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Complexity of the Stable Marriage and Stable Roommate Problems in Three Dimensions

Cheng Ng and Daniel S. Hirschberg

Technical Report No. 88–28

Abstract

The stable marriage problem is a matching problem that pairs members of two sets. The objective is to achieve a matching that satisfies all participants based on their preferences. The stable roommate problem is a variant involving only one set, which is partitioned into pairs with a similar objective. There exist asymptotically optimal algorithms that solve both problems.

In this paper, we investigate the complexity of three dimensional extensions of these problems. This is one of twelve research directions suggested by Knuth in his book on the stable marriage problem. We show that these problems are $\mathcal{NP}$-complete, and hence it is unlikely that there exist efficient algorithms for their solutions.

Applying the polynomial transformation developed in this paper, we extend the $\mathcal{NP}$-completeness result to include the problem of matching couples—who are both medical school graduates—to pairs of hospital resident positions. This problem is important in practice and is dealt with annually by NRMP, the centralized program that matches all medical school graduates in the United States to available resident positions.

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Complexity of the Stable Marriage and Stable Roommate Problems in Three Dimensions

Introduction

Consider the problem of assigning $3n$ students to n disjoint work groups of three students each. The students must guard against any three individuals abandoning their assignments and instead conspiring to form a new group that they consider more desirable.

The following procedure is followed: each student ranks all $(3n-1)(3n-2)$ possible pairs of fellow students according to her preference for working with the pairs. A *destabilizing triple* for an assignment M consists of three students such that each ranks the remaining two (as a pair) more desirable than the pair that she is assigned to in M. The students' task, the 3-*person stable assignment problem* (or 3PSA for short), is to find a *stable assignment*, one that is free of all destabilizing triples, if such an assignment exists.

Readers will recognize that 3PSA is a three dimensional generalization of the *stable roommate problem*, which partitions $2n$ persons into n pairs of stable roommates. A better known variation is the *stable marriage problem*, which divides the participants into two disjoint sets, male and female. Each pair in a stable marriage must include a male and a female. The stable marriage problem has a similar generalization in three dimensions, which we name the 3-*gender stable marriage problem* (or 3GSM for short) and define in the next section.

The stable roommate and stable marriage problems have been studied extensively. There exist efficient algorithms for both problems that run in $O(n^2)$ time [GS62] [MW71] [Ir85]. Ng and Hirschberg have obtained lower bound results proving that these algorithms are asymptotically optimal [NH88]. Since no significant improvement is possible on the original problems, it is then natural to consider their three dimensional generalizations, 3GSM and 3PSA. This is one of twelve research directions suggested by Knuth in his treatise on the stable marriage problem [Kn76].

In this paper, we show that both 3GSM and 3PSA are $\mathcal{NP}$-complete. Hence, it is unlikely that fast algorithms exist for these problems. We then apply techniques developed in the proofs to two problems dealing with the task of matching married couples to jobs, showing that these problems are also $\mathcal{NP}$-complete. Before proceeding, we first give some preliminary definitions.

Definitions

An instance of 3GSM involves three finite sets A, B, and D. These sets have equal cardinality k, which is the *size* of the problem instance. A *marriage* in 3GSM is a complete matching of the three sets, i.e., a subset of $A \times B \times D$ with cardinality k such that each element of A, B, and D appears exactly once.

For each element a of A, we define its *preference*, denoted by $\geq_a$, to be a linear order on the elements of $B \times D$. The intuitive meaning of $(\beta_1, \delta_1) \geq_a (\beta_2, \delta_2)$ is that a prefers (β_1, δ_1) to (β_2, δ_2) in a marriage. For $b \in B$ and $d \in D$, there are also analogous definitions $\geq_b$ and $\geq_d$ on the Cartesian products $A \times D$ and $A \times B$ respectively. When the subscript in the relation is evident from context, we omit it from the $\geq$ notation.

A marriage is *unstable* if there exists a triple $t \in A \times B \times D$ such that t is not in the marriage and each component of t prefers the pair that it is matched with in t to the pair that it is matched with in the actual marriage. A *stable marriage* is a marriage where no such *destabilizing triple* can be found. Formally, a stable marriage is a marriage M, such that, $\forall (a, b, d) \notin M$ and for the triples (a, β_1, δ_1), (α_2, b, δ_2), $(\alpha_3, \beta_3, d) \in M$; either $(\beta_1, \delta_1) \geq_a (b, d)$, $(\alpha_2, \delta_2) \geq_b (a, d)$, or $(\alpha_3, \beta_3) \geq_d (a, b)$.

A 3PSA instance of size n involves a set S of cardinality $n = 3k$, where k is an integer. The *preference* of $s \in S$, denoted $\geq_s$, is a linear order on the set of unordered pairs $\{ (s_1, s_2) \mid s_1 \neq s_2 \text{ and } s_1, s_2 \in S - \{s\} \}$. A *stable assignment* M in 3PSA is a partition of S into k disjoint triplet subsets, such that, $\forall \{s_1, s_2, s_3\} \notin M$ and for the subsets $\{s_1, \sigma_{11}, \sigma_{12}\}$, $\{s_2, \sigma_{21}, \sigma_{22}\}$, $\{s_3, \sigma_{31}, \sigma_{32}\} \in M$; either $(\sigma_{11}, \sigma_{12}) \geq_{s_1} (s_2, s_3)$, $(\sigma_{21}, \sigma_{22}) \geq_{s_2} (s_1, s_3)$, or $(\sigma_{31}, \sigma_{32}) \geq_{s_3} (s_1, s_2)$.

When referring to preferences, we adopt the convention that items are listed in decreasing order of favor. For example, the listing $p_1 \; p_2 \; \ldots \; p_k$, where each p_i denotes a pair, represents the preference $p_1 \geq p_2 \geq \cdots \geq p_k$. We also use the simpler notation xyz to denote the triple (x, y, z) and xy to denote the pair (x, y).

Although 3GSM is similar to its 2-gender counterpart in that an instance can have more than one stable marriage,[1] it differs from the 2-gender counterpart in that there exist instances that have no stable marriage. Figure 1 shows a 3GSM instance with $A = \{\alpha_1, \alpha_2\}$, $B = \{\beta_1, \beta_2\}$ and $D = \{\delta_1, \delta_2\}$. A complete list of all possible marriages, each shown with a corresponding destabilizing triple, confirms that no stable marriage exists for this instance of 3GSM.

α_1	$\beta_1\delta_2$	$\beta_1\delta_1$	$\beta_2\delta_2$	$\beta_2\delta_1$
α_2	$\beta_2\delta_2$	$\beta_1\delta_1$	$\beta_2\delta_1$	$\beta_1\delta_2$
β_1	$\alpha_2\delta_1$	$\alpha_1\delta_2$	$\alpha_1\delta_1$	$\alpha_2\delta_2$
β_2	$\alpha_2\delta_1$	$\alpha_1\delta_1$	$\alpha_2\delta_2$	$\alpha_1\delta_2$
δ_1	$\alpha_1\beta_2$	$\alpha_1\beta_1$	$\alpha_2\beta_1$	$\alpha_2\beta_2$
δ_2	$\alpha_1\beta_1$	$\alpha_2\beta_2$	$\alpha_1\beta_2$	$\alpha_2\beta_1$

Possible Marriage	Destabilizing Triple
$\{\alpha_1\beta_1\delta_1,\ \alpha_2\beta_2\delta_2\}$	$\alpha_1\beta_1\delta_2$
$\{\alpha_1\beta_1\delta_2,\ \alpha_2\beta_2\delta_1\}$	$\alpha_2\beta_1\delta_1$
$\{\alpha_1\beta_2\delta_1,\ \alpha_2\beta_1\delta_2\}$	$\alpha_1\beta_1\delta_2$
$\{\alpha_1\beta_2\delta_2,\ \alpha_2\beta_1\delta_1\}$	$\alpha_2\beta_2\delta_2$

Figure 1. An instance of 3GSM that has no stable marriage.

$\mathcal{NP}$-Completeness of 3GSM

In the previous section, we noted that some instances of 3GSM do not have stable marriages. In this section, we will show that deciding whether a given instance of 3GSM has a stable marriage is an $\mathcal{NP}$-complete problem. This is accomplished by giving a polynomial transformation from the 3-dimensional matching problem (or 3DM for short) to 3GSM. A proof that 3DM is $\mathcal{NP}$-complete is first given in Karp's [KA72] landmark paper.

An instance of 3DM involves three finite sets of equal cardinality—which we denote by A', B', and D', relating them to A, B, and D of 3GSM. Given a set of triples $T' \subseteq A' \times B' \times D'$, the 3DM problem is to decide if there exists an $M' \subseteq T'$

[1] In fact, the number of stable marriages in many instances is exponential in the instances' size. Irving and Leather [IL86] give a proof of this for the 2-gender case. Extending the proof to cover the 3-gender case is straightforward.

such that M' is a complete matching, i.e., each element of A', B', and D' appears exactly once in M'.

Given a 3DM instance I', we construct a corresponding 3GSM instance I. Although our construction can be adapted to work for any 3DM instance in general; we shall assume, in order to simplify the presentation, that no element of A', B', or D' appears in more than three triples of T'. This assumption is made without loss of generality. In their reference work on $\mathcal{NP}$-completeness, Garey and Johnson [GJ79, p 221] mention that 3DM remains $\mathcal{NP}$-complete with this restriction.

We construct I by first building a "frame" consisting of the elements $\alpha_1, \alpha_2 \in A$, $\beta_1, \beta_2 \in B$, and $\delta_1, \delta_2 \in D$. The preferences of these elements do not depend on the structure of I' and are displayed in Figure 2. In Figure 2 and subsequent figures, we are only interested in the roles played by a few items in each preference list. Therefore, we use the notation Π_{Rem} to denote any fixed but arbitrary permutation of the remaining items.

α_1	$\beta_1\delta_1$	$\beta_2\delta_1$	$\beta_1\delta_2$	$\ldots$	Π_{Rem}	$\ldots$	
α_2	$\beta_2\delta_2$			$\ldots$	Π_{Rem}	$\ldots$	
$\vdots$							
β_1	$\alpha_1\delta_2$			$\ldots$	Π_{Rem}	$\ldots$	$\alpha_1\delta_1$
β_2	$\alpha_2\delta_2$	$\alpha_1\delta_1$		$\ldots$	Π_{Rem}	$\ldots$	
$\vdots$							
δ_1	$\alpha_1\beta_2$			$\ldots$	Π_{Rem}	$\ldots$	$\alpha_1\beta_1$
δ_2	$\alpha_1\beta_1$	$\alpha_2\beta_2$		$\ldots$	Π_{Rem}	$\ldots$	
$\vdots$							

Figure 2. Preferences of the elements $\alpha_1, \alpha_2, \beta_1, \beta_2, \delta_1, \delta_2$.

One property of the frame we prove later in Lemma 2 is that the triples $\alpha_1\beta_1\delta_1$ and $\alpha_2\beta_2\delta_2$ must be included in any stable marriage. Note that $\alpha_1\beta_1\delta_1$ is the weakest link in such a marriage because it represents the least preferred match for both β_1 and δ_1. Consequently, if any element $a \in A$ is matched in marriage with a pair that it prefers less than $\beta_1\delta_1$, then $a\beta_1\delta_1$ becomes a destabilizing triple.

The above observation gives us a strategy that uses the pair $\beta_1\delta_1$ as a "boundary" in the preferences of A's remaining elements. A necessary condition for a stable marriage in I is that all remaining elements of A must match with a pair located left of the boundary, i.e., $\geq \beta_1\delta_1$. Using information from T' to construct the set of items to be positioned left of the boundary, we ensure that this condition for stable marriage can be met only if T' contains a complete matching. The remaining difficulty is to ensure that matching all elements of A left of the boundary is sufficient to yield a stable marriage. Before giving details of the construction that provides the solution, we first prove the lemmas that establish the frame's properties.

Lemma 1:
If a stable marriage M exists for I constructed by extending the frame in Figure 2, then $\alpha_1\beta_2\delta_1 \notin M$.

Proof: By contradiction. Suppose $\alpha_1\beta_2\delta_1 \in M$. Since $\alpha_1\beta_2\delta_1 \in M$, δ_2's match cannot be $\alpha_1\beta_1$ or $\alpha_2\beta_2$. From δ_2's preference, $\alpha_1\beta_1$ is the only pair $\geq_{\delta_2} \alpha_2\beta_2$. Therefore, $\alpha_2\beta_2 \geq_{\delta_2} \delta_2$'s match in M. Moreover, $\beta_2\delta_2$ and $\alpha_2\delta_2$ are the first preference choices of α_2 and β_2 respectively. Hence, $\alpha_2\beta_2\delta_2$ is a destabilizing triple for M, a contradiction. ∎

Lemma 2:
If a stable marriage M exists for I constructed by extending the frame in Figure 2, then $\alpha_1\beta_1\delta_1 \in M$ and $\alpha_2\beta_2\delta_2 \in M$.

Proof: We first prove $\alpha_1\beta_1\delta_1 \in M$. Suppose β_1 is not matched with $\alpha_1\delta_1$ in M, we can then find a destabilizing triple for M. There are two cases:

Case 1: β_1 is matched with $\alpha_1\delta_2$. $\alpha_1\beta_1\delta_2 \in M$ implies that $\alpha_2\beta_2\delta_2$, $\alpha_1\beta_1\delta_1$, and $\alpha_1\beta_2\delta_1 \notin M$. By an argument similar to that of Lemma 1, $\alpha_1\beta_2\delta_1$ is a destabilizing triple.

Case 2: β_1 is not matched with $\alpha_1\delta_2$ nor $\alpha_1\delta_1$. $\alpha_1\beta_2\delta_1 \notin M$ by Lemma 1. $\alpha_1\beta_1\delta_1$ is also $\notin M$, which implies that $\alpha_1\beta_1\delta_2$ is a destabilizing triple in this case.

Hence, we conclude that $\alpha_1\beta_1\delta_1 \in M$, which implies that $\alpha_1\beta_1\delta_2 \notin M$. It is now easy to verify that if $\alpha_2\beta_2\delta_2 \notin M$, then it is a destabilizing triple. ∎

If the sets of I' (A', B', and D') each has k elements, then the sets of I (A, B, and D) each has $3k+2$ elements. The α's, β's, or δ's, which are in the frame, account for two elements. The remaining $3k$ elements are defined as follows.

Suppose $A' = \{a_1, a_2, \ldots, a_k\}$, $B' = \{b_1, b_2, \ldots, b_k\}$, and $D' = \{d_1, d_2, \ldots, d_k\}$. According to an earlier assumption, each element $a_i \in A'$ appears in no more than three triples of T'. We clone three copies of a_i and replace a_i with the clones $a_i[1]$, $a_i[2]$, and $a_i[3]$ in A. These clones' preferences are set up to make it possible for **exactly one** of their matches in a stable marriage to correspond to a triple in T'.

To prevent the two remaining clones from interfering with the above setup, we add elements w_{a_i},y_{a_i} to B and x_{a_i},z_{a_i} to D. In a stable marriage, the pairs $w_{a_i}x_{a_i}$ and $y_{a_i}z_{a_i}$ are required to match with two of a_i's clones, putting them out of action. We complete the sets B and D by adding to them the elements of B' and D' respectively. To summarize, $A = \{\alpha_1, \alpha_2\} \cup \bigcup_{a_i \in A'} \{a_i[1], a_i[2], a_i[3]\}$, $B = B' \cup \{\beta_1, \beta_2\} \cup \bigcup_{a_i \in A'} \{w_{a_i}, y_{a_i}\}$, and $D = D' \cup \{\delta_1, \delta_2\} \cup \bigcup_{a_i \in A'} \{x_{a_i}, z_{a_i}\}$.

The preferences that achieve our objectives are shown in Figure 3. In this particular instance, we have assumed that $a_i b_{j_1} d_{l_1}$, $a_i b_{j_2} d_{l_2}$, and $a_i b_{j_3} d_{l_3}$ are the triples containing a_i in T'. When there exists fewer than three such triples, we equate two or more of the j's and l's.

The following lemma establishes the roles of w_{a_i}, x_{a_i}, y_{a_i} and z_{a_i}.

Lemma 3:
If a stable marriage M exists for I constructed with the preferences shown in Figure 3, then for every $a_i \in A'$, there exists $j_1, j_2 \in \{1,2,3\}$, $j_1 \neq j_2$ such that

a) $a_i[j_1]\, w_{a_i} x_{a_i} \in M$, and

b) $a_i[j_2]\, y_{a_i} z_{a_i} \in M$.

Proof: Consider the triple $a_i[1]\, w_{a_i} x_{a_i}$, which represents the third preference choice of x_{a_i} and the first preference choices of $a_i[1]$ and w_{a_i}. It becomes a destabilizing triple unless x_{a_i} is matched with one of its first three preference choices, proving part (a) of the lemma.

α_1						
α_2						
$\vdots$						
$a_i[1]$	$w_{a_i} x_{a_i}$	$y_{a_i} z_{a_i}$	$b_{j_1} d_{l_1}$	$\beta_1 \delta_1$	$\ldots$	Π_{Rem}
$a_i[2]$	$w_{a_i} x_{a_i}$	$y_{a_i} z_{a_i}$	$b_{j_2} d_{l_2}$	$\beta_1 \delta_1$	$\ldots$	Π_{Rem}
$a_i[3]$	$w_{a_i} x_{a_i}$	$y_{a_i} z_{a_i}$	$b_{j_3} d_{l_3}$	$\beta_1 \delta_1$	$\ldots$	Π_{Rem}
$\vdots$						
β_1						
β_2						
$\vdots$						
w_{a_i}	$a_i[1]\, x_{a_i}$	$a_i[2]\, x_{a_i}$	$a_i[3]\, x_{a_i}$	$\ldots$	Π_{Rem}	
y_{a_i}	$a_i[1]\, z_{a_i}$	$a_i[2]\, z_{a_i}$	$a_i[3]\, z_{a_i}$	$\ldots$	Π_{Rem}	
$\vdots$						
b_i	$\ldots$	Π_{Rem}				
$\vdots$						
δ_1						
δ_2						
$\vdots$						
x_{a_i}	$a_i[3]\, w_{a_i}$	$a_i[2]\, w_{a_i}$	$a_i[1]\, w_{a_i}$	$\ldots$	Π_{Rem}	
z_{a_i}	$a_i[3]\, y_{a_i}$	$a_i[2]\, y_{a_i}$	$a_i[1]\, y_{a_i}$	$\ldots$	Π_{Rem}	
$\vdots$						
d_i	$\ldots$	Π_{Rem}				
$\vdots$						

Figure 3. Preferences in the 3GSM instance I. The column of $\beta_1\delta_1$'s represents the boundary. Preferences of α's, β's, and δ's are those shown in Figure 2.

Similarly, z_{a_i} must be matched with one of its first three preference choices. Otherwise, $y_{a_i} z_{a_i}$ forms a destabilizing triple with $a_i[1]$ or $a_i[2]$, depending on which a_i clone is matched in part (a). ∎

We are now ready to prove the $\mathcal{NP}$-completeness of 3GSM by showing that I has a stable marriage if and only if T' has a complete matching of I'.

Theorem 1:
If T' contains a complete matching M' of the 3DM instance I', then the constructed 3GSM instance I has a stable marriage M.

Proof: We show that it is possible to construct a stable marriage M. Begin by adding $\alpha_1\beta_1\delta_1$ and $\alpha_2\beta_2\delta_2$ to M.

For each element $a_i \in A'$, the only triples in T' containing a_i are $a_ib_{j_1}d_{l_1}$, $a_ib_{j_2}d_{l_2}$, and $a_ib_{j_3}d_{l_3}$ using the notations found in Figure 3. One of these triples is in M'.

$$\text{Add to } M \begin{cases} a_i[1]\, b_{j_1}d_{l_1}, & a_i[2]\, w_{a_i}x_{a_i}, \quad \text{and} \quad a_i[3]\, y_{a_i}z_{a_i} \qquad \text{if } a_ib_{j_1}d_{l_1} \in M'; \\ a_i[1]\, w_{a_i}x_{a_i}, & a_i[2]\, b_{j_2}d_{l_2}, \quad \text{and} \quad a_i[3]\, y_{a_i}z_{a_i} \qquad \text{if } a_ib_{j_2}d_{l_2} \in M'; \\ a_i[1]\, w_{a_i}x_{a_i}, & a_i[2]\, y_{a_i}z_{a_i}, \quad \text{and} \quad a_i[3]\, b_{j_3}d_{l_3} \qquad \text{if } a_ib_{j_3}d_{l_3} \in M'. \end{cases}$$

Since M' is a complete matching, the above construction guarantees that those elements of B and D that originate from B' and D' are used exactly once in M. It is easy to verify that all other elements of A, B, and D are also used exactly once. Hence, M is a marriage.

To show that M is stable, it is sufficient to show that no element of A is a component of a destabilizing triple. α_1 and α_2 satisfy this condition immediately because they are matched with their first preference choices.

Referring to Figure 3, each of the remaining elements of A is matched with a pair located left of the boundary. Hence, the only pairs that can form destabilizing triples are $w_{a_i}x_{a_i}$ and $y_{a_i}z_{a_i}$. However, w_{a_i}'s (y_{a_i}'s) match is one of its first three preference choices. These three choices are in exact reverse order of x_{a_i}'s (z_{a_i}'s). This eliminates w_{a_i} and y_{a_i} from participating in any destabilizing triple. ∎

Theorem 2:
If the 3GSM instance I has a stable marriage, then T' contains a complete matching of the 3DM instance I'.

Proof: Suppose I has a stable marriage M. Lemma 2 requires M to include $\alpha_1\beta_1\delta_1$ and $\alpha_2\beta_2\delta_2$. Lemma 3 requires that, for each $a_i \in A'$, two of the a_i clones match with $w_{a_i}x_{a_i}$ and $y_{a_i}z_{a_i}$. Let M' represent the matching that results when M is restricted to the remaining elements that are without predetermined matches.

For each $a_i \in A'$, only one a_i clone remains to be matched in M'. Therefore, we shall drop the distinction between an a_i clone and the a_i it represents, without the risk of introducing any ambiguity in M'. The elements that participate in M' can then be characterized as exactly those elements of A', B', and D'. Since M' is a subset of a marriage, it represents a complete matching.

Due to the absence of destabilizing triples, every a_i in M' must match with a preference choice located left of the boundary. The construction of I, as illustrated in Figure 3, restricts this choice to the third item in the preference list since the first two items are already matched. Moreover, the triple formed by a_i and this item is contained in T'. Hence, every triple in M' is also a triple in T' and M' is the desired complete matching contained in T'. ∎

Theorem 3:
3GSM is $\mathcal{NP}$-complete.

Proof: It is easy to verify that the construction of I from I' can be accomplished within a polynomial time bound. Therefore, Theorems 1 and 2 establish that 3GSM is $\mathcal{NP}$-hard. It is also possible to check the stability of a given marriage in polynomial time, establishing 3GSM's membership in $\mathcal{NP}$. ∎

$\mathcal{NP}$-Completeness of 3PSA

The $\mathcal{NP}$-completeness of 3PSA follows from that of 3GSM because the former is a generalization of the latter. Given a 3GSM instance I where $A = \{a_1, a_2, \dots, a_k\}$, $B = \{b_1, b_2, \dots, b_k\}$, and $D = \{d_1, d_2, \dots, d_k\}$; we can extend it into a 3PSA instance $\hat{I}$ by defining $S = A \cup B \cup D$. Each element of S retains its entire preference list from I as the first k^2 preference items in $\hat{I}$. We refer to these k^2 items as *inherited items*. All remaining items are inconsequential in $\hat{I}$ and are arranged in fixed but arbitrary permutations following the inherited items. The result is illustrated in Figure 4.

Theorem 4:
3PSA is $\mathcal{NP}$-complete.

Proof: Any stable marriage M in I is an assignment in $\hat{I}$. Any destabilizing triple for M in I is simultaneously a destabilizing triple for M in $\hat{I}$. Therefore, the stability of M in I implies its stability in $\hat{I}$.

We claim that any stable assignment $\hat{M}$ in $\hat{I}$ involves only inherited items and is therefore a marriage in I. This is equivalent to claiming that $\hat{M}$ is a complete

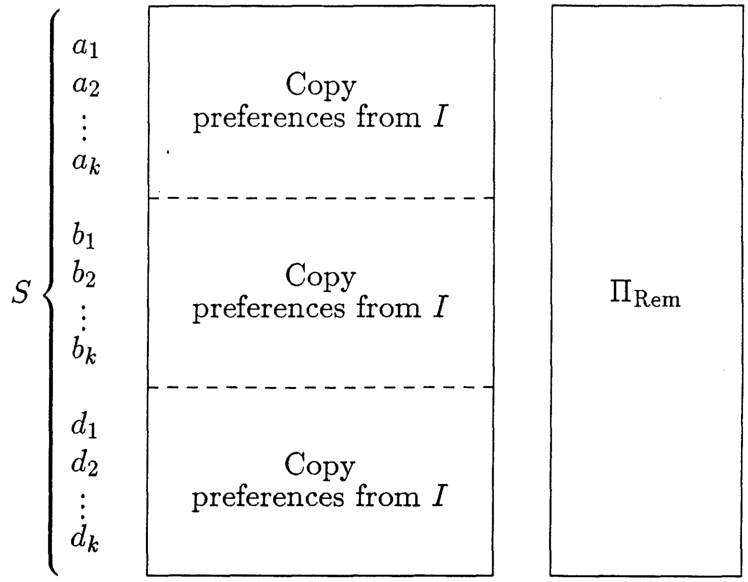

Figure 4. Preferences in the 3PSA instance $\hat{I}$.

matching of $A \times B \times D$. Otherwise, there exist elements $a_i \in A, b_j \in B, d_l \in D$ not matched to inherited items, which implies that $a_i b_j d_l$ is a destabilizing triple.

Since $\hat{M}$ involves only inherited items, any destabilizing triple for $\hat{M}$ in $\hat{I}$ is simultaneously a destabilizing triple for $\hat{M}$ in I. Therefore, the stability of $\hat{M}$ in $\hat{I}$ implies its stability in I. ∎

Applications

In addition to the interest generated amongst computer scientists, the stable marriage problem has also received substantial attention from game theorists. It is used to model economic problems that require matching representatives from different market forces, such as matching labor to the job market. Since 1951, an algorithm that solves the stable marriage problem has been the basis for the success of NRMP, the centralized national program that matches medical school graduates to hospital resident positions [Ro84].

In recent years, NRMP administrators have recognized that, increasingly, medical schools are graduating married couples in the same year. In 1983, they instituted a "couples program" which allows a participating couple to increase the probability of their being matched with two positions in close proximity. To participate in this special program, a couple submits a combined preference list that ranks pairs of resident positions.

We now apply the analysis of this paper to two problems involving dual-career couples, one of which models exactly the NRMP matching program involving couples.

Our first problem, the dual-career job matching problem (or DCJM for short), involves k couples seeking careers in two disjoint job markets. The couples are represented by the set $A = \{ a_i \mid 1 \leq i \leq k \}$. Two sets of k employers, each with a single job offer, are represented by $B = \{ b_i \mid 1 \leq i \leq k \}$ and $D = \{ d_i \mid 1 \leq i \leq k \}$. We retain the $\geq$ notation for preferences. In this case, a_i's preference remains a linear order on the elements of $B \times D$. However, b_i's and d_i's preferences are linear orders involving only elements of A. This simplification is due to the absence of direct interaction between B and D; for each couple, employers in B offer jobs only to one member while employers in D offer jobs only to the other member.

A *job assignment* M in DCJM is a complete matching of the sets A, B, and D. A *destabilizing triple* for M is a triple $abd \notin M$ such that, a's employers in M are $\beta_1\delta_1$, b employs α_1, d employs α_2, and the conditions (i) $bd \geq_a \beta_1\delta_1$, (ii) $a \geq_b \alpha_1$, and (iii) $a \geq_d \alpha_2$ are satisfied. Note that $\geq_b$ or $\geq_d$ may represent equality when $a = \alpha_1$ (which implies $b = \beta_1$) or $a = \alpha_2$ ($d = \delta_1$), but not both. A *stable job assignment* is a job assignment in which no destabilizing triple exists.

We now prove that deciding whether an instance of DCJM has a stable job assignment is $\mathcal{NP}$-complete. The proof mirrors that for 3GSM—by giving a polynomial transformation of a 3DM instance I' to a DCJM instance I. We begin by constructing a new frame tailored to the current problem, which requires three new elements for each of the sets A, B, and D. Their preferences are displayed in Figure 5.

In Lemma 2, we actually proved a stronger result than is required for subsequent uses of the lemma. For the DCJM frame, the only property we need is that any stable job assignment must contain $\alpha_1\beta_1\delta_1$. This property is sufficient to validate, in the preferences of A's remaining elements, that $\beta_1\delta_1$ is the boundary separating items that yield stable matches from those that yield unstable matches.

α_1	$\beta_1\delta_1$	$\beta_3\delta_2$	$\beta_2\delta_3$	$\ldots$	Π_{Rem}	$\ldots$	
α_2	$\beta_2\delta_2$	$\beta_3\delta_3$		$\ldots$	Π_{Rem}	$\ldots$	
α_3	$\beta_3\delta_3$			$\ldots$	Π_{Rem}	$\ldots$	
$\vdots$							
β_1	$\ldots$			$\ldots$	Π_{Rem}	$\ldots$	α_1
β_2	α_1	α_2		$\ldots$	Π_{Rem}	$\ldots$	
β_3	α_1	α_2		$\ldots$	Π_{Rem}	$\ldots$	
$\vdots$							
δ_1	$\ldots$			$\ldots$	Π_{Rem}	$\ldots$	α_1
δ_2	α_2	α_1		$\ldots$	Π_{Rem}	$\ldots$	
δ_3	α_2	α_1		$\ldots$	Π_{Rem}	$\ldots$	
$\vdots$							

Figure 5. Preferences of the frame elements for DCJM.

Lemma 4:
If a stable job assignment M exists for I constructed by extending the frame in Figure 5, then $\alpha_1\beta_1\delta_1 \in M$.

Proof: Suppose $\alpha_1\beta_1\delta_1 \notin M$. We prove the following assertions (a)–(d) in succession, resulting in a contradiction.

a) $\alpha_1\beta_3\delta_2 \notin M$. Otherwise, β_2's employee is not α_1. Hence, $\alpha_2\beta_2\delta_2$ is a destabilizing triple.

b) $\alpha_1\beta_2\delta_3 \notin M$. Otherwise, β_2's employee is α_1, which implies that $\alpha_2\beta_2\delta_2 \notin M$ and β_3's employee is not α_1. Hence, $\alpha_2\beta_3\delta_3$ is a destabilizing triple.

c) δ_2's employee is α_2. Otherwise, δ_2's employee is not α_2. Hence, $\alpha_1\beta_3\delta_2$, which is not in M by assertion (a), is a destabilizing triple.

d) δ_2's employee is not α_2. Otherwise, δ_3's employee is not α_2. Hence, $\alpha_1\beta_2\delta_3$, which is not in M by assertion (b), is a destabilizing triple.

Statements (c) and (d) contradict each other. ∎

$$
\begin{array}{l|llllll}
\vdots & & & & & & \\
a_i[1] & w_{a_i}x_{a_i} & y_{a_i}z_{a_i} & b_{j_1}d_{l_1} & \beta_1\delta_1 & \ldots & \Pi_{\text{Rem}} \\
a_i[2] & w_{a_i}x_{a_i} & y_{a_i}z_{a_i} & b_{j_2}d_{l_2} & \beta_1\delta_1 & \ldots & \Pi_{\text{Rem}} \\
a_i[3] & w_{a_i}x_{a_i} & y_{a_i}z_{a_i} & b_{j_3}d_{l_3} & \beta_1\delta_1 & \ldots & \Pi_{\text{Rem}} \\
\vdots & & & & & & \\
\hline
\vdots & & & & & & \\
w_{a_i} & a_i[1] & a_i[2] & a_i[3] & \ldots & \Pi_{\text{Rem}} & \\
y_{a_i} & a_i[1] & a_i[2] & a_i[3] & \ldots & \Pi_{\text{Rem}} & \\
\vdots & & & & & & \\
b_i & \ldots & \Pi_{\text{Rem}} & & & & \\
\vdots & & & & & & \\
\hline
\vdots & & & & & & \\
x_{a_i} & a_i[3] & a_i[2] & a_i[1] & \ldots & \Pi_{\text{Rem}} & \\
z_{a_i} & a_i[3] & a_i[2] & a_i[1] & \ldots & \Pi_{\text{Rem}} & \\
\vdots & & & & & & \\
d_i & \ldots & \Pi_{\text{Rem}} & & & & \\
\vdots & & & & & &
\end{array}
$$

Figure 6. Preferences of the DCJM instance I. The column of $\beta_1\delta_1$'s represents the boundary.

With minor modifications, the construction of 3GSM illustrated in Figure 3 is adapted to work for DCJM. The resulting DCJM instance I, shown in Figure 6, is obtained from Figure 3 by removing the second component in w_{a_i}, x_{a_i}, y_{a_i}, and z_{a_i}'s first three preference choices (only these choices are relevant).

Theorem 5:
DCJM is $\mathcal{NP}$-complete.

Proof: Suppose I has a stable job assignment. We claim a result identical to Lemma 3, although the differences between 3GSM and DCJM require a technical adjustment in the proof. It remains valid that x_{a_i} must match with one of its first three preference choices. In Lemma 3, the w_{a_i} component in these three preference choices places w_{a_i} in x_{a_i}'s match automatically. This is no longer guaranteed in the present case. To patch the proof, we observe that matching an a_i clone with x_{a_i}, but not w_{a_i}, yields a destabilizing triple immediately because the a_i clone is matched with an item located right of the boundary. Hence, one of the three

a_i clones must match with $w_{a_i}x_{a_i}$. Extending the same argument to y_{a_i} and z_{a_i} proves that a second a_i clone must match with $y_{a_i}z_{a_i}$.

The proof in Theorem 2 now follows. For each $a_i \in A'$, the remaining a_i clone must match with its third preference item because this is the only unmatched item located left of the boundary. The job assignment that results from matching the remaining a_i clone, for all $a_i \in A'$, corresponds to a complete matching contained in T'. Hence, if the DCJM instance I has a stable job assignment, then T' contains a complete matching of I'.

Conversely, if T' contains a complete matching, we can find a stable job assignment by applying the technique of Theorem 1 to the DCJM instance I. In this case, $\alpha_1\beta_1\delta_1$, $\alpha_2\beta_2\delta_2$, and $\alpha_3\beta_3\delta_3$ are the assignments in the frame. For each $a_i \in A'$, an a_i clone is matched with a pair in $B \times D$ such that the resulting triple is in T'. We can always find such a pair among those preference items located left of the boundary. The remaining a_i clones are matched with $w_{a_i}x_{a_i}$ and $y_{a_i}z_{a_i}$. The job assignment that results is stable. ▮

We now turn our attention to the problem involving couples such that both members are seeking employment in the same job market. The dual-career single job market matching problem (or DCSJMM for short) models exactly a matching problem handled annually by NRMP since 1983, when it first permitted couples to rank pairs of hospital resident positions.

In a case study on NRMP's role in the medical resident labor market, Roth calls attention to a potentially serious problem with NRMP's "couples program" [Ro84]. Roth gave an example involving four hospitals and two couples and demonstrated that no stable job assignment exists for the given example [Ro84, p 1008]. We will show, from the perspective of computational complexity, that the problem is more serious than Roth had envisioned. Specifically, we show that deciding whether a DCSJMM instance has a stable job assignment is $\mathcal{NP}$-complete. It follows that, even if a DCSJMM instance I has a stable job assignment, it is unlikely an efficient algorithm that finds it exists. Otherwise, we can execute the algorithm on I, and in polynomial time check the outcome for stability,[2] thus deciding if I has a stable job assignment.

[2] The polynomial time bound follows from DCSJMM's membership in $\mathcal{NP}$, a property that is easy to verify.

An instance of DCSJMM involves k couples represented by the set $C = \{ c_i \mid 1 \leq i \leq k \}$ and $2k$ employers, each with a single job offer, represented by the set $S = \{ s_i \mid 1 \leq i \leq 2k \}$. Each $c_i = f_i m_i$, where f_i and m_i are the female and male members of c_i respectively. Preferences of C's elements are linear orders on the set of ordered pairs $\{ s_i s_j \mid i \neq j \text{ and } s_i, s_j \in S \}$. The pair $s_i s_j$ is ordered such that f_i's potential employer is listed first. Preferences of S's elements are linear orders on the set $\{f_1, m_1, f_2, m_2, \ldots, f_k, m_k\}$.

A *job assignment* M is a set of triples $\{ c s_i s_j \mid c = fm \in C;\ s_i, s_j \in S;\ i \neq j \}$ such that each element of C and S appears exactly once in M. Note that $c s_i s_j \in M$ implies that s_i employs f and s_j employs m. A *destabilizing triple* for M is a triple $c s_i s_j \notin M$, $c = fm \in C$; $s_i, s_j \in S$; $i \neq j$; such that, c's employers in M are $\sigma_1 \sigma_2$, s_i employs γ_1, s_j employs γ_2; and the conditions (i) $s_i s_j \geq_c \sigma_1 \sigma_2$, (ii) $f \geq_{s_i} \gamma_1$, and (iii) $m \geq_{s_j} \gamma_2$ are satisfied. A *stable job assignment* is a job assignment in which no destabilizing triple exists.

We apply the same technique used in the $\mathcal{NP}$-completeness proof of 3PSA to prove that DCSJMM is $\mathcal{NP}$-complete. Given a DCJM instance I where $A = \{a_1, a_2, \ldots, a_k\}$, $B = \{b_1, b_2, \ldots, b_k\}$, and $D = \{d_1, d_2, \ldots, d_k\}$; we extend it into the DCSJMM instance $\hat{I}$ shown in Figure 7. In $\hat{I}$, $C = A$ and $S = B \cup D$. The first k^2 preference items of C's elements and the first k preference items of S's elements are inherited from I with the exact ordering retained. For those elements of S that originate from B, an inherited preference item a_i is further changed to f_i to reflect B's strict policy in I of hiring only a couple's female member. Similarly, elements that originate from D modify a_i to m_i. All remaining items are inconsequential in $\hat{I}$ and are arranged in fixed but arbitrary permutations following the inherited items.

Theorem 6:
DCSJMM is $\mathcal{NP}$-complete.

Proof: Any stable job assignment M in I is a job assignment in $\hat{I}$. Any destabilizing triple for M in I is simultaneously a destabilizing triple for M in $\hat{I}$. Therefore, the stability of M in I implies its stability in $\hat{I}$.

We claim that any stable assignment $\hat{M}$ in $\hat{I}$ consists entirely of matches that involve only inherited items. Otherwise, there exists elements $a_i \in A$, $b_j \in B$,

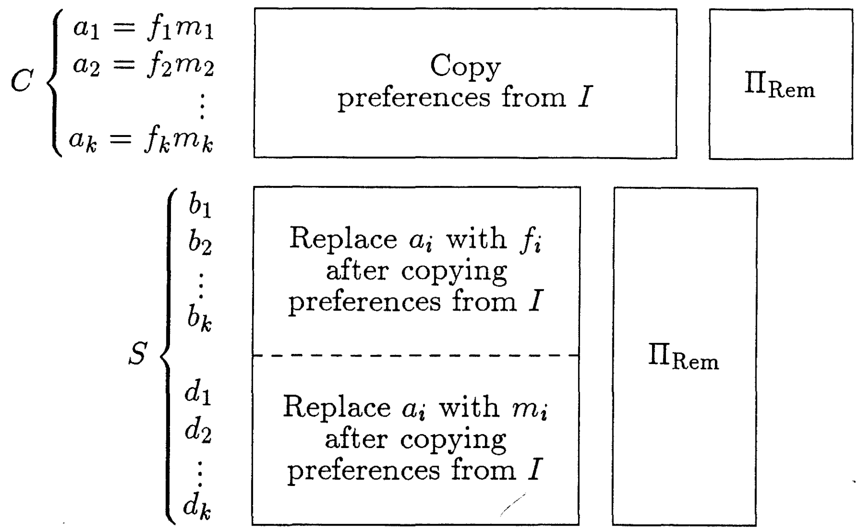

Figure 7. Preferences in the DCSJMM instance $\hat{I}$.

and $d_l \in D$ that are not matched to inherited items in $\hat{M}$. In such cases, it is easy to verify that $a_i b_j d_l$ is a destabilizing triple for $\hat{M}$.

Since $\hat{M}$ involves only inherited items, the stability of $\hat{M}$ in $\hat{I}$ implies its stability in I. ∎

Conclusions

We have shown that three dimensional generalizations of the stable marriage and stable roommate problems are $\mathcal{NP}$-complete. Our result also applies to the problem of finding stable job assignments for dual-career couples. Such a problem is dealt with annually by NRMP, when it assigns couples who are both medical school graduates to hospital resident positions. We show that this assignment problem is $\mathcal{NP}$-complete, and hence we cannot expect NRMP to have an efficient solution.

It may be interesting, as a topic for further research, to investigate the possibility of applying our result to other matching problems and their variants.

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