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HYDROMAGNETIC IONIZING FRONTS

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ABSTRACT

One of the techniques by which highly ionized plasmas can be generated in the laboratory makes use of strong, electromagnetically driven shock waves propagating into a cold gas. If a magnetic field already exists in the undisturbed region these shocks will in general not be gasdynamic in character but the current-carrying interface will coalesce with the ionizing front. The process has certain features in common with detonation waves, and differs from previously analyzed hydromagnetic shocks in that the electric field in the undisturbed region need not vanish. If the initial magnetic field has a longitudinal component the gas must be permitted to acquire a transverse velocity. Moreover, since such shocks are almost always compressive, the plasma will usually also have a forward velocity. In closed-end tubes, therefore, the front must be followed by a rarefaction wave in which the longitudinal flow is brought to rest again.

In this paper the phenomenon is analyzed as a one-dimensional single-fluid hydromagnetic problem, neglecting dissipation behind the wave. Zero conductivity is assumed for the region in front of the wave, and thermodynamic equilibrium is required behind. The problem is not determined unless an additional condition is imposed. We hypothesize that the rarefaction wave remains attached to the front. In the limit of essentially complete ionization behind the front, the problem can be

solved analytically as long as the transverse magnetic field there remains small compared with the longitudinal field. In this case the front velocity, plasma density and temperature, and the electric fields—as well as the structure of the rarefaction wave—can be expressed as simple functions of the initial magnetic field, the discharge current, the ionization energy, and the initial gas density. It is of particular interest to note that in this limit the compression is found to be very modest [$\rho_2 = \rho_1 (\gamma + 1)/\gamma$], and the trailing edge of the rarefaction wave propagates at half the speed of the front. It is also possible to generate non-compressive ionizing waves, provided that the magnetic field in the undisturbed region has a transverse component that is being appropriately reduced by the driving current flowing in the ionizing front.

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INTRODUCTION

In recent years it has become convenient to produce and heat highly ionized plasmas by means of electromagnetically driven shock waves. A great variety of shock tubes have been developed, and actually many pinch discharges and rapid-compression experiments fall into the same category. In the analysis of the dynamics of the phenomena it is usually assumed that the current-carrying region can be regarded as an impenetrable piston. This assumption is strictly justified only if the conductivity there is essentially infinite, and if no magnetic field exists in the undisturbed region. If a finite magnetic field is present ahead of the disturbance, however, in cases of interest some of the current will have to flow in the shock front itself. This is true even if the conductivity is infinite, and irrespective of whether the gas is already conducting or whether it is ionized by the shock itself.

This means that the shock is a hydromagnetic phenemenon, and the first current interface does not represent an impenetrable piston at all. Moreover, if the initial magnetic field has a component parallel to the direction of propagation of the disturbance, no real magnetic piston can exist anywhere. The piston-like discontinuity, or driving interface, in that case is replaced by a continuously expanding region of nonsteady flow, a rarefaction wave, in which the applied magnetic field spreads at a finite velocity through the propelled plasma. The flow pattern of plasma in a shock tube under these conditions has recently been analyzed for the case in which the gas ahead of the shock is already highly conducting.[1] In this paper we investigate the phenomenon for cases in which the gas ahead of the shock is not yet ionized, i.e., where the undisturbed region has essentially zero conductivity, and the ionization is assumed to take place in the front itself. We will use the term hydromagnetic ionizing front.

From the theory of gasdynamics it is well known that the speed of a plane shock, or the ratio in which the energy is distributed between internal energy and mass motion, is not uniquely determined by the conservation laws alone. In addition to the state of the undisturbed gas, either the shock speed or the flow velocity or the pressure of the gas behind the shock must be specified. The energy driving the shock and heating the gas can then be considered as being supplied by the flow itself (or by the piston). If, on the other hand, the shock is driven primarily by an independent energy release in the front itself, as for instance in the case of detonations, neither the shock speed nor the flow velocity nor the gas pressure behind the shock can be specified as given conditions. Therefore some other criterion must be found to render the problem unique. In the theory of gaseous detonations the Chapman-Jouguet hypothesis is used, according to which the gas leaves the combustion zone at exactly sonic speed.[2]

In hydromagnetics the electromagnetic energy driving a shock is likewise released in the front itself. In these phenomena either the flow conditions or the magnetic field behind the shock, but not both, may be considered as specified. The additional constraint needed to determine the hydromagnetic shock flow uniquely, in almost all analyses to date, has been the requirement that the electric field must vanish in the frame of the medium ahead of the shock as well as behind it. In other words, the gas has been assumed conducting in the undisturbed as well as in the shocked region. A very complete discussion of all the different types of shocks that may exist under these conditions has been given by Bazer and Ericson.[3]

If, on the other hand, the undisturbed gas has negligible conductivity, the electric field there may be finite and cannot be specified a priori. We conclude that in the analyses of hydromagnetic ionizing fronts, just as in the theory of detonations, another criterion must exist that determines the phenomenon uniquely. We repeat: hydromagnetic ionizing fronts differ from ordinary hydromagnetic shocks not only because some energy must be invested in ionization (and perhaps dissociation of molecules), but primarily because the electric field in the undisturbed region cannot be directly related to the shock velocity and the magnetic field. The last statement is equivalent to pointing out that the un-ionized medium ahead of the ionizing front does not permit any propagation of hydromagnetic signals. It is these latter features, and not the energy conversion in the ionization process, which make the phenomenon similar to gaseous detonations.

In this paper, then, we limit our discussion to magnetically driven ionizing shocks under the condition that a magnetic field exists in the undisturbed region ahead. Moreover, we focus our attention on cases where the field is not parallel to the plane of the ionizing front. It is certainly possible to devise experiments in the laboratory in which a hydromagnetic driver is constrained to move in a direction with a component parallel to a magnetic field existing ahead of it;[4] in some experiments the propagation is exactly along the magnetic field ahead of it.[5] We will show that such an ionizing wave may provide a unique and very useful way of producing a magnetized uniform plasma if certain requirements are fulfilled. In fact, this latter aspect has motivated the present investigation.

THE MODEL

In the analysis we restrict ourselves to a simplified one-dimensional model. The geometry is best explained with the help of Fig. 1. The gas is considered to be confined between two infinite conducting planes, both parallel to the xz plane. The initial magnetic field is also parallel to the xz plane, the applied electric field is always parallel to the y axis, and everything is assumed to be independent of both the y and z coordinates. This means we are looking at plane wave motion and are choosing our x coordinate along the direction of propagation. It also implies that we ignore the viscous drag at the flow boundaries, and any variation of the fluid properties, such as the electrical conductivity, that might appear in the neighborhood of the surfaces.

The gas ahead of the wave is assumed to be at rest, in equilibrium, and nonconducting. Furthermore, we assume that immediately behind the shock the gas is again in thermodynamic equilibrium, so that it obeys an equation of state and so that its relevant physical properties such as composition, electrical conductivity, etc. can be computed from equilibrium considerations. This means we are limiting ourselves to densities high enough to ensure sufficiently rapid equilibration rates. We need not make any assumptions concerning the shock structure in this case, other than requiring that the shock thickness is finite and constant. The exact mechanism of ionization is not under discussion here. The requirement of equilibrium behind the front implies that the current there is zero if the flow is steady. This means that the electric field must be zero in the frame of the moving gas behind the front, even if the gas has finite resistivity there. Therefore, the shock jump relations are always automatically independent of the transport properties, such as the conductivity.[6]

It is not immediately obvious that a steady wave should propagate in a shock-tube experiment in which, for instance, the current input is kept constant. Because shocks are usually compressive, the front must ordinarily be followed by an expansion wave with its nonsteady flow, unless a suitable additional driving piston is provided. However, it has been shown that in the limit of negligible dissipstion, i.e., isentropic conditions in the expansion region, the flow there can be described as a "centered rarefaction wave."[7] This means that, in this approximation at least, the entire flow pattern spreads at a uniform rate and draws constant total current, so that a steady shock can indeed be driven ahead of it. Accordingly, we treat the problem in two steps. First we discuss the shock relations under the assumptions of steady flow. Here we include the effects of dissociation and ionization and point out the conditions under which steady propagation should be possible. Then we look at the expansion wave, assuming negligible resistivity, viscosity, and thermal conductivity. Finally we combine the two regions to describe the entire phenomenon. The model is depicted schematically in Fig. 2. The situation and the analyses here are therefore very similar to those treated by Kemp and Petschek,[1] the only difference being that the latter assume complete dissociation and ionization ahead of the wave, whereas we require negligible electrical conductivity.

SHOCK RELATIONS

In accordance with Fig. 2, we distinguish quantities in the regions R_1 and R_2 ahead of an behind the shock by the subscripts 1 and 2, respectively. Since we assume the shock to be steady, it is most convenient to start out by describing the flow in a frame of reference in which the front is stationary (see Fig. 3a). The basicequations are then independent of time and, in our one-dimensional problem, may be integrated immediately to give the familiar symmetric jump conditions connecting the quantities in regions R_1 and R_2. It is easily shown that these relations do not depend explicitly on any of the irreversible processes occurring in the transition as long as no energy is lost by radiation; i.e., they are true conservation laws. These are then readily transformed to any other frame of reference in uniform motion with respect to the shock. It is instructive, and in fact algebraically economical, to express the shock jump conditions in a coordinate system fixed in the undisturbed un-ionized gas, which we shall call the laboratory frame. If we denote the speed of the shock in this frame by U and the velocity of the plasma by $\underline{V}$, i.e., $\underline{V}_1 = (0,0,0)$ and $\underline{V}_2 = (v_2, 0, w_2)$, (see Fig. 3), where all speeds are assumed to be small compared to the speed of light, these shock relations are

$$H_{x1} = H_{x2} = H_x, \tag{1}$$

$$\rho_1 U = \rho_2 (U - v_2), \tag{2}$$

$$\rho_1 U v_2 = p_2 - p_1 + \frac{1}{2}\mu (H_{z2}^{\,2} - H_{z1}^{\,2}), \tag{3}$$

$$-\rho_1 U w_2 = \mu H_x (H_{z2} - H_{z1}), \tag{4}$$

$$\rho_1 U \left(e_2 - e_1 + \frac{1}{2} v_2^{\,2} + \frac{1}{2} w_2^{\,2}\right) + \frac{1}{2}\mu U (H_{z2}^{\,2} - H_{z1}^{\,2})$$
$$= p_2 v_2 + E_2 H_{z2} - E_1 H_{z1}, \tag{5}$$

$$E_2 = E_1 + \mu U (H_{z2} - H_{z1}) = \mu (v_2 H_{z2} - w_2 H_x). \tag{6}$$

We have retained the symbols E_1 and E_2 for the electric fields on purpose, because these may be observables and because we want to point out the relationship to ordinary hydromagnetic theory. We have also written e for the internal energy per unit mass so that $e_1 = p_1/(\gamma_1-1)\rho_1$. Since the gas is being ionized in the shock transition we propose the following equation of state for the gas in region R_2:

$$e_2 = e_0 + [p_2/(\gamma_2-1)\rho_2] . \tag{7}$$

This means that we are assuming we can describe the plasma as a polytropic ideal gas with an additional "frozen-in" internal energy e_0, as, for instance, stored in dissociation and ionization. In general, both γ_2 and e_0 will be functions of p_2 and ρ_2, of course, and the system (1) through (7) must be supplemented by a set of equations which determines these relations. This requires numerical means,[8] and does not add any profound insight into our problem. In the analysis discussed here we simply consider both e_0 and γ_2 as given fixed quantities. The latter is, in fact, a valid approximation if the gas in R_2 is hot enough to be practically completely dissociated and fully ionized. In that case we simply have $e_0 = e_i + e_d$, the total energy of ionization and dissociation per unit mass, and $\gamma_2 = 5/3$. For hydrogen the approximation is good if, for instance, p_2 is less than one atmosphere, and the temperature exceeds 30 000°K, i.e., p_2/ρ_2 is greater than 5×10^8 m^2/sec^2.[8]

Equation (5), in the form given here, is most readily derived from the complete energy equation as given by Pai.[9] The form is interesting because it states that the work done on a unit volume of the undisturbed gas, including the energy change in the magnetic field, has to be provided by both a piston moving with the gas velocity v_2 and the negative divergence of the Poynting vector in the tube. It is the divergence of the Poynting vector which, at least in part, takes the place of the chemical energy released in a detonation wave. The piston, of which either p_2 or v_2 may be specified as the additional datum mentioned before, is necessary to ensure the assumed steady flow. We shall show, however, that here, as in the case of detonation waves, the flow is only maintained uniform by such a piston if its speed equals or exceeds a certain minimum.[2] If no such piston is provided, or if the piston is too slow, a region of nonsteady flow in the manner of a rarefaction wave appears between it and the propagating shock front, and the quantity p_2v_2 in Eq. (5) is not determined by the piston but by the dynamics of the expansion wave; i.e., the energy is taken from the expanding gas itself.

If we set $E_1 = 0$, $e_0 = 0$, and $\gamma_2 = \gamma_1$, the system of equations (1) through (7) is identical with the one derived very elegantly by Lüst for ordinary nonrelativistic hydromagnetic shocks,[10] the solutions of which have been adequately studied.[3] Since we have to abandon the condition $E_1 = 0$ for our ionising fronts, it is obvious that the set of algebraic equations (2) and (7) is insufficient to determine the solutions completely. Just as is done in the discussion of gaseous detonations, we can derive a relation between any two dependent variables—eliminating all the others with the help of the shock equations. This yields the locus of all possible solutions, thus affording us considerable insight into the nature of the phenomenon.

SIMPLIFIED SOLUTION

In the treatment of detonations, relations between p_2 and $1/\rho_2$, the so-called Hugoniots, are usually derived for purposes of discussion. In our case it is more instructive and convenient to find the relationship between U, the shock velocity, and v_2, the x component of the flow velocity behind the front. We use Eqs. (2) to (7) to express U, w_2, p_2, ρ_2, E_2, and hence also E_1, as functions of $\rho_1, p_1, \gamma_1, H_x, H_{z1}$, and of H_{z2}, γ_2, and e_0, as well as of v_2. Physically, this means that we are specifying the conditions in the undisturbed gas, and the current—but not the electric field. If we eliminate in Eq. (5) the quantities w_2, e_2, ρ_2, p_2, E_2 and E_1 with the help of Eqs. (2), (3), (4), (6), and (7), we obtain a relation of the fourth degree which is cubic in U and quadratic in v_2. We could solve this for v_2, and study the behavior of $v_2(U)$. Algebraically, however, it turns out to be much more convenient to introduce a set of new dimensionless variables which simplify the expressions considerably, and permit a much more direct inspection of the character of the solutions.

Let us define the following new variables:

$$\Delta H = H_{z2} - H_{z1} \neq 0 \qquad (8)$$

$$X = \frac{\rho_1 U v_2}{\mu(\Delta H)^2} \qquad (9)$$

$$Y = \frac{\rho_1 U^2}{\mu(\Delta H)^2} \qquad (10)$$

$$Z = \frac{\rho_1 U w_2}{\mu(\Delta H)^2} \qquad (11)$$

$$\Pi = \frac{p}{\mu(\Delta H)^2} \qquad (12)$$

$$\epsilon = \frac{\rho_1 e_0}{\mu(\Delta H)^2} \qquad (13)$$

$$a = \frac{H_x}{\Delta H} \tag{14}$$

$$\beta = \frac{H_{z1} + H_{z2}}{\Delta H} . \tag{15}$$

We are not interested in the case $\Delta H = 0$ because this is the ordinary gasdynamic shock. The parameter β can have any value, in principle. In particular, $\beta = 1$ implies $H_{z1} = 0$, $B = -1$ means $H_{z2} = 0$, and $\beta = 0$ refers to $H_{z2} = -H_{z1}$. In analogy to the nomenclature introduced for ordinary hydromagnetic shocks,[7] we shall call these cases magnetic "switch-on," "switch-off," and "transverse" ionizing fronts, respectively. With the above substitutions, the solution takes on the form

$$Y = \frac{(\gamma_2 + 1)X^2 + (\gamma_2 - 1 - \beta + 2\gamma_2 \Pi_1)X + (\gamma_2 - 1)a^2}{2X + 2(\gamma_2 - 1)\epsilon + \gamma_2 - 1 + \beta + 2\Pi_1(\gamma_2 - \gamma_1)/(\gamma - 1)} \tag{16}$$

$$Z = -a \tag{17}$$

$$\rho_2/\rho_1 = Y/(Y - X) \tag{18}$$

$$\Pi_2 = X - \beta/2 + \Pi_1 \tag{19}$$

$$\frac{E_2}{\mu U \Delta H} = \frac{E_1}{\mu U \Delta H} + 1 = \frac{a^2 + \frac{1}{2}(1+\beta)X}{Y} . \tag{20}$$

Although this form is still implicit, since X contains the dependent variable U, many features of the solutions are easily demonstrated. When E_1, ϵ, and $\gamma_2 - \gamma_1$ are all set equal to zero, these equations are again reduced, of course, to the ones investigated by Bazer and Ericson.[3] In particular, it is readily shown that in such a case X cannot be negative if the entropy is not supposed to diminish across the shock. Also, it is easily seen that under those circumstances X

can only be zero if $\beta = 0$, and then we have $Y = a^2$, and $\Pi_2 = \Pi_1$; i.e., the noncompressive so-called symmetrical or Alfvén shock.

None of these inferences can be drawn from Eqs. (16) to (20) if E_1 is allowed to differ from zero. This is the first important conclusion.

We shall now point out some of the general features of Eq. (16), which is plotted for various a's in Fig. 4. Of course we are only interested in the region $Y < a^2 + 1/2(1+\beta)X$ so that E_1 never vanishes.

(a) Equation (16) describes hyperbolas in the x-y plane. The asymptotes are:

$$X = \frac{1}{2}(1 + \beta - \gamma_2) - (\gamma_2 - 1)\epsilon + \frac{\gamma_2 - \gamma_1}{\gamma_1 - 1}\Pi_1 \tag{21a}$$

and

$$Y = \frac{1}{2}(\gamma_2 + 1) X + \frac{1}{4}(\gamma_2 - 1)(1 + \beta - \gamma_2) - \frac{1}{2}(\gamma_2^2 - 1)\epsilon + (\gamma_2 - 1)\frac{\gamma_1 + \gamma_2}{\gamma_1 - 1}\Pi_1 , \tag{21b}$$

i.e., they do not depend on the parameter a.

(b) When X is very large compared to $a, \beta, \gamma, \epsilon$, and Π_1, we have $Y \to \frac{1}{2}(\gamma_2+1)X$. This is the ordinary gasdynamic strong shock. We should expect this property because it is clear that the piston in Eq. (5) is doing practically all the work in this case.

(c) The curves $Y(X)$ have minima. The minima have as loci the straight lines

$$Y_m = (\gamma_2 + 1) X - \frac{1}{2}(1+\beta-\gamma_2) + \gamma_2 \Pi_1 . \tag{22}$$

These are seen to be independent of both α and ϵ. The fact that the Y(X) have minima means that for each set of given conditions ρ_1, p_1, ΔH, etc. the resulting relation $U(v_2)$ has a minimum. Again, this feature is reminiscent of the behavior of detonation waves. One might, for instance, be tempted to identify the minimum with the familiar Chapman-Jouquet point in the theory of gaseous detonations, although the analogy should not be stretched too far.

The analysis of gaseous detonation waves shows that at the point of minimum propagation speed the flow velocity of the gas behind the front relative to the front is always exactly sonic. That is, at that point the rarefaction wave follows the front immediately. Moreover, the entropy behind the front is a minimum when compared to values of entropy on other points along the $U(v_2)$ curve. The analogous conditions are generally not fulfilled for the propagation speeds Y_m of our hydromagnetically driven ionizing fronts. However, in the special case $\beta = -1$, the magnetic switch-off wave, we can show that the analogy is almost complete. This is the second important conclusion.

The proof is elementary. We merely have to express the relative velocity $u_2 = -(U-v_2)$ in terms of our new variables:

$$\frac{\rho_2 u_2^2}{\mu(\Delta H)^2} = Y - X. \tag{23}$$

Substitution from Eqs. (19) and (22) yields for the relative gas speed at the minimum of U

$$(u_2^2)_m = (U-v_2)^2_m = (1/\rho_2)\left[\gamma_2 p_2 + \frac{1}{2}(1+\beta)(\gamma-1)\mu(\Delta H)^2\right]. \tag{24}$$

If dissipation can be neglected, the propagation speeds c_2 along the x direction for small disturbances in the plasma in region R_2 are given by the relation[7,11]

$$c_2^2 \left[\frac{\mu}{\rho_2} (H_x^2 + H_{z2}^2) - c_2^2\right] = \frac{\gamma_2 p_2}{\rho_2} \left(\frac{\mu}{\rho_2} H_x^2 - c_2^2\right). \quad (25)$$

Obviously for $H_{z2} = 0$, we have $\beta = -1$, and hence

$$(u_2^2)_m = \frac{\gamma_2 p_2}{\rho_2} = c_2^2 .$$

Likewise, it can be readily shown that the change of entropy per unit mass $ds = (1/T)[de + pd(1/\rho)]$, taken along the curve $Y(X)$ at the point where $dY = 0$, is given by

$$(T_2 ds_2)_m = \frac{\mu}{2\rho_1} (1+\beta) (H_{z2} - H_{z1})^2 \frac{dX}{Y}. \quad (26)$$

which again is zero for $\beta = -1$. We shall therefore call this point in this special case the C-J (Chapman-Jouguet) point, and the mode of operation of the ionizing front at this point the C-J ionizing process.

This result is not too surprising because here the magnetic field has no transverse component behind the front, so that the gas flow in the x direction is purely accustic. The entropy produced in a switch-off ionizing wave can be shown to be a maximum at the C-J point rather than a minimum; it is therefore not clear whether the phenomenon is stable at this point.

In the theory of simple gaseous detonation, as pointed out before, it is usually argued that the C-J process must occur whenever there is no piston added that moves with a speed $v_2 > (v_2)_m$, the gas flow velocity in the x direction corresponding to the C-J point.[2] The same can be demonstrated here. It is easily verified that, in the case of $\beta = -1$, we have $\gamma_2 p_2 > \rho_2(U-v_2)^2$ for $v_2 > (v_2)_m$. This means that any rarefaction wave existing behind the shock will catch up with and weaken the shock, reducing both U and v_2—either until the flow behind the front is uniform, or until v_2 equals $(v_2)_m$, whichever is reached first. In that case, therefore, the situation $v_2 < (v_2)_m$ is never obtained. Besides, situations with $v_2 < (v_2)_m$, $\beta = -1$ are believed to be unstable, because they involve supersonic flow normal to the front on both sides of the shock. As a result, we can use Eq. (22) for $\beta = -1$ to express the additional condition for the C-J process. Hence we can eliminate either Y or X from Eq. (16) so that the problem of the switch-off wave is completely determined provided the C-J process itself is stable. It should be noted that White[12] has recently observed turbulence inC-J detonations.

In order to extend the solution to the general case $-\infty < \beta < +\infty$, we shall postulate here that the relevant physical condition determining the mode of operation according to the arguments in the previous paragraph is

$$U - v_2 = c_2 . \tag{27}$$

This means region R_2 in Fig. 2 is assumed always to be shrunk to zero length. Here c_2 is given by the smallest positive root of Eq. (25), because a magnetosonic expansion is a slow wave.[1]

It does, of course, seem possible that the actual propagation of ionizing fronts is governed by the ionization rate rather than by the magnetosonic conditions analyzed here. In particular it may be argued that E_1 must be sufficiently small so that no electric breakdown occurs in region R_1. In that case, however, a steady phenomenon can only result if v_2 is equal to or smaller than given by condition (27), because otherwise the expansion wave will overtake the front, causing a nonsteady or nonequilibrium flow. Equation (27) can therefore be regarded as a limiting condition on v_2 for steady propagation. In experiments where steady ionizing switch-on fronts have actually been observed,[5] one of the two electrodes (conducting plates) shown in Fig. 1 does not extend into region R_1, so that the electric field E_1 is, as it were, convected along with the speed U, and attenuates with increasing distance from the front. This means that the gas in region R_1 is exposed to electric fields of the magnitude of E_1 only for a very short time and a finite ionization rate is consistent with a steady propagation speed. Since a completely self-consistent calculation of ionization rates, and hence of the structure of the hydromagnetic front, is an exceedingly complex problem, we keep this discussion simple by assuming that condition (27) can be used as a good approximation for all cases of interest.

Equation (27) can be combined with Eq. (25) and rewritten, with the help of our new variables (8) to (15), to read

$$(Y-X)\left[\alpha^2 + \frac{1}{4}(1+\beta)^2 - Y + X\right] = \gamma_2(\alpha^2 - Y + X) . \quad (28)$$

Because of Eq. (19), and after some rearrangement, we finally obtain our general subsidiary equation

$$(1+\beta)^2 (Y - X) = 4(\gamma_2 X + X - Y - \frac{1}{2}\gamma_2\beta + \gamma_2 \Pi_1)(\alpha^2 + X - Y) . \quad (29)$$

The algebraic solution of the simultaneous equations (16) and (29) is still cumbersome unless $\beta = -1$ or $\alpha = 0$.

First we examine the case where $\alpha = 0$, i.e., $H_x = 0$. In this case the smallest root of Eq. (25) vanishes and $c_2 = 0$. This means that $\rho_2 = \infty$, and $Y = X$; i.e., we get a so-called "snowplow" solution and, of course, there is no expansion wave. In particular we find, neglecting p_1,

$$U = v_2 = (2e_0)^{1/2} , \quad (30)$$

$$p_2 = 2\rho_1 e_0 - \frac{1}{2}\mu (H_{z2}^{\ 2} - H_{z1}^{\ 2}) , \quad (31)$$

$$E_2 = \mu H_{z2} U, \quad (32)$$

$$E_1 = \mu H_{z1} U . \quad (33)$$

These results differ from previous snowplow solutions, because in the earlier treatments the energy equation was not used. We obtain the conventional form of the plane snowplow solution if we eliminate e_0 between Eqs. (30) and (31), and arbitrarily set $p_2 = 0$. In view of the predicted infinite density and the possible negative pressures, according to Eq. (31), it is quite clear that our model is not any better than the earlier one. In fact, we must conclude that an ionizing front will not be steady if $\alpha = 0$ and Eq. (27) applies.

If Eq. (27) is abandoned, steady solutions are possible of course. Since there cannot be an expansion wave when $\alpha = 0$, the flow is similar to that driven by a conventional impenetrable piston. This situation has recently been studied in more detail by Lyubimov and Kulikovskii,[13, 14, 15] who decided that they had to supply information concerning the shock structure and dissipative effects in order to arrive at unique solutions. It is interesting to note that they found conditions under which current-free, i.e., ordinary gasdynamic ionizing shocks should propagate ahead of the current-carrying interface. The question of stability was not yet considered, however. Clearly, the special case of $\alpha = 0$ and $\beta = +1$, i.e., $H_x = H_{z1} = 0$, gives no trouble if the conductivity is sufficiently high behind the front, because in that case E_1 is certainly vanishingly small and the usual model of the idealized flow in a magnetically driven shock tube should be valid. The expressions for the velocities, pressures, and electric fields in all these cases differ somewhat from those given in Eqs. (30) to (33), of course. We shall not discuss these here, but rather limit the treatment to the range of values of $\alpha > \alpha_{crit}$ for which the speed of the expansion wave is fast enough to rule out the possibility of purely gasdynamic shocks even if the conductivity were infinite. In this case Eq. (27) can certainly be used as a limiting condition.

For simplicity we examine the important case where

$$\alpha^2 \gg (1 + \beta)^2, \tag{34}$$

so that we can use as a good approximation

$$Y = (\gamma_2 + 1)\, X - \frac{1}{2}\gamma_2\beta + \gamma_2 \Pi_1 . \tag{35}$$

A plot of Eq. (35) is also included in the example on Fig. 4. For $\beta = -1$, both Eqs. (29) and (35) are identical with Eq. (22), and then Eq. (35) is valid for all $\alpha > 0$. Certainly for experiments in which $H_x \gg H_{z1}$ and $H_x \gg H_{z2}$, Eq. (35) is adequate. We may, moreover, always neglect Π_1, because we will certainly need $\Pi_1 \ll 1$ in ionizing hydromagnetic waves; Π_1 was only carried in our equations for the sake of completeness. The subscript of γ_2 may then also be dropped. If we now use Eq. (35) to eliminate X from Eq. (16) we obtain the solution for the wave speed

$$Y = (A+B^2)^{1/2} - B , \tag{36}$$

where

$$A = (\gamma^2 - 1)\,\alpha^2 + \frac{1}{2}\beta\gamma\left(\frac{1}{2}\beta\gamma + \gamma - 1 - \beta\right)$$

and

$$B = (\gamma^2 - 1)\,\epsilon + \frac{1}{2}\gamma\,(\gamma - 1 - \beta) .$$

The terms containing β in this expression are strictly justified only for $(1+\beta)^2 << 1$, because of condition (34). For $A >> B^2$, i.e., $\mu H_x \Delta H >> \rho_1 e_0$, we find

$$U^2 \approx (\mu/\rho_1) H_x \Delta H \; (\gamma^2 - 1)^{1/2} . \tag{37}$$

For $B^2 >> A$, on the other hand, we have

$$U \approx \frac{\mu H_x \Delta H}{\rho_1 (2e_0)^{1/2}} . \tag{38}$$

In Fig. 5 we show a plot of Y as a function of α for $\beta = \pm 1$, $\gamma = 5/3$, and a variety of values for ϵ.

The other quantities of interest—v_2, ρ_2, p_2, and E_2—are most easily expressed in terms of U, the wave speed, by using Eqs. (35), (18), (19), and (20). In these, too, we shall ignore p_1 everywhere and drop the subscript of γ_2. From Eqs. (35), we immediately obtain

$$v_2 = \frac{U}{\gamma+1} \left(1 + \frac{\beta\gamma}{2Y}\right) \tag{39}$$

and, by using Eq. (18),

$$\rho_2 = \rho_1 \, (1 + 1/\gamma) \left(1 - \frac{\beta}{2Y}\right)^{-1} . \tag{40}$$

According to Eq. (19), p_2 is given by

$$p_2 = \frac{\rho_1 U^2}{\gamma + 1} \left(1 - \frac{\beta}{2Y}\right) . \tag{41}$$

This also determines the temperature behind the front as

$$(RT)_2 = \frac{p_2}{\rho_2} = \frac{\gamma U^2}{(\gamma+1)^2} \left(1 - \frac{\beta}{2Y}\right)^2 . \tag{42}$$

Finally, the electric field in the region R_2 is determined from Eq. (20) to be

$$E_2 = \frac{\mu \Delta H}{U}\left[\frac{\mu}{\rho_1} H_x^2 + \frac{(1+\beta)U^2}{2(\gamma+1)}\left(1 + \frac{\beta\gamma}{2Y}\right)\right]. \tag{43}$$

SPECIFIC CONCLUSIONS

We have shown that many of the previously drawn conclusions concerning hydromagnetic shock jump properties cannot be carried over to the important case where E_1 is allowed to differ from zero. Also we have shown that the magnetic switch-off ionizing wave is almost in complete analogy with Chapman-Jouguet detonation theory.

From the set of relations (36) to (43) further conclusions concerning these hydromagnetic ionizing fronts may be drawn immediately. First of all, it is easily demonstrated, with the help of Eq. (16), that $a^2 \gg Y \gg 1$ if both $a^2 \gg (1+\beta)^2$ and $a^2 \gg 1$ are fulfilled. Equations (36) to (43) therefore show that under these circumstances v_2, ρ_2, p_2, and E_2 do not depend strongly on β. Also, we see that in this case the difference between conditions (22) and (35) is negligible. In other words, if the longitudinal magnetic field H_x is much stronger than both H_{z1} and H_{z2}, Eqs. (36) through (43) can be expected to describe the phenomenon rather well, even if the postulate (27) is not the correct one. This is the third important conclusion.

Furthermore, certain interesting features pertaining to the extreme case mentioned above are worth pointing out. Equation (40) in this limit states that ρ_2/ρ_1 is remarkably insensitive to changes in the independent variables, the value being surprisingly low. For example, for $\gamma = 5/3$ we have $\rho_2/\rho_1 \approx 1.6$.

Substitution for U from Eq. (36) in Eq. (43) shows that E_2 varies only slowly with ΔH. In fact, for $\mu H_x \Delta H \ll \rho_1 e_0$ Eq. (38) applies, and we have

$$E_2 \approx \mu H_x (2e_0)^{1/2}, \qquad (44)$$

which is independent of the current and gas density. Equation (44) as well as Eq. (30) resemble the findings by Alfvén[16] and Fahleson,[17] although the experiments described by them apparently did not involve distinct fronts producing full ionization, as assumed in our model. Equation (38), when combined with Eq. (4), can also be written

$$w_2^2 = 2e_0 . \qquad (45)$$

Actually, when Eq. (38) applies, the temperature T_2 is often too low to justify the original assumption of complete ionization.

In Fig. 6, Eq. (43)—for the case of $\beta = +1$—is plotted in a non-dimensional form, i.e., expressing the quantity $E_2/\mu H_x (2e_0)^{1/2}$ as a function of $\Delta H \left[\mu/(\rho_1 e_0)\right]^{1/2}$ for various values of $H_x \left[\mu/(\rho_1 e_0)\right]^{1/2}$. The solid curves are fair approximations also for $\beta \neq 1$, provided that $(1+\beta)^2 \ll \alpha^2$. The predictions of Eqs. (36) through (43) may be compared with the experimental findings of Wilcox et al., in which $\beta = +1$.[5] Although their geometry is not one-dimensional but cylindrical, their observations agree fairly well with some of the major conclusions arrived at here (slow uniform propagation speed of a distinct front, voltage regulations, etc.).[18] More extensive comparison between theory and experiment is planned for the near future.

Whereas the magnetic "switch-on" wave is of particular interest to the experimentalist because of the simplicity in instrumentation, the "switch-off" wave is more attractive from the analytical point of view. In addition to the close correspondence to gaseous detonation waves, in the switch-off case, we note that both Eqs. (16) and (20) become simplified. In particular, it is interesting to see that, for $\beta = -1$, Eqs. (36) through (43) are exact, the only restriction being $a > 0$.

Finally we investigate under what conditions v_2 can be zero, i.e., $\rho_2 = \rho_1$. As pointed out before, Eqs. (16) through (20) do not restrict X to values greater than zero if β is permitted to take on values less than zero. In our model of a closed input end of the tube, v_2 can never be negative. If conditions in the front call for $v_2 < 0$, a precompression shock is set up, violating the assumption of gas at rest in region R_1. If the precompression shock is strong enough to ionize the gas, the front will change its character so that v_2 is greater than zero. In a very similar manner, deflagrations are changed into detonations in the case of closed gas-combustion tubes. We certainly may set $X = 0$ in both Eqs. (16) and (29), and obtain two simultaneous equations in Y, β, and a:

$$Y_0 = \frac{(\gamma-1)a^2}{2(\gamma-1)\epsilon + \gamma - 1 - \beta} \tag{46}$$

and

$$-(1+\beta)^2 Y_0 \gtrsim 2(2Y_0 + \beta\gamma)(a^2 - Y_0) . \tag{47}$$

We use the symbol $\gtrsim$ to allow values of $c_2 \gtrsim U$ in Eq. (27). If we eliminate Y_0 between Eqs. (46) and (47), we find the minimum condition for $-\beta$ as a function of a and ϵ that makes $v_2 = 0$ possible. We shall not do this here, because it is lengthy and not particularly instructive. However, we may also ask what can be the maximum α for which a switch-off wave, $\beta = -1$, does not yet bring about a compression. This means that, after imposing $\beta + 1 = 0$ in Eqs. (46) and (47), we solve for α. The result is

$$\alpha^2 \leqslant \gamma \left[\epsilon + \frac{\gamma}{2(\gamma - 1)} \right] . \qquad (48)$$

We may, of course, express this relation as a condition for the minimum admissible value of H_{z1} if H_x, e_0, ρ, and γ are all given:

$$H_{z1}{}^2 \geqslant (2/\gamma^2)(\gamma - 1) \left[H_x{}^2 - (\gamma/\mu)\, \rho e_0 \right]. \qquad (49)$$

The propagation speed of the front is then given directly by Eq. (46). The transverse velocity becomes independent of H_x:

$$w_2{}^2 = 2e_0 + \frac{\gamma\mu}{(\gamma - 1)\rho} H_{z1}{}^2 . \qquad (50)$$

The expression for the pressure is simply

$$p_2 = \frac{1}{2} \mu H_{z1}{}^2 , \qquad (51)$$

which imposes a required minimum on H_{z1} to ensure adequate ionization. The electric fields are

$$E_2 = - \mu w_2 H_x \qquad (52)$$

and

$$E_1 = E_2 \left(1 - \frac{\rho U^2}{\mu H_x{}^2}\right) .$$

The situation is particularly simple for $\mu H_x{}^2 >> \gamma \rho e_0$. In that case, Eq. (49) reduced to

$$H_{z1}/H_x \gtrsim \left[2(\gamma-1)\right]^{1/2}/\gamma \tag{53}$$

for $\gamma = 5/3$. Moreover, both U and the impedance $-E_2/H_{z1}$ become independent of current (the minus sign refers to the fact that, for $\beta < 0$, E is negative if H_{z1} is positive):

$$U^2 \approx \frac{(\gamma-1)\mu}{\gamma\rho} H_x^2 , \tag{54}$$

$$E_2 \approx \gamma E_1 \approx -\left[\frac{\gamma\mu}{(\gamma-1)\rho}\right]^{1/2} \mu H_x H_{z1} , \tag{55}$$

whereas

$$w_2^2 \approx \frac{\gamma\mu}{(\gamma-1)\rho} H_{z1}^2 = \frac{2}{\gamma-1} \frac{\gamma p_2}{\rho} . \tag{56}$$

We feel that such a switch-off ionizing wave would be a very suitable means of generating a uniform magnetized plasma. After the plasma is formed, the resulting transverse motion is easily arrested by shorting out E_2 through a suitable resistor so that a simple Alfvén-wave relaxation will take place without disturbing the state of the gas. It would be interesting to try to realize this situation experimentally, and to test the various conclusions arrived at in this analysis.

For $v_2 > 0$, however, the front must be followed by a rarefaction wave. A very brief discussion of this phenomenon is presented in the next section.

RAREFACTION WAVE

In the treatment of the rarefaction wave we must assume that dissipation is negligible. This was already necessary when Eq. (25) was introduced; otherwise, the analysis becomes extremely complicated. Even so, in general the quantitative description of one-dimensional isentropic rarefaction waves requires numerical integration because they are nonlinear phenomena. Such computations have been carried out for a number of examples by Kemp and Petschek[1] and, in principle, their results could be used in conjunction with our shock solutions to describe the entire flow completely. We shall not go into such detail here. The equations of motion for the centered rarefaction wave become very much simplified, however, if the value of H_x/H_{z2} is large. In that case the situation can be approximated by the familiar isentropic acoustic solution, and analytic treatment is possible.[19] In particular, it can be shown that in region R_4 (see Fig. 2), where we require that $v_4 = 0$, the speed of sound c_4 is given by

$$c_4 \approx \frac{1}{2} U \left\{ 1 - (\beta\gamma/2\gamma) \right], \tag{57}$$

if v_2 and c_2 obey Eqs. (39) and (27), respectively. This means that the tail of the expansion wave moves at roughly half the speed of the front. The density ρ_4, accordingly, is

$$\rho_4 \approx \rho_2 \left(\frac{\gamma+1}{2\gamma}\right)^{2/(\gamma-1)} \approx 2\rho_1 \left(\frac{\gamma+1}{2\gamma}\right)^{(\gamma+1)/(\gamma-1)} . \tag{58}$$

where the value of ρ_1 was substituted from Eq. (40). For $\gamma = 5/3$, this yields $\rho_4 \approx 0.8\ \rho_1$.

Therefore it appears that the expansion produced by a hydromagnetic ionizing wave is very mild, and about half the length of the generated plasma is uniform and without longitudinal motion if H_z is much less than H_x.

Pressure and temperature in region R_4 may also be computed. The results are

$$p_4 \approx p_2 \left(\frac{\gamma+1}{2\gamma}\right)^{2\gamma/(\gamma-1)} \approx \frac{\rho_1 U^2}{2\gamma} \left(\frac{\gamma+1}{2\gamma}\right)^{(\gamma+1)/(\gamma-1)} \tag{59}$$

and

$$(RT)_4 \approx (RT)_2 \left(\frac{\gamma+1}{2\gamma}\right)^2 \approx \frac{U^2}{4\gamma} , \tag{60}$$

where the values of p_2 and $(RT)_2$ are substituted from Eqs. (41) and (42).

Finally we calculate H_{z4} and E_4 (or w_4) in this approximation. We find

$$\mu H_x^2 \, dH_z \approx - H_z dp,$$

so that we have

$$H_{z4} \approx H_{z2} \exp\left[\frac{\mu}{H_x^2}(p_2-p_4)\right] \tag{61}$$

$$\approx H_{z2} \left[1 + \frac{\mu}{H_x^2}(p_2-p_4)\right].$$

Similarly, we deduce the approximate solution

$$w_4 \approx w_2 - v_2 H_{z2}/H_x$$

so that we have

$$E_4 = - \mu w_4 H_x \approx E_2 . \tag{62}$$

For large H_x/H_{z4}, the net impedance of the shock tube, which we may express as $E_4 (H_{z4}-H_{z1})^{-1}$, is then essentially computed from Eq. (43), where U must be evaluated from Eq. (36). That is, the expansion wave does not contribute appreciably to the electrical behavior. In retrospect, this is fortunate because large current densities at finite conductivity in region R_3 would certainly conflict violently with the assumption of isentropic flow there. We conclude that the major deviation from this idealized model will be caused by the finite viscosity of the plasma, which must definitely cause considerable dissipation. It is therefore essential that the channel in which such a plasma is generated is not too narrow in the direction of the electric field.

This discussion may suffice to outline the principal features of hydromagnetic ionizing waves and of the plasma which can be generated by them.

FOOTNOTES AND REFERENCES

[*] Work done under auspices of the U. S. Atomic Energy Commission.

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FIGURE LEGENDS

Fig. 1. Idealized experiment with plane hydromagnetic ionizing waves.

Fig. 2. Model for analysis of hydromagnetic ionizing waves.

Fig. 3 Schematic for shock conditions. Note that in this example the current is in the +y direction so that the velocity w_2 is negative (-z direction).

Fig. 4. Plot of Y(X), Eq. (16), for various values of α^2. This includes plots of Eqs. (21) and (35).

Fig. 5. Plot of Y(α) for various values of ε and β.

Fig. 6. Plot of $E_2(\Delta H)$, Eq. (43), for various values of $\mu H_x^2/\rho_1$ (made nondimensional).

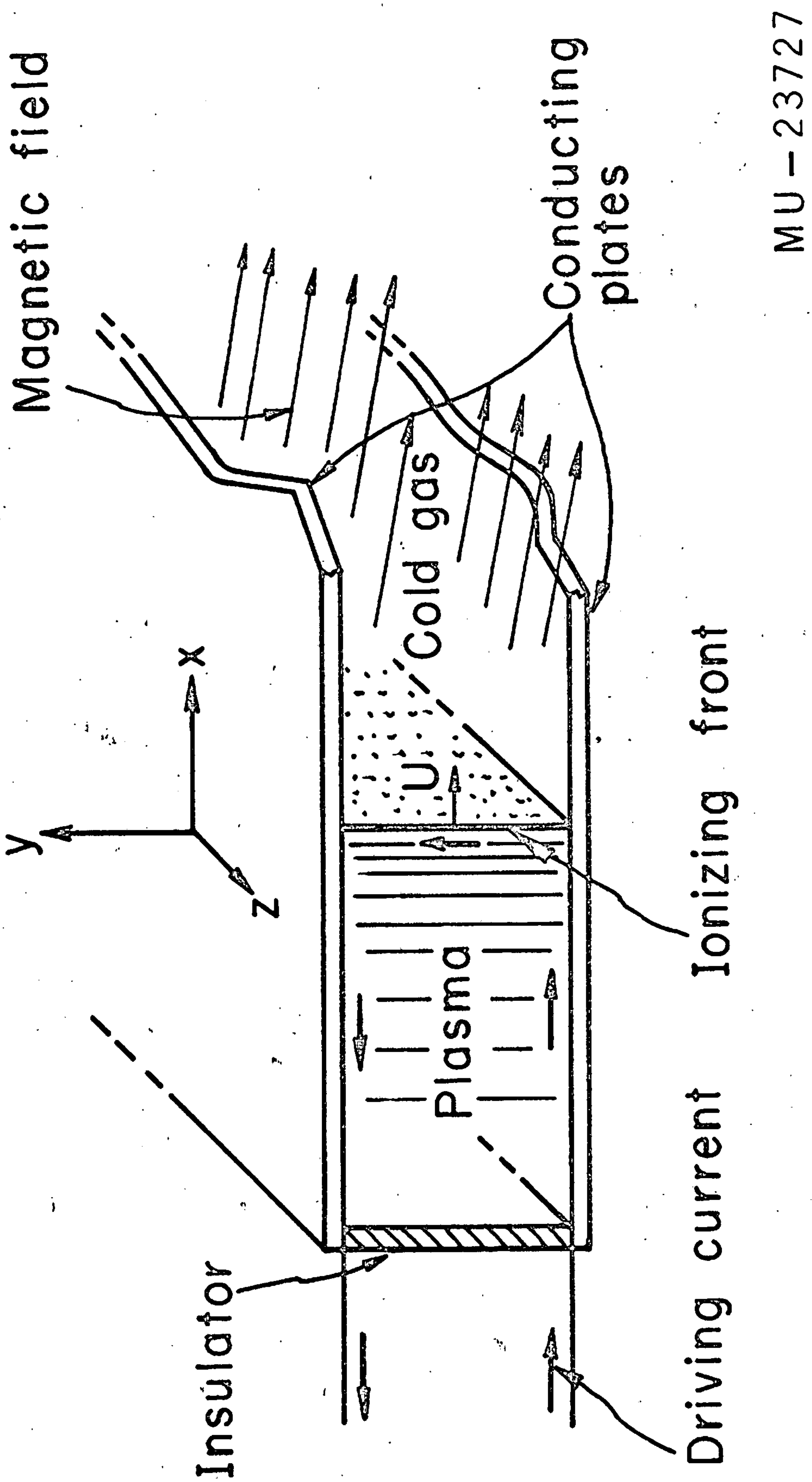

Magnetic field
Conducting plates
Cold gas
x
y
z
U
Ionizing front
Plasma
Insulator
Driving current
MU-23727

Fig. 1.

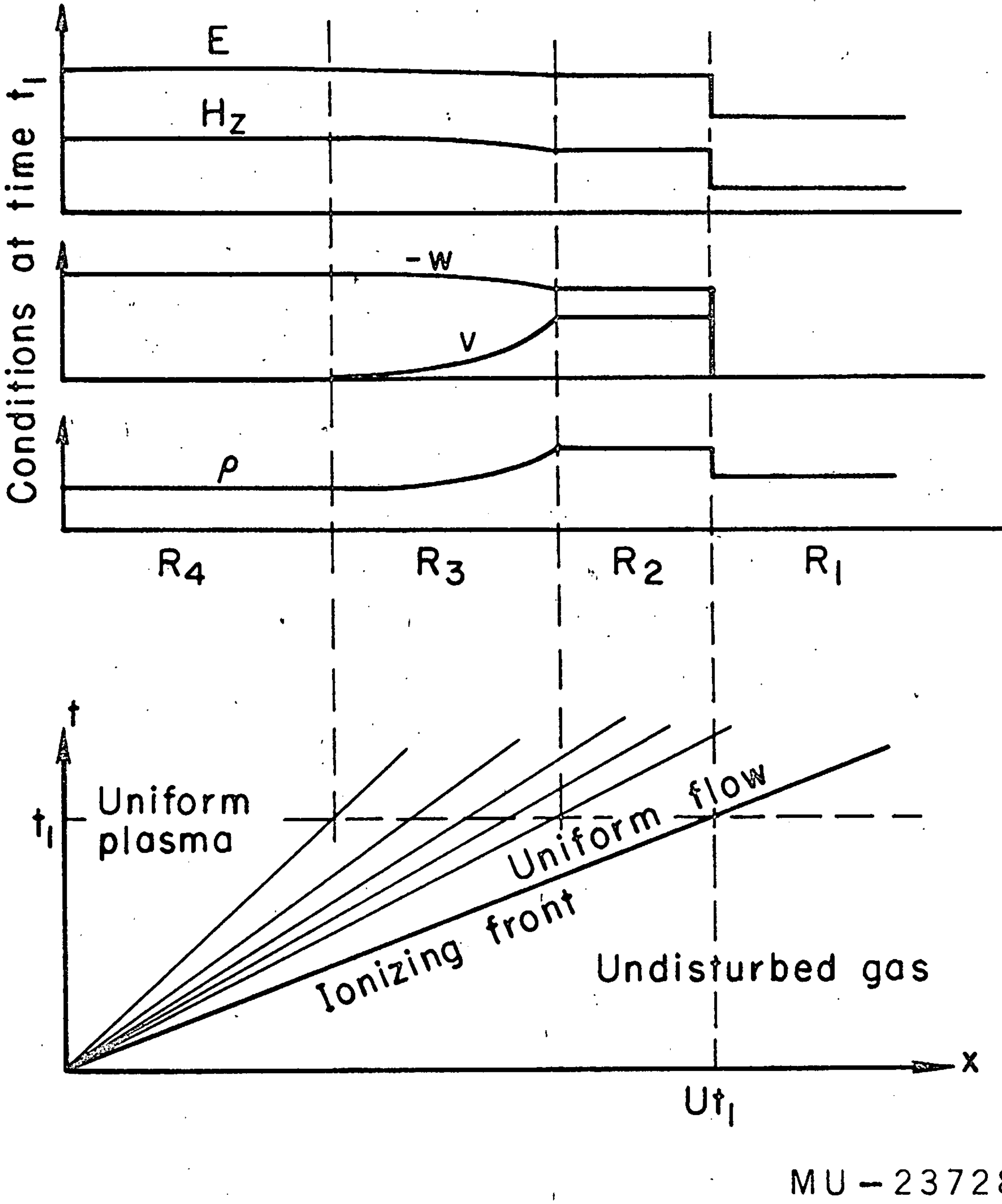

E
H_z
Conditions at time t_1
-w
v
ρ
R_4
R_3
R_2
R_1
t
t_1
Uniform plasma
Uniform flow
Ionizing front
Undisturbed gas
x
Ut_1
MU-23728

Fig. 2.

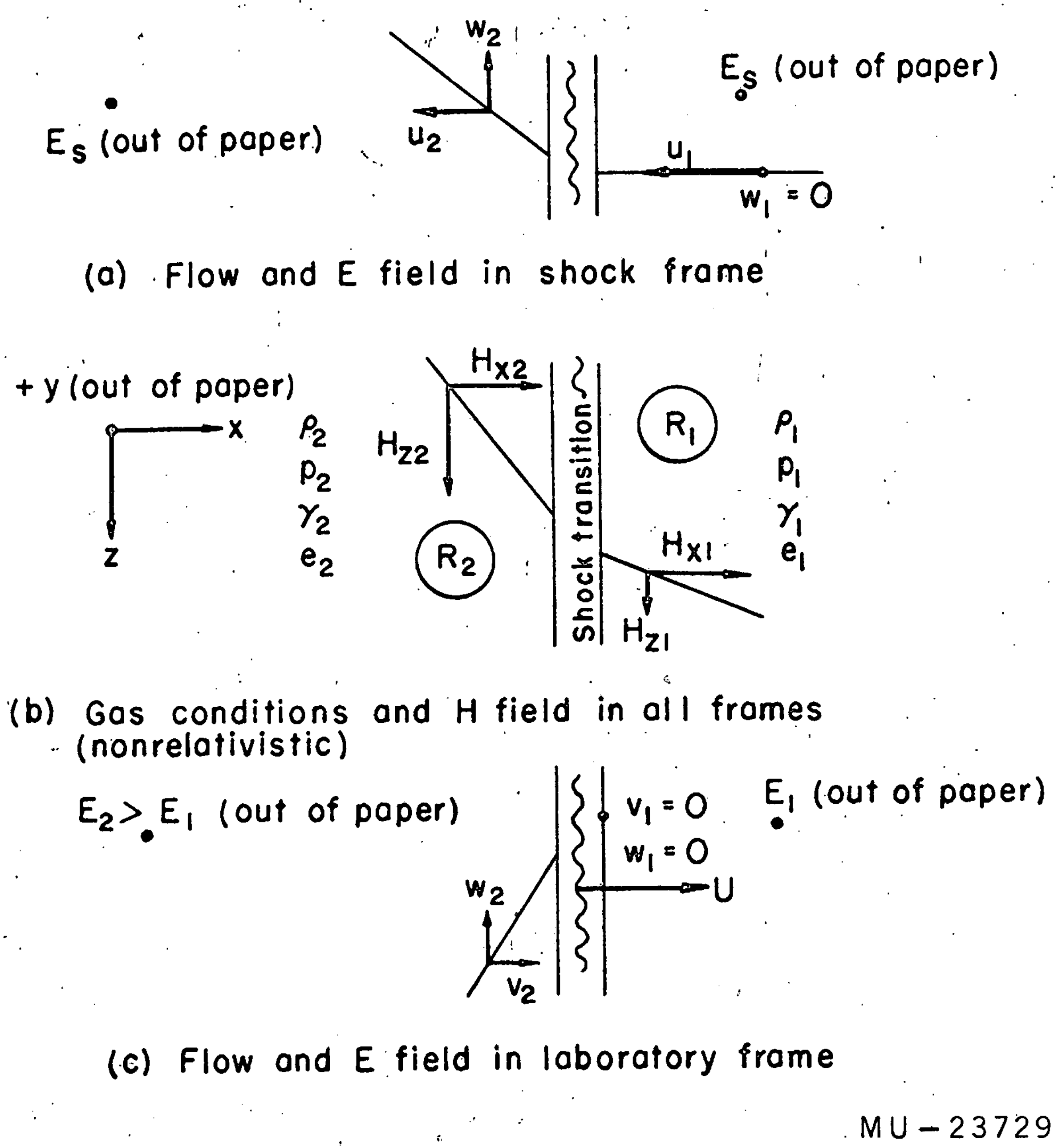
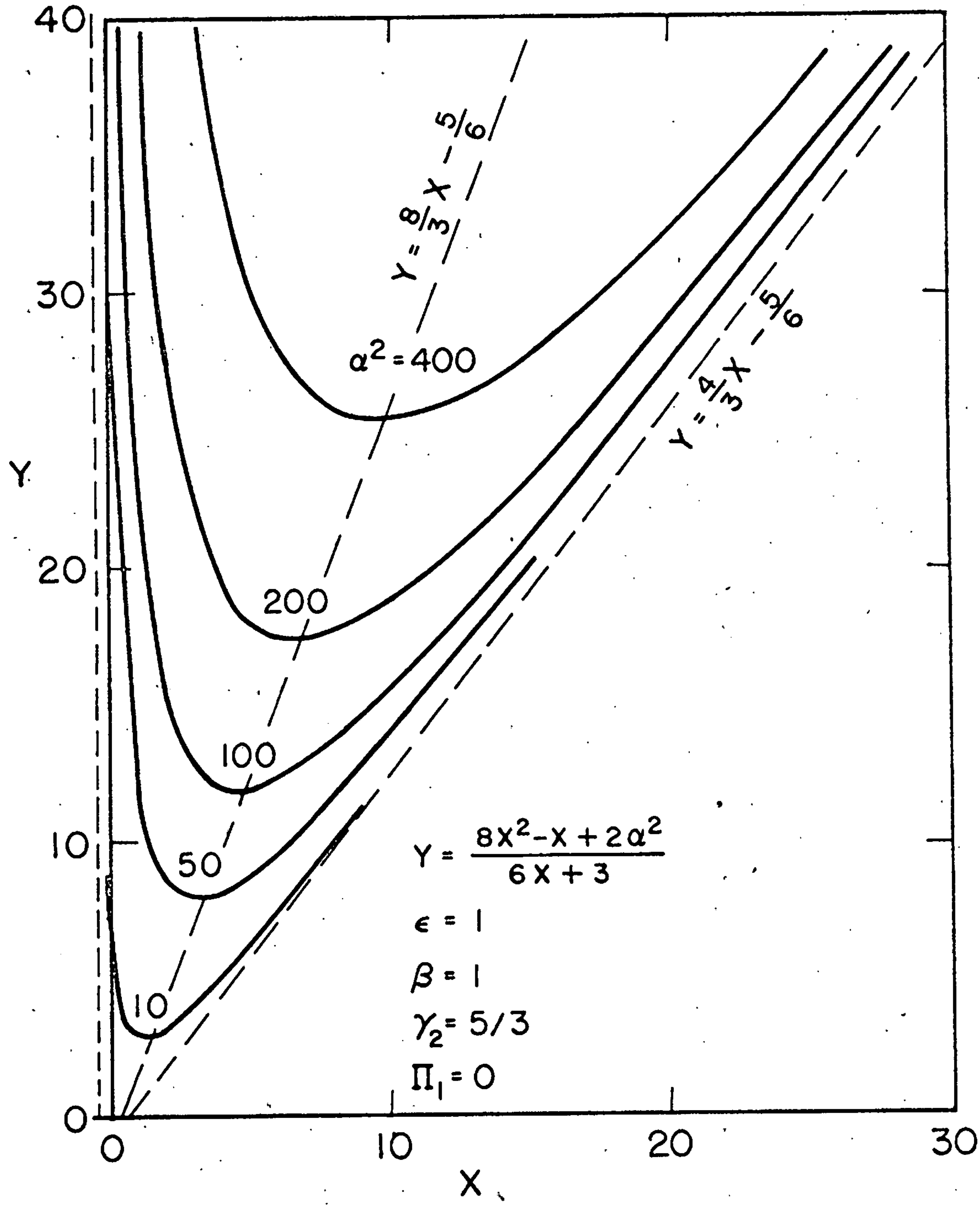
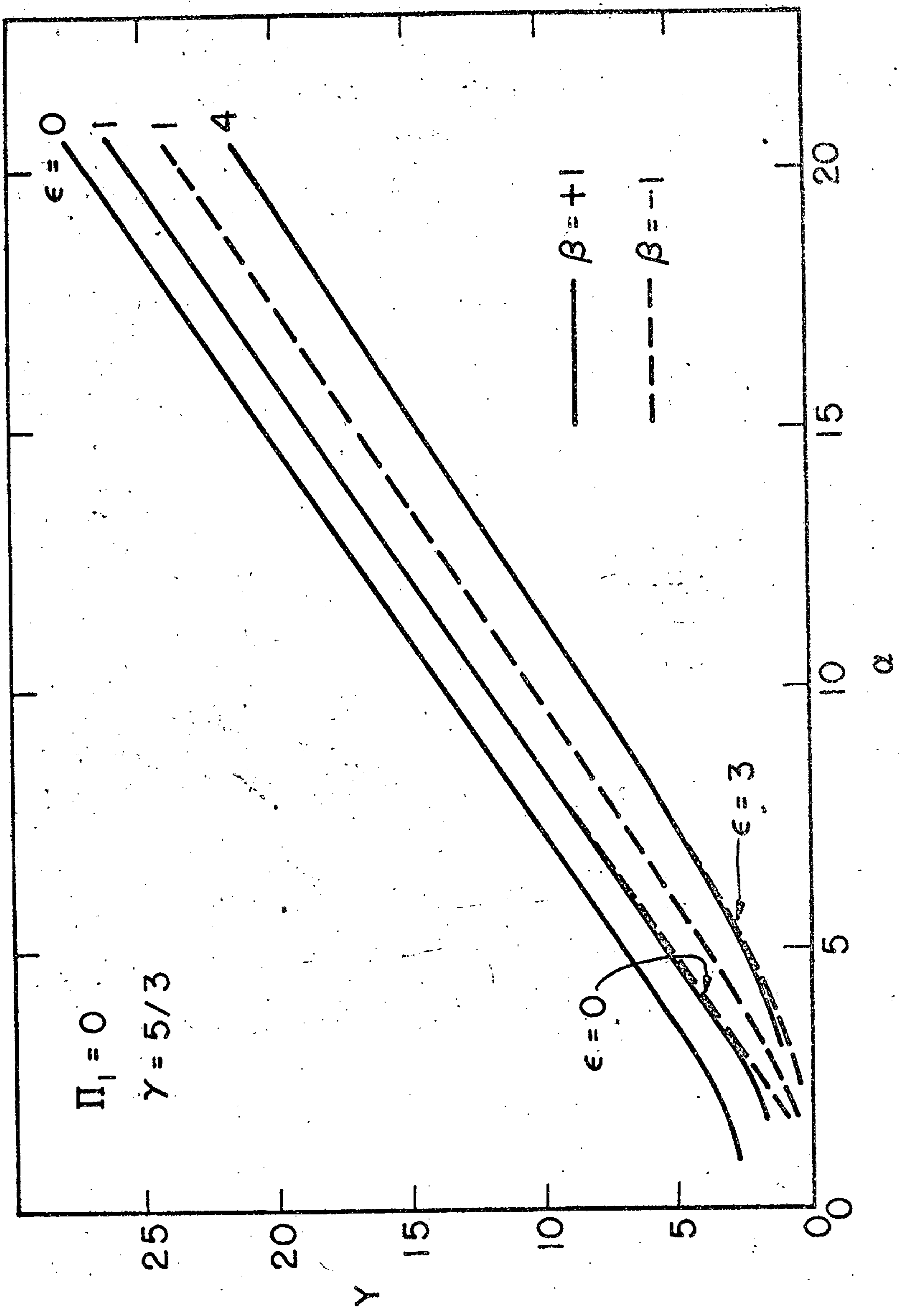

w2
Es (out of paper)
Es (out of paper)
u2
u1
w1 = 0
(a) Flow and E field in shock frame
Hx2
+ y (out of paper)
x
z
ρ2
p2
γ2
e2
Hz2
R2
Shock transition
R1
ρ1
p1
γ1
e1
Hx1
Hz1
(b) Gas conditions and H field in all frames (nonrelativistic)
E2 > E1 (out of paper)
v1 = 0
w1 = 0
E1 (out of paper)
w2
U
v2
(c) Flow and E field in laboratory frame
MU-23729

Fig. 3.



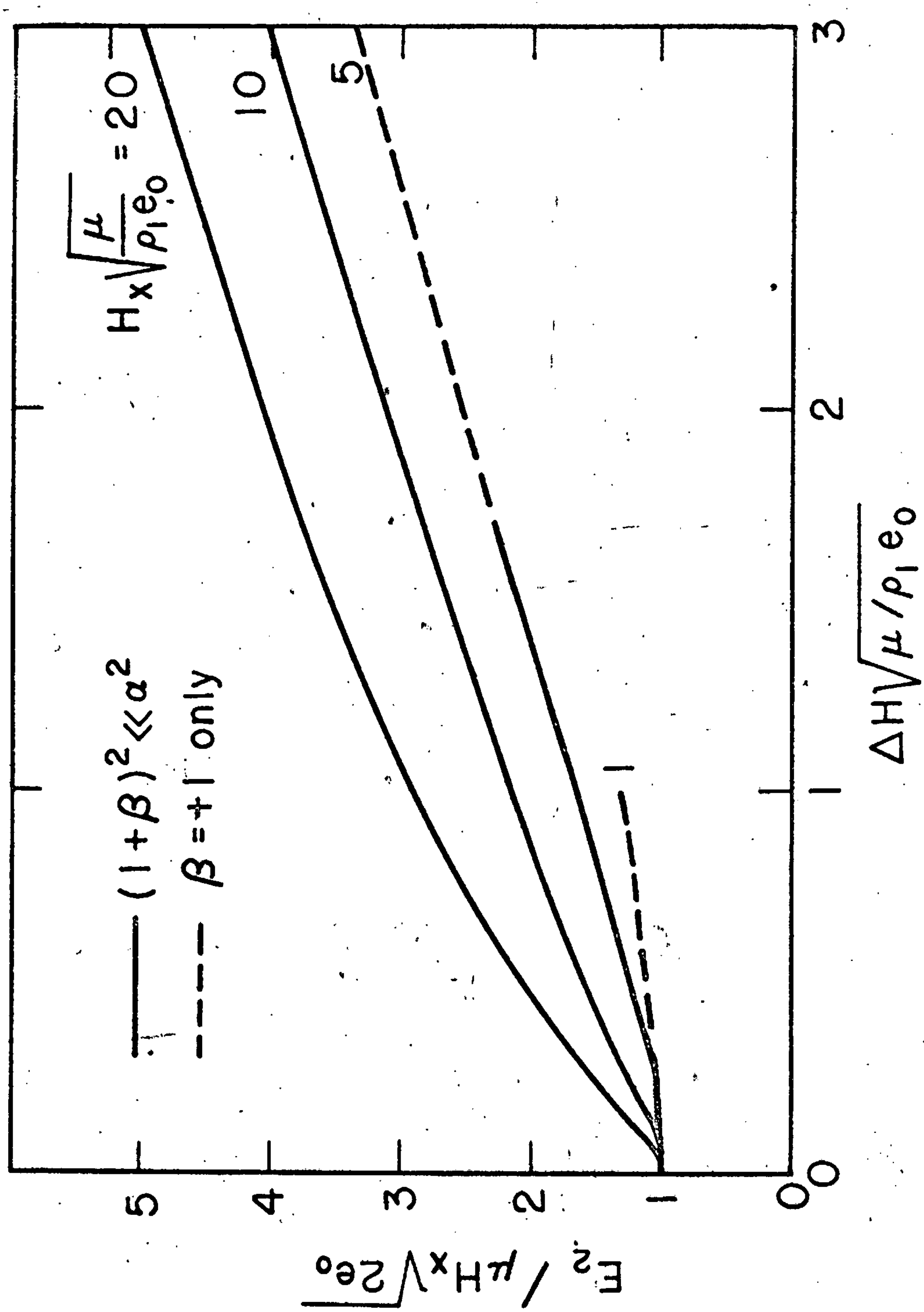
MU-23730

Fig. 4



MU-25397

Fig. 5.



MU-23732

Fig. 6.