Title
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Authors
Basdevant, Jean-Louis
Kreps, Rodney E.

Publication Date
1965-07-12

UCRL-16255

University of California

Ernest O. Lawrence Radiation Laboratory

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Berkeley, California

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AEC Contract No. W-7405-eng-48

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Jean-Louis Basdevant and Rodney E. Kreps

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Jean-Louis Basdevant+

Lawrence Radiation Laboratory
University of California
Berkeley, California

and

Rodney E. Kreps‡

Department of Physics
University of California
Berkeley, California

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ABSTRACT

The calculation of the $I = 0$ $J^P = 1^-$ three-pion state is done on the basis of previously derived equations.[1] The two-body amplitude is assumed to be well represented by the pole of the ρ. Particular choices for the resonance form factors appearing in the equations are discussed. The behavior of the eigenvalues of the Faddeev kernel is exhibited and it is explained why the ω meson does not appear in such a model. A resonance is found at relatively high energy which does not depend on the choice of the form factors. This resonance seems to correspond to the experimental E meson.

I. INTRODUCTION

Previously,[1] we have described the reduction to a simple form of a set of relativistic three-body equations given by Alessandrini and Omnes.[2] This paper describes a calculation of the $I = 0$, $J^P = 1^-$ three-pion state, based on the resulting one-dimensional integral equation. We have explained in I why this channel is interesting, both for the simplicity of the resulting equations and for the experimental variety of resonances. Since there can be no two-body s-wave interaction, the π-π amplitude is dominated by the ρ, and thus the three-pion amplitude is dominated by π-ρ states.

Section II discusses the choice of the resonance form factors, which are the undetermined functions appearing in the kernel. The parameters occurring in the form factors are determined phenomenologically from the on-shell two-body amplitude. Section III describes the singularities of the kernel and the lack of Fredholm character as the three-body energy becomes real above threshold. Section IV gives the method that we use to analyze the equation and look for resonances, and the specific numerical results. Finally, Section V is a discussion of these results, in which we examine their interpretation and the general meaningfulness of our model.

II. FORM FACTORS

The kernel given by Eq. (4:6) of I involves an integration. A natural variable change in this integration puts the kernel in the form

$$K(x,y) = -\frac{3}{(2\pi)^3}\frac{(y^2-1)^{\frac{1}{2}}(x^2-1)}{D\left[\sigma(s,y)\right]}\int_{-1}^{+1}\frac{(1-\xi^2)}{a(\xi)}\frac{\left[x+y+a(\xi)\right]}{\left[x+y+a(\xi)\right]^2-s}\,F\Big(x,y,a(\xi)\Big)\,d\xi \tag{2:1}$$

$$\text{where}\quad a(\xi) = \left(x^2+y^2-1+2(x^2-1)^{\frac{1}{2}}(y^2-1)^{\frac{1}{2}}\xi\right)^{\frac{1}{2}} \tag{2:2}$$

$$\text{and}\quad F(x,y,a) = \frac{g\Big(p(x,y,a)\Big)\,g\Big(p(y,x,a)\Big)}{p(x,y,a)\;\;p(y,x,a)}\,. \tag{2:3}$$

All our notations follow I : $g(p)$ is the form-factor and

$$p(x,y,a) = \tfrac{1}{2}\left[2(y^2+a^2)-x^2-3\right]^{\frac{1}{2}}, \tag{2:4a}$$

$$\sigma(s,y) = s-2\,y\,s^{\frac{1}{2}}+1, \tag{2:4b}$$

while $D(\sigma)$ is the two-body D function

$$D(\sigma) = c+\frac{1}{16\pi^2}\int_4^\infty\left(\frac{x-4}{x}\right)^{\frac{1}{2}}\frac{\left[g\left(\left(\frac{x-4}{4}\right)^{\frac{1}{2}}\right)\right]^2}{x-\sigma}\,dx, \tag{2:5}$$

where c is a real constant as yet undetermined.

The remaining undetermined quantity in the kernel (2:1) is the form factor $g(p)$. As pointed out in I we know some of its

properties; in particular the asymptotic behavior is given by Eq. (4:5) of I. In addition we require that it have no singularities in the physical region $p > 0$. The simplest form which satisfies all these conditions is

$$g(p) = p/(p^2 + a^2)^2 , \tag{2:6}$$

where a is a real parameter as yet undetermined. A normalization constant in $g(p)$ is superfluous, since $D(\sigma)$ as determined by Eq. (2:5) contains an arbitrary constant.

Since we want the two-body amplitude to have a pole corresponding to the existence of the ρ resonance, we shall require $D(\sigma)$ to give us a resonance curve of the correct width at the observed mass. For $D(\sigma)$ given by Eq.(2:5) and Eq. (2:6), this determines all the parameters appearing in the problem. Specifically, we require the usual conditions

$$\text{Re } D(m_\rho^2) = 0 \tag{2.7a}$$

and

$$\text{Im } D(m_\rho^2)/\text{Re } D'(m_\rho^2) = m_\rho \Gamma_\rho . \tag{2.7b}$$

The resulting resonance curve is shown in Fig. 1, and it in fact has its peak in the right position. The curve is not perfectly symmetric but an evaluation of its approximate width gives a reasonable value. An interesting property of $D(\sigma)$ is that its second-sheet zero is in the present case approximately twice as far from the real axis as would be

expected from the usual Breit-Wigner representation of the resonance curve. This and the nonsymmetry of the curve are due to the fact that the width of the ρ is relatively large.

As was pointed out in I , the basic reason for assuming separability of the two-body off-shell amplitude is that it is, at present, the only choice which allows one to carry out the three-body calculation with reasonable numerical accuracy. It is certain that the function $g(p)$ defined in Eq. (2:6) does satisfy the requirements for a resonance form factor in the Blankenbecler-Sugar[3] equation, but its essential role in our model is to simmulate the off-shell dependence of the two-body amplitude. In order to test the importance of the exact form of the off-shell behavior we have taken several forms for $g(p)$ with different asymptotic properties. Specifically, besides (2:6) we also considered

$$g(p) = \frac{p}{(p^2 + a^2)^{3/2}} \qquad (2:8a)$$

and

$$g(p) = \frac{p}{(p^2 + a^2)^{5/2}} \quad . \qquad (2:8b)$$

These two other forms have the desired behavior at threshold, but at infinity (2:8a) behaves like an s wave and (2:8b) like a d wave. Whereas the value of a^2 in (2:6) which fits the ρ resonance is $a^2 = 0.74\ m_\pi^2$, in (2:8a) it is $a^2 = 0.11\ m_\pi^2$ and in (2:8b) $a^2 = 1.62\ m_\pi^2$.

At this point it must be noted that our full on-shell amplitude is expected to be good only in the vicinity of the resonance. The resonance curve for the full amplitude is more symmetric than the curve for the D function alone. However, as can be seen from the small values obtained for a^2, the on-shell amplitude shows a marked rise just above threshold. We do not consider this as a fundamental drawback; $g(p)$ with p on the energy shell is not meant to reproduce the amplitude except in the vicinity of the pole, and furthermore it will be seen in Section IV that the three-body results are relatively independent of the form taken for $g(p)$.

III. SINGULARITIES OF THE KERNEL

In their original form, the Faddeev equations are not amenable to numerical calculations, for several reasons. First, they are integral equations in three variables and, unless the kernel is exceptionally smooth, the present capability of computers is inadequate. Second, the kernel is disconnected and must be iterated at least once to remove the δ functions. Third, the kernel must be iterated five times in order for it to be compact in the physical region.[4] From a computational point of view, any iteration is undesirable. However, the separable approximation to the two-body amplitude eliminates the first two difficulties. It must be noted that this is a particular property of the three-body problem; if such an approximation was made in four-body Faddeev equations,[5] for instance, the resulting equations would not have the simplicity of the present case.

The kernel of our one-dimensional equation, given by Eq. (2:1), is compact in the complex energy plane off the real axis. We could iterate a number of times to obtain a kernel that is compact everywhere, but we shall prefer to study the noniterated kernel with the energy complex.

In the complex plane of the integration variable y appearing in Eq. (2:1), the kernel has several singularities which collapse onto the real axis as the energy becomes real. In order to do the integration accurately we shall need to know the position and nature of these singularities.

Apart from a square-root-type singularity due to the presence of the factor $(y^2 - 1)^{\frac{1}{2}}$ in Eq. (2:1), which can be taken into account explicitly by using the proper Jacobi polynomials in Gauss-Mehler integration, the singularities of the kernel can be put in two categories:

(a) singularities of the two-body amplitude,

(b) singularities arising from the presence of the three-body Green's function.

The important singularities of the two-body amplitude are the branch point of the elastic cut and the pole in the second sheet. As their position in the complex y plane depends only on s, the total energy, they are easy to handle. In particular the difficulty created by the pole can be eliminated by making one subtraction. The influence of the branch point can be allowed for by breaking up the y integration into appropriate intervals.

The three-body Green's function leads to a much more involved situation, as the positions of the singularities will also depend on the variable x. It is clear that the integration in Eq. (2:1) will yield a term of the form

$$K_1 = \log\left(\frac{x + y + a_+ - z}{x + y + a_- - z}\right), \tag{3:1}$$

where $z = s^{\frac{1}{2}}$, and a_+ and a_- are given by

$$a_{\pm} = \left[1 + \left[(x^2 - 1)^{\frac{1}{2}} \pm (y^2 - 1)^{\frac{1}{2}}\right]^2\right]^{\frac{1}{2}} . \tag{3:2}$$

The singularities of (3:1) are the branch point of $(y^2 - 1)^{\frac{1}{2}}$, the points $a_{\pm} = 0$, and the points where

$$F_{\pm} = x + y + a_{\pm} - z = 0 . \qquad (3:3)$$

The controlling singularities for our calculation are the latter; and a plot of the location of these branch points in the complex y plane, as x varies from 1 to ∞ and z is held constant, is given in Fig. 2. The relevant features of this curve are that it essentially parallels the integration path at a distance Im $z/2$, and terminates at the point $\frac{1}{2}(z + 1/z)$. This singularity can thus be handled in the same way as the two-body branch cut.

If the presence of these singularities is not taken into account carefully, our experience has shown that the error of the final result is increased by several orders of magnitude, and can in fact make the calculation meaningless.

IV. METHODS AND RESULTS

Although our equations can be solved to obtain the three-body amplitude, which would be necessary, for example, if one were interested in making Dalitz plots, our interest has been in determining the possible existence of resonances and bound states. These will appear, as usual, as poles of the three-body amplitude in the total energy, resulting from an eigenvalue of the kernel equal to one.

At this point it must be recalled that the equations we have written are valid in the first Riemann sheet of the energy. For determining bound states this is quite satisfactory; however, it is well known that resonances correspond to second-sheet poles. It is possible in principle to continue the Faddeev equations through the three-body cut. This operation is complicated by the fact that the cuts of both the three-body Green's function and the two-body amplitude must be taken into account. Further, it is not even necessary, since, as in the two-body case, the object of interest is the physical amplitude just above the real axis. The amplitude will of course not have an actual pole, but a peak, which is caused by an eigenvalue of the kernel passing near one. It has proved convenient, in looking for resonances, to compute the Fredholm determinant rather than the eigenvalues; the inverse of the absolute value of this determinant will show a peak which is essentially that of the amplitude.

We have explained in the previous section that it is computationally difficult to operate just above the real axis and consequently we

shall examine the behavior of the Fredholm determinant at the minimum distance from the real axis compatible with these requirements. We expect that this procedure will not obscure resonances, since its effect should be primarily to increase the width of resonance curves by approximately twice the distance in energy from the real axis.

The mass and the width are determined in principle by examining the full amplitude on the real axis. However, such a procedure is somewhat tedious, and in a model calculation of this type yields information no more meaningful than that obtained by extrapolating the behavior of the determinant down to the real axis. The extrapolation yields an indetermination in the mass of less than two parts per thousand. The indetermination on the width is much greater and of the order of 10% ; however, the relative values of the various widths that we quote have a greater accuracy.

The first result is that in our model it is impossible to obtain a low-energy resonance or a bound state; in particular the physical ω does not appear.[6] However, a high-energy resonance is found. The behavior of the inverse of the absolute square of the Fredholm determinant is shown in Fig. 3. The important property of this result is that the position and width of this resonance are independent of which form factor is chosen. As can be seen from Table I, where the mass and width are given for each of our form-factors, the masses differ by only 2.5% and the widths by 10% in spite of the completely different asymptotic behaviors of the form-factors.

The mass of the resonance varies considerably and the width much less, when the input ρ mass is changed and its width kept

constant, as can be seen from Table II. Similarly Table III shows that when the input width is changed and the mass held constant, the width of the resonance is affected considerably more than the mass.

These results can be most easily understood by looking at the behavior of the largest eigenvalue, as a function of energy, as shown in Fig. 4. The kernel has an accumulation point of eigenvalues at the origin, therefore only the few largest eigenvalues are important. In the present case the largest one effectively exhausts the trace of the kernel, and it is the only one which need be considered in looking for resonances. It is known[7] that the analytic structure of the Fredholm determinant consists only of the various right hand cuts. Furthermore the Fredholm determinant is related to the eigenvalues by

$$\Delta(E) = \prod_i [1 - \lambda_i(E)] , \qquad (4:1)$$

where E is the energy. We therefore expect the largest eigenvalue to have roughly the same analytic structure as the determinant.

The eigenvalues are known to have the three-body cut. Figure 4, where the eigenvalue is evaluated above the real axis, exhibits this property quite clearly. On the real axis below threshold the computed eigenvalues were real, as they should be in this case.

Furthermore, Fig. 4 shows a typical dispersion form which suggests strongly that the dominant influence comes from the complex $\pi - \rho$ normal threshold.

The absence of a low-energy resonance is due to the large imaginary part of the eigenvalue in the vicinity of the $\pi - \rho$ threshold. When one changes the form factor, the eigenvalue curve

retains the same general shape. More importantly, the curves are essentially identical in the high-energy region, which explains the previously mentioned independence of our results from the form factors. The effect of changing the input mass and width of the ρ can be qualitatively understood by the assumption that the location of the $\pi - \rho$ normal threshold determines the effective shape of the curve.

The calculation has also been carried out with a two-body D function with a linear asymptotic form, as described in I . The resulting eigenvalue then depends on the free parameter a^2 occurring in the form factor. The results were completely different; they were extremely dependent on the parameter and on the form factor chosen. In order to obtain a resonance, a very large value of the parameter was needed. The reason for this is that the eigenvalue curve was completely different from that of Fig. 4. The real and imaginary part of the eigenvalue were practically constant and decreased very slowly as the energy increased. Further the real part could be made to pass through one at any energy by adjusting the parameter.

The accuracy of the eigenvalues was better than one part in a thousand everywhere of interest, and matrices of order 25×25 were sufficient for this accuracy. The accuracy was checked by increasing the matrix size up to 35×35 . The high order of accuracy obtained for the relatively small matrix size is due to the use of Gaussian integration techniques.[8] The accuracy decreases as the energy increases, but, for example, at an energy of $5m_\pi$ a matrix 15×15 gave an error of less than one part in 10^5 for the largest eigenvalue.

V. CONCLUSION

The first comment that we can make on our results is that they do not suggest the dynamical origin of the ω. We regard the eigenvalue curve, Fig. 4, as being the essential representation of our model. Its most salient feature is the strong influence of the π-ρ normal threshold, which in particular produces a large imaginary part of the eigenvalue in the region around this threshold. It is therefore illusory to hope that the resonance obtained represents the ω, and that its large mass and width are due to the crudeness of the approximations.

However, it has been pointed out that the ω seems strongly coupled to π - ρ,[9] and should appear as a three-pion resonance.[10] It may still be possible to get the ω, in additon to the resonance found here, in a Faddeev-type approach by refining the approximations to the two-body amplitude. In fact, as was mentioned in I, when we look for the ω the region where we need to know the two-body amplitude lies well below the pole of the ρ. In this region the separable approximation has no reason to be valid and the exact off-shell dependence has a greater importance than at higher energies.

On the other hand, as mentioned in I, the Faddeev approach has several defects in relativistic calculations. Certain classes of diagrams are omitted, crossing is ignored, and inelastic effects are not considered. If any of these are important in the mechanism which produces the ω, a Faddeev-type approach may not be able to predict it at all.

It is clear by now that our model is a restricted version of the Faddeev approach to the three-body problem. It exhibits exactly three-body elastic unitarity and relativistic invariance. Because of our approximation to the two-body amplitude, the model has its best chance to give sensible results in energy regions where the ρ is physical and the third pion sufficiently energetic to interact strongly with it. Thus we are treating systems of three pions not just interacting but resonating by pairs. We feel that our model is in fact probably meaningful in this energy region because of the independence of the calculation from the form factor. This is essentially equivalent to the statement that in this energy range the two-body left-hand singularities, which the form factors represent, are important only insofar as they produce the ρ. Thus the three-body amplitude in that energy region seems dominated by the $\pi - \rho$ normal threshold.

Since the spirit of this calculation has been to look for results independent of the form factors, we regard as probably unreliable any predictions in the bound-state region.

As to the resonance that we have found, we believe that even an "exact" two-body off-shell amplitude would still give it, since the two-body amplitude must exhibit the pole of the ρ. A not very well established experimental resonance, the E meson,[11] does exist with the quantum numbers $I^G = 0^-$, $J^P = 1^-$. Its mass is 1415 ± 15 MeV and its width is 70 ± 15 MeV. It has been seen in the $K\bar{K}\pi$ system, dominated by $K^* \bar{K}$. The agreement with our resonance is striking, and we feel that the fact that it has not been observed in a $\pi^+\pi^-\pi^0$

system may very well be due to the difficulty of seeing such a state at that energy.

Obviously the best test of our model is to examine other channels, for example those of the A_1 and A_2 mesons,[12] which are relatively well established.[11] These mesons have been seen in three-pion states dominated by $\pi - \rho$. Work is now in progress on these calculations.

ACKNOWLEDGMENTS

We are greatly indebted to Dr. V. A. Alessandrini and Dr. R. L. Omnes for their constant interest in this work and for many enlightening discussions. We are grateful to Dr. I. T. Drummond for several crucial remarks.

We also wish to thank Dr. D. L. Judd for his hospitality at the theoretical group of the Lawrence Radiation Laboratory.

REFERENCES AND FOOTNOTES

* This work was done under the auspices of the United States Atomic Energy Commission.

+ On leave of absence from Laboratoire de Physique Théorique at Hautes Energies, Orsay, France. Work supported in part by a NATO fellowship. Address after October 1, 1965: Centre de Recherches Nucléaires, Strasbourg-Cronenbourg, Bas-Rhin, France.

‡ Present address: University of Pittsburgh, Pittsburgh, Pennsylvania.

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6. Here it must be noted that our results disagree with various earlier calculations, for example A. N. Mitra, Phys. Rev. 127, 1342 (1962) and A. Ahmadzadeh, Lawrence Radiation Laboratory Report UCRL-11749, Nov. 1964. Among other things, these calculations took insufficient account of the difficulties caused by incorrect analyticity.

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10. G. F. Chew, Phys. Rev. Letters 4, 142 (1960).

11. See, for instance, G. Goldhaber, Second Corral Gables Conference Symmetry Principles at High Energy (W. H. Freeman and Co., San Francisco, 1965), p. 34.

12. There is some evidence that the A_1 is a kinematic enhancement and not a resonance. Therefore it might not appear in our model.

Table I. Mass and width in MeV of the resonance vs the form factors, for the actual ρ mass (763 MeV) and width (106 MeV).

$g(p)$	$p/(p^2+a^2)^{3/2}$	$p/(p^2+a^2)^{2}$	$p/(p^2+a^2)^{5/2}$
M	1435	1465	1470
Γ	50	55	60

Table II. Mass and width in MeV of the resonance vs the ρ mass, with the ρ width at its actual value.

m_ρ	560	763	840
M	1090	1470	1600
Γ	80	60	50

Table III. Mass and width in MeV of the resonance vs the ρ width, with the ρ mass at its actual value.

Γ_ρ	40	106	280
M	1425	1470	1500
Γ	25	60	130

FIGURE CAPTIONS

Figure 1. Plot of the absolute value of $1/D(\sigma)$ versus σ in m_π^2, the square of the two-body c.m. energy. $D(\sigma)$ the two-body D function is normalized to one at infinity.

Figure 2. Locus of the logarithmic singularities of the kernel in the complex y - plane for $z = (10 + i\ 0.4)m_\pi$. The point A is the location of the singularities for $x = 1$; as x increases to ∞ the singularity F_- moves to B and the singularity F_+ goes to $-\infty$. The heavy line from 1 to ∞ is the path over which the kernel is integrated. The energy is expressed in pion masses.

Figure 3. Plot of the inverse absolute square of the three-body Fredholm determinant versus Re $(s^{\frac{1}{2}})$, for $\mathrm{Im}(s^{\frac{1}{2}}) = 0.4$. The energy is expressed in pion masses.

Figure 4. Plot of the largest eigenvalue versus Re $(s^{\frac{1}{2}})$ in pion masses for $\mathrm{Im}(s^{\frac{1}{2}}) = 0.4\ m_\pi$. The dashed curve is the real part; the solid curve the imaginary part. If $\mathrm{Im}(s^{\frac{1}{2}}) = 0$ the eigenvalue would be strictly real below threshold, $s^{\frac{1}{2}} = 3$, as the kernel is hermitian.

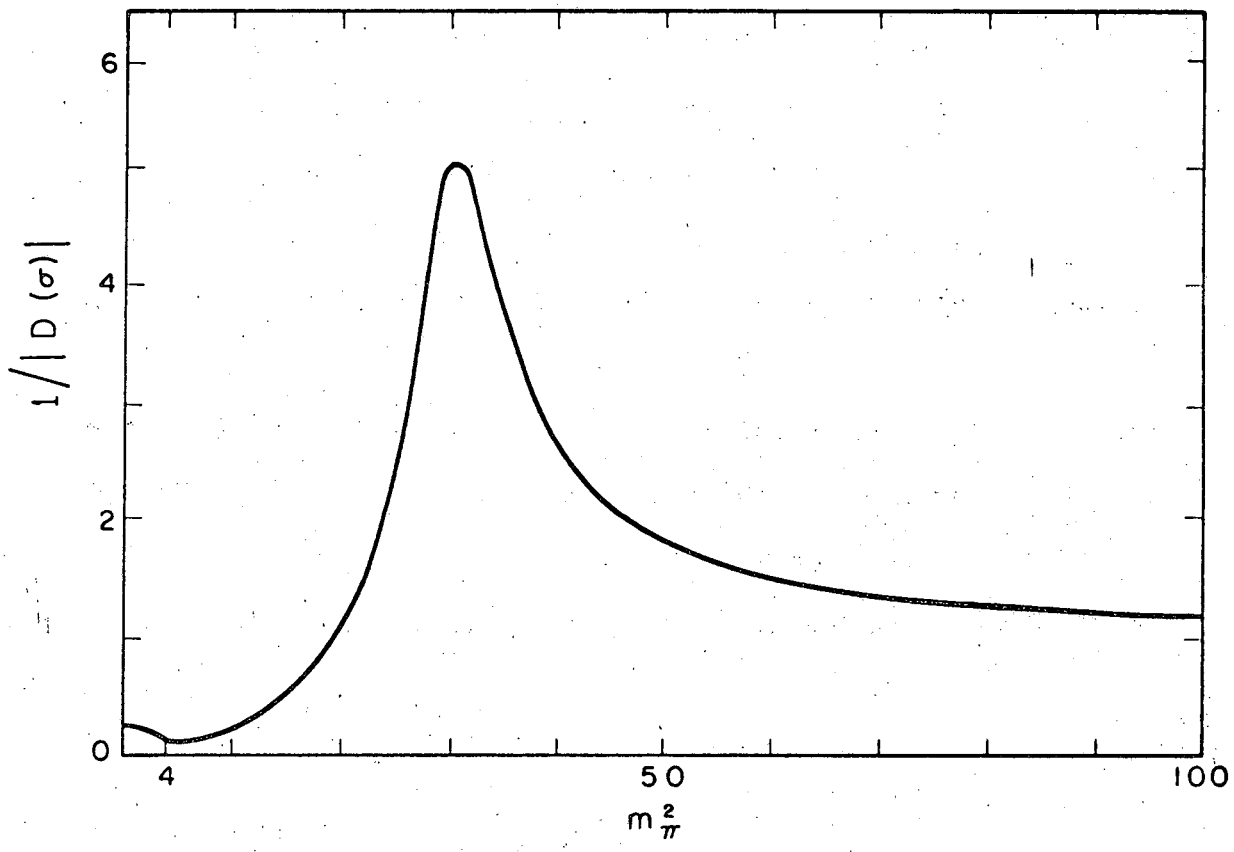

Fig. 1

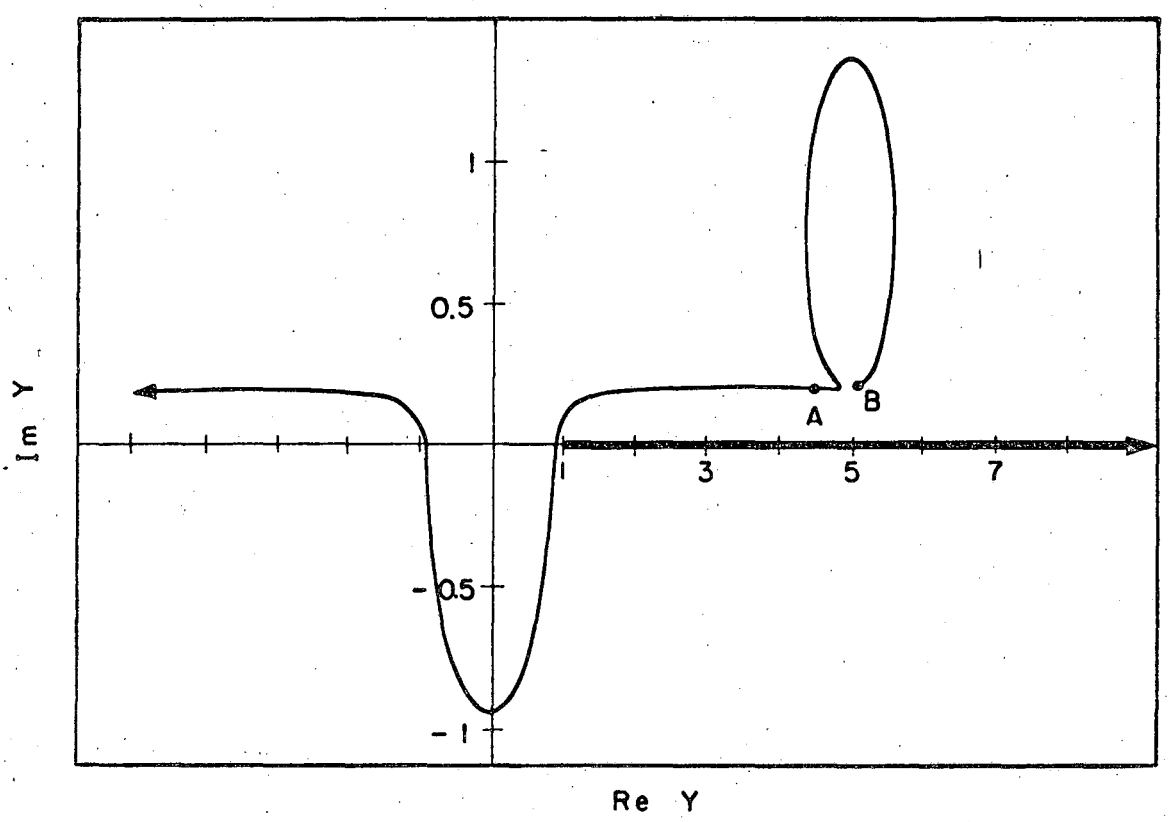

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Fig. 2

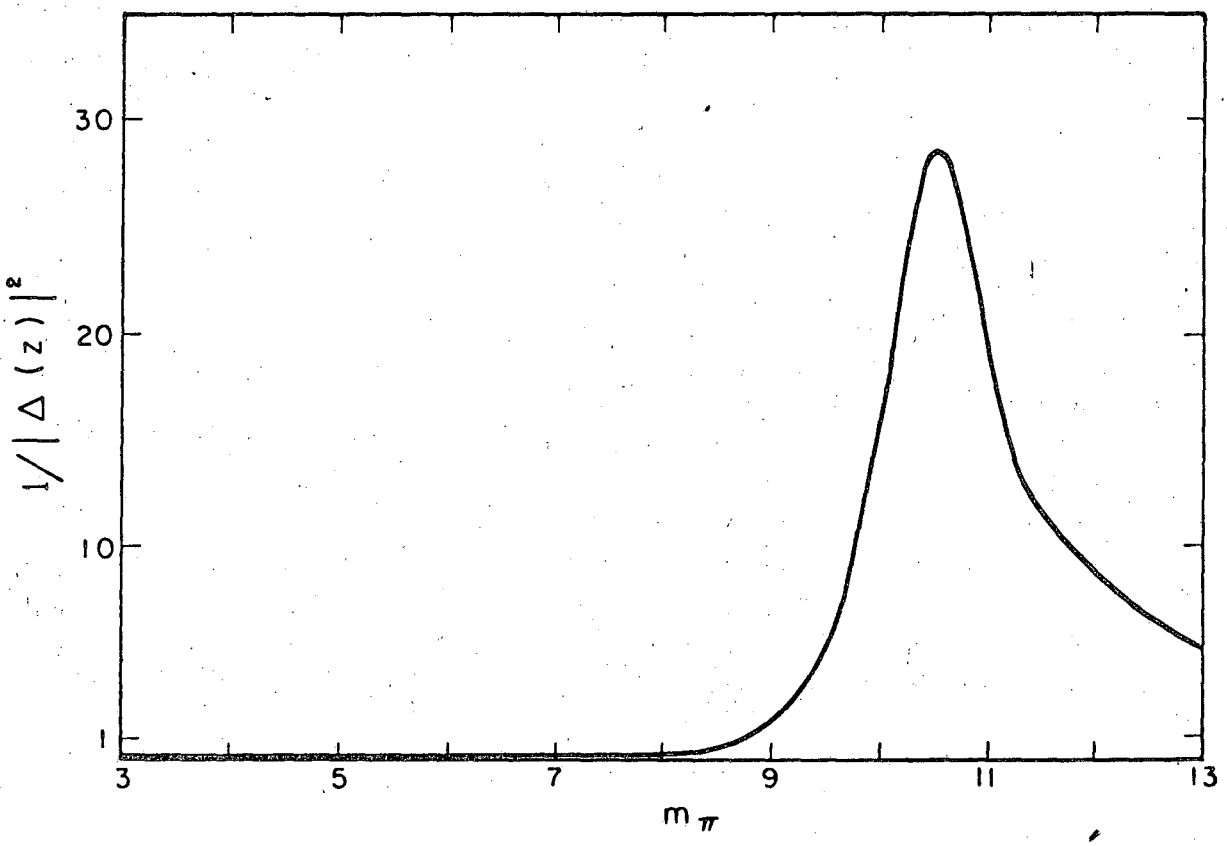

Fig. 3

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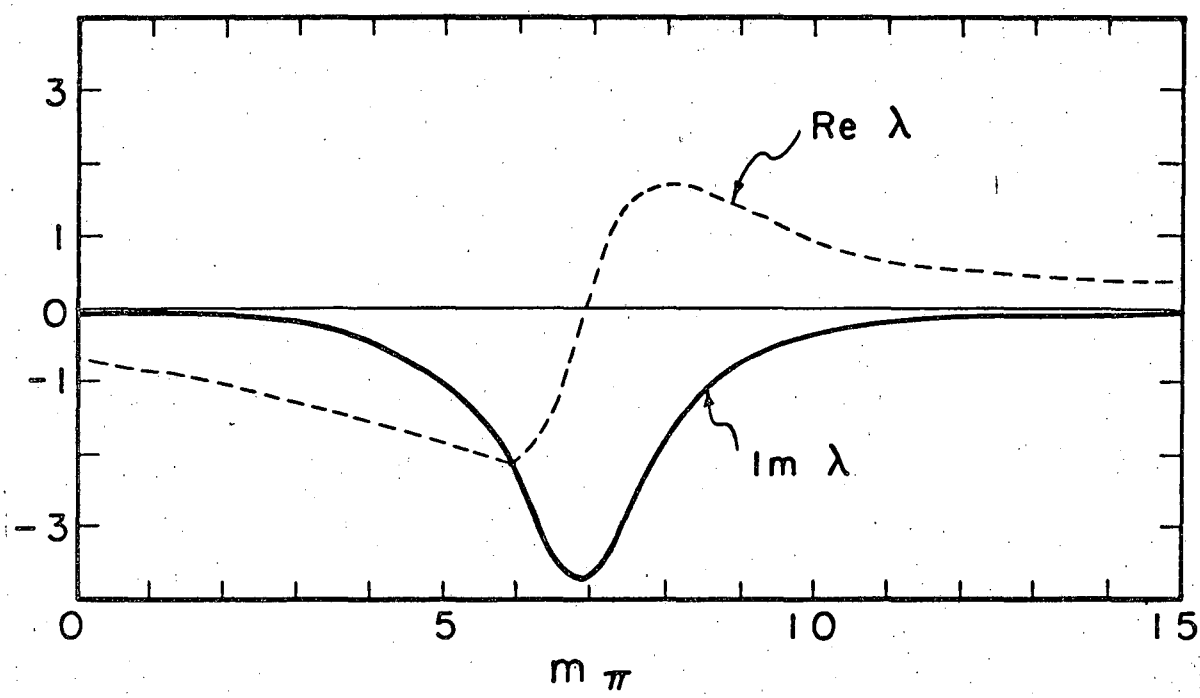

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Fig. 4