EXTREMAL METRICS ON RULED MANIFOLDS

ZHIQIN LU AND REZA SEYYEDALI

Abstract. In this paper, we consider a compact Kähler manifold with extremal Kähler metric and a Mumford stable holomorphic bundle over it. We proved that, if the holomorphic vector field defining the extremal Kähler metric is liftable to the bundle and if the bundle is relatively stable with respect to the action of automorphisms of the manifold, then there exist extremal Kähler metrics on the projectivization of the dual vector bundle.

Contents

1. Introduction

Let (M, ω) be a Kähler manifold of dimension m and L be an ample line bundle over M such that $\omega \in 2\pi c_1(M)$. Let $\pi : E \to M$ be a holomorphic vector bundle of rank $r \geq 2$. This gives a holomorphic fibre bundle $\mathbb{P}E^*$ over M with fibre $\mathbb{P}^{r-1}$. We denote the tautological line bundle on $\mathbb{P}E^*$ by $\mathcal{O}_{\mathbb{P}E^*}(-1)$ and its dual bundle by $\mathcal{O}_{\mathbb{P}E^*}(1)$. By the Kodaira embedding theorem, for $k \gg 0$, the line bundles $\mathcal{O}_{\mathbb{P}E^*}(1) \otimes \pi^* L^k$ on $\mathbb{P}E^*$ are very ample.

In [10, 11], Hong proved that if E is Mumford stable; ω has constant scalar curvature; and M does not admit any nontrivial holomorphic vector fields, then $\mathbb{P}E^*$ admits

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cscK metric in the class of $\mathcal{O}_{\mathbb{P}E^*}(1)\otimes\pi^*L^k$ for $k\gg 0$. In [12], he generalized the result to the case that the base manifold has nontrivial automorphism group. He proved that if all Hamiltonian holomorphic vector fields on M can be lifted to holomorphic vector fields on $\mathbb{P}E^*$ and the corresponding Futaki invariants vanish, then $\mathbb{P}E^*$ admits cscK metrics in the class of $\mathcal{O}_{\mathbb{P}E^*}(1)\otimes\pi^*L^k$ for $k\gg 0$. The result was further generalized by replacing the liftiblity of holomorphic vector fields by a stability condition(cf. [13]). Hong considered the action of Aut(M) on the space of holomorphic structures on E and showed that if E is stable under this action, then there exist cscK metrics on $(\mathbb{P}E^*,\mathcal{O}_{\mathbb{P}E^*}(1)\otimes\pi^*L^k)$ for $k\gg 0$. The stability assumption is used to perturb approximation solutions to genuine cscK metrics.

In this article, we generalize Hong's result to the case that the base admits an extremal metric. Our main theorem is the following

Theorem 1.1. *Let (M,L) be a compact polarized manifold and $\omega_\infty\in c_1(L)$ be an extremal Kähler metric. Let X_s be the gradient vector field of the scalar curvature of ω_∞, i.e. $dS(\omega_\infty)=\iota_{X_s}\omega_\infty$. Let E be a Mumford stable holomorphic vector bundle over M. Suppose that the holomorphic vector field X_s can be lifted to a holomorphic vector field on $\mathbb{P}E^*$. If E is relatively stable under the action of* $\mathrm{Aut}(M)$ *in the sense of Definition 6.5, then there exist extremal metrics on $(\mathbb{P}E^*,\mathcal{O}_{\mathbb{P}E^*}(1)\otimes\pi^*L^k)$ for $k\gg 0$.*

We follow the ideas of [13, 22]. Let $G=\mathrm{Ham}(M,\omega_\infty)$ be the group of Hamiltonian isometries of (M,ω_∞) and $\mathfrak{g}$ be its Lie algebra. Let G_E be the subgroup of all Hamiltonian isometries of (M,ω_∞) that can be lifted to automorphisms of $\mathbb{P}E^*$. Let $\mathfrak{g}_E$ be the Lie algebra of G_E, i.e., space of all Hamiltonian holomorphic vector fields X on M that are liftable to holomorphic vector fields $\tilde{X}$ on $\mathbb{P}E^*$. Fix $T\subseteq G_E$ a maximal torus and $K\subseteq G$ the subgroup of all elements in G that commute with T. Let $\mathfrak{t}$ and $\mathfrak{k}$ be the Lie algebras of T and K respectively. We denote the space of all Hamiltonians whose gradient vector fields are in $\mathfrak{t}$ and $\mathfrak{k}$ by $\bar{\mathfrak{t}}$ and $\bar{\mathfrak{k}}$ respectively (including constant functions). Suppose that E is Mumford stable. Then the Donaldson-Uhlenbeck-Yau Theorem implies that E admits a Hermitian-Einstein metric h. The metric h induces a hermitian metric $g=\hat{h}$ on $\mathcal{O}_{\mathbb{P}E^*}(1)$. The restriction of the $(1,1)$-form

$$\omega_g=i\bar{\partial}\partial\log g=i\bar{\partial}\partial\log\widehat{h}$$

on fibres are Fubini-Study metrics and therefore $\omega_g|_{\mathrm{Fiber}}$ is non-degenerate. Hence for $k\gg 0$, the $(1,1)$-forms $\omega_k=\omega_g+k\omega_\infty$ define Kähler metrics. Finding extremal metrics on $(\mathbb{P}E^*,\mathcal{O}_{\mathbb{P}E^*}(1)\otimes L^k)$ is equivalent to finding $\phi\in\mathcal{C}^\infty(\mathbb{P}E^*)^T$ and $f\in\bar{\mathfrak{t}}$ such

that

$$S(\omega_k + \sqrt{-1}\,\bar{\partial}\partial\phi) + \frac{1}{2}\langle \nabla f, \nabla \phi\rangle = f, \tag{1.1}$$

where ∇ and $\langle\ ,\ \rangle$ are taken with respect to ω_k, and $\mathcal{C}^\infty(\mathbb{P}E^*)^T$ is the space of smooth functions on $\mathbb{P}E^*$ that are invariant under the action of T. To see that $\omega_k + \sqrt{-1}\,\bar{\partial}\partial\phi$ is an extremal metric, we assume that

$$df = \iota(X)\omega_k$$

for some holomorphic vector field X. We write $X = X_1 + \bar{X}_1$ for holomorphic $(1,0)$ vector field X_1. Then a straightforward computation shows that

$$\bar{\partial} S = \iota(X_1)(\omega_k + \sqrt{-1}\,\bar{\partial}\partial\phi).$$

Our strategy is to replace equation (1.1) with the one that is easier to solve and is relating it to a finite dimensional GIT problem (cf. [13, 22]). The first step is to find $\phi \in \mathcal{C}^\infty(\mathbb{P}E^*)^T$ and $b \in \bar{\mathfrak{k}}$ such that

$$S(\omega_k + \sqrt{-1}\,\bar{\partial}\partial\phi) + \frac{1}{2}\langle \nabla l_k(b), \nabla \phi\rangle = l_k(b), \tag{1.2}$$

where ∇ and $\langle\ ,\ \rangle$ are taken with respect to ω_k and $l_k(b)$ is a lift of b to $\mathbb{P}E^*$ defined in Definition 4.5. Note that if $b \in \bar{\mathfrak{t}}$, then $\omega_k + \sqrt{-1}\,\bar{\partial}\partial\phi$ is an extremal metric. Allowing b to be in a slightly larger space makes it easier to solve the equation. In order to solve equation (1.2), we first construct Kähler forms $\omega_{k,p}$ in the class of ω_k for any positive integer p and $k \gg 0$ as approximation solutions. We then apply contraction mapping theorem.

Theorem 1.2. *Let $p \geq 6$ be an integer. Suppose that $X_s \in \mathfrak{t}$. Then for any $k \gg 0$, we can find $\phi \in \mathcal{C}^\infty(\mathbb{P}E^*)^T$ and $b \in \bar{\mathfrak{k}}$ such that*

$$S(\omega_{k,p} + \sqrt{-1}\,\bar{\partial}\partial\phi) + \frac{1}{2}\langle \nabla l_{k,p}(b), \nabla \phi\rangle = l_{k,p}(b).$$

Here ∇ and $\langle\ ,\ \rangle$ are taken with respect to $\omega_{k,p}$. Moreover b has the following expansion:

$$b = r(r-1) + k^{-1}S(\omega) - k^{-2}\pi_N(\Sigma_E) + O(k^{-3}),$$

where $\pi_N : \mathcal{C}^\infty(M) \to \ker(\mathcal{D}^*\mathcal{D})$ and

$$\Sigma_E = \frac{2}{r}\Lambda^2(\mathrm{Ric}(\omega) \wedge \mathrm{tr}(iF_h)) - \frac{2}{r(r+1)}\Lambda^2(\mathrm{tr}(iF_h) \wedge \mathrm{tr}(iF_h))$$
$$+ \frac{2}{r+1}\Lambda^2 \mathrm{tr}(iF_h \wedge iF_h) - \mu S(\omega) + \frac{\mathrm{tr}(u_{X_s})}{r}.$$

See Proposition 4.3 and 4.13 for the definition of u_{X_s} and $l_{k,p}$ respectively.

The metric $\omega_{k,p} + \sqrt{-1}\,\bar{\partial}\partial\phi$ would have been an extremal metric if b were in $\bar{\mathfrak{t}}$. If not, we perturb the holomorphic structure on E so that the Hamiltonian b lies in $\bar{\mathfrak{t}}$ after the perturbation. This can be done by applying implicit function theorem using the stability assumption. In a recent paper, Brönnle [1], using the similar method, proved that if the base is cscK without holomorphic vector fields and the bundle is a direct sum of stable bundles with different slopes, then the projectivization admits extremal metrics.

The outline of the paper is as follows: In section 2, we go over some basic facts and definitions. In section 3, we compute an expansion for the scalar curvature of the metrics ω_k. Section 4 is devoted to the construction of Kähler metrics $\omega_{k,p}$. In Section 5, we prove Theorem 1.2. In the last section, we adopt Hong's moment map setting to our situation and prove the main theorem.

2. Preliminaries

Let V be a hermitian vector space of dimension r. The projective space $\mathbb{P}V^*$ can be identified with the space of hyperplanes in V via $f \in V^* \to \ker(f) = V_f \subseteq V$. There is a natural isomorphism between V and $H^0(\mathbb{P}V^*, \mathcal{O}_{\mathbb{P}V^*}(1))$ which sends $v \in V$ to $\hat{v} \in H^0(\mathbb{P}V^*, \mathcal{O}_{\mathbb{P}V^*}(1))$ such that for any $f \in V^*, \hat{v}(f) = f(v)$.

Definition 2.1. For any hermitian inner product h on V, we use $\langle \cdot, \cdot \rangle_h$ to denote the hermitian inner product induced by h and we use $\| \cdot \|_h$ to denote the norm with respect to h on both V and V^*. The hermitian inner product h induces a hermitian metric on $\mathcal{O}_{\mathbb{P}V^*}(1)$, which can be explicitly represented as follows: for $v, w \in V$ and $f \in V^*$ we define

$$\langle \hat{v}, \hat{w} \rangle_{\hat{h}} = \frac{f(v)\overline{f(w)}}{\|f\|_h^2}. \tag{2.1}$$

We denote the induced metric on $\mathcal{O}_{\mathbb{P}V^*}(1)$ by $\widehat{h}$.

The following is a straightforward computation.

Proposition 2.2. *For any $v, w \in V$ we have*

$$\langle v, w \rangle_h = C_r^{-1} \int_{\mathbb{P}V^*} \langle \hat{v}, \hat{w} \rangle_{\widehat{h}} \frac{\omega_{FS}^{r-1}}{(r-1)!}$$

where C_r is a constant defined by

$$C_r = \int_{\mathbb{C}^{r-1}} \frac{(\sqrt{-1})^{r-1} d\xi \wedge d\overline{\xi}}{(1 + \sum_{j=1}^{r-1} |\xi_j|^2)^{r+1}} = \frac{(2\pi)^{r-1}}{r!}, \tag{2.2}$$

and $(\sqrt{-1})^{r-1} d\xi \wedge d\overline{\xi} = (\sqrt{-1}\, d\xi_1 \wedge d\overline{\xi}_1) \wedge \cdots \wedge (\sqrt{-1}\, d\xi_{r-1} \wedge d\overline{\xi}_{r-1})$.

Definition 2.3. For any $v \in V$ and any hermitian inner product h on V, we define an endomorphism $\lambda(h) = \lambda(v, h)$ of V by

$$\lambda(v, h) = \frac{1}{\|v\|_h^2} v \otimes v^{*_h},$$

where $v^{*_h}(\cdot) = h(\cdot, v)$ is the dual element of v with respect to the inner product h.

The above settings can be made into the following family version. Let (M, ω) be a Kähler manifold of dimension m and E be a holomorphic vector bundle on M of rank $r \geq 2$. Let L be an ample line bundle on M endowed with a hermitian metric σ so that $i\bar{\partial}\partial \log \sigma = \omega$. The configuration (M, ω, L, σ) is called a polarized Kähler manifold. Let $\mathbb{P}E^*$ be the projectivization of the dual bundle E^* of E. A hermitian metric h on E induces a hermitian metric $\widehat{h}$ on the line bundle $\mathcal{O}_{\mathbb{P}E^*}(1)$ by (2.1).

Let ω_g be the $(1,1)$-form on $\mathbb{P}E^*$ defined by

$$\omega_g = i\bar{\partial}\partial \log \widehat{h}.$$

Let $\pi : \mathbb{P}E^* \to M$ be the projection map. Define the smooth functions $f_1, \dots f_m \in \mathcal{C}^\infty(\mathbb{P}E^*)$ by

$$\frac{\omega_g^{r-1+j}}{(r-1+j)!} \wedge \frac{\pi^*\omega^{m-j}}{(m-j)!} = f_j \frac{\omega_g^{r-1}}{(r-1)!} \wedge \frac{\pi^*\omega^m}{m!}. \tag{2.3}$$

Alternatively, f_j's can be generated by the following equation

$$\omega_k^{m+r-1} = \frac{(m+r-1)!}{m!(r-1)!} \sum_{j=1}^{m} k^{m-j} f_j \omega_g^{r-1} \wedge \pi^*\omega^m, \tag{2.4}$$

where

$$\omega_k = \omega_g + k\pi^*\omega.$$

Definition 2.4. Let X be a compact Kähler manifold with the Kähler metric ω. Assume that the complex dimension of X is N. For any (j, j)-form α on X, we define the contraction $\Lambda_\omega^j \alpha$ of α with respect to the Kähler form ω by

$$\frac{N!}{j!(N-j)!} \alpha \wedge \omega^{N-j} = (\Lambda_\omega^j \alpha)\, \omega^N.$$

In particular, we define

$$\Lambda_\omega \alpha = \Lambda_\omega^1 \alpha.$$

Definition 2.5. We define the vertical subbundle V and the horizontal subbundle H of the holomorphic tangent bundle $T(\mathbb{P}E^*)$ of $\mathbb{P}E^*$ as follows: let $u \in \mathbb{P}E^*$ and let

$\pi(u) = x$.

$$V_u = T_u(\mathbb{P}E_x^*);$$
$$H_u = \{\xi \in T_u(\mathbb{P}E^*) \mid \omega_g(\xi, w) = 0, \forall w \in V_u\}.$$

Since the restriction of ω_g to the fibre is the Fubini-Study metric of $\mathbb{P}E_x^*$, $\omega_g|_{\text{Fibre}}$ is non-degenerate. As a result, H is indeed a vector bundle of rank m, and we have the following (holomorphic bundle) decomposition

$$T(\mathbb{P}E^*) = H \oplus V.$$

By the dimension consideration, we have $H^* = \pi^*(T^*M)$, where H^* is the dual bundle of H. Let V^* be the dual bundle of V. Then we have

$$T^*(\mathbb{P}E^*) = V^* \oplus \pi^*(T^*M). \tag{2.5}$$

Let $\bigwedge(T^*(\mathbb{P}E^*))$ be the bundle of differential forms of $\mathbb{P}E^*$. Write

$$\bigwedge(T^*(\mathbb{P}E^*)) = C_H \oplus C_V \oplus C_m,$$

where C_H, C_V and C_m are the bundles of horizontal, vertical, and mixed forms, respectively. Note that $C_H = \pi^*(\bigwedge(T^*M))$. For any differential form α on $\mathbb{P}E^*$, we write $\alpha = \alpha_H + \alpha_V + \alpha_m$, where $\alpha_H, \alpha_V, \alpha_m$ are the horizontal, vertical, and mixed components of α respectively.

Using the above notation, we have

Lemma 2.6. *There is no mixed component of ω_g.*

Proof. This follows from (2.5).

□

If we write

$$\omega_g = (\omega_g)_H + (\omega_g)_V$$

as its horizontal and vertical parts. Then (2.3) can be written as

$$f_j\, \pi^*(\omega^m) = \frac{m!}{j!(m-j)!}(\omega_g)_H^j \wedge \pi^*\omega^{m-j}. \tag{2.6}$$

Let $F_h \in \bigwedge^{1,1}(\mathrm{Hom}\,(E, E))$ be the curvature tensor of h

$$F_h = \bar{\partial}\,(\partial h \cdot h^{-1}).$$

From (2.6), we can prove the following

Lemma 2.7. *For any $v \in E^*$, we have*

$$f_j([v]) = \Lambda_\omega^j \Big(\sqrt{-1}\, \mathrm{Tr}\big(\lambda(v,h) F_h\big) \Big)^j,$$

where $[v] \in \mathbb{P}E^*$ is the class of v in $\mathbb{P}E^*$.

Proof. Let

$$\beta = \sqrt{-1}\, \mathrm{tr}\big(\lambda(v,h) F_h\big) = \sqrt{-1} \|v\|_h^{-2} \langle F_h(v), v\rangle_h. \tag{2.7}$$

Let $x = \pi(u)$. We assume that at x, $\{e_1, \cdots, e_r\}$ is a *normal* frame. That is, under this frame

$$h_{i\bar{j}}(x) = \delta_{ij}, \quad dh_{i\bar{j}}(x) = 0.$$

Since there are no connection terms, by a straightforward computation, we obtain[1]

$$\omega_g = \pi^*(\beta) + \omega_g|_{\mathbb{P}E^*{}_x}. \tag{2.8}$$

Therefore,

$$(\omega_g)_H = \pi^* \beta. \tag{2.9}$$

The lemma follows from Definition 2.4.

□

Let α be a $(1,1)$-form on $\mathbb{P}E^*$. Define $\tilde{\Lambda}_{\omega_g} \alpha_V$ by

$$\tilde{\Lambda}_{\omega_g} \alpha_V \wedge ((\omega_g)_V)^{r-1} = (r-1) \alpha_V \wedge ((\omega_g)_V)^{r-2}.$$

Therefore, we have

$$(\tilde{\Lambda}_{\omega_g} \alpha_V)\, \omega_g^{m+r-j-1} \wedge \pi^* \omega^j = (m+r-j-1) \alpha_V \wedge \omega_g^{m+r-j-2} \wedge \pi^* \omega^j$$

for $j \geq 0$.

Definition 2.8. For any smooth function $f \in \mathcal{C}^\infty(\mathbb{P}E^*)$, define the operators Δ_V, Δ_H and $\tilde{\Delta}_H$ (and call them the Laplacians) by the following equations

$$\begin{aligned}
&(r-1)\sqrt{-1}\, \bar{\partial}\partial f \wedge \omega_g^{r-2} \wedge \pi^* \omega^m = \Delta_V f \omega_g^{r-1} \wedge \pi^* \omega^m, \\
&m\sqrt{-1}\, \bar{\partial}\partial f \wedge \omega_g^{r-1} \wedge \pi^* \omega^{m-1} = \Delta_H f\, \omega_g^{r-1} \wedge \pi^* \omega^m, \\
&\tilde{\Delta}_H f = \Delta_H f - f_1 \Delta_V f.
\end{aligned}$$

Remark 1. The Laplacians Δ_H and Δ_V are the same as ones defined in [10].

[1]Strictly speaking, β is a section of the sheaf $\mathcal{C}^\infty(\mathbb{P}E^*) \otimes \mathcal{A}^{1,1}(M)$. So the π^* operation is only acting on the second component.

Definition 2.9. For any $x \in M$, we define W_x as the space of all eigenfunctions of the Laplacian (on functions) on $\mathbb{P}E_x$ (with respect to the metric $\omega_g|_{\mathbb{P}E_x}$) associated to the first nonzero eigenvalue. Define the vector bundle W whose fibers are W_x(c.f. [11]).

Let $\mathrm{End}_0(E_x)$ be the space of traceless endomorphisms of E_x for any $x \in M$. The first nonzero eigenvalue of the Laplacian is r. As is well-known,

$$\Phi \in \mathrm{End}_0(E_x) \to \mathrm{Tr}\big(\lambda(h)\Phi\big) \in W_x$$

is a 1-1 correspondence. Define $\mathrm{End}_0(E)$ to be the smooth vector bundle whose fibers are $\mathrm{End}_0(E_x)$ for any $x \in M$. Thus we have $W = \mathrm{End}_0(E)$.

3. Scalar curvature

The goal of this section is to find the asymptotic expansion for the scalar curvature of the Kähler form $\omega_k = \omega_g + k\pi^*\omega$. The main result of this section is

Theorem 3.1. *Let ω be a Kähler metric on M and h be a hermitian metric on E. Let*

$$\omega_k = \omega_g + k\pi^*(\omega),$$

where k is a large positive integer. Then we have the following expansion of the scalar curvature $\mathrm{Scal}(\omega_k)$ *of ω_k*

$$\begin{aligned}
&\mathrm{Scal}(\omega_k) = r(r-1) + k^{-1}(\pi^*S(\omega) + 2r\,\Lambda_\omega(\mathrm{Tr}(\lambda(h)F_h^\circ)))\\
&+ k^{-2}\Big(2\Lambda_\omega^2((\pi^*(Ric(\omega) - \mathrm{Tr}(iF_h)) \wedge \omega_g)_H) - f_1(\pi^*(S(\omega) - \Lambda_\omega(\mathrm{Tr}(iF_h))))\\
&+ \Delta_V(f_2 - \frac{1}{2}f_1^2) + \tilde{\Delta}_H f_1 - r f_1^2 + 2r f_2\Big) + O(k^{-3}),
\end{aligned}$$

where $S(\omega)$ is the scalar curvature of ω and $F_h^\circ = F_h - \frac{1}{r}tr(F_h)$ is the trace-less part of the curvature tensor of h. (For the definition of $f_1, \dots f_m$, $\lambda(h)$, $\tilde{\Delta}_H$, Δ_V, Λ_ω and Λ_ω^2, see (2.3)*, Definition 2.3, Definition 2.4 and Definition 2.8).*

Let $\alpha = \pi^*\alpha_1$ be a horizontal form of $\mathbb{P}E^*$ (see footnote 1). Then we define

$$\Lambda_\omega \alpha = \pi^*(\Lambda_\omega \alpha_1).$$

First we prove the following purely algebraic lemmas.

Lemma 3.2. *Let α be a $(1,1)$-form on $\mathbb{P}E^*$. Then*

$$\Lambda_{\omega_k}\alpha = \tilde{\Lambda}_{\omega_g}\alpha_V + k^{-1}\Lambda_\omega\alpha_H + k^{-2}\big(2\Lambda_\omega^2(\alpha \wedge \omega_g)_H - (\Lambda_\omega\alpha_H)f_1\big) + O(k^{-3}).$$

In particular if $\alpha \in \bigwedge^{1,1}(M)$, *then*

$$\Lambda_{\omega_k}\alpha = k^{-1}\pi^*(\Lambda_\omega\alpha) + k^{-2}\Big(2\Lambda_\omega^2(\alpha\wedge\omega_g)_H - f_1\,\pi^*(\Lambda_\omega\alpha)\Big) + O(k^{-3}).$$

Proof. By definition, we have

$$(\Lambda_{\omega_k}\alpha)\omega_k^{m+r-1} = (m+r-1)\alpha\wedge\omega_k^{m+r-2}.$$

We define $g_j = g_j(\alpha)$ by the equation

$$\frac{1}{(m+r-2)!}\,\alpha\wedge\omega_k^{m+r-2} = \frac{1}{(r-1)!m!}\,k^m(\sum_{j=0}^{m} k^{-j}g_j)\,\omega_g^{r-1}\wedge\omega^m.$$

Let

$$\alpha = \alpha_V + \alpha_H + \alpha_m$$

be the decomposition of α into its vertical, horizontal, and mixed components. Then we have

$$\begin{aligned}
&\frac{(r-1)!\,m!}{(m+r-2)!}\,\alpha\wedge\omega_k^{m+r-2}\\
&= (\tilde\Lambda_{\omega_g}\alpha_V)((\omega_g)_H + k\pi^*\omega)^m\wedge((\omega_g)_V)^{r-1}\\
&+ m\alpha_H\wedge((\omega_g)_H + k\pi^*\omega)^{m-1}\wedge((\omega_g)_V)^{r-1}\\
&= \sum_j k^{m-j}g_j((\omega_g)_V)^{r-1}\wedge\pi^*\omega^m.
\end{aligned}$$

Simple calculation shows that

$$\begin{aligned}
g_0 &= \tilde\Lambda_{\omega_g}\alpha_V;\\
g_1 &= \Lambda_\omega\alpha_H + (\tilde\Lambda_{\omega_g}\alpha_V)f_1;\\
g_2 &= 2\Lambda_\omega^2(\alpha\wedge\omega_g)_H + (\tilde\Lambda_{\omega_g}\alpha_V)f_2.
\end{aligned}$$

By (2.4), the above equation implies

$$\begin{aligned}
\Lambda_{\omega_k}\alpha &= \frac{\sum k^{-j}g_j}{\sum k^{-j}f_j}\\
&= g_0 + k^{-1}(g_1 - g_0f_1) + k^{-2}(g_2 - g_1f_1 - g_0f_2 + g_0f_1^2) + O(k^{-3}).
\end{aligned}$$

The lemma is proved.

□

Let Δ_k be the Laplacian with respect to the metric ω_k. That is,

$$\Delta_k f = \Lambda_{\omega_k}(\sqrt{-1}\,\bar\partial\partial f)$$

for smooth functions f on $\mathbb{P}E^*$. Then we have the following asymptotics:

Lemma 3.3. *For any $f \in \mathcal{C}^\infty(\mathbb{P}E^*)$, we have*

$$\Delta_k f = \Delta_V f + k^{-1}\tilde{\Delta}_H f + k^{-2}\Big(- f_1\tilde{\Delta}_H f + 2\Lambda_\omega^2(\sqrt{-1}\,\bar{\partial}\partial f \wedge \omega_g)_H\Big) + O(k^{-3})$$

as $k \to \infty$.

Proof. Let $\alpha = \sqrt{-1}\,\bar{\partial}\partial f$. Then we have

$$\Delta_k f = \Lambda_{\omega_k}\alpha.$$

By Lemma 3.2, we have

$$\Delta_k f = \tilde{\Lambda}_{\omega_g}\alpha_V + k^{-1}\Lambda_\omega\alpha_H + k^{-2}\big(2\Lambda_\omega^2(\alpha\wedge\omega_g)_H - (\Lambda_\omega\alpha_H)f_1\big) + O(k^{-3}).$$

By Definition 2.8, we have

$$(\tilde{\Lambda}_{\omega_g}\alpha_V)\omega_g^{r-1}\wedge\pi^*\omega^m = (r-1)\alpha\wedge\omega_g^{r-2}\wedge\pi^*\omega^m = (\Delta_V f)\omega_g^{r-1}\wedge\pi^*\omega^m.$$

Thus

$$\tilde{\Lambda}_{\omega_g}\alpha_V = \Delta_V f.$$

Similarly, we have

$$\tilde{\Lambda}_{\omega_g}\alpha_V\, f_1 + \Lambda_\omega\alpha_H = \Delta_H f.$$

Thus we have

$$\Lambda_\omega\alpha_H = \tilde{\Delta}_H f.$$

The lemma is proved. □

Proof of Theorem 3.1. we have the following exact sequence of holomorphic vector bundles on $\mathbb{P}E^*$.

$$0 \to V \to T\mathbb{P}E^* \to \pi^*TM \to 0.$$

The hermitian metric h on E induces a Fubini-Study metric h_{FS} on V. The positive $(1,1)$-forms ω_k and $(\omega_g)_H + k\pi^*\omega$ induce hermitian metrics on vector bundles $T\mathbb{P}E^*$ and π^*TM respectively. As holomorphic hermitian vector bundles, the above exact sequence splits in the smooth category:

$$(T\mathbb{P}E^*, \omega_k) = (V, h_{FS})\bigoplus(\pi^*TM, (\omega_g)_H + k\pi^*\omega)$$

and in addition, we have

$$\mathrm{Ric}(\omega_k) = \mathrm{Tr}(iF_{h_{FS}}) + \mathrm{Ric}((\omega_g)_H + k\pi^*\omega).$$

On the other hand, we have the following Euler sequence of holomorphic vector bundles on $\mathbb{P}E^*$.

$$0 \to \mathbb{C} \to \pi^*E^*\otimes\mathcal{O}_{\mathbb{P}E^*}(1) \to V \to 0.$$

This gives the following isometric isomorphism of holomorphic line bundles on $\mathbb{P}E^*$.

$$(\det(V), \det(h_{FS})) \cong (\det(\pi^* E \otimes \mathcal{O}_{\mathbb{P}E^*}(1)), \det(\pi^* h \otimes \hat{h})).$$

Therefore (cf. (2.8)), $\mathrm{Tr}(iF_{h_{FS}}) = r\omega_g - \pi^*\mathrm{Tr}(iF_h)$, and we have

$$\mathrm{Ric}(\omega_k) = r\omega_g + \mathrm{Ric}((\omega_g)_H + k\pi^*\omega) - \pi^*\mathrm{Tr}(iF_h).$$

On the other hand, by (2.6), we have

$$k^{-m}\big((\omega_g)_H + k\pi^*\omega\big)^m = (1 + k^{-1}f_1 + \cdots + k^{-m}f_m)\,\pi^*\omega^m.$$

As a result,

$$\begin{aligned}\mathrm{Ric}((\omega_g)_H + k\pi^*\omega) &= \sqrt{-1}\,\bar\partial\partial \log\big((\omega_g)_H + k\pi^*\omega\big)^m \\ &= \sqrt{-1}\,\bar\partial\partial \log(\sum_{j=0}^{m} k^{-j} f_j) + \pi^*(\mathrm{Ric}(\omega)).\end{aligned}$$

Consequently,

$$\mathrm{Ric}\,(\omega_k) = r\omega_g - \pi^*\mathrm{Tr}(iF_h) + \pi^*(\mathrm{Ric}(\omega)) + \sqrt{-1}\,\bar\partial\partial \log(\sum_{j=0}^{m} k^{-j} f_j). \tag{3.1}$$

Taking trace of (3.1) with respect to ω_k, we get

$$\mathrm{Scal}(\omega_k) = \Lambda_{\omega_k}\alpha + \Delta_k \log(\sum_{j=0}^{m} k^{-j} f_j),$$

where

$$\alpha = \pi^*(\mathrm{Ric}(\omega) - \mathrm{Tr}(iF_h)) + r\omega_g.$$

Let

$$b = \pi^*(S(\omega) - \Lambda_\omega(\mathrm{Tr}(iF_h))).$$

Using Lemma 3.2, we get

$$\begin{aligned}&\Lambda_{\omega_k}\alpha = r(r-1) + k^{-1}(b + rf_1) \\ &+ k^{-2}(2\Lambda_\omega^2(\pi^*(\mathrm{Ric}(\omega) - \mathrm{Tr}(iF_h)) \wedge \omega_g)_H - f_1 b - r f_1^2 + 2r f_2) + O(k^{-3}).\end{aligned}$$

By Lemma 3.3, we have

$$\Delta_k \log(\sum_{j=0}^{m} k^{-j} f_j) = k^{-1}(\Delta_V f_1) + k^{-2}(\Delta_V(f_2 - \frac{1}{2}f_1^2) + \tilde\Delta_H f_1) + O(k^{-3}).$$

Therefore, we have

$$\begin{aligned}\mathrm{Scal}(\omega_k) &= r(r-1) + k^{-1}(b + rf_1 + \Delta_V f_1)\\ &+ k^{-2}(2\Lambda_\omega^2(\pi^*(\mathrm{Ric}(\omega) - \mathrm{Tr}(iF_h)) \wedge \omega_g)_H - f_1 b\\ &+ \Delta_V(f_2 - \frac{1}{2}f_1^2) + \tilde{\Delta}_H f_1 - rf_1^2 + 2rf_2) + O(k^{-3}).\end{aligned}$$

On the other hand, by the discussion at the end of the last section, we have $\Delta_V f_1 = rf_1 - \Lambda_\omega \mathrm{Tr}(iF_h)$. This concludes the proof.

□

An easy computation shows the following

Corollary 3.4 (c.f. [13])**.** *Suppose that h is a Hermitian-Einstein metric on E with respect to ω, i.e. $\Lambda_\omega(iF_h) = \mu I_E$, where μ is the $\omega-$slope of the bundle E. Then for any $x \in M$, we have*

$$\frac{1}{(2\pi)^{r-1}}\int_{\mathbb{P}E_x^*} \mathrm{Scal}(\omega_k)\,\omega_g^{r-1} = C(k) + k^{-1}S(\omega) + k^{-2}\Big(\frac{2}{r}\Lambda_\omega^2(\mathrm{Ric}(\omega)\wedge\mathrm{Tr}(iF_h)) -$$
$$\frac{2}{r(r+1)}\Lambda_\omega^2(\mathrm{Tr}(iF_h)\wedge\mathrm{Tr}(iF_h)) + \frac{2}{r+1}\Lambda_\omega^2\mathrm{Tr}(iF_h\wedge iF_h) - \mu S(\omega)\Big) + O(k^{-3}),$$

where $C(k)$ is a constant depends on k.

4. Construction of approximate solutions

In this section, we first compute the linearization of the scalar curvature operator at the Kähler metrics ω_k.

Proposition 4.1. [9] *Let (Y,ω) be a Kähler manifold of dimension n. Then the linearization of the scalar curvature operator at the Kähler metric ω is given by the following formula.*

$$L(\phi) = (\Delta^2 - S(\omega)\Delta)\phi + n(n-1)\frac{\sqrt{-1}\,\bar{\partial}\partial\phi\wedge\mathrm{Ric}(\omega)\wedge\omega^{n-2}}{\omega^n},$$

where ϕ is a smooth function on Y.

Applying the above proposition to $(\mathbb{P}E^*,\omega_k)$, we obtain the following.

Proposition 4.2. *Let L_k be the linearization of the scalar curvature operator at Kähler metrics ω_k. Then we have the following*

$$L_k = \Delta_V(\Delta_V - r) + O(k^{-1}).$$

Proof. By (3.1), we have

$$\sqrt{-1}\,\bar{\partial}\partial\phi \wedge \mathrm{Ric}(\omega_k) \wedge \omega_k^{n-2} = C^n_{r-3+n}\sqrt{-1}\,\bar{\partial}\partial\phi \wedge \omega_g \wedge \pi^*\omega^n + O(k^{n-1}).$$

Since $\mathrm{Scal}(\omega_k) = r(r-1) + O(k^{-1})$ by Theorem 3.1, we have

$$(n+r-1)(n+r-2)\frac{\sqrt{-1}\bar{\partial}\partial\phi \wedge \mathrm{Ric}(\omega_k) \wedge \omega_k^{n+r-3}}{\omega_k^{n+r-1}} = r(r-2)\Delta_V + O(k^{-1}).$$

The result follows from Proposition 4.1.

□

We make the following definition of a holomorphic vector field. Let X be a $(1,0)$-vector field such that $\bar{\partial}X = 0$. Then $X + \bar{X}$ is a real vector field and it is called a holomorphic vector field. A holomorphic vector field generates a one-parameter group of holomorphic automorphisms.

Let ω_∞ be an extremal metric on M and X_s be the holomorphic vector field such that $dS(\omega_\infty) = \iota_{X_s}\omega_\infty$.

Let $G = \mathrm{Ham}(M, \omega_\infty)$ be the group of Hamiltonian isometries of (M, ω_∞) and $\mathfrak{g}$ be its Lie algebra. Let G_E be the subgroup of all Hamiltonian isometries of (M, ω) that can be lifted to automorphisms of $\mathbb{P}E^*$ and let $\mathfrak{g}_E$ be its Lie algebra. $\mathfrak{g}_E$ is the space of holomorphic vector fields X on M such that

(1) there exist holomorphic vector fields $\tilde{X}$ of $\mathbb{P}E^*$ such that $\pi_*\tilde{X} = X$;
(2) there exist real valued functions f such that $df = \iota_X\omega_\infty$.

Let $\mathfrak{h}$ be a Lie sub algebra of $\mathfrak{g}$. We denote the space of all Hamiltonians (including constant functions) whose gradient vector fields are in $\mathfrak{h}$ by $\bar{\mathfrak{h}}$. Fix $T \subseteq G_E$ a maximal torus and $K \subseteq G$ the subgroup of all elements in G that commute with T. Let $\mathfrak{t}$ and $\mathfrak{k}$ be the Lie algebras of T and K respectively. Suppose that $b \in \bar{\mathfrak{k}}$. By definition, there exists a holomorphic vector field X on M such that $db = \iota_X\omega_\infty$. If we further assume that $b \in \bar{\mathfrak{t}}$, then there exists a unique holomorphic vector field $\tilde{X}$ on $\mathbb{P}E^*$ such that $\pi_*\tilde{X} = X$. As a result, we are able to define the Hamiltonian functions $l_k(b)$ on $\mathbb{P}E^*$ such that $kd(l_k(b)) = \iota_{\tilde{X}}\omega_k$. However, if b does not belong to $\bar{\mathfrak{t}}$, then the corresponding holomorphic vector field does not lift to a holomorphic vector field on $\mathbb{P}E^*$. Nevertheless, we are still able to define $l_k(b)$. In order to do that, we use the following proposition proved in [13].

Proposition 4.3. *For any holomorphic vector field X on M, there exists a unique smooth $u_X \in \Gamma(\mathrm{End}(E))$ such that*

$$\Lambda_{\omega_\infty}\partial(\bar\partial u_X - \iota_X F_h) = 0,$$
$$\int_M \mathrm{tr}(u_X)\omega_\infty^m = 0.$$

Moreover, there exists a holomorphic vector field $\tilde X$ on $\mathbb{P}E^*$ such that $\pi_*\tilde X = X$ if and only if $\bar\partial u_X - \iota_X F_h = 0$.

If $f \in \bar{\mathfrak{g}}_E$ and X be the gradient vector field corresponding to b, we can explicitly compute $l_k(b)$ in terms of u_X. Indeed, we have the following.

Lemma 4.4. *Suppose that holomorphic vector field X has a holomorphic lift $\tilde X$ to $\mathbb{P}E^*$, then $\iota_{\tilde X}\omega_g = d\theta_X$, where $\theta_X = \mathrm{Tr}(u_X\lambda(h))$. Moreover, if $f \in \bar{\mathfrak{g}}_E$ such that $df = \iota_X\omega_\infty$, then $d(\theta_X + kf) = \iota_{\tilde X}\omega_k$.*

Inspired by the proceeding lemma, we define the lift of elements of $\bar{\mathfrak{g}}$ to $\mathbb{P}E^*$.

Definition 4.5. We define

$$l_k : \bar{\mathfrak{g}} \to \mathcal{C}^\infty(\mathbb{P}E^*)$$
$$f \in \bar{\mathfrak{g}} \mapsto l_k(f) = f + k^{-1}\theta_X,$$

where X is the holomorphic vector field on M such that $\iota_X\omega = df$ and $\theta_X = \mathrm{Tr}(u_X\lambda(h))$.

Suppose that $X_s \in \mathfrak{t}$. Then there exists a holomorphic vector field $\tilde X_s$ on $\mathbb{P}E^*$ so that $\pi_*\tilde X_s = X_s$. Moreover from the definition of the function $f \to l_k(f)$, we conclude that $dl_k(S(\omega_\infty)) = k^{-1}\iota_{\tilde X_s}\omega_k$.

Let A be a vector space on which the group T acts. Let A^T be the subspace of T invariant elements of A. The main goal of this section is to prove the following proposition.

Proposition 4.6. *Let h_{HE} be the Hermitian-Einstein metric on E with respect to ω_∞, i.e. $\Lambda_{\omega_\infty}F_{(E,h_{HE})} = \mu I_E$, where μ is the slope of the bundle E. Then there exist $\eta_0, \eta_1, \cdots \in \mathcal{C}^\infty(M)^T$, $\Phi_0, \Phi_1, \cdots \in \Gamma(M, W)^T$, $\varphi_0, \varphi_1, \cdots \in \mathcal{C}^\infty(\mathbb{P}E^*)^T$ and $b_0, b_1, \cdots \in \bar{\mathfrak{k}}$ such that for any positive integer p, if*

$$\varphi_{k,p} = \sum_{j=2}^{p}\eta_j k^{-j+2} + \sum_{j=2}^{p}\Phi_j k^{-j+1} + \sum_{j=2}^{p}\varphi_j k^{-j},$$

and

$$b_{k,p} = \sum_{j=0}^{p} k^{-j} b_j,$$

then

$$S(\omega_k + \sqrt{-1}\,\bar\partial\partial\varphi_{k,p}) + \frac{1}{2}\langle \nabla l_k(b_{k,p}), \nabla\varphi_{k,p}\rangle - l_k(b_{k,p}) = O(k^{-p-1}).$$

Here the gradient and inner product are computed with respect to the Kähler metrics ω_k*. Moreover* $b_0 = r(r-1)$ *and* $b_1 = S(\omega_\infty)$*.*

Define $A_1(\omega, h) = S(\omega) I_E + \frac{i}{2\pi}\Lambda_\omega F_h^0$ and $S_1(\omega, h) = \mathrm{Tr}(A_1(h,\omega)\lambda(h))$, where F_h^0 is the traceless part of F_h.

Proposition 4.7. *Suppose that* $\omega_\infty \in 2\pi c_1(L)$ *is an extremal Kähler metric on* M *and* h_{HE} *is a Hermitian-Einstein metric on* E *with the* ω_∞*-slope* μ*. Then we have*

$$\begin{aligned} A_{1,1} &:= \frac{d}{dt}\Big|_{t=0} A_1(\omega_\infty + it\overline{\partial}\partial\eta, h_{HE}(I + t\phi)) \\ &= \big(\mathcal{D}^*\mathcal{D}\eta - \frac{1}{2}\langle \nabla_{\omega_\infty} S(\omega_\infty), \nabla_{\omega_\infty}\eta\rangle_{\omega_\infty}\big) I_E \\ &+ \frac{i}{2\pi}\Big\{(\Lambda_{\omega_\infty}\overline{\partial}\partial\Phi + 2\Lambda^2_{\omega_\infty}(F_{h_{HE}} \wedge (i\overline{\partial}\partial\eta)))\Big\}^0, \end{aligned}$$

where $\mathcal{D}^*\mathcal{D}$ *is Lichnerowicz operator (cf.* [7, Page 515]*) and* $\{\Sigma\}^0$ *is the traceless part of* Σ*, i.e.* $\{\Sigma\}^0 = \Sigma - \frac{1}{r}tr(\Sigma)$*. Note that we use the operator* ∂ *to denote the covariant derivative of sections of the bundle* $\mathrm{End}(E)$*.*

Proof. Define $f(t) = \Lambda_{\omega_\infty + it\overline{\partial}\partial\eta} F_{(h_{\mathrm{HE}}(I+t\phi))}$. Then we have

$$mF_{(h_{\mathrm{HE}}(I+t\phi))} \wedge (\omega_\infty + it\overline{\partial}\partial\eta)^{m-1} = f(t)(\omega_\infty + it\overline{\partial}\partial\eta)^m.$$

Differentiating with respect to t at $t = 0$, we obtain

$$m\overline{\partial}\partial\phi \wedge \omega_\infty^{m-1} + m(m-1)F_{h_{\mathrm{HE}}} \wedge (i\overline{\partial}\partial\eta) \wedge \omega_\infty^{m-2} = f'(0)\omega_\infty^m + mf(0)(i\overline{\partial}\partial\eta) \wedge \omega_\infty^{m-1}.$$

Since $f(0) = \mu I_E$, we get $f'(0) = \Lambda_{\omega_\infty}\overline{\partial}\partial\phi + 2\Lambda^2_{\omega_\infty}(F_{h_{\mathrm{HE}}} \wedge (i\overline{\partial}\partial\eta)) - \mu\Lambda_{\omega_\infty}(i\overline{\partial}\partial\eta) I_E$. On the other hand (cf. [7, pp. 515, 516].)

$$\frac{d}{dt}\Big|_{t=0} S(\omega_\infty + it\overline{\partial}\partial\eta) = \mathcal{D}^*\mathcal{D}\eta - \frac{1}{2}\langle \nabla_{\omega_\infty} S(\omega_\infty), \nabla_{\omega_\infty}\eta\rangle_{\omega_\infty}.$$

The proposition follows from the above two equations. □

Lemma 4.8. *Suppose that* $\omega_\infty \in 2\pi c_1(L)$ *is an extremal metric on* M *and* h_{HE} *be a Hermitian-Einstein metric on* E*, i.e.* $\Lambda_{\omega_\infty} F_{(E,h_{HE})} = \mu I_E$*, where* μ *is the* ω_∞*-slope of*

the bundle E. We have

$$\begin{aligned} S_{1,1} &:= \frac{d}{dt}\Big|_{t=0} S_1(\omega_\infty + it\overline{\partial}\partial\eta, h_{HE}(I+t\phi)) \\ &= \mathcal{D}^*\mathcal{D}\eta - \frac{1}{2}\langle \nabla_{\omega_\infty} S(\omega_\infty), \nabla_{\omega_\infty}\eta\rangle_{\omega_\infty} \\ &+ \frac{i}{2\pi}\mathrm{Tr}\Big(\big\{\Lambda_{\omega_\infty}\overline{\partial}D\phi + 2\Lambda^2_{\omega_\infty}(F_{h_{HE}} \wedge (i\overline{\partial}\partial\eta))\big\}^0 \lambda(h_{HE})\Big). \end{aligned}$$

Proof. The proof follows from the previous proposition and the fact that

$$\{\Lambda_{\omega_\infty} F_{(E,h_{\mathrm{HE}})}\}^0 = 0.$$

Note that

$$\begin{aligned} &\frac{d}{dt}\Big|_{t=0} \mathrm{Tr}\Big(\{\Lambda_{\omega_\infty + it\overline{\partial}\partial\eta} F_{(h_{\mathrm{HE}}(I+t\phi))}\}^0 \lambda(h_{\mathrm{HE}}(I+t\phi))\Big) \\ &\quad = \mathrm{Tr}\Big(\frac{d}{dt}\Big|_{t=0} \{\Lambda_{\omega_\infty + it\overline{\partial}\partial\eta} F_{(h_{\mathrm{HE}}(I+t\phi))}\}^0 \lambda(h_{\mathrm{HE}})\Big) \\ &\quad + \mathrm{Tr}\Big(\{\Lambda_{\omega_\infty} F_{h_{\mathrm{HE}}}\}^0 \frac{d}{dt}\Big|_{t=0} \lambda(h_{\mathrm{HE}}(I+t\phi))\Big) \\ &\quad = \mathrm{Tr}\Big(A_{1,1}(\eta,\phi)\lambda(h_{\mathrm{HE}})\Big). \end{aligned}$$

□

Since T is a compact group, by the uniqueness of the Hermitian-Einstein metric, h is invariant under T.

Lemma 4.9. *Suppose that E is Mumford stable and h is a Hermitian-Einstein metric with respect to ω_∞, i.e. $\Lambda_{\omega_\infty} F_h = \mu I_E$. Then h is invariant under the action of T.*

Corollary 4.10. *The scalar curvature of ω_k is invariant under the action of T.*

The above two results follows from the uniqueness of the Hermitian-Einstein metric.

Corollary 4.11. *The map*

$$\begin{aligned} &\mathcal{C}^\infty(M) \oplus \Gamma(M,W) \oplus \bar{\mathfrak{g}} \to \mathcal{C}^\infty_0(M) \oplus \Gamma(M,W) \\ &(\eta, \Phi, b) \mapsto S_{1,1}(\eta,\Phi) + \frac{1}{2}\langle \nabla_{\omega_\infty} S_\infty, \nabla_{\omega_\infty}\eta\rangle_{\omega_\infty} - b \end{aligned}$$

is surjective. Here $S_\infty = S$ is the scalar curvature of ω_∞ and $\mathcal{C}^\infty_0(M)$ is the space of smooth functions η on M such that

$$\int_M \eta\omega^m_\infty = 0.$$

Moreover, the equivariant version is also valid, that is,

$$\mathcal{C}^\infty(M)^T \oplus \Gamma(M,W)^T \oplus \bar{\mathfrak{k}} \to \mathcal{C}_0^\infty(M)^T \oplus \Gamma(M,W)^T$$
$$(\eta,\Phi,b) \mapsto S_{1,1}(\eta,\Phi) + \frac{1}{2}\langle \nabla_{\omega_\infty} S_\infty, \nabla_{\omega_\infty}\eta\rangle_{\omega_\infty} - b$$

is surjective.

Lemma 4.12. *Let $\eta \in \mathcal{C}^\infty(M)$ and $\varphi \in \mathcal{C}^\infty(\mathbb{P}E^*)$. Then*

$$\langle \nabla_{\omega_k}\varphi, \nabla_{\omega_k}\eta\rangle_{\omega_k} = O(k^{-1}).$$

Moreover if $\varphi \in \mathcal{C}^\infty(M)$, then

$$\langle \nabla_{\omega_k}\varphi, \nabla_{\omega_k}\eta\rangle_{\omega_k} = k^{-1}\langle \nabla_{\omega_\infty}\varphi, \nabla_{\omega_\infty}\eta\rangle_{\omega_\infty} + O(k^{-2}).$$

Before we give the proof of proposition, we explain how to find $\varphi_{k,2}$ and $b_{k,2}$. We can write

$$S(\omega_k) = r(r-1) + S_1 k^{-1} + S_2 k^{-2} + \dots.$$

Note that Corollary 4.10 implies that $S(\omega_k)$ is invariant under the action of T. Thus, $S_i \in \mathcal{C}^\infty(\mathbb{P}E^*)^T$, where $S_1 = S_1(\omega_\infty, h_{\mathrm{HE}})$. For any smooth function φ on $\mathbb{P}E^*$, we have

$$S(\omega_k + k^{-1}\sqrt{-1}\,\bar\partial\partial\varphi) = r(r-1) + (S_1 + \triangle_V(\triangle_V - r)\varphi)k^{-1} + O(k^{-2}),$$

$$S(\omega_k + k^{-2}\sqrt{-1}\,\bar\partial\partial\varphi) = r(r-1) + S_1 k^{-1} + (S_2 + \triangle_V(\triangle_V - r)\varphi)k^{-2} + O(k^{-3}).$$

Hence for $\eta \in \mathcal{C}^\infty(M)$, $\Phi \in \Gamma(M,E)$ and $\varphi \in \mathcal{C}^\infty(\mathbb{P}E^*)$, we have

$$\begin{aligned} &S(\omega_k + \sqrt{-1}\,\bar\partial\partial\eta + k^{-1}\sqrt{-1}\,\bar\partial\partial\Phi + k^{-2}\sqrt{-1}\,\bar\partial\partial\varphi) \\ &= r(r-1) + S_1 k^{-1} + (S_2 + S_{1,1}(\eta,\Phi) + \triangle_V(\triangle_V - r)\varphi)k^{-2} + O(k^{-3}). \end{aligned}$$

Therefore

$$\begin{aligned} &S(\omega_k + \sqrt{-1}\,\bar\partial\partial\eta + k^{-1}\sqrt{-1}\,\bar\partial\partial\Phi + k^{-2}\sqrt{-1}\,\bar\partial\partial\varphi) - l(r(r-1) + k^{-1}S(\omega) + k^{-2}b_2) \\ &= k^{-2}\Big(S_2 + \triangle_V(\triangle_V - r)\varphi + S_{1,1}(\eta,\Phi) - b_2 - \Theta_s\Big) + O(k^{-3}) \end{aligned}$$

for some smooth function Θ_s. On the other hand

$$\begin{aligned} &\langle \nabla l(r(r-1) + k^{-1}S(\omega_\infty) + k^{-2}b_2), \nabla(\eta + k^{-1}\Phi + k^{-2}\varphi_2)\rangle \\ &= k^{-2}\langle \nabla_{\omega_\infty} S(\omega_\infty), \nabla_{\omega_\infty}\eta\rangle_{\omega_\infty} + O(k^{-3}). \end{aligned}$$

Now we can find $\varphi_2 \in \mathcal{C}^\infty(\mathbb{P}E^*)^T$ such that

$$\triangle_V(\triangle_V - r)\varphi_2 - b_2 - \Theta_s \in \mathcal{C}^\infty(M) \oplus \Gamma(M,W).$$

Lemma 4.12 implies that Θ_s is invariant under the action of T. Applying Lemma 4.8 implies that there exist $\eta_2 \in \mathcal{C}^\infty(M)^T$ and $\phi_2 \in \Gamma(M,W)^T$ and $b_2 \in$such that

$$S_{1,1}(\eta_2, \Phi_2) + \frac{1}{2}\langle \nabla_{\omega_\infty} S(\omega_\infty), \nabla_{\omega_\infty}\eta_2\rangle_{\omega_\infty} = \triangle_V(\triangle_V - r)\varphi_2 - b_2 - \Theta_s.$$

Hence

$$S(\omega_k) - l(r(r-1) + k^{-1}S(\omega)) = O(k^{-2}),$$
$$S(\omega_k + \sqrt{-1}\,\bar\partial\partial\varphi_{k,2}) + \frac{1}{2}\langle \nabla l(b_{k,2}), \nabla(\varphi_{k,2})\rangle - l(b_{k,2}) = O(k^{-3}),$$

where $\varphi_{k,2} = \eta_2 + k^{-1}\Phi_2 + k^{-2}\varphi_2$ and $b_{k,2} = r(r-1) + k^{-1}S(\omega) + k^{-2}b_2$. Note that $\varphi_{k,2} \in \mathcal{C}^\infty(\mathbb{P}E^*)^T$, since η_2, ϕ_2 and φ_2 are invariant under the action of T.

Proof of Proposition 4.6. We prove it by induction on p. Suppose that we have chosen $\eta_2, \dots \eta_{p-1} \in \mathcal{C}^\infty(M)^T$, $\Phi_2, \dots \Phi_{p-1} \in \Gamma(M,W)^T$, $\varphi_2, \dots \varphi_{p-1} \in \mathcal{C}^\infty(\mathbb{P}E^*)^T$ and $b_0, \dots b_{p-1} \in$ such that

$$S(\omega_k + \sqrt{-1}\,\bar\partial\partial\varphi_{k,p-1}) + \langle \nabla l(b_{k,p-1}), \nabla\varphi_{k,p-1}\rangle - l(b_{k,p-1}) = k^{-p}\epsilon_p + O(k^{-p-1}).$$

We have

$$\begin{aligned} &S(\omega_{k,p-1} + k^{-p+2}\sqrt{-1}\,\bar\partial\partial\eta_p + k^{-p+1}\sqrt{-1}\,\bar\partial\partial\Phi_p + k^{-p}\sqrt{-1}\,\bar\partial\partial\varphi_p) \\ &\quad = S(\omega_{k,p-1}) + k^{-p}(\triangle_V(\triangle_V - r)\varphi_p + S_{1,1}(\eta_p, \Phi_p)) + O(k^{-p-1}). \end{aligned}$$

On the other hand,

$$\begin{aligned} &\langle \nabla l(b_{k,p-1} + k^{-p}b_p), \nabla(\varphi_{k,p-1} + k^{-p+2}\eta_p + k^{-p+1}\Phi_p + k^{-p}\varphi_p)\rangle - l(b_{k,p-1} + k^{-p}b_p) \\ &= \langle \nabla l(b_{k,p-1}), \nabla\varphi_{k,p-1}\rangle - l(b_{k,p-1}) + k^{-p}(\langle \nabla_{\omega_\infty} S(\omega_\infty), \nabla_{\omega_\infty}\eta\rangle_{\omega_\infty} - b_p) + O(k^{-p-1}). \end{aligned}$$

Corollary implies that we there exist $\eta_p \in \mathcal{C}^\infty(M)^T, \Phi_p \in \Gamma(M,W)^T, \varphi_p \in \mathcal{C}^\infty(\mathbb{P}E^*)^T$ and $b_p \in$ such that

$$\triangle_V(\triangle_V - r)\varphi_p + S_{1,1}(\eta_p, \Phi_p) + \frac{1}{2}\langle \nabla_{\omega_\infty} S(\omega_\infty), \nabla_{\omega_\infty}\eta\rangle_{\omega_\infty} - b_p - \epsilon_p = \text{Constant}.$$

This concludes the proof.

□

Definition 4.13. Define $\omega_{k,p} = \omega_k + \sqrt{-1}\,\bar\partial\partial\varphi_{k,p}$. For any positive integer p and any $b \in \bar{\mathfrak{g}}$, we define $l_{k,p}(b) = l_k(b) - \frac{1}{2}\langle \nabla l_k(b), \nabla\varphi_{k,p}\rangle_{\omega_{k,p}}$.

The following lemma is straightforward.

Lemma 4.14. *Let $b \in \bar{\mathfrak{g}}_E$ and X be the holomorphic vector fields on M such that $db = \iota_X\omega$. Suppose that $\tilde X$ is the holomorphic lift of X to $\mathbb{P}E^*$. Then $k^{-1}dl_{k,p} = \iota_{\tilde X}\omega_{k,p}$.*

Corollary 4.15. *We have*

$$S(\omega_{k,p}) - l_{k,p}(b_{k,p}) = O(k^{-p-1}).$$

5. Proof of Theorem 1.2

The goal of this section is to prove Theorem 1.2. We closely follow [1,12,22]. Before we give the proof, we go over some estimates from Hong and Brönnle. Let's fix a large positive integer p. In this section, the operators $l = l_{k,p}$, $\mathcal{D}^*\mathcal{D}$ and ∇ and inner products are with respect to the metrics $\omega_{k,p}$.

Proposition 5.1 (c.f. [1])**.** *Let $L^2_4 = H^{4,2}$ be the Sobolev space of functions whose up to 4-th derivatives are in L^2 and $(L^2_4)^T$ is the subspace of T-invariant functions.*

(1) *Let p be a fixed positive integer. There exists a constant C independent of k such that the operators*

$$G_{k,p} : (L^2_4)^T \times \bar{\mathfrak{k}} \to (L^2)^T,$$
$$G_k(\phi, b) = \mathcal{D}^*\mathcal{D}\phi - \frac{1}{2}\langle \nabla S(\omega_{k,p}), \nabla\phi\rangle - \frac{1}{2}\langle \nabla l_{k,p}(b_{k,p})), \nabla\phi\rangle - l_{k,p}(b)$$

has right sided inverses P_k satisfying $||P_k||_{op} \leq Ck^3$. Note that $G_{k,p}$ is the linearization of the extremal operator at $(\omega_{k,p}, b_{k,p})$.

(2) *There exists a constant C independent of k such that*

$$\begin{aligned}&||Q_{k,p}(\phi, b) - Q_{k,p}(\psi, b')||_{L^2}\\ &\leq C\max(||(\phi, b)||_{L^2_4}, ||(\psi, b')||_{L^2_4})||(\phi, b) - (\psi, b')||_{L^2_4},\end{aligned}$$

where

$$Q_{k,p}(\phi, b) = S(\omega_{k,p} + \sqrt{-1}\,\bar{\partial}\partial\phi) + \frac{1}{2}\langle \nabla l_{k,p}(b_{k,p} + b), \nabla\phi\rangle - l_{k,p}(b_{k,p} + b) - G_{k,p}$$

is the nonlinear part of the extremal operator at $(\omega_{k,p}, b_{k,p})$.

Remark 2. Our setting is slightly different from the setting in [1]. In [1], Brönnle studied non-simple bundles over a base that does not admit nontrivial holomorphic vector field. However, the same proof as in [1] works in our setting.

Proof of Theorem 1.2 . We want to solve the following equation for $\phi \in \mathcal{C}^\infty(\mathbb{P}E^*)^T$ and $b \in \bar{\mathfrak{k}}$.

$$S(\omega_{k,p} + \sqrt{-1}\,\bar{\partial}\partial\phi) + \frac{1}{2}\langle \nabla l(b_{k,p} + b), \nabla\phi\rangle = l(b_{k,p} + b).$$

We can write it as the sum of linear and non-linear parts.

$$G_{k,p}(\phi, b) + Q_{k,p}(\phi, b) = 0.$$

Then in order to solve the equation, it suffices to solve the fixed point problem

$$\mathcal{Q}_k(\phi, b) = (\phi, b),$$

where $\mathcal{Q}(\phi, b) = -P_k(Q_{k,p}(\phi, b))$. We prove that the map $\mathcal{Q}$ is a contraction on the set

$$B := \{(\phi, b) \in L^2_4 \times \bar{\mathfrak{k}} \ \mid \ ||(\phi, b)||_{L^2_4} \leq 2C_1 k^{-p+2}\}$$

for $p \geq 6$ and $k \gg 0$. First note that

$$\begin{aligned} ||\mathcal{Q}(0,0)||_{L^2} &= ||P_k(Q_{k,p}(0,0))||_{L^2} \leq Ck^3 ||Q_{k,p}(0,0)||_{L^2} \\ &= Ck^3 ||S(\omega_{k,p}) - l_{k,p}(b_{k,p})||_{L^2} \leq C_1 k^{-p+2}. \end{aligned}$$

Let $(\phi, b), (\phi', b') \in B$. We have

$$\begin{aligned} ||\mathcal{Q}(\phi, b) - \mathcal{Q}(\phi', b')||_{L^2} &\leq ||P_k||_{op} ||Q_{k,p}(\phi, b) - Q_{k,p}(\phi', b')||_{L^2} \\ &\leq Ck^3 ||Q_{k,p}(\phi, b) - Q_{k,p}(\phi', b')||_{L^2} \\ &\leq Ck^{5-p} ||(\phi, b), (\phi' - b')||_{L^2_4}. \end{aligned}$$

Therefore,

$$||\mathcal{Q}(\phi, b) - \mathcal{Q}(0,0)||_{L^2} \leq Ck^{5-p} ||(\phi, b)||_{L^2}.$$

This implies that

$$||\mathcal{Q}(\phi, b)||_{L^2} \leq ||\mathcal{Q}(0,0)||_{L^2} + Ck^{5-p} ||(\phi, b)||_{L^2} \leq 2C_1 k^{-p+2},$$

for $k \gg 0$ and $p \geq 6$. Hence $\mathcal{Q}(\phi, b) : B \to B$ is a contraction for $k \gg 0$ and $p \geq 6$. Therefore, we can solve the equation for $\phi \in L^2_4$ and $b \in \bar{\mathfrak{k}}$. Now elliptic regularity implies that ϕ is smooth.

□

An immediate consequence of Theorem 1.2 is the following.

Corollary 5.1. Let (M, L) be a compact polarized manifold and $\omega_\infty \in c_1(L)$ be an extremal Kähler metric. Let X_s be the gradient vector field of the scalar curvature of ω_∞, i.e. $dS(\omega_\infty) = \iota_{X_s}\omega_\infty$. Let E be a Mumford stable holomorphic vector bundle over M. Suppose that all holomorphic vector fields on M can be lifted to holomorphic vector fields on $\mathbb{P}E^*$. Then there exist extremal metrics on $(\mathbb{P}E^*, \mathcal{O}_{\mathbb{P}E^*}(1) \otimes L^k)$ for $k \gg 0$.

6. Hong's moment map setting and proof of Theorem 1.1

In this section, we follow [13] to prove Theorem 1.1. As before, let (M, ω_∞) be a Kähler manifold of dimension m and G be the group of Hamiltonian isometries of (M, ω_∞). Note that the Lie algebra of G is the space of Hamiltonian vector fields on (M, ω_∞). Define

$$\mathcal{N} = \{f \in \mathcal{C}^\infty(M) \mid \iota_X \omega_\infty = df \text{ for some } X \in \mathfrak{g}\} = \mathrm{Ker}(\mathcal{D}^*\mathcal{D}).$$

Any $\xi \in \mathfrak{g}$ defines a holomorphic vector field $\xi^\#$ on M. For any $\xi \in \mathfrak{g}$, there exists a unique smooth function $f_\xi \in \mathcal{C}^\infty(M)$ such that

$$\iota_{\xi^\#}\omega_\infty = df_\xi \quad \text{and} \quad \int_M f_\xi \omega_\infty^m = 0. \tag{6.1}$$

The following is a straightforward computation.

Proposition 6.1. *The map $\xi \in \mathfrak{g} \to f_\xi$ is an isomorphism of Lie algebras. Moreover, for any $g \in G$ and $\xi \in \mathfrak{g}$, we have*

$$f_{Ad(g)\xi} = f_\xi \circ \sigma_{g^{-1}},$$

where $\sigma_g : M \to M$ is defined by $\sigma_g(x) = g.x$.

Corollary 6.2. *A function $f \in \mathcal{N}$ is in the center of $\mathcal{N}$ if and only if f is G-invariant. Moreover, if $f \in \mathcal{C}^\infty(M)$ is a G-invariant function, then $\pi_{\mathcal{N}}(f)$ is in the center of $\mathcal{N}$, where $\pi_{\mathcal{N}} : \mathcal{C}^\infty(M) \to \mathcal{N}$ is the orthogonal projection.*

Let Y be a Kähler manifold (open or compact without boundary). Suppose that the Lie algebra $\mathfrak{g}$ acts on Y. Then $[\xi_1^\sharp, \xi_2^\sharp] = [\xi_1, \xi_2]^\sharp$ for all $\xi_1, \xi_2 \in \mathfrak{g}$. Integrating the action of $\mathfrak{g}$, we obtain an action of G (an open neighborhood of identity in G) on Y. Therefore, there exists an equivariant moment map $\mu^Y : Y \to \mathfrak{g}$. Compose μ^Y with the map $\xi \in \mathfrak{g} \to f_\xi$ defined by (6.1), we have an equivariant moment map $\mu^Y : Y \to \mathcal{N}$. We apply this setting to the case when Y is the smooth locus of moduli space of Hermitian-Einstein connections $\mathcal{M}$ on a smooth complex vector bundle $\mathcal{E}$ and the action of G on Y. More precisely, let $\mathcal{E}$ be a smooth complex vector bundle of rank r on M and h be a fixed hermitian metric on $\mathcal{E}$. We fix a holomorphic structure on $\det(\mathcal{E})$. Let $\mathcal{G}$ be the group of unitary gauge transformations of $(\mathcal{E}, h)$. Let $\mathcal{A}$ be the space of Hermitian-Einstien connections A on $\mathcal{E}$ such that $(\mathcal{E}, \bar\partial_A)$ is a simple holomorphic vector bundle and A induces the fixed holomorphic structure on $\det(\mathcal{E})$ modulo the action of the unitary gauge group $\det(\mathcal{E})$.

Now we define the moduli space of simple Hermitian-Einstein metrics on $\mathcal{E}$ as follows:

$$\mathcal{M} = \frac{\mathcal{A}}{\mathcal{G}}.$$

One can compute the tangent space to $\mathcal{A}$ and the moduli space $\mathcal{M}$ (c.f. [15]): for any $A \in \mathcal{A}$, we have

$$T_A\mathcal{A} = \{\alpha \in \Omega^1(M, \mathrm{End}(\mathcal{E}, h)) \mid \nabla^A\alpha \in \Omega^{1,1} \text{ and } \Lambda\nabla^A\alpha = 0\}.$$

Moreover if $[A] \in \mathcal{M}$ is a smooth point of $\mathcal{M}$, then

$$T_{[A]}\mathcal{M} = \frac{\{\alpha \in \Omega^1(M, \mathrm{End}(\mathcal{E}, h)) \mid \nabla^A\alpha \in \Omega^{1,1} \text{ and } \Lambda\nabla^A\alpha = 0\}}{\{\nabla_A s \mid s \in \Gamma(M, \mathrm{End}(\mathcal{E}, h))\}}.$$

Note that the moduli space $\mathcal{M}$ is not smooth in general. However, one can define an action of $\mathfrak{g}$ on $\mathcal{A}$ as follows: Proposition 4.3 tells that for any $A \in \mathcal{A}$ and any $X \in \mathfrak{g}$, there exists a unique $u_X \in \Gamma(M, \mathrm{End}(\mathcal{E}))$, depending on A, such that

$$\Lambda\partial_A(\bar\partial_A u_X - \iota_X F_A) = 0 \text{ and } \int_M \mathrm{tr}(u_X)\omega^m = 0.$$

For any $A \in \mathcal{A}$ and $X \in \mathfrak{g}$, define

$$\theta_X(A) = -(-\partial_A g_X^* + \bar\partial_A g_X - \iota_X F_A) \in T_A\mathcal{A}.$$

Note that the vector field θ_X is the infinitesimal vector field on $\mathcal{A}$ induced by the action of X. Hong proved that the vector field θ_X can be descended to the moduli space $\mathcal{M}$. Moreover, he proved that

$$[\theta_X, \theta_Y] - \theta_{[X,Y]} \in d_A\Gamma(M, \mathrm{End}(\mathcal{E})).$$

This implies that on the moduli space $\mathcal{M}$, we have $[\theta_X, \theta_Y] = \theta_{[X,Y]}$. Therefore we have an action of the Lie algebra $\mathfrak{g}$ on $\mathcal{M}$.

Proposition 6.3. *([13]) The map $\mu^e : \mathcal{M} \to \mathcal{N}$ given by*

$$\mu^e([A]) = \pi_{\mathcal{N}}\big(\Lambda^2\mathrm{tr}(iF_A \wedge iF_A)\big)$$

is an equivariant moment map for the action of G on $\mathcal{M}$.

By the definition of $\mathcal{A}$, any connection $A \in \mathcal{A}$ induces the fixed holomorphic structure on $\det(\mathcal{E})$ up to the action of the gauge group. Therefore the function

$$\frac{2}{r}\Lambda^2(\mathrm{Ric}(\omega) \wedge \mathrm{tr}(iF_A)) - \frac{2}{r(r+1)}\Lambda^2(\mathrm{tr}(iF_A) \wedge \mathrm{tr}(iF_A)) - \mu S(\omega) + \frac{\mathrm{tr}(u_{X_s})}{r}$$

is independent of the choice of $A \in \mathcal{A}$. We define the moment map $\mu : \mathcal{M} \to \mathcal{N}$ as follows:

$$\mu([A]) = \mu^e([A])$$
$$+ \pi_{\mathcal{N}}\Big(\frac{2}{r}\Lambda^2(\mathrm{Ric}(\omega) \wedge \mathrm{tr}(iF_A)) - \frac{2}{r(r+1)}\Lambda^2(\mathrm{tr}(iF_A) \wedge \mathrm{tr}(iF_A)) - \mu S(\omega) + \frac{\mathrm{tr}(u_{X_s})}{r}\Big).$$

Lemma 6.4. *The moment map μ is equivarent if $c_1(L) = \lambda c_1(E)$ for some constant $\lambda \in \mathbb{Z}$.*

Proof. Since $c_1(L) = \lambda c_1(E)$, there exists a smooth function φ on M such that $\lambda \mathrm{tr}(iF_A)) - \omega_\infty = \sqrt{-1}\,\bar{\partial}\partial\varphi$. Taking the trace with respect to ω_∞, we have $r\mu\lambda - m = \Delta\varphi$, where μ is the slope of E. Using the Hermitian-Einstein condition, we obtain that φ is constant and therefore $\lambda \mathrm{tr}(iF_A)) = \omega_\infty$. Therefore

$$\frac{1}{r}\Lambda^2(\mathrm{Ric}(\omega_\infty) \wedge \mathrm{tr}(iF_A)) - \frac{1}{r(r+1)}\Lambda^2(\mathrm{tr}(iF_A) \wedge \mathrm{tr}(iF_A)) - \mu S(\omega_\infty) + \frac{\mathrm{tr}(u_{X_s})}{r}$$

is invariant under the action of G since ω_∞ is invariant under the action of G.

□

Following [13], we define the following.

Definition 6.5. A holomorphic structure A is called stable relative to the maximal torus T if there exists a connection A_∞ in the orbit of $[A] \in \mathcal{M}$ such that $[A_\infty]$ is a smooth point of $\mathcal{M}$, $\mu([A_\infty]) \in \mathfrak{t}$ and

$$\frac{\partial \mu}{\partial A}([A_\infty]) : T_{[A_\infty]} \to \frac{\bar{\mathfrak{k}}}{\bar{\mathfrak{t}}}$$

is surjective.

Remark 6.6. The moment map μ is not equivariant in general. If $c_1(L) = \lambda c_1(E)$, then the notion of relative stability defined above is a GIT notion of stability introduced by Székelyhidi ([21]). However, it is not clear how the notion of stability defined above is related to a GIT notion of stability in general .

Proof of Theorem 1.1. Theorem 1.2 implies that for any $A \in \mathcal{A}$, we can find $\phi^A \in \mathcal{C}^\infty(\mathbb{P}{E^*}_A)$ and $b^A \in \bar{\mathfrak{k}}$ such that

$$S(\omega^A_{k,p} + \sqrt{-1}\,\bar{\partial}\partial\phi^A) + \frac{1}{2}\langle \nabla l(b^A), \nabla \phi^A \rangle = l(b^A).$$

Moreover b^A has the following expansion.

$$b^A = r(r-1) + k^{-1}S(\omega) - k^{-2}\mu(A) + k^{-3}R^A.$$

One can easily see through the computation that $\omega_{k,p}^A$, ϕ^A and b^A depend smoothly on A. Suppose that A_∞ is in the orbit of E and $\mu(A_\infty) \in \mathfrak{t}$. Define $\Phi(A,t) = \mu(A) + tR^A$. Then $\Phi(A_\infty, 0) \in \bar{\mathfrak{t}}$ and $p_1(\frac{\partial \Phi}{\partial A}(A_\infty, 0))$ is surjective, where $p_1 : \bar{\mathfrak{k}} \to \frac{\bar{\mathfrak{k}}}{\bar{\mathfrak{t}}}$ is the quotient map. Therefore applying the implicit function theorem, we find A_t for small t such that $\mu(A_t) + tR^A \in \bar{\mathfrak{t}}$ for t small enough; hence for $k = t^{-1} \gg 0$, we have $b^{A_t} = r(r-1) + tS(\omega) - t^2\mu(A_t) + t^3 R^{A_t} = r(r-1) + tS(\omega) \in \bar{\mathfrak{t}}$. This implies that $\omega_{k,p}^{A_{k^{-1}}} + \sqrt{-1}\,\bar{\partial}\partial\phi^{A_{k^{-1}}}$ are extremal metrics for $k \gg 0$. Note that $A_{k^{-1}}$ are compatible with the holomorphic structure of E since they are all in the orbit of A_∞.

□

An holomorphic vector bundle $\mathcal{E}$ is called projectively flat, if the curvature of E is of the form $c \cdot \mathrm{Id} \otimes \omega$, where c is a constant and ω is the Kähler metric of the base manifold (M, L). Assume that (M, L) is an extremal Kähler manifold. Then by our result, for $k \gg 0$, there are extremal Kähler metrics in the classes $\mathcal{O}_{\mathbb{P}E^*(1)} \otimes \pi^*(L^k)$ on $\mathbb{P}E^*$.

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University of California, Irvine, Department of Mathematics

Department of Mathematics, University of California, Irvine, Irvine, CA 92697, USA
E-mail address, Zhiqin Lu: zlu@uci.edu

Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario, N2L 3G1, Canada
E-mail address, Reza Seyyedali: rseyyeda@uwaterloo.ca