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SURFACES, SUBMANIFOLDS, AND ALIGNED FOX REIMBEDDING IN NON-HAKEN 3-MANIFOLDS

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ABSTRACT. Understanding non-Haken 3-manifolds is central to many current endeavors in 3-manifold topology. We describe some results for closed orientable surfaces in non-Haken manifolds, and extend Fox’s theorem for submanifolds of the 3-sphere to submanifolds of general non-Haken manifolds. In the case where the submanifold has connected boundary, we show also that the ∂ -connected sum decomposition of the submanifold can be aligned with such a structure on the submanifold’s complement.

1. INTRODUCTION

A closed orientable irreducible 3-manifold N is called *Haken* if it contains a closed orientable incompressible surface; otherwise N is *non-Haken*. In Section 2 we describe some results for surfaces in non-Haken manifolds. Generalizing a theorem of Fox ([F]), we show in Section 3 that a 3-dimensional submanifold of a non-Haken manifold N is homeomorphic either to a handlebody complement in N or the complement of a handlebody in S^3 . Sections 2 and 3 are independent, but both represent progress towards understanding submanifolds of non-Haken manifolds. In Section 4 we combine the techniques from Section 2 with the results from Section 3 to show that if the submanifold $M \subset N$ is ∂ -reducible and has connected boundary, then the embedding can be chosen to align a full collection of separating ∂ -reducing disks in M with similar disks in the complement of M .

2. HANDLEBODIES IN NON-HAKEN MANIFOLDS

Let N be a closed orientable 3-manifold, F a closed orientable surface of non-trivial genus imbedded in N . Recall that F is *compressible* if there exists an essential simple closed curve on F which bounds an imbedded disk D in N with interior disjoint from F . D is a *compressing disk* for F .

Definition 1. *Suppose F is a separating closed surface in an orientable irreducible closed 3-manifold N . F is reducible if there exists an essential simple closed curve on F which bounds compressing disks on both sides of F . The union of the two compressing disks is a reducing sphere for F .*

Suppose \mathbf{S} is a collection of disjoint reducing spheres for F . A reducing sphere $S \in \mathbf{S}$ is redundant if a component of $F - \mathbf{S}$ that is adjacent to $S \cap F$ is planar. \mathbf{S} is complete if, for any disjoint reducing sphere S' , S' is redundant in $\mathbf{S} \cup S'$.

Let $\sigma(\mathbf{S})$ denote the number of components of $F - \mathbf{S}$ that are not planar surfaces.

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Since N is irreducible, any sphere in N is necessarily separating. Suppose a reducing sphere S' is added to a collection \mathbf{S} of disjoint reducing spheres. If S' is redundant, the number of non-planar complementary components in F is unchanged, since S' necessarily separates the component of $F - \mathbf{S}$ that it intersects and the union of two planar surfaces along a single boundary component is still planar. If S' is not redundant then the number of non-planar complementary components in F increases by one. Thus we have:

Lemma 2. *Suppose $\mathbf{S} \subset \mathbf{S}'$ are two collections of disjoint reducing spheres for F in N . Then $\sigma(\mathbf{S}) \leq \sigma(\mathbf{S}')$. Equality holds if and only if each sphere S' in $\mathbf{S}' - \mathbf{S}$ is redundant in $S' \cup \mathbf{S}$. In particular, \mathbf{S} is complete if and only if for every collection \mathbf{S}' such that $\mathbf{S} \subset \mathbf{S}'$, $\sigma(\mathbf{S}) = \sigma(\mathbf{S}')$.*

□

Let H be a handlebody imbedded in N . H has an *unknotted core* if there exists a pair of transverse simple closed curves $c, d \subset \partial H$ such that $c \cap d$ is a single point, d bounds an embedded disk in H and c (the *core*) bounds an imbedded disk in N transverse to ∂H . Note that the interior of the latter imbedded disk may intersect H .

Lemma 3. *Let F be a connected, closed, separating, orientable surface in a closed orientable irreducible 3-manifold N . Suppose that F has compressing disks to both sides. Then at least one of the following must hold:*

- (1) F is a Heegaard surface for N .
- (2) N is Haken.
- (3) There exist disjoint compressing disks for F on opposite sides of F .

Proof. The proof is an application of the generalized Heegaard decomposition described in [ST]. As F is compressible to both sides, we can construct a handle decomposition of N starting at F so that F appears as a “thick” surface in the decomposition. If F is not a Heegaard surface, then this decomposition contains a “thin” surface G adjacent to F . If G is incompressible in N , then N is Haken. If G is compressible we apply [CG] to obtain the required disjoint compressing disks for F . □

Theorem 4. *Let H be a handlebody of genus g imbedded in a closed orientable irreducible non-Haken 3-manifold N . Let G be the complement of H in N . Let $F = \partial H = \partial G$. Suppose F is compressible in G . Then at least one of the following must hold:*

- (1) The Heegaard genus of N is less than or equal to g .
- (2) F is reducible.
- (3) H has an unknotted core.

Proof. The proof is by induction on the genus of H . If $g = 1$, then the result of compressing F into G is a 2-sphere, necessarily bounding a ball in N . If a ball it bounds lies in G then the Heegaard genus of N is ≤ 1 . If a ball it bounds contains H then H is an unknotted solid torus in N and so it has an unknotted core.

Suppose then that $genus(H) = g > 1$ and assume inductively that the theorem is true for handlebodies of genus $g - 1$. Suppose that G , the complement of H , has compressible boundary. If G is a handlebody then $G \cup_F H$ is a Heegaard splitting of genus g and we are done. So suppose G is not a handlebody. Then by

Lemma 3 there are disjoint compressing disks on opposite sides of F , say D in H and E in G . Without loss of generality we can assume that D is non-separating. Compress H along D to obtain a new handlebody H_1 with boundary F_1 ; let G_1 be the complement of H_1 .

If ∂E is inessential in F_1 then it bounds a disk in $H_1 \subset H$ as well, so F is reducible.

If ∂E is essential in F_1 then E is a compressing disk in G_1 so we can apply the inductive hypothesis to H_1 . If 1 or 3 holds then it holds for H , and we are done. Suppose instead F_1 is reducible. Let \mathbf{S} be a collection of disjoint reducing spheres for F_1 chosen to maximize σ among all possible such collections and then, subject to that condition, further choose \mathbf{S} to minimize $|E \cap \mathbf{S}|$. Clearly $E \cap \mathbf{S}$ contains no closed curves, else replacing a subdisk lying in the disk collection $\mathbf{S} \cap G_1$ with an innermost disk of $E - \mathbf{S}$ would reduce $|E \cap \mathbf{S}|$. Similarly, we have

Claim 1 Suppose ϵ is an arc component of $\partial E - \mathbf{S}$ and F_0 is the component of $F_1 - \mathbf{S}$ in which ϵ lies. If ϵ separates F_0 (so the ends of ϵ necessarily lie on the same component of ∂F_0) then neither component of $F_0 - \epsilon$ is planar.

Proof of claim 1: Let c_0 be the closed curve component of $\partial F_0 \subset \mathbf{S} \cap F_1$ on which the ends of ϵ lie and, of the two arcs into which the ends of ϵ divide c_0 , let α be adjacent to a planar component of $F_0 - \epsilon$. Then the curve $\epsilon \cup \alpha$ clearly bounds a disk in both G_1 and H_1 and then so does the curve $c' = \epsilon \cup (c_0 - \alpha)$. Let S' be a sphere in N intersecting F_1 in c' and S_0 be the reducing sphere in \mathbf{S} containing c_0 . Replacing S_0 with S' (or just deleting S_0 if c' is inessential in F_1) gives a new collection \mathbf{S}' of disjoint reducing spheres, intersecting ∂E in at least two fewer points. Moreover $\sigma(\mathbf{S}') = \sigma(\mathbf{S})$ since the only change in the complementary components in F_1 is to add to one component and delete from another a planar surface along an arc in the boundary. Then the collection \mathbf{S}' contradicts our initial choice for \mathbf{S} , a contradiction that proves the claim.

Let H' be the closed complement of \mathbf{S} in H_1 , so H' is itself a collection of handlebodies.

Claim 2 Either F is reducible or $\partial H'$ is compressible in $N - H'$.

Proof of claim 2: If ∂E is disjoint from \mathbf{S} and is inessential in $\partial H'$, then ∂E bounds a disk in H' , hence in H , so F is reducible. If ∂E is disjoint from \mathbf{S} and is essential in $\partial H'$, then E compresses $\partial H'$ in $N - H'$, verifying the claim. Finally, if E intersects \mathbf{S} , consider an outermost disk A cut off from E by \mathbf{S} . According to Claim 1, this disk, together with a subdisk of \mathbf{S} , constitute a disk E' that compresses $\partial H'$ in $N - H'$, proving the claim.

Following Claim 2, either F is reducible or the inductive hypothesis applies to a component H_0 of H' . If 2 holds for H_0 then consider a reducing sphere S for H_0 , isotoped so that the curve $c = S \cap \partial H_0$ is disjoint from the disks $\mathbf{S} \cap H_0$. The disk $S - H_0$ may intersect H_1 ; by general position with respect to the dual 1-handles, each component of intersection is a disk parallel to a component of $\mathbf{S} \cap H_1$. But each such disk can be replaced by the corresponding disk in $\mathbf{S} - H_1$ so that in the end c also bounds a disk in $N - H_1$. After this change, S is a reducing sphere for F_1 in N and, since c is essential in H_0 , $\sigma(\mathbf{S} \cup S) > \sigma(\mathbf{S})$, contradicting our initial choice for \mathbf{S} . Thus in fact 1 or 3 holds for H_0 , hence also for H . \square

In the specific case $N = S^3$, we apply precisely the same argument, combined with Waldhausen's theorem [W] on Heegaard splittings of S^3 , to obtain:

Corollary 5. *Let H be a handlebody imbedded in S^3 , and suppose G , the complement of H , has compressible boundary. Then either H has an unknotted core or the boundary of H is reducible.*

This corollary is similar to ([MT], Theorem 1.1), but no reembedding of $S^3 - H$ is required.

3. COMPLEMENTS OF HANDLEBODIES IN NON-HAKEN MANIFOLDS

In [F] (see also [MT] for a brief version) Fox showed that any compact connected 3-dimensional submanifold M of S^3 is homeomorphic to the complement of a union of handlebodies in S^3 . We generalize this result to non-Haken manifolds, showing that a submanifold M of a non-Haken manifold N has an almost equally simple description, that is, M is homeomorphic to the complement of handlebodies either in S^3 or in N .

Definition 6. *Let N be a compact irreducible 3-manifold, and let M be a compact 3-submanifold of N . We will say the complement of M in N is standard if it is homeomorphic to a collection of handlebodies or to $N \# (\text{collection of handlebodies})$. (We regard B^3 as a handlebody of genus 0.)*

Note that in the latter case M is actually homeomorphic to the complement of a collection of handlebodies in S^3 .

Theorem 7. *Let N be a closed orientable irreducible non-Haken 3-manifold, and let M be a connected compact 3-submanifold of N with non-empty boundary. Then M is homeomorphic to a submanifold of N whose complement is standard.*

Proof. The proof will be by induction on $n+g$ where n is the number of components of ∂M and g is the genus of ∂M , that is, the sum of the genera of its components. If $n+g = 1$ then ∂M is a single sphere. Since N is irreducible, the sphere bounds a 3-ball in N . So either M or its complement is a 3-ball and in either case the proof is immediate.

To verify the inductive step, suppose first that ∂M has multiple components $T_1, \dots, T_n, n \geq 2$. Each component T_i must bound a distinct component J_i of $N - M$ since each must be separating in the non-Haken manifold N . Let $M' = M \cup J_n$; by inductive assumption M' can be reembedded so that its complement is standard. After the reembedding, remove J_n from M' , to recover a homeomorph of M and adjoin J_1 (now homeomorphic either to a handlebody or to $N \# (\text{handlebody})$) instead. Reembed the resulting manifold so that its complement is standard and remove J_1 to recover M , now with standard complement.

Henceforth we can therefore assume that ∂M is connected and not a sphere. Since N is non-Haken there exists a compressing disk D for ∂M in $N - \partial M$; the compressing disk lies either in M or in its closed complement J .

Case 1. ∂D is non-separating on ∂M .

If D lies inside M , compress M along D to obtain M' and use the induction hypothesis to find an imbedding of M' with standard complement. Reconstruct M by attaching a trivial 1-handle to M' , thus simultaneously attaching a trivial 1-handle to the complement.

If D lies outside M , attach a 2-handle to M corresponding to D to obtain M' , whose connected boundary has lower genus. Invoking the inductive hypothesis, imbed M' in N with standard complement. Reconstruct M from M' by removing a co-core of the attached 2-handle, thus adding a 1-handle to the complement of M' .

Case 2. ∂D is separating on ∂M .

Suppose D lies outside M . Then D also separates the closed complement J of M into two components, J_1 and J_2 , since $H_2(N) = 0$. Denote the components of $\partial M - \partial D$ by $\partial_1 \subset J_1$ and $\partial_2 \subset J_2$, both of positive genus. Let $M' = M \cup J_2$. Reimbed M' so that its complement is standard. The boundary of M' consists of ∂_1 together with a disk. Since the complement of M' is standard, there is a non-separating compressing disk D' for $\partial M'$ contained in the complement of M' . D' is also a non-separating compressing disk for the reimbedded ∂M (which is contained in M'). Apply case 1 to this new imbedding of M .

We can now suppose that the only compressing disks for ∂M are separating compressing disks lying inside M . Choose a family \mathbf{D} of such ∂ -reducing disks for M that is maximal in the sense that no component of $M' = M - \mathbf{D}$ is itself ∂ -compressible. Since each compressing disk is separating, $\text{genus}(\partial M') = \text{genus}(\partial M) > 0$ so $\partial M'$ is compressible in N . Such a compressing disk E can't lie inside M' , by construction, so it lies in the connected manifold $N - M'$; let M_1 be the component of M' on whose boundary ∂E lies. Since each disk in \mathbf{D} was separating, M has the simple topological description that it is the boundary-connect sum of the components of M' . So M can easily be reconstructed from M' in $N - M'$ by doing boundary connect sum along arcs connecting each component of $M' - M_1$ to M_1 in $N - (M' \cup E)$. After this reimbedding of M , E is a compressing disk for ∂M that lies outside M , so we can conclude the proof via one of the previous cases. \square

4. ALIGNED FOX REIMBEDDING

Now we combine results from the previous two sections and consider this question: If M is a connected 3-submanifold of a non-Haken manifold N and M is ∂ -reducible, to what extent can a reimbedding of M , so that its complement is standard, have its ∂ -reducing disks aligned with meridian disks of its complement. Obviously non-separating disks in M cannot have boundaries matched with meridian disks of $N - M$, since N contains no non-separating surfaces. But at least in the case when ∂M is connected, this is the only restriction.

Definition 8. For M a compact irreducible orientable 3-manifold, define a disjoint collection of separating ∂ -reducing disks $\mathbf{D} \subset M$ to be full if each component of $M - \mathbf{D}$ is either a solid torus or is ∂ -irreducible.

For M reducible, $\mathbf{D} \subset M$ is full if there is a prime decomposition of M so that for each summand M' of M containing boundary, $\mathbf{D} \cap M'$ is full in M' .

$M \subset N$ a 3-submanifold is aligned to a standard complement if the complement of M is standard and there is a (complete) collection of reducing spheres \mathbf{S} for ∂M so that $\mathbf{S} \cap M$ is a full collection of ∂ -reducing disks for M .

There is a uniqueness theorem, presumably well-known, for full collections of disks, which is most easily expressed for irreducible manifolds:

Lemma 9. Suppose M is an irreducible orientable 3-manifold with boundary and M is expressed as a boundary connect sum in two ways: $M = M_1 \natural M_2 \natural \dots \natural M_n =$

$M_1^* \natural M_2^* \natural \dots \natural M_{n^*}^*$, where each M_i, M_j^* is either a solid torus or ∂ -irreducible. Then, after rearrangement, $n^* = n$ and $M_i \cong M_i^*$.

Proof. One can easily prove the theorem from first principles, along the lines of e. g. [H, Theorem 3.21], the standard proof of the corresponding theorem for connected sum. But a cheap start is to just double M along its boundary to get a manifold DM . The decompositions above double to give connected sum decompositions of DM in which each factor consists of either $S^1 \times S^2$ or the double of an irreducible, ∂ -irreducible manifold which is then necessarily irreducible. Then [H, Theorem 3.21] implies that $n = n^*$ and that the two original decompositions of M also each contain the same number of solid tori. After removing these, we are reduced to the case in which the only ∂ -reducing disks in M are separating and $n^* = n$.

Following the outline suggested by the proof of [H, Theorem 3.21], choose a disk D that separates M into the component M_n and the component $M_1 \natural M_2 \natural \dots \natural M_{n-1}$. Choose disks E_1, \dots, E_{n-1} that separate M into the components $M_1^*, M_2^*, \dots, M_{n^*}^*$. Choose the disks to minimize the number of intersection components in $D \cap (\cup \{E_i\})$. Since each manifold is irreducible and ∂ -irreducible, a standard innermost disk, outermost arc argument (in D) shows that in fact D is then disjoint from $\{E_i\}$, so $D \subset M_n^*$ (say). Since M_n^* is ∂ -irreducible, D is ∂ -parallel in M_n^* so in fact (with no loss of generality) $M_n \cong M_n^*$ and $M_1 \natural M_2 \natural \dots \natural M_{n-1} \cong M_1^* \natural M_2^* \natural \dots \natural M_{n-1}^*$. The result follows by induction. \square

Theorem 10. *Let N be a closed orientable irreducible non-Haken 3-manifold, and M be a connected compact 3-submanifold of N with connected boundary. Then M can be reimbedded in N with standard complement so that M is aligned to the standard complement.*

Proof. The proof is by induction on the genus of ∂M . Unless M has a separating ∂ -reducing disk, there is nothing beyond the result of Theorem 7 to prove. So we assume that M does have a separating ∂ -reducing disk; in particular the genus of ∂M is $g \geq 2$. We inductively assume that the theorem has been proven whenever the genus of ∂M is less than g .

The first observation is that it suffices to find an embedding of M in N so that there is some reducing sphere S for ∂M in N . For such a reducing sphere divides $J = N - M$ into two components J_1 and J_2 . Apply the inductive hypothesis to $M \cup J_1$ to reimbed it with aligned complement J'_2 . Notice that by a standard innermost disk argument, the reducing spheres can be taken to be disjoint from S . After this reimbedding, apply the inductive hypothesis to $M \cup J'_2$ to reimbed it so that its complement J'_1 is aligned. After this reimbedding, M has aligned complement $J'_1 \cup_{S-M} J'_2$.

Our goal then is to find a reimbedding of M so that afterwards ∂M has a reducing sphere. First use Theorem 7 to reimbed M in N so that its complement J is standard, i. e. either a handlebody or $N\#$ (handlebody). Since M is ∂ -reducible, Lemma 3 applies: either M is itself a handlebody (in which case the required reimbedding of M is easy) or there are disjoint compressing disks D in J and E in M . Since J is standard, D can be chosen to be non-separating in J . Then ∂E is not homologous to ∂D in ∂M so ∂E is either separating in ∂M or non-separating in $\partial M - \partial D$. In the latter case, two copies of E can be banded together along an arc in $\partial M - \partial D$ to create a separating essential disk in M that is disjoint

from D . The upshot is that we may as well assume that $D \subset J$ is non-separating and $E \subset M$ is separating.

Add a 2-handle to M along D to get M' , still with standard complement J' . Dually, M can be viewed as the complement of the neighborhood of an arc $\alpha \subset M'$. If ∂E is inessential in $\partial M'$, it bounds a disk D' in $J' \subset J$. Then the sphere $D' \cup E$ is a reducing sphere for M as required. So we may as well assume that ∂E is essential in $\partial M'$ and of course still separates M' . By inductive assumption M' can be embedded in N so that its complement is aligned, but note that this does not immediately mean that ∂E itself bounds a disk in $N - M'$. Let \mathbf{S} be a complete collection of reducing spheres for $\partial M'$ intersecting M' in a full collection of disks.

E divides M' into two components, U and V with, say, $\alpha \subset U$. If M' is reducible (i.e. contains a punctured copy of N) an innermost (in E) disk argument ensures that the reducing sphere is disjoint from E . By possibly tubing E to that reducing sphere, we can ensure that the N -summand, if it lies in M' , lies in $U \subset M'$. That is, we can arrange that V is irreducible. E extends to a full collection of disks in M' , with the new disks dividing U and V into ∂ -connected sums: $U = U_1 \natural \dots \natural U_m$, $V = V_1 \natural \dots \natural V_n$, $m, n \geq 1$, with each U_i, V_j either ∂ -irreducible or a solid torus (with one of the U_i possibly containing N as a connect summand). By Lemma 9, some component V' of $M' - \mathbf{S}$ is homeomorphic to V_n . Tube together all components of \mathbf{S} incident to V' along arcs in $\partial V'$ to get a reducing sphere S' dividing M' into two components, one homeomorphic to V_n and the other homeomorphic to $U \natural V_1 \natural V_2 \natural \dots \natural V_{n-1}$. The latter homeomorphism carries $\alpha \subset U$ to an arc α' that is disjoint from the reducing sphere S' . Then $M' - \eta(\alpha')$ is homeomorphic to M and admits the reducing sphere S' . In other words, the reimbedding of M that replaces $M' - \eta(\alpha)$ with $M' - \eta(\alpha')$ makes ∂M reducible in N , completing the argument. \square

Corollary 11. *Given $M \subset N$ as in Theorem 10, suppose \mathbf{D} is a full set of disks in M . Then, with at most one exception, each component of $M - \mathbf{D}$ embeds in S^3 .*

Proof. Following Theorem 10 reimbed M in N with standard complement so that M is aligned to the standard complement. Then there is a collection \mathbf{S} of disjoint spheres in N so that, via Lemma 9, $M - \mathbf{S}$ and $M - \mathbf{D}$ are homeomorphic. Since N is irreducible, each component but at most one of $N - \mathbf{S}$ is a punctured 3-ball. Finally, each component of $N - \mathbf{S}$ contains at most one component of $M - \mathbf{S}$ since each component of \mathbf{S} is separating. \square

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