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NOTES

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Eigenvalues, Almost Periodic Functions, and the Derivative of an Integral

Mark Finkelstein and Robert Whitley

In [4], examples are given of functions $g(x)$, bounded and continuous on $(0, 1]$, for which the indefinite integral $\int_0^x g(t) dt$ is not differentiable from the right at zero. The purpose of this note is to show that such examples arise in a natural way and that the class of such examples is quite large.

First, to see how such examples can come up, consider the operator

$$Hf(x) = \frac{1}{x} \int_0^x f(t) dt.$$

A result due to Hardy [5, Equation 9.9.1], [7, Exercise 8.14] asserts that H is a bounded linear operator on the Banach space $L^p[0, 1]$ for $p > 1$ with norm less than or equal to $p/(p-1)$. Actually Hardy gives the result for $L^p[0, \infty)$, but it is somewhat simpler to work with $L^p[0, 1]$. Considering the function $f(x) = x^a$ (a real), which is in $L^p[0, 1]$ when $a > -1/p$, computing $Hf(x)$, and letting $a \rightarrow -1/p$ shows that Hardy's bound for the norm of H is sharp and also shows that H is unbounded on $L^1[0, 1]$. The case $p = \infty$ is a limiting case in which the norm of H is one.

In working with the function $f(x) = x^a$ of the preceding paragraph, it will not have escaped the reader's notice that this function is an eigenfunction for H with eigenvalue $1/(1+a)$. A basic operator theory question is: What are the eigenvalues of H ? If λ is a nonzero eigenvalue with associated eigenvector f , then

$$\frac{1}{x} \int_0^x f(t) dt = \lambda f(x), \tag{1}$$

where equality holds *almost everywhere*. But equation (1) shows that f is absolutely continuous on $(0, 1]$, and (1) can be taken to hold not just almost everywhere but for all x in $(0, 1]$. That is, a continuous f satisfying (1) for all x in $(0, 1]$ can be chosen from the equivalence class of functions in $L^p[0, 1]$ that satisfy (1) almost everywhere. It is legitimate to differentiate equation (1) and, by rearranging, to get

$$\lambda x f'(x) + (\lambda - 1)f(x) = 0, \tag{2}$$

which is an Euler differential equation [2, chap. 4, sec. 2] on $(0, 1]$. A priori (2) also holds only almost everywhere, but on solving for f' we see, as earlier, that a continuous representative from the equivalence class of f' can be chosen so that (2) holds for all x in $(0, 1]$. It has solutions of the form $f(x) = x^\beta$, where $\beta = a + ib$, with f in $L^p[0, 1]$ if $a > -1/p > -1$. The associated eigenvalue is $\lambda = 1/(1+\beta)$. It is immediate from (1) that the eigenfunction f is continuous at 0 (from the right, since f has domain $[0, 1]$) if and only if $F(x) = \int_0^x f(t) dt$ is differentiable (from the right) at zero, and this occurs exactly when $a > 0$.

This leads to the observation that for the case $a = 0$, $f(x) = x^{ib} = \cos(b \log x) + i \sin(b \log x)$ is a bounded continuous function with $F(x) = \int_0^x f(t) dt$ not differentiable from the right at zero, the same being true of its real and imaginary parts $\cos(b \log x)$ and $\sin(b \log x)$. These two examples occur in [4], where it is remarked that aside from $\sin x$ and $\cos x$ the authors are not aware of any other functions g , say continuous and bounded on the real line, with the property that $\int_0^x g(\log t) dt$ is not differentiable at zero. We ask: For what functions g , continuous and bounded on $(0, \infty)$, is it true that $G(x) = \int_0^x g(-\log t) dt$ is not differentiable at zero? We make the successive changes of variables

$$u = -\log t, \quad y = u + \log x, \quad z = -\log x$$

to obtain

$$\begin{aligned} \frac{1}{x} \int_0^x g(-\log t) dt &= \frac{1}{x} \int_{-\log x}^{+\infty} g(u) du = \int_0^{\infty} g(y - \log x) e^{-y} dy \\ &= \int_0^{\infty} g(y + z) e^{-y} dy = \phi(z). \end{aligned}$$

Thus the function $x \mapsto \int_0^x g(-\log t) dt$ is differentiable from the right at zero if and only if

$$\lim_{z \rightarrow \infty} \phi(z) = \lim_{z \rightarrow \infty} \int_0^{\infty} g(y + z) e^{-y} dy = A \quad (3)$$

exists.

Recall that a function g is (*uniformly*) *almost periodic* on $\mathbf{R}$ [6], [3] if it is continuous and for each $\epsilon > 0$ there is a number L , depending on g and ϵ , with the property that every interval of length L in $\mathbf{R}$ contains an “ ϵ -period” p for g (i.e., contains a point p with the property that

$$\sup_{x \in \mathbf{R}} |g(x + p) - g(x)| \leq \epsilon.) \quad (4)$$

A periodic function is certainly almost periodic, for if L is larger than a period of the function, p can be found so that (4) holds with $\epsilon = 0$.

Lemma 1. *If g is nonconstant and almost periodic on $\mathbf{R}$, then the limit in (3) does not exist.*

Proof. Since almost periodic functions are bounded, the integral in (3) exists. Assume that the limit A in (3) exists. For given $\epsilon > 0$, points p_n ($n = 1, 2, \dots$) can be found such that $p_n \rightarrow \infty$ and $|g(x + p_n) - g(x)| < \epsilon$ for all x and n . For any t , $|g(y + t + p_n) - g(y + t)| < \epsilon$ holds for all y . Thus

$$A = \lim_{z \rightarrow \infty} \phi(z) = \lim_{n \rightarrow \infty} \phi(t + p_n).$$

Since

$$\left| \int_0^{\infty} g(t + y + p_n) e^{-y} dy - \int_0^{\infty} g(t + y) e^{-y} dy \right| < \epsilon, \quad (5)$$

it follows that

$$\int_0^\infty g(y+t)e^{-y} dy = A. \quad (6)$$

In (6), we make the change of variable $w = y + t$ to obtain

$$\int_t^\infty g(w)e^{-w} dw = \int_0^\infty g(w)e^{-w} dw - \int_0^t g(w)e^{-w} dw = Ae^{-t}. \quad (7)$$

Differentiating (7) with respect to t shows that $g(t)e^{-t} = -Ae^{-t}$ and therefore that g is constant, contrary to our assumption. ■

The lemma can be improved in several ways. Trivially, g need be defined only on $(0, \infty)$ and satisfy the obvious variant of almost periodicity. The aspect of almost periodicity that is used in the proof is that g has arbitrarily large ϵ -periods. Bohr remarks that this is a weaker condition than almost periodicity [3, p. 32]. The condition that would suffice to make (5) hold is that g be almost periodic with respect to a norm other than the supremum norm (for this see [1]).

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An Elementary Proof That Every Singular Matrix Is a Product of Idempotent Matrices

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In this note we give an elementary proof of a theorem first proved by J. A. Erdos [3]. This theorem, which is the main result of [3], states that every noninvertible $n \times n$ matrix is a finite product of matrices M with the property that $M^2 = M$. (These are known as *idempotent matrices*. Noninvertible matrices are also called *singular matrices*.) An alternative formulation of this result reads: every noninvertible linear mapping of a finite dimensional vector space is a finite product of idempotent linear mappings α , linear mappings that satisfy $\alpha^2 = \alpha$. This result was motivated by a result of J. M. Howie asserting that each self-mapping α of a nonempty finite set X with image size at most $|X| - 1$ (which occurs precisely when α is noninvertible) is a product of idempotent mappings. We shall see that Erdos's theorem is a consequence of Howie's result.