

# The Stable non-Gaussian Asset Allocation: A Comparison with the Classical Gaussian Approach

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## Abstract

We analyze a multistage stochastic asset allocation problem with decision rules. The uncertainty is modeled using economic scenarios with Gaussian and stable Paretian non-Gaussian innovations. The optimal allocations under these alternative hypothesis are compared. If the agent has very low or very high risk aversibility, then the Gaussian and stable non-Gaussian scenarios result in similar allocations. When the risk aversion of the agent is between these two extreme cases, then the two distributional assumptions result in very different asset allocations. Our calculations suggest that the allocations may be up to 85% different depending on the level of risk aversion of the agent.

*Keywords:* dynamic portfolio optimization, stable distribution, scenario generation.

JEL code: C53, C61, G11

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# 1 Introduction

Strategic investment planning is the allocation of portfolio across broad asset classes such as bonds, stocks, cash and real estate considering the legal and policy constraints facing the institution. Empirical evidence by Culp et al. [14] suggests that asset allocation is the most important factor in determining investment performance.

Most of the early models in this field are either myopic or represent deterministic formulations of multiperiod problems. Myopic models can not capture long-term investment goals in the presence of transaction costs. Investment options with maturities exceeding a single period can not be included. The effects of transaction costs, liquidity considerations are not accurately accounted for. Moreover, these models tend to produce high portfolio turnovers and opportunistic asset trades.

There has been a growing interest in the development of multiperiod stochastic models for asset and liability management (ALM). Kusy and Ziemba [26] developed a multiperiod stochastic linear programming model for Vancouver City Savings Credit Union for a 5-year planning period. Their work suggests that their stochastic ALM model is superior to 5-year deterministic models. Another successful application of multistage stochastic programming is the Russell-Yasuda Kasai model by Carino et al. [10]. The investment strategy suggested by the model resulted in extra income of \$79 million during the first two years of its application (1991 and 1992). An ALM model designed by Mulvey [39] has been implemented by the Pacific Financial Asset Management Company. Boender [4] reported the success of a hybrid simulation/optimization scenario model for ALM of pension funds in the Netherlands. The application of the model to a particular pension fund lead to a reduction of the yearly expected contributions of \$100 million. See Rachev and Tokat [51] for a review of recent advances in ALM.

The ALM models that have gained applicability are based on stochastic programming with or without decision rules. In these models, the future uncertainty is modeled using discrete scenarios. A representative set of scenarios describes the possible future environmental situations facing the institution. There are two scenario generation techniques used in the literature: time series analysis and stochastic differential equations Following Sims [53], Dert [17] used VAR and Boender et al. [5] used VECM to generate economic scenarios for pension plans. Boender et al. creates future price inflation, wage growth, bond return, cash return, equity return, real estate return and nominal GNP growth scenarios. The innovations are assumed to follow normal distribution. Mulvey [40] generates economic scenarios using stochastic differential equations, where the innovations follow Brownian motion.

However, there is contrary evidence in the literature. Fama [19] and Mandelbrot [27,28] found excess kurtosis in their investigation of the returns on financial assets, which led them to reject the normal assumption and propose stable distribution as an alternative. Balke and Fomby [2] show that most macroeconomic time series exhibit non-Gaussian behavior. They analyze fifteen post World War

II US macroeconomic time series which includes Consumer Price Index (a measure of price inflation), nominal compensation per hour in manufacturing (a measure of wage growth), yields on AAA bonds, Standard and Poors 500 stock price index, and GNP deflator. They report that residuals from autoregressive models indicate that these series show significant evidence of excess kurtosis and skewness. They conclude that even after estimating GARCH models, significant excess kurtosis and/or skewness still remains.

Using stable distributions in portfolio optimization is not a new idea. For instance, Ziemba [58] utilized symmetric stable distribution in an approximate portfolio optimization problem. The interest in this approached faded due to theoretical and computational difficulties, which outweighed its advantages. The recent theoretical contributions and increase in the computational power facilitates another look at the potential benefits. A recent paper by Ortobelli, Rachev and Schwartz [45] compares the optimal allocation between a risk-free asset and a risky asset under the alternative hypothesis of normal and stable returns. They report that the allocation might change up to 40% depending on the risk aversion level of the agents.

A multistage stochastic asset allocation problem with decision rule is analyzed in this study. The optimal allocations achieved under scenarios that are generated by Gaussian and stable non-Gaussian innovations are compared. We find that if the agent is very risk averse, then the normal and stable scenarios result in similar allocations. Similarly, if the agent has very low risk aversibility, then the normal and stable scenarios again result in similar allocations. However, when the coefficient of risk aversion is somewhere between the two cases, the two distributional assumptions may result in very different asset allocations depending on the utility function and the risk aversion of the decision maker. The allocations may be up to 85% different depending on the level of risk aversion. Since stable economic scenarios capture risk more realistically, they suggest more conservative asset allocations.

Section 2 reviews the literature on multistage stochastic ALM programming with decision rules, and Section 3 reviews the literature on scenario generation. Stable distribution is introduced in Section 4. Our model is set up in Section 5, with the discussion of the scenario generation and asset allocation modules. The asset allocation results are reported in Section 6. Section 7 concludes.

## 2 Multistage Stochastic ALM Programming with Decision Rules

In this method, time is discretized into  $n$ -stages across the planning horizon, and investments are made using a decision rule, e.g. fixed mix, at the beginning of each time period. The decision rule can easily be tested with out-of-sample scenarios, and confidence limits on the recommendations can be constructed. The use of this approach hinges on discovering policies that are intuitive and that will produce superior results. Decision rules may lead to non-convexities

and highly nonlinear functions. Some decision rules used in the literature are fixed mix, buy-and-hold, life cycle mix (Berger and Mulvey, 1998), constant proportional portfolio insurance (Perold and Sharpe, 1988), and target wealth path tracking (Mulvey and Ziemba, 1998). Boender (1997) and Boender et al. (1998) describe an ALM model designed for Dutch pension funds. Their goal is to find efficient frontiers of initial asset allocations which minimize the value of downside risk for certain given values of average contribution rates. The scenarios are generated across the time horizon of interest. The management selects a funding policy, an indexation policy of the earned pension rights, and an investment decision rule. These strategies are simulated against generated scenarios. Then, the objective function of the optimization problem is a completely specified simulation model except for the initial asset mix. The hybrid simulation/optimization model requires the following three steps:

1. Randomly generate initial asset mixes, simulate them and evaluate their contribution rates and downside risks.
2. Select the best performing initial asset mixes that are located at a minimal critical distance from each other.
3. Use a local search algorithm to identify the optimal initial asset mix.

Maranas et al., (1997) adopted another approach to stochastic programming with decision rules. They determine the optimal parameters of the decision rule by means of a global optimization algorithm. They propose a dynamically balanced investment policy which is specified by the following parameters:

$w_0$ : initial dollar wealth,  
 $r_{it}^s$ : percentage return of asset  $i \in \{1, 2, \dots, I\}$  in time period  
 $t \in \{1, 2, \dots, T\}$  under scenario  $s \in \{1, 2, \dots, S\}$ ,  
 $p^s$ : probability of occurrence of scenario  $s$

The decision variables are:

$w_t^s$ : dollar wealth at time  $t$  in scenario  $s$ ,  
 $\lambda_i$ : fraction of wealth invested in asset category  $i$  (note that it is constant over time).

The model is a multiperiod extension of the mean-variance method. The multi-period efficient frontier is obtained by varying  $\beta$  ( $0 \leq \beta \leq 1$ ). The formulation is as follows:

$$\begin{aligned} \max_{\lambda_i, w_t^s} \quad & \beta \text{mean}(w_T) - (1 - \beta) \text{var}(w_T) \\ \text{subject to} \quad & \end{aligned}$$

$$w_T^s = w_0 \prod_{t=1}^T \left[ \sum_{i=1}^I (1 + r_{it}^s) \lambda_i \right], \quad s = 1, \dots, S \quad (1)$$

$$\sum_{i=1}^I \lambda_i = 1 \quad (2)$$

$$0 \leq \lambda_i \leq 1, \quad i = 1, \dots, I$$

The wealth accumulation is governed by (1). When (1) is substituted into the objective function, we get a nonconvex multivariable polynomial function in  $\lambda_i$  involving multiple local minima. A global optimization tool which obtains the above efficient frontier, has been developed.

There are other models constructed using similar methodologies. Berger and Mulvey (1988) describe Home Account Advisor<sup>TM</sup> which assists individual investors in ALM using decision rules. Sweeney et al. (1998) applies a simulation/optimization scenario approach to optimal insurance asset allocation in a multi-currency environment.

### 3 Scenario Generation

A scenario gives a single set of outcomes for the random variables in the model over the planning horizon. A representative set of scenarios describes the possible future environmental situations. Traditional quantitative forecasting methods extrapolate new ideas about future developments based on the knowledge of the past and present. However, the economic environment may change invalidating the past assumptions. Hence, subjective beliefs of the management has become an essential part of scenario building. See Bunn and Salo (1993) for a review of qualitative scenario generation techniques.

The earlier ALM models used few independent scenarios to describe uncertainty. The recent models have become more sophisticated in scenario generation methods. The following are the widely accepted techniques in the literature.

#### 3.1 Discrete Time Series Models

##### 3.1.1 Multivariate Approach

Using Vector Autoregression (VAR) in the analysis of economic systems became popular after an influential paper by Sims (1980). Following Sims, Dert (1998) used this methodology to generate scenarios for a pension plan. He created future price inflation, wage inflation, stock returns, bond returns, cash return and real estate returns that are consistent with historical patterns in means, standard deviations, autocorrelations and cross correlations between state variables. A Markov model was used in determining future development of each individual participating in the pension plan. Carino et al. (1998) also employed VAR in generating scenarios for the Yasuda Kasai model.

VAR may sometimes diverge from long-term equilibrium. Boender et al. (1998) extended the VAR model to a Vector Error Correction Model (VECM) which additionally takes economic regime changes and long term equilibria into account. First, a submodel generates future economic scenarios. Then, a liability submodel determines the earned pension rights and payments corresponding to each economic scenario.

The economic scenario submodule uses time series analysis. The vector of the lognormal transformations of inflation, wage growth, bond return, cash return, equity return, real estate return and nominal GNP growth is  $y_t$ . Diagnostic tests revealed the order of the VAR process as 1.

$$y_t \sim N(\mu + \Omega * \{y_{t-1} - \mu\}, \Sigma),$$

where  $N(\mu, \Sigma)$  denotes a Gaussian distribution with mean  $\mu$  and covariance matrix  $\Sigma$ .

The extended VECM is given as

$$y_t \sim N(\Omega_1 y_{t-1} + \Omega_2 C^T (x_{t-1} - \mu_1 I_{\{T_1\}} - \mu_2 I_{\{T_2\}}), \Sigma),$$

where the  $\Omega_1$  corresponds to the short term dynamics and the  $\Omega_2$  corresponds to the long term correction. The index set  $T_1$  specifies the period of an economic regime with growth vector  $\mu_1$ , and  $T_2$  gives the period of another economic regime with growth vector  $\mu_2$ . The second term,  $C^T (x_{t-1} - \mu_1 I_{\{T_1\}} - \mu_2 I_{\{T_2\}})$ , generates the error correction to restore violations of the equilibria, while  $\Omega_2$  determines the speed of the response.

The parameters of the model were estimated using historical data. Then, scenarios were generated iteratively using the parameter estimates. They reported that the VECM improves the explanatory power of the model. The VECM has a clearer economic interpretation which incorporates regime changes and long run equilibrium.

The liability submodule uses a push Markov model to determine the future status of each individual plan member depending on age, gender, and employee category. Given this information, the pull part of the model is used to determine additional promotions and new employees. Then, the pension rules are applied to compute the guaranteed pension payments and earned pension rights.

### 3.1.2 Cascade Approach

Wilkie (1986) suggested using a cascade structure rather than a multivariate model in which each variable could affect all of the others. He considers inflation, ordinary shares, and fixed interest securities as the main economic determinants of a stochastic investment model. The model includes the following variables: inflation, an index of share of dividends, the dividend yield on this index, (the dividend index divided by the corresponding price index), and the yield on consols (as a measure of the general level of fixed interest yields in the market).

Wilkie's investigations and actuarial experience lead him to the conclusion that inflation is the driving force for the other investment variables. Figure 1<sup>1</sup>, depicts the cascade structure of the model. In the figure, the arrows indicate the direction of influences.

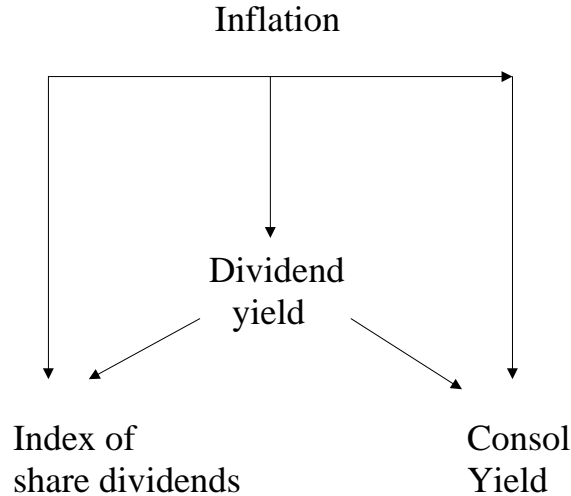


Figure 1: Wilkie's cascade structure

The inflation is described using a first order autoregressive model. The dividend yield depends on both the current level of inflation and the previous values of itself. The index of share dividends depends on inflation and the residual of the yield model. The consol yield also depends on inflation and the residual of the yield model along with the previous values of itself. Then, the estimated parameters are used to generate future economic scenarios. Wilkie (1995) improves this basic model.

### 3.2 Continuous Time Models

Mulvey (1996) designed an economic projection model for Towers Perrin using stochastic differential equations. The model has a cascade structure as depicted in Figure 2<sup>2</sup>. In this model, the Treasury yield curve is considered as the driving force for the other investment variables. Government bond returns, price and wage inflation, stock dividend yield, and stock dividend growth rate are analyzed within the framework of the cascade structure. Returns on primary asset categories such as large cap stocks, small cap stocks and corporate bonds consistent with the investment variables, are generated.

<sup>1</sup>Source: Wilkie (1986)

<sup>2</sup>Resource: Mulvey (1996)

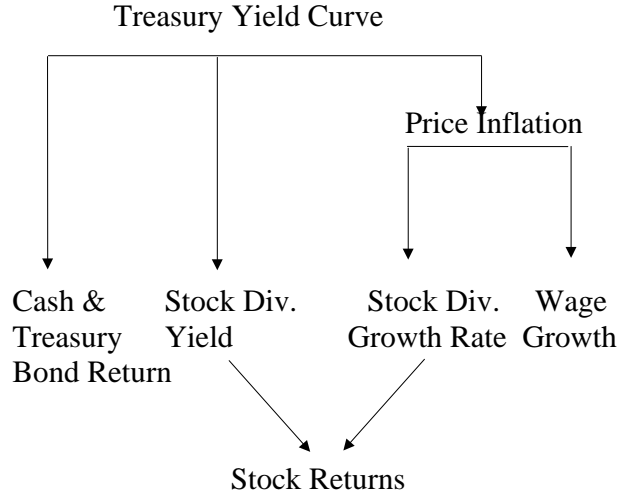


Figure 2: Mulvey's cascade structure

It is assumed that short and long-term interest rates (denoted by  $r_t$  and  $l_t$ , respectively), are linked through a correlated white noise term. The spread between the two is kept under control by using a stabilizing term. This variant of the two-factor Brennan and Schwartz (1982) model is as follows:

$$\begin{aligned} dr_t &= a(r_0 - r_t)dt + b\sqrt{r_t}dz_1, \\ dl_t &= c(l_0 - l_t)dt + e\sqrt{l_t}dz_2, \end{aligned}$$

where  $a$  and  $c$  are functions that depend on the spread between the long and short rates,  $b$  and  $e$  are constants, and  $dz_1$  and  $dz_2$  are correlated Weiner terms.

The price inflation rate is modeled as a diffusion process that depends on short term interest rates:

$$dp_t = ndr_t + g(p_0 - p_t)dt + h(v_{pt})dz_3,$$

where  $p_t$  is the price inflation at time  $t$ , and  $v_{pt}$  is the stochastic volatility at time  $t$ . Since the volatility of inflation persists, it is represented using Autoregressive Conditional Heteroskedasticity (ARCH) model. The equation for the stochastic volatility is given by:

$$dv_{pt} = k(v_{p0} - v_{pt})dt + m\sqrt{v_{pt}}dz_4,$$



where  $g$  and  $k$  are functions that handle the independent movement of the underlying prices at time  $t$  for the price inflation and stochastic volatility, respectively, and  $h$  and  $m$  are constants.

Real yields are related to interest rates, current inflation, and expectations for future inflation. The diffusion equation for long-term yield is

$$dy_t = n(y_u, y_t, l_u, l_t, p_u, p_t)dl + q(y_u, y_t, l_u, l_t, p_u, p_t)dt + u(y_t)dz_5,$$

where  $y_u$  is the normative level of real yields, and  $n$  and  $q$  are vector functions that depend upon various economic factors.

The wage inflation is connected to price inflation in a lagged and smoothed fashion. The stock returns are broken down into two components: dividends and capital appreciation, and they are estimated independently. Mulvey found that the decomposed structure provides more accurate linkages to the key economic factors such as inflation and interest rates.

The parameters of the model are calibrated by considering the overall market trends in light of historical evidence and subjective beliefs of the management. This model has been in use at Towers Perrin since 1992. Mulvey and Thorlacius (1998) extended the model to a global environment that links the economies of individual countries within a common framework.

## 4 Stable Distribution

There are several important reasons for modeling financial variables using stable distributions. Stable distributions are leptokurtotic. When compared to normal distribution, they typically have fatter tails and higher peaks around the center. Asymmetry is allowed. Hence, they fit the empirical distribution of the financial data better.

Any distribution in the domain of attraction of a specified stable distribution will have properties which are close to the ones of stable distribution. Even if the observed data does not exactly follow the ideal distribution specified by the modeler, in principle, the resulting decision is not affected. The wide use of a normal distribution is mainly due to the fact that it is the only distribution with finite variance that is stable with respect to the summation scheme.

Each stable distribution has an index of stability which remains the same regardless of the sampling interval adopted. The index of stability can be regarded as an overall parameter that can be employed in inference and decision making. However, we should note that for some financial data, empirical analysis shows that the index of stability increases as the sampling interval increases.

It is possible to check whether or not a distribution is in the domain of attraction of a stable distribution by examining the tails of the distribution. The tails dictate the properties of the distribution.

## 4.1 Description of Stable Distribution

If the sums of linear functions of independent identically distributed (iid) random variables belong to the same family of distributions, the family is called stable. Formally, a random variable  $r$  has stable distribution if for any  $a > 0$  and  $b > 0$ , there exist constants  $c > 0$  and  $d \in \mathcal{R}$ , such that

$$ar_1 + br_2 \stackrel{d}{=} cr + d,$$

where  $r_1$  and  $r_2$  are independent copies of  $r$ , and  $\stackrel{d}{=}$  denotes equality in distribution. The distribution is described by the following parameters:  $\alpha \in (0, 2]$  (index of stability),  $\beta \in [-1, 1]$  (skewness parameter),  $\mu \in \mathcal{R}$  (location parameter), and  $\sigma \in [0, \infty)$  (scale parameter). The variable is then represented as  $r \sim S_{\alpha, \beta}(\mu, \sigma)$ . Gaussian distribution is actually a special case of stable distribution when  $\alpha = 2$ ,  $\beta = 0$ . The smaller the stability index is, the stronger the leptokurtic nature of the distribution becomes, i.e., with a higher peak and fatter tails. If the skewness parameter is equal to zero, as in the case of Gaussian distribution, the distribution is symmetric. When  $\beta > 0$  ( $\beta < 0$ ), the distribution is skewed to the right (left). If  $\beta = 0$  and  $\mu = 0$ , then the stable random variable is called symmetric  $\alpha$ -stable ( $S\alpha S$ ). The scale parameter generalizes the definition of standard deviation. The stable analog of variance is variation,  $v_\alpha$ , which is given by  $\sigma^\alpha$ .

Stable distributions generally do not have closed form expressions for density and distribution functions. They are more conveniently described by characteristic functions. The characteristic function of random variable  $r$ ,  $\Phi_r(\theta) = E[\exp(ir\theta)]$ , is given by

$$\begin{aligned} \Phi_r(\theta) &= \exp \left\{ -\sigma^\alpha |\theta|^\alpha \left( 1 - i\beta \text{sign}(\theta) \tan \frac{\pi\alpha}{2} \right) + i\mu\theta \right\}, \text{ if } \alpha \neq 1, \\ &= \exp \left\{ -\sigma |\theta| \left( 1 - i\beta \frac{2}{\pi} \text{sign}(\theta) \ln \theta \right) + i\mu\theta \right\}, \text{ if } \alpha = 1. \end{aligned}$$

The  $p^{\text{th}}$  absolute moment of  $r$ ,  $E|X|^p = \int_0^\infty P(|X|^p > y) dy$ , is finite if  $0 < p < \alpha$ , and infinite otherwise. Hence, when  $\alpha < 1$ , the first moment is infinite. When  $\alpha < 2$ , the second moment is infinite. The only stable distribution that has finite first and second moments is the Gaussian distribution.

In models that use financial data, it is generally assumed that  $\alpha \in (1, 2]$ . There are several reasons for this:

- 1) When  $\alpha > 1$ , the first moment of the distribution is finite. It is convenient to be able to speak of expected returns.
- 2) Empirical studies support this parametrization.
- 3) Although the empirical distributions of the financial data sometimes depart from normality, the deviation is not “too much”.

In the scenario generation, one may need to use multivariate stable distributions. The extension to the multivariate case is nontrivial. Although most of the literature concentrates on the univariate case, recently, some new results have become available. See for example Samorodnitsky and Taquq (1994), and Mittnik and Rachev (1999).

If  $R$  is a stable  $d$ -dimensional stable vector, then any linear combination of the components of  $R$  is also a stable random variable. However, the converse is true under certain conditions (Samorodnitsky and Taquq, 1994). The characteristic function of  $R$  is given by:

$$\begin{aligned} \Phi_Y(\theta) &= \exp\left\{-\int_{S_d} |\theta^T s| \left(1 - i \operatorname{sign}(\theta^T s) \tan \frac{\pi\alpha}{2}\right) \Gamma(ds) + i\theta^T \mu\right\}, \\ &\quad \text{if } \alpha \neq 1, \\ &= \exp\left\{-\int_{S_d} |\theta^T s| \left(1 + i \frac{2}{\pi} \operatorname{sign}(\theta^T s) \ln |\theta^T s|\right) \Gamma(ds) + i\theta^T \mu\right\}, \\ &\quad \text{if } \alpha = 1, \end{aligned}$$

where  $\Gamma$  is the spectral measure which replaces the scale and skewness parameters that enter the description of the univariate stable distribution. It is a bounded nonnegative measure on the unit sphere  $S_d$ , and  $s \in S_d$  is the integrand unit vector. The index of stability is again  $\alpha$ , and  $\mu$  is the vector of locations.

In some financial applications, one needs to model the dependence between variables. Stable distributions have infinite second moment: covariance is not defined. However, subordinated Gaussian can be used to model dependence between stable variables<sup>3</sup>. Subordinated Gaussian is defined as follows: Let  $X \sim N(0, 2\sigma^2)$ , and  $A \sim S_{\alpha/2,0}(1, c)$ ,  $X$  and  $A$  being independent. Then, one can generate  $Z = A^{1/2} X \sim S_{\alpha,0}(0, \sigma^*)$ , where  $c = (\frac{\sigma^{*2}}{\sigma^2}) [\cos(\pi\alpha/4)]^{2/\alpha}$ .

The 'truncated' covariance matrix can be used to capture the dependence by leaving out the very extreme events. Let  $\Sigma$  be the truncated covariance matrix. It can be estimated by exponential smoothing<sup>4</sup> as follows:

$$c_{j,t+1|t}^2 = (1 - \lambda) \sum_{i=1}^{\infty} \lambda^i R_{j|t-i}^2$$

is the diagonal element of the truncated covariance matrix, and

$$c_{jk,t+1|t}^2 = (1 - \lambda) \sum_{i=1}^{\infty} \lambda^i R_{j|t-i} R_{k|t-i},$$

where  $\lambda$  is the smoothing parameter, measures the truncated covariance between  $j$  and  $k$ . Hence,  $\Sigma = \{c_{ik}\}$  and  $c_{jj} = 2\sigma_j^2$ . Suppose the truncation points are  $x$  and  $-x$ , then  $R_{j|t-i}$  is defined as the following:

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<sup>3</sup>See Mercury (1999) for a more detailed discussion.

<sup>4</sup>See Riskmetrics (1996) for a discussion of exponential smoothing.

$$R_{j|t-i} = \left\{ \begin{array}{l} r_t^j, \text{ if } r_t^j \text{ is in } (-x, x) \\ -x, \text{ if } r_t^j < -x \\ x, \text{ if } r_t^j > x \end{array} \right\}.$$

## 4.2 Financial Modeling and Estimation

Financial modeling involves information on past market movements. In such cases, it is not the unconditional return distribution which is of interest, but the conditional distribution, which is conditioned on information contained in past return data, or a more general information set. The class of autoregressive moving average (ARMA) models is a natural candidate for conditioning on the past of a return series. These models have the property that the conditional distribution is homoskedastic. In view of the fact that financial markets frequently exhibit volatility clusters, the homoskedasticity assumption may be too restrictive. As a consequence, conditional heteroskedastic models are now common in empirical finance. Engle's (1982) autoregressive conditional heteroskedastic (ARCH) models, and the generalization (GARCH) of Bollerslev (1986), possibly in combination with an ARMA model, referred to as an ARMA-GARCH models are widely used. It turns out that ARCH-type models driven by normally distributed innovations imply unconditional distributions, which actually possess heavier tails. Thus, in this respect, ARCH models and stable distributions can be viewed as competing hypotheses.

Mittnik et al. (1997), present empirical evidence favoring the stable hypothesis over the normal assumption as a model for unconditional, homoskedastic conditional, and heteroskedastic conditional distributions of several asset return series.

### 4.2.1 Maximum Likelihood Estimation

We use an approximate conditional maximum-likelihood (ML) estimation procedure suggested by Mittnik et al. (1996). The unconditional ML estimate of  $\theta = (\alpha, \beta, \mu, \sigma)$  is obtained by maximizing the logarithm of the likelihood function

$$L(\theta) = \prod_{t=1}^T S_{\alpha,\beta} \left( \frac{r_t - \mu}{\sigma} \right) \sigma^{-1}.$$

The estimation of all stable models is approximate in the sense that the stable density function,  $S_{\alpha,\beta}(\mu, \sigma)$ , is approximated via fast Fourier transformation (FFT) of the stable characteristic function,

$$\int_{-\infty}^{\infty} e^{itx} dH(x) = \begin{cases} \exp\{-\sigma^\alpha |t|^\alpha [1 - i\beta \text{sign}(t) \tan \frac{\pi\alpha}{2}] + i\mu t\}, & \text{if } \alpha \neq 1, \\ \exp\{-\sigma |t| [1 + i\beta \frac{2}{\pi} \text{sign}(t) \ln |t|] + i\mu t\}, & \text{if } \alpha = 1, \end{cases}$$

where  $H$  is the distribution function corresponding to  $S_{\alpha,\beta}(\mu, \sigma)$ .

This ML estimation method essentially follows that of DuMouchel (1987), but differs in that the stable density is approximated numerically by an FFT of the characteristic function, rather than some series expansion. As DuMouchel shows, the resulting estimates are consistent and asymptotically normal with the asymptotic covariance matrix of  $T^{1/2}(\hat{\theta} - \theta_0)$ , being given by the inverse of the Fisher information matrix. The standard errors of the estimates are obtained by evaluating the Fisher information matrix at the ML point estimates. For details on stable ML estimation see Mittnik et al. (1999), Mittnik and Rachev (1999), and Paulauskas and Rachev (1999).

### 4.3 Comparison of Estimation Methods

When the residuals of the ARMA model have Gaussian distribution, Least Squares (LS) estimation is equivalent to conditional ML estimation. Furthermore, Whittle estimator is asymptotically equivalent to LS and ML estimation methods. However, when the innovations have stable distribution, the properties of conventional estimation methods may change due to the infinite variance property. In the stable case, ML estimates are still consistent and asymptotically normal (DuMouchel, 1987); LS and Whittle estimates are consistent but they are not asymptotically normal. The LS and Whittle estimates have infinite variance limits with a convergence rate that is faster than that of the Gaussian case (Mikosch, Gadrich, Kluppelberg, and Adler (1995). Calder and Davis (1998) compare LS, Least Absolute Deviation (LAD), and ML methods for the estimation of ARMA model with stable innovations. Their simulations reveal that the difference between the estimates of the three methods is insignificant when the index of stability of the residuals is 1.75. However, when  $\alpha = 1$  or  $\alpha = .75$ , they report that the LAD and ML estimation procedures are superior to LS estimation. ML estimation has desirable properties in both the Gaussian and stable setting, but it is computationally very demanding. Since the variables of interest in this paper have indices of stability greater than 1.5, nonlinear LS estimation method has been utilized in this study. Our parameter estimates are consistent, but they are not asymptotically normal. However, due to the high index of stability, the parameter estimates are comparable to those that would be achieved if ML estimation were to be used.

## 5 Model Setup

### 5.1 Asset Allocation Model

The dynamic asset allocation approach used in this study is very similar to that of Boender<sup>5</sup> (1998). A number of alternative initial asset allocations are generated. These allocations are then simulated into the future by using the economic scenarios, which are generated under the Gaussian and Stable

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<sup>5</sup>See Section 2 for a brief review.

assumptions for the innovations of the time series models. While the initial allocations are simulated, the asset allocation is updated every month according to fixed mix decision rule<sup>6</sup>. In general, fixed mix strategy requires the purchase of stocks as they fall in value, and the sale of stocks as they rise in value. Fixed mix strategy does not have much downside protection, and tends to do very well in flat but oscillating markets. However, it tends to do relatively poorly in bullish markets (Perold and Sharpe, 1998).

Once the initial asset allocations are simulated, the risk and reward are calculated corresponding to these initial allocations at the end of the horizon, say, one year. The decision-maker then, chooses the initial asset allocation (fixed mix proportions) today that results in the best risk-reward combination at the end of the horizon of interest for the given decision rule.

The initial asset allocation that is selected by the decision-maker depends on the assumptions made about the innovations of the economic scenarios. The economic scenarios driven by stable innovations result in a different risk-reward profile than the economic scenarios driven by Gaussian innovations.

While we follow the general structure of Boender (1997), the objective functions used are different. We use three alternative objective functions: the first two are analogs of mean-variance analysis, and the third is power utility function.

Konno and Yamazaki (1991) advocate mean-mean absolute deviation analysis as an alternative to mean-variance method. Mean absolute deviation accords less importance to outliers, it is computationally easier to calculate, and it can be used to model the asymmetric perception of risk around the mean return. They show that if the returns are multivariate normally distributed, then the two measures are essentially the same<sup>7</sup>. Since mean absolute deviation is well defined for both normal and stable distributions, we use mean-mean absolute deviation analysis.

The mean compound portfolio return of fixed mix rule  $i \in \{1, 2, \dots, I\}$  at the final date is:

$E[\hat{R}_T^i] = \frac{1}{S} \sum_{s=1}^S \hat{R}_{s,T}^i$ , where  $\hat{R}_{s,T}^i$  is compound return of allocation  $i$  in time period of 1 through  $T$  under scenario  $s \in \{1, 2, \dots, S\}$ . It is calculated as

$$\hat{R}_{s,T}^i = \prod_{t=1}^T (1 + R_{s,t}^i) - 1,$$

where  $R_{s,t}^i$  is the return of the portfolio  $i$  under scenario  $s \in \{1, 2, \dots, S\}$  in time period  $t \in \{1, 2, \dots, T\}$ :

---

<sup>6</sup>Perold and Sharpe (1988) suggest constant proportion portfolio insurance as an alternative strategy. In this strategy, one sells stocks as they fall in value and buy stocks as they rise in value. We will report the optimal allocations under this strategy in a later version of this paper.

<sup>7</sup>Konno and Yamazaki (1991) compare efficient frontiers for NIKKEI 225 index generated by standard deviation and mean absolute deviation risk measures. They report that the difference of the optimal portfolio generated by the two risk measures is at most 10 %. They suggest that this difference can be mainly attributed to the non-normality of the data.

$$R_{s,t}^i = \sum_{j=1}^J w_j^i r_{jst},$$

where  $r_{jst}$  is the percentage return of asset  $j \in \{1, 2, \dots, J\}$  under scenario  $s$  in time period  $t$ , and  $w_j^i$  is the proportion of funds<sup>8</sup> of portfolio  $i$  invested in asset  $j$ .

The mean absolute deviation of asset allocation  $i$  at the final date is:

$$MAD(\hat{R}_T^i) = \frac{1}{S} \sum_{s=1}^S \left| \hat{R}_{s,T}^i - E[\hat{R}_T^i] \right|.$$

The utility function defined over the mean final return and the mean absolute deviation of final return is given as:

$$U(\hat{R}_T^i) = E[\hat{R}_T^i] - c \cdot MAD(\hat{R}_T^i),$$

where  $c$  is the coefficient of risk aversion.

We consider another risk measure which gives importance to outliers more than mean absolute deviation but less than variance:

$$MD(\hat{R}_T^i) = \frac{1}{S} \sum_{s=1}^S \left| \hat{R}_{s,T}^i - E[\hat{R}_T^i] \right|^r, \text{ where } 1 < r < 2.$$

Notice that when  $r = 2$ , the above risk measure becomes the variance. Since variance is not defined for non-Gaussian stable variables, we use those values of  $r < 2$  for which  $MD(\hat{R}_T^i)$  is finite, such as  $r = 1.5$ . The utility function defined over the mean final return and this new risk measure is:

$$U(\hat{R}_T^i) = E[\hat{R}_T^i] - c \cdot MD(\hat{R}_T^i),$$

where  $c$  is the coefficient of risk aversion.

Finally, we consider the more traditional power utility function which has linear risk tolerance:

$$U(W^i) = \frac{1}{S} \sum_{s=1}^S \frac{1}{(1-\gamma)} (W_s^i)^{(1-\gamma)}, \gamma > -1^9$$

where  $\gamma$  is the coefficient of relative risk aversion, and  $W_s^i$  is the final wealth. Assuming that the initial wealth is 1, we compute the final wealth as follows:

$$W_s^i = 1 \cdot (1 + \hat{R}_{s,T}^i).$$

## 5.2 Scenario Generation

A cascade structure similar to Mulvey<sup>10</sup> (1996) has been adopted in this study, (see Figure 3). However, the analysis is done in discrete time as in Wilkie<sup>11</sup> (1995). Although both studies use annual data, we have used monthly data. 3-month Treasury bill rate and 10-year Treasury bond rate are modeled first as measures of short term and long term interest rates. The price inflation depends on the Treasury bond rate and the previous values of inflation. Following Wilkie's and Mulvey's approaches, stock returns are analyzed in two components: dividend growth and dividend yield growth.

The relationship of economic variables does not denote a one way casual relationship, but rather indicates the sequencing of the modules. The economic

<sup>8</sup>Fix mix rule requires that  $w_j^i$  does not depend on time.

<sup>9</sup>Note that  $U(W^i)$  is finite if  $(1-\gamma) < 2$

<sup>10</sup>See Section 3.2 for a brief description.

<sup>11</sup>See Section 3.1.2 for a brief review.

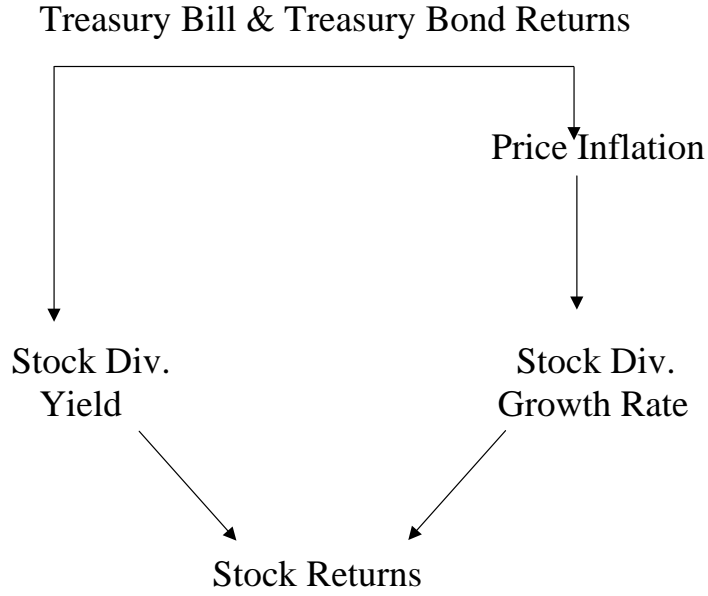


Figure 3: A cascade structure similar to Mulvey

variables are modeled using Box-Jenkins methodology. The standard Gaussian Box-Jenkins techniques carry over to the stable setting with some possible changes. There are two criteria that we used in the model section:

- 1) Autocorrelation function (ACF) and Partial Autocorrelation function (PACF) were used to determine the order of the autoregressive and moving average terms and to detect the significance of the serial correlation of the residuals. Adler, Feldman, and Gallagher (1998) compare stable, Cauchy and Gaussian limits in the construction of confidence interval for ACF and PACF. The simulations show that when  $\alpha \geq 1.7$ , Gaussian and Cauchy limits are better than stable limits. However, for  $\alpha < 1.7$ , while Gaussian limits still perform in the acceptable range, Cauchy and stable limits are better than the Gaussian limits. Gaussian limits were used in our analysis since the indices of stability of all the residuals except for one are greater than 1.7.
- 2) Akaike Information Criteria was used to trade between extra explanatory power and parsimonious parameter selection. It is valid in both the Gaussian and the stable setting (Adler, Feldman, and Gallagher, 1998).

We do not model the time varying volatility of the economic variables. Fitting ARMA-GARCH models may reduce the kurtosis in the residuals. However, Balke and Fomby (1994) show that even after estimating GARCH models, significant excess kurtosis and/or skewness still remains. Mittnik et al. (1997) present empirical evidence favoring stable hypothesis over the normal assump-



tion as a model for ARMA-GARCH residuals. We postpone, to a future paper, modeling the time varying volatility in the generation of economic scenarios.

### 5.2.1 Level 1: Short Term and Long Term Interest Rates

3-month Treasury bill rate is used as a proxy for short term risk free interest rate. The 10-year Treasury bond yield is used as a proxy for the long term interest rate. Dickey-Fuller and Phillips Perron tests for unit root suggest that both Treasury bill and Treasury bond are first order unit root processes. There is no agreement in economic theory on whether short term and long term interest rates have a long-term equilibrium relationship, or not. We analyze the data to decide on this issue on empirical grounds for our data set. One option is to ignore the nonstationarity, and simply estimate in levels. However, the classical asymptotic theory for test statistics is nonstandard. Another option is to difference the apparently nonstationary variables before estimating the regression equations. If the true processes are regressions in first differences, then this approach is the right one. However, the series may have been, in fact, stationary, or a linear combination of the series might be stationary. In such circumstances, the analysis is misspecified. One needs to test for possible cointegration among series. The disadvantage of this approach is that alternative tests for unit roots and cointegration can produce conflicting results. One practical solution suggested by Hamilton (1994), is to employ parts of all three approaches. If the regression for the data in the levels form yields similar inferences with those in the stationary first difference form, then one can be satisfied that the results were not governed by the assumptions made about unit roots; confidence in the specification increases.

When short term and long term interest rates are allowed to have linear trends, and the cointegration equation is allowed to have intercept and trend, Johansen Cointegration test (1991, 1995) suggests that there is no cointegration relationship between the two series. However, if one imposes the restriction that the individual series have no trend and the cointegration equation has no trend, but possibly an intercept, then Johansen Cointegration test finds one cointegration equation. These results are very sensitive to the time period analyzed, as well as the assumed lag length (see Table 2). We conclude that it is likely that there is no linear combination of short term and long term interest rates that is stationary.

We analyze the reduced form equations by fitting a vector autoregression (VAR) to first differences of Treasury bond and Treasury bill rates. The Akaike Information Criteria suggests VAR(2), in which case the residuals exhibit no significant serial correlation. VAR in levels suggests a similar form which increases the confidence in the specification.

$$d(Tbill)_t = \gamma^1 d(Tbill)_{t-1} + \gamma^2 d(Tbill)_{t-2} + \gamma^3 d(Tbond)_{t-1} + \gamma^4 d(Tbond)_{t-2} + \epsilon_t^{Tbill}, \quad (5)$$

$$d(Tbond)_t = \beta^1 d(Tbill)_{t-1} + \beta^2 d(Tbill)_{t-2} + \beta^3 d(Tbond)_{t-1} + \beta^4 d(Tbond)_{t-2} + \epsilon_t^{Tbond}, \quad (6)$$

The adjusted  $R^2$  of model (5) is 0.16, and the adjusted  $R^2$  of model (6) is

Table 1: Johansen Cointegration Test Summary

Time period	2/1965-12/1999		
Lag interval	1		
Individual Data Series	No trend	No trend	Linear trend
Cointegration Eqn.	No intercept/ No trend	Intercept/ No trend	Intercept/ Trend
L.R. Test	Rank=1	Rank=1	Rank=0

Time period	2/1965-12/1999		
Lag interval	2		
Individual Data Series	No trend	No trend	Linear trend
Cointegration Eqn.	No intercept/ No trend	Intercept/ No trend	Intercept/ Trend
L.R. Test	Rank=0	Rank=0	Rank=0

Time period	2/1968-12/1999		
Lag interval	1		
Individual Data Series	No trend	No trend	Linear trend
Cointegration Eqn.	No intercept/ No trend	Intercept/ No trend	Intercept/ Trend
L.R. Test	Rank=1	Rank=0	Rank=0

0.14. The correlation between the residuals of the two variables is 0.56. The residuals cannot be assumed independent in the scenario generation process.

A problem of this approach is that the second moment does not exist for stable random variables with  $\alpha < 2$ . The dependence structure between the innovations of Treasury bill and Treasury bond rates cannot be modeled by using covariance measure in the generation of stable scenarios. We derive the dependence structure from truncated observations by leaving out the virtually impossible values. Mercury (1998) package is used to estimate the 'truncated' covariance matrix by exponential smoothing<sup>12</sup> and to simulate the residuals.

Once the truncated covariance matrix for  $d(Tbill)$  and  $d(Tbond)$ ,  $\Sigma = \{c_{ik}\}$ , is estimated, we generate

$$Z = A^{1/2} X \tilde{S}_{\alpha,0}(0, \sigma^*),$$

where  $c = (\frac{\sigma^{*2}}{\sigma^2})[\cos(\pi\alpha/4)]^{2/\alpha}$ ,  $X \sim N(0, 2\sigma^2)$ , and  $A \sim S_{\alpha/2,0}(1, c)$ ,  $X$  and  $A$  being independent<sup>13</sup>. The dependence structure between risk factors still remains: The stable random variable  $Z = A^{1/2}X$  can be viewed informally as  $N(0, 2\sigma^2A)$ -distributed, i.e., normal with random variance  $2\sigma^2A$ .

The future correlated residuals for  $d(Tbill)$  and  $d(Tbond)$  are then simulated as follows:

<sup>12</sup>Refer to Section 4.1 for estimation of the truncated covariance matrix.

<sup>13</sup>

See Section 4.1.

- 1) We simulate  $N$  independent multivariate normal random variables with the truncated covariance matrix  $\Sigma$  between components of each vector-column:

$$G = \begin{bmatrix} X_{1,1} & X_{1,2} \\ \vdots & \vdots \\ X_{N,1} & X_{N,2} \end{bmatrix}, \text{ where every column is } N(0, \Sigma).$$

- 2) We simulate  $N$  independent identically distributed stable random variables,  $A_{j,i} \sim S_{\alpha_j/2,0}(1, c_j)$

$$S = \begin{bmatrix} A_{1,1}^{1/2} & A_{1,2}^{1/2} \\ \vdots & \vdots \\ A_{N,1}^{1/2} & A_{N,2}^{1/2} \end{bmatrix}.$$

- 3) The matrix  $T$  which is the dot product of  $G$  and  $S$  will contain  $N$  simulations for  $d(Tbill)$  and  $d(Tbond)$  with the desired stable parameters:

$$T = G \circ S, \text{ where } T_{j,i} \sim S_{\alpha_j,0}(0, \sigma_j), \forall i = 1, 2, \dots, N, \text{ and } j = 1, 2.$$

$T$  can informally be viewed as  $N(0, \Sigma^*)$  - distributed, with random covariance matrix  $\Sigma^*$ , where  $\Sigma^* = \{\sigma_{ij}^*\}$ ,  $\sigma_{jj}^* = c_{jj} \cdot A_j$ , and  $\sigma_{ij} = c_{ij} \cdot (A_i A_j)^{1/2}$ . Note that  $A_j$  is the square of the realization from  $S$  for the  $j$ -th variable (corresponding to  $d(Tbill)$  or  $d(Tbond)$ ), and  $c_{ij}$  is an element of  $\Sigma$ .

### 5.2.2 Level 2: Inflation

Mulvey finds that inflation depends on previous inflation rates and the current yield curve. Since we avoided modeling the yield curve, we checked whether the inflation rate can be explained by changes of short term and long term interest rates. Changes in short rate do not explain inflation rates at 1 or 5 percent significance levels. However, changes in long rate have significant explanatory power at the 5 percent level. The residuals exhibit ARMA(1,1) structure. There is a very significant peak in the partial autocorrelation function at lag 9. When ninth order autoregressive term is added, the serial correlation in the residuals becomes insignificant. However, if a different time horizon is considered, there is no longer a significant peak in the partial autocorrelation function at lag 9. Since there is no particular reason for its existence, we conclude that it is an outlier.

We use the following time series model for price inflation:

$$\begin{aligned} Inf_t &= c^{Inf} + \gamma^{Inf} d(Tbond)_t + resInf_t \\ resInf_t &= \beta^{Inf} resInf_{t-1} + \alpha^{Tbond} \epsilon_{t-1}^{Inf} + \epsilon_t^{Inf}, \end{aligned} \quad (7)$$

where  $Inf$  : log differences of seasonally adjusted monthly CPI values.

This model gives the highest Akaike Information Criterion without leaving any significant serial correlation in the residuals. The Jarque-Bera statistic rejects that  $\epsilon_t^{Inf}$  comes from normal distribution at 1% and 5% significance levels. The residuals have a kurtosis of 8.15 and a skewness of 0.49. The adjusted  $R^2$  of the estimated model is .51.

### 5.2.3 Level 3: Stock Dividend Growth Rate and Stock Dividend Yield

Mulvey and Thorlacius (1998) suggests dividing the stock returns into two components: dividend and capital appreciation. They argue that by separating the base components as dividend growth and dividend yield, one can accurately depict cash income. The decomposed structure provides more accurate linkages to key economic factors such as interest rates and the inflation level. We adopt their approach.

Mulvey (1996) observes that growth of dividends net of inflation has been fairly stable over the last several decades. He suggests that dividend growth can be linked to inflation and past dividend growth.

The data reveals that the dividend growth rate can be explained by dividend growth rate of the previous two years and second order autoregressive terms:

$$\begin{aligned} Divgt &= \gamma^{Divg} Inf_t + resDivgt \\ resDivgt &= \beta_1^{Divg} resDivgt_{t-1} + \beta_2^{Divg} resDivgt_{t-2} + \epsilon_t^{Divg}, \end{aligned} \quad (8)$$

where  $Divg$  : log differences of dividend index of S&P 500.

The inclusion of changes in short and long rate directly in the model does not have any significant explanatory power. The residuals have no significant serial correlation left over. The kurtosis of the residuals is 6.63, and the skewness is -.002. The Jarque-Bera statistic rejects that  $\epsilon_t^{Divg}$  comes from normal distribution at 1% and 5% significance levels. The adjusted  $R^2$  of the estimated model is .23.

The Dickey-Fuller test for unit root suggests dividend yield is a first order integrated process. Hence, we model the change in the dividend yield rather than dividend yield itself. Mulvey suggests that dividend yield depends on the movement of short-term and long-term interest rates. However, our analysis shows that short term interest rate as proxied by 3 month Treasury rate has no significant explanatory power for explaining dividend yield movements. The current and the previous month's long rates have significant explanatory power (at 5% level) in explaining the change in dividend yield.

Using the Akaike Information Criteria, the time series model we suggest for change in the dividend yield is as follows:

$$d(Divy)_t = \gamma_1^{Divy} d(Tbond)_t + \gamma_2^{Divy} d(Tbond)_{t-1} + \epsilon_t^{Divy} \quad (9)$$

where  $Divy$  : logarithm of monthly dividend yield of S&P 500.

This model leaves no significant serial correlation in the residuals. The Jarque-Bera statistic rejects that  $\epsilon_t^{Divy}$  comes from normal distribution at 1% and 5% significance levels. The residuals are kurtotic and skewed. The kurtosis is 12.02, and the skewness is 1.56. The adjusted  $R^2$  of the estimated model is .13.

### 5.2.4 Simulation of Future Scenarios

Future economic scenarios are simulated at monthly intervals. One set of scenarios is generated by assuming that the residuals of each variable is identical

normally distributed. This is the classical assumption made in the literature. Another set of scenarios is generated by assuming that the residuals are identical stable distributed. The estimated normal and stable parameters<sup>14</sup> for the innovations of the time series models are given in Table 3. See Figures 4-8 for graphical comparison of stable and normal fit to the residuals.

Table 2: The estimated normal and stable parameters for the innovations

innovations of	normal dist.		stable dist.			
	$\mu$	$\sigma$	$\alpha$	$\beta$	$\mu$	$\sigma$
Price Inflation ( <i>Inf</i> )	6.15 e-06	.0021	1.7072	0.1073	6.15 e-06	0.0012
Dividend gr. ( <i>Divg</i> )	9.89 e-4	.0195	1.7505	-0.0229	9.89 e-4	0.0114
Dividend yield ( <i>d(Divy)</i> )	-.002551	.0407	1.8076	0.2252	-.002551	0.0239
Treasury bill ( <i>d(Tbill)</i> )	.000336	.0579	1.5600	0	0	.0308
Treasury bond ( <i>d(Tbond)</i> )	.000818	.0339	1.9100	0	0	.0230

The scenarios have a tree structure. At each stage (month) we generate  $n$  possible scenarios. For each scenario, we first generate a normal or stable residual for Treasury bill, and calculate the corresponding Treasury bill rate for next month. Then, given this short rate, we generate Treasury bond rate, price inflation, dividend growth rate and dividend yield for next month according to the cascade structure and the time series models we have built. For instance, the inflation rate for next month is generated by using the Treasury bond rate, inflation rate and the surprise to expected inflation this month, and the normal or stable innovation of inflation rate next month. Note that we allow for innovation of each economic variable in each simulated month.

At the next stage,  $n$  new offspring scenarios are generated from the parent scenarios. This continues until the final time of interest. If the horizon of interest is three quarters, then we generate  $n^9$  alternative economic scenarios. In this study, we generate 2 scenarios for each month, so 512 possible economic scenarios are considered over the next three quarters.

We expect that the simulation results exhibit the long run characteristics of the data. Monthly data from 2/1965 through 12/1999 is used for the estimation of the time series models. The historical annualized averages (over the full estimation period and over the recent four years), and the simulation annualized averages under the normal and stable hypothesis for Treasury bond rate, Treasury bill rate, dividend growth rate, dividend yield, inflation, return on S&P 500, are given in Table 4. The 9-month scenario tree is repeated 99 times. The values in the brackets are the 5% and 95% sample quartiles of the simulated variables.

<sup>14</sup>Stable parameters are estimated using maximum likelihood estimation method. See Section 4.2.1.

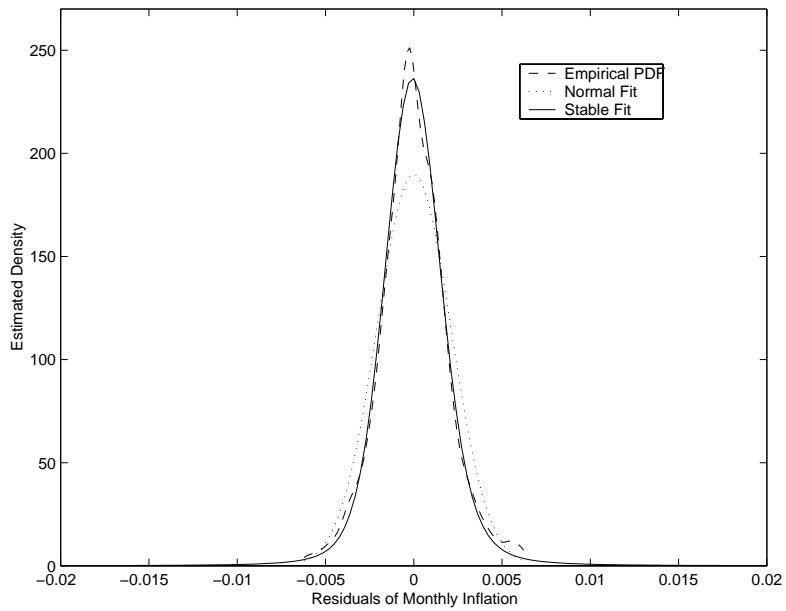


Figure 4: Residuals of Monthly Inflation

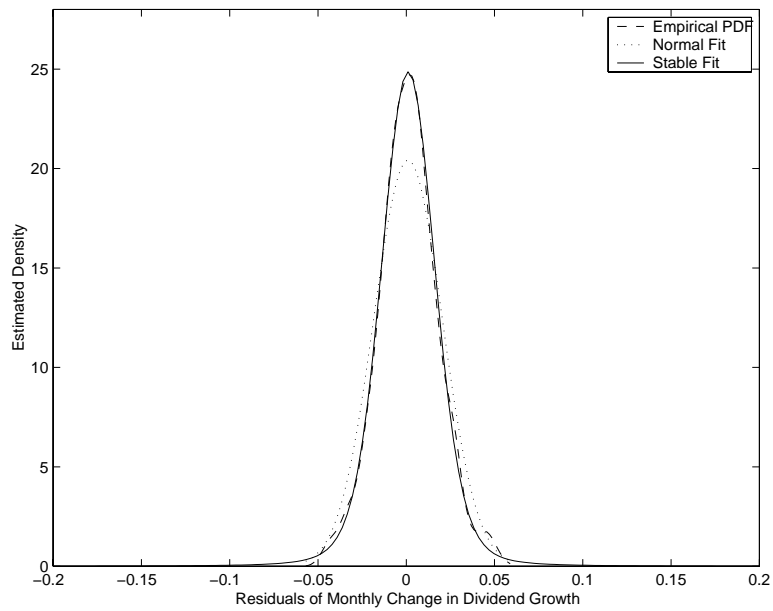


Figure 5: Residuals of Monthly Change in Dividend Growth

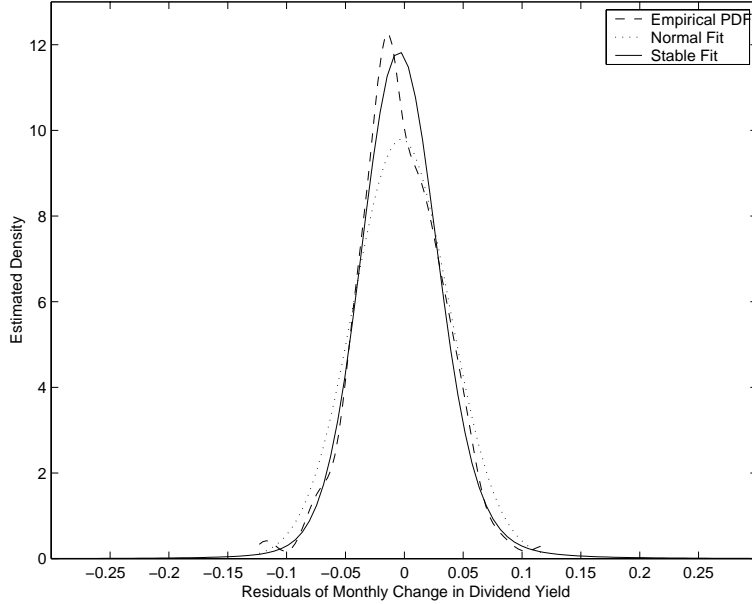


Figure 6: Residuals of Monthly Change in Dividend Yield

The annualized averages of the simulated data are close to the historical data over the last four years: the historical averages are within the confidence bounds. However, the simulation results cannot capture the excessive returns on the stock market over the last four years, as depicted in the third column of Table 4.

We have not investigated the forecasting power of our model. Backtesting results will be reported in a later study. The parameters of the time series model can be adapted to reflect the subject beliefs and expectations of the management about how the economy will be performing over the next year. See Bunn and Salo (1993) for a review of qualitative scenario generation techniques.

### 5.2.5 Valuation of Assets

A portfolio that is composed of Treasury bill and S&P 500 is analyzed. The monthly return on Treasury bill is already simulated. We need to derive the monthly return of S&P using the dividend yield and the dividend index. The dividend index is calculated by multiplying price index with the dividend yield:

$$DI_t = P_t * DY_t ,$$

where  $DI_t$  is the dividend index for period  $t$ ,  $P_t$  is the price index for period  $t$ , and  $DY_t$  is the dividend yield for period  $t$ . The dividend growth is just log differences of dividend indices.

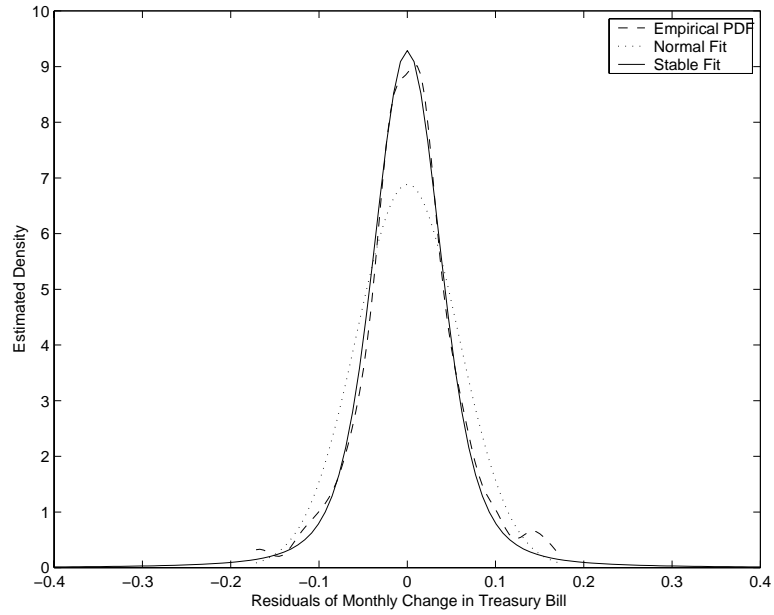


Figure 7: Residuals of Monthly Change in Treasury Bill

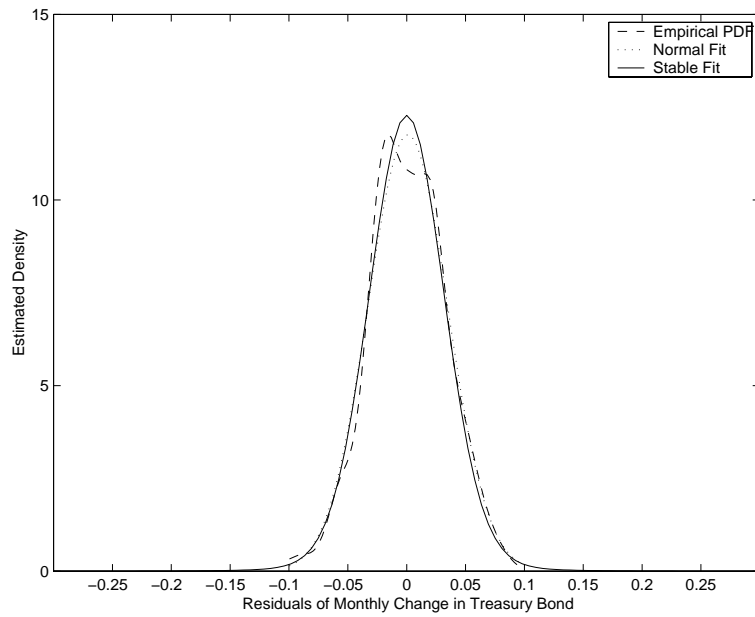


Figure 8: Residuals of Monthly Change in Treasury Bond



Table 3: The Historical Averages and the Simulation Averages of Economic Variables

Average (%) Annualized	Historical ('65-'99)	Historical ('96-'99)	Normal Scenarios	Stable Scenarios
Treasury Bond Rate	7.84	5.92	6.4148 [6.0193, 6.8232]	6.3483 [5.939, 6.8195]
Treasury Bill Rate	6.43	4.87	5.1319 [4.6389, 5.6776]	4.9515 [4.2795, 5.4755]
Dividend Growth	5	3	4.2664 [1.4969, 6.9644]	3.8991 [-1.1176, 7.1615]
Dividend Yield	3.57	1.69	1.1480 [1.0659, 1.2383]	1.1589 [1.08, 1.25]
Inflation	5	4	3.8899 [1.4842, 6.5171]	3.6106 [ 0.08, 7.2405]
Return on S&P 500	12.6	23.88	9.7248 [5.5153, 13.9584]	10.8114 [4.907, 16.955]
Return on T-Bill	6.43	4.87	5.1268 [4.6338, 5.6725]	4.9462 [4.274, 5.4655]

The dividend yield and dividend growth rate are simulated as explained in the previous section. Hence, we can get back simulated future price index in period  $t$  under scenario  $s$  from the simulated dividend growth and dividend yield indices by

$$P_{st} = \frac{DI_{st}}{DY_{st}}.$$

Then, we can calculate the return for holding S&P 500 for a month under scenario  $s$  as

$$r_{st} = \frac{P_{st} - P_{s(t-1)} + DI_{st}}{P_{s(t-1)}}.$$

## 6 Computational Results

The differences in the allocations under the Gaussian and stable scenarios depend critically on the utility function used. The optimal allocations under the three utility functions considered are discussed in this section.

The first utility function analyzed is:  $U(\hat{R}_T^i) = E[\hat{R}_T^i] - c.MAD(\hat{R}_T^i)$ , where coefficient  $c$  is the risk aversion coefficient. The mean absolute deviation and the mean final compound portfolio return at the end of 3 quarters is presented in Figure 9. The stable economic scenarios result in higher mean absolute deviation of final compound return for a given mean return.

The optimal allocation depends on the risk aversion of the agents. Table 4 presents the optimal allocations under the normal and stable scenarios. If the agents are very risk averse,  $c = .550$  for instance, the normal and stable

scenarios result in the same allocation (See Figure 10). If the agent has very low risk aversibility,  $c = .350$  for instance, then the normal and stable scenarios again result in the same allocation (See Figure 11).

However, when the coefficient of risk aversion is between .350 and .550, the two distributional assumptions result in very different allocations. The stable scenarios capture the extreme events better, and hence model the risk more realistically. Stable scenario simulation suggest more conservartive asset allocation. When  $c = .450$ , normal scenarios suggest 60% in S&P 500 and 40% in Treasury bill, whereas stable scenarios suggest 10% in S&P 500 and 90% in Treasury bill (See Figure 12).

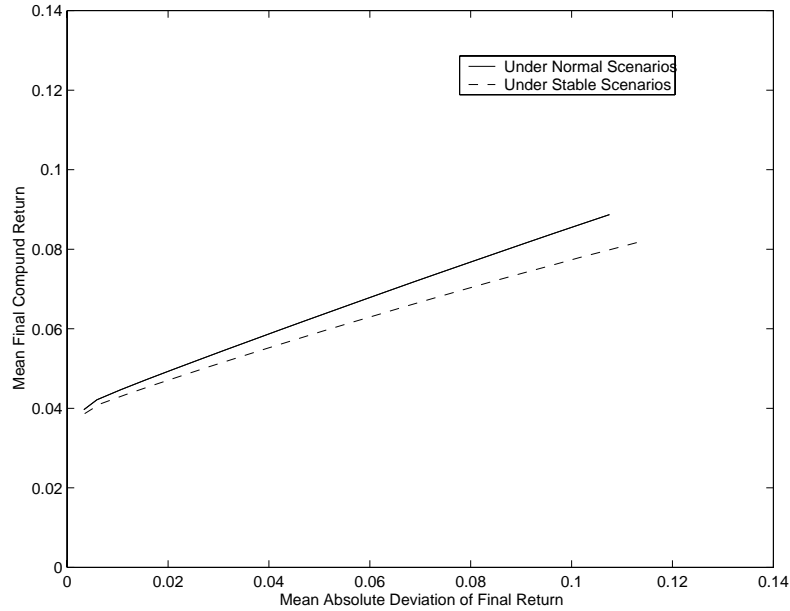


Figure 9: Risk-Reward Diagram under Normal and Stable Scenarios

The other utility function considered is:  $U(\hat{R}_T^i) = E[\hat{R}_T^i] - c.MD(\hat{R}_T^i)$ . The  $MD(\hat{R}_T^i)$  and the mean final compound portfolio return at the end of 3 quarters is depicted in Figure 13. The stable economic scenarios result in higher risk for given mean return. When the agent is highly risk averse both Gaussian and stable scenarios suggest to put all the money in Treasury bills. If the agent has very low risk aversion, then both scenario sets result in investing 100% in the stock. When  $c = 1$ , Gaussian scenarios suggest to put 60% in S&P 500 and 40% in Treasury bill, whereas the stable scenarios suggest to invest 30% in S&P 500 and 70% in Treasury bill. Table 5 presents the optimal allocations under the normal and stable scenarios.

Table 4: Optimal Allocations under the Normal and Stable Scenarios

c	Normal Scenarios		Stable Scenarios	
	Optimal Percentage Invested		Optimal Percentage Invested	
	S&P 500	Treasury Bill	S&P 500	Treasury Bill
.350	100%	0%	100%	0%
.401	100	0	45	55
.403	100	0	40	60
.405	100	0	30	70
.420	100	0	20	80
.430	100	0	15	85
.440	80	20	10	90
.450	60	40	10	90
.460	40	60	10	90
.490	20	80	10	90
.510	10	90	10	90
.550	5	95	5	95

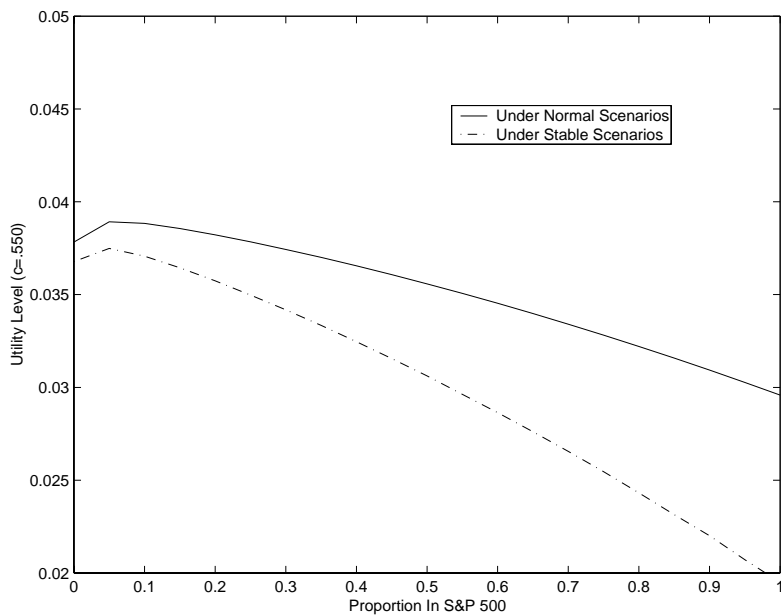


Figure 10: Utility Level When Risk Aversion is High

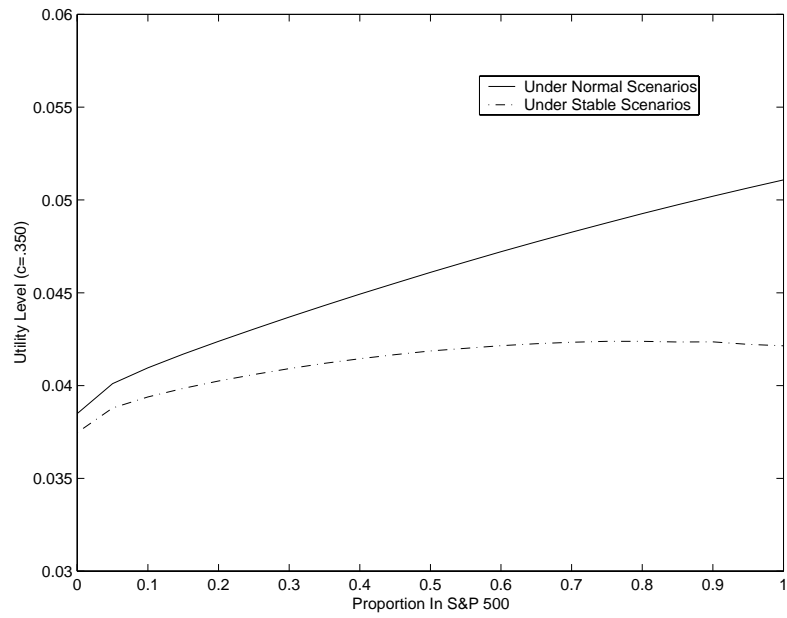


Figure 11: Utility Level When Risk Aversion is Low

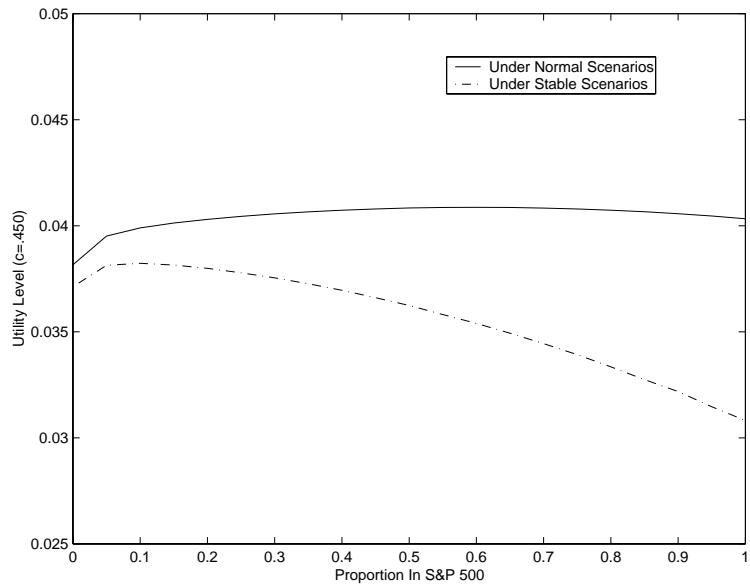


Figure 12: Utility Level For "Average" Investor

Table 5: Optimal Allocations under the Normal and Stable Scenarios

$c$	Normal Scenarios		Stable Scenarios	
	Optimal Percentage Invested		Optimal Percentage Invested	
	S&P 500	Treasury Bill	S&P 500	Treasury Bill
.4	100%	0%	100	0%
.5	100	0	90	10
.6	100	0	70	30
.7	100	0	55	45
.8	85	15	45	55
.9	70	30	35	65
1	60	40	30	70
1.5	30	70	15	85
2	15	85	10	90
4	5	95	5	95

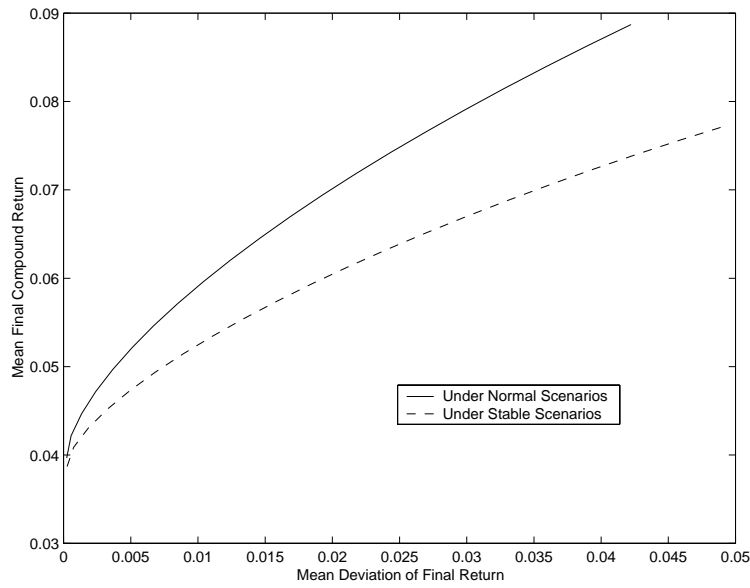


Figure 13: Risk-Reward Diagram under Normal and Stable Scenarios

Table 6: Optimal Allocations under the Normal and Stable Scenarios

$\gamma$	Normal Scenarios		Stable Scenarios	
	Optimal Percentage Invested		Optimal Percentage Invested	
	S&P 500	Treasury Bill	S&P 500	Treasury Bill
.7	100%	0%	100%	0%
.9	100	0	100	0
.99	100	0	100	0
1.5	100	0	95	5
1.8	100	0	80	20
2	90	10	75	25
3	60	40	55	45
4	45	55	40	60
5	40	60	35	65
10	20	80	20	80

The last utility function analyzed is power utility of final wealth:

$$U(W^i) = \frac{1}{S} \sum_{s=1}^S \frac{1}{(1-\gamma)} (W_s^i)^{(1-\gamma)}.$$

This utility function is not very sensitive to extreme events. Similar to the previous utility functions, both scenario generation methods result in the same optimal allocation when the risk aversion is very high or very low. However, when the level of risk aversion is in between these two extremes, the difference in optimal allocations is not as large as we had before, only up to 20%. When normal scenarios suggest to put 60 % in S&P, stable scenarios suggest to put 55% in S&P. Table 6 presents the optimal allocations under the normal and stable scenarios.

## 7 Conclusion

The ALM models that are based on stochastic programming with or without decision rules are starting to gain applicability in the industry. In these models, the future uncertainty is modeled using discrete scenarios. A representative set of scenarios describes the possible future economic situations facing the institution.

Generating scenarios that realistically represent the future uncertainty is important for the validity of the results of stochastic programming based ALM models. The assumption underlying the scenario generation models used in the literature is the normal distribution. The validity of normal distribution has been questioned in the finance and macroeconomics literature. The leptokurtic (heavy tailed and peaked) and asymmetric nature of the economic variables can be better captured by using stable distribution as opposed to normal distribution.

We analyze the effects of the distributional assumptions on optimal asset allocation. A multistage dynamic asset allocation model with decision rules has been set up. The optimal asset allocations found under normal and stable scenarios are compared. The analysis suggests that the normal scenarios may greatly underestimate risks depending on the utility function of the decision maker. Stable scenario modeling may lead to asset allocations that are up to 85% different from those of normal scenario modeling.

One drawback of the multistage stochastic asset allocation problem with decision rule is that once a decision rule, such as fixed mix, is adopted, it is used to update the asset allocation every period. This greatly reduces the computational complexity of the problem. However, it lacks realism due to the assumption of no recourse decision. In a later paper, we will analyze the asset allocation problem using stochastic programming with recourse. We will also evaluate the out of the sample gain of using the stable allocation instead of the normal allocation to see if and how much the agents benefit .

The financial data exhibit heavy tails, time varying volatility, and long range dependence. This study has only considered explicit modeling of heavy tails in the financial data. The conditional heteroskedastic models (ARMA-GARCH) utilizing stable distributions can be used to describe the time varying volatility along with the asymmetric and leptokurtic behavior. In addition to these, the long-range dependence can also be modeled if fractional-stable GARCH models are employed. These aspects of financial data will be considered in a later paper.

*Acknowledgements: We would like to thank Boryana Racheva-Jotova from Bravo Consulting for making Mercury Software available to us and for her computational assistance. We would also like to thank Carlo Marinelli for his computational assistance.*

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## 8 Appendix

Appendix 1: Time Series Model

Approach 1:

$$d(Tbill)_t = .321d(Tbond)_{t-1} + .320\epsilon_{t-1}^{Tbill} + \epsilon_t^{Tbill}, \quad (3)$$

$$d(Tbond)_t = .326d(Tbill)_t + resTbond_t,$$

$$resTbond_t = -.350resTbond_{t-1} + .699\epsilon_{t-1}^{Tbond} + \epsilon_t^{Tbond}, \quad (4)$$

Approach 2:

$$d(Tbill)_t = .292d(Tbill)_t - .119d(Tbill) + .335d(Tbond)_{t-1} + \epsilon_t^{Tbill}, \quad (5)$$

$$d(Tbond)_t = .387d(Tbond)_{t-1} - .225d(Tbond)_{t-2} + \epsilon_t^{Tbond}, \quad (6)$$

$$Inf_t = .004 + .007d(Tbond)_t + resInf_t$$

$$resInf_t = .965resInf_{t-1} - .705\epsilon_{t-1}^{Inf} + \epsilon_t^{Inf}, \quad (7)$$

$$Divg_t = .965Inf_t + resDivg_t$$

$$resDivg_t = -.527resDivg_{t-1} - .297resDivg_{t-2} + \epsilon_t^{Divg}, \quad (8)$$

$$d(Divy)_t = -.147d(Tbond)_t + .459d(Tbond)_{t-1} + \epsilon_t^{Divy} \quad (9)$$

All of the data are monthly and they cover the period of 2/1965-12/1999.

*Tbond* : logarithm of 10-year Treasury constant maturity yield- averages of business (source: Federal Reserve Board) ,

*Tbill* : logarithm of 3-month T bill rate, secondary market- averages of business days on a bank discount basis (source: Federal Reserve Board),

*Inf* : log differences of seasonally adjusted monthly CPI values (source: Federal Reserve Board),

*Divg* : log differences of dividend index, where dividend index is price index times dividend yield (source:Datastream),

*Divy* : logarithm of monthly dividend yield calculated from the annualized dividend yield (source:Datastream),

*d(.)*: first difference operator