show that some fundamental properties of a normed space Z are inherited by the generalized Peirce subspaces. We adopt the following notation: if Y is a closed subspace of a normed space Z, the collections of norm exposed faces and symmetric faces in ∂Y_1 will be denoted by $\mathscr{F}_Y$ and $\mathscr{S}\mathscr{F}_Y$ respectively. Similarly for $\mathscr{U}_Y, \mathscr{G}\mathscr{T}_Y$, and if $K \in \mathscr{S}\mathscr{F}_Y$, then $S(K,Y)$ and $P_k(K,Y)$ denote a symmetry associated to K and the corresponding generalized Peirce projections.

For any $K \in \mathscr{F}_Y$, by the Hahn–Banach theorem, $K = F_x \cap Y$ for some $x \in Z^*$. If Z is WFS then a symmetry S, corresponding to the face F_x of Z_1, fixes $\overline{\mathrm{sp}}\, F_x + F_x^\diamond$. If S leaves Y invariant then $S|_Y$ is a symmetry of Y fixing

$$(\overline{\mathrm{sp}}\, F_x + F_x^\diamond) \cap Y = (\overline{\mathrm{sp}}\, F_x \cap Y) + (F_x^\diamond \cap Y).$$

Therefore, in order to show that $K \in \mathscr{S}\mathscr{F}_Y$, it suffices to prove that S leaves Y invariant and that $\overline{\mathrm{sp}}\, F_x \cap Y = \overline{\mathrm{sp}}\, K$ and $F_x^\diamond \cap Y = K^\diamond \cap Y$. The latter is done in Lemma 3·5 if $Y = Q(Z)$ and Q is any contractive projection on Z. Specializing to $Q = P_k(v)$ and using some earlier commutation formulae then leads to the fact that the generalized Peirce spaces $Z_k(v)$ are WFS and neutral, whenever Z is WFS and neutral.

LEMMA 3·5. *Let Z be any normed space, let $Q: Z \to Z$ be a contractive linear projection, and let $w \in \mathscr{G}\mathscr{T} \cap Q^*(U)$. Then, with $Y = Q(Z)$,*

$$(F_w \cap Y)^\diamond \cap Y = F_w^\diamond \cap Y, \tag{16}$$

and
$$\overline{\mathrm{sp}}\,(F_w \cap Y) = \overline{\mathrm{sp}}\,(F_w) \cap Y. \tag{17}$$

Proof. Obviously $(F_w \cap Y)^\diamond \cap Y \supset F_w^\diamond \cap Y$. To prove equality, let ρ belong to $(F_w \cap Y)^\diamond \cap Y$, where $\|\rho\| = 1$, and let $\phi \in F_w$. Since $w = Q^*w$, we have $Q\phi \in F_w$ and $Q\phi \diamond \rho$. Hence

$$2 = \|Q\phi\| + \|\rho\| = \|Q\phi + \rho\| = \|Q(\phi \pm \rho)\| \leqslant \|\phi \pm \rho\| \leqslant 2.$$

Thus $\phi \diamond \rho$, and $\rho \in F_w^\diamond \cap Y$, proving (16).

Obviously $\overline{\mathrm{sp}}\,(F_w \cap Y) \subset \overline{\mathrm{sp}}\,(F_w) \cap Y$. To prove equality, suppose first that $\rho = \Sigma\, a_i \rho_i$ belongs to $(\mathrm{sp}\, F_w) \cap Y$, so that $\rho_i \in F_w$. Then $\rho = Q\rho = \Sigma\, a_i\, Q\rho_i$ and $Q\rho_i \in F_w \cap Y$. Therefore $(\mathrm{sp}\, F_w) \cap Y \subset \mathrm{sp}\,(F_w \cap Y)$. Now (17) follows by approximation. ∎

THEOREM 3·6. *Let Z be a neutral WFS space. Let* $v \in \mathcal{GT}$. *Then* $Z_k(v)$ *is a neutral WFS space, for* $k \in \{0, 1, 2\}$. *More precisely, for every norm exposed face K in* $Z_k(v)$, *there is a norm exposed face F in Z such that* $K = F \cap Z_k(v)$ *and the generalized Peirce projections corresponding to a symmetric face in* $Z_k(v)$ *are the restrictions of the global generalized Peirce projections in Z, i.e.* $P_j(K, Z_k(v)) = P_j(F)|_Y$, *(where* $Y = Z_k(v)$*) for* $j = 0, 1, 2$. *Furthermore* $P_k(v)(F) = K$.

Proof. Let $K \in \mathcal{F}_Y$ where $Y = Z_k(v)$. Then $K = F_x \cap Z_k(v)$ for some x in $U_k(v)$ with $\|x\| = 1$. It is clear that $K \subset P_k(v)(F)$ and conversely, if $\rho \in F$, then

$$\langle P_k(v)\rho, x\rangle = \langle \rho, P_k(v)^* x\rangle = \langle \rho, x\rangle = 1,$$

so $P_k(v)(F) \subset K$. By Theorems 2·3 and 2·5 there exists $w \in \mathcal{GT} \cap U_k(v)$ with $F_x = F_w$. By Theorem 3·3, the symmetry S_F (with $F = F_x$) leaves $Z_k(v)$ invariant. Therefore $S_K := S_F|_Y$ is an isometric symmetry fixing $(\overline{\mathrm{sp}}\, F + F^\diamond) \cap Z_k(v)$, which equals $\overline{\mathrm{sp}}\, K + (K^\diamond \cap Z_k(v))$ by Lemma 3·5. Therefore $K \in \mathcal{SF}_Y$ and $Z_k(v)$ is WFS.

Moreover, since $P_j(F)$ leaves $Z_k(v)$ invariant and its restriction has the 'correct' range, we may define the generalized Peirce projections $P_j(K, Z_k(v))$ to be $P_j(F)|_Y$. It follows immediately that $Z_k(v)$ is neutral if Z is neutral and hence, by Theorem 2·4, that the generalized Peirce projections are unique. ∎

We have just proved a hereditary property for certain contractive projections, namely the generalized Peirce projections. Since it is known that the category of JB^*-triples is stable under arbitrary contractive projections, the following is a natural question.

Problem 1. If Z is a WFS space and $Q: Z \to Z$ is an arbitrary contractive projection, is $Q(Z)$ a WFS space?

4. *Facial structure in SFS spaces*

In the previous sections, we obtained properties of WFS spaces. To proceed further, we restrict our attention to the subclass of strongly facially symmetric spaces. This class was introduced by the authors in [7] to obtain a geometric spectral theorem. In [7], these were simply called facially symmetric spaces.

In this section we show that the facial structure of the unit balls of a strongly facially symmetric space and its dual behave almost exactly as they do in the case of a JBW^*-triple and its predual (cf. [4]). This is further evidence that the category of dual spaces of strongly facially symmetric spaces, which satisfy certain physically meaningful axioms, will coincide with the category of JBW^*-triples.

Definition 2. A WFS space Z is said to be a *strongly facially symmetric space* (abbreviation *SFS* space) if for every norm exposed face F in Z_1 and every $y \in Z^*$ with $\|y\| = 1$ and $F \subset F_y$ we have $S_F^* y = y$, where S_F denotes a symmetry associated with F.

This property implies that $y = v_F + P_0(F)^* y$ and is analogous to orthomodularity in a lattice (the definition of which is recalled below).

Note that the WFS space Z in Example 1·7 is also neutral. However, it is easy to see that it is not an SFS space. For $u = (1,0)$ and $v = (1, \frac{1}{3}\sqrt{3})$, we have $F_u \subset F_v$, $F_u \neq F_v$ and $F_u^\diamond \cap F_y = \varnothing$ since $F_u^\diamond = \{0\}$. This shows that [7], lemma 2·7, which is needed below, fails in neutral WFS spaces. Moreover this example shows that the spectral theorem ([7], theorem 1) and the uniqueness of the generalized tripotent occurring in the polar decomposition (Proposition 1·4) are not valid in the category of neutral WFS spaces.

The only result in the previous sections that does not hold automatically for SFS spaces is the stability under generalized Peirce projections, which we now prove.

Note that in an SFS space, every projective unit is a generalized tripotent (i.e. $\mathcal{U} = \mathcal{GT}$), since for $u \in \mathcal{U}$, the condition $S_u^* u = u$ is included in the definition of SFS space.

PROPOSITION 4·1. *Let Z be a neutral SFS space. Then for any $v \in \mathcal{GT}$, and $k \in \{0,1,2\}$, $Z_k(v)$ is an SFS space.*

Proof. Let $F \in \mathcal{F}$, $y \in U_k(v)$ and suppose $K := F \cap Z_k(v) \subset F_y \cap Z_k(v)$. We must show that $S(K, Z_k(v))^* y = y$. For any $\rho \in F$, by Theorem 3·6 we have $P_k(v)\rho \in K$ and

$$\langle \rho, y\rangle = \langle \rho, P_k(v)^* y\rangle = \langle P_k(v)\rho, y\rangle = 1.$$

Thus $F \subset F_y$, and $S(K, Z_k(v))^* y = S_F^* y = y$. ∎

Although a main theme of our theory is to be free from a global order structure, a key role will be played by an order structure on the set of generalized tripotents.

A priori, there are three ways to define an order structure on $\mathcal{GT}_Z$, where Z is a WFS space, namely

(1) by using the bijection of $\mathcal{GT}$ with $\mathcal{SF}$ given in Proposition 1·8, and set-theoretic inclusion of faces;

(2) by using the fact that the sum of orthogonal generalized tripotents is a generalized tripotent;

(3) by using generalized Peirce projections.

Example 1·7 shows that for WFS spaces these definitions are not equivalent. The following lemma shows this equivalence in SFS spaces.

LEMMA 4·2. *Let Z be a strongly facially symmetric space and suppose that $u, v \in \mathcal{GT}_Z$. The following are equivalent:*

(1) $F_u \subset F_v$;

(2) *$v - u$ is either 0 or a generalized tripotent with $(v-u) \diamond u$;*

(3) $P_2(u)^* v = u$.

Proof. (1) implies (2). Since Z is SFS, by [7], lemma 2·8, $F_u \subset F_v$ implies that $v = u + b$ with $b = P_0(u)^* v$. To show that $b \in \mathcal{GT}$, if $b \neq 0$, it suffices to prove that b is a projective unit, i.e. $\|b\| = 1$ and $\langle b, F_b^\diamond\rangle = 0$. Since $b \neq 0$, $F_u \neq F_v$ so by [7], lemma 2·7 there exists $\phi \in F_u^\diamond \cap F_v$. Thus $\phi(b) = \phi(v-u) = 1$ and $\|b\| = 1$. Next, let $\rho \in F_b^\diamond$. Since $F_u \diamond F_b$, we have $P_0(u)F_b^\diamond \subset F_b^\diamond$ by Theorem 3·3. Therefore $P_0(u)\rho \in F_b^\diamond \cap F_u^\diamond = F_v^\diamond$ by Lemma 1·8, and so

$$\langle b, \rho\rangle = \langle b, P_0(u)\rho\rangle = \langle b+u, P_0(u)\rho\rangle = \langle v, P_0(u)\rho\rangle = 0.$$

(2) implies (3). If $v = u+b$ with $b \in U_0(u)$, then $P_2(u)^*v = P_2(u)^*(u+b) = u$.

(3) implies (1). If $\rho \in F_u$, then $\langle v, \rho\rangle = \langle v, P_2(u)\rho\rangle = \langle P_2(u)^*v, \rho\rangle = \langle u, \rho\rangle = 1$, so $\rho \in F_v$. ▮

Definition 3. Let Z be a strongly facially symmetric space and suppose that $u, v \in \mathscr{GT}_Z$. If one of the conditions in Lemma 4·2 is satisfied, we write $u \leqslant v$.

We are now able to obtain more detailed information on the generalized tripotent arising in the polar decomposition.

THEOREM 4·3. *Let Z be a neutral SFS space. For any $f \in Z$ with $\|f\| = 1$, there is a unique generalized tripotent v such that*

(*a*) *$f \in F_v$, and*

(*b*) $\langle v, \{f\}^\diamond\rangle = 0$.

Moreover

(*c*) *F_v is the smallest norm exposed face containing f, and*

(*d*) *f is faithful on $U_2(v)$ in the following sense*:

$$f(u) = \|P_2(u)f\| > 0 \quad \textit{for any } u \in \mathscr{GT} \textit{ with } u \leqslant v.$$

Proof. The existence of v which satisfies (*a*) and (*b*) is given in Proposition 1·4.

To show uniqueness, let $w \in \mathscr{GT}$ satisfy $f \in F_w$. Then $F := F_w \cap F_v$ is a norm exposed face, hence $F = F_u$ for some $u \in \mathscr{GT}$. Obviously $u \leqslant v$ and $f \in F_u$. If $u \neq v$, then $\tilde{u} := v-u \in \mathscr{GT}$, $\tilde{u} \diamond u$, and $F_{\tilde{u}} \diamond F_u$, which implies that $F_{\tilde{u}} \subset F_u^\diamond \subset \{f\}^\diamond$, hence $\langle v, F_{\tilde{u}}\rangle = 0$ by (*b*). Thus

$$0 = \langle v, F_{\tilde{u}}\rangle = \langle u+\tilde{u}, F_{\tilde{u}}\rangle = \langle \tilde{u}, F_{\tilde{u}}\rangle,$$

a contradiction. Therefore $F_u = F_v$, and $F_v \subset F_w$, which proves (*c*) and the uniqueness.

To prove (*d*), write $v = u+w$ with $u \diamond w$ and $w \in \mathscr{GT}$. Then

$$1 = f(u)+f(w) = \langle P_2(u)f, v\rangle + \langle P_0(u)f, v\rangle \leqslant |\langle P_2(u)f, v\rangle| + |\langle P_0(u)f, v\rangle|$$

$$\leqslant \|P_2(u)f\| + \|P_0(u)f\| = \|(P_2(u)+P_0(u))f\| \leqslant \|f\| = 1.$$

Therefore $f(u) = \|P_2(u)f\| \geqslant 0$. If $f(u) = 0$, then $f(w) = 1$ and $w \neq v$, contradicting (*c*). ▮

In a neutral SFS space, for $f \neq 0$ in Z we denote by $v(f)$ the unique generalized tripotent v with $f(v) = \|f\|$ and $\langle v, \{f\}^\diamond\rangle = 0$. If $f, g \in Z$, then $f \diamond g$ if and only if $v(f) \diamond v(g)$, as follows from corollary 1·3(*b*) and lemma 2·1 of [**7**].

An important construct in the study of the facial structure of any convex set is the notion of a *norm semi-exposed face*, which by definition is the intersection of an arbitrary family of norm exposed faces. In the predual of a JBW^*-triple, these objects coincide with the norm exposed faces and the norm closed faces of the unit ball [**4**]. The next result proves one of these assertions for SFS spaces.

THEOREM 4·4. *Let Z be a neutral strongly facially symmetric space. Then every norm semi-exposed face of Z_1 is norm exposed.*

Proof. By Proposition 1·6, any norm semi-exposed face G has the form $\bigcap\{F_\alpha : \alpha\in T\}$ where for $\alpha\in T$, F_α is the norm exposed face associated with the generalized tripotent v_α. Fix $\beta\in T$ and set $H = F_\beta$. For any finite subset $A\subset T$ with $\beta\in A$, define $F_A\in\mathcal{SF}$ by $F_A = \bigcap\{F_\alpha : \alpha\in A\}$. By Theorem 4·3(*d*) we have $\langle\rho, v_A\rangle = \|P_2(v_A)\rho\|$. Moreover if $A\subset B$ then $F_A\supset F_B$, so by Corollary 3·4 we have $P_2(v_B) = P_2(v_B)P_2(v_A)$. Thus for $\rho\in H$,

$$\rho(v_B) = \|P_2(v_B)\rho\| = \|P_2(v_B)P_2(v_A)\rho\| \leqslant \|P_2(v_A)\| = \rho(v_A).$$

It follows that $\lim_A\langle\rho, v_A\rangle$ exists for $\rho\in H$. Now define a functional Φ on $\operatorname{sp} H$ by

$$\Phi(\sum_i a_i\rho_i) = \lim_A(\sum_i a_i\rho_i(v_A)).$$

Clearly Φ is linear and of norm 1 so extends to $Z_2(H)$. Let $x\in U$ be defined by $\langle x,\sigma\rangle = \Phi(P_2(H)\sigma)$, for $\sigma\in Z$. Then $\|x\| = 1$ and $G = F_x$. ▮

The following consequence of Theorem 4·4 provides a local structure of complete orthomodular lattice in SFS spaces.

Let L be a lattice, i.e. a partially ordered set any two of whose elements a and b have a least upper bound $a\vee b$ and a greatest lower bound $a\wedge b$. The least and greatest elements of L, if they exist, are denoted by 0 and 1 respectively.

Recall that if $u\in L$, where L is a lattice, then $u'\in L$ is a *complement* for u if $u\wedge u' = 0$ and $u\vee u' = 1$. The lattice L is said to be *orthocomplemented* if there is an order reversing map $u\mapsto u^\perp$ (called *orthocomplementation*) on L satisfying $u^{\perp\perp} = u$ and such that $u^\perp$ is a complement for u; and *orthomodular* if in addition $u\leqslant v$ implies $v = u\vee(v\wedge u^\perp)$. Also L is a *complete lattice* if every non-empty subset of L has a least upper bound and a greatest lower bound.

PROPOSITION 4·5. *Let Z be a neutral strongly facially symmetric space and fix a generalized tripotent w. The set $L_w := \{v\in\mathcal{GT} : v\leqslant w\}\cup\{0\}$ is a complete orthomodular lattice with smallest element* 0, *largest element w, and orthocomplementation $v\mapsto v^\perp = w-v$.*

Proof. Since, for $u,v\in L_w$, $F := F_u\cap F_v$ is either empty or a norm exposed face with $F\subset F_w$, it is clear that the generalized tripotent corresponding to F is the greatest lower bound of u and v. Also, the order reversing map $v\mapsto w-v$ defines a structure of orthocomplemented lattice on L_w.

To show completeness it suffices to show that every non-empty family has a greatest lower bound (= GLB). For an arbitrary family $\{v_\alpha\}\subset L_w$, if $\bigcap_\alpha F_\alpha = \varnothing$, where F_α is the face corresponding to v_α, then $0 = \operatorname{GLB} v_\alpha$. On the other hand, if $\bigcap_\alpha F_\alpha \neq \varnothing$, then by Theorem 4·4, $\bigcap_\alpha F_\alpha = F_u$ for some $u\in\mathcal{GT}$, and clearly $u = \operatorname{GLB} v_\alpha$ in this case.

To prove orthomodularity, let $u\leqslant v$. By Lemma 4·2 we have $v = u+(v-u)$ and $u\diamond(v-u)$. Also $F_{v-u} = F_v\cap F_{w-u}$, i.e. $v-u = v\wedge(w-u) = v\wedge u^\perp$. Finally, since $u\diamond(v-u)$,

$$v = u+(v-u) = u\vee(v-u) = u\vee(v\wedge u^\perp).$$ ▮

Each $f\in Z$ with $\|f\| = 1$ defines a weak*-exposed face in U_1 which will be denoted by F^f, i.e. $F^f = \{x\in U : \|x\| = 1 = f(x)\}$. Note that $x\in F^f$ if and only if $f\in F_x$.

For any generalized tripotent v let $F(v)$ denote the convex set $v+U_0(v)_1$. Our next theorem states that the collection $\{F(v): v\in\mathcal{GT}\}$ coincides with the collection of weak*-semi-exposed faces of U_1.

THEOREM 4·6. *Let Z be a neutral strongly facially symmetric space. For each $v\in\mathcal{GT}$*

$$F(v) = \bigcap\{F^f : f\in F_v\},$$

hence $F(v)$ is a weak-semi-exposed face. Moreover, for any weak*-semi-exposed face G of U_1, there exists $v\in\mathcal{GT}$ with $G = F(v)$.*

Proof. For $x\in F(v)$ and $f\in F_v$ we have $f(x) = f(P_2(v)x) = f(v) = 1$. Thus $x\in F^f$ and $F(v)\subset\bigcap\{F^f : f\in F_v\}$. On the other hand, if x lies in F^f for all $f\in F_v$, then $F_v\subset F_x$, so that by [7], lemma 2·8 we have $x = v+P_0(v)^*x\in F(v)$.

Now, let $G=\bigcap\{F^f : f\in T\}$ be a weak*-semi-exposed face in U_1. By Theorem 4·4, the norm semi-exposed face $F := \bigcap\{F_x : x\in G\}$ is norm exposed, say $F = F_v$ for some $v\in\mathcal{GT}$. To prove that $G = F(v)$, note first that if $x\in G$ and $f\in T$ then $x\in F^f$, $f\in F_x$ and therefore $T\subset F_x$. By the first statement, $F(v) = \bigcap\{F^g : g\in F_v\}$, and therefore, $G\supset F(v)$, since $T\subset F_v$. Conversely, if $x\in G$, then $F_v = \bigcap\{F_y : y\in G\}\subset F_x$ and therefore $x = v+P_0(v)^*x\in F(v)$. ∎

In general, a weak*-semi-exposed face in U_1 is not weak*-exposed. However, if the space Z is not 'too big', for example if it is σ-finite, we do have the following result:

PROPOSITION 4·7. *Let Z be a neutral strongly facially symmetric space which is σ-finite, i.e. every orthogonal family of generalized tripotents is at most countable. Then every weak*-semi-exposed face of U_1 is weak*-exposed.*

Proof. Let G be a weak*-semi-exposed face of U_1. By Theorem 4·6, there is a $v\in\mathcal{GT}$ with $G = F(v)$. It suffices to prove that $v = v(\psi)$ for some $\psi\in Z$, for then we will have $G = F^{\psi}$.

Let $\{\phi_\alpha\}$ be a maximal orthogonal family of elements of F_v. Then $\{v(\phi_\alpha)\}$ is an orthogonal family of generalized tripotents in L_v and therefore $u = \text{LUB}\, v(\phi_\alpha)$ exists in L_v. If $u\neq v$ then there exists $\phi\in F_v\cap F_u^{\diamond}$ which implies $\phi\diamond\phi_\alpha$ for all α, contradicting maximality. Thus $u = v$.

On the other hand, by σ-finiteness, $\{\phi_\alpha\} = \{\psi_n\}_{n=1}^{\infty}$, say, is countable. We now have, with $\psi := \sum_{n=1}^{\infty} 2^{-n}\psi_n$, that $\psi\in F_v$.

Moreover, if $g\in\{\psi\}^{\diamond}$, then $g\in\{\psi_n\}^{\diamond}$ for all n, by Remark 1·3. This implies that $\langle v(\psi_n), g\rangle = 0$ for all n and therefore, since

$$v = w^*\text{-lim}\,(v(\psi_1)+\ldots+v(\psi_n))$$

that $\langle v, g\rangle = 0$. Thus $\langle v, \{\psi\}^{\diamond}\rangle = 0$, and $v = v(\psi)$. ∎

Two important questions remain open concerning the facial structure of an SFS space and its dual. The answer to each of these questions is 'yes' in the category of JBW^*-triples.

Problem 2. Let Z be an SFS space, let $f\in Z$, and let $v = v(f)$. Is the face generated by f norm dense in F_v?

Problem 3. Let Z be an SFS space. Is every proper norm closed face of Z_1 automatically a norm exposed face? In particular, is every extreme point of Z_1 a norm exposed point of Z_1?

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