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On the Graph Homology with Integral Coefficients

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On the Graph Homology with Integral Coefficients

DISSERTATION

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by

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VITA

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ABSTRACT OF THE DISSERTATION

On the Graph Homology with Integral Coefficients

By

Matthew Levy

Doctor of Philosophy in Mathematics

University of California, Irvine, 2021

Professor Vladimir Baranovsky, Chair

In [BS], the authors construct a spectral sequence which converges to the homology groups of the graph configuration space. This construction requires a characteristic 0 field to ensure a commutative model for the cochain algebra. For arbitrary coefficients a commutative model may not exist and we suggest a different approach. Each vertex of a graph G is colored by a copy of $C^*_N(M;R)$, the normalized cochains, where R is a commutative ring with unity of any characteristic. We construct a complex similar to the Bendersky-Gitler complex in [BG]. Its differential involves sums over sequences of collapsing edges of the graph: for a single collapsed edge multiplication of tensor factors in $C^*_N(M;R)$ is used, while for general sequences one uses the sequence operations of McClure and Smith, cf. [MS]. If the graph G has at most 5 vertices with a planar-type labelling and $Z_G \subset M^{\times n}$ is the closed subspace of diagonals built from the graph G, we show this computes the relative cohomology groups $H^*(M^{\times n}, Z_G; R)$. These are isomorphic to the homology groups of the graph configuration space $H_{nm-*}(M^G;R)$ if M is a compact oriented manifold. When G is the complete graph on n vertices, these are the homology groups of the usual (labelled) configuration space.

Chapter 1

Introduction

In [BS], the authors construct a spectral sequence converging to the cohomology groups $H^*(M^{\times n}, Z_G; k)$ in the case when k is a field of characteristic zero. Later, in [BZ] the authors have defined a purely algebraic complex that depends on a planar/planted labelling of the graph and an algebra A over the brace operad. They have conjectured that their complex computes the homology of the graph configuration space in the case when A is the normalized cochain algebra and has arbitrary characteristic. We prove their conjecture in the case when the graph has at most 5 vertices and show that when the labeling of the graph is not induced from a planar embedding then an appropriate complex must use a more general structure on A, that of an algebra over an E_3 part of the surjections operad of McClure and Smith, cf. [MS].

First we introduce cochains on a topological space or simplicial set, contractions of complexes, and discuss the Eilenberg-Zilber theorem. We analyze a Bendersky-Gitler type complex [BG] of direct sums of cochains on the cartesian product(s) of a topological space or simplicial set. After application of the Eilenberg-Zilber contractions we work in a complex formed by direct sums of tensor products of cochains and work through the perturbed differential.

When pushed down to this smaller complex, the simultaneous collapse of multiple edges becomes important and the order in which we collapse them as well.

In Chapter 2 we analyze graphs with up to 5 vertices which are planar/planted trees and the labelling agrees with this structure. We describe what happens for different orders of edge collapses; in some orders of collapse we are left only with degeneracies and thus the contribution to the differential is 0, whereas for other orders we get sequence operations of a particularly nice type; they are described by a certain vertical and horizontal order and in the "E_2" filtered part of the surjection operad.

In Chapter 3 we generalize our results to larger trees with planar structure and refine a conjecture connecting our perturbed Bendersky-Gitler type complex with the Chromatic Graph Homology complex in the case when the brace algebra is the normalized cochains on a topological space or a simplicial set. The refined conjecture explains the differential in the Chromatic Graph Homology complex in terms of a sum over all orders of edge collapses in the tree just as we have for our perturbed complex.

In Chapter 4 we analyze more general graphs where it is possible that some subtrees do not have a labelling agreeing with any possible planar embedding. These are trees where either the vertical or horizontal orderings (or both) cannot be made to be in alignment with the canonical (natural) ordering. For a particularly simple tree made of only 4 vertices we argue that E_3 sequence operations must be involved in the d_3 formula representing the collapse of the 3 edges connecting the vertices. This is interesting because it seems to diverge from the pattern of E_2 operations for graphs with planar labelling.

Finally, in Chapter 5 we summarize results, think about applications, and look to future work.

The paper focuses on the overlapping partitions that make up the definition of the sequence operations of [MS]. We prove a theorem that the *Shih Homotopy* takes the form

$$h(a_m \times b_m) = \sum_{i=0}^{m} (-1)^{i+1}(w_0, \ldots, w_i, fg(w_i, \ldots, w_m))$$

where g is the Alexander-Whitney map and f is the Eilenberg-MacLane shuffle map. This is a particularly enlightening formula for us because it explains how a set of overlapping partitions gets further broken down under applications of the homotopy. Everything up to i remains untouched, then g isolates pieces of each factor creating subdivisions (subpartitions) when it can. We prove that this is actually the correct homotopy we want in Theorem 1.2. This says that Shih is a homotopy between the identity and fg; the proof is clean and simple.

The notions of algebraic contraction and homotopy play main roles in this paper. In simplest terms, this paper deals with what happens when you have one (co)chain complex with two types of differentials (the special one we call a perturbation datum) and a contraction to a smaller complex. How does the perturbation affect calculating the (co)homology? We use the perturbation lemma to create a perturbed differential on the smaller complex which allows us to correct for the perturbation and compute the correct (co)homology of the large total complex using only the smaller complex.

1.1 Background

We start with an explanation of the underlying mathematical tools. The foundational tool is the dg-coalgebra of chains $C_*(M; R)$ with coefficients in a commutative ring, R. If M is a manifold or topological space, then in degree $n \geq 0$ this is defined to be the free abelian group on the set of continuous maps $\triangle^n \to M$ tensored with R over $\mathbb{Z}$. If M is a simplicial set, then $C_n(M; R)$ is the free abelian group on the set of n-simplicies tensored with R.

Important maps are the socalled face maps $\partial_i : C_n(M; R) \to C_{n-1}(M; R)$ and degeneracy maps $s_i : C_n(M;R) \to C_{n+1}(M;R)$ for $0 \le i \le n$.

The face maps are induced by the pre-composition with the $(n-1)$-simplex which arises by forgetting the i^{th} vertex of $\triangle^n$. Similarly, the degeneracy maps arise by pre-composition with the $(n+1)$-simplex which repeats the i^{th} vertex. These operations satisfy certain *simplicial relations* that are well known in the literature.

For a simplex $\sigma \in C_n(M;R)$ we will use the notation $\sigma(a_0, \dots, a_t)$ to mean the t-simplex with vertices at the a_i locations. For example, the standard 3-simplex (tetrahedron) contains both the 2-simplex $\sigma(0,1,3)$ and the degenerate 4-simplex $\sigma(0,1,2,2,3)$.

We have a differential $d_n : C_n(M;R) \to C_{n-1}(M;R)$ defined by

$$d_n := \sum_{i=0}^{n} (-1)^i \partial_i$$

with the property that $d_{n-1} \circ d_n = 0$.

For each n, we have the subset $D_n \subseteq C_n(M;R)$ defined by $D_n :=< s_i(C_{n-1}(M;R) >$, for all i, which contains all the degenerate simplices. From this we may construct the normalized singular chain complex $C_*^N(M;R) := C_*(M;R)/D_*$ where the differential d is the induced differential. We have an associative coproduct $\triangle : C_*^N(M;R) \to C_*^N(M;R) \otimes C_*^N(M;R)$ which acts on generators by

$$\sigma(0,\dots,n) \mapsto \sum_{0 \le i \le n} \sigma(0,\dots,i) \otimes \sigma(i,\dots,n)$$

where the degree of σ is n.

The cochain algebra is the dual:

$$C_N^*(M;R) := \mathrm{Hom}(C_*^N(M;R), R)$$

1.2 Graph Configuration Space and its Bendersky-Gitler Complex

Let G be a finite graph without loops (=edges that connect a vertex with itself) or multiple edges, with a fixed labeling of vertices by $\{1, \ldots, n\}$ and the set of edges $E(G)$. Let M be a compact oriented manifold or, more generally, a simplicial set.

We define *the graph configuration space* as the complement

$$M^G = M^{\times n} \setminus Z$$

where $Z = \cup_{\alpha \in E(G)} Z_\alpha$ is the union of diagonals Z_α given by the condition $x_i = x_j$ if α is the edge connecting the vertex labeled by i with the vertex labeled by j. Note that we only remove the diagonals corresponding to edges of G, so when G is the complete graph on n vertices we get the usual ordered configuration space, cf. [FH], but in general the graph configuration space will be a larger subset of the cartesian product $M^{\times n}$.

We will construct a certain complex (Bendersky-Gitler type complex) that computes the relative cohomology groups $H^*(M^{\times n}, Z; R)$. In the special case with M a compact oriented manifold, the relative cohomology is isomorphic to the usual homology of M^G by Lefschetz duality, but most of the work will deal with the relative cohomology case where M can be assumed just a simplicial set.

As the Bendersky-Gitler type complex is very "large" it turns out it is hard to work with directly, but we can try to fix this by replacing cartesian products with tensor products of the cochain complexes. In a sense, $C^*_N(M \times M; R)$ is much larger than $C^*_N(M; R) \otimes C^*_N(M; R)$ although "up to homotopy" they are not much different. In fact, their cohomologies are the same in a strong sense. There are maps between the two so that in one direction of composition it is the identity, and the other way it is almost the identity (identity up to homotopy). In other words, the complex $C_N^*(M;R)^{\otimes n}$ is a *contraction* of $C_N^*(M^{\times n};R)$, cf. [EZ] and [Re].

As a result of a similar contraction of the Bendersky-Gitler complex to a smaller complex we pick up a more complicated differential which takes into account the homotopy "fixing" this lack of isomorphism. Apriori, it seems the new differential is computationally infeasible, but due to the many degeneracies that arise in the formulae (which are 0 in the normalized cochains) the story is much nicer. In fact, we end up with a description in terms of the sequence operations of [MS].

Let R be a commutative ring with 1. The authors of [BS], citing [BG], define a Bendersky-Gitler type bicomplex that computes the relative cohomology $H^*(M^{\times n}, Z; R)$. For any subset $s \subset E(G)$ define $Z_s = \cup_{\alpha \in s} Z_\alpha$. Then the bicomplex is

$$C_N^*(Z_\emptyset;R) \to \bigoplus_{\alpha \in E} C_N^*(Z_\alpha;R) \to \bigoplus_{s \subset E; |s|=2} C_N^*(Z_s;R) \to \cdots$$

We identify each Z_s with $M^{\times l(s)}$ where $l(s)$ is the number of connected components of the graph only containing the edges in s. The horizontal differential arises from a simplicial construction for the open cover $Z = \cup_\alpha Z_\alpha$. Essentially, we reintroduce edges of the graph one at a time and use the inclusion data of the sets Z_s as the simplicial topological data (see [BG] for more on simplicial spaces); the usual alternating sum over face maps is used as the horizontal differential.

We wish to construct a complex

$$C_N^*(M;R)^{\otimes n} \to \cdots \to \bigoplus_{s \subset E; |s|=p} C_N^*(M;R)^{\otimes l(s)} \to \cdots$$

such that the total complex computes the same cohomology, thus computing $H^*(M^{\times n}, Z; R)$.

As a technical note: it will be useful to view elements of $C^*_N(M;R)^{\otimes n}$ as their images under the embedding

$$i : C^*_N(M;R)^{\otimes n} \hookrightarrow \left(C^N_*(M;R)^{\otimes n}\right)^\vee$$
$$a_1 \otimes \cdots \otimes a_n \mapsto \left(\sigma_1 \otimes \cdots \otimes \sigma_n \mapsto \prod_{i=1}^{n} a_i(\sigma_i)\right)$$

Many times we will define or describe maps on chains; the corresponding maps on cochains can be defined by this type of evaluation.

1.3 Contracting Homotopy and Perturbation Lemma

Following [GR] we introduce a contracting homotopy as cochain maps (f, g, h) where $f : N \to M$, $g : M \to N$, and $h : N \to N$ (last map is of degree 1). Here, we have the "large" complex N and the "small" complex M. These must satisfy the identities:

$$\textbf{(c1)}\ fg = 1_M$$

$$\textbf{(c2)}\ hd + dh + gf = 1_N$$

$$\textbf{(c3 - c5)}\ fh = hg = hh = 0$$

We will refer to $(c3)$ through $(c5)$ as the *side conditions*. They are satisfied in many specific examples and a contracting homotopy that does not satisfy the side condition may always be adjusted to give another contracting homotopy that does satisfy them.

Here, when N is the Bendersky-Gitler complex it actually has two differentials: the boundary map on (co)-chains and the combinatorial differential from the covering of the closed boundary. When we look at the pullbacks of the f and g maps below we call this an

Eilenberg-Zilber contraction. In the simplest form this is a contraction of $C_*^N(X \times Y; R) \to C_*^N(X; R) \otimes C_*^N(Y; R)$.

Theorem 1.1. *Let $c : \{N, M, f, g, h\}$ be a contraction and $\delta : N \to N$ a perturbation datum of c. Then, there is a new contraction*

$$c_\delta : \{(N, d_N + \delta), (M, d_M + d_\delta), f_\delta, g_\delta, h_\delta\}$$

where $d_\delta = f\delta \left(\sum_{i\geq 0}(-1)^i(h\delta)^i\right) g$, $f_\delta = f\left(1 - \delta\sum_{i\geq 0}(-1)^i(h\delta)^i\right) h$, $g_\delta = \left(\sum_{i\geq 0}(-1)^i(h\delta)^i\right) g$, and $h_\delta = \left(\sum_{i\geq 0}(-1)^i(h\delta)^i\right) h$, provided that the infinite sums make sense.
Thus we can compute the cohomology of $(N, d_N + \delta)$ using M and the new differential.

This is the basic perturbation lemma.

For us, N is the Bendersky-Gitler type cohomology complex described above and M is an analogous complex made of tensors with only the usual tensor product differential. The perturbation lemma says we can extend the differential so that it computes the same cohomology as the large total complex by changing the differential to $D = \pm d_0 \pm d_1 \ldots$ where $d_0 = d$ the usual tensor product differential, and $d_i = f\delta(h\delta)^{i-1}g$ for $i \geq 1$ where δ is the combinatorial differential in N arising from collapsing edges. It may be convenient to view d_i as arising from all the ways of collapsing i edges from the graph G.

We now define two maps well-known in the literature, cf. [EZ]. The first is the Alexander-Whitney map:

$$g : C_*^N(X \times Y; R) \to C_*^N(X; R) \otimes C_*^N(Y; R)$$

$$a_m \times b_m \mapsto \sum_{i=0}^{m} \partial_{i+1} \cdots \partial_m a_m \otimes \partial_0 \cdots \partial_{i-1} b_m$$

and the second is the Eilenberg-MacLane shuffle map:

$$f : C_*^N(X;R) \otimes C_*^N(Y;R) \to C_*^N(X \times Y;R)$$

$$a_p \otimes b_q \mapsto \sum (-1)^{\mathrm{sig}(\alpha,\beta)} s_{\beta_q} \cdots s_{\beta_1} a_p \times s_{\alpha_p} \cdots s_{\alpha_1} b_q$$

which is a sum over all (p,q)-shuffles (α,β) of $\{0,\ldots,p+q-1\}$ where the sign is determined by the signature of the shuffle permutation which is $\sum_{i=1}^{p} \alpha_i - (i-1)$.

The most interesting map for us is the Shih homotopy operator. Real's Formula, cf. [Re], gives a direct definition involving face and degeneracy maps as follows:

$$h : C_m^N(X \times Y;R) \to C_{m+1}^N(X \times Y;R)$$

$$a_m \times b_m \mapsto$$

$$\sum (-1)^{\overline{m}+\mathrm{sig}(\alpha,\beta)} s_{\beta_q+\overline{m}} \cdots s_{\beta_1+\overline{m}} s_{\overline{m}-1} \partial_{m-q+1} \cdots \partial_m a_m \times s_{\alpha_{p+1}+\overline{m}} \cdots s_{\alpha_1+\overline{m}} \partial_{\overline{m}} \cdots \partial_{m-q-1} b_m$$

where $\overline{m} = m-p-q$ and the sum is taken over all indices $0 \leq q \leq m-1$, $0 \leq p \leq m-q-1$, and all $(p+1,q)$ shuffles (α,β) of the set $\{0,\ldots,p+q\}$ (signs are based on permutation and $\overline{m}$).

For example, if $m=5$, $p=1$, $q=2$ ($\overline{m}=2$), and the shuffle is $\{\alpha_1,\beta_1,\alpha_2,\beta_2\} = \{0,1,2,3\}$ then the term we get is:

$$-a(0,1,1,2,2,3,3) \times b(0,1,3,3,4,4,5)$$

A more interesting formula shows that the Shih homotopy is actually built from both the Alexander-Whitney map and the Eilenberg-MacLane shuffle map which we prove next. It is based on a particular case explained in [BF2].

Theorem 1.2. *a) Writing* $a_m \times b_m = (w_0, \ldots, w_m)$ *we can express the Shih homotopy as follows:*

$$h(a_m \times b_m) = \sum_{i=0}^{m} (-1)^{i+1}(w_0, \ldots, w_i, fg(w_i, \ldots, w_m))$$

where $g : C_*^N(X \times Y; R) \to C_*^N(X; R) \otimes C_*^N(Y; R)$ *is the Alexander-Whitney map and* $f : C_*^N(X; R) \otimes C_*^N(Y; R) \to C_*^N(X \times Y; R)$ *is the Eilenberg-MacLane shuffle map. Note: the composition* $fg(w_i, \ldots, w_m)$ *gives us a sum of* $(m-i)$*-simplices which we then concatenate at the end to get a sum of* $(m+1)$*-simplices.*

b) For any topological space or simplicial set A *and an operator of dg-modules* $T : C_*^N(A; R) \to C_*^N(A; R)$ *which upon its restriction to* $C_0^N(A; R)$ *is the identity map, there exists a canonical homotopy:*

$$h(w_0, \ldots, w_m) := \sum_{i=0}^{m} (-1)^{i+1}(w_0, \ldots, w_i, T(w_i, \ldots, w_m))$$

such that

$$hd + dh + T = 1_{C_*^N(A;R)}$$

c) The Shih homotopy resolves the difference between the identity map and the composition fg.

Proof. For a), it is clear that $\overline{m}$ in the Shih formula plays the role of $i+1$ in this new formula, and q plays the role of the breaking point for the Alexander-Whitney map.

For b), using the fact that T is a map of dg-modules (ie. it commutes with the differential)

$$hd + dh = \sum_{k=0}^{n-1} (w_0, \ldots, w_k, T(w_{k+1}, \ldots, w_n)) - (w_0, \ldots, w_{k-1}, T(w_k, \ldots, w_n))$$

which is a telescoping series. The only surviving terms are

$$hd + dh = (w_0, \ldots, w_n) - T(w_0, \ldots, w_n).$$

For c), we notice that fg is the identity on 0-simplicies then we apply b).

$\square$

Abusing notation, we also have maps $g : C_*^N(M^{\times n}; R) \to C_*^N(M; R)^{\otimes n}$ and $f : C_*^N(M; R)^{\otimes n} \to C_*^N(M^{\times n}; R)$ for all $n \geq 1$ which just involve compositions of the previous Alexander-Whitney/Eilenberg MacLane maps; we either break off tensor factors one at a time or add cartesian factors on one at a time. The order we do this in fact does not matter (as long as we follow a particular convention consistently), but we can always work from left to right for simplicity.

It is apparent that fg is the identity on $C_0^N(M^{\times n}; R)$ and we apply Theorem 1.2 to get the "canonical" Shih homotopy.

Using the pull-back of these maps we satisfy the conditions of a contraction and the perturbation datum is the combinatorial differential from Bendersky-Gitler. Thus it should be enough to compute the cohomology of this total complex using the complex of tensors, the tensor product differential, and d_δ. It is the aim of this paper to make more clear and explicit what is the map d_δ in Theorem 1.1 for this contraction and perturbation datum.

1.4 Sequence Operations

In [MS] we view $C^*_N(M; R)$ as an algebra over the surjection operad (also referred to as sequence operations). This extends the usual notion of cochains as an algebra; the sequence

12 representing the usual product of cochains and the sequence 21 the opposite product. More precisely, the surjection 12 acts by sending $a \otimes b \mapsto a \cdot b$ and 21 acts by sending $a \otimes b \mapsto b \cdot a$. The multiplication map in $C^*_N(M; R)$ involves a sum over all overlapping partitions of $\{0, \ldots, \deg(a) + \deg(b)\}$ with two parts ($A_1 = \{0, \ldots, i\}$ and $A_2 = \{i, \ldots, \deg(a) + \deg(b)\}$). Then $[a \cdot b](\sigma) := \sum a(\sigma(A_1)) \cdot b(\sigma(A_2))$. For example, if $\deg(a) = 2$ and $\deg(b) = 3$ then we have the following overlapping partitions:

$$\{\{0\}, \{0, 1, 2, 3, 4, 5\}\}, \{\{0, 1\}, \{1, 2, 3, 4, 5\}\}, \{\{0, 1, 2\}, \{2, 3, 4, 5\}\},$$

$$\{\{0, 1, 2, 3\}, \{3, 4, 5\}\}, \{\{0, 1, 2, 3, 4\}, \{4, 5\}\}, \{\{0, 1, 2, 3, 4, 5\}, \{5\}\}.$$

Although many of these partitions involve evaluating a or b (or both) on simplices of the wrong degree we include them all for completeness (also, it makes more sense for the coalgebra over an operad structure).

We recall now the notion of a (differential graded) operad.

A *dg-operad* [LV] is an operad in the category of dg-modules over R. That is, there is a dg-module of n-ary operations $P(n)$ for each $n \geq 0$ with a symmetric group action. These must satisfy the usual insertion and associativity properties. The idea is that the elements of $P(n)$ encode different possible operations with n inputs and one output in a vector space with some alegebrac structure (corresponding to P). For instance, in the case of Lie algebras we have three ternary operations $[[x, y], z]$, $[[y, z], x]$, $[[z, x], y]$, but the Jacobi identity says they are linearly dependent, i.e. we have a linear relation between the corresponding elements of $P(3)$. In this work we will deal almost exclusively with the surjection operad, where n fold operations are encoded by finite sequences of integers in $\{1, \ldots, n\}$ (possibly with repetitions but all n integers must occur in a sequence.

An *algebra over a dg-operad* will mean a dg-module M over R with a map of operads $A : P \to \mathrm{End}(M)$. This is a map respecting the operadic axioms from P to the endomorphism operad of M. Or in other words it is a bunch of maps

$$P(n) \otimes M^{\otimes n} \to M$$

that respect the same insertion and associativity conditions. We can think of this as a description of the action of an element in $P(n)$ on n inputs from M. When P is the dg-operad which is the trivial dg-module in all numbers of inputs then the operad is the commutative operad. An algebra over this operad is a commutative dg-algebra. For any number of inputs there is only one way to multiply them in a commutative algebra (whereas for an associative, non-commutative algebra, the order of the inputs may give different results when multiplied).

As we have algebras, we also have coalgebras. Coalgebras over an operad are governed by maps

$$P(n) \otimes M \to M^{\otimes n}$$

Although $C_N^*(M; R)$ is not a commutative algebra, [MS] show that there are sequence operations that fix this lack of commutativity, in some sense. Commutativity holds up to homotopy, and in fact we have two different homotopies, corresponding to sequences 121 and 212. Again, these binary operations are not necessarily commutative, but admit higher homotopies 1212 and 2121, and so on. The corresponding operations on e.g. integral cohomology are known as Steenrod operations. The full structure of the (co)algebra over the surjection operad is a combinatorial extension of this.

In more detail: for a sequence of length $d + r$ with values in $\{1, \ldots, d\}$ (all values must appear in this sequence) and any $n \geq 1$, following McClure and Smith, cf. [MS], we consider *overlapping partitions* $\{0, 1, \ldots, n\} = A_1 \cup A_2 \cup \ldots \cup A_{d+r}$ of the type

$$A_1 = \{n_0, 1, \ldots, n_1\}, A_2 = \{n_1, n_1 + 1, \ldots, n_2\}, \ldots, A_{d+r} = \{n_{d+r-1}, n_{d+r-1} + 1, \ldots, n_{d+r}\}$$

with $0 = n_0 \leq n_1 \leq \ldots \leq n_{d+r} = n$.

Any choice $\overline{A}$ of such partition gives an operation

$$C_*^N(M;R) \to C_*^N(M;R)^{\otimes d}$$

which raises homological degree by r.

If the sequence is written as $t_1, \ldots, t_{d+r}$ with $t_j \in \{1, \ldots, d\}$ and σ is an n-simplex,

$$\sigma[\overline{A}, t] := \bigotimes_{i=1}^{d} \sigma \left(\bigsqcup_{t_j = i} A_j \right)$$

where an index in $\bigsqcup_{t_j=i} A_j$ may repeat several times if it comes from more than one A_j (so the corresponding simplex ends up being degenerate). Summing up over all partitions, we get

$$\sigma[t] = \sum_{\overline{A}} \pm \sigma[\overline{A}, t]$$

where the signs are explained in [MS].

This gives the chains $C_*^N(M;R)$ a structure of a coalgebra over the surjection operad. Dually, for cochains we can define a map $\langle t \rangle : C_N^*(M;R)^{\otimes d} \to C_N^{*-r}(M;R)$

$$a_1 \otimes \cdots a_d \mapsto (-1)^r (a_1 \otimes \cdots \otimes a_d)(\sigma[t])$$

We call $\langle t \rangle$ the sequence operation corresponding to the surjection t. Any degenerate surjection (one that repeats a number in a row) will act by degeneracies thus we can mod out by these degenerate surjections (for example 112 is a degenerate surjection). Then the maps $\langle t \rangle$ give $C_N^*(M;R)$ the structure of an algebra over the dg-operad of non-degenerate surjections.

For example, we have a 4-simplex $\sigma = \sigma(0,1,2,3,4) \in C_4^N(M;R)$ and the surjection 121 suggests we split up $\{0,1,2,3,4\}$ into three overlapping subsets A_1, A_2, A_3. This is completely determined by the 2 points of overlaps (=end points of the intervals) and we can create a sum of simplices in $\left(C_*^N(M;R)^{\otimes 2}\right)_5$. In particular we have the sum

$$\pm\sigma(0,1,2,3,4)\otimes\sigma(0,1)\pm\sigma(0,1,2,3,4)\otimes\sigma(1,2)\pm\sigma(0,1,2,3,4)\otimes\sigma(2,3)\pm$$
$$\sigma(0,1,2,3,4)\otimes\sigma(3,4)\pm\sigma(0,2,3,4)\otimes\sigma(0,1,2)\pm\sigma(0,1,3,4)\otimes\sigma(1,2,3)\pm$$
$$\sigma(0,1,2,4)\otimes\sigma(2,3,4)\pm\sigma(0,3,4)\otimes\sigma(0,1,2,3)\pm\sigma(0,1,4)\otimes\sigma(1,2,3,4)\pm$$
$$\sigma(0,4)\otimes\sigma(0,1,2,3,4)$$

Now we can use evaluation on $a\otimes b$ to get a binary operation of cohomological degree (-1) on $C_N^*(M;R)$.

There is a filtration on the space of surjection operations based on the complexity of a surjection. Essentially, the complexity is determined by the length of the longest subsequence in t of the type $ijijiji\cdots$ (not necessarily formed by consecutive elements of t). For example, the surjection 1212 has complexity 3 and 12134343 has complexity 4 making these E_3 and E_4 operations respectively. If we take just the space of surjections of complexity $\leq n$ then we get a dg-operad which is quasi-isomorphic to the little n-discs dg-operad.

In [GR], the authors prove a computational theorem involving the Shih homotopy and the twist map t that exchanges factors. In particular, he shows that $(1212\ldots)\circ\sigma = g\circ(t\circ h)^j\Delta(\sigma)$ where j is the number of excess $1's$ and $2's$; so it appears at least for cases of two factors that it has been proven that sequence operations come from this repeated action of a permutation and homotopy.

Remark. We expect that another operad, the Barratt-Eccles operad BE, to be involved in the story as well. This is an operad for which $BE(n)$ is the normalized bar construction of the symmetric group S_n. It admits a morphism onto the operad of surjections. See [BF1] for the definitions.

Conjecturally, there are morphisms of complexes

$$BE(n) \otimes C_*^N(M, R) \to C_*^N(M^{\times n}; R)$$

such that the surjection operations are obtained by composing with the Alexander-Whitney map $C_*^N(M^{\times n}; R) \to C_*^N(M; R)^{\otimes n}$. In fact, there may be more than one such map but our conjecture is that all our combinatorial formulas in later sections can be written as linear combinations of operations $C_*^N(M, R) \to C_*^N(M^{\times n}; R)$ obtained from $BE(n)$, for a particular choice of the above morphism.

Chapter 2

Case of Collapses into Vertex 1

For the Bendersky-Gitler complex the combinatorial differential arises from reintroducing edges one at a time and collapsing the corresponding vertices, but after perturbing to our complex of tensors we use the perturbation lemma to extend this to multiple edges. We concentrate first on collapses into the vertex labelled 1.

2.1 Setup

Let G be a connected graph with the set of vertices $V(G)$ labeled by $\{1, \ldots, n\}$ and set of edges $E(G)$. The perturbation lemma recipe says that the contracted version of Bendersky-Gitler sequence will acquire a total differential $D = \pm d_0 \pm d_1 \pm \ldots$ where each term d_i is constructed from the Eilenberg-Zilber homotopy and a collapse of i edges in G. If we assume edges are collapsed involving vertex 1, the edge collapses correspond to pull-backs of

$$\Delta_{1,j} : M^{\times(n-1)} \to M^{\times n}, \quad (x_1, \ldots, x_{n-1}) \mapsto (x_1, \ldots, x_{j-1}, x_1, x_j, \ldots, x_{n-1})$$

where $j = 1$ means the identity map and $j \geq 2$ if we assume no loops or cycles. We can think about this as the part of the perturbation datum that collapses edge e.

$$\delta_e = \Delta^*_{1,j} : C^*_N(M^{\times n}; R) \to C^*_N(M^{\times (n-1)}; R)$$

Theorem 2.1. *Each d_i in the perturbed differential $D = \pm d_0 \pm d_1 \pm \ldots$ is itself a sum over permutations of subsets of i edges.*

Proof. By Basic Pertrubtaion Lemma, each d_i is by definition written as $d_i = f\delta\,(h\delta)^{i-1}\,g$ and each time we apply $\delta = \sum \pm\delta_e$ we get a sum over collapsing all the remaining edges. In total, after i applications we have summed over every way to collapse i edges and all orders of doing this as well. □

Let $e \subset E(G)$ be a subset of edges and $\pi \in S_e$ be a permutation then $d_{e,\pi}$ is our notation for the contribution to $d_{|e|}$. For completeness, this is the map

$$d_{e,\pi} := f\delta_{\pi(e_k)}h\delta_{\pi(e_{k-1})}h \cdots h\delta_{\pi(e_1)}g$$

where $k = |e|$. That is we collapse edge $\pi(e_1)$ first, then $\pi(e_2)$ second, ect...

If G only has two vertices connected by an edge, then the map corresponds to the usual multiplication. To see this, the perturbation lemma gives us a differential $D := \pm d_0 \pm d_1$ where d_0 is the usual differential and $d_1 = f\delta g$. Since g is the pullback of the Alexander-Whitney map, δ is the pullback of the diagonal map, and f is the identity in this case, this is the usual multiplication map.

Theorem 2.2. *If the subgraph made from a subset of edges $e \subset E(G)$ contains a cycle then $d_{e,\pi} = 0$ for any permutation π.*

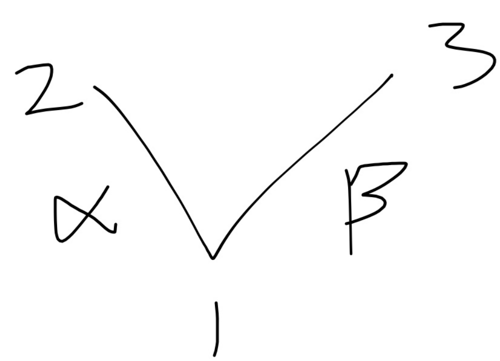
Proof. A cycle implies δ_{e_i} will be the identity for some edge $e_i \in e$ then the side conditions imply that $d_{e,\pi} = 0$. □

So if the subset of edges forms a subgraph with a cycle the contribution to the differential will be null.

2.2 Case of 3 Vertices

When there are 3 vertices in G and two edges going into vertex 1, we have two orders of edge collapses (from the 2 ways we can permute $\{2,3\}$). This contributes to d_2 in the following way: it is a sum of two maps of degree 1, involving one application of the Shih homotopy for each map.

Let us label the edge connecting 1 with 2 by α and the edge connecting 1 with 3 by β. For collapse order $\{2,3\}$ the contribution to d_2 is $f\Delta^*_{1,2}h\Delta^*_{1,2}g$; for the order $\{3,2\}$ the contribution is $f\Delta^*_{1,2}h\Delta^*_{1,3}g$. Again, f is the identity map and g is the pullback of the Alexander-Whitney map on 3 tensor factors this time. h is the pullback of the Shih homotopy operator on $C^*_N(M \times M; R)$. Let $a \otimes b \otimes c \in C^*_N(M;R)^{\otimes 3}$, we wish to describe how $\sigma \in C^N_*(M;R)$ is altered by the maps before it is evaluated at $a \otimes b \otimes c$. Visually, σ passes through the following maps before it is evaluated:



$$
\begin{tikzcd}
 & C^*_N(M \times M) \arrow[dr] & \\
C^*_N(M \times M \times M) \arrow[ur, "\Delta^*_{1,2}"] \arrow[dr, "\Delta^*_{1,3}"'] & \oplus & C^*_N(M) \\
 & C^*_N(M \times M) \arrow[ur] &
\end{tikzcd}
$$

$$C^N_*(M;R) \to_f C^N_*(M;R) \to_{\Delta_{1,2}} C^N_*(M \times M;R) \to_h$$

$$C^N_{*+1}(M \times M;R) \to_{\Delta_{1,2}} C^N_*(M \times M \times M;R) \to_g C^N_*(M;R)^{\otimes 3}$$

Theorem 2.3. *Assume* $\deg(\sigma) = n$*. Then* $h(\sigma \times \sigma)$ *is a sum over all shuffles of all the overlapping partitions* A_1, A_2, A_3 *of* $\{0, \dots, n\}$*:*

$$h(\sigma \times \sigma) = \sum \pm \sigma(A_1A_2) \times \sigma(A_1A_3)$$

where by A_1A_2 *we mean the concatenation of* A_1 *and* A_2 *shuffled into an* $(n+1)$*-simplex (using the* s_i *maps as in the definition of* f*) and* A_1 *in the first factor has the same shuffles as the* A_1 *in the second factor (although most of these shuffles give degenerate simplices in the product).*

Proof. Let us write $\sigma \times \sigma = (w_0, \dots, w_n)$. Then according to Theorem 1.2

$$h(\sigma \times \sigma) = \sum_i (-1)^{i+1}(w_0, \dots, w_i, fg(w_i, \dots, w_n))$$

Let us define for each i, $A_1 = \{0, \dots, i\}$ and since the Alexander-Whitney map g involves two overlapping partitions of $\{i, \dots, n\}$ we denote these by A_2 and A_3. Since we have a sum over all i and the map g is a sum over all overlapping partitions of $\{i, \dots, n\}$ together we can rewrite this as a single sum over all the overlapping partitions of $\{0, \ldots, n\}$ with 3 overlapping pieces. The Eilenberg-MacLane shuffle map f involves shuffles of A_2 and A_3.

$\square$

For example, if $n = 5$, $A_1 = \{0, 1\}$, $A_2 = \{1, 2, 3\}$, and $A_3 = \{3, 4, 5\}$ then one of the shuffles is $\pm\sigma(0, 1, 1, 2, 2, 3, 3) \times \sigma(0, 1, 3, 3, 4, 4, 5)$.

Theorem 2.4. *The map d_2 corresponding to collapse of edge α and edge β is the sequence operation ± 1232 applied to the cochain algebra $C^*_N(M; R)$ in the perturbed complex of tensors.*

Proof. We have a sum over two orders of edge collapse. If we collapse α first and then β second we use Theorem 2.3 to say we are evaluating $a \otimes b \otimes c$ on:

$$\sum \pm g(\sigma(A_1 A_2) \times \sigma(A_1 A_2) \times \sigma(A_1 A_3))$$

We argue g of these terms is always 0 because if the first and second tensor factor do not contain degeneracies then the third must (due to $A_1 A_3$ being a $\leq m$-simplex shuffled into an $(m+1)$-simplex by s_i maps). A_1 and A_2 being adjacent in the overlapping partition means we cannot use numbers from A_3 until the last tensor factor.

Now if we collapse β first and then α it corresponds to evaluating $a \otimes b \otimes c$ on:

$$\sum \pm g(\sigma(A_1 A_2) \times \sigma(A_1 A_3) \times \sigma(A_1 A_2))$$

it is clear that the third factor of the Alexander-Whitney map must contain all of A_2 to be non-degenerate. This leaves A_1 split into two overlapping partitions B_1 and B_2 and

$$\sum \pm g(\sigma(A_1 A_2) \times \sigma(A_1 A_3) \times \sigma(A_1 A_2)) = \sum \pm \sigma(B_1) \otimes \sigma(B_2 \sqcup A_3) \otimes \sigma(A_2)$$

or after relabelling we have a sum over all overlapping partitions A_1, A_2, A_3, A_4 of $\{1, \ldots, n\}$

$$\sum \pm\sigma(A_1) \otimes \sigma(A_2 \sqcup A_4) \otimes \sigma(A_3)$$

giving us a sequence operation of ± 1232. $\square$

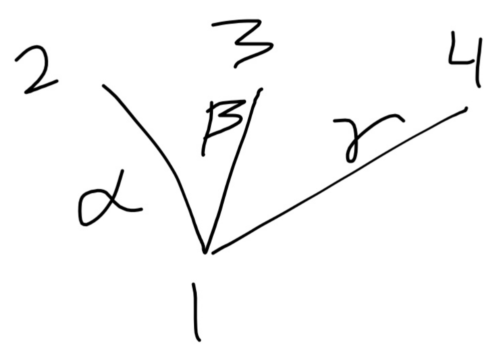
For example: let $n = 5$ and $A_1 = \{0, 1, 2\}, A_2 = \{2, 3, 4\}, A_3 = \{4, 5\}$ then one of the terms in the sum is

$$g(\sigma(0, 1, 2, 2, 2, 3, 4) \times \sigma(0, 1, 2, 4, 5, 5, 5) \times \sigma(0, 1, 2, 2, 2, 3, 4)) =$$

$$\sigma(0) \otimes \sigma(0, 1, 2, 4, 5) \otimes \sigma(2, 3, 4) + \sigma(0, 1) \otimes \sigma(1, 2, 4, 5) \otimes \sigma(2, 3, 4)+$$

$$\sigma(0, 1, 2) \otimes \sigma(2, 4, 5) \otimes \sigma(2, 3, 4)$$

As we vary i in the Shih map h and vary the breaking point for the Alexander-Whitney map, we can get all the sequence operation simplices defined for operation 1232.



2.3 Case of 4 Vertices

For the case of graphs with 4 vertices there are many more options.

The simplest case is when we have 4 vertices all connected in a corolla coming into vertex 1 and the collapse is in order. We collapse vertex 2 into 1, then 3 into 1 and then 4 into 1. We call this collapse order $\{2,3,4\}$ into vertex 1.

Theorem 2.5. *We can express the part of the differential that comes from collapse order* $\{2,3,4\}$ *by:*

$$\sum \pm g(\sigma(A_1A_2A_3) \times \sigma(A_1A_2A_3) \times \sigma(A_1A_2A_4) \times \sigma(A_1A_5)) = 0$$

collapse order $\{3,4,2\}$ *by:*

$$\sum \pm g(\sigma(A_1A_2A_3) \times \sigma(A_1A_5) \times \sigma(A_1A_2A_3) \times \sigma(A_1A_2A_4))$$

$$\pm g(\sigma(A_1A_2) \times \sigma(A_1A_3A_5) \times \sigma(A_1A_2) \times \sigma(A_1A_4)) =$$

$$(0 \pm 123242) \circ \sigma$$

and collapse order $\{4,3,2\}$ *by:*

$$\sum \pm g(\sigma(A_1A_2A_3) \times \sigma(A_1A_5) \times \sigma(A_1A_2A_4) \times \sigma(A_1A_2A_3))$$

$$\pm g(\sigma(A_1A_2) \times \sigma(A_1A_3A_5) \times \sigma(A_1A_4) \times \sigma(A_1A_2)) =$$

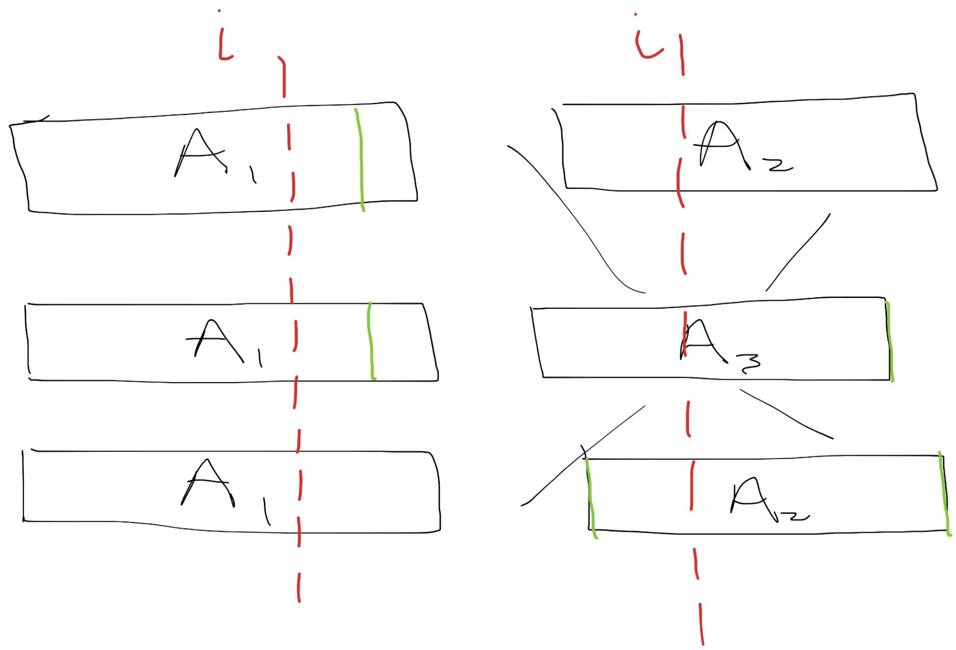
$$(\pm 123432 \pm 124232) \circ \sigma$$

for the result of the first order to possibly be non-degenerate after application of Alexander-Whitney, it must be the case that A_2, A_3, A_4 are all 0-simplices (because non-zero degrees implies degeneracies in the last factor) and thus A_5 begins with the same number that A_1 ends with. Since A_1 cannot be cleared before the last factor (by the fact that A_1 and A_2 are adjacents), this means that the last factor will always contain a degenerate if the first, second, and third do not. So, this order of collapses does not add anything to the differential.

The reason that there are more for the second and third orders is that we can split A_1 or A_2 in the previous configuration with 3 vertices. We start with a term coming from the map $\Delta_{1,3}h\Delta_{1,2}$ which is a shuffled form of $\sigma(A_1A_2) \times \sigma(A_1A_3) \times \sigma(A_1A_2)$ where A_1, A_2, A_3 is an overlapping partition of $\{0, \ldots, p\}$ for $\sigma \in C_p^N(M; R)$. If we then apply h again, we can "split" A_1 or A_2 at the i position of h and then collect some non-degenerate simplex inside of either A_1 or A_2. The Alexander-Whitney map splits the chosen piece into further overlapping partitions.

This can be seen through the definition of h. It doesn't change the simplex before the i^{th} position where it interrupts the sequence at least at one more place due to the Alexander-Whitney map. Remember: if the Alexander-Whitney map returns a degenerate simplex, then when we shuffle it we can be sure it will be degenerate (it is a well-defined linear map). We can ignore such terms. Therefore, if we split A_1 at the i position we get something like the picture; or we could split A_2 at i and get more overlapping pieces that way. In total there are two different configurations coming from the two different types of splitting points.

Definition By the term *A-block configuration* we mean a sum over all shuffles of the corresponding concatenated A-blocks of an overlapping partition of $\{0, \ldots, p\}$. We assume shuffles are done so that the correct degree is achieved agreeing with the number of applications of h. So the A-block configurations for the collapse order $\{4, 3, 2\}$ are



$A_1 \quad A_2$

$A_1 \quad A_3 \quad A_5$

$A_1 \quad A_4$

$A_1 \quad A_2$

and

$A_1 \quad A_2 \quad A_3$

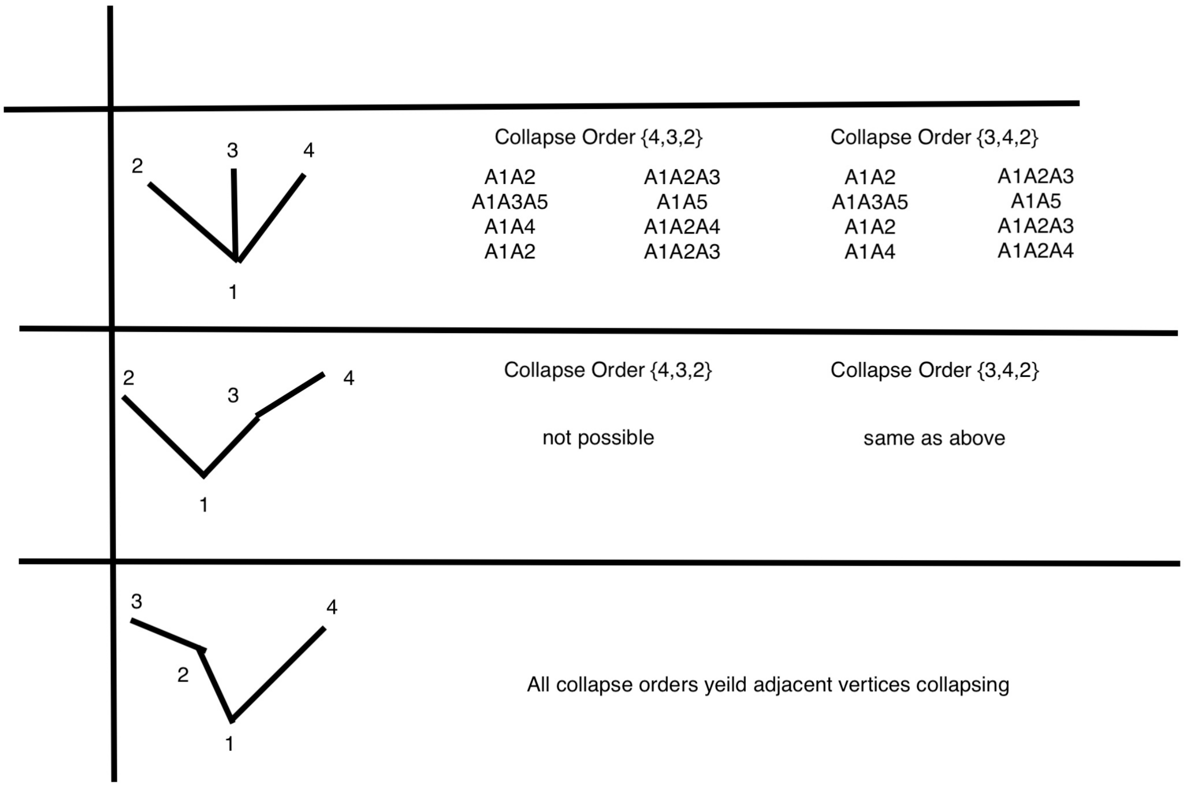
$A_1 \quad A_5$

$A_1 \quad A_2 \quad A_4$

$A_1 \quad A_2 \quad A_3$

When we apply the Alexander-Whitney map to these configurations the first one splits into the sequence operation 124232 and the second into 123432. Notice that both of these two operations are of the form $1\ldots$ where the ... contain no more $1'$s and the rest is equivalent to a sequence operation of complexity ≤ 2 with $2 <_v$ (3 and 4) and either $4 <_h 3$ or $3 <_v 4$ in the induced horizontal and vertical order. We discuss this horizontal and vertical order on some sequences in Chapter 3.

The following is a table of a few trees with 4 vertices labelling the graph agreeing with its planar/planted structure along with the A-block configurations of their corresponding maps before application of Alexander-Whitney. In general, there are $3! = 6$ orders of collapse for each graph but orders where vertices do not collapse into vertex 1 or those that have vertex 2 collapse "too early" are not pictured.



To wrap your head around what one of the terms from these configurations looks like lets set $n = 6$ with $A_1 = \{0, 1\}$, $A_2 = \{1, 2\}$, $A_3 = \{2, 3\}$, $A_4 = \{3, 4\}$, and $A_5 = \{4, 5, 6\}$. One shuffled term of the configuration

$A_1 \quad A_2 \quad A_3$

$A_1 \quad A_5$

$A_1 \quad A_2 \quad A_4$

$A_1 \quad A_2 \quad A_3$

is the following

$$\sigma(0,1,1,1,1,2,2,2,3) \times \sigma(0,1,4,5,6,6,6,6,6) \times$$

$$\sigma(0,1,1,1,1,2,3,4,4) \times \sigma(0,1,1,1,1,2,2,2,3)$$

and if you apply Alexander-Whitney you get $\pm\sigma(0) \otimes \sigma(0,1,4,5,6) \otimes \sigma(1,2,3,4) \otimes \sigma(2,3) \pm \sigma(0,1) \otimes \sigma(1,4,5,6) \otimes \sigma(1,2,3,4) \otimes \sigma(2,3)$.

For collapse order $\{3,4,2\}$ one of the A-blocks yields 123242 after Alexander-Whitney while the other is a tensor product of at least one degenerate simplex for all shuffles.

We deal with non-planar labellings and collapsing into other vertices besides 1 in Chapter 4. In terms of planar/planted graphs with labelling agreeing with the structure, we sum up our findings so far:

1) The perturbed differential d_i comes from collapsing orders of subsets of i edges in the graph.

2) If our order is $\{4,3,2\}$ of vertices collapsing into 1 we get operations of ± 124232 and ± 123432.

3) If our order is $\{3,4,2\}$ of vertices collapsing into 1 we get the operation ± 123242.

4) Orders of edges that collapse adjacent vertices (ignoring the inevitable final collapse of 2 into 1) and those whose union is not a tree contribute 0 to d_i.

The last fact is due to the side conditions for our (f,g,h) maps.

2.4 Case of 5 Vertices

In the case of 5 vertices on a tree with planar/planted structure and labelling agreeing with this structure we get further A-block configurations yielding sequence operations. For certain orders of edge collapses, some of the A_i in the top row can be "split" like before. The i position determines where one splits the overlapping partition and then the Alexander-Whitney map further splits the block when you move down in the rows to its next appearance. Each time we pick up two new overlapping pieces and one new factor.

For instance, given that the collapse order is $\{5,4,3,2\}$ we interrupt the A-block

$$\begin{array}{llll} A_1 & A_2 & A_3 \\ A_1 & A_5 \\ A_1 & A_2 & A_4 \\ A_1 & A_2 & A_3 \end{array}$$

coming from collapse order $\{4,3,2\}$ at the top A_3 block with i. Then we obtain a new factor (new row) and a total configuration of:

$$\begin{array}{llll} A_1 & A_2 & A_3 & A_4 \\ A_1 & A_7 \\ A_1 & A_2 & A_6 \\ A_1 & A_2 & A_3 & A_5 \\ A_1 & A_2 & A_3 & A_4 \end{array}$$

after shifting over the index of previous partition parts (like $A_4 \mapsto A_6$ and $A_5 \mapsto A_7$).

Remember this is only showing one of the overlapping pieces that i interrupts but in general there will be others. We now list some A-block configurations for planar/planted trees with 5 vertices that will yield sequence operations after applying the Alexander-Whitney map.

We display A-block configurations that can happen for various collapse orders into vertex 1 that occur on planar graphs with 5 vertices.

{5,4,3,2}	A1A2 A1A3A5A7 A1A6 A1A4 A1A2	A1A2A3 A1A5A7 A1A6 A1A2A4 A1A2A3	A1A2 A1A3A7 A1A4A6 A1A4A5 A1A2	A1A2A3 A1A7 A1A2A4A6 A1A2A5 A1A2A3	A1A2A3A4 A1A7 A1A2A6 A1A2A3A5 A1A2A3A4
{5,3,4,2}	A1A2 A1A3A5A7 A1A4 A1A6 A1A2	A1A2A3 A1A5A7 A1A2A4 A1A6 A1A2A3	A1A2A3A4 A1A7 A1A2A3A5 A1A2A6 A1A2A3A4		
{4,3,5,2}	A1A2 A1A3A5A7 A1A4 A1A2 A1A6	A1A2A3 A1A5A7 A1A2A4 A1A2A3 A1A6	A1A2A3A4 A1A7 A1A2A3A5 A1A2A3A4 A1A2A6		
{4,5,3,2}	A1A2 A1A3A5A7 A1A6 A1A2 A1A4	A1A2A3 A1A5A7 A1A6 A1A2A3 A1A2A4	A1A2A3A4 A1A7 A1A2A6 A1A2A3A4 A1A2A3A5	A1A2 A1A3A7 A1A4A6 A1A2 A1A4A5	A1A2A3 A1A7 A1A2A4A6 A1A2A3 A1A2A5
{3,4,5,2}	A1A2 A1A3A5A7 A1A2 A1A4 A1A6	A1A2A3 A1A5A7 A1A2A3 A1A2A4 A1A6	A1A2A3A4 A1A7 A1A2A3A4 A1A2A3A5 A1A2A6		
{3,5,4,2}	A1A2 A1A3A5A7 A1A2 A1A6 A1A4	A1A2A3 A1A5A7 A1A2A3 A1A6 A1A2A4	A1A2 A1A3A7 A1A2 A1A4A6 A1A4A5	A1A2A3 A1A7 A1A2A3 A1A2A4A6 A1A2A5	A1A2A3A4 A1A7 A1A2A3A4 A1A2A6 A1A2A3A5

Notice that when the tree G is not a corolla it restricts the collapse order. In particular, if $x <_v y$ in the vertical order of the tree then it must be true that x collapses into vertex 1 before y. So for the second tree pictured, collapse order $\{5, 4, 3, 2\}$ is not possible.

For collapse order $\{5, 4, 3, 2\}$ these configurations yield the sequence operations (in order of A-configuration) 12524232, 12454232, 12523432, 12353432, and 12345432.

For collapse order $\{3, 4, 5, 2\}$ we get $12324252 + 0 + 0$.

For collapse order $\{3,5,4,2\}$ we get $12325242 + 0 + 12324542 + 0 + 0$

and so on...

By following the same pattern of splitting the top row of A-blocks and relabelling later ones we can calculate all the block patterns and thus sequence operations for the relevant collapse orders. Any order where vertex 2 does not collapse last yields A-block configurations that are killed by the Alexander-Whitney map.

Theorem 2.6. *We can express the part of the differential that comes from collapse order*
$\{3,4,5,2\}$ *by* $(\pm 12324252) \circ \sigma$
$\{3,5,4,2\}$ *by* $(\pm 12325242 \pm 12324542) \circ \sigma$
$\{4,5,3,2\}$ *by* $(\pm 12425232 \pm 12423532 \pm 12343532) \circ \sigma$
$\{4,3,5,2\}$ *by* $(\pm 12423252 \pm 12343252) \circ \sigma$
$\{5,3,4,2\}$ *by* $(\pm 12523242 \pm 12353242) \circ \sigma$
$\{5,4,3,2\}$ *by* $(\pm 12524232 \pm 12454232 \pm 12523432 \pm 12353432 \pm 12345432) \circ \sigma$

Proof. Construct A-block configurations for all orders of collapse using splittings of previous configurations. All configurations are pictured in the table above. Then apply a split coming from the final application of the Alexander-Whitney map and relabel overlapping partition pieces. Check that for some A-configurations they are killed by Alexander-Whitney while others give the desired sequence operations. □

At this point we see a pattern emerging which is similar to the definition of the Chromatic Graph Homology complex in [BZ]. The difference is that from our work with small trees we see that the operations in the [BG] complex can be separated by the order that the edges collapse.

Chapter 3

Chromatic Graph Homology for Brace Algebras

We aim to show that the complex constructed in [BZ] computes the same homology as that coming from the Bendersky-Gitler complex when the brace algebra chosen is $C_N^*(M;R)$. In doing so, we also see that the brace operations in their paper can be categorized in a certain way. In particular, we will see that we can classify them based on the order that we collapse the edges of the connected tree. In the Bendersky-Gitler complex, the differential comes from collapsing single edges at a time, but for the perturbed complex we are forced to collapse multiple edges at a time. We conjecture that the differential in the Chromatic Graph Homology is the same as the perturbed differential coming from orders of collapsing edges of subgraphs of a tree with labelling that respects a planar/planted structure. For trees of small size it has already been shown that this pattern is apparent. To prove the conjecture we would need better control over the structure of A-block configurations coming from orders of edge collapse. At the current moment it seems all we have is an algorithm to construct them but no general expression for them.

3.1 Baranovsky-Zubkov

In their paper, [BZ] deal with the case that the graph G is a planar planted tree. That is, it is a tree with a certain fixed labelling of vertices that agrees with the planar structure and a "tail" for vertex 1. They correctly note for non-commutative brace algebras that even for a tree with 3 vertices (vertex 1 in the middle), $d^2 \neq 0$ for the naive choice of d that only contracts one edge at a time. This is because if we collapse 2 into 1 first and then 3 into 1 second we get a different result than first collapsing 3 into 1 and subsequently 2 into 1. This is equivalent to the statement that $a \cdot b \cdot c \neq a \cdot c \cdot b$ in the brace algebra.

As hinted at previously, there is a surjection 1232 that represents the homotopy between these two multiplications and so that is the operation to which the authors assign to this graph with 3 vertices.

The Chromatic Graph Homology complex starts with a complex made of tensors just as our perturbed complex and assigns to any planar planted tree G with i edges a map d_i of total degree 1 based on the tree. They use the concept of horizontal and vertical order to determine the recursive definition.

First we need to acknowledge the equivalence of the surjection suboperad of complexity ≤ 2 (which means that for any pair of distinct indices i and j the sequence does not have a subsequence $ijij$ or $jiji$) and so-called brace trees. These are planar trees with white and black nodes. Every white vertex is labelled by a number and there are certain rules about where black vertices can be. See [BZ]. For example, our favorite surjection 1232 can be visualized as a tree with one white node labelled 1 connected to one black node attaching to vertex 2 and 3 connected vertically (with $2 <_v 3$). In fact, every surjection with small enough complexity, i.e. we don't have run-on terms for too long like 12121212, can be represented as a certain brace tree.

Since a brace tree has a planar structure there is a planar labelling for the edges and we can discuss if two vertices are related by vertical or horizontal order. We write $x <_h y$ and $x <_v y$ in those cases. For a planar/planted labelling of a planar/planted tree it is always true that $x <_h y$ or $x <_v y$ implies $x < y$ in the natural ordering. But for trees with a labelling not agreeing with any planar/planted structure this may not be the case.

Since we only need the two partial orderings to describe sequence operations, we are not discussing the brace trees in more detail, but rather give a direct definition of the horizontal and vertical orders of a sequence.

Definition. For a sequence with values in $\{1, \ldots, n\}$ and two distinct indices i, j we write:

(a) $i <_h j$ if all occurrences of i are to the left of all occurrences of j in the sequence;

(b) $i <_v j$ if all occurrences of j are between two consecutive occurrences of i.

Note that for any $i \neq j$ we have exactly one of the possibilities: $i <_h j$, $j <_h i$, $i <_v j$, $j <_v i$ (since otherwise there will be a subsequence $ijij$ or a subsequence $jiji$).

Definition. (Chromatic Graph Homology) From [BZ], we let G be a planar/planted tree with standard labelling and i edges. Let A be a brace algebra or equivalently an algebra over the E_2 part of the surjection operad. The non-recursive definition for d_i is more pertinent to us; we write it as a sum over sequence operations on A of the following type:

1) They have a single occurrence of 1 which is first.

2) $2 <_v y$ for all $y \geq 3$ in the sequence operation.

3) For $x, y \geq 3$ if $x <_v y$ in G then $x <_h y$ in the sequence operation.

4) For $x, y \geq 3$ if $x <_h y$ in G then $x <_h y$ or $y <_h x$ or $x <_v y$ in the sequence operation.

For example, the sequence operation 12524232 has a single 1 at the beginning, 2 is smaller in vertical order than the larger factors and $5 <_h 4 <_h 3$; so it is of this type assuming $3 <_h 4 <_h 5$ in G.

3.2 Relation with BG Complex

We state a conjecture connecting the Bendersky-Gitler complex to the Chromatic Graph Homology complex for brace algebras (sequence operations of complexity ≤ 2).

Conjecture 3.1. *Let G be a tree with planar planted structure and vertices labelled agreeing with this structure. Fix an order of edge collapses into the vertex 1. This is a permutation of the remaining edges (ie. a permutation of the set $\{2, 3, \ldots, k\}$). Then the contribution to the differential in the perturbed complex of differentials for this order is a sum over sequence operations (surjection operations) of the following type:*

1) They have a single 1 at the start.

2) $2 <_v y$ for all $y \geq 3$.

3) Let $x < y \in \{3, \ldots, k\}$ then $x <_v y$ or $y <_h x$ in the sequence operation if $y < x$ in the collapse order.

4) $x <_h y$ in the sequence operation if $x < y$ in the collapse order.

If vertex 2 doesn't collapse last then we don't assign a sequence operation to this order of edge collapses. The contribution is 0 in that case of order of edge collapse.

For example, an easy case to deal with would be one we have already seen. We have 4 vertices and we collapse in the order $\{4, 3, 2\}$. In this case there are only two brace trees that satisfy this type, that is the ones corresponding to the surjections 123432 and 124232. For the collapse order $\{3, 4, 2\}$ we only have 123242. As we have already seen for three vertices, the only interesting edge collapse order is $\{3, 2\}$ which produces sequence operation 1232.

Notice: if we sum over all edge collapse orders into 1 we get all the sequence operations predicted by [BZ]; for if $x <_v y$ in the tree to be collapsed (not the sequence) then it must be the case that x collapses before y. And so we don't end up with terms involving $y <_h x$ in the sequence operation.

Conjecture 3.2. *Let G be a planar planted tree with a labelling that respects the planar/planted structure then orders of collapses of i edges do not contribute to the differential d_i in the perturbed BG complex unless all collapsed edges are connected and the order has them all collapsing vertices into the vertex of smallest labelling connecting the subtree. When they do all collapse into the smallest vertex, then we use a standard labelling ($\{1, \ldots, n\}$), the order of collapse, and Conjecture* 3.1 *to get the contribution to d_i.*

Using Conjecture 3.1 and 3.2 we can show that the Chromatic Graph Homology differential coincides with perturbed BG differential.

Theorem 3.3. *When G is a planar planted tree with at most 5 vertices with the labelling induced by the planar structure, then The Chromatic Graph Homology for a brace algebra computes $H^*(M^{\times n}, Z; R)$ coinciding with the cohomology of the Bendersky-Gitler complex when the brace algebra chosen is $C^*_N(M; R)$.*

Proof. The perturbed differential in the complex of tensors has the same differential as the complex of tensors in the Chromatic complex. We have a sum over edge collapse orders, but many of them do not contribute to the differential. Only the order of collapse for collapses into vertex 1 matters and for these we get exactly the sequence operations as the conjecture which coincide with the sequence operations in the definition of the differential for the Chromatic Graph Homology complex. □

Chapter 4

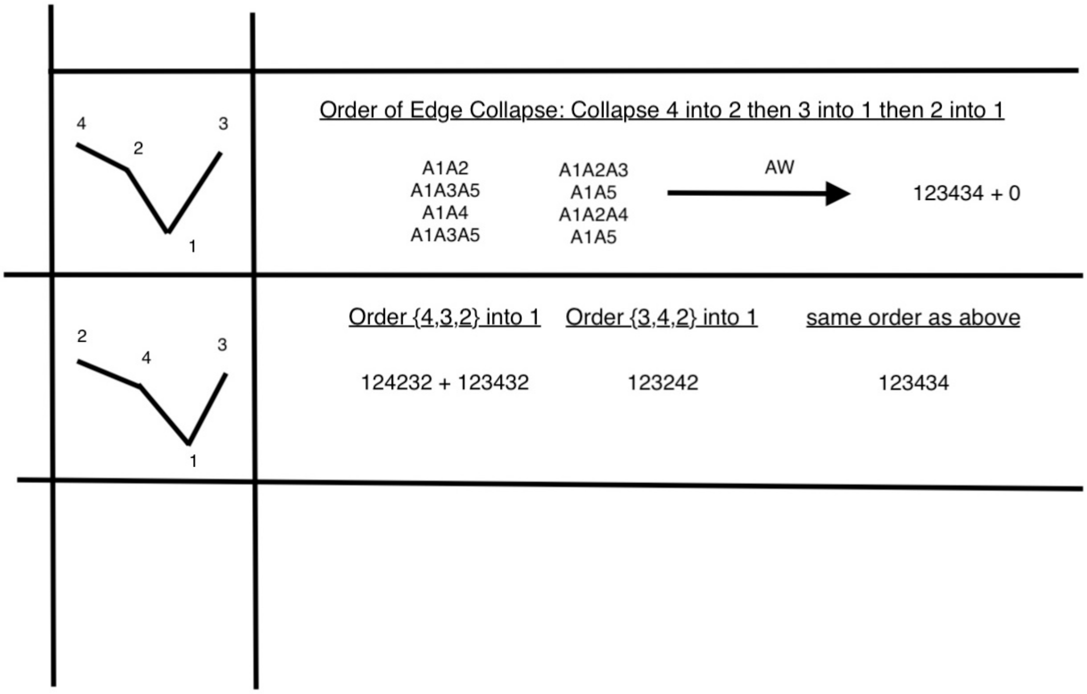
General Case

In general, there is no reason that a subtree of a graph should have a labelling that respects any sort of planar/planted structure. In particular, the vertical order or the horizontal order may not be respected in conjuntion with the natural ordering of the vertices.

For general graphs, vertices can collapse into others besides the one labelled by 1. For instance in the case of 4 vertices, one interesting order of collapses is 4 into 2 first, followed by 3 into 1, and finally 2 into 1. This gives us maps on chains $\Delta_{i,j}$ of $\Delta_{1,2}$, $\Delta_{1,3}$, and $\Delta_{2,4}$ with two Shih homotopies wedged inbetween. From the case of 3 vertices we know how the first few maps work.

4.1 E_3 Operations from Graphs

Examples of graphs that could have this order of edge collapse and do not have a labelling that agrees with its planar structure are the following:



The top row shows what happens if we apply $\Delta_{2,4}h\Delta_{1,3}h\Delta_{1,2}(\sigma)$ and then apply the Alexander-Whitney map. For the first A-block configuration since A_2 is not used in lower rows and A_1, A_2 are adjacents it must be the case that A_2 is a 0-simplex for the term to possibly be non-degenerate after Alexander-Whitney. If we look at the cases where A_2 is a 0-simplex then we may ignore A_2 and relabel like:

A_1

$A_1 \quad A_2 \quad A_4$

$A_1 \quad A_3$

$A_1 \quad A_2 \quad A_4$

after we apply the Alexander-Whitney map, since A_1, A_2 are adjacent, we split A_1 into 3 overlapping pieces, include all of A_3 in the third factor and include all of A_2A_4 in the forth factor. Relabeling again, we get:

$$\sigma(B_1) \otimes \sigma(B_2) \otimes \sigma(B_3 \sqcup A_3) \otimes \sigma(A_2 \sqcup A_4)$$

which is the operation 123434. Remember this is in the E_3 part of the filtration on the surjection operad but is not part of the E_2 part. All of the operations for the Chromatic Graph homology algebraic construction are by definition E_2 operations and are described by vertical and horizontal rules that only make sense for the E_2 filtration of the surjection dg-operad.

4.2 Complete Graph on 4 Vertices

Of course we can always view any graph on 4 vertices as a subgraph of the complete graph on 4 vertices. This is the graph G with 4 vertices and 6 edges connecting all of the vertices to one another. We have already shown above that for some subgraphs, the contribution to d_3 contains E_3 operations.

A good question that needs to be answered is: *do these E_3 operations not all cancel for the complete graph or does d_3 still live in the E_2 world?*

If one would write out all possible subgraphs which are trees and look at all permutations of edge collapses, it seems reasonable that the reader should be able to compute $H^*(M, Z; R)$ and thus the homology of the labelled configuration space of M with 4 distinct points, in the case of a compact manifold M ($R = \mathbb{Z}$ coefficients is an interesting case).

4.3 Graphs of 5 Vertices or More

It is reasonable to believe that graphs that are "out of order" based on its planar/planted embedding may contribute E_3 operations or even operations E_n for $n \geq 4$. When we have these out of order edge collapses and not all the collapses happen into vertex 1 it seems to give these higher operations like in the case drawn in the previous section.

Ideally, we would like a formula that spits out sequence operations for any graph G no matter what its labelling is and without an embedding. If the graph G has cycles then only the subgraphs which are trees (because forests cancel out) contribute to the differentials d_i (this is Theorem 2.2).

So, the general framework should start with a graph G with any labelling and assign every connected subtree a sum of sequence operations. The complex that computes $H^*(M^{\times n}, Z; R)$ involves a differential $D = \pm d_0 \pm d_1 \pm \cdots$ where each d_i is a sum over these connected subtrees with i edges, and thus a sum of sequence operations.

Chapter 5

Conclusions

We tabulate known facts and probable conjecture towards the understanding of a more computationally feasible Bendersky-Gitler type complex. In the case of graphs with a labelling agreeing with its planar planted structure we think this is just the Chromatic Graph homology for Brace Algebras (E_2-algebras) but for graphs with subtrees not respecting one of these orders there may be E_3 operations or higher operations of the E_∞-algebra $C_N^*(M; R)$. An important case to study would be the complete graph on 4 vertices because this contains every tree with or without planar labelling. We believe a more all-encompassing theory will explain what is happening in the non-planar case and we hope to return to this in future work.

5.1 Planar Planted Trees

For a sequence of edge collapses that connect a tree where the vertex labels agree with both horizontal and vertical order, we understand the Bendersky-Gitler perturbed complex differential to be altered by surjection operations (of complexity ≤ 2). The surjection operations are only dependent on the order in which the edges collapse vertices into the vertex labelled by 1 or the smallest labelled vertex (in the natural order). Orders of collapse that include cycles, collapsing into higher vertices, and disjoint subgraphs do not contribute to the differential.

When the graph G is itself a planar/planted tree then all connected subgraphs are also planar/planted trees. In this case, the Chromatic Graph Homology complex is well defined and it is our conjecture that this computes $H^*(M^{\times n}, Z; R)$ in the case that the brace algebra is $C_N^*(M; R)$. We showed that the conjecture holds true for graphs with 5 or less vertices.

5.2 Applications

Of course we could always directly use the Shih homotopy formula to explicitly calculate the d_i contribution to the perturbed differential. This formula includes lots of shuffle maps where we need to sum over all possible shuffle permutations of a certain type. There are way too many of these permutations to explicitly compute using this formula even for a computer. What we see is the sensitivity to degeneracies in the Alexander-Whitney map means that most shuffles in the Shih homotopy formula end up useless in the final formula when we deal with normalized cochains.

A general formula in terms of sequence operations for the maps that compute the homology of the graph configuration space M^G could be computationally advantageous. Those interested should be anybody involved in computational algebraic topology and in particular persistent homology. [Re] and [GR] are just a sample of papers devoted to the computation aspects of algebraic topology. The idea is that homology groups are only as useful as our computational ability to deal with them is. More efficient algorithms are needed to advance the field.

Those interested in physics will find applications to configuration spaces of manifolds and their homological properties. The topology of configuration spaces also intertwines with Knot Theory; the homotopy groups of the configuration space of $\mathbb{C}$ are the Braid Groups. In fact, the Chromatic Homology, Jones Polynomial, and Khavonov Homology are all related in the case of an Alternating Link.

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