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On the Computability of Immersions

A dissertation submitted in partial satisfaction
of the requirements for the degree

Doctor of Philosophy
in
Mathematics

by

Daniel Patrick Epelbaum

Committee in charge:

Professor Fedor Manin, Chair
Professor Stephen Bigelow
Professor Daryl Cooper

September 2024

The Dissertation of Daniel Patrick Epelbaum is approved.

Professor Stephen Bigelow

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July 2024

On the Computability of Immersions

This thesis dedicated to those who perished on the climb

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Abstract

On the Computability of Immersions

by

Daniel Patrick Epelbaum

Suppose someone hands you a pair of smooth manifolds, and a smooth map between them. Can you decide whether there is an immersion homotopic to this map? We will reduce this to a question about lifting a certain bundle, and then use rational homotopy theoretic techniques to provide an algorithm when the codimension of the manifolds is odd.

Contents

Chapter 1

Introduction

Given a pair of smooth manifolds, when can we immerse one in the other? A lot is known in the case of immersibility into $\mathbb{R}^n$. The Whitney immersion theorem tells us that any manifold of dimension m can be immersed in R^{2m-1}. In 1985, Cohen strengthened this result proving the immersion conjecture: any manifold of dimension m can be immersed in $\mathbb{R}^{2m-\alpha(m)}$ where $\alpha(m)$ is the number of 1s in the binary expansion of m [1]. This bound is tight: for any m there are manifolds of dimension m that cannot be immersed in dimension $2m - \alpha m$.

In smaller codimension, we can ask when questions of immersibility can be answered by a computer algorithm. For example, [2] investigates immersibility of both smooth and PL manifolds in $\mathbb{R}^n$, and concludes that for smooth manifolds, immersibility of an m-manifold into $\mathbb{R}^n$ is decidable when $n-m$ is odd, and it is this result we will generalize in this thesis.

More generally, we might be interested in the question of when M can be immersed in N, for arbitrary smooth manifolds M, N. Here we might hope to compute the set of immersions, up to regular homotopy (homotopy through immersions.) Here we will end up with difficulties arising from the difficulty of studying homotopy classes of maps

between manifolds in general. To control this we will discuss a modification of the problem in which we are looking for immersions with prescribed homotopy behavior. That is, we will consider the following decision problem: given a pair of oriented closed smooth manifolds (M, N) and a smooth map $f : M \to N$, is there an immersion homotopic to f? The main result of this thesis will be a generalization of the above result for this problem. In particular we prove the following theorem:

Theorem 1.0.1. *There is an algorithm that on input (M, N, f), where M, N are smooth oriented manifolds of dimension m, n respectively and $f : M \to N$ is a smooth map between them, decides whether there is an immersion $g : M \to N$ such that $g \simeq f$, as long as $n - m$ is odd.*

This proof will have several steps. First, we use the h-principle of Hirsch and Smale to reduce the question to a homotopy-theoretic lifting problem.

To decide on the existence of a lift then, we will use some tools from rational homotopy theory. The idea here is that if we ignore finite homotopy group obstructions, we can put an algebraic structure on the set of possible lifts on each stage of the relevant Moore-Postnikov tower, and this allows us to construct a lift, if one exists, using obstruction theory. The algorithm here is very much like that in [3].

In the next section, we will discuss briefly the notion of a decision problem and review some of the building blocks of our homotopy-theoretic computation. Then, in chapter 2, we introduce the tools we will need from rational homotopy theory, as well as define the notion of a *rational fibrewise HM-space*. In chapter 3, we will use these tools to provide an algorithm for a certain class of homotopy-lifting problem. Finally, in chapter 4, we will see how to apply this to investigate the decidability of immersions.

1.1   Computational Preliminaries

In this thesis, when we look at the question of what a computer can decide, we are concerned only with the question of what is computable, not what is *effectively computable*. This definition is quite robust under changes of computational model (see [4]). In fact, in the algorithm of chapter 3, we will have several instances of choosing an integer for which we know *a* choice exists but with no particular upper bound, and this will make any attempt to discuss efficiency or runtime nearly impossible in principle. We will therefore proceed in building our algorithm out of known algorithms without paying attention to their runtimes.

We will use as building blocks for our algorithm the following results from computational homotopy theory. Proofs for each of these can be found in [2]

Theorem 1.1.1. *(a) There is an algorithm which on input a simplicial complex X can compute generators and relations for $\pi_k(X)$, as well as simplicial representatives for each generator*

(b) There is an algorithm which on input a map of simply connected finite simplicial complexes $Y \to B$ computes the relative Moore-Postnikov tower to any finite stage, as well as the cohomology of each stage, and the maps of cohomology induced by each $P_n \to P_{n-1}$.

(c) There is an algorithm such that given a diagram

$$\begin{array}{ccc} A & \longrightarrow & P_n \\ \big\uparrow & \nearrow & \big\downarrow \\ X & \longrightarrow & P_{n-1}, \end{array}$$

where (X, A) is a finite simplicial pair and $P_n \twoheadrightarrow P_{n-1}$ is a Moore-Postnikov stage,

computes the obstruction to filling in the dotted arrow (in $H^n(X, A; \pi_n(P^n))$) and if the obstruction vanishes, constructs a lifting extension.

We will also need to specify how we will model smooth manifolds algorithmically. In general, it is not decidable whether a given n-dimensional simplicial complex is homeomorphic to a smooth manifold. Again following [2], we will input a manifold as a finite simplicial complex together with a choice of polynomial map with rational coefficients to $\mathbb{R}^N$ for each top dimensional simplex, fixing some large N, and such that the derivatives of each map are nonsingular and agree on the boundaries of adjacent simplices. This gives a C^1 triangulation, and since the category of C^1-manifolds is equivalent to the category of C^∞-manifolds, and C^1 immersions can be approximated by C^∞ ones, this is sufficient. We also note that given such a collection of data, whether or not it represents a manifold is decidable.

We will need one more theorem from [2] which will help convert our geometric problem into a homotopy theoretic one.

Theorem 1.1.2. *Given a manifold M as above, there is an algorithm that computes the classifying map of the tangent bundle. In particular, it is possible to compute a simplicial complex structure on $BSO(n)$, and a simplicial approximation of the classifying map.*

Chapter 2

Rational Homotopy Theory

2.1 Relative Minimal Models

The main tools we will need from rational homotopy theory are the notion of a minimal model, and its relative version, developed by Sullivan in [5]. For a more thorough introduction, see [6]. The key idea is that given a simply connected CW complex X, we can construct a rational graded commutative differential algebra in a way that institutes a duality between the category of simply connected CW complexes up to rational equivalence and the category of 1-connected rational commutative differential graded algebras (dgas) up to quasi-isomorphism. In particular, we construct the *minimal model* $\mathcal{M}_X$ as follows. We begin with a graded vector space W whose $i^{(th)}$ graded piece $W^{(i)}$ is given by $(\pi_i(X) \otimes \mathbb{Q})^*$. Note that since we started with a simply connected space, the resulting vector space is trivial below dimension 2, and the resulting dga is said to be 1-*connected* or *simply connected.* Then as an algebra, the minimal model of X is $\wedge W$, the free graded commutative algebra generated by W. Then a differential on $\mathcal{M}_X$ is determined by maps $d_i : W^{(i)} \to \mathcal{M}_X^{(i+1)}$, which can be constructed by dualizing the k-invariants of the Postnikov tower, after tensoring with $\mathbb{Q}$. Crucially this will result in maps which land in the

subalgebra generated by elements of $W^{(k)}$ for $k < i$, and so for each $w \in W^{(i)}$ we have $d(w) \in \wedge^{\geq 2} \oplus_{j=2}^{i-1} W^{(j)}$. Such a dga is said to be *minimal*.

A space is modelled by a minimal model, and a fibration is modelled by a *relative minimal model*. Suppose we have a fibration of simply connected CW-complexes $p : Y \twoheadrightarrow B$, with fiber F which is also simply connected. Then we can construct a *relative minimal model*. This consists of the following:

- $\mathcal{M}_B = (\wedge W_B, d_B)$ a minimal model for the base space B

- A graded vector space W_F where each ith graded piece is given by $(\pi_i(F) \otimes \mathbb{Q})^*$

- A differential d on $\mathcal{M}_X \otimes_{\mathbb{Q}} \wedge W_F$ which restricts to d_B under the standard inclusion $\mathcal{M}_X \to \mathcal{M}_X \otimes_{\mathbb{Q}} \wedge W_F$ given by $x \mapsto x \otimes 1$.

Again this should be *minimal* which amounts to that $d(W^{(i)}) \subset \mathcal{M}_X \otimes_{\mathbb{Q}} \wedge \oplus_{j=2}^{i-1} W_F^{(j)}$, and for this to be a model of the fibration the cohomology of the dga $(\mathcal{M}_X \otimes \wedge W_F, d)$ should be the cohomology of Y. Similar to the minimal model, the differential can be constructed by dualizing the Moore-Postnikov tower.

We prove a few lemmas.

Lemma 2.1.1. *Let (A, d) be a minimal rational finitely generated simply connected dga. Suppose we are given a set of k equations*

$$dx_i = a_i + \sum_{j=1}^{k} a_j^i x_j$$

with each a_i and $a_j^i \in A$ and a prescribed sequence $2 \leq n_1 \leq ... \leq n_k$. Then there is an algorithm to determine the (possibly empty) affine space of solutions with the condition $|x_i| = n_i$, presented as a solution $s = (s_1, ..., s_k)$ and a basis of a space W such that the affine space $s + W$ is the set of solutions in the $\mathbb{Q}$-vector space $V = \oplus_{i=1}^{k} A^{(n_i)}$.

Proof. This is simply a matter of solving a matrix equation. Indeed, with V as given in the statement of the lemma, we construct a pair of maps $D, T : V \to \oplus_{i=1}^k A$ as follows: on $v = (v_1, \ldots, v_k)$ define

$$Dv = (dv_1, ..., dv_k)$$

$$Tv = (\sum_{i=1}^k a_i^1 v_i, ..., \sum_{i=1}^k a_i^k v_i)$$

and set

$$C = (a_1, ..., a_k)$$

so that the system of equations is equivalent to

$$Dv = Tv + C$$

which we can now turn into a matrix equation. In particular, as each $A^{(n_i)}$ can be represented as a rational vector space, as can A itself, and since the differential is $\mathbb{Q}$-linear, so is D, as is T by construction. Then picking a basis for V and A as $\mathbb{Q}$-vector spaces, we have the above equation as a matrix equation, and the space of solutions can be constructed via row reduction. $\qquad\qquad\square$

Given a square

$$
\begin{array}{ccc}
\mathcal{M}_A & \longleftarrow & \mathcal{M}_Y \\
\uparrow & & \uparrow \\
\mathcal{M}_X & \longleftarrow & \mathcal{M}_B
\end{array}
$$

and a relative minimal model $(\mathcal{M}_B \otimes \wedge W, d)$ with a differential d which is linear through dimension n, where n is the cohomological dimension of a CW pair (X, A) for which $\mathcal{M}_X$ and $\mathcal{M}_A$ are minimal models, and $\mathcal{M}_X \to \mathcal{M}_A$ models the inclusion $A \hookrightarrow X$.

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Taking the pushout of bottom right triangle, and replacing $\mathcal{M}_Y$ with the given relative model we arrive at the following triangle, for which we would like to construct the dashed line:

$$
\begin{array}{ccc}
 & \mathcal{M}_X \otimes \wedge W & \\
 {}^{f^*}\nearrow & \quad \uparrow \phi^* & \\
\mathcal{M}_A \xleftarrow{\quad i^* \quad} & \mathcal{M}_X &
\end{array}
$$

Lemma 2.1.2. *Given a diagram as above, there is an algorithm to determine whether there exists a map ψ such that $i^* \circ \psi = f^*$.*

Proof. Fix a basis for W, written as a choice of elements $w_j^{(i)}$ where i ranges over the positive dimensional degrees, and j ranges from 1 to the dimension of $W^{(i)}$. Constructing a map for the dashed line as in the above diagram is simply a matter of picking a target for each element $w_j^{(i)}$, as $\mathcal{M}_X \otimes \wedge W$ is the free commutative graded $\mathcal{M}_X$-algebra. The only conditions then to check on such a map ψ are $d_{\mathcal{M}_X} \circ \psi = \psi \circ d$ and $f^* = i^* \circ \psi$. Then for each $w_j^{(i)}$ we have the equations $d_{\mathcal{M}_X}(\psi(w_j^{(i)})) = \psi(d(w_j^{(i)}))$ and $f^*(w_j^{(i)}) = i^*(\psi(w_j^{(i)}))$.

By assumption, d is linear through the cohomological dimension of (X, A), and in higher dimensions there is a unique lift of any element in $\mathcal{M}_A$. Then we have a fixed $\psi(w_j^{(i)})$ for $i > n$.

Otherwise we know that $dw_j^{(i)}$ takes the form

$$
m_j^i + \sum_{k < i, 1 \leq k \leq dim(W^{(i)})} m_j^{i,k,l} w_l^{(k)}
$$

where each m_j^i and $m_j^{i,k,l} \in \mathcal{M}_X$. Putting all of this together then, we obtain the set of equations:

$$
d_{\mathcal{M}_X}(\psi(w_j^{(i)})) = m_j^i + \sum_{k < i, 1 \leq k \leq dim(W^{(i)})} m_j^{i,k,l} \psi(w_l^{(k)})
$$

and by lemma 2.1.1 we can construct the affine space of solutions to these equations as a subspace of $V = \oplus_{i,j} A^{(i)}$ (note that we have a copy of $A^{(i)}$ for each generator in $W^{(i)}$.) We can call this space S.

We consider the affine space $\tilde{V} \subset V$ given by:

$$\oplus_{i,j} i^{*-1}(f^*(w_j^{(i)}))$$

and we simply have to compute $S \cap \tilde{V}$, which is the intersection of two affine subspaces of a $\mathbb{Q}$-vector space, which can be computed.

$\square$

2.2   Rational Fibrewise HM-Spaces

Before we are able to prove the lifting result, we will need two different generalizations of the notion of an H-space.

Definition 2.2.1. A fibrewise H-space is a fibration $p : Y \twoheadrightarrow B$ with a section $e : B \to Y$ and a multiplication map $m : Y \times_B Y \to B$ which is associative up to fibrewise homotopy, and for which the section acts as an identity, that is the maps $m \circ (\mathrm{id} \times (e \circ p)) \circ \Delta$ and $m \circ ((e \circ p) \times \mathrm{id}) \circ \Delta$ are fibrewise homotopic to the identity on Y, where $\Delta : Y \to Y \times_B Y$ is the diagonal map.

We note that this is stronger than simply requiring an H-space structure on each fiber—the fibers need to have an H-space in some strong uniform sense. We want to weaken this definition slightly however, to include spaces which may not have a section. A motivating example here is the Hopf fibrations. Given a map $f : X \to S^4$ from some manifold X, the lifting algorithm we construct will decide whether the map f can be lifted across the fibration $p : S^7 \twoheadrightarrow S^4$ with fiber S^3, despite the fact that this doesn't

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admit a fibrewise H-space structure. This fibration admits an H-space structure on each fiber, and this choice is continuous in a precise sense: for any map $f : X \to S^4$ for which the pullback of p across f admits a section, the H-space structures on each fiber can be chosen on the pullback so that the section is the identity. We then want to think of this Hopf fibration as a sort of *group without identity*. One way of looking at a group without identity is to look at the notion of a heap.

Definition 2.2.2 (Heap). A *Mal'cev operation* on a set H is a ternary operation τ which satisfies

- $\forall a, b, c, d, e \in H, \tau(\tau(a, b, c), d, e) = \tau(a, \tau(b, c, d), e) = \tau(a, b, \tau(c, d, e))$

- $\forall a, b \in H, \tau(a, a, b) = b = \tau(b, a, a)$

The first of these is a kind of associativity, and the second is often referred to as the Mal'cev condition. A *Heap* is a set with a Mal'cev operation.

If (H, τ) is a heap, it is a straightforward exercise to check that for any $e \in H$, the operation $a * b = \tau(a, e, b)$ turns H into a group. On the other hand if we have a group $(G, *)$ the operation $\tau(a, b, c) = a * b^{-1} * c$ is a Mal'cev operation which turns the group into a heap. It is natural then to consider a heap to be a group with a forgotten identity element. The choice of any distinguished element recovers the structure of a group.

If an H-space is a 'grouplike' space, we want to look at something 'heaplike.' This motivates the next definition.

Definition 2.2.3. A fibrewise HM-space (for Hopf-Mal'cev) is a fibration $p : Y \twoheadrightarrow B$ together with a fibrewise homotopy Mal'cev operation, i.e. a map $\tau : Y \times_B Y \times_B Y \to Y$ for which the following diagrams are commutative, up to fibrewise homotopy:

$$Y \times_B Y \times_B Y \times_B Y \times_B Y \xrightarrow{\tau \times id \times id} Y \times_B Y \times_B Y$$

$$id \times id \times \tau \Big\downarrow \qquad\qquad\qquad\qquad \Big\downarrow \tau$$

$$Y \times_B Y \times_B Y \xrightarrow{\qquad\qquad \tau \qquad\qquad} Y$$

$$Y \times_B Y \xrightarrow{id \times \Delta} Y \times_B Y \times_B Y \xleftarrow{\Delta \times id} Y \times_B Y$$

$$\pi_1 \qquad\qquad \Big\downarrow \tau \qquad\qquad \pi_2$$

$$Y$$

where π_i denotes projection onto the $i^{(th)}$ coordinate. These are exactly the same axioms as for a heap of sets, we simply require only that they commute up to fibrewise homotopy rather than on the nose.

On sets, the choice of an identity element turns a heap into a group. An analogous result holds here.

Lemma 2.2.4. *Let* $p : Y \twoheadrightarrow B$, τ *be a fibrewise HM-space, and let* $e : B \to Y$ *be a section of* p*. Then with multiplication map* $m : Y \times_B Y \to Y$ *given by* $\tau \circ (\mathrm{id} \times (e \circ p) \times \mathrm{id}) \circ (id \times \Delta)$*,* p *is a fibrewise H-space.*

Proof. We begin by observing that since m is defined on elements of the fibrewise product $(a, b) \in Y \times_B Y$, we have that $p(a) = p(b)$ so that we have $(\mathrm{id} \times (e \circ p) \times \mathrm{id}) \circ (\mathrm{id} \times \Delta) = (\mathrm{id} \times (e \circ p) \times \mathrm{id}) \circ (\Delta \times \mathrm{id})$. Then checking associativity of multiplication is a straightforward calculation. Indeed we have:

$$m \circ (m \times \mathrm{id}) =$$

$$\tau \circ (\mathrm{id} \times (e \circ p) \times \mathrm{id}) \circ (\mathrm{id} \times \Delta) \circ ((\tau \circ (\mathrm{id} \times (e \circ p) \times \mathrm{id}) \circ (\mathrm{id} \times \Delta)) \times \mathrm{id}) =$$

$$\tau \circ (\tau \times \mathrm{id} \times \mathrm{id}) \circ (\mathrm{id} \times (e \circ p) \times \mathrm{id} \times (e \circ p) \times \mathrm{id}) \circ (\mathrm{id} \times \Delta \times \Delta)$$

simply by substitution, and then applying the commutativity of operations on different copies of the product. Since the fibrewise Mal'cev operation has to also be homotopy

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associative, this is homotopic to

$$\tau \circ (\mathrm{id} \times \mathrm{id} \times \tau) \circ (\mathrm{id} \times (e \circ p) \times \mathrm{id} \times (e \circ p) \times \mathrm{id}) \circ (\mathrm{id} \times \Delta \times \Delta)$$

Applying the equality above we have that this is equal to

$$\tau \circ (\mathrm{id} \times \mathrm{id} \times \tau) \circ (\mathrm{id} \times (e \circ p) \times \mathrm{id} \times (e \circ p) \times \mathrm{id}) \circ (\Delta \times \mathrm{id} \times \Delta)$$

and finally this can be rewritten as

$$\tau \circ (\mathrm{id} \times (e \circ p) \times \mathrm{id}) \circ (\Delta \times \mathrm{id}) \circ (\mathrm{id} \times m)$$

and again applying the above equality we have this is equal to

$$\tau \circ (\mathrm{id} \times (e \circ p) \times \mathrm{id}) \circ (\mathrm{id} \times \Delta) \circ (\mathrm{id} \times m) = m \circ (\mathrm{id} \times m)$$

To check that the section acts as an identity, we first observe that $((e \circ p) \times \mathrm{id}) \circ \Delta \circ (e \circ p) = \Delta \circ (e \circ p) = (\mathrm{id} \times (e \circ p)) \circ \Delta \circ (e \circ p)$ since $(e \circ p) \circ (e \circ p) = (e \circ p)$ and for any map f we have $(f \times f) \circ \Delta = \Delta \circ f$.

Now we compute

$$m \circ (\mathrm{id} \times (e \circ p)) \circ \Delta =$$
$$\tau \circ (\mathrm{id} \times (e \circ p) \times \mathrm{id}) \circ (id \times \Delta) \circ (\mathrm{id} \times (e \circ p)) \circ \Delta =$$
$$\tau \circ (id \times \Delta) \circ (\mathrm{id} \times (e \circ p)) \times \Delta$$

and by the axioms of a fibrewise HM-space, $\tau \circ (\mathrm{id} \times \Delta)$ is fibrewise homotopic to π_1

and the above map is fibrewise homotopic to

$$\pi_1 \circ (\mathrm{id} \times (e \circ p)) \circ \Delta = \mathrm{id}$$

An analogous argument shows that $m \circ ((e \circ p) \times \mathrm{id}) \circ \Delta$ is also fibrewise homotopic to the identity, and hence p with section e and multiplication m is a fibrewise H-space.

$\square$

When dealing with fibrations, we have two different ways of generalizing rationalizations. The first is to work with the rationalization of the base and total space. That is, given a fibration $p : Y \to B$ we can construct a rationalization $p_{\mathbb{Q}} : Y_{\mathbb{Q}} \to B_{\mathbb{Q}}$. This satisfies that for any commutative square

$$\begin{array}{ccc} Y & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ B & \longrightarrow & B' \end{array}$$

where Y', B' are rational spaces, there is a (unique up to homotopy) factorization through the rationalization. If we take this fibration and pull back along the rationalization $p : B \to B_{\mathbb{Q}}$ we get a fibration which we will denote $p^{\mathbb{Q}} : Y^{\mathbb{Q}} \twoheadrightarrow B$ which is called the 'fibrewise rationalization' of p. It satisfies a universal property in the category of fibrations over B: given a map $f : Y \to R$ over B, where $R \twoheadrightarrow B$ is a fibration with fiber a rational space, f factors uniquely up to homotopy through the fibrewise rationalization. In particular, the homotopy groups of the fiber are rationalizations of the homotopy groups of the fiber of p.

We will need the following lemma about fibrewise rationalizations.

Theorem 2.2.5. *Let $p : Y \twoheadrightarrow B$ be a fibration of simply connected spaces with simply connected fiber, with a relative minimal model $(M_B \otimes M_F, d)$ which is linear through dimension k. Then for any $n \le k$ the nth Moore-Postnikov stage $L_n^{\mathbb{Q}}$ of the fibrewise*

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rationalization $p^{\mathbb{Q}} : Y^{\mathbb{Q}} \twoheadrightarrow B$ is a fibrewise HM-space, and the maps $L_n^{\mathbb{Q}} \twoheadrightarrow L_{n-1}^{\mathbb{Q}}$ are all fibrewise HM-maps, as are the classifying maps $k_n : L_{n-1} \to B \times K(\pi_n(F) \otimes \mathbb{Q}, n+1)$.

Proof. We will proceed in two stages. We start by showing the rationalization $p_{\mathbb{Q}} : Y_{\mathbb{Q}} \twoheadrightarrow B_{\mathbb{Q}}$ is a fibrewise HM-space, and then we lift that structure to the fibrewise rationalization. To see that the rationalization is a fibrewise HM-space is straightforward: starting with the relative minimal model, we can construct a coMal'cev operation. In particular we have the relative minimal model presented as $M_B \otimes \wedge W_F$ where W_F is the graded vector space built out of the rational homotopy groups of F. The linearity condition on the differential ensures that the map $M_B \otimes \wedge W_F \to (M_B \otimes \wedge W_F)^{\otimes_{M_B} 3}$ induced by the map sending $w \in W$ to $w \otimes 1 \otimes 1 - 1 \otimes w \otimes 1 + 1 \otimes 1 \otimes w$ is a map of dgas. This map satisfies properties dual to those of the Mal'cev operation, and so it gives us a fibrewise heap structure on the rationalization. We want to first show that the k-invariants on each Postnikov stage of the rationalization are HM-maps. The spaces $K(\pi_n(F) \otimes \mathbb{Q}, n+1)$ are H-spaces, and so the operation $(a, b, c) = a - b + c$ endows it with the structure of an HM-space. This gives the trivial $B_{\mathbb{Q}}$ fibration $B_{\mathbb{Q}} \times K(\pi_n(F) \otimes \mathbb{Q}, n+1)$ the structure of a relative HM-space. A simple computation shows that the coMal'cev operation this induces on the relative minimal model on this fibration is precisely the map sending α to $\alpha \otimes 1 \otimes 1 - 1 \otimes \alpha \otimes 1 + 1 \otimes 1 \otimes \alpha$, from which it follows that the k-invariant on the corresponding stage of the Moore-Postnikov tower is an HM-map.

$$
\begin{array}{ccc}
L_{n,\mathbb{Q}} & \longrightarrow & B \times E(\pi_n(F) \otimes \mathbb{Q}, n) \\
\downarrow & \lrcorner & \downarrow \\
L_{n-1,\mathbb{Q}} & \xrightarrow{k_{n,\mathbb{Q}}} & B \times K(\pi_n(F) \otimes \mathbb{Q}, n+1)
\end{array}
$$

Because the maps on the bottom and right of this pullback square are HM-maps, so is the map $L_{n,\mathbb{Q}} \to L_{n-1,\mathbb{Q}}$. Them we need to show that this structure pulls back appropriately to the fibrewise rationalization. In particular since the fibrewise rationalization

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is a pullback, we can define the Mal'cev operation on $L_n^{\mathbb{Q}}$ by pulling back the operation on the rationalization:

$$
\begin{array}{ccc}
(L_n^{\mathbb{Q}})^{\times_B 3} & \longrightarrow & L_{n,\mathbb{Q}}^{\times_{B_{\mathbb{Q}}} 3} \\
& \searrow^{\tau_n} & \downarrow^{\tau_{n,\mathbb{Q}}} \\
& L_n^{\mathbb{Q}} \longrightarrow L_{n,\mathbb{Q}} \\
& \downarrow & \downarrow \\
& B \longrightarrow B_{\mathbb{Q}}
\end{array}
$$

The associativity and Mal'cev properties are satisfied trivially since this is a pullback of a fibrewise HM-space structure on the rationalization.

$\square$

Finally, we will need one more technical lemma about fibrewise H-spaces. This is a fibrewise version of lemma 3.6 in [3] which tells us that the multiplication by r map on a fibrewise H space kills torsion elements of cohomology.

Lemma 2.2.6. *Let $p : H \to B$ be a fibrewise H-space of finite type, A a finitely generated coefficient group, and $\alpha \in H^{(n)}(H; A)$ a cohomology class with the property that $t\alpha \in p^*(H^n(B; A))$ for some positive t. Then there is an $r > 0$ such that $\chi_r^* \alpha \in p^*(H^n(B; A))$, where $\chi_r : H \to H$ is the 'multiplication by r map,' i.e. $\chi_2(a) = m(a, a)$, and $\chi_n(a) = m(a, \chi_{n-1}(a))$.*

Proof. Note that in the language of the Serre spectral sequence we can rewrite this condition as saying that we have an element in $\alpha \in \bigoplus_{p+q=n} E_\infty^{p,q}$ such that $t\alpha \in E_\infty^{n,0}$ for some t. Then it suffices to show the following: suppose we have some torsion element of $\beta \in E_\infty^{p,q}$ where $q \geq 1$, then there exists some r such that $\chi_r^* \alpha = 0$. Indeed, since the direct sum above is finite, then we can take the direct sum decomposition of the element α into a finite sum of terms β_q in $E_{p,q}^\infty$, for which some r_q will suffice, and then $\chi_{r_1 \ldots r_n}$

will kill all but β_0 as desired. Recall that the E_1 page of the Serre spectral sequence has terms $E_1^{p,q} = C^p(B; H^q(F; A))$.

Suppose then that we have some torsion cohomology class β as above, there is some corresponding cocycle γ which survives to $E_\infty^{p,q}$, but $t\gamma$ does not survive. By lemma 3.5 of [3] we know that there is some s such that $\chi_s^*(H^q(F; A)) \subseteq tH^q(F; A)$, from which we can conclude that $\chi_s^*(C^p(B; H^q(F; A))) \subseteq tC^p(B; H^q(F; A))$, and hence $\chi_s^*(\gamma)$ does not survive to $E_\infty^{p,q}$, as desired.

$\square$

Chapter 3

The Lifting Algorithm

We are now ready to provide the algorithm for a specific collection of homotopy lifting problems.

In particular, we have the following theorem:

Theorem 3.0.1. *There is an algorithm that takes as input:*

- *simply connected finite type CW complexes X, B, Y*

- *a fibration $p : Y \twoheadrightarrow B$ with simply connected fiber and a map $f : X \to B$ as in the diagram below*

$$
\begin{array}{ccc}
 & & Y \\
 & & \downarrow p \\
X & \xrightarrow{\ f\ } & B
\end{array}
$$

- *a relative minimal model $(M_B \otimes M_F, d_L)$ for $p : Y \twoheadrightarrow B$ whose differential d_L is linear through the dimension of X*

and decides whether there exists a lift $g : X \to Y$ of f.

Proof. Let d be the dimension of X. To simplify the proof a bit, we pullback p across f to obtain a fibration $\hat{p} : \hat{Y} \to X$, for which we will determine if a section exists.

We will denote by P_n the nth Moore-Postnikov stage of $\hat{p} : \hat{Y} \twoheadrightarrow X$. The linear relative minimal model pulls back to model $\hat{p}$ so we still have a relative minimal model for this fibration which is linear through the dimension of X. Using this minimal model, we can apply lemma 2.1.2 to construct a map of dgas dual to a lift. Here the algorithm might fail to find a lift, in which case we know none exists. Indeed, if a lift existed, applying the equivalence would produce a dual map on dgas, so that if no such map exists, no section can exist.

We assume then that we found such a map. In particular we have $\epsilon : \mathcal{M}_X \otimes \wedge W_F \to \mathcal{M}_X$ which is a map of dgas, where $\mathcal{M}_X \otimes \wedge W_F$ with differential d_L^* is the relative minimal model of $\hat{p}$.

We will attempt to construct a section for $\hat{p}$ inductively as follows: for each n through dimension d we will construct a fibration $h_n : L_n \twoheadrightarrow X$, as well as a section e_n and rational equivalences $\phi_n : L_n \to P_n$, $\theta_n : P_n \to L_n$ over X, where the composition $\theta_n \circ \phi_n$ is multiplication by some integer s_n under the fibrewise H-space structure on L_n with the section e_n. In particular, these will be constructed so that each h_n is a fibrewise HM-space with operation τ_n, and there are maps $r_n : L_n \to L_{n-1}$ which commute with the fibrewise HM-structure and form a Moore-Postnikov tower for h_d. At each stage we will also fix a fibrewise rationalization u_n to the corresponding Moore-Postnikov stage $L_n^{\mathbb{Q}}$ which is isomorphic to $P_n^{\mathbb{Q}}$. By lemma 2.2.5 each $L_n^{\mathbb{Q}}$ has the structure of a fibrewise HM-space, and we will ensure that the maps u_n are each HM-maps.

At each stage, in the construction of ϕ_n, it is possible for the construction to fail, and this indicates that no section of $\hat{p}$ exists. We will see why when we construct ϕ_n.

Since the space is simply connected, we set $h_1 : X \twoheadrightarrow X$ to the identity and the other data is trivial.

Then we assume we have constructed h_{n-1}, e_{n-1}, ϕ_{n-1}, θ_{n-1} and u_{n-1}. We start by constructing the map h_n. We do this by picking $L_n \twoheadrightarrow L_{n-1}$ as a $K(\pi_n(F), n)$ fibration. This will then serve as the top stage on the Moore-Postnikov tower for $h_n : L_n \twoheadrightarrow X$. This is equivalent to picking a classifying map in $[L_{n-1}, K(\pi_n(F), n+1)]_B$, or alternatively, a cohomology class in $H^{n+1}(L_{n-1}; \pi_n(F))$. By the universal coefficient theorem this is isomorphic to

$$\mathrm{Hom}(H_{n+1}(L_{n-1}), \pi_n(F)) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(H_n(L_{n-1}), \pi_n(F))$$

From the relative minimal model of $\hat{p}$ we have a map

$$d_L^*|_{W_F^{(n)}} : W_F^{(n)} \to H^{n+1}(L_{n-1}; \mathbb{Q})$$

and since $W_F^{(n)}$ is simply $(\pi_n(F) \otimes \mathbb{Q})^*$ or equivalently $\mathrm{Hom}(\pi_n(F), \mathbb{Q})$ we have an element of

$$\mathrm{Hom}(\mathrm{Hom}(\pi_n(F), \mathbb{Q}), H^{(n+1)}(L_{n-1}; \mathbb{Q}))$$

which we will denote $k_{n,\mathbb{Q}}$.

Now since $\mathrm{Ext}(H_n(X), \mathbb{Q})$ is trivial, the universal coefficient theorem lets us view the above as isomorphic to

$$\mathrm{Hom}(\mathrm{Hom}(\pi_n(F), \mathbb{Q}), \mathrm{Hom}(H_{(n+1)}(L_{n-1}), \mathbb{Q}))$$

by taking the dual map then, we obtain an element

$$\mathrm{Hom}(H_{(n+1)}(L_{n-1}) \otimes \mathbb{Q}, \pi_n(F) \otimes \mathbb{Q})$$

From this element $d_L^*|_{W_F^{(n)}} \in \mathrm{Hom}(H_{n+1}(L_{n-1}) \otimes \mathbb{Q}, \pi_n(F) \otimes \mathbb{Q})$ we will construct an element of $\mathrm{Hom}(H_{n+1}(L_{n-1}), \pi_n(F))$. In particular, there is a natural map $H_{n+1}(L_{n-1}) \to$

$H_{n+1}(L_{n-1} \otimes \mathbb{Q})$ given by $a \mapsto a \otimes 1$, and so by composing with this map we can construct $\widetilde{d_L^*|_{W_F^{(n)}}} \in \mathrm{Hom}(H_{n+1}(L_{n-1}), \pi_n(F) \otimes \mathbb{Q})$. Next we fix a minimal generating set for both $H_{n+1}(L_{n-1})$ and $\pi_n(F)$. In particular, from the basis for $W^{(n)}$ we have a minimal set of generators for the free part of $\pi_n(F)$ so we simply adjoin a minimal generating set for the torsion part. Next we consider $\widetilde{d_L^*|_{W_F^{(n)}}}(a)$ for each a in the generating set for $H_{n+1}(L_{n-1})$, and we can write each of these as linear combinations of pure tensors $b \otimes \frac{p}{q}$ for b in the minimal generating set of $\pi_n(F)$. Since the torsion part is killed in the tensor with $\mathbb{Q}$ we know that only elements of the free part of $\pi_n(F)$ show up in these linear combinations, and these we can identify with basis elements for $W^{(n)}$. We consider the collection of all $\frac{p}{q}$ arising as coefficients in these terms, and we can pick the least common multiple Q of the qs. Then the subgroup $QH_{n+1}(L_{n-1})$ lands in the image of $\pi_n(F)$ under the map $\pi_n(F) \to \pi_n(F) \otimes \mathbb{Q}$. Then precomposing with the multiplication by $Q \times N$ map gives us a map which lifts to $\tilde{k}_n : H_{n+1}(L_{n-1}) \to \pi_n(F)$, where N is an integer we will determine shortly.

Since $H^{n+1}(L_{n-1}; \pi_n(F))$ can be decomposed as

$$\mathrm{Hom}(H_{n+1}(L_{n-1}), \pi_n(F)) \oplus Ext_{\mathbb{Z}}^1(H_n(L_{n-1}), \pi_n(F))$$

from the inclusion

$$\mathrm{Hom}(H_{n+1}(L_{n-1}), \pi_n(F)) \hookrightarrow \mathrm{Hom}(H_{n+1}(L_{n-1}), \pi_n(F)) \oplus \mathrm{Ext}_{\mathbb{Z}}^1(H_n(L_{n-1}), \pi_n(F))$$

on $\tilde{k}_n$ we get an element $k_n \in H^{n+1}(L_{n-1}; \pi_n(F))$.

We now have a fibration $h_n : L_n \twoheadrightarrow X$. We fix a choice of u_n, a rationalization along the dashed line in the square below:

$$L_n \dashrightarrow^{u_n} L_n^{\mathbb{Q}}$$
$$\downarrow r_n \qquad\qquad \downarrow r_n^{\mathbb{Q}}$$
$$L_{n-1} \xrightarrow[u_{n-1}]{} L_{n-1}^{\mathbb{Q}}$$

Next we construct the relative Mal'cev operation τ_n. To do this, we will consider the following diagram.

$$
\begin{array}{ccccc}
L_n^{\times_X 3} & \xrightarrow{\tau_n^{\mathbb{Q}} \circ u_n^{\times_X 3}} & L_n^{\mathbb{Q}} & \xrightarrow{\hat{k}_n^{\mathbb{Q}}} & X \times E(\pi_n(F) \otimes \mathbb{Q}, n)
\end{array}
$$

Our aim is to construct a τ_n along the dashed line making the diagram commute. Essentially, we want to lift the Mal'cev operation from the fibrewise rationalization. We note that if we construct a map τ_n making the above diagram commute, it will commute with the map u_n.

L_n is a pullback, so we can build τ_n by constructing a map to $X \times E(\pi_n(F), n)$ that commutes with the rest of the pullback square. Starting at $L_n \times_X L_n \times_X L_n$ we can follow along the map $\hat{k}_n^{\mathbb{Q}} \circ \tau_n^{\mathbb{Q}} \circ u_n^{\times_X 3}$. We want to lift this to a map to $X \times E(\pi_n(F), n+1)$, in such a way that it is also a lift of the map $k_n \circ \tau_{n-1} \circ h_n^{\times_X 3} : L_n^{\times_X 3} \to X \times K(\pi_n(F), n+1)$. We note that $X \times E(\pi_n(F), n)$ is also an HM-space. Then we have a map $\hat{\tau}_n : X \times E(\pi_n(F), n)^3 \to X \times E(\pi_n(F), n)$, and composing this with $\hat{k}_n^{\times_X 3}$ provides a map $\psi_n : L_n^{\times_X 3} \to X \times E(\pi_n(F), n)$. Since $\hat{k}_n^{\mathbb{Q}}$ commutes with the fibrewise HM-space structure, we know that ψ_n commutes with the diagram:

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and so it remains to show that it commutes with

which is where we determine N as mentioned above. Commutativity of the diagram above hinges on simply the two maps to $X \times K(\pi_n(F), n+1)$ agreeing, or equivalently equality of the pair of cohomology classes in $H^{(n+1)}(L_n^{\times X3}; \pi_n(F))$. One of these maps factors through $E(\pi_n(F), n)$ and so the corresponding cohomology class is trivial. Then we only need to look at the cohomology class from the bottom path in the diagram. Since the diagram commutes after rationalization, we know this class is a torsion element. If we construct a $\tilde{k}_n$ with $N = 1$ then look at the corresponding torsion class, we set N to be m times the order of this class, where m will be determined by an analogous argument for extending the section. Then considering the classes κ_n and $\tilde{\kappa}_n$ in $H^{(n+1)}(L_{n-1}; \pi_n(F))$

represented by k_n and $\tilde{\kappa}_n$, we have $\kappa_n = N\tilde{\kappa}_n$, and so pulling back across the map

$\tau_{n-1} \circ r_n^{\times X^3}$ we get that the torsion element from $\tilde{\kappa}_n$ will be killed by this multiplication.

Then the class we get pulling back k_n is trivial, and so the above diagram commutes.

Then we can pull back ψ_n across k_n and we have a τ_n as desired.

That τ_n satisfies the conditions for a fibrewise Mal'cev operation is a straightforward

consequence of the fact that we are defining τ_n by pulling back a Mal'cev operation across

a map that respects the fibrewise Mal'cev operation.

Now we construct the section.

Again we are looking for a map along the dashed line in the following diagram:
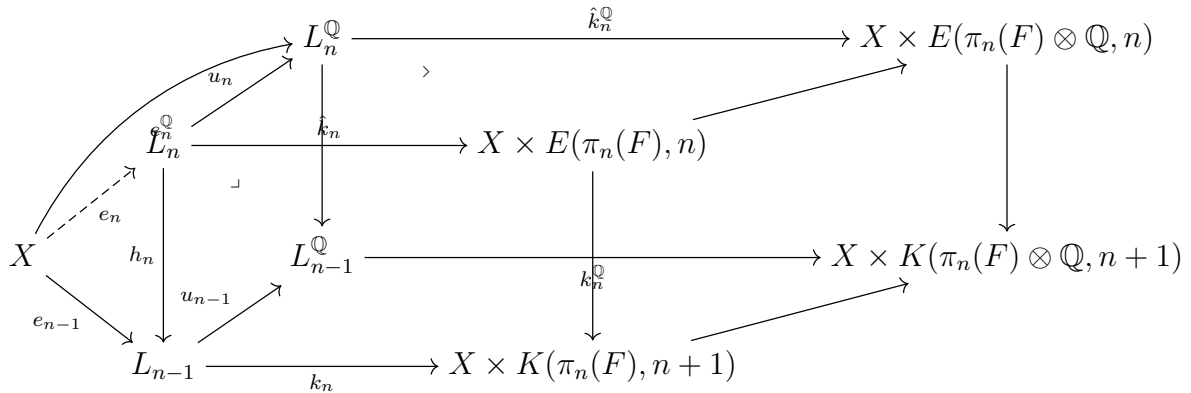
and by an analogous argument to the one above, the obstructions to making such

a lift lie in the torsion part of $H^{n+1}(X; \pi_n(F))$, and so constructing this diagram with

$m = 1$ above will give us such a torsion element, and setting m to be the order of this

element will kill the obstruction, allowing us to pick an e_n.

We are now ready to construct $\phi_n : L_n \to P_n$. By construction, L_n is fibrewise

rationally equivalent to P_n, since the fibers are rationally equivalent, and the k-invariants

are the same up to torsion. Then it remains only to show that we can actually compute

such a rational equivalence. Suppose we try to build a map along the dashed line in the

following diagram, making it commute:

$$
\begin{array}{ccccc}
L_n & \dashrightarrow^{\phi_n} & P_n & \longrightarrow & E(\pi_n(F), n) \\
\downarrow{\scriptstyle r_n} & & \downarrow & & \downarrow \\
L_{n-1} & \xrightarrow{\phi_{n-1}} & P_{n-1} & \longrightarrow & K(\pi_n(F), n+1)
\end{array}
$$

Since the map ϕ_{n-1} is a rational equivalence, and r_n followed by k_n is 0, we know that the obstruction to lifting $\phi_{n-1} \circ r_n$ is torsion. With the choice of section, we can endow L_n with a fibrewise H-space structure, and then an application of lemma 2.2.6 tells us that if we precompose r_n in the above diagram with χ_k for some integer k, we can push the obstruction to lifting to an element of $H^{n+1}(B; \pi_n(F))$, which will in particular be an obstruction to lifting $\phi_{n-1} \circ r_n \circ e_n$ to P_n. We claim here that either this obstruction vanishes, or there is no section to P_n.

Indeed, suppose there is a section $\gamma : X \to P_n$. Then consider the following diagram.

$$
\begin{array}{ccccc}
& & X & & \\
& & \downarrow{\scriptstyle \gamma} \quad \searrow^{\tilde{\gamma}} & & \\
L_n & & P_n & & L_n \\
\downarrow{\scriptstyle r_n} & & \downarrow{\scriptstyle p_n} & & \downarrow{\scriptstyle r_n} \\
L_{n-1} & \xrightarrow{\phi_{n-1}} & P_{n-1} & \xrightarrow{\theta_{n-1}} & L_{n-1} \\
\downarrow{\scriptstyle k_n} & & \downarrow{\scriptstyle k_n^p} & & \downarrow{\scriptstyle k_n} \\
K(\pi_n(F), n+1) & \xrightarrow{\alpha} & K(\pi_n(F), n+1) & \xrightarrow{\beta} & K(\pi_n(F), n+1)
\end{array}
$$

We want to show that the map γ lifts to a section $\tilde{\gamma}$. First, we note that we have yet to define the maps α and β. These can be described by maps on $\pi_n(F)$, and in order to make the diagram commute, we simply have to make sure we pull back the correct cohomology classes on L_n and P_n in each spot. In particular, by construction of k_n we can set α to be the map which kills torsion elements and multiplies the free part by QN. Similarly, β can be the map which kills torsion and multiplies the free part by $\frac{s_n}{QN}$.

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Since r_n is a pullback of the universal $K(\pi_n(F), n)$ fibration, whether $\theta_{n-1} \circ p_n \circ \gamma$ lifts to L_n depends only on the map to $K(\pi_n(F), n+1)$, but by commutativity, this is simply $\beta \circ k_n^P \circ p_n \gamma$ and since this factors through the homotopically trivial map $k_n^P \circ p_n$ we know it is homotopically trivial. Then we have a section $\tilde{\gamma} : X \to L_n$. An identical argument on the other side of the diagram allows us to lift $\phi_{n-1} \circ r_n \circ \tilde{\gamma}$ to P_n, and create a section $\gamma' : X \to P_n$. Then we have the following commutative diagram:

$$
\begin{array}{ccc}
 & X & \\
 \swarrow^{\tilde{\gamma}} & & \searrow^{\gamma'} \\
L_n \dashrightarrow^{\phi_n} & & P_n \\
\downarrow^{r_n} & & \downarrow^{p_n} \\
L_{n-1} \xrightarrow{\phi_{n-1}} & & P_{n-1}
\end{array}
$$

Using the same argument above, this time endowing L_n with a fibrewise H-space structure with the section $\tilde{\gamma}$, lemma 2.2.6 together with the existence of γ' guarantees that the obstruction to lifting $r_n \circ \phi_{n-1}$ to ϕ_n vanishes. This contradicts that $e_{n-1} \circ r_n \circ e_n$ does not lift since $e_n \circ \phi_n$ now provides a lift.

Finally then it remains to construct θ_n. In particular we want to construct a θ_n so that $\theta_n \circ \phi_n$ is χ_{s_n} on L_n for some integer s_n. For each such integer then we have a diagram:

$$
\begin{array}{ccccc}
L_n & \xrightarrow{\quad\quad\quad \chi_{s_n} \quad\quad\quad} & & & L_n \\
\downarrow^{\phi_n} & & & & \downarrow \\
P_n \dashrightarrow_{p_n} P_{n-1} & \xrightarrow{\theta_{n-1}} & L_{n-1} & \xrightarrow{\chi_S} & L_{n-1} \\
& & \downarrow^{k_n} & & \downarrow^{k_n} \\
& K(\pi_n(F), n+1) & \xrightarrow{\chi_S} & K(\pi_n(F), n+1) &
\end{array}
$$

In particular, the obstruction to lifting $\theta_{n-1} \circ p_n$ is given by the cohomology class $k_n \circ \theta_{n-1} \circ p_n$ and since θ_{n-1} is a rational equivalence, the obstruction to such a lift is torsion. Then for some S the obstruction vanishes, and we can produce a lift θ_n.

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Having completed our induction then, we can simply compose the final section e_n and rational equivalence ϕ_n to obtain a section for $\hat{p}$ as desired.

$\square$

Chapter 4

Immersions as a Lifting Problem

With the main algorithm for lifting problems provided, we now turn to the decision problem discussed in the introduction. In particular we want investigate when we can algorithmically reduce the question of manifold immersibility to a lifting extension problem which can be plugged into the algorithm from the previous chapter.

4.1 Reducing Immersibility to Lifting

The first step is to use the h-principle of Hirsch and Smale—the existence of an immersion homotopic to $f : M \to N$ is equivalent to the existence of a tangent bundle monomorphism $F : TM \to TN$ sitting over $f : M \to N$ [7].

To convert this to a lifting problem, we will construct a bundle, which we will denote by $\pi : \mathrm{Mono}(TM, TN) \twoheadrightarrow M \times N$, for any given manifolds M, N of dimension m, n. The fibers of this bundle will be homotopic to the real Stiefel manifolds $V(m, n)$, the space of orthonormal m-frames in $\mathbb{R}^n$. Over each (p, q) we want to think of this as the space of immersions (we use here heavily the fact that $GL(n)$ deformation retracts onto $O(n)$ to speak of an orthogonal structure and simplify some computation) of $T_p(M)$ into $T_q(N)$.

To construct this space explicitly we will describe a system of transition maps on local trivializations.

Here we have to set up some notation. We will consider the tangent bundle over M constructed as a collection of charts with transition maps, in particular a collection of opens $\mathcal{U}$ over M with transition maps φ_{ij} on $(U_i \cap U_j) \times V(m,n) \simeq \pi^{-1}(U_i \cap U_j)$ satisfying that $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ on $\pi^{-1}(U_i \cap U_j \cap U_k)$. Similarly we have the tangent bundle over N, with local trivializations $\mathcal{V}$ and transition maps ψ_{ij}. Then we construct a $V(m,n)$-bundle on $M \times N$ as follows: we construct an open cover $\mathcal{W}$ on $M \times N$ by taking the open sets $W = U \times V$ for each U, V in $\mathcal{U}, \mathcal{V}$ respectively. An intersection $W_i \cap W_j$ can be written as $(U_{i_1} \cap U_{j_1}) \times (V_{i_2} \cap V_{j_2})$ and then the transition map can be defined on orthogonal frames O by $\zeta_{i,j}(O) = \psi^*_{i_2 j_2} O \varphi_{i_1, j_1}$, which provides a linearly independent frame. A straightforward calculation shows that for any triple intersection we have $\zeta_{jk} \zeta_{ij} = \zeta_{ik}$. It is here that we use that we can, for example by singular value decomposition, retract $GL(n)$ onto $O(n)$ to consistently create orthonormal frames from linearly independent ones. Then we have constructed an open cover on $M \times N$ and described a coherent system of transition maps between the trivialization at each open, and hence we have a $V(m,n)$ bundle on $M \times N$. It remains to show that it correctly parametrizes the tangent bundle monomorphisms.

Lemma 4.1.1. *Fix a smooth map $f : M \to N$. The set of homotopy classes of orthogonal tangent bundle monomorphisms $TM \to TN$ over f is in bijective correspondence with homotopy classes of lifts of the triangle:*

$$
\begin{array}{ccc}
 & & \mathrm{Mono}(TM, TN) \\
 & \nearrow & \downarrow \\
M & \xrightarrow{\ id \times f\ } & M \times N
\end{array}
$$

Proof. Fix open covers $\mathcal{U}$ and $\mathcal{V}$ of M and N admitting orthonormal frames.

Let $[\phi]$ be a homotopy class of tangent bundle monomorphism containing a specific

monomorphism ϕ. Then pick some $\phi \in [\phi]$ and we will construct a lift $\psi : M \to$ Mono(TM, TN). Then for each $u \in \mathcal{U}$ and each $v \in \mathcal{V}$ with $f(u) \cap v$ nonempty ϕ provides, for each $p \in u$ with $f(p) \in v$, an m-frame O over $(p, f(p))$ viewed as a point in the local trivialization over $u \times v$, and again using a retraction to $O(n)$ we can make the frame orthonormal. If we had some u' such that $p \in u'$ and v' such that $f(p) \in v'$ then we would produce a frame O', and these would be related precisely by $O' = \psi^* O \varphi$ where φ is the transition map from u to u' for the tangent bundle on M and ψ is the transition map from v to v' for the tangent bundle on N. This is exactly the transition map from $u \times v$ to $u' \times v'$ for Mono(TM, TN) on $M \times N$, so this produces a lift. If we had picked a different $\phi' \in [\phi]$ they would be related by a homotopy, and again reducing to coordinates over $\mathcal{U} \times \mathcal{V}$, we could produce a homotopy of lifts.

Conversely, given a homotopy class $[g]$ of lift of the above diagram, we pick some lift g in the class and produce a tangent bundle monomorphism. Again, for each p in M, each $u \in U$ containing p and each $v \in \mathcal{V}$ containing $f(p)$ the lift g provides an m-frame over $(p, f(p))$ which we can view as a map from $T_p M$ to $T_{f(p)} N$ in the coordinates of u and v. Again that this is a coherent choice of frame across all choices of u, v follows from the fact that the transition maps agree. Finally, a different $g' \in [g]$ would be related by a homotopy to g, and this can again be written out in coordinates. $\qquad \square$

We want to take this a step further however, as we would like to produce a lifting problem where the fibration is uniform over all N, once dimensions are fixed. To do this, we construct a bundle, in the same way as Mono(TM, TN) over $BSO(m) \times BSO(n)$, which we will denote $p_{m,n} :$ Mono$(m - \text{planes}, n - \text{planes}) \twoheadrightarrow BSO(m) \times BSO(n)$. In particular, we can again start with the universal bundles $ESO(m) \twoheadrightarrow BSO(m)$ and $ESO(n) \twoheadrightarrow BSO(n)$, written out in coordinates over some systems of local trivializations. We prove the following theorem.

Lemma 4.1.2. *There is a bundle $p_{n.m} : \mathrm{Mono}(m - \mathrm{planes}, n - \mathrm{planes}) \twoheadrightarrow M \times N$ such that for all smooth manifolds M, N of dimension m, n respectively, there is a commutative diagram:*

$$
\begin{array}{ccc}
Mono(TM, TN) & \longrightarrow & \mathrm{Mono}(m - \mathrm{planes}, n - \mathrm{planes}) \\
\downarrow & & \downarrow {\scriptstyle p_{m,n}} \\
M \times N & \xrightarrow[\ \kappa_m \times \kappa_n\ ]{} & BSO(m) \times BSO(n)
\end{array}
$$

where κ_m, κ_n are the classifying maps for the tangent bundles of M, N respectively. Furthermore, the above diagram is a pullback square.

Proof. Again we pick a covering collection of opens $\mathcal{U}$ on $BSO(m)$ and $\mathcal{V}$ on $BSO(n)$ admitting local trivializations, and consider the open cover $\mathcal{W}$ of $BSO(m) \times BSO(n)$ by taking sets of the form $W = U \times V$. Then over each of these sets we have a trivial $V(m, n)$ bundle and we construct transition maps from $U \times V$ to $U' \times V'$ as before: if φ is the transition map from U to U' and ψ is the transition map from V to V' then over each point in $(U \times V) \cap (U' \times V') = (U \times U') \cap (V \times V')$ we have the transition map $\zeta(O) = \psi^* O \varphi$.

The map $\mathrm{Mono}(TM, TN) \to \mathrm{Mono}(m - \mathrm{planes}, n - \mathrm{planes})$ works as follows: we can pull back the collection $\mathcal{U}$ on $BSO(m)$ across κ_m to form a collection of opens on M, and the local trivializations of $ESO(m) \twoheadrightarrow BSO(m)$ pull back to trivializations of the tangent bundle of M. Similarly, the collection of opens $\mathcal{V}$ pulls back across κ_n along with the trivializations of $ESO(n) \twoheadrightarrow BSO(n)$. Then for any point $(p, q) \in M \times N$ we can pick some $\kappa_m^{-1}(u) \times \kappa_n^{-1}(v)$ containing (p, q) and we can use this choice of trivialization to map the fiber over (p, q) to the fiber over $\kappa_m(p) \times \kappa_n(q)$. That this describes a map coherently over all choices of trivialization of $BSO(m)$ and $BSO(n)$ follows from the fact that the tangent bundles of M and N are pullbacks of $ESO(m) \twoheadrightarrow BSO(m)$ and $ESO(n) \twoheadrightarrow BSO(n)$ respectively.

Finally, this is a pullback square because firstly it would factor through the pullback,

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and the map to the pullback would be a map of fiber bundles with the same base, and

hence a projection of fibers, and since the fiber is the same it is hence a homeomorphism.

$\square$

We can put this together with lemma 4.1.1 to get the following theorem:

Theorem 4.1.3. *Given M, N, f where M is a smooth m-dimensional manifold, N is a smooth n-dimensional manifold and f is a smooth map between them, with $p_{m,n}$: $\mathrm{Mono}(m-\mathrm{planes}, n-\mathrm{planes}) \twoheadrightarrow BSO(m) \times BSO(n)$ as in lemma 4.1.2, then there is a bijective correspondence between homotopy classes of lifts of the diagram:*

$$
\begin{array}{c}
Mono(m-planes, n-planes) \\
\Big\downarrow{\scriptstyle p_{m,n}} \\
M \xrightarrow[(\kappa_m \times \kappa_n) \circ (id \times f)]{} BSO(m) \times BSO(n)
\end{array}
$$

and homotopy classes of tangent bundle monomorphisms over f.

4.2   Immersibility in Odd Codimension

We prove the following:

Theorem 4.2.1. *There is an algorithm that on input (M, N, f), where M, N are smooth oriented manifolds of dimension m, n respectively and $f : M \to N$ is a smooth map between them, decides whether there is an immersion $g : M \to N$ such that $g \simeq f$, as long as $n - m$ is odd.*

Proof. Suppose then we are given the triple (M, N, f) as in the statement of the theorem. Then applying theorem 4.1.3 we want to prove that the existence of such a lift is decidable.

There are two obstructions to using theorem 3.0.1 then, firstly we need to construct a relative minimal model of the bundle $\mathrm{Mono}(m-\mathrm{planes}, n-\mathrm{planes}) \twoheadrightarrow BSO(m) \times BSO(n)$

which has linear differential and secondly need to address the possibility that M is not simply connected, (preventing the use of any of the lifting algorithms.)

To construct the relative minimal model the first step is to determine a minimal model for both the base and the fiber. We consider two cases, based on the parity of n. In the case that n is even then, the rational cohomology of $BSO(n)$ has a generator for each Pontrjagin class, and one for the Euler class which squares to the top Pontrjagin class. Since the codimension is odd, so is m and hence $BSO(m)$ has rational cohomology generated only by the Pontrjagin classes. Then we have for the base the minimal model

$$\mathcal{B} = \mathbb{Q}\langle \alpha_i^{(4i)}, \beta_j^{(4j)}, \varepsilon^{(n)} \rangle$$

where $i \in \{1, ..., \frac{n}{2} - 1\}$, $j \in \{1, ..., \frac{m-1}{2}\}$.

The fiber is the Stiefel manifold $V_m(\mathbb{R}^n)$ which is a homogeneous space $SO(n)/SO(n-m)$. To find the minimal model of this we will use the Cartan-Weil model for a homogeneous space, as in [8]. In particular this allows us to model $V_m(\mathbb{R}^n)$ up to homotopy via a fibration $SO(n) \hookrightarrow V_m(\mathbb{R}^n) \twoheadrightarrow BSO(n-m)$, which is given by the pullback of the universal $SO(n)$ fibration across the map $BSO(n-m) \to BSO(n)$ induced by the inclusion $SO(n-m) \hookrightarrow SO(n)$. Putting this together the underlying graded vector space generating the minimal model for the fiber is

$$V_F = \mathbb{Q}\{\gamma_k^{(4k-1)}, \sigma^{(n-1)}\}$$

with $k \in \{\frac{n-m+1}{2}, ..., \frac{n}{2} - 1\}$.

Then the relative minimal model $(\mathcal{B} \otimes \wedge V_F, \tilde{d})$ is determined by the restriction of the

differential to V_F. In order to compute this, we start by constructing a map

$$ESO(m) \times ESO(n) \xrightarrow{\quad f \quad} \text{Mono}(m - \text{planes}, n - \text{planes})$$

$$BSO(m) \times BSO(n)$$

which we will define as follows. Over a particular local trivialization U_i of $ESO(m) \times ESO(n) \to BSO(m) \times BSO(n)$ we consider a point (p, O_m, O_n) where $p \in U_i$, O_m is an orthogonal $m \times m$ matrix and O_n is an orthogonal $n \times n$ matrix. Then the columns of $O_m I_{m,n} O_n^\dagger$ will provide an orthogonal frame, where $I_{m,n}$ is the $m \times n$ matrix sending the standard basis of $\mathbb{R}^m$ to the first m elements of the standard basis of $\mathbb{R}^n$. To check that this defines a coherent map is simply a matter of checking that it behaves the same on an intersection of local trivializations, but this follows immediately from the definition of $\text{Mono}(m - \text{planes}, n - \text{planes})$.

In the world of $\mathbb{Q}$-dgas then this map is dual to a map over $\mathcal{B}$

$$(\mathcal{B} \otimes \wedge V_F, \tilde{d}) \xrightarrow{f^*} \mathcal{B}\langle a_i^{(4i-1)}, e^{(n-1)}, b_j^{(4j-1)} | da_i = \alpha_i, db_i = \beta_i, de = \varepsilon \rangle$$

and since this is a DGA map over B, it is determined by its action on elements of V_F. We then have to determine both $f^*(\gamma_k)$ and $f^*(\sigma)$.

γ_k is an element of $(\pi_{(4k-1)}(V_m(\mathbb{R}^n)) \otimes \mathbb{Q})^*$ and in this context f^* is the dual of the map induced by f, $\pi_{(4k-1)}(SO(m) \times SO(n)) \to \pi_{(4k-1)}(V_m(\mathbb{R}^n))$. To understand what f does to the fiber $SO(m) \times SO(n)$ we decompose it as a sequence of steps:

$$SO(m) \times SO(n) \xrightarrow{\text{include}\times -1} SO(n) \times SO(n) \xrightarrow{\text{multiply}} SO(n) \xrightarrow{\text{project}} V_m(\mathbb{R}^n)$$

and dualizing this allows us to write

$$f^*(\gamma_k) = \begin{cases} a_k - b_k & k \leq m \\ \\ a_k & \text{otherwise} \end{cases}$$

and

$$f^*(\sigma) = \varepsilon$$

which allows us to determine f^*. Because this map has to commute with the differential, we can conclude that for $k \leq m$

$$f^*(\tilde{d}(\gamma_k)) = \beta_k - \alpha_k$$

but the only preimage of $\beta_k - \alpha_k$ under f^* is $\beta_k - \alpha_k$ and so $d(\gamma_k) = \beta_k - \alpha_k$. Similarly for $k > m$ we conclude $\tilde{d}(\gamma_k) = a_k$ and $\tilde{d}(\sigma) = \varepsilon$. Then we have computed a relative minimal model $(\mathcal{B} \otimes \wedge V_F, \tilde{d})$ for the bundle in our lifting problem.

In the case where n is odd, and so m is even, we have for the base

$$\mathcal{B} = \mathbb{Q}\langle \alpha^{(4i)}, \beta^{(4j)}, \varepsilon(m) \rangle$$

with $i \in \{1, \ldots, \frac{n-1}{2}\}, j \in \{1, \ldots \frac{m}{2} - 1\}$. For the fiber we have the vector space

$$V_F = \mathbb{Q}\{\gamma_k^{4k-1}\}$$

where $k \in \{\frac{n-m+1}{2}, \ldots, \frac{n-1}{2}\}$ (the argument is the same as the other case, we simply don't have any Euler class in the fiber since both n and $n - m$ are odd.) Again we construct the map f as before, and it still dualizes a map which decomposes on the fiber in the same way.

Finally, we address the case where M is not simply connected. Essentially we are

going to replace M by a simply connected complex and create a lifting problem so that a lift exists exactly when one exists over M. Since the construction is nearly identical to the plus construction introduced in [9], we will call this space M^+ (the only difference is that for the plus construction we want a space with perfect fundamental group so we get a space with actually identical cohomology).

We construct M^+ in two steps, following fairly directly the idea for the plus construction. First, we pick a generating set for $\pi_1(M)$, (for instance choosing the complement of a spanning tree of the 1-skeleton.) We then add 2-cells with attaching maps along each such generator. We call this space $\tilde{M}$, and note that it is simply connected. We then consider the homology long exact sequence of the pair $(\tilde{M}, M)$. In particular since the space $\tilde{M}/M$ is a bouquet of 2-spheres we have the sequence

$$0 \to H_2(M) \to H_2(\tilde{M}) \to H_2(\tilde{M}, M) \xrightarrow{\delta} H_1(M) \to 0$$

Which gives the short exact sequence

$$0 \to H_2(M) \to H_2(\tilde{M}) \to \ker \delta \to 0$$

Since $H^2(\tilde{M}, M)$ is free abelian, so is $\ker \delta$. Then $H_2(\tilde{M})$ decomposes as a direct sum $H_2(M) \oplus F$ for a free abelian group F. Since $\tilde{M}$ is simply connected, the Hurewicz homomorphism gives us that each element of $H_2(M)$ is represented by a map $S^2 \to \tilde{M}$. Then to obtain M^+ we attach 3-cells with attaching maps representing a basis of F. Since the attaching maps form a basis of a subgroup of the free part of $H^2(\tilde{M})$ when we look at the homology sequence of the pair $(M^+, \tilde{M})$ the map $H_3(M^+, \tilde{M}) \to H_2(\tilde{M})$ has trivial kernel, and so the homology groups above degree 2 are all isomorphic. In particular then we have a space M^+ which is simply connected but in degree 2 and higher has isomorphic

homology to M.

Then consider the following diagram:

$$
\begin{array}{c}
Mono(m-planes, n-planes) \\
\end{array}
$$

We construct the lifting problem according to lemma 4.1.3, and we want to decide if an f exists. Note that since $BSO(m) \times BSO(n)$ is rationally an H-space, we can use the main result of [3] to construct an extension. We can pick a g and then look for a lift $\tilde{f}$ and if such a lift exists we are done since restricting to M provides a lift f. Then suppose no such lift exists. Then in particular no such lift exists rationally, and the first obstruction to finding such a lift lies in $H^n(M^+; \pi_{n-1}(F) \otimes \mathbb{Q})$ where F is the fiber $V_m(\mathbb{R}^n)$ of the fibration over $BSO(m) \times BSO(n)$. By the universal coefficient theorem, $H^n(M^+; \pi_{n-1}(F) \otimes \mathbb{Q}) \cong \mathrm{Hom}(H_n(M^+), \pi_{n-1}(F) \otimes \mathbb{Q})$ and since the inclusion $i : M \hookrightarrow M^+$ induces an isomorphism between $H_n(M^+)$ and $H_n(M)$ above degree 2 (and $H_1(M^+)$ is trivial) we can conclude that the obstruction is an obstruction to lifting f as well.

To conclude then we summarize the steps of the algorithm:

Input:

- A pair of closed oriented smooth manifolds, M,N as a pair of simplicial complexes with C^1-triangulations with $\dim N - \dim M$ odd.

- A smooth map $f : M \to N$

Output: 'YES' if there is an immersion homotopic to f, 'NO' otherwise.

Steps:

1. Using the algorithms in section, compute simplicial approximations of the classifying maps $\kappa_M : M \to BSO(M)$ and $\kappa_N : N \to BSO(N)$ for the corresponding tangent bundles.

2. Construct the map $\phi : M \to BSO(M) \times BSO(N)$ where $\phi = (\kappa_M \times \kappa_N) \circ (\mathrm{id} \times f)$.

3. Construct M^+ and pick an extension ϕ^+ of ϕ to M^+ (we note here that we have to include this step in general not only in the case that M is simply connected because it is not in general decidable if M is simply connected.)

4. Using the relative minimal model for the appropriate codimension, use the algorithm from theorem 3.0.1 to decide if the map ϕ^+ lifts to $\mathrm{Mono}(m - \mathrm{planes}, n - \mathrm{planes})$.

5. Output the result of the algorithm from the previous step.

$\square$

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