# Degrees of Freedom of the Two-Way Relay MIMO Interference Channel


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***Abstract*—We investigate the symmetric degrees of freedom (DoF) of the $K$-pair ($2K$ users) two-way relay Multiple-Input Multiple-Output (MIMO) Gaussian interference channel for $K = 2, 3$ where each user is equipped with $M$ antennas and the relay node is equipped with $N$ antennas. The two users of each pair communicate with each other via the help of the relay only. Expressing the DoF characterization as a function of the ratio $\gamma = M/N$, we find that the DoF value per user is piecewise linear depending on $M$ and $N$ alternately. As we will show in this paper, while the DoF achievability only needs linear beamforming transmission and zero-forcing reception, inter-pair signal subspace alignment is essential at the relay node as well as the users. In addition, the DoF converse is first developed based on the linear dimension counting approach, which can be further translated to the information theoretic statement.**


## I. Introduction

In wireless networks with multiple sources and destinations (users), concurrent transmissions give rise to competition for channel resources, e.g., frequency and time, between different information flows. How to deal with the interference caused by different concurrent transmissions is attractive and essential to understand the fundamental capacity limit of wireless networks. Recently, a number of interesting signaling schemes have been investigated to deal with the interference problem so as to improve the achievable rate of wireless networks. In particular, the notion of interference alignment is one of the most attractive ideas that have emerged out in recent work.

The idea of interference alignment in wireless networks was inspired by study of degrees of freedom (DoF) of networks. The DoF characterization is quite interesting and important, because not only it motivates a number of attractive ideas such as interference alignment, but also it implies the number of independent signaling that can be communicated in networks, which is a capacity approximation characterization. Interference alignment was first proposed for the two-user $X$ channel in [9], and then was shown by Cadambe et. al. in the $K$-user interference channel in [8]. In the past years, a variety of interference alignment schemes have been proposed for a number of multiuser networks such as interference channels, $X$ channels, broadcast channels and multi-hop networks, as summarized in [7], but the central new insight to emerge from those schemes is always to *align* interference as much as possible rather than avoid interference, while keeping the desired signal distinguishable from the interference. So far, the DoF results that have already been known are almost for *one-way* communication networks only. If the communication networks allow *two-way* transmission, then even the DoF characterization of the networks remain unknown in general.

Recently, Lee et. al. have shown the DoF for a 3-user MIMO $Y$ channel in [3] where each user has 2 antennas, and each user sends one independent message to each of the others via the help a 3-antenna relay node. The transmission scheme shown in [3], incorporating the ideas of interference alignment and network coding, achieves the cut-set DoF bound of the network. The key to this surprising result was the new idea of alignment for network coding. That is, only the signal vectors carrying the messages of paired users talking to each other can be aligned at the relay node, and the signal vectors of different pairs remain distinguishable. This idea was then applied to two-way or multi-way networks in [5], [6], [4] to find their DoF. However, all those results only account for a special network where each node in the network has the same number of antennas.

In this work, we investigate the DoF of the $K$-pair ($2K$ users) two-way relay MIMO interference channel where each user has $M$ antennas and the relay has $N$ antennas where $M$ and $N$ can be arbitrary positive integers. Since the paired two users may not project a common subspace the relay, the idea of alignment for network coding proposed in [3] cannot be applied in general. In addition, no matter if signals of the paired two users can be aligned or not at the relay, the joint signal alignment among multiple pairs is possible. Since there are many parameters and signal alignment coming into the play, the DoF characterization of the two-way relay MIMO interference channel is difficult in general. In this paper, in order to solve this problem, we propose a new idea of *inter-pair signal subspace alignment*. With this new idea, we show that the DoF per user is piecewise linear depending on $M$ and $N$ alternatively, implying that there are antenna redundancies at either the user side or the relay node.

## II. System Model

Consider a two-way relay MIMO Gaussian interference channel where there are $K$ pairs of users, each pair consisting of two users, for a total of $2K$ users. As shown in Fig. 1, each user is equipped with $M$ antennas and the relay is equipped with $N$ antennas. For brevity, we label the users on the left-hand-side with indices $1, 2, \cdots, K$ and the users on the right-hand-side with indices $K+1, K+2, \cdots, 2K$. User $i$ and User

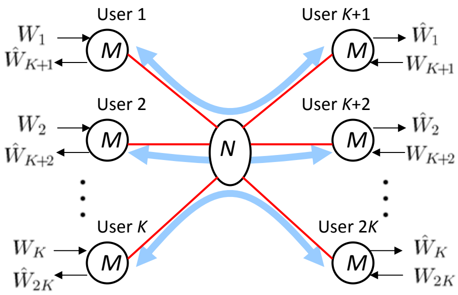


Fig. 1. The $K$ user-pair ($2K$ users) two-way relay MIMO Gaussian interference channel where each user has $M$ antennas and the relay has $N$ antennas.

$i+K$ comprise of the $i^{th}$ pair where $i = 1, 2, \cdots, K$. In the $i^{th}$ pair, each user sends one independent message to the other via the help of the relay node only. We denote by $\mathbf{H}^{[Rk]}$ the $N \times M$ channel matrix from User $k$ to the relay, and $\mathbf{H}^{[kR]}$ the $M \times N$ channel matrix from the relay node to User $k$ where $k = 1, 2, \cdots, 2K$. In this paper, we assume that the channel coefficients are independently drawn from continuous distributions, and the channel coefficients stay constant during the entire transmission once they are drawn. Notice that for the same User $k$, while our results are valid regardless of whether the channel matrices $\mathbf{H}^{[Rk]}$ and $\mathbf{H}^{[kR]}$ are identical, without loss of generality we assume they are generated independently. We assume that the global channel knowledge is available at all nodes. In this work, we assume that the user nodes and the relay node *all* work in the full-duplex mode[1], i.e., they can hear or transmit simultaneously. Since the relay node hears from all $2K$ users, the received signal vector at the relay at time $t \in \mathbb{Z}^+$ is given by:

$$\mathbf{y}^{[R]}(t) = \textstyle\sum_{k=1}^{2K} \mathbf{H}^{[Rk]}\mathbf{x}^{[k]}(t) + \mathbf{z}^{[R]}(t) \quad (1)$$

where $\mathbf{x}^{[k]}(t)$ is the transmitted signal vector from User $k$, which is represented by a complex-valued $M \times 1$ vector satisfying an average power constraint $\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}[\|\mathbf{x}^{[k]}(t)\|^2] \le P$ for $T$ channel uses. The $N \times 1$ column vector $\mathbf{z}^{[R]}(t)$ represents the i.i.d. circularly symmetric complex additive white Gaussian noise (AWGN) at the relay, each entry of which is an i.i.d. Gaussian random variable with zero-mean and unit-variance, i.e., $\mathbf{z}^{[R]}(t) \sim \mathcal{CN}(\mathbf{0}, \mathbf{I})$. At the user side, each user *only* hears from the relay, and thus the received signal vector at User $k$ at time $t$ is given by:

$$\mathbf{y}^{[k]}(t) = \mathbf{H}^{[kR]}\mathbf{x}^{[R]}(t) + \mathbf{z}^{[k]}(t) \quad (2)$$

where $\mathbf{x}^{[R]}(t)$ is the complex-valued $N \times 1$ transmitted signal vector from the relay which satisfies an average power constraint $\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}[\|\mathbf{x}^{[R]}(t)\|^2] \le P$ for $T$ channel uses, and the $M \times 1$ column vector $\mathbf{z}^{[k]}(t) \sim \mathcal{CN}(\mathbf{0}, \mathbf{I})$ represents the i.i.d. circularly symmetric complex AWGN at User $k$.

Let $R_k(P) = R(P)$ denote the symmetric achievable rate of each user. Also, we define $d(K, M, N) \triangleq \lim_{\rho\to\infty} R(P)/\log(P)$ as the symmetric DoF per user. In this paper, we investigate the cases of $K = 2$ and $K = 3$. The dependence on $K, M, N$ may be dropped for compact notation when no ambiguity would be caused. Moreover, we use "a", "**a**" and "**A**" to denote a scalar, a column vector and a matrix, respectively.

[1]If all nodes work in the half-duplex node, then the DoF value per user that we show in this exposition will be scaled by a factor $1/2$.

## III. Main Results

In this section, we present our DoF results of the network that we defined in Section II, and show the intuition behind our new results.

*Theorem 1:* For the two-way relay two-user interference channel where each user has $M$ antennas and the relay node has $N$ antennas, the number of DoF per user is given by:

$$d = \begin{cases} M, & M/N \le 1/3 \\ N/3, & 1/3 < M/N \le 1/2 \\ 2M/3, & 1/2 < M/N \le 3/4 \\ N/2, & 3/4 < M/N. \end{cases} \quad (3)$$

*Proof:* All the insights behind the $K = 2$ setting are basically included in the $K = 3$ setting. The proof for this theorem is presented [1] in detail. ■

*Theorem 2:* For the two-way relay three-user interference channel where each user has $M$ antennas and the relay node has $N$ antennas, the number of DoF per user is given by:

$$d = \begin{cases} M, & M/N \le 1/5 \\ N/5, & 1/5 < M/N \le 1/4 \\ 4M/5, & 1/4 < M/N \le 5/18 \\ 2N/9, & 5/18 < M/N \le 1/3 \\ 2M/3, & 1/3 < M/N \le 3/8 \\ N/4, & 3/8 < M/N \le 1/2 \\ M/2, & 1/2 < M/N \le 2/3 \\ N/3, & 2/3 < M/N. \end{cases} \quad (4)$$

*Proof:* Due to the space limitation, we only show two interesting cases in the next section, and present all the analysis and rigorous proof for this theorem in our full paper [1]. ■

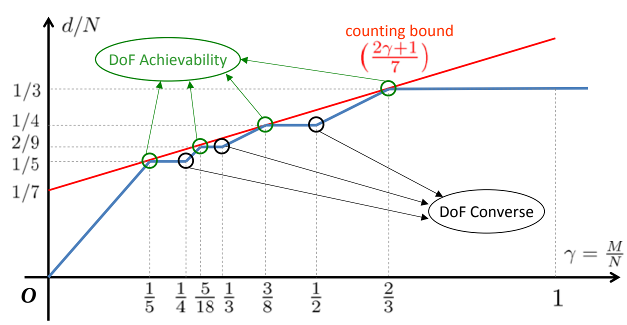


Fig. 2. The DoF per user of the $K = 3$ pair two-way relay MIMO Gaussian interference channel as a function of $\gamma = M/N$

In Fig. 2, we plot the value of DoF per user as shown in (4) normalized by $N$ denoted as the blue curve, with respect to the ratio $\gamma = M/N$. It can be seen that the DoF curve is piecewise linear, depending on the parameters $M$ and $N$, alternatively. This observation may remind us of the three-user $M \times N$ Gaussian MIMO interference channel, where the DoF curve is also piecewise linear, bouncing between the DoF counting bound and the decomposition bound [2]. The piecewise linearity implies antenna redundancies at either transmitters or receivers in general. For example, in Fig. 2, each user is able to achieve $d = N/5$ DoF when $\gamma = 1/5$ (denoted as the first green circle). Then suppose we increase the number of antennas at each transmitter such that the ratio $\gamma = M/N$ increases. Intuitively, increasing the number of antennas to the network cannot decrease the channel capacity. Thus, the DoF value per user, $d = N/5$, should still be achievable. On the other hand, consider the case of $\gamma = 1/4$ where Theorem 2 implies that each user cannot achieve more than $d = N/5$ DoF (denoted as the first black circle). If we decrease the number of antennas at each transmitter such that the ratio $\gamma$ decreases, then $d = N/5$ DoF value per user is still the outer bound because decreasing the number of antennas cannot increase the channel capacity. Therefore, for any case where $\gamma \in [1/5, 1/4]$, each user has $d = N/5$ DoF, which depends on the value of $N$ only, i.e., there are antenna redundancies at the transmitter side.

With the antenna redundancies argument, intuitively it suffices to first show the DoF achievability at $\gamma = \frac{1}{5}, \frac{5}{18}, \frac{3}{8}, \frac{2}{3}$ with green circles, and the DoF converse at $\gamma = \frac{1}{4}, \frac{1}{3}, \frac{1}{2}$ with black circles, and then extend to the regimes between every two adjacent transition points. Moreover, the DoF converse for the regimes $\gamma < 1/5$ and $\gamma > 2/3$ are determined by the single-user DoF bound and the cut-set DoF bound of the network, respectively.

## IV. 3-Pair: DoF Achievability

Among the four transmission points $\gamma = \frac{1}{5}, \frac{5}{18}, \frac{3}{8}, \frac{2}{3}$ leading to the DoF achievability, we only show the cases of $\gamma = 5/18$ and $\gamma = 3/8$ in this section due to the space limitation. Rigorous proof for all DoF achievability is deferred to [1].

### A. Example: $(M, N) = (5, 18) \Rightarrow d = 4$

As implied by Theorem 2, we will show that each user is able to achieve $d = 4$ DoF in this case. The entire communication consists two phases, from users to the relay, and then from the relay to users.

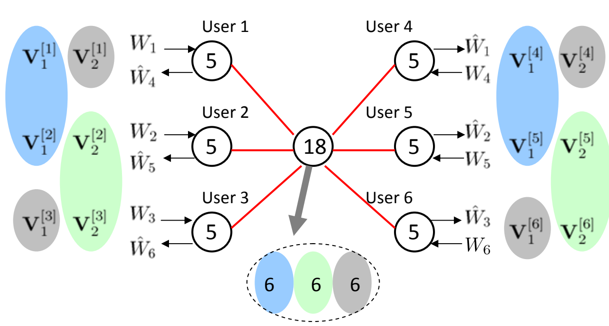


Fig. 3. DoF achievability for $(K, M, N) = (3, 5, 18)$

*1) From Users to the Relay:* In the first phase, each User $k$ encodes its 4 symbols using a $5 \times 4$ beamforming matrix, and the transmitted signal vector of User $k$ can be written as

$$\mathbf{x}^{[k]} = \mathbf{V}^{[k]}\mathbf{u}^{[k]} = \mathbf{V}_1^{[k]}\mathbf{u}_1^{[k]} + \mathbf{V}_2^{[k]}\mathbf{u}_2^{[k]} \quad (5)$$

where the $4 \times 1$ column vector $\mathbf{u}_1^{[k]}$ represents the 4 independent symbols. For convenience, we denote $\mathbf{u}_1^{[k]}$, $\mathbf{u}_2^{[k]}$ as two $2 \times 1$ column vectors, consisting of the first two and last two entries of $\mathbf{u}^{[k]}$, respectively, and $\mathbf{V}_1^{[k]}$ $\mathbf{V}_2^{[k]}$ are the corresponding $5 \times 2$ matrices.

Now let us consider the signals at the relay node. Our goal is that for each pair, the signal subspace spanned by interference contributed by the other two pairs has as lower dimensions as possible, while $d$ mixing data streams for each pair can still be separable. Here, let us consider pair 3 as an example first. In the 18-dimensional space at the relay, in order to protect a 4-dimensional subspace for pair 3, the signal coming from pair 1 and pair 2 cannot span more than 14 dimensions. Since they have a total of $4d = 16$ symbols, we need to align 2 symbols. Thus, we align $\mathbf{u}_1^{[5]}$ in the signal subspace spanned by the six symbols $\mathbf{u}_1^{[1]}, \mathbf{u}_1^{[2]}, \mathbf{u}_1^{[4]}$ (shown in the blue color in Fig. 3). This can be done since User 5 projects at the relay a 5-dimensional subspace which has a 2-dimensional intersection with the 15-dimensional subspace projected by User 1, User 2, User 4, i.e., the number of common dimensions is $5 \times 4 - 18 = 2$. Thus, we have the following alignment equation:

$$\mathbf{H}^{[R1]}\mathbf{V}_1^{[1]} + \mathbf{H}^{[R2]}\mathbf{V}_1^{[2]} + \mathbf{H}^{[R4]}\mathbf{V}_1^{[4]} = -\mathbf{H}^{[R5]}\mathbf{V}_1^{[5]} \quad (6)$$

which can be solved by finding the solution of the equation:

$$\begin{bmatrix}\mathbf{H}^{[R1]} & \mathbf{H}^{[R2]} & \mathbf{H}^{[R4]} & \mathbf{H}^{[R5]}\end{bmatrix}\begin{bmatrix}\mathbf{V}_1^{[1]T} & \mathbf{V}_1^{[2]T} & \mathbf{V}_1^{[4]T} & \mathbf{V}_1^{[5]T}\end{bmatrix}^T = \mathbf{O}. \quad (7)$$

Once we guarantee interference alignment for pair 3, we continue to look into pair 2 and pair 1, respectively. For pair 2, the $4d = 16$ symbols of pair 1 and pair 3 are interference. Similarly, in order to protect a 4-dimensional subspace at the relay for pair 2, we need to ensure that the 16 interfering symbols span no more than 14 dimensions. Hence, we align the two symbols $\mathbf{u}_2^{[4]}$ in the signal subspace spanned by the six symbols $\mathbf{u}_1^{[3]}, \mathbf{u}_2^{[1]}, \mathbf{u}_1^{[6]}$ (shown in the gray color). Again, this can be done since the 5-dimensional subspace projected by User 4 at the relay has a 2-dimensional intersection with the 15-dimensional subspace projected by User 1, User 3, User 6. Similarly, for pair 3, the 16 symbols of pair 1 and pair 2 are interference, and we align the two symbols $\mathbf{u}_2^{[6]}$ in the space spanned by the six symbols $\mathbf{u}_2^{[2]}, \mathbf{u}_2^{[3]}, \mathbf{u}_2^{[5]}$ (shown in the green color). These two operations produce the following two alignment equations:

$$\mathbf{H}^{[R3]}\mathbf{V}_1^{[3]} + \mathbf{H}^{[R1]}\mathbf{V}_2^{[1]} + \mathbf{H}^{[R6]}\mathbf{V}_1^{[6]} = -\mathbf{H}^{[R4]}\mathbf{V}_2^{[4]}, \quad (8)$$

$$\mathbf{H}^{[R2]}\mathbf{V}_2^{[2]} + \mathbf{H}^{[R3]}\mathbf{V}_2^{[3]} + \mathbf{H}^{[R5]}\mathbf{V}_2^{[5]} = -\mathbf{H}^{[R6]}\mathbf{V}_2^{[6]}. \quad (9)$$

Both equations above can be solved by finding the solutions of the following two equations:

$$\begin{bmatrix}\mathbf{H}^{[R3]} & \mathbf{H}^{[R1]} & \mathbf{H}^{[R6]} & \mathbf{H}^{[R4]}\end{bmatrix}\begin{bmatrix}\mathbf{V}_1^{[3]T} & \mathbf{V}_2^{[1]T} & \mathbf{V}_1^{[6]T} & \mathbf{V}_2^{[4]T}\end{bmatrix}^T = \mathbf{O}, \quad (10)$$

$$\begin{bmatrix}\mathbf{H}^{[R2]} & \mathbf{H}^{[R3]} & \mathbf{H}^{[R5]} & \mathbf{H}^{[R6]}\end{bmatrix}\begin{bmatrix}\mathbf{V}_2^{[2]T} & \mathbf{V}_2^{[3]T} & \mathbf{V}_2^{[5]T} & \mathbf{V}_2^{[6]T}\end{bmatrix}^T = \mathbf{O}. \quad (11)$$

So far, it can be easily seen that for all $4d = 16$ symbols of arbitrary two pairs, we align 2 symbols into the subspace spanned by the other 14 symbols at the relay. What remains to be guaranteed is: (a) those 14 symbols span a 14-dimensional signal subspace, i.e., to ensure the linear independencies among the beamforming vectors carrying the 4 symbols per user, and then (b) the vector subspace projected by each user of the remaining pair and that 14-dimensional subspace span the entire 18-dimensional signal space at the relay, i.e., to ensure the separability of the 4 mixing (linear combinations) symbols of the remaining pair from interference caused by the other two pairs. Due to the symmetry of the design, it suffices to show that the first two pairs consisting of User 1, User 2, User 4 and User 5 project a 14-dimensional subspace at the relay, which has only null intersection with the 4-dimensional subspace projected by User 3. That is, we need to ensure that the following $18\times18$ square matrix has full rank almost surely:

$$\Big[\mathbf{H}^{[R1]}\mathbf{V}^{[1]} \;\; \mathbf{H}^{[R2]}\mathbf{V}^{[2]} \;\; \mathbf{H}^{[R4]}\mathbf{V}^{[4]} \;\; \mathbf{H}^{[R5]}\mathbf{V}_2^{[5]} \;\; \mathbf{H}^{[R3]}\mathbf{V}^{[3]}\Big].$$

As a matter of fact, this can be verified through a simple numerical test, and we defer the rigorous proof to [1].

*2) From the Relay to Users:* The transmission in the second phase, from the relay to users, basically follows a *dual* approach for the reciprocal channel. We introduce the scheme in the following.

Since we already reserve a 4-dimensional inter-pair interference-free subspace for each pair at the relay, we can obtain 4 mixing symbols for each pair. Due to the symmetry of the design, we consider User 3 in pair 3 only. We define the following $18\times14$ matrix

$$\mathbf{G}_3 \triangleq \Big[\mathbf{H}^{[R1]}\mathbf{V}^{[1]} \;\; \mathbf{H}^{[R2]}\mathbf{V}^{[2]} \;\; \mathbf{H}^{[R4]}\mathbf{V}^{[4]} \;\; \mathbf{H}^{[R5]}\mathbf{V}_2^{[5]}\Big], \quad (12)$$

whose column subspace is the 14-dimensional subspace projected from pair 1 and pair 2. Also, we denote $\mathbf{G}_3^c$ as a $18\times4$ (randomly picked) full rank matrix which lies in the null space of $\mathbf{G}_3^T$, i.e., $(\mathbf{G}_3^c)^T\mathbf{G}_3=\mathbf{O}$. Recall that the received signal at the relay is given by:

$$\begin{aligned}\mathbf{y}^{[R]} &= \textstyle\sum_{k=1}^{6}\mathbf{H}^{[Rk]}\mathbf{x}^{[k]}+\mathbf{z}^{[R]} && (13)\\ &= \textstyle\sum_{k=1}^{6}\mathbf{H}^{[Rk]}(\mathbf{V}_1^{[k]}\mathbf{u}_1^{[k]}+\mathbf{V}_2^{[k]}\mathbf{u}_2^{[k]})+\mathbf{z}^{[R]} && (14)\\ &= \Big[\mathbf{H}^{[R1]}\mathbf{V}_1^{[1]}\cdots\mathbf{H}^{[R6]}\mathbf{V}_2^{[6]}\Big]\Big[\mathbf{u}_1^{[1]^T}\cdots\mathbf{u}_2^{[6]^T}\Big]^T+\mathbf{z}^{[R]}. && (15)\end{aligned}$$

To obtain the 4 mixing symbols of pair 3, denoted by a $4\times1$ vector $\mathbf{u}^{R3}$, the relay uses the receive beamforming matrix $\mathbf{G}_3^c$ to obtain the four noisy mixing symbols as:

$$\begin{aligned}\mathbf{u}^{R3} = (\mathbf{G}_3^c)^T\mathbf{y}^{[R]} &= (\mathbf{G}_3^c)^T\mathbf{H}^{[R3]}(\mathbf{V}_1^{[3]}\mathbf{u}_1^{[3]}+\mathbf{V}_2^{[3]}\mathbf{u}_2^{[3]})+\\ &+(\mathbf{G}_3^c)^T\mathbf{H}^{[R6]}(\mathbf{V}_1^{[6]}\mathbf{u}_1^{[6]}+\mathbf{V}_2^{[6]}\mathbf{u}_2^{[6]})+(\mathbf{G}_3^c)^T\mathbf{z}^{[R]}. \quad (16)\end{aligned}$$

Following the similar approach, the relay can also obtain the four noisy mixing symbols $\mathbf{u}^{R1}$ for pair 1 and $\mathbf{u}^{R2}$ for pair 2, respectively. Now we use $\mathbf{u}_1^{Ri}$, $\mathbf{u}_2^{Ri}$ to represent the two column vectors consisting of the first two and last two entries of $\mathbf{u}^{Ri}$ for $i=1,2,3$.

In the second phase, the transmitted signal vector of the relay is given by:

$$\mathbf{x}^{[R]} = \eta\textstyle\sum_{i=1}^{3}\mathbf{V}^{Ri}\mathbf{u}^{Ri} \quad (17)$$

where $\mathbf{V}^{Rk}$ is the $18\times4$ beamforming matrix for the mixing symbols $\mathbf{u}^{Rk}$ which will be determined later, and $\eta$ is the scaling parameter to normalize the mixing symbols power, which can be easily seen to be a bounded value in the scale of $1/P$, i.e., $\eta=O(1/P)$.

Next, let us consider the user side. Each User $k$ applies a $5\times4$ receiver beamforming matrix $\mathbf{U}^{[k]}=[\mathbf{U}_1^{[k]}\;\mathbf{U}_2^{[k]}]$ where $\mathbf{U}_1^{[k]}$ and $\mathbf{U}_2^{[k]}$ are two $5\times2$ matrices. We design each receiver beamforming matrix in such a approach as in the first phase, by replacing $\mathbf{V}^{[k]}$, $\mathbf{H}^{[Rk]}$ with $\mathbf{U}^{[k]}$ and $\mathbf{H}^{[kR]}$, respectively. That is, as a dual approach of phase 1, if we project the receiver (user) subspaces back to the transmitter (relay), any two pairs project a 14-dimensional subspace which has only null intersection with the 4-dimensional subspace projected by each user of the remaining pair. Thus, by using the design above, we already reserve a 4-dimensional subspace for pair 3 at the relay, leaving the other 14 dimensions for the first two pairs. The 14-dimensional subspace is given by the column space of the following $18\times14$ full rank matrix

$$\mathbf{G'}_3 \triangleq \Big[(\mathbf{U}^{[1]^T}\mathbf{H}^{[1R]})^T\;(\mathbf{U}^{[2]^T}\mathbf{H}^{[2R]})^T\;(\mathbf{U}^{[4]^T}\mathbf{H}^{[4R]})^T\;(\mathbf{U}_2^{[5]^T}\mathbf{H}^{[5R]})^T\Big], (18)$$

which has the same form as $\mathbf{G}_3$ in (12) by replacing $\mathbf{V}^{[k]}$ with $\mathbf{U}^{[k]}$, and $\mathbf{H}^{[Rk]}$ with $\mathbf{H}^{[kR]^T}$ due to duality. Recall that User 3 desires the 4 symbols $\mathbf{u}^{[6]}$. In order for User 3 to decode $\mathbf{u}^{[6]}$, we design the receiver beamforming matrices $\mathbf{V}^{R3}=\mathbf{V}^{R6}$ as a $18\times4$ (randomly picked) full rank matrix which lies in the null space of $\mathbf{G'}_3$, i.e., $\mathbf{G'}_3^T\mathbf{V}^{R3}=\mathbf{O}$. Thus, the 4 mixing symbols $\mathbf{u}^{R3}$ of pair 3 are zero-forced at the four users of pair 1 and pair 2. With the similar approach, we design the beamforming matrices for pair 1 at the relay so that the 4 mixing symbols $\mathbf{u}^{R1}$ are zero-forced at the four users of pair 2 and pair 3, and the beamforming matrices of pair 2 so that $\mathbf{u}^{R2}$ are zero-forced at the four users of pair 1 and pair 3, respectively. Therefore, with its receiver beamforming matrix, User 3 observes the signal vector

$$\begin{aligned}\mathbf{y}^{[3]'} &\triangleq \mathbf{U}^{[3]^T}\mathbf{y}^{[3]} = \mathbf{U}^{[3]^T}(\mathbf{H}^{[3R]}\mathbf{x}^{[R]}+\mathbf{z}^{[3]}) && (19)\\ &= \mathbf{U}^{[3]^T}\mathbf{H}^{[3R]}\eta\textstyle\sum_{i=1}^{3}\mathbf{V}^{Ri}\mathbf{u}^{Ri}+\mathbf{U}^{[3]^T}\mathbf{z}^{[3]} && (20)\\ &= \gamma\mathbf{U}^{[3]^T}\mathbf{H}^{[3R]}\mathbf{V}^{R3}\mathbf{u}^{R3}+\mathbf{U}^{[3]^T}\mathbf{z}^{[3]} && (21)\\ &= \underbrace{\eta\mathbf{U}^{[3]^T}\mathbf{H}^{[3R]}\mathbf{V}^{R3}(\mathbf{G}_3^c)^T\mathbf{H}^{[R3]}\mathbf{V}^{[3]}\mathbf{u}^{[3]}}_{\text{self-interference}}\\ &\quad+\underbrace{\eta\mathbf{U}^{[3]^T}\mathbf{H}^{[3R]}\mathbf{V}^{R3}(\mathbf{G}_3^c)^T\mathbf{H}^{[R6]}\mathbf{V}^{[6]}\mathbf{u}^{[6]}}_{\text{desired signal}}\\ &\quad+\underbrace{\eta\mathbf{U}^{[3]^T}\mathbf{H}^{[3R]}\mathbf{V}^{R3}(\mathbf{G}_3^c)^T\mathbf{z}^{[R]})+\mathbf{U}^{[3]^T}\mathbf{z}^{[3]}}_{\triangleq\,\mathbf{z}^{[3]'}:\text{ noise}}. && (22)\end{aligned}$$

Since User 3 knows its own signals, it can subtract the self-interference from $\mathbf{y}^{[3]'}$ to obtain an effective AWGN MIMO channel from User 6, and the equivalent MIMO channel output is given by:

$$\mathbf{y}^{[3]''} = \eta\mathbf{U}^{[3]^T}\mathbf{H}^{[3R]}\mathbf{V}^{R3}(\mathbf{G}_3^c)^T\mathbf{H}^{[R6]}\mathbf{V}^{[6]}\mathbf{u}^{[6]}+\mathbf{z}^{[3]'} \quad (23)$$

where $\mathbf{U}^{[3]^T}\mathbf{H}^{[3R]}\mathbf{V}^{R3}(\mathbf{G}_3^c)^T\mathbf{H}^{[R6]}\mathbf{V}^{[6]}$ is the equivalent $4\times 4$ full rank MIMO channel matrix. Since the vector $\mathbf{V}^{R3}$ characterizes the power constraint, i.e., $||\mathbf{V}^{R3}||^2 = O(P)$, we have $||\eta\mathbf{V}^{R3}||^2 = O(1)$, which implies that the cumulative noise $\mathbf{z}^{[3]'}$ is transmission power independent. Therefore, User 3 can decode its desired 4 symbols from User 6 to achieve 4 DoF. Similarly, symmetric argument can be carried out for other users to show each user achieves 4 DoF.

*Remark:* Once we finish the transmission design for the first phase, the scheme for the second phase can be automatically designed owing to duality. That is, from the DoF perspective, we only need to ensure the separability of the $d$ mixing symbols for each pair from interference at the relay caused by the other pairs.

### B. Example: $(M, N) = (3, 8) \Rightarrow d = 2$

We study the case of $(M, N) = (3, 8)$ as another example, to show that each user achieves $d = 2$ DoF. With the remark message at the send of Section IV-A, and due to the space limitation as well, we only show the signal transmission and reception in the first phase.

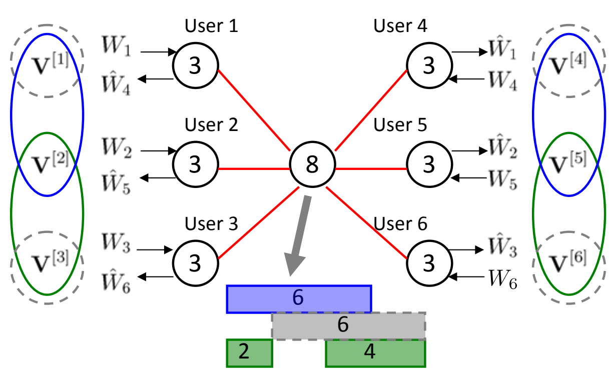


Fig. 4. DoF Achievability for $(K, M, N) = (3, 3, 8)$

In the first phase, User $k$ encodes its two symbols using a $3 \times 2$ beamforming matrix $\mathbf{V}^{[k]}$. Then the transmitted signal vector of User $k$ is given by:

$$\mathbf{x}^{[k]} = \mathbf{V}^{[k]}\mathbf{u}^{[k]} \tag{24}$$

where $\mathbf{u}^{[k]}$ is a $2 \times 1$ vector representing 2 independent symbols.

Now consider the signals at the relay. Again, we first consider pair 3. Since the relay has 8 antennas, it has a 8-dimensional space. In order to protect a 2-dimensional subspace for pair 3, the $4d = 8$ symbols of pair 1 and pair 2 needs to be accommodated into the remaining 6-dimensional subspace. Thus, we align the two symbols $\mathbf{u}^{[5]}$ in the subspace spanned by $\mathbf{u}^{[1]}, \mathbf{u}^{[2]}, \mathbf{u}^{[4]}$ (shown in the blue circle in Fig. 4). This can be done since User 1, User 2, User 4 and User 5, each with 3 antennas, will jointly have a 4-dimensional subspace in common at the relay, i.e., the number of common dimensions is $3 \times 4 - 8 = 4$. Therefore, we have the first alignment equation:

$$\mathbf{H}^{[R1]}\mathbf{V}^{[1]} + \mathbf{H}^{[R2]}\mathbf{V}^{[2]} + \mathbf{H}^{[R4]}\mathbf{V}^{[4]} = -\mathbf{H}^{[R5]}\mathbf{V}^{[5]}. \tag{25}$$

With the similar argument, when we reserve a 2-dimensional subspace for pair 1 at the relay, we align two symbols $\mathbf{u}^{[6]}$ in the subspace spanned by $\mathbf{u}^{[2]}, \mathbf{u}^{[3]}, \mathbf{u}^{[5]}$ (shown in the green circle) at the relay, to produce the second alignment equation:

$$\mathbf{H}^{[R2]}\mathbf{V}^{[2]} + \mathbf{H}^{[R3]}\mathbf{V}^{[3]} + \mathbf{H}^{[R5]}\mathbf{V}^{[5]} = -\mathbf{H}^{[R6]}\mathbf{V}^{[6]}. \tag{26}$$

Subtracting (26) from (25), we obtain

$$\mathbf{H}^{[R1]}\mathbf{V}^{[1]} - \mathbf{H}^{[R3]}\mathbf{V}^{[3]} + \mathbf{H}^{[R4]}\mathbf{V}^{[4]} = \mathbf{H}^{[R6]}\mathbf{V}^{[6]}, \tag{27}$$

which implies that the two symbols $\mathbf{u}^{[6]}$ are in the subspace spanned by $\mathbf{u}^{[1]}, \mathbf{u}^{[3]}, \mathbf{u}^{[4]}$ (shown in the gray circle). That is, when we reserve a 2-dimensional subspace for pair 2 at the relay, the requirement of aligning 2 interfering symbols is already satisfied. In fact, this is quite interesting since once we have the first two alignment equations (25) and (26), the third alignment constraint (27) is automatically satisfied.

Finally, rewriting the alignment equations (25) and (26) in a compact form, we obtain:

$$\underbrace{\begin{bmatrix} \mathbf{H}^{[R1]} & \mathbf{H}^{[R2]} & \mathbf{O} & \mathbf{H}^{[R4]} & \mathbf{H}^{[R5]} & \mathbf{O} \\ \mathbf{O} & \mathbf{H}^{[R2]} & \mathbf{H}^{[R3]} & \mathbf{O} & \mathbf{H}^{[R5]} & \mathbf{H}^{[R6]} \end{bmatrix}}_{\triangleq \mathbf{H}^{[R\cdot]}:\ 16\times 18} \begin{bmatrix} \mathbf{V}^{[1]} \\ \vdots \\ \mathbf{V}^{[6]} \end{bmatrix} = \mathbf{O}. \tag{28}$$

Notice that $\mathbf{H}^{[R\cdot]}$ is a $16 \times 18$, and it can be proved to have full rank almost surely. Thus, the beamforming matrices of each user can be determined in the null space of $\mathbf{H}^{[R\cdot]}$ as in (28), and each of them has full rank to ensure the linear independencies among the 2 beamforming vectors carrying its 2 symbols.

So far, all signals from arbitrary two pairs span a 6-dimensional subspace at the relay. What remains to be shown is for every pair, the 6-dimensional interfering subspace and the 2-dimensional subspace projected by each user of the remaining pair span the entire 8-dimensional signal space at the relay, to ensure the 2-dimensional desired subspace for the remaining pair is reserved. Due to symmetry of the design, it suffices to show that the 6-dimensional interfering subspace projected by User 1, User 2, User 4 and User 5, and the 2-dimensional desired subspace projected from User 3 has only null intersection. That is, we need to guarantee that the following $8 \times 8$ square matrix has full rank:

$$\begin{bmatrix} \mathbf{H}^{[R1]}\mathbf{V}^{[1]} & \mathbf{H}^{[R2]}\mathbf{V}^{[2]} & \mathbf{H}^{[R3]}\mathbf{V}^{[3]} & \mathbf{H}^{[R4]}\mathbf{V}^{[4]} \end{bmatrix}. \tag{29}$$

Again, this can be verified through a simple numerical test, and we defer the rigorous proof to [1].

This completes the design in the first phase.

In the second phase, the beamforming approach is dual to that in the first phase. As similar as in the $(M, N) = (5, 18)$ setting, after separating the mixing symbols for each pair at the relay in the first phase, and then after post-precessing and subtracting its own signal in the second phase, each user only sees an equivalent AWGN $3\times 3$ full rank MIMO channel from the paired user without inter-pair interference. Thus, each user is able to achieve 2 DoF.

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