Title
Emittance growth due to random force error

Author
Qiang, Ji

Publication Date
2019-12-01

DOI
10.1016/j.nima.2019.162844

Peer reviewed

Emittance growth due to random force error

Ji Qiang

Lawrence Berkeley National Laboratory, Berkeley, CA 94720

Abstract

In references [1] and [2], the authors showed that the emittance growth rate due to the force error of macroparticle sampling will be proportional to the step size of simulation. In this note, we show that for the random force error, the emittance growth rate will be independent of the step size, which is consistent with the simulation observation.

In the Eq. 17 of reference [1] and the Eq. 27 of reference [2], the authors showed that the emittance growth rate due to the error from sampling a smooth particle distribution function with finite number of macroparticles in the particle-in-cell (PIC) simulation is proportional to the step size used in the simulation. This is puzzling since if the step size becomes smaller and tends to zero, the artificial emittance growth rate would become smaller and vanish eventually. Using the same example of Fig. 2 in reference [1], for a linear focusing-drift-defocusing-drift (FODO) lattice, we ran the simulation using $25,000$ macroparticles with the nominal integration step size and half of the nominal step size. The four dimensional emittance growth evolution of the 1 GeV, 30 A current proton beam using the two step sizes is shown in Fig. 1 It is seen that the emittance growths from both step sizes closely follow each other. The emittance growth rate is independent of the step size. In the following, we will show that for the random force error, the artificial emittance growth rate is independent of step size.

In the PIC simulation of high intensity beams, the motion of macroparticle

Preprint submitted to NIMA *September 10, 2019*

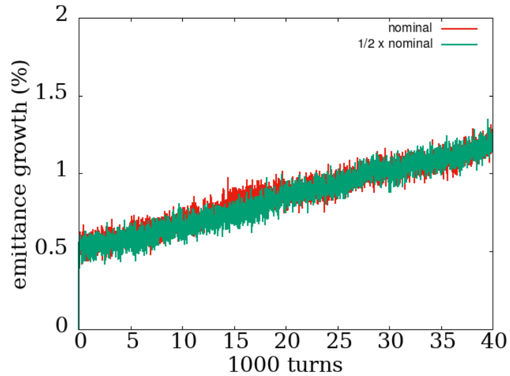

Figure 1: The 4D emittance growth evolution in a FODO lattice using 25 thousand macroparticles with nominal and half nominal step sizes in the simulation.

will be subject to both the force from external focusing/acceleration and the space-charge force from charged particle Coulomb interactions. Due to the use of finite number of macroparticles to sample the particle distribution function and the use of finite number of grid points, the space-charge force calculated from the numerical solution of the Poisson equation in PIC simulation will not be exact and contains a numerical error. Following the concept of splitting operator, here, we focus only on the effect of numerical space-charge force error in the simulation. If the error of space-charge force is a random error, the equations of motion governing macroparticles will be a group of stochastic differential equations. Using the Euler method for stochastic differential equations, after one step τ, the new coordinates (in one-dimension) can be approximated as [3]:

$$x_2 = x_1 \quad (1)$$

$$x_2' = x_1' + \delta F\sqrt{\tau} \quad (2)$$

where δF denotes the random Gaussian force error with zero mean and standard deviation of $\sqrt{<(\delta F)^2>}$. The new emittance under the effect of this force will be:

$$\begin{aligned}
\epsilon_2^2 &= <x_2^2><x_2'^2> - <x_2 x_2'>^2 \\
&= <x_1^2><x_1'^2> - <x_1 x_1'>^2 + 2(<x_1^2><x_1'\delta F> - <x_1 x_1'><x_1 \delta F>)\sqrt{\tau} \\
&\quad +(<x_1^2><\delta F^2> - <x_1 \delta F>^2)\tau \qquad (3)
\end{aligned}$$

where $<>$ denotes the average with respect to the particle distribution. The above equation can be rewritten as:

$$\begin{aligned}
\epsilon_2^2 &= \epsilon_1^2 + 2(<x_1^2><x_1'\delta F> - <x_1 x_1'><x_1 \delta F>)\sqrt{\tau} \\
&\quad +(<x_1^2><\delta F^2> - <x_1 \delta F>^2)\tau \qquad (4)
\end{aligned}$$

Using the assumption that δF is a random Gaussian error and independent of x and x', i.e. $<x\delta F>= 0$, and $<x'\delta F>= 0$, we obtain:

$$\epsilon_2^2 = \epsilon_1^2 + <x_1^2><\delta F^2> \tau \qquad (5)$$

The emittance growth will be:

$$\Delta\epsilon \approx \frac{1}{2}(<x^2><(\delta F)^2> \tau/\epsilon, \qquad (6)$$

and the emittance growth rate will be:

$$\frac{\Delta\epsilon}{\tau} \approx \frac{1}{2}\frac{<x^2><(\delta F)^2>}{\epsilon} \qquad (7)$$

This emittance growth rate due to the random force error is independent of the time step size and is proportional to the variance of the random force error that is inversely proportional to the number of macroparticles used in the simulation.

1. ACKNOWLEDGEMENTS

This work was supported by the U.S. Department of Energy under Contract No. DE-AC02-05CH11231 and used computer resources at the National Energy Research Scientific Computing Center.

References

[1] J. Qiang, Nuclear Inst. and Methods in Physics Research, A 918, p. 1 (2019).

[2] F. Kesting and G. Franchetti, Phys. Rev. ST Accel. Beams 18, 114201 (2015).

[3] T. Sauer, Numerical solution of stochastic differential equations in finance, in Handbook of Computational Finance, edited by J. C. Duan, W. Hardle, and J. Gentle, pp. 529 - 550, Springer, Berlin, (2012).