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DESCENT FOR n-BUNDLES

JESSE WOLFSON

Abstract. Given a Lie group G, one constructs a principal G-bundle on a manifold X by taking a cover $U \to X$, specifying a transition cocycle on the cover, and descending the trivialized bundle $U \times G$ along the cover. We demonstrate the existence of an analogous construction for local n-bundles for general n. We establish analogues for simplicial Lie groups of Moore's results on simplicial groups; these imply that bundles for strict Lie n-groups arise from local n-bundles. Our construction leads to simple finite dimensional models of Lie 2-groups such as String(n) from cocycle data.

1. Introduction

The nerve of a group is a simplicial set satisfying Kan's horn-filling conditions. Grothendieck observed that the nerve provides an equivalence between the category of groups and the category of reduced Kan simplicial sets whose horns have unique fillers above dimension one. More generally, he showed that the nerve extends to an equivalence between the category of groupoids and the category of Kan simplicial sets whose horns have unique fillers above dimension one. Inspired by this, Duskin [6] defined an n-groupoid to be a Kan simplicial set whose horns have unique fillers above dimension n.

In the last decade, Henriques [12], Pridham [18] and others have begun the study of Lie n-groupoids: simplicial manifolds whose horn-filling maps are surjective submersions in all dimensions, and isomorphisms above dimension n. Examples are common. (See also Getzler [10].) Lie 0-groupoids are precisely smooth manifolds. Lie 1-groups are nerves of Lie groups. Abelian Lie n-groups are equivalent to chain complexes of abelian Lie groups supported between degrees 0 and $n-1$.

Simplicial Lie groups whose underlying simplicial set is an $(n-1)$-groupoid give rise to Lie n-groups, by the $\overline{W}$-construction (Section 6). We call this special class of Lie n-groups *strict Lie n-groups*.

Much of the theory of principal bundles for Lie groups generalizes naturally to principal bundles for strict Lie n-groups. In Theorems 5.7 and 6.7, we show that the construction of a fiber bundle from a cocycle on a cover has a close analogue for cocycles for strict Lie n-groups. As an application, we show how this allows for the construction of finite dimensional Lie 2-groups, such as String(n), from cohomological data. The method works equally well for $n > 2$.

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Outline. We develop our results within a category C which has a terminal object $*$ and a subcategory of covers. We require the subcategory of covers to be stable under pullback, to contain the maps $X \longrightarrow *$ for every object X, to satisfy an axiom of right-cancelation, and to be contained within the class of effective epimorphisms. Our motivating example is the category of finite dimensional smooth manifolds, with surjective submersions as covers. Other examples include Banach manifolds, analytic manifolds over a complete normed field, and of course sets.

In Section 2, we recall the definitions and basic properties of higher stacks. This section parallels the discussion in Behrend and Getzler [2], the main difference being that in that paper, the category C is assumed to possess finite limits. Here, we work with categories of manifolds, so this assumption does not hold. (Conversely, they do not impose the assumption that the maps $X \longrightarrow *$ are covers, which fails in the setting of not-necessarily smooth analytic spaces.)

In Section 3, we show that the collection of k-morphisms in a Lie n-groupoid forms a Lie $(n-k)$-groupoid (Theorem 3.6). In Section 4, we apply this to study Duskin's n-strictification functor τ_n. (For a discussion in the absolute case, see [11, Section 3.1, Example 5], [12, Definition 3.5] or [10, Section 2].) In the Lie setting, the functor τ_n does not always exist. However, when it does, it provides a partial left adjoint to the inclusion of n-stacks into the category of ∞-stacks. We recall the relevant properties and give a necessary and sufficient criterion for existence. In Section 5, we impose the additional axiom that quotients of regular equivalence relations in C exist. (This was first established by Godemont for smooth manifolds, or analytic manifolds over a complete normed field.) Under this assumption, we introduce local n-bundles and prove our main result on descent (Theorem 5.7).

In Section 6, we extend results of Moore on simplicial groups in Set to simplicial Lie groups. These results provide a ready supply of examples satisfying the hypotheses of Theorem 5.7. We conclude in Section 7, by applying our results to construct finite dimensional Lie 2-groups from cocycle data. We describe the resulting model of String(n) and compare it to the model constructed by Schommer-Pries [19].

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2. Higher Stacks

We work in a category C with a subcategory of "covers".

Axiom 1. The category C has a terminal object $*$, and the map $X \longrightarrow *$ is a cover for every object $X \in \mathsf{C}$.

Axiom 2. Pullbacks of covers along arbitrary maps exist and are covers.

Axiom 3. If g and f are composable maps such that fg and g are covers, then f is also a cover.

If it exists, the *kernel pair* of a map $f : X \longrightarrow Y$ in a category C is the pair of parallel arrows

$$X \times_Y X \rightrightarrows X$$

The map $f : X \longrightarrow Y$ is an *effective epimorphism* if f is the coequalizer of this pair:

$$X \times_Y X \rightrightarrows X \xrightarrow{f} Y$$

Axiom 4. Covers are effective epimorphisms.

Axioms 1 and 2 ensure that isomorphisms are covers, that C has finite products, that projections along factors of products are covers, and that covers form a pretopology on C. Axiom 3, which we borrow from Behrend and Getzler [2], ensures that being a cover is a local property: it is preserved *and* reflected under pullback along covers. Likewise, Axiom 4 ensures that being an isomorphism is a local property.

We will be interested in the following examples of categories with covers satisfying these assumptions:

(1) Set, the category of sets, with surjections as covers;
(2) Smooth, the category of finite-dimensional smooth manifolds, with surjective submersions as covers;
(3) the category of Banach manifolds, with surjective submersions as covers (see [12]);
(4) the category of analytic manifolds over a complete normed field, with surjective submersions as covers (see [20, Chapter III]).

Denote by sC the category of simplicial objects in C. In particular, we have the category of simplicial sets sSet. The category C embeds fully faithfully in sC as the category of constant simplicial diagrams. We do not distinguish between the category C and its essential image under this embedding.

Let Δ^k denote the standard k-simplex, that is, the simplicial set $\Delta^k = \Delta(-, [k])$. A simplicial set S is the colimit of its simplices

$$\operatorname*{colim}_{\Delta^k \longrightarrow S} \Delta^k \cong S$$

Definition 2.1. Let $X_\bullet$ be a simplicial object in C. Let S be a simplicial set. Denote by $\hom(S, X)$ the limit

$$\hom(S, X) := \lim_{\Delta^k \longrightarrow S} X_k$$

Note that such limits do not exist in general.

By a Lie group, we will mean a group internal to the category C, that is, an object G with product $m : G \times G \longrightarrow G$, inverse $i : G \longrightarrow G$, and identity $e : * \longrightarrow G$, satisfying the usual axioms. We may associate to a Lie group its *nerve* $N_\bullet G \in \mathsf{sC}$, which is the simplicial object

$$N_k G = G^k.$$

The face maps $d_i : G^k \longrightarrow G^{k-1}$ are defined for $i = 0$ and $i = k$ by projection away from the first and last factor respectively, and for $0 < i < k$ by

$$d_i = G^{i-1} \times m \times G^{k-1}.$$

The degeneracy maps $s_i : G^k \longrightarrow G^{k+1}$ are defined by

$$s_i = G^i \times e \times G^{k-i}.$$

In fact, the above construction does not use the existence of an inverse, and works if G is only a monoid in C.

We will use the following simplicial subsets of Δ^k:

(1) the boundary $\partial\Delta^k$ of Δ^k;
(2) the i^{th} horn $\Lambda^k_i \subset \partial\Delta^k$, obtained from $\partial\Delta^k$ by omitting its i^{th} face.

Definition 2.2. Let $f : X \longrightarrow Y$ be a map in sC. The *matching object* $M_k(f)$ is the limit

$$\hom(\partial\Delta^k, X) \times_{\hom(\partial\Delta^k,Y)} Y_k.$$

Denote by $\mu_k(f)$ the induced map from X_k to $M_k(f)$.

The object of *relative* Λ^k_i*-horns* $\Lambda^k_i(f)$ is the limit

$$\hom(\Lambda^k_i, X) \times_{\hom(\Lambda^k_i,Y)} Y_k.$$

Denote by $\lambda^k_i(f)$ the induced map from X_k to $\Lambda^k_i(f)$.

A section of $M_k(f)$ is a k-simplex of Y together with a lift of its boundary to X, and $\mu_k(f)$ measures the extent to which these relative spheres are filled by k-simplices of X. Similarly, a section of $\Lambda^k_i(f)$ is a k-simplex of Y together with a lift of the Λ^k_i-horn to X, and $\lambda^k_i(f)$ measures the extent to which these relative horns are filled by k-simplices of X.

In the absolute case, where the target Y of the simplicial map f is the terminal object $*$, we write $M_k(X)$ and $\Lambda^k_i(X)$ instead of $M_k(f)$ and $\Lambda^k_i(f)$, and similarly for the induced maps $\mu_k(X)$ and $\lambda^k_i(X)$.

As an example, we have

$$\Lambda^k_i(N_\bullet G) \cong \begin{cases} *, & k = 0, 1, \\ G^k, & k > 1. \end{cases}$$

This is easily seen if $0 < i < k$; for $i = 0$ or $i = k$, the proof requires the existence of the inverse for G. In fact, the isomorphisms $\Lambda^k_i(N_\bullet G) \cong N_k G$, $k > 1$, together with the condition $N_0 G \cong *$, characterize the nerves of groups, and indeed give an alternative axiomatization of the theory of groups.

Grothendieck extended this observation, omitting the condition $N_0 G \cong *$.

Definition 2.3. A Lie groupoid $\mathcal{G}$ in C is an internal groupoid in C, with morphisms $\mathcal{G}_1$ and objects $\mathcal{G}_0$, source and target maps $s, t : \mathcal{G}_1 \longrightarrow \mathcal{G}_0$, multiplication

$$m\colon \mathcal{G}_1 \times^{t,s}_{\mathcal{G}_0} \mathcal{G}_1 \longrightarrow \mathcal{G}_1,$$

unit $e : \mathcal{G}_0 \longrightarrow \mathcal{G}_1$, and inverse $i : \mathcal{G}_1 \longrightarrow \mathcal{G}_1$, such that s and t are covers.

The *nerve* $N_\bullet\mathcal{G}$ of a groupoid is the simplicial object $N_\bullet\mathcal{G} \in \mathsf{sC}$,

$$N_k\mathcal{G} = \begin{cases} \mathcal{G}_0, & k = 0, \\ \mathcal{G}_1, & k = 1, \\ \underbrace{\mathcal{G}_1 \times^{t,s}_{\mathcal{G}_0} \cdots \times^{t,s}_{\mathcal{G}_0} \mathcal{G}_1}_{k}, & k > 1. \end{cases}$$

On 1-simplices, the face maps d_0 and d_1 correspond to the target t and source s. The degeneracy $s_0\colon \mathcal{G}_0 \longrightarrow \mathcal{G}_1$ corresponds to the unit. On k-simplices for $k > 1$, the face maps

$$d_i : \underbrace{\mathcal{G}_1 \times^{t,s}_{\mathcal{G}_0} \cdots \times^{t,s}_{\mathcal{G}_0} \mathcal{G}_1}_{k} \longrightarrow \underbrace{\mathcal{G}_1 \times^{t,s}_{\mathcal{G}_0} \cdots \times^{t,s}_{\mathcal{G}_0} \mathcal{G}_1}_{k-1}$$

are defined for $i = 0$ and $i = k$ by projection away from the first and last factor respectively, and for $0 < i < k$ by

$$d_i = \underbrace{\mathcal{G}_1 \times^{t,s}_{\mathcal{G}_0} \cdots \times^{t,s}_{\mathcal{G}_0} \mathcal{G}_1}_{i-1} \times m \times \underbrace{\mathcal{G}_1 \times^{t,s}_{\mathcal{G}_0} \cdots \times^{t,s}_{\mathcal{G}_0} \mathcal{G}_1}_{k-i}.$$

The degeneracy maps

$$s_i : \underbrace{\mathcal{G}_1 \times^{t,s}_{\mathcal{G}_0} \cdots \times^{t,s}_{\mathcal{G}_0} \mathcal{G}_1}_{k} \longrightarrow \underbrace{\mathcal{G}_1 \times^{t,s}_{\mathcal{G}_0} \cdots \times^{t,s}_{\mathcal{G}_0} \mathcal{G}_1}_{k+1}$$

are defined by

$$s_i = \underbrace{\mathcal{G}_1 \times^{t,s}_{\mathcal{G}_0} \cdots \times^{t,s}_{\mathcal{G}_0} \mathcal{G}_1}_{i-1} \times e \times \underbrace{\mathcal{G}_1 \times^{t,s}_{\mathcal{G}_0} \cdots \times^{t,s}_{\mathcal{G}_0} \mathcal{G}_1}_{k-i}.$$

Grothendieck's observation, generalized to Lie groupoids from his setting of discrete groupoids, is as follows.

Proposition 2.4 (Grothendieck)**.** *A simplicial object $X_\bullet \in \mathsf{sC}$ is isomorphic to the nerve of a Lie groupoid if and only if the horn-filler maps*

$$\lambda^k_i(X) : X_k \longrightarrow \Lambda^k_i(X)$$

are covers for $k = 1$, and isomorphisms for $k > 1$.

In particular, a simplicial object $X_\bullet \in \mathsf{sC}$ is isomorphic to the nerve of a Lie group if and only if the above conditions are fulfilled and $X_0 = *$.

Maps between Lie groupoids are in bijection with simplicial maps between their nerves. As a result, the full subcategory of sC consisting of those simplicial objects satisfying the above conditions is equivalent to the category of Lie groupoids.

Motivated by Grothendieck's observation, Duskin [6] introduced a notion of n-groupoid valid in any topos. Duskin's notion was adapted by Henriques [12] (see also [10]) to cover higher Lie groupoids.

Definition 2.5. Let $n \in \mathbb{N} \cup \{\infty\}$. A *Lie n-groupoid* is a simplicial object $X_\bullet \in \mathsf{sC}$ such that for all $k > 0$ and $0 \le i \le k$, the limit $\Lambda^k_i(X)$ exists in C, the map

$$\lambda^k_i(X) \colon X_k \longrightarrow \Lambda^k_i(X)$$

is a cover, and it is an isomorphism for $k > n$.

A *Lie n-group* $X_\bullet$ is a Lie n-groupoid such that $X_0 = *$.

As an example, a Lie 0-groupoid is the same as an object of C (viewed as a constant simplicial diagram).

Definition 2.6 (Verdier)**.** Let $n \in \mathbb{N} \cup \{\infty\}$. A map $f : X_\bullet \longrightarrow Y_\bullet$ of Lie ∞-groupoids is an *n-hypercover* if, for all $k \ge 0$, the limit $M_k(f)$ exists in C, the map

$$\mu_k(f) \colon X_k \longrightarrow M_k(f)$$

is a cover for all k, and it is an isomorphism for $k \ge n$.

Hypercovers in sSet are the same as trivial fibrations, that is, Kan fibrations which are also weak homotopy equivalences. Hypercovers play much the same role in the theory of Lie n-groupoids. We refer to an ∞-hypercover simply as a "hypercover."

A 0-hypercover is an isomorphism, while a 1-hypercover of a Lie 0-groupoid is isomorphic to the nerve of the cover $f_0 : X_0 \longrightarrow Y_0$. In other words,

$$X_k \cong \underbrace{X_0 \times_{Y_0} \times \cdots \times_{Y_0} X_0}_{k+1}$$

Definition 2.7. An *augmentation* of a simplicial object $X_\bullet \in \mathsf{sC}$ is a simplicial map to an object $Y \in \mathsf{C} \subset \mathsf{sC}$.

This amounts to the same thing as a map $\varepsilon : X_0 \longrightarrow Y$ that renders the diagram

$$X_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} X_0 \xrightarrow{\varepsilon} Y$$

commutative.

Definition 2.8. The *orbit space* $\pi_0(X)$ of a Lie ∞-groupoid $X_\bullet$ is a cover

$$X_0 \longrightarrow \pi_0(X)$$

which coequalizes the fork

$$X_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} X_0 \longrightarrow \pi_0(X)$$

In other words, for any augmentation $\varepsilon : X_0 \longrightarrow Y$ of $X_\bullet$, there is an induced map

$$\begin{array}{ccccc} X_1 & \underset{d_1}{\overset{d_0}{\rightrightarrows}} & X_0 & \longrightarrow & \pi_0(X) \\ & & & \searrow & \vdots \\ & & & & Y \end{array}$$

Furthermore, if ε is a cover, then so is the induced map from $\pi_0(X)$ to Y, by Axiom 3.

It is characteristic of the theory of Lie ∞-groupoids that the orbit space $\pi_0(X)$ need not exist. One case in which the orbit space of $X_\bullet$ exists, however, is when $X_\bullet$ admits an augmentation $\varepsilon : X_\bullet \longrightarrow Y$ which is a hypercover.

Proposition 2.9.

(1) *An augmentation $\varepsilon : X_\bullet \longrightarrow Y$ is an n-hypercover if and only if the maps $\varepsilon : X_0 \longrightarrow Y$ and $\mu_1(\varepsilon) : X_1 \longrightarrow X_0 \times_Y X_0$ are covers, and the maps*

$$\mu_k(X)\colon X_k \longrightarrow M_k(X)$$

are covers for $k > 1$ and isomorphisms for $k \geq n$.

(2) *If the augmentation $\varepsilon : X_\bullet \longrightarrow Y$ is a hypercover, then $\pi_0(X) \cong Y$.*

Proof. If $\varepsilon : X_\bullet \longrightarrow Y$ is an augmentation, we have

$$M_k(\varepsilon) = \begin{cases} Y, & k = 0, \\ X_0 \times_Y X_0, & k = 1, \\ M_k(X), & k > 1. \end{cases}$$

The first part follows by inspection.

The second part is a restatement of Axiom 4. $\square$

Let $\Delta_{\leq n} \subseteq \Delta$ be the full subcategory with objects $\{[m] \mid m \leq n\}$. An *n-truncated* simplicial object is a functor

$$X_{\leq n} : \Delta^{\circ}_{\leq n} \longrightarrow \mathsf{C}.$$

Denote by $\mathsf{s}_{\leq n}\mathsf{C}$ the category of n-truncated simplicial objects in C. Restriction along $\Delta_{\leq n} \hookrightarrow \Delta$ induces the functor of *n-truncation*:

$$\mathrm{tr}_n : \mathsf{sC} \longrightarrow \mathsf{s}_{\leq n}\mathsf{C}.$$

When S is a simplicial set of dimension less than or equal to n, we abuse notation and write S for $\mathrm{tr}_n S$.

When C has finite limits, n-truncation $\mathrm{tr}_n : \mathsf{sC} \longrightarrow \mathsf{s}_{\leq n}\mathsf{C}$ admits a right-adjoint $\mathrm{cosk}_n : \mathsf{s}_{\leq n}\mathsf{C} \longrightarrow \mathsf{sC}$, called the n-coskeleton. The composition

$$\mathrm{cosk}_n \circ \mathrm{tr}_n : \mathsf{sC} \longrightarrow \mathsf{sC}$$

is denoted Cosk_n. When C does not possess finite limits, the functor Cosk_n is only partially defined.

In the category of simplicial sets, there is also a left-adjoint $\mathrm{sk}_n : \mathsf{s}_{\leq n}\mathsf{Set} \longrightarrow \mathsf{sSet}$ to n-truncation, called the n-skeleton. The composition

$$\mathrm{sk}_n \circ \mathrm{tr}_n : \mathsf{sSet} \longrightarrow \mathsf{sSet}$$

is denoted Sk_n.

Special cases of the next two lemmas first appeared in [5]. Let $S \hookrightarrow T$ be a monomorphism of finite simplicial sets.

Lemma 2.10. *Suppose that S is n-dimensional. Let $Y_\bullet \in \mathsf{sC}$ be a simplicial object such that the limit $\hom(T, Y)$ exists. Let $f : \mathrm{tr}_n X_\bullet \longrightarrow \mathrm{tr}_n Y_\bullet$ be a map in $\mathsf{s}_{\leq n}\mathsf{C}$ such that the matching object $M_k(f)$ exists for all $k \leq n$, and the map*

$$\mu_k(f) : X_k \longrightarrow M_k(f)$$

is a cover for all $k \leq n$. Then the limit

$$\hom(S, X) \times_{\hom(S,Y)} \hom(T, Y)$$

exists.

Proof. Filter the simplicial set S

$$\emptyset = S_0 \hookrightarrow \ldots \hookrightarrow S_N = S$$

where

$$S_\ell \cong S_{\ell-1} \cup_{\partial\Delta^{n_\ell}} \Delta^{n_\ell}.$$

Here, $n_\ell \leq n$ for all ℓ.

Suppose that the limit

$$Z_j = \hom(S_j, X) \times_{\hom(S_j,Y)} \hom(T, Y).$$

exists for $j < \ell$. This is true for $\ell = 1$, since $Z_0 \cong \hom(T, Y)$. The limit Z_ℓ is the pullback

$$\begin{array}{ccc} Z_\ell & \longrightarrow & X_{n_\ell} \\ \big\downarrow & & \big\downarrow{\scriptstyle \mu_{n_\ell}(f)} \\ Z_{\ell-1} & \longrightarrow & M_{n_\ell}(f) \end{array}$$

This pullback exists because $\mu_{n_\ell}(f)$ is a cover. $\square$

Lemma 2.11. *Let* $f : X_\bullet \longrightarrow Y_\bullet$ *be a hypercover such that the limit*

$$\hom(S, X) \times_{\hom(S,Y)} \hom(T, Y)$$

exists. Then the limit $\hom(T, X)$ *exists and the map*

$$\hom(T, X) \longrightarrow \hom(S, X) \times_{\hom(S,Y)} \hom(T, Y)$$

is a cover.

Proof. Filter the simplicial set T

$$S = S_0 \hookrightarrow \ldots \hookrightarrow S_N = T$$

where

$$S_\ell \cong S_{\ell-1} \cup_{\partial\Delta^{n_\ell}} \Delta^{n_\ell}.$$

Suppose that the limit

$$Z_j = \hom(S_j, X) \times_{\hom(S_j,Y)} \hom(T, Y).$$

exists for $j < \ell$, and that the map $Z_j \longrightarrow Z_0$ is a cover. The limit

$$Z_0 = \hom(S, X) \times_{\hom(S,Y)} \hom(T, Y)$$

exists by hypothesis. The limit Z_ℓ is the pullback

$$\begin{array}{ccc} Z_\ell & \longrightarrow & X_{n_\ell} \\ \big\downarrow & & \big\downarrow {\scriptstyle \mu_{n_\ell}(f)} \\ Z_{\ell-1} & \longrightarrow & M_{n_\ell}(f) \end{array}$$

This pullback exists because $\mu_{n_\ell}(f)$ is a cover.

We conclude that the limit $Z_N = \hom(T, X)$ exists, and that the morphism $Z_N \longrightarrow Z_0$ is a cover. □

Theorem 2.12.

(1) *The composition of two* n*-hypercovers is an* n*-hypercover.*
(2) *The pullback of an* n*-hypercover along a map of Lie* ∞*-groupoids exists in* sC*, and is an* n*-hypercover.*

Proof. Consider a composable pair of n-hypercovers

$$X_\bullet \xrightarrow{g} Y_\bullet \xrightarrow{f} Z_\bullet$$

Suppose that the matching object $M_j(fg)$ exists and that the map $\mu_j(fg)$ is a cover for $j < k$. This is certainly the case for $k = 1$, since $M_0(fg) \cong Z_0$, and $\mu_0(fg) = f_0 g_0$ is the composition of the two covers f_0 and g_0. Lemma 2.10 now shows that the matching object $M_k(fg)$ exists. The square in the commuting diagram

$$\begin{array}{ccccc} X_k & \xrightarrow{\mu_k(g)} & M_k(g) & \longrightarrow & Y_k \\ & {\scriptstyle \mu_k(fg)} \searrow & \big\downarrow & & \big\downarrow {\scriptstyle \mu_k(f)} \\ & & M_k(fg) & \longrightarrow & M_k(f) \end{array}$$

is a pullback. Since f and g are n-hypercovers, we see that $\mu_k(fg)$ is a (composition of) cover(s) for all k, and an isomorphism if $k > n$.

We turn to the second statement. Consider an n-hypercover $f : X_\bullet \longrightarrow Z_\bullet$ and a map $g : Y_\bullet \longrightarrow Z_\bullet$ of Lie ∞-groupoids. Suppose that the limits g^*X_j and $M_j(g^*f)$ exist and that the maps $\mu_j(g^*f)$ are covers for $j < k$. Lemma 2.10 shows that the matching object $M_k(g^*f)$ exists. The limit g^*X_k is the pullback

$$\begin{array}{ccc} g^*X_k & \longrightarrow & X_k \\ {\scriptstyle \mu_k(g^*f)}\downarrow & & \downarrow{\scriptstyle \mu_k(f)} \\ M_k(g^*f) & \longrightarrow & M_k(f) \end{array}$$

The map $\mu_k(f)$ is a cover for all k because f is an n-hypercover. This shows that the pullback g^*X_k exists, that the map $\mu_k(g^*f)$ is a cover for all k, and that it is an isomorphism for $k \geq n$. □

There is also a relative version of the notion of a Lie n-groupoid, modeled on the definition of a Kan fibration in the theory of simplicial sets.

Definition 2.13. Let $n \in \mathbb{N} \cup \{\infty\}$. A map $f : X_\bullet \longrightarrow Y_\bullet$ of Lie ∞-groupoids is an *n-stack* if for all $k > 0$ and $0 \leq i \leq k$, the limit $\Lambda^k_i(f)$ exists, the map

$$\lambda^k_i(f) : X_k \longrightarrow \Lambda^k_i(f)$$

is a cover, and it is an isomorphism if $k > n$.

There are analogues of Lemmas 2.10 and 2.11 for n-stacks, due to Henriques [12], but only under certain additional conditions on the simplicial sets S and T.

Definition 2.14. An inclusion of finite simplicial sets $S \hookrightarrow T$ is an *expansion* if it can be written as a composition

$$S = S_0 \hookrightarrow \cdots \hookrightarrow S_N = T$$

where

$$S_\ell \cong S_{\ell-1} \cup_{\Lambda^{n_\ell}_{i_\ell}} \Delta^{n_\ell}.$$

A finite simplicial set S is *collapsible* if the inclusion of some, and hence any, vertex is an expansion.

Let $S \hookrightarrow T$ be a monomorphism of finite simplicial sets.

Lemma 2.15. *Suppose that S is n-dimensional and collapsible. Let $Y_\bullet \in \mathsf{sC}$ be a simplicial object such that the limit* $\hom(T, Y)$ *exists and the restriction to any vertex*

$$\hom(T, Y) \longrightarrow Y_0$$

is a cover. Let $f \colon \operatorname{tr}_n X_\bullet \longrightarrow \operatorname{tr}_n Y_\bullet$ be a map in $\mathsf{s}_{\leq n}\mathsf{C}$ such that the limit $\Lambda^k_i(f)$ exists for all $0 < k \leq n$ and $0 \leq i \leq n$, and the map

$$\lambda^k_i(f) : X_k \longrightarrow \Lambda^k_i(f)$$

is a cover for all $0 < k \leq n$ and $0 \leq i \leq n$. Then the limit

$$\hom(S, X) \times_{\hom(S,Y)} \hom(T, Y)$$

exists.

Proof. Filter the simplicial set S

$$\Delta^0 = S_0 \hookrightarrow \ldots \hookrightarrow S_N = S$$

where

$$S_\ell \cong S_{\ell-1} \cup_{\Lambda^{n_\ell}_{i_\ell}} \Delta^{n_\ell}.$$

Here, $n_\ell \leq n$ for all ℓ.

Suppose that the limit

$$Z_j = \hom(S_j, X) \times_{\hom(S_j,Y)} \hom(T,Y).$$

exists for $j < \ell$. This is true for $\ell = 1$, by the hypotheses on $\hom(T,Y)$. We have the pullback diagram

$$\begin{array}{ccc} Z_\ell & \longrightarrow & X_{n_\ell} \\ \downarrow & & \downarrow{\scriptstyle \lambda^{n_\ell}_{i_\ell}(f)} \\ Z_{\ell-1} & \longrightarrow & \Lambda^{n_\ell}_{i_\ell}(f) \end{array}$$

This pullback exists because $\lambda^{n_\ell}_{i_\ell}(f)$ is a cover. □

Lemma 2.16. *Let $f : X_\bullet \longrightarrow Y_\bullet$ be an ∞-stack such that the limit*

$$\hom(S, X) \times_{\hom(S,Y)} \hom(T,Y)$$

exists. Suppose that the inclusion $S \hookrightarrow T$ is an expansion. Then the limit $\hom(T,X)$ exists, and the map

$$\hom(T,X) \longrightarrow \hom(S,X) \times_{\hom(S,Y)} \hom(T,Y)$$

is a cover.

Proof. Filter the simplicial set T

$$S = S_0 \hookrightarrow \ldots \hookrightarrow S_N = T$$

where

$$S_\ell \cong S_{\ell-1} \cup_{\Lambda^{n_\ell}_{i_\ell}} \Delta^{n_\ell}.$$

Suppose that the limit

$$Z_j = \hom(S_j, X) \times_{\hom(S_j,Y)} \hom(T,Y).$$

exists for $j < \ell$, and the map $Z_j \longrightarrow Z_0$ is a cover. The limit

$$Z_0 = \hom(S, X) \times_{\hom(S,Y)} \hom(T,Y)$$

exists by hypothesis. We have the pullback diagram

$$\begin{array}{ccc} Z_\ell & \longrightarrow & X_{n_\ell} \\ \downarrow & & \downarrow{\scriptstyle \lambda^{n_\ell}_{i_\ell}(f)} \\ Z_{\ell-1} & \longrightarrow & \Lambda^{n_\ell}_{i_\ell}(f) \end{array}$$

This pullback exists because $\lambda^{n_\ell}_{i_\ell}(f)$ is a cover.

We conclude that the limit $Z_N = \hom(T,X)$ exists, and that the morphism $Z_N \longrightarrow Z_0$ is a cover. □

Theorem 2.17.

(1) *An n-hypercover is an n-stack.*
(2) *A hypercover which is an n-stack is an n-hypercover.*
(3) *The composition of two n-stacks is an n-stack.*
(4) *Let $f\colon X_\bullet \longrightarrow Y_\bullet$ be an n-stack and let $g\colon Z_\bullet \longrightarrow Y_\bullet$ be a map of Lie ∞-groupoids. If the pullback of f_0 along g_0 exists in C, then the pullback of f along g exists in sC and this pullback is an n-stack.*

Proof. Let $f\colon X_\bullet \longrightarrow Y_\bullet$ be an ∞-hypercover. We see that f is an ∞-stack by considering the finite inclusions $\Lambda^k_i \hookrightarrow \Delta^k$ and applying Lemma 2.11. It remains to show that if f is an n-hypercover, then $\lambda^k_i(f)$ is an isomorphism when $k > n$ (and $0 \le i \le k$).

The square in the commuting diagram

$$\begin{array}{ccccc} X_k & \xrightarrow{\mu_k(f)} & M_k(f) & \longrightarrow & X_{k-1} \\ & \searrow^{\lambda^k_i(f)} & \downarrow & & \downarrow{\scriptstyle \mu_{k-1}(f)} \\ & & \Lambda^k_i(f) & \longrightarrow & M_{k-1}(f) \end{array} \tag{2.1}$$

is a pullback. If f is an n-hypercover and $k > n$, then the maps $\mu_k(f)$ and $\mu_{k-1}(f)$ are isomorphisms, and we see that $\lambda^k_i(f)$ is an isomorphism.

To prove the second part, we consider (2.1) in the case where f is a hypercover and an n-stack. If $k > n$, then $\lambda^k_i(f)$ is an isomorphism. The map $\mu_k(f)$ is a cover, and, by Axiom 4, an epimorphism. The diagram (2.1) now implies that $\mu_k(f)$ is an isomorphism. Similarly, the map $M_k(f) \longrightarrow \Lambda^k_i(f)$ is an isomorphism.

The map $\Lambda^k_i(f) \longrightarrow M_{k-1}(f)$ is induced by the inclusion $\partial\Delta^{k-1} \hookrightarrow \Lambda^k_i$. Lemma 2.11 shows that it is a cover. The pull-back of $\mu_{k-1}(f)$ along this cover is the map $M_k(f) \longrightarrow \Lambda^k_i(f)$. This map is an isomorphism for $k > n$. Axiom 4 therefore implies that $\mu_{k-1}(f)$ is an isomorphism for $k > n$. Hence f is an n-hypercover.

Turning to the third part of the theorem, consider a composable pair of n-stacks

$$X_\bullet \xrightarrow{\ g\ } Y_\bullet \xrightarrow{\ f\ } Z_\bullet$$

Suppose that the limit $\Lambda^j_i(fg)$ exists and that the map $\lambda^j_i(fg)$ is a cover, for all $0 < j < k$ (and $0 \le i \le j$). Lemma 2.16 now shows that the limit $\Lambda^k_i(fg)$ exists. The square in the commuting diagram

$$\begin{array}{ccccc} X_k & \xrightarrow{\lambda^k_i(g)} & \Lambda^k_i(g) & \longrightarrow & Y_k \\ & \searrow^{\lambda^k_i(fg)} & \downarrow & & \downarrow{\scriptstyle \lambda^k_i(f)} \\ & & \Lambda^k_i(fg) & \longrightarrow & \Lambda^k_i(f) \end{array}$$

is a pullback. Since f and g are n-stacks, we see that $\lambda^k_i(fg)$ is a (composition of) cover(s) for all $k > 0$ and $0 \le i \le k$, and an isomorphism if $k > n$.

We turn to the fourth statement. Consider an n-stack $f : X_\bullet \longrightarrow Z_\bullet$ and a map $g : Y_\bullet \longrightarrow Z_\bullet$ of Lie ∞-groupoids. Suppose that, for $j < k$ (and $0 \le i \le j$), the pullback g^*X_j exists, the limit $\Lambda^j_i(g^*f)$ exists, and the map $\lambda^j_i(g^*f)$ is a cover. Lemma 2.15 shows that the limit $\Lambda^k_i(g^*f)$ exists for $0 \le i \le k$. The limit g^*X_k is the pullback

$$\begin{array}{ccc} g^*X_k & \longrightarrow & X_k \\ \downarrow{\scriptstyle \lambda_i^k(g^*f)} & & \downarrow{\scriptstyle \lambda_i^k(f)} \\ \Lambda_i^k(g^*f) & \longrightarrow & \Lambda_i^k(f) \end{array}$$

The map $\lambda_i^k(f)$ is a cover for all $k > 0$ because f is an n-hypercover. This shows that the pull-back g^*X_k exists, that the map $\lambda_i^k(g^*f)$ is a cover for all $k > 0$ and $0 \leq i \leq k$, and that this map is an isomorphism for $k > n$. □

3. Higher Morphism Spaces in Higher Stacks

Order preserving maps of all finite ordinals, including 0, form a category Δ_+ extending Δ. An augmented simplicial set is a functor

$$\Delta_+^\circ \longrightarrow \mathsf{Set}$$

Such a functor consists of a simplicial set S equipped with a map to a constant simplicial set S_{-1}.

The ordinal sum

$$[n] + [m] := \{0 \leq \ldots \leq n \leq 0' \leq \ldots \leq m'\} = [n + m + 1]$$

endows the category Δ_+ with a monoidal structure. This structure extends along the Yoneda embedding

$$\Delta_+ \longrightarrow \mathsf{sSet}_+$$

to give a closed monoidal structure on sSet_+ called the *join*, and denoted $\star$. Given an augmented simplicial set K, denote the right adjoint to $K \star (-)$ by

$$\mathsf{sSet}_+ \xrightarrow{(-)^{K\star}} \mathsf{sSet}_+$$

Example 3.1. The functor $(-)^{\Delta^{n-1}\star}$ is Illusie's $\mathrm{Dec}_n(-)$ (c.f. [13, Chapter VI]).

The inclusion $i : \Delta \hookrightarrow \Delta_+$ provides a forgetful functor

$$\mathsf{sSet}_+ \xrightarrow{i^*} \mathsf{sSet}$$

Its right adjoint

$$\mathsf{sSet} \xrightarrow{i_*} \mathsf{sSet}_+$$

augments a simplicial set by a point.

Definition 3.2.

(1) Let S and T be simplicial sets. Denote by $S \star T$ the simplicial set

$$S \star T = i^*((i_*S) \star (i_*T)).$$

(2) Let $X_\bullet$ be in sC and let S be a finite simplicial set. Denote by $X_\bullet^{S\star}$ the putative simplicial object with k-simplices

$$X_k^{S\star} := \hom(S \star \Delta^k, X)$$

Face and degeneracy maps are given by $1 \star d_i$ and $1 \star s_i$.

In [7], Duskin gave a construction of the collection of morphisms in a higher category.[1]

Definition 3.3. Let $X_\bullet$ be an ∞-groupoid in Set. Define $P^{\geq 1}X_\bullet$ to be the pullback

$$\begin{array}{ccc} P^{\geq 1}X_\bullet & \longrightarrow & X_\bullet^{\Delta^0 \star} \\ \big\downarrow & & \big\downarrow \\ X_0 & \longrightarrow & X_\bullet \end{array}$$

The simplicial set $P^{\geq 1}X$ is an ∞-groupoid which models the "space" of 1-morphisms in $X_\bullet$.[2]

We will generalize $P^{\geq 1}X_\bullet$ to higher morphism spaces $P^{\geq k}X_\bullet$, for all $k > 0$, and at the same time, we will define a relative version of the construction. The discussion follows the lines of Joyal's proof of [15, Theorem 3.8], except that we restrict attention to the case

$$(S \hookrightarrow T) = (\partial\Delta^{k-1} \hookrightarrow \Delta^{k-1}),$$

and work in the Lie setting.

Given a proper, non-empty subset J of $[n]$, let

$$\Lambda^n_J = \bigcup_{i \in J} \partial_i \Delta^n \subset \partial\Delta^n.$$

An induction shows that Λ^n_J is collapsible.

Let $f : X_\bullet \longrightarrow Y_\bullet$ be an ∞-stack. The ℓ-simplices of the simplicial object

$$X_\bullet^{\partial\Delta^{k-1}\star} \times_{Y_\bullet^{\partial\Delta^{k-1}\star}} Y_\bullet^{\Delta^{k-1}\star} \tag{3.1}$$

are given by the limit

$$\hom(\partial\Delta^{k-1} \star \Delta^\ell, X) \times_{\hom(\partial\Delta^{k-1}\star\Delta^\ell, Y)} Y_{k+\ell} \\ \cong \hom(\Lambda^{k+\ell}_{\{0,\ldots,k-1\}}, X) \times_{\hom(\Lambda^{k+\ell}_{\{0,\ldots,k-1\}}, Y)} Y_{k+\ell}.$$

Lemma 2.15 implies that this limit exists. In the special case of vertices $\ell = 0$, we obtain a natural identification with the space of relative horns $\Lambda^k_k(f)$.

Denote by $f^{\partial\Delta^{k-1}\star}$ the induced map

$$f^{\partial\Delta^{k-1}\star} \colon X^{\Delta^{k-1}\star} \longrightarrow X_\bullet^{\partial\Delta^{k-1}\star} \times_{Y_\bullet^{\partial\Delta^{k-1}\star}} Y_\bullet^{\Delta^{k-1}\star}.$$

Lemma 3.4. *Let $f : X_\bullet \longrightarrow Y_\bullet$ be an ∞-stack. For $\ell > 0$, the map $\Lambda^\ell_i(f^{\partial\Delta^{k-1}\star})$ is canonically isomorphic to $\Lambda^{k+\ell}_{k+i}(f)$.*

Proof. An exercise in the combinatorics of joins shows that

$$(\Delta^{k-1} \star \Lambda^\ell_i \hookrightarrow \Delta^{k-1} \star \Delta^\ell) \cong (\Lambda^{k+\ell}_{\{k,\ldots,\widehat{k+i},\ldots,k+\ell\}} \hookrightarrow \Delta^{k+\ell}), \text{ and}$$
$$(\partial\Delta^{k-1} \star \Delta^\ell \hookrightarrow \Delta^{k-1} \star \Delta^\ell) \cong (\Lambda^{k+\ell}_{\{0,\ldots,k-1\}} \hookrightarrow \Delta^{k+\ell}).$$

[1] Duskin called this the "path-homotopy complex". His construction is hinted at in the earlier treatment of nerves in [13], and of weak Kan complexes in [4].

[2] Lurie [16] denotes this construction '$\hom^L_X$'. Given $x, y \in X_0$, Lurie's $\hom^L_X(x, y)$ is the fiber at (x, y) of a canonical map $P^{\geq 1}X \to \hom(\partial\Delta^1, X)$.

In this way, we obtain a pushout square

$$\begin{array}{ccc} \partial\Delta^{k-1}\star\Lambda^\ell_i & \longrightarrow & \partial\Delta^{k-1}\star\Delta^\ell \\ \downarrow & & \downarrow \\ \Delta^{k-1}\star\Lambda^\ell_i & \longrightarrow & \Lambda^{k+\ell}_{k+i} \end{array}$$

which gives rise to the pullback square

$$\begin{array}{ccc} \Lambda^{k+\ell}_{k+i}(f) & \longrightarrow & (X^{\partial\Delta^{k-1}\star}\times_{Y^{\partial\Delta^{k-1}\star}} Y^{\Delta^{k-1}\star})_\ell \\ \downarrow & & \downarrow \\ \Lambda^\ell_i(X^{\Delta^{k-1}\star}) & \longrightarrow & \Lambda^\ell_i(X^{\partial\Delta^{k-1}\star}\times_{Y^{\partial\Delta^{k-1}\star}} Y^{\Delta^{k-1}\star}) \end{array}$$

But this is also the pullback square defining $\Lambda^\ell_i(f^{\partial\Delta^{k-1}\star})$. $\square$

It follows that if f is an ∞-stack, then so is $f^{\partial\Delta^{k-1}\star}$, and if f is an n-stack, $f^{\partial\Delta^{k-1}\star}$ is an $(n-k)$-stack.

Definition 3.5. For $k > 0$, the *relative higher morphism space* $P^{\geq k}(f)_\bullet$ is the pullback

$$\begin{array}{ccc} P^{\geq k}(f)_\bullet & \longrightarrow & X_\bullet^{\Delta^{k-1}\star} \\ \downarrow & & \downarrow \\ \Lambda^k_k(f) & \longrightarrow & X_\bullet^{\partial\Delta^{k-1}\star}\times_{Y_\bullet^{\partial\Delta^{k-1}\star}} Y_\bullet^{\Delta^{k-1}\star} \end{array}$$

The following result is an immediate consequence of Lemma 3.4 and Theorem 2.17.

Theorem 3.6. *If $f : X_\bullet \longrightarrow Y_\bullet$ is an n-stack, the relative higher morphism space $P^{\geq k}(f)_\bullet$ is a Lie $(n-k)$-groupoid.*

The vertices of $P^{\geq k}(f)_\bullet$ are the k-simplices of $X_\bullet$. Its 1-simplices correspond to $(k+1)$-simplices $x \in X_{k+1}$ such that

$$f(x) = s_k d_k f(x)$$

and, for $i < k$,

$$d_i x = s_{k-1}d_{k-1}d_i x.$$

We interpret x as a path rel boundary in the fiber of f, which begins at $d_{k+1}x$ and ends at $d_k x$.

When the matching object $M_k(f)$ exists, it provides a natural augmentation

$$\pi\colon P^{\geq k}(f)_\bullet \longrightarrow M_k(f). \tag{3.2}$$

The map underlying $\pi\colon P^{\geq k}(f)_0 \cong X_k \longrightarrow M_k(f)$ is $\mu_k(f)$. This determines an augmentation because the diagram

$$P^{\geq k}(f)_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} P^{\geq k}(f)_0 \xrightarrow{\pi} M_k(f)$$

commutes.

The augmentation (3.2) encodes the idea that vertices in the relative higher morphism space are lifts of a k-morphism in $Y_\bullet$, that edges are paths rel boundary between these lifts such that the paths live in the fiber of f, etc.

Theorem 3.7. *If f is an n-hypercover, then the augmentation* (3.2) *exists and is an $(n-k)$-hypercover.*

Proof. We prove that π is an $(n-k)$-hypercover by showing that the square

$$\begin{array}{ccc} P^{\geq k}(f)_\ell & \longrightarrow & X_{k+\ell} \\ {\scriptstyle \mu_\ell(\pi)}\downarrow & & \downarrow{\scriptstyle \mu_{k+\ell}(f)} \\ M_\ell(\pi) & \longrightarrow & M_{k+\ell}(f) \end{array}$$

is a pullback. For $\ell = 0$, this is evident: both horizontal maps are isomorphisms in this case.

For $\ell > 0$, the above square is the top half of a commutative diagram

$$\begin{array}{ccc} P^{\geq k}(f)_\ell & \longrightarrow & X_{k+\ell} \\ {\scriptstyle \mu_\ell(\pi)}\downarrow & & \downarrow{\scriptstyle \mu_{k+\ell}(f)} \\ M_\ell(\pi) & \longrightarrow & M_{k+\ell}(f) \\ \downarrow & & \downarrow \\ \Lambda^k_k(f) & \longrightarrow & (X^{\partial\Delta^{k-1}\star} \times_{Y^{\partial\Delta^{k-1}\star}} Y^{\Delta^{k-1}\star})_\ell \end{array}$$

The outer rectangle is a pullback by definition. Thus, it suffices to prove that the bottom square is a pullback.

For $\ell = 1$, this may be checked directly. For $\ell > 1$, an exercise in the combinatorics of joins shows that we have a pushout square

$$\begin{array}{ccc} \partial\Delta^{k-1} \star \partial\Delta^\ell & \longrightarrow & \partial\Delta^{k-1} \star \Delta^\ell \\ \downarrow & & \downarrow \\ \Delta^{k-1} \star \partial\Delta^\ell & \longrightarrow & \partial\Delta^{k+\ell} \end{array}$$

which gives rise to a pullback square

$$\begin{array}{ccc} M_{k+\ell}(f) & \longrightarrow & M_\ell(X^{\Delta^{k-1}\star}) \\ \downarrow & & \downarrow \\ (X^{\partial\Delta^{k-1}\star} \times_{Y^{\partial\Delta^{k-1}\star}} Y^{\Delta^{k-1}\star})_\ell & \longrightarrow & M_\ell(X^{\partial\Delta^{k-1}\star} \times_{Y^{\partial\Delta^{k-1}\star}} Y^{\Delta^{k-1}\star}) \end{array}$$

This square embeds in a commutative diagram

$$\begin{array}{ccccc} M_\ell(P^{\geq k}(f)) & \longrightarrow & M_{k+\ell}(f) & \longrightarrow & M_\ell(X^{\Delta^{k-1}\star}) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \Lambda^k_k(f) & \longrightarrow & (X^{\partial\Delta^{k-1}\star} \times_{Y^{\partial\Delta^{k-1}\star}} Y^{\Delta^{k-1}\star})_\ell & \longrightarrow & M_\ell(X^{\partial\Delta^{k-1}\star} \times_{Y^{\partial\Delta^{k-1}\star}} Y^{\Delta^{k-1}\star}) \end{array}$$

The outer rectangle is a pullback because $M_\ell(-)$ commutes with limits. We observed above that the right square is a pullback. We conclude that the left square is a pullback, completing the proof that $\mu_\ell(\pi)$ is a cover. □

4. Strictification

In this section we recall Duskin's "n-strictification" functor τ_n for $n \geq 0$. This is a partially defined left-adjoint to the inclusion of the category of n-stacks into the category of ∞-stacks. We establish its main properties in Propositions 4.5 and 4.6.

Let $f\colon X_\bullet \longrightarrow Y_\bullet$ be an ∞-stack such that the orbit space $\pi_0(P^{\geq n}(f))$ exists. We define a map

$$\operatorname{tr}_n \tau_n(f)\colon \operatorname{tr}_n \tau_n(X, f)_\bullet \longrightarrow \operatorname{tr}_n Y_\bullet \tag{4.1}$$

On k-simplices, for $k < n$, $\operatorname{tr}_n \tau_n(f)$ is the map

$$f_k\colon X_k \longrightarrow Y_k$$

On n-simplices, $\operatorname{tr}_n \tau_n(f)$ is the canonical map

$$\pi_0(P^{\geq n}(f)) \longrightarrow Y_n$$

Lemma 4.1. *Let $f\colon X_\bullet \longrightarrow Y_\bullet$ be an ∞-stack such that the orbit space $\pi_0(P^{\geq n}(f))$ exists. The maps $\lambda^k_i(\tau_n(f))$ are covers for $k \leq n$. For all i, the limit $\Lambda^{n+1}_i(\tau_n(f))$ exists and the map $\Lambda^{n+1}_i(f) \longrightarrow \Lambda^{n+1}_i(\tau_n(f))$ is a cover.*

Proof. For $k < n$, the natural map $\operatorname{tr}_n X_\bullet \longrightarrow \operatorname{tr}_n \tau_n(X, f)_\bullet$ induces an isomorphism between the maps $\lambda^k_i(f)$ and $\lambda^k_i(\operatorname{tr}_n \tau_n(f))$. For $k = n$, we have a commuting square

$$\begin{array}{ccc} X_n & \longrightarrow & \tau_n(X, f)_n \\ {\scriptstyle \lambda^n_i(f)}\big\downarrow & & \big\downarrow{\scriptstyle \lambda^n_i(\tau_n(f))} \\ \Lambda^n_i(f) & \xrightarrow[\cong]{} & \Lambda^n_i(\tau_n(f)) \end{array}$$

This square guarantees that the map $\lambda^n_i(\tau_n(f))$ is a cover for all i. Indeed, the top horizontal map is the cover $X_n \longrightarrow \pi_0(P^{\geq n}(f))$, the map $\lambda^n_i(f)$ is a cover by assumption, and the bottom horizontal map is an isomorphism. Axiom 3 implies that $\lambda^n_i(\tau_n(f))$ is a cover.

Lemma 2.15 guarantees that, for all i, the limit $\Lambda^{n+1}_i(\tau_n(f))$ exists. It remains to show that the map $\Lambda^{n+1}_i(f) \longrightarrow \Lambda^{n+1}_i(\tau_n(f))$ is a cover for all i.

Recall that given a proper, non-empty subset J of $[n+1]$,

$$\Lambda^{n+1}_J = \bigcup_{i \in J} \partial_i \Delta^{n+1} \subset \partial\Delta^{n+1}.$$

For each J, there is a map

$$\Lambda^{n+1}_J(f) = \hom(\Lambda^{n+1}_J, X) \times_{\hom(\Lambda^{n+1}_J, Y)} Y_{n+1} \longrightarrow \Lambda^{n+1}_J(\tau_n(f)) = \hom(\Lambda^{n+1}_J, \tau_n(X,f)) \times_{\hom(\Lambda^{n+1}_J, Y)} Y_{n+1}.$$

We will show that it is a cover, by induction on $|J|$.

Let $J_+ = J \cup \{j\}$, where $j \notin J$, and let

$$\Lambda^{n+1}_{J\cap j}(f) = \hom(\Lambda^{n+1}_J \cap \partial_j \Delta^{n+1}, X) \times_{\hom(\Lambda^{n+1}_J \cap \partial_j \Delta^{n+1}, Y)} Y_{n+1}. \tag{4.2}$$

We have a pair of pullback diagrams in which the vertical maps are covers:

$$\begin{array}{ccc} \Lambda^{n+1}_J(f) \times_{\Lambda^{n+1}_{J\cap j}(f)} \pi_0(P^{\leq n}(f)) & \longrightarrow & \Lambda^{n+1}_J(f) \\ \downarrow & & \downarrow \\ \Lambda^{n+1}_{J_+}(\tau_n(f)) & \longrightarrow & \Lambda^{n+1}_J(\tau_n(f)) \end{array}$$

and

$$\begin{array}{ccc} \Lambda^{n+1}_{J_+}(f) & \longrightarrow & X_n \\ \downarrow & & \downarrow \\ \Lambda^{n+1}_J(f) \times_{\Lambda^{n+1}_{J\cap j}(f)} \pi_0(P^{\leq n}(f)) & \longrightarrow & \pi_0(P^{\leq n}(f)) \end{array}$$

Composing the left vertical arrows, we see that the map

$$\Lambda^{n+1}_{J_+}(f) \longrightarrow \Lambda^{n+1}_{J_+}(\tau_n(f))$$

is a cover; this completes the induction step. □

Our goal is now to construct, for any i, a "missing face map"

$$d_i \colon \Lambda^{n+1}_i(\tau_n(f)) \longrightarrow \pi_0(P^{\geq n}(f)) = \tau_n(X,f)_n.$$

Compose the covers $\Lambda^{n+1}_i(f) \longrightarrow \Lambda^{n+1}_i(\tau_n(f))$ and $\lambda^{n+1}_i(f)$ to obtain a cover $X_{n+1} \longrightarrow \Lambda^{n+1}_i(\tau_n(f))$. Denote by qd_i the composite

$$X_{n+1} \xrightarrow{d_i} X_n \longrightarrow \pi_0(P^{\geq n}(f))$$

Lemma 4.2. *The diagram*

$$X_{n+1} \times_{\Lambda^{n+1}_i(\tau_n(f))} X_{n+1} \rightrightarrows X_{n+1} \xrightarrow{qd_i} \pi_0(P^{\geq n}(f)) \tag{4.3}$$

commutes.

Remark 4.3. Recall that a *point* of C is a functor $p\colon \mathsf{C} \longrightarrow \mathsf{Set}$ which preserves finite limits, which preserves arbitrary colimits, and which takes covers to surjections. We say C has *enough points* if for every pair of maps $f \neq g$ in C, there exists a point p such that $p(f) \neq p(g)$.

We prove the lemma under the assumption that C has enough points. This assumption is satisfied in many examples of interest, and has the benefit of allowing for an elementary proof.

One could proceed without this assumption by using Ehresman's theory of sketches in combination with Barr's theorem on the existence of a Boolean cover of the topos of sheaves on C. The latter approach is discussed in [3, Section 2 – "For Logical Reasons"] or in more depth in [14, Chapter 7, especially 7.5].

Proof. To show that 4.3 commutes, we construct an epi

$$\mathbb{K} \xrightarrow{g} X_{n+1} \times_{\Lambda_i^{n+1}(\tau_n(f))} X_{n+1}$$

and we show that the diagram

$$\mathbb{K} \rightrightarrows X_{n+1} \xrightarrow{qd_i} \pi_0(P^{\geq n}(f)) \tag{4.4}$$

commutes. This implies that 4.3 commutes.

Step 1: Construct g.

By way of motivation, the fork

$$\mathbb{K} \rightrightarrows X_{n+1}$$

can be understood as a "fattened version" of the kernel pair of the cover $X_{n+1} \longrightarrow \Lambda_i^{n+1}(\tau_n(f))$.

In more detail, sections of $X_{n+1} \times_{\Lambda_i^{n+1}(\tau_n(f))} X_{n+1}$ consist of pairs of $n+1$-simplices of X which live over the same $n+1$-simplex of Y and such that their i^{th}-horns are homotopic rel boundary over Y. We will construct $\mathbb{K}$ by adding in the data of the homotopies. We start by defining a space of homotopies $\mathbb{H}_i^{n+1}(f)$ as the pullback

$$\begin{array}{ccccc}
\mathbb{H}_i^{n+1}(f) & \longrightarrow & & & (P^{\geq n}(f)_1)^{\times(n+1)} \\
 & & & & \downarrow{\scriptstyle (d_0)^{\times(n+1)}} \\
\downarrow{\scriptstyle (d_0)^{\times(n+1)}} & & & & (X_n)^{\times(n+1)} \\
 & & & & \downarrow \\
\Lambda_i^{n+1}(f) & \longrightarrow & (X_n)^{\times(n+1)} & \longrightarrow & (\pi_0(P^{\geq n}(f)))^{\times(n+1)}
\end{array}$$

Observe that the pullback exists because the right vertical maps are both covers. Observe that a section of $\mathbb{H}_i^{n+1}(f)$ consists of a pair of relative i^{th} $(n+1)$-horns of f along with choices of homotopies rel boundary over Y between the j^{th}-faces of each relative horn for $j \neq i$.

We now glue this data in to form $\mathbb{K}$. For this, note that in addition to the map $(d_0)^{\times(n+1)} \colon \mathbb{H}_i^{n+1}(f) \longrightarrow \Lambda_i^{n+1}(f)$, there is another map $(d_1)^{\times(n+1)} \colon \mathbb{H}_i^{n+1}(f) \longrightarrow \Lambda_i^{n+1}(f)$. Define $\mathbb{K}$ to be the iterated pullback

$$\begin{array}{ccccc}
\mathbb{K} & \longrightarrow & & & X_{n+1} \\
\downarrow & & & & \downarrow{\scriptstyle \lambda_i^{n+1}(f)} \\
X_{n+1} \times_{\Lambda_i^{n+1}(f)} \mathbb{H}_i^{n+1}(f) & \longrightarrow & \mathbb{H}_i^{n+1}(f) & \xrightarrow{(d_1)^{\times(n+1)}} & \Lambda_i^{n+1}(f) \\
\downarrow & & \downarrow{\scriptstyle (d_0)^{\times(n+1)}} & & \\
X_{n+1} & \xrightarrow[\lambda_i^{n+1}(f)]{} & \Lambda_i^{n+1}(f) & &
\end{array}$$

The limit $\mathbb{K}$ exists because $\lambda_i^{n+1}(f)$ is a cover (f is an ∞-stack). Observe that $\mathbb{K}$ encodes the data of pairs (x_0, x_1) of $(n+1)$-simplices in the same fiber of f and explicit fiber homotopies rel boundary $(p_j)_{0\le j\neq i}^{n+1}$ between all but their i^{th} faces. The projections along the left and right X_{n+1} factors induce a map

$$\mathbb{K} \xrightarrow{g} X_{n+1} \times_{\Lambda_i^{n+1}(\tau_n(f))} X_{n+1}$$

Step 2: Prove that g is an epi.

Because we assume that C has enough points, it suffices to check that the map $p(g)$ is a surjection for any point $p\colon \mathsf{C} \longrightarrow \mathsf{Set}$. A point p preserves finite limits and arbitrary colimits, and takes covers to surjections. As a result, p takes each construction we have been considering to its analogue in the category Set.

It suffices to show that if $f\colon X_\bullet \longrightarrow Y_\bullet$ is a Kan fibration of simplicial sets, then the map

$$\mathbb{K} = X_{n+1} \times_{\Lambda_i^{n+1}(f)} \mathbb{H}_i^{n+1}(f) \times_{\Lambda_1^{n+1}(f)} X_{n+1} \xrightarrow{g} X_{n+1} \times_{\Lambda_i^{n+1}(\tau_n(f))} X_{n+1}$$

is surjective. This map fits into a pullback square

$$\begin{array}{ccc} \mathbb{K} & \longrightarrow & (P^{\ge n}(f)_1)^{\times(n+1)} \\ {\scriptstyle g}\big\downarrow & & \big\downarrow \\ X_{n+1} \times_{\Lambda_i^{n+1}(\tau_n(f))} X_{n+1} & \longrightarrow & (X_n \times_{\pi_0(P^{\ge n}(f))} X_n)^{\times(n+1)} \end{array}$$

The map

$$P^{\ge n}(f)_1 \longrightarrow X_n \times_{\pi_0(P^{\ge n}(f))} X_n$$

is surjective because the simplicial set $P^{\ge n}(f)_\bullet$ is Kan. As a result, the map g is surjective, because surjections of sets are preserved under products and pullbacks.

Step 3: Show that the square (4.4) commutes.

We will construct a sequence of covers

$$K_{n+3} \longrightarrow \cdots \longrightarrow K_0 = \mathbb{K} \tag{4.5}$$

fitting into a commuting square

$$\begin{array}{ccc} K_{n+3} & \xrightarrow{h} & P^{\ge n}(f)_1 \\ {\scriptstyle c}\big\downarrow & & \big\downarrow{\scriptstyle (d_0,d_1)} \\ \mathbb{K} & \xrightarrow[(d_i,d_i)g]{} & X_n \times X_n \end{array} \tag{4.6}$$

By way of motivation, we note that the restriction of this sequence of covers to a given section $(x_0, x_1, (p_j)_{0\le j\neq i}^{n+1})$ of $\mathbb{K}$ will encode the process of homotoping the i^{th} horn of x_1 to the i^{th}-horn of x_0 one face at a time, using the specified homotopies p_j, and then using this to produce a homotopy between the i^{th} faces of x_0 and x_1.

Granting the existence of the sequence (4.5) and the square (4.6), we conclude the lemma as follows. Recall that q denotes the map $X_n \longrightarrow \pi_0(P^{\ge n}(f))$. By definition,

$$qd_0 = qd_1\colon P^{\ge n}(f)_1 \longrightarrow \pi_0(P^{\ge n}(f))$$

The square 4.6 implies that

$$\begin{aligned} qd_i \operatorname{pr}_1 gc &= qd_0 h \\ &= qd_1 h \\ &= qd_i \operatorname{pr}_2 gc \end{aligned}$$

The map $c\colon K_{n+3} \longrightarrow \mathbb{K}$ is an epimorphism, because it is a cover (Axiom 4). We conclude that

$$qd_i \operatorname{pr}_1 g = qd_i \operatorname{pr}_2 g,$$

or equivalently, that 4.4 commutes.

Step 3a: the induction.

We now construct 4.5. In detail, a section of $\mathbb{K}$ is a tuple $(x_0, x_1, (p_j)_{0\leq j\neq i}^{n+1})$ where

$$\begin{aligned} x_k &\in X_{n+1}, \text{ and} \\ p_j &\in P^{\geq n}(f)_1, \end{aligned}$$

such that

$$f(x_0) = f(x_1),$$

and, for $k = 0, 1$, and all j,

$$d_{n+k} p_j = d_j x_k.$$

Equivalently, a section of $\mathbb{K}$ is a tuple $(x_0, x_1, (p_j)_{j=0}^{n+1})$ where $(x_0, x_1, (p_j)_{0\leq j\neq i}^{n+1})$ satisfies the conditions above, and $p_i = s_n d_i x_1$. Note further that, for $i < n$

$$\begin{aligned} d_i p_j &= s_{n-1} d_{n-1} d_i p_j \\ &= s_{n-1} d_n d_i p_j \\ &= s_{n-1} d_i d_{n+1} p_j \\ &= s_{n-1} d_i d_j x_1. \end{aligned} \tag{4.7}$$

For the base of the induction, Equation (4.7) implies that the assignment

$$(x_0, x_1, (p_j)_{j=0}^{n+1}) \mapsto ((d_0 s_n x_1, \ldots, d_{n-1} s_n x_1, -, x_1, p_{n+1}), s_n f(x_1))$$

defines a map $\mathbb{K} \longrightarrow \Lambda_n^{n+2}(f)$. Denote by K_1 the pullback

$$K_1 := \mathbb{K} \times_{\Lambda_n^{n+2}(f)} X_{n+2}$$

The projection $K_1 \longrightarrow \mathbb{K}$ is a cover, because it is the pullback of the cover $\lambda_n^{n+2}(f)$. Similarly, if we denote a section of K_1 by $(x_0, x_1, (p_j)_{j=0}^{n+1}, z_{n+1})$, then Equation (4.7) implies that the assignment

$$(x_0, x_1, (p_j)_{j=0}^{n+1}, z_{n+1}) \mapsto ((d_0 s_{n+1} x_1, \ldots, d_{n-1} s_{n+1} x_1, p_n, -, d_n z_{n+1}), s_n f(x_1))$$

defines a map $K_1 \longrightarrow \Lambda_{n+1}^{n+2}(f)$. Denote by K_2 the pullback

$$K_2 := K_1 \times_{\Lambda_{n+1}^{n+2}(f)} X_{n+2}$$

The projection $K_2 \longrightarrow K_1$ is a cover, because it is the pullback of the cover $\lambda_{n+1}^{n+2}(f)$.

For the inductive step suppose that we have constructed a cover $K_\ell \longrightarrow K_{\ell-1}$, such that sections of K_ℓ are tuples $(x_0, x_1, (p_j)_{j=0}^{n+1}, (z_j)_{j=n+2-\ell}^{n+1})$ with

$$(x_0, x_1, (p_j)_{j=0}^{n+1}, (z_j)_{n+3-\ell}^{n+1}) \in K_{\ell-1}, \text{ and}$$
$$z_{n+2-\ell} \in X_{n+2},$$

such that

$$f(z_{n+2-\ell}) = s_{n+1} d_{n+2} f(z_{n+2-\ell}), \text{ and}$$
$$d_i z_{n+2-\ell} = \begin{cases} d_i s_{n+1} d_{n+1} z_{n+3-\ell} & i \neq n+2-\ell, n+1 \\ p_{n+2-\ell} & i = n+2-\ell \\ d_n z_{n+3-\ell} & i = n+1 \end{cases}$$

Then Equation (4.7) implies that the assignment

$$(x_0, x_1, (p_j)_{j=0}^{n+1}, (z_j)_{j=n+2-\ell}^{n+1}) \longmapsto$$

$$((d_0 s_{n+1} d_{n+1} z_{n+2-\ell}, \dots, d_{n-\ell} s_{n+1} d_{n+1} z_{n+2-\ell}, p_{n+1-\ell}, d_{n+2-\ell} s_{n+1} d_{n+1} z_{n+2-\ell},$$
$$\dots, d_n s_{n+1} d_{n+1} z_{n+2-\ell}, -, d_{n+1} z_{n+2-\ell}), s_{n+1} d_{n+1} f(z_{n+2-\ell}))$$

defines a map $K_\ell \longrightarrow \Lambda_{n+1}^{n+2}(f)$. Denote by $K_{\ell+1}$ the pullback

$$K_{\ell+1} := K_\ell \times_{\Lambda_{n+1}^{n+2}(f)} X_{n+2}$$

This completes the induction step for $\ell < n+2$.

The induction above demonstrates the existence of the sequence of covers

$$K_{n+2} \longrightarrow \cdots \longrightarrow K_0 = \mathbb{K}$$

The construction allows us to denote a section of K_{n+2} by

$$(x_0, x_1, (p_j)_{0 \leq j \neq i}^{n+1}, (z_j)_{j=0}^{n+1})$$

where

$$s_{n+1} f(x_0) = f(z_0),$$

and, for $j \neq i$,

$$d_j x_0 = d_j d_{n+1} z_0.$$

For $i \neq n+1$, the assignment

$$(x_0, x_1, (p_j)_{0 \leq j \neq i}^{n+1}, (z_j)_{j=0}^{n+1}) \mapsto$$
$$((d_0 s_{n+1} x_0, \dots, d_{i-1} s_{n+1} x_0, -, d_{i+1} s_{n+1} x_0, \dots, d_n s_{n+1} x_0, x_0, d_{n+1} z_0), s_{n+1} f(x_0))$$

defines a map $K_{n+2} \longrightarrow \Lambda_i^{n+2}(f)$. Denote by K_{n+3} the pullback

$$K_{n+3} := K_{n+2} \times_{\Lambda_i^{n+2}(f)} X_{n+2}$$

Similarly, for $i = n+1$, the assignment

$$(x_0, x_1, (p_j)_{j=0}^{n}, (z_j)_{j=0}^{n}) \mapsto ((d_0 s_n x_0, \dots, d_{n-1} s_n x_0, x_0, d_{n+1} z_0, -), s_n f(x_0))$$

defines a map $K_{n+2} \longrightarrow \Lambda_{n+2}^{n+2}(f)$. Denote by K_{n+3} the pullback

$$K_{n+3} := K_{n+2} \times_{\Lambda_{n+2}^{n+2}(f)} X_{n+2}$$

Note that, for any i, the map $K_{n+3} \longrightarrow K_{n+2}$ is a cover, because it is a pullback of the cover $\lambda_j^{n+2}(f)$ (where $j = i$ if $i < n+1$ and $j = n+2$ if $i = n+1$). Denote a section of K_{n+3} by $(\mathbf{x}, w)$ where $\mathbf{x} \in K_{n+2}$ and $w \in X_{n+2}$. The construction, in all cases, guarantees that the assignment

$$(\mathbf{x}, w) \mapsto d_i w$$

defines a map $h\colon K_{n+3} \longrightarrow P^{\geq n}(f)_1$ which, along with $c\colon K_{n+3} \longrightarrow \mathbb{K}$, gives 4.6. $\square$

By Axiom 4, 4.3 determines a map

$$\Lambda_i^{n+1}(\tau_n(f)) \xrightarrow{d_i} \tau_n(X, f)_n,$$

We now extend 4.1 to a map

$$\operatorname{tr}_{n+1} \tau_n(f)\colon \operatorname{tr}_{n+1} \tau_n(X, f)_\bullet \longrightarrow \operatorname{tr}_{n+1} Y_\bullet$$

On k-simplices, for $k \leq n$, the map $\operatorname{tr}_{n+1} \tau_n(f)$ equals the map $\operatorname{tr}_n \tau_n(f)$. On $(n+1)$-simplices, $\operatorname{tr}_{n+1} \tau_n(f)$ is the canonical map

$$\Lambda_1^{n+1}(\tau_n(f)) \longrightarrow Y_{n+1}$$

The missing face map

$$\operatorname{tr}_{n+1} \tau_n(X, f)_{n+1} = \Lambda_1^{n+1}(\tau_n(f)) \xrightarrow{d_1} \pi_0(P^{\geq n}(f)) = \operatorname{tr}_{n+1} \tau_n(X, f)_n$$

makes $\operatorname{tr}_{n+1} \tau_n(X, f)_\bullet$ into an $(n+1)$-truncated simplicial object.

Definition 4.4. Let $f\colon X_\bullet \longrightarrow Y_\bullet$ be an ∞-stack such that $\pi_0(P^{\geq n}(f))$ exists. Define $\tau_n(X, f)_\bullet$ to be the limit

$$\tau_n(X, f)_\bullet := \operatorname{cosk}_{n+1} \operatorname{tr}_{n+1} \tau_n(X, f)_\bullet \times_{\operatorname{Cosk}_{n+1} Y_\bullet} Y_\bullet$$

The *n-strictification of f* is the map

$$\tau_n(f)\colon \tau_n(X, f)_\bullet \longrightarrow Y_\bullet$$

Proposition 4.5. *Let $f\colon X_\bullet \longrightarrow Y_\bullet$ be an ∞-stack such that $\pi_0(P^{\geq n}(f))$ exists. The n-strictification $\tau_n(f)\colon \tau_n(X, f)_\bullet \longrightarrow Y_\bullet$ is an n-stack. The maps f and $\tau_n(f)$ are isomorphic if and only if f is an n-stack.*

Proof. We show that $\tau_n(f)$ is an n-stack. Lemma 4.1 established that $\lambda_i^k(\tau_n(f))$ is a cover for $k \leq n$ and all i. We now show that the map $\lambda_i^{n+1}(\tau_n(f))$ is an isomorphism for all i. The inclusion $\Lambda_i^{n+1} \hookrightarrow \partial\Delta^{n+1}$ induces a map

$$d_{\hat{i}}\colon M_{n+1}(\tau_n(f)) \longrightarrow \Lambda_i^{n+1}(\tau_n(f))$$

Observe that, in the notation of 4.2, if $J = [n+1] \setminus \{i\}$, then

$$M_{n+1}(\tau_n(f)) \cong \Lambda_i^{n+1}(\tau_n(f)) \times_{\Lambda_{J\cap i}^{n+1}(f)} \pi_0(P^{\geq n}(f))$$

The missing face map $d_i\colon \Lambda_i^{n+1}(\tau_n(f)) \longrightarrow \pi_0(P^{\geq n}(f))$ induces a map

$$(1, d_i)\colon \Lambda_i^{n+1}(\tau_n(f)) \longrightarrow M_{n+1}(\tau_n(f))$$

Note that the map $(1, d_i)$ is a right inverse for the map $d_{\hat{i}}$.

For all i, the map $\lambda_i^{n+1}(\tau_n(f))$ factors as the composite

$$\tau_n(X, f)_{n+1} = \Lambda_1^{n+1}(\tau_n(f)) \xrightarrow{(1, d_1)} M_{n+1}(\tau_n(f)) \xrightarrow{d_{\hat{i}}} \Lambda_i^{n+1}(\tau_n(f))$$

These maps fit into a commuting diagram

$$\begin{array}{ccccc} & & X_{n+1} & & \\ & \swarrow & \downarrow & \searrow & \\ \Lambda_1^{n+1}(\tau_n(f)) & \underset{d_{\hat{1}}}{\overset{(1,d_1)}{\rightleftarrows}} & M_{n+1}(\tau_n(f)) & \underset{(1,d_i)}{\overset{d_{\hat{i}}}{\rightleftarrows}} & \Lambda_i^{n+1}(\tau_n(f)) \end{array}$$

The proof of Lemma 4.2 shows that

$$(1, d_1) \circ d_{\hat{1}} \circ (1, d_i) = (1, d_i)$$
$$(1, d_i) \circ d_{\hat{i}} \circ (1, d_1) = (1, d_1)$$

We conclude that

$$\begin{aligned} (d_{\hat{1}}(1, d_i)) \circ \lambda_i^{n+1}(\tau_n(f)) &= (d_{\hat{1}}(1, d_i)) \circ (d_{\hat{i}}(1, d_1)) \\ &= d_{\hat{1}} \circ (1, d_1) \\ &= 1_{\Lambda_1^{n+1}(\tau_n(f))}. \end{aligned}$$

Similarly,

$$\begin{aligned} \lambda_i^{n+1}(\tau_n(f)) \circ d_{\hat{1}}(1, d_i) &= d_{\hat{i}}(1, d_1) \circ d_{\hat{1}}(1, d_i) \\ &= 1_{\Lambda_i^{n+1}(\tau_n(f))}. \end{aligned}$$

We have shown that $\lambda_i^{n+1}(\tau_n(f))$ is an isomorphism for all i.

Lemma 2.15 guarantees that the limit $\Lambda_i^{n+2}(\tau_n(f))$ exists. Note that a section of $\Lambda_i^{n+2}(\tau_n(f))$ consists of a tuple

$$\{x_j\}_{j \neq i} \in \tau_n(X, f)_{n+1}^{\times(n+2)} = \Lambda_1^{n+1}(\tau_n(f))^{\times(n+2)} \cong M_{n+1}(\tau_n(f))^{\times(n+2)}$$

such that, for $k < j$, $d_k x_j = d_{j-1} x_k$, while a section of $\tau_n(X, f)_{n+2} = M_{n+2}(\tau_n(f))$ consists of a tuple

$$\{x_j\} \in \tau_n(X, f)_{n+1}^{\times(n+3)} = M_{n+1}(\tau_n(f))^{\times(n+3)}$$

such that, for $k < j$, $d_k x_j = d_{j-1} x_k$. In particular, the map

$$\Lambda_i^{n+2}(\tau_n(f)) \xrightarrow{\partial d_i} M_{n+1}(\tau_n(f))$$

induces an embedding

$$\begin{aligned} \Lambda_i^{n+2}(\tau_n(f)) &\hookrightarrow M_{n+2}(\tau_n(f)) = \tau_n(X, f)_{n+2} \\ \{x_j\}_{j \neq i} &\mapsto \{x_j\}_{j \neq i} \cup \{x_i := \partial d_i(\{x_j\}_{j \neq i})\} \end{aligned}$$

Because $\lambda_i^{n+1}(\tau_n(f))$ is an isomorphism for all i, this map is an isomorphism, with inverse given by $\lambda_i^{n+2}(\tau_n(f))$.

For $k > n + 2$, the map $\lambda_i^k(\tau_n(f))$ is an isomorphism because the map $\tau_n(f)$ is $(n+1)$-coskeletal, and the inclusion $\Lambda_i^k \hookrightarrow \Delta^k$ is the identity on $(n+1)$-skeleta. By inductively applying Lemma 2.15, we conclude that $\tau_n(X, f)_k$ is an object of C for all k and that $\tau_n(f)$ is an n-stack.

We have shown that if f is isomorphic to $\tau_n(f)$, then f is an n-stack. Conversely, suppose that f is an n-stack. For any k and any map, the $(k-1)$-skeleton of the map determines its Λ_i^k-horns. The horn-filling maps for $\tau_n(f)$ are isomorphisms above dimension n. If f is an n-stack, then the horn-filling maps for f are also isomorphisms above dimension n. We conclude that the canonical map from f to

$\tau_n(f)$ is an isomorphism if f is an n-stack and if the map from f to $\tau_n(f)$ induces an isomorphism on n-skeleta.

When f is an n-stack, the map from f to $\tau_n(f)$ is automatically an isomorphism on n-skeleta. Indeed, if f is an n-stack, then $P^{\geq n}(f)_\bullet$ is a Lie 0-groupoid (Theorem 3.6). As a result, the map from X_n to $\pi_0(P^{\geq n}(f))$ is an isomorphism. $\square$

If $f\colon X_\bullet \longrightarrow Y_\bullet$ is a hypercover, then Proposition 2.9 and Theorem 3.7 imply that

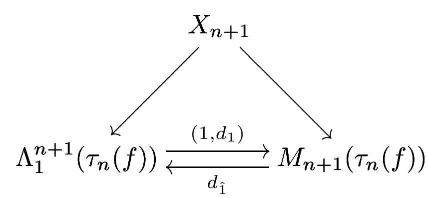
$$\pi_0(P^{\geq n}(f)) \cong M_n(f)$$

In particular, the n-strictification $\tau_n(f)$ is an n-stack.

Proposition 4.6. *If $f : X_\bullet \to Y_\bullet$ is a hypercover, then*

(1) *the map $\tau_n(f)$ is an n-hypercover, and*
(2) *the map $X_\bullet \xrightarrow{q} \tau_n(X,f)_\bullet$ is a hypercover.*

Proof. We show that $\tau_n(X,f)_{n+1} = \Lambda_1^{n+1}(\tau_n(f)) \cong M_{n+1}(\tau_n(f))$. Consider the commuting triangle



We showed in Lemma 4.1 that the map $X_{n+1} \longrightarrow \Lambda_1^{n+1}(\tau_n(f))$ is a cover. A similar argument shows that, because f is a hypercover, the map $X_{n+1} \longrightarrow M_{n+1}(\tau_n(f))$ is a cover. Axiom 3 implies that both $d_{\hat{1}}$ and $(1,d_1)$ are covers. Axiom 4 implies that they are both epimorphisms. By construction

$$d_{\hat{1}}(1,d_1) = 1_{\Lambda_1^{n+1}(\tau_n(f))}$$

As a result,

$$\begin{aligned}(1,d_1)d_{\hat{1}}(1,d_1) &= (1,d_1)\\ &= 1_{M_{n+1}(\tau_n(f))}(1,d_1)\end{aligned}$$

Because $(1,d_1)$ is an epimorphism, we conclude that $(1,d_1)d_{\hat{1}} = 1_{M_{n+1}(\tau_n(f))}$.

This isomorphism combines with the isomorphism

$$\begin{aligned}\tau_n(X,f)_n &\cong M_n(f)\\ &= M_n(\tau_n(f))\end{aligned}$$

to show that

$$\tau_n(X,f)_\bullet \cong \mathrm{Cosk}_{n-1}(X)_\bullet \times_{\mathrm{Cosk}_{n-1}(Y)_\bullet} Y_\bullet$$

We have shown that $\tau_n(f)$ is an n-hypercover.

For $k < n$, $M_k(q) \cong X_k$ by inspection. A similar check shows that the map from $M_n(q)$ to $\tau_n(X,f)_n$ is an isomorphism. We observed above that $\tau_n(X,f)_n \cong M_n(f)$ when f is a hypercover. This implies that the map $X_n \longrightarrow M_n(q)$ is isomorphic to the cover $X_n \longrightarrow M_n(f)$. For $k > n$, an exercise in combinatorics shows that the map $X_k \longrightarrow M_k(q)$ is isomorphic to the cover $X_k \longrightarrow M_k(f)$. We conclude the proof. $\square$

Proposition 4.7. *Let $X_\bullet$ be a Lie ∞-groupoid, $U_\bullet \xrightarrow{f} X_\bullet$ a hypercover, and $Y_\bullet$ a Lie n-groupoid. Then any span*

$$X_\bullet \xleftarrow{f} U_\bullet \longrightarrow Y_\bullet$$

factors uniquely through $\tau_n(f)$ as in the diagram

$$\begin{array}{ccccc} X_\bullet & \xleftarrow{f} & U_\bullet & \longrightarrow & Y_\bullet \\ & {\scriptstyle \tau_n(f)}\nwarrow & \downarrow & \nearrow & \\ & & \tau_n(U,f)_\bullet & & \end{array}$$

Further, the vertical map in this diagram is a hypercover.

Proof. The data of a span is equivalent to a commuting triangle

$$\begin{array}{ccc} U_\bullet & \longrightarrow & X_\bullet \times Y_\bullet \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \pi_X} & \\ & X_\bullet & \end{array}$$

Theorem 2.17 shows that both diagonal maps are ∞-stacks over $X_\bullet$.

We apply τ_n to obtain the commuting diagram

$$\begin{array}{ccc} \tau_n(U,f)_\bullet & \longrightarrow & \tau_n(X \times Y, \pi_X)_\bullet \\ \uparrow & & \uparrow \\ U_\bullet & \longrightarrow & Y_\bullet \\ & \searrow \quad X_\bullet \quad \swarrow & \end{array}$$

Proposition 4.5 shows that the map

$$X_\bullet \times Y_\bullet \longrightarrow \tau_n(X \times Y, \pi_X)_\bullet$$

is an isomorphism. Proposition 4.6 shows that $\tau_n(f)$ is an n-hypercover and that the map

$$U_\bullet \longrightarrow \tau_n(U,f)_\bullet$$

is a hypercover. $\square$

5. n-Bundles and Descent

A classical construction produces a principal bundle for a Lie group from local data on the base. This local data is frequently presented in the form of a cocycle φ on a cover $U \to X$. If we pass to the nerve of the cover $f : U_\bullet \to X$, we can encode the cocycle as a 0-stack

$$E^\varphi_\bullet \xrightarrow{p} U_\bullet \tag{5.1}$$

The principal bundle corresponding to the cocycle is the 0-strictification

$$\tau_0(E^\varphi, fp) \xrightarrow{\tau_0(fp)} X$$

From this perspective, the principal bundle is representable because the 0-stack p has the structure of a *twisted Cartesian product.* Twisted Cartesian products were studied by Barratt, Guggenheim and Moore in their work on principal and associated bundles for simplicial groups [1].

Definition 5.1. A map $p : E_\bullet \to X_\bullet$ is a *twisted Cartesian product*, if there exists $Y_\bullet \in \mathsf{sC}$, and there exist isomorphisms

$$\begin{array}{ccc} E_k & \xrightarrow[\cong]{\varphi_k} & X_k \times Y_k \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \pi_{X_k}} & \\ & X_k & \end{array}$$

for each $k \in \mathbb{N}$, such that, for $i < k$,

$$\varphi_{k-1} d_i^E = (d_i^X \times d_i^Y)\varphi_k,$$

and, for all i,

$$\varphi_k s_i^E = (s_i^X \times s_i^Y)\varphi_k.$$

Definition 5.2. A *local n-bundle* is a twisted Cartesian product which is also an n-stack.

In analogy with 5.1, we consider local n-bundles on the total spaces of $(n+1)$-hypercovers

$$E_\bullet \xrightarrow{p} U_\bullet \xrightarrow{f} X_\bullet$$

Our goal in this section is to show that the n-strictification

$$\tau_n(E, fp)_\bullet \xrightarrow{\tau_n(fp)} X_\bullet$$

exists.

From the definition, it suffices to show that the orbit space $\pi_0 P^{\geq n}(fp)$ exists. Because the map fp is an $(n+1)$-stack, the relative higher morphism space $P^{\geq n}(fp)$ is a Lie 1-groupoid (Theorem 3.6). We see that the existence of the n-strictification is equivalent to the existence of the orbit space of a Lie groupoid.

Theorem 5.3 (Godemont)**.** *Let $\mathcal{G}$ be a Lie groupoid in the category of analytic manifolds over a complete normed field. If the map $(s,t) : \mathcal{G}_1 \to \mathcal{G}_0 \times \mathcal{G}_0$ is a closed embedding, then $\pi_0(\mathcal{G})$ is an analytic manifold and the map $\mathcal{G}_0 \to \pi_0(\mathcal{G})$ is a surjective submersion.*

Serre gives a proof in [20, Theorem III.12.2] which applies mutatis mutandi to the category of smooth manifolds. Inspired by this, we formulate an analogue of Godemont's Theorem for categories with covers. We begin by defining an analogue of closed embeddings.

Definition 5.4. Let $f \colon X \longrightarrow Y$ be a morphism in C. The *graph* Γ_f of f is the inclusion

$$X \times_Y Y \xhookrightarrow{\Gamma_f} X \times Y$$

The subcategory of *regular embeddings* is the smallest sub-category of C which is closed under pullback along covers and which contains all graphs.

An isomorphism is a regular embedding, because it is a pullback of the graph

$$* \xrightarrow{\Delta} * \times *$$

along a cover $X \to *$. A regular embedding in the category of smooth manifolds is a certain type of closed embedding.

Definition 5.5. A *regular* Lie n-groupoid is a Lie n-groupoid $X_\bullet$ such that the map $\mu_1(X) : X_1 \longrightarrow M_1(X) \cong X_0 \times X_0$ is a regular embedding.

Axiom 5. (Godemont's Theorem) Let $X_\bullet$ be a regular Lie n-groupoid. The orbit space $\pi_0(X_\bullet)$ exists.

Remark 5.6. In the setting of smooth or analytic manifolds, this axiom is just a restatement of Godemont's theorem: the definition of a Lie n-groupoid ensures that the map $(d_0, d_1)\colon X_1 \longrightarrow X_0 \times X_0$ defines an equivalence relation on X_0, and that $\pi_0(X_\bullet)$ is the quotient of X_0 by this relation. Serre's treatment in [20] shows that $\pi_0(X_\bullet)$ is a manifold if $X_1 \longrightarrow X_0 \times X_0$ is a closed embedding.

Theorem 5.7 (Descent for n-bundles)**.** *Suppose that Godemont's Theorem holds in* C*. If* $p : E_\bullet \to U_\bullet$ *is a local* n*-bundle and* $f : U_\bullet \to X_\bullet$ *is an* $(n+1)$*-hypercover, then the* n*-strictification*

$$\tau_n(E, fp)_\bullet \xrightarrow{\tau_n(fp)} X_\bullet$$

exists.

Remark 5.8. To study bundles for simplicial Lie groupoids, one should use twisted *fiber* products rather than twisted Cartesian products in the definition of local n-bundles. The theorem also holds for this more general notion.

The following lemma is the crux of the proof.

Lemma 5.9. *There exists an isomorphism*

$$P^{\geq n}(fp)_1 \xrightarrow[\cong]{\psi} P^{\geq n}(f)_1 \times Y_n$$

such that the maps $\phi_n d_n$ *and* $(d_n \times 1_{Y_n})\psi$ *are equal.*

Proof. The definition of $P^{\geq n}(fp)_1$ allows us to view it as a sub-object of E_{n+1}. Since p is a twisted Cartesian product, there exists $Y_\bullet \in \mathsf{sC}$ and isomorphisms

$$E_k \xrightarrow[\cong]{\varphi_k} U_k \times Y_k$$

such that, for $i < k$,

$$\varphi_{k-1} d_i^E = (d_i^U \times d_i^Y)\varphi_k,$$

and, for all i,

$$\varphi_k s_i^E = (s_i^U \times s_i^Y)\varphi_k.$$

Using this, we see that sections of $P^{\geq n}(fp)_1$ consist of pairs

$$(u, y) \in U_{n+1} \times Y_{n+1}$$

such that, for $i < n$,

$$d_i y = s_{n-1} d_{n-1} d_i y,$$
$$d_i u = s_{n-1} d_{n-1} d_i u,$$

and

$$f(u) = s_n d_n f(u).$$

Similarly, sections of $P^{\geq n}(p)_1$ consist of pairs

$$(u, y) \in U_{n+1} \times Y_{n+1}$$

such that, for $i < n$,

$$d_i y = s_{n-1} d_{n-1} d_i y,$$

and

$$u = s_n d_n u.$$

These equations say that if (u, y) is a section of $P^{\geq n}(fp)_1$, then u is a section of $P^{\geq n}(f)_1$ and the natural map

$$\begin{aligned} P^{\geq n}(fp)_1 &\longrightarrow P^{\geq n}(f)_1 \times_{U_n} P^{\geq n}(p)_1 \\ (u, y) &\longmapsto (u, (s_n d_n u, y)) \end{aligned} \tag{5.2}$$

is an isomorphism.

Because p is an n-stack, $P^{\geq n}(p)_\bullet$ is a Lie 0-groupoid (Theorem 3.6). The target map

$$P^{\geq n}(p)_1 \longrightarrow P^{\geq n}(p)_0$$

is therefore an isomorphism. Using this isomorphism and the map 5.2, we obtain the desired isomorphism

$$P^{\geq n}(fp)_1 \xrightarrow[\cong]{\psi} P^{\geq n}(f)_1 \times Y_n$$

□

Proof of Theorem 5.7. Axiom 5 reduces the proof to showing that

$$P^{\geq n}(fp)_1 \longrightarrow P^{\geq n}(fp)_0 \times P^{\geq n}(fp)_0$$

is a regular embedding. The map

$$P^{\geq n}(f)_\bullet \xrightarrow{\pi} M_n(f)$$

is a 1-hypercover, because f is an $(n+1)$-hypercover (Theorem 3.7). In particular, the map $\mu_1(\pi)$ gives an isomorphism

$$P^{\geq n}(f)_1 \xrightarrow[\cong]{\mu_1(\pi)} U_n \times_{M_n(f)} U_n$$

The canonical map

$$U_n \times_{M_n(f)} U_n \xhookrightarrow{\imath} U_n \times U_n$$

is a regular embedding. Composing this with the map above, we obtain a regular embedding

$$P^{\geq n}(f)_1 \xrightarrow{\imath\mu_1(\pi)} U_n \times U_n$$

The isomorphisms

$$P^{\geq n}(fp)_0 \xrightarrow{\cong} E_n \xrightarrow{\cong} U_n \times Y_n$$

and

$$P^{\geq n}(fp)_1 \xrightarrow[\cong]{\psi} P^{\geq n}(f)_1 \times Y_n$$

allow us to factor the map

$$P^{\geq n}(fp)_1 \longrightarrow P^{\geq n}(fp)_0 \times P^{\geq n}(fp)_0$$

as the composite

$$P^{\geq n}(fp)_1 \xrightarrow{\Gamma_{d^E_{n+1}}} P^{\geq n}(fp)_1 \times Y_n \xrightarrow{\psi \times 1_Y} Y_n \times P^{\geq n}(f)_1 \times Y_n$$
$$\xrightarrow{1_Y \times (\imath\mu_1(\pi)) \times 1_Y} Y_n \times U_n \times U_n \times Y_n$$

Each map in this sequence is a regular embedding. □

6. Strict Lie n-Groups and their Actions

Principal and associated bundles for discrete simplicial groups provide examples of local n-bundles in sSet. This theory was developed by Barratt, Guggenheim and Moore [1]. In this section, we develop analogous results for simplicial Lie groups. While we restrict to simplicial groups for ease of exposition, the results and proofs carry over to simplicial Lie groupoids. We refer the reader to [17, Section 4] for a related treatment of principal bundles for simplicial Lie groups.

Definition 6.1. A *simplicial Lie group* $G_\bullet$ in C is a simplicial diagram in the category of group objects in C. Denote by sGroup(C) the category of simplicial Lie groups.

Eilenberg and Mac Lane [8] introduced a pair of functors W and $\overline{W}$ from simplicial groups to simplicial sets which generalize the universal bundle and nerve of a group.

Definition 6.2. Let $G_\bullet$ be a simplicial group.

(1) The *total space* $W_\bullet G$ *of the universal* $G_\bullet$*-bundle* is the simplicial set with

$$W_n G := G_0 \times \cdots \times G_n$$
$$d_i(g_0, \ldots, g_n) := (g_0, \ldots, g_{i-2}, g_{i-1} d_i g_i, d_i g_{i+1}, \ldots, d_i g_n)$$
$$s_i(g_0, \ldots, g_n) := (g_0, \ldots, g_{i-1}, e, s_i g_i, \ldots, s_i g_n)$$

(2) The *nerve* $\overline{W}_\bullet G$ of $G_\bullet$ is the simplicial set with

$$\overline{W}_0 G := *,$$

and, for $n > 0$,

$$\overline{W}_n G := G_0 \times \cdots \times G_{n-1}$$
$$d_i(g_0, \ldots, g_{n-1}) := (g_0, \ldots, g_{i-2}, g_{i-1} d_i g_i, d_i g_{i+1}, \ldots, d_i g_{n-1})$$
$$s_i(g_0, \ldots, g_{n-1}) := (g_0, \ldots, g_{i-1}, e, s_i g_i, \ldots, s_i g_{n-1})$$

(3) The assignment which sends $(g_0, \ldots, g_n) \in W_n G$ to $(g_0, \ldots, g_{n-1}) \in \overline{W}_n G$ defines a twisted Cartesian product $W_\bullet G \to \overline{W}_\bullet G$. This is the *universal* $G_\bullet$*-bundle*.

Because C has finite products, the same formulas give functors W and $\overline{W}$ from the category of simplicial Lie groups to the category sC.

Definition 6.3. If $G_\bullet$ is a simplicial Lie group and $X_\bullet$ is a simplicial object in C, then a left *left action* $G_\bullet \circlearrowleft X_\bullet$ consists of maps

$$\begin{aligned} G_k \times X_k &\longrightarrow X_k \\ (g, x) &\longmapsto gx \end{aligned}$$

for each k, such that, in all dimensions and for all i:

$$\begin{aligned} g_1(g_2x) &= (g_1g_2)x, \\ ex &= x, \\ d_i(gx) &= (d_ig)(d_ix), \text{ and} \\ s_i(gx) &= (s_ig)(s_ix). \end{aligned}$$

Right actions are defined analogously.

When $G_\bullet$ and $X_\bullet$ are constant simplicial diagrams, this is the usual notion of a left action of a Lie group. The right action of $G_\bullet$ on itself induces a right $G_\bullet$-action on $W_\bullet G$.

Definition 6.4. Suppose we have a left action $G_\bullet \circlearrowleft X_\bullet$. The *homotopy quotient* $(WG \times_G X)_\bullet$ is defined by

$$\begin{aligned} (WG \times_G X)_n &:= \overline{W}_n G \times X_n \\ s_i(g_0, \dots, g_{n-1}, x) &:= (g_0, \dots, g_{i-1}, e, s_ig_i, \dots, s_ig_{n-1}, s_ix) \\ d_n(g_0, \dots, g_{n-1}, x) &:= (g_0, \dots, g_{n-2}, g_{n-1}d_nx) \end{aligned}$$

and, for $i < n$

$$d_i(g_0, \dots, g_{n-1}, x) := (g_0, \dots, g_{i-1}d_ig_i, \dots, d_ig_{n-1}, d_ix).$$

If $G_\bullet$ acts on $X_\bullet$ and $Y_\bullet$ and $f : X_\bullet \to Y_\bullet$ is $G_\bullet$-equivariant, then f induces a map of homotopy quotients

$$(WG \times_G X)_\bullet \xrightarrow{1 \times_G f} (WG \times_G Y)_\bullet$$

given on n-simplices by $1_{\overline{W}G} \times f_n$.

Definition 6.5. A *strict Lie n-group* is a simplicial Lie group $G_\bullet$ such that the horn-filling maps $\lambda_i^k(G)$ are isomorphisms for $k \geq n$.

We might have defined a strict Lie n-group as a simplicial Lie group such that the maps $\lambda_i^k(G)$ were also covers for all $k < n$. An argument due to Moore shows that this follows from our definition.

Proposition 6.6. *Let $G_\bullet$ be a strict Lie n-group. The simplicial object underlying $G_\bullet$ is a Lie $(n-1)$-groupoid.*

Proof. We perform an induction on the dimension of the horns to show that the maps $\lambda_i^k(G)$ are covers for $k < n$.

Suppose that for $l < k$ and all i, the limit $\Lambda_i^l(G)$ exists and the map $\lambda_i^l(G)$ is a cover. Lemma 2.15 shows that the limit $\Lambda_i^k(G)$ exists for all i.

Sections of $\Lambda_i^k(G) \times G_k$ are tuples

$$((g_0, \dots, -, \dots, g_k), g) \in \Lambda_i^k(G) \times G_k$$

such that, for $j < m \neq i$,

$$d_{m-1} g_j = d_j g_m.$$

We perform an induction on $0 \leq \ell \leq k+1$ to construct sections $g^\ell \in G_k$ such that for $j < \ell$ we have

$$d_j g^\ell = g_j.$$

Fix $((g_0, \ldots, -, \ldots, g_k), g) \in \Lambda_i^k(G) \times G_k$, and set

$$g^0 := g.$$

Now suppose that for $0 \leq \ell$, we have $g^\ell \in G_k$ such that, for $j < \ell$,

$$d_j g^\ell = g_j.$$

We define

$$a_j^\ell := g_j (d_j g^\ell)^{-1} \in G_{k-1}.$$

The horn relations on the g_j ensure that $(a_0^\ell, \ldots, -, \ldots, a_k^\ell)$ defines a section of $\Lambda_i^k(G)$, so we have

$$((a_0^\ell, \ldots, -, \ldots, a_k^\ell), g^l) \in \Lambda_i^k(G) \times G.$$

We define

$$g^{l+1} := (s_\ell a_\ell^\ell) g^\ell.$$

A short exercise shows that, for $j < \ell + 1$,

$$d_j g^{\ell+1} = g_j.$$

This completes the induction step. We now define

$$\begin{array}{ccc} \Lambda_i^k(G) \times G_k & \xrightarrow{\quad\varphi\quad} & \Lambda_i^k(G) \times G_k \\ ((g_0, \ldots, -, \ldots, g_k), g) & \longmapsto & ((g_0(d_0 g^{k+1})^{-1}, \ldots, -, \ldots, g_k(d_k g^{k+1})^{-1}), g^{k+1}) \end{array}$$

By construction, φ is an isomorphism. It factors the projection

$$\Lambda_i^k(G) \times G_k \longrightarrow \Lambda_i^k(G)$$

as

$$\begin{array}{ccc} \Lambda_i^k(G) \times G_k & \xrightarrow[\cong]{\varphi} & \Lambda_i^k(G) \times G_k \\ \downarrow & & \downarrow \pi_{G_k} \\ \Lambda_i^k(G) & \xleftarrow[\lambda_i^k(G)]{} & G_k \end{array}$$

Axiom 3 guarantees that $\lambda_i^k(G)$ is a cover. Lemma 2.15 shows that the limit $\Lambda_i^{k+1}(G)$ exists for all i. This concludes the induction step. □

Observe that a strict Lie 1-group is a Lie group viewed as a constant simplicial diagram. A strict Lie 2-group is a simplicial Lie group $G_\bullet$ such that the simplicial object underlying $G_\bullet$ is the nerve of a Lie groupoid. Therefore, a strict Lie 2-group could be equivalently described as a Lie group in the category of Lie groupoids. Strict Lie 2-groups are relatively abundant. For example, the Lie 2-group associated to a finite-dimensional nilpotent differential graded Lie algebra concentrated in degrees $[-1, 0]$ is a strict Lie 2-group (see [9] for details).

Theorem 6.7.

(1) *The nerve of a strict Lie n-group is a Lie n-group.*
(2) *The homotopy quotient of a $G_\bullet$-equivariant n-stack is an n-stack.*

Proof. Let $G_\bullet$ be a strict n-group, or let $\varphi : X_\bullet \to Y_\bullet$ be a $G_\bullet$-equivariant n-stack. We show that, in the first case, $\overline{W}_\bullet G$ is a Lie n-group, and, in the second, that $1 \times_G \varphi$ is an n-stack.

Denote sections of $\overline{W}_k G$ by

$$\boldsymbol{g}^j = (g_0^j, \dots, g_{k-1}^j) \in G_0 \times \cdots \times G_{k-1} = \overline{W}_k G.$$

Denote sections of $\Lambda_i^k(\overline{W}G)$ by

$$(\boldsymbol{g}^0, \dots, -, \dots, \boldsymbol{g}^k) \in \Lambda_i^k(\overline{W}G).$$

We now give a series of isomorphisms which relate horns in the nerve or homotopy quotient to horns in the strict Lie n-group or equivariant n-stack. The formulas proceed from the observation that the highest face in these horns determines all but the last coordinates of the lower ones; these last coordinates themselves determine a horn in the original simplicial Lie group or $G_\bullet$-map.

For $k > 0$, the maps $\Lambda_i^k(\overline{W}G) \longrightarrow \overline{W}_{k-1}G \times \Lambda_i^{k-1}(G)$ given by

$$(\boldsymbol{g}^0, \dots, \widehat{\boldsymbol{g}^i}, \dots, \boldsymbol{g}^k) \mapsto \begin{cases} (\boldsymbol{g}^k, (g_{k-2}^0, \dots, -, \dots, g_{k-2}^{k-2}, (g_{k-2}^k)^{-1} g_{k-2}^{k-1})) & i < k-1, \\ (\boldsymbol{g}^k, (g_{k-2}^0, \dots, g_{k-2}^{k-2}, -)) & i = k-1, \\ (\boldsymbol{g}^{k-1}, (g_{k-2}^0, \dots, g_{k-2}^{k-2}, -)) & i = k, \end{cases}$$

are isomorphisms. For $k > 0$ and $i < k$, the maps $\Lambda_i^k(WG \times_G \varphi) \longrightarrow \overline{W}_k G \times \Lambda_i^k(\varphi)$ given by

$$\begin{aligned} &(((\boldsymbol{g}^0, x_0), \dots, \widehat{(\boldsymbol{g}^i, x_i)}, \dots, (\boldsymbol{g}^k, x_k)), (\boldsymbol{h}, y)) \\ &\qquad\qquad \mapsto \begin{cases} (\boldsymbol{h}, ((x_0, \dots, \widehat{x}_i, \dots, x_{k-1}, (h_{k-1})^{-1} x_k), y)) & i < k, \\ (\boldsymbol{h}, ((x_0, \dots, x_{k-1}, -), y)) & i = k, \end{cases} \end{aligned}$$

are isomorphisms. For $k > 1$, the isomorphisms for horns in the nerve fit into the following commuting squares:

$$\begin{array}{ccc} \overline{W}_k G & \xrightarrow{\lambda_i^k(\overline{W}G)} & \Lambda_i^k(\overline{W}G) \\ \Vert & & \downarrow \cong \\ \overline{W}_{k-1}G \times G_{k-1} & \xrightarrow[1 \times \lambda_i^{k-1}(G)]{} & \overline{W}_{k-1}G \times \Lambda_i^{k-1}(G) \end{array}$$

$$\begin{array}{ccc} \overline{W}_k G & \xrightarrow{\lambda_k^k(\overline{W}G)} & \Lambda_k^k(\overline{W}G) \\ \Vert & & \downarrow \cong \\ \overline{W}_{k-1}G \times G_{k-1} & \xrightarrow[1 \times \lambda_{k-1}^{k-1}(G)]{} & \overline{W}_{k-1}G \times \Lambda_{k-1}^{k-1}(G) \end{array}$$

For $k = 1$, we have

$$\begin{array}{ccc} \overline{W}_1 G & \xrightarrow{\lambda^1_i(\overline{W}G)} & \Lambda^1_i(\overline{W}G) \\ \| & & \downarrow \cong \\ G_0 & \longrightarrow & * \end{array}$$

Similarly, for $k \geq 1$, the isomorphisms for horns in the homotopy quotients fit into the commuting squares

$$\begin{array}{ccc} WG \times_G X_k & \xrightarrow{\lambda^k_i(1 \times_G \varphi)} & \Lambda^k_i(WG \times_G \varphi) \\ \| & & \downarrow \cong \\ \overline{W}_k G \times X_k & \xrightarrow[1 \times \lambda^k_i(\varphi)]{} & \overline{W}_k G \times \Lambda^k_i(\phi) \end{array}$$

These squares show that the relevant horn-filling maps for $\overline{W}_\bullet G$ and $1 \times_G \varphi$ are covers for all k and isomorphisms for $k > n$. □

Remark 6.8. While we do not need it for this paper, one could define a strict n-stack as a homomorphism of simplicial Lie groups such that the relative horn-filling maps are isomorphisms in dimensions at least n. The analogues of the results above hold in the relative case, with minimal changes to the proofs. One could also make analogous definitions for simplicial Lie groupoids. The analogues of the results above hold, with minimal changes to the proofs.

7. A Finite Dimensional String 2-Group

In this section, we specialize to the category of smooth manifolds and apply our results to construct finite dimensional Lie 2-groups. Let A be an abelian group. For each natural number n, Eilenberg and MacLane introduced a simplicial abelian group $K(A, n)_\bullet$ whose geometric realization represents the cohomology functor $H^n(-; A)$. They further observed that $\overline{W}_\bullet K(A, n)$ is isomorphic to $K(A, n+1)_\bullet$. This construction and identification also exist for abelian Lie groups.

Definition 7.1. Let G be a Lie group. Let A an abelian Lie group.

(1) An *A-valued n-cocycle* on $\overline{W}_\bullet G$ is a span

$$\overline{W}_\bullet G \longleftarrow U_\bullet \longrightarrow K(A, n)_\bullet$$

such that $U_\bullet \to \overline{W}_\bullet G$ is a hypercover.

(2) An *equivalence of cocycles* is a commuting diagram of cocycles

$$\begin{array}{ccccc} & \swarrow & U^0_\bullet & \searrow & \\ & & \uparrow & & \\ \overline{W}_\bullet G & \longleftarrow & V_\bullet & \longrightarrow & K(A, n)_\bullet \\ & & \downarrow & & \\ & \nwarrow & U^1_\bullet & \nearrow & \end{array}$$

such that the maps $V_\bullet \to U^i_\bullet$ are hypercovers.

The connected 2-types $\mathcal{G}_\bullet \in \mathsf{sSmooth}$ which have arisen in the literature are determined by

(1) a Lie group G,
(2) an abelian Lie group A, and
(3) an equivalence class of 3-cocycles

$$\overline{W}_\bullet G \longleftarrow U_\bullet \longrightarrow K(A,3)_\bullet$$

Much work has gone into finding geometric models for smooth 2-types. By pulling back the universal twisted $K(A,2)$-bundle along a cocycle, we obtain a local 2-bundle

$$E_\bullet \longrightarrow U_\bullet$$

The composite

$$E_\bullet \longrightarrow U_\bullet \longrightarrow \overline{W}_\bullet G$$

is an ∞-stack. This shows that connected smooth 2-types can be realized as finite dimensional Lie ∞-groups.

Over the last decade there have been many attempts to do better. The most relevant of these is provided by Schommer-Pries [19] who showed that connected smooth 2-types can be realized as weak group objects in the bicategory of finite dimensional Lie groupoids. Zhu [21], drawing on ideas of Duskin, constructed a nerve for such weak group objects and showed that the nerve is a Lie 2-group.

The tools in this article allow us to construct a Lie 2-group $X_\bullet$ directly from the data above. The object produced is equivalent to the one obtained by Zhu from Schommer-Pries. Our methods extend to $n > 2$.

By Theorem 6.7, Proposition 4.7 specializes to the following.

Corollary 7.2. *Any A-valued n-cocycle*

$$\overline{W}_\bullet G \xleftarrow{\ f\ } U_\bullet \longrightarrow K(A,n)_\bullet$$

factors uniquely through $\tau_n(f)$ as in the diagram

$$\begin{array}{ccccc} \overline{W}_\bullet G & \xleftarrow{\ f\ } & U_\bullet & \longrightarrow & K(A,n)_\bullet \\ & {\scriptstyle \tau_n(f)}\nwarrow & \downarrow & \nearrow & \\ & & \tau_n(U,f)_\bullet & & \end{array}$$

This factorization is an equivalence of cocycles.

We can now use Theorem 5.7 to produce a Lie 2-group from an A-valued 3-cocycle on $\overline{W}_\bullet G$. We can assume, without loss of generality, that the 3-cocycle

$$\overline{W}_\bullet G \xleftarrow{\ f\ } U_\bullet \xrightarrow{\ \varphi\ } K(A,3)_\bullet \tag{7.1}$$

has $U_0 = *$ and f a 3-hypercover. We pull back the universal $K(A,2)$-bundle along φ to obtain a local 2-bundle

$$\varphi^* WK(A,2) \xrightarrow{\ p\ } U_\bullet$$

We descend this local 2-bundle along the 3-hypercover f, as in Theorem 5.7. We obtain a 2-stack

$$X_\bullet := \tau_2(\varphi^* WK(A,2), fp)_\bullet \longrightarrow \overline{W}_\bullet G$$

The object $X_\bullet$ is the desired Lie 2-group.

We now examine $X_\bullet$ in more detail. For simplicity, we ignore degeneracies. The hypercover in the cocycle 7.1 is determined by its 2-skeleton (Proposition 4.6). The 2-skeleton consists of

(1) a cover $f_1 : U \to G$, which we view as a 1-truncated hypercover, and
(2) a cover $f_2 : V \to (\operatorname{cosk}_1 U \times_{\operatorname{Cosk}_1 \overline{W}G} \overline{W}G)_2$.

The Lie 2-group $X_\bullet$ has the same 1-skeleton as this hypercover.

The 2-simplices of $X_\bullet$ are determined by the Lie 1-groupoid $P^{\geq 2}(fp)_\bullet$. Its vertex manifold, $P^{\geq 2}(fp)_0$, is isomorphic to $V \times A$. We showed that

$$\begin{aligned} P^{\geq 2}(fp)_1 &\cong (V \times_{(\operatorname{cosk}_1 U \times_{\operatorname{Cosk}_1 \overline{W}G} \overline{W}G)_2} V) \times K(A,2)_2 \\ &\cong (V \times_{(\operatorname{cosk}_1 U \times_{\operatorname{Cosk}_1 \overline{W}G} \overline{W}G)_2} V) \times A \end{aligned}$$

at the end of the proof of Theorem 5.7. The target map of $P^{\geq 2}(fp)_\bullet$ is given by

$$(v_2, v_3, a) \mapsto (v_2, a)$$

We abuse notation and denote by φ the restriction of the map

$$U_3 \xrightarrow{\varphi} K(A,3)_3$$

to $P^{\geq 2}(f)_1 \subset U_3$. The source in the Lie groupoid $P^{\geq 2}(f)_\bullet$ is given by

$$(v_2, v_3, a) \mapsto (v_3, \varphi(v_2, v_3) + a)$$

The Lie 1-groupoid structure on $P^{\geq 2}(fp)_\bullet$ ensures that φ is an A-valued 1-cocycle on the cover

$$V \longrightarrow (\operatorname{cosk}_1 U \times_{\operatorname{Cosk}_1 \overline{W}G} \overline{W}G)_2$$

The orbit space

$$X_2 \longrightarrow (\operatorname{cosk}_1 U \times_{\operatorname{Cosk}_1 \overline{W}G} \overline{W}G)_2$$

is the principal A-bundle determined by this data.

We now describe the higher simplices. The map $\Lambda^k_i(\tau_2(fp))$ is an isomorphism for $k > 2$ and all i (Proposition 4.5). For $k = 3$, the data of these isomorphisms can be reduced to a trivialization ζ of the bundle

$$d_3^* P^\vee \otimes d_2^* P \otimes d_1^* P^\vee \otimes d_0^* P \longrightarrow (\operatorname{cosk}_1 U \times_{\operatorname{Cosk}_1 \overline{W}G} \overline{W}G)_3$$

Here d_i denotes the face map from 3-simplices to 2-simplices in the simplicial object $(\operatorname{cosk}_1 U \times_{\operatorname{Cosk}_1 \overline{W}G} \overline{W}G)_\bullet$, and $P^\vee$ denotes the dual bundle of P.

Proposition 4.5 implies that $X_\bullet$ is determined by its 3-skeleton. In the present context, this is equivalent to requiring that ζ satisfy a pentagonal coherence condition coming from the 1-skeleton of the 4-simplex.

Summing up, we see that the Lie 2-group $X_\bullet$ is reducible to the following data:

(1) a cover $f \colon U \longrightarrow G$,
(2) a principal A-bundle $P \longrightarrow (\operatorname{cosk}_1 U \times_{\operatorname{Cosk}_1 \overline{W}G} \overline{W}G)_2$, and
(3) a trivialization ζ of

$$d_3^* P^\vee \otimes d_2^* P \otimes d_1^* P^\vee \otimes d_0^* P \longrightarrow (\operatorname{cosk}_1 U \times_{\operatorname{Cosk}_1 \overline{W}G} \overline{W}G)_3$$

which satisfies a pentagonal coherence condition coming from the 1-skeleton of the 4-simplex.

If we take $\mathrm{Spin}(n)$ for G, $U(1)$ for A, and a cocycle representing the fractional first Pontrjagin class $\frac{p_1}{2}$, then the resulting Lie 2-group is the nerve, *à la* Duskin–Zhu, of Schommer-Pries's model for $\mathrm{String}(n)$.

The discussion above gives a construction of higher central extensions of Lie groups. More generally, we might consider higher *abelian* extensions. We consider an abelian Lie group A on which the Lie group G acts by automorphisms. The data which specifies a higher abelian extension of a Lie group, and the construction which produces a Lie 2-group from this data, are analogous to the case of higher central extensions above. We sketch the necessary changes.

For each $n > 0$, the G-action on A induces G-actions on $W_\bullet K(A, n-1)$ and $K(A, n)_\bullet$ such that the universal $K(A, n-1)$-bundle

$$W_\bullet K(A, n-1) \longrightarrow K(A, n)_\bullet$$

is G-equivariant with respect to these actions. We define the *twisted universal bundle*

$$W^G_\bullet K(A, n-1) \longrightarrow K^G(A, n)_\bullet$$

by taking the homotopy quotient of the universal $K(A, n-1)$-bundle with respect to the G-action. An A-valued n-cocycle on $\overline{W}_\bullet G$ is now a commuting triangle

$$\begin{array}{ccc} U_\bullet & \longrightarrow & K^G(A,n)_\bullet \\ & \searrow \quad \swarrow & \\ & \overline{W}_\bullet G & \end{array}$$

such that $U_\bullet \to \overline{W} G_\bullet$ is a hypercover. Equivalences of cocycles are defined analogously. The analogue of Corollary 7.2 holds, and the construction proceeds just as above. The only difference is that, if we unpack the construction of the Lie 2-group $X_\bullet$ produced from this more general notion of cocycle, we observe that instead of the 2-simplices X_2 being the total space of a principal A-bundle, X_2 is now the total space of a G-twisted principal A-bundle. We leave the remaining details to the interested reader.

The methods above additionally allow for the construction of Lie n-groups for $n > 2$. In particular, there exists a finite dimensional model of $\mathrm{Fivebrane}(n)$ as a Lie 7-group extending $\mathrm{String}(n)$ by $K(\mathbb{Z}, 7)$. We expect that a model of $\mathrm{Fivebrane}(n)$ also exists as a Lie 6-group extending $\mathrm{String}(n)$ by $K(U(1), 6)$.

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