Title
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I. Principal Assumptions Necessary for a Dispersion Relation

A dispersion relation relates a dispersive process to an absorptive process. In an elastic scattering problem a dispersion relation is an integral relation between the real and imaginary parts of the scattering amplitude. We will denote these parts by D and A, i.e., $T = D + iA$, where T is the scattering amplitude and D and A are real. The symbol A is used because the imaginary part of T is related to the absorption cross section.

An example of a dispersion relation, which is valid for the coherent amplitude for the forward scattering of gamma rays from nucleons, or for the forward amplitude for nucleon-neutral pion scattering, is

$$D(\omega) - D(\mu) = 2\,\frac{k^2}{\pi}\,P\int_0^\infty d\omega'\,\frac{\omega'\,A(\omega')}{k'^2(k'^2 - k^2)}, \tag{1}$$

where k, ω, and μ are the momentum, energy, and mass of the scattered particle in the laboratory system, and the symbol P denotes that the principal part of the integral is to be taken. The constants $\hbar$ and c are taken to be equal to 1.

The derivation of this equation was discussed in Lecture 7. The principal assumptions necessary to this derivation are:

1. Causality

The essential assumption on which dispersion relations are based is that the outgoing wave at any time is determined by the incoming wave at preceding times. In a relativistic theory it is required that no signal may propagate faster than the speed of light in a vacuum.

2. Convergence at High Energy

In order for a dispersion relation to be valid, the scattering amplitude must not diverge too badly at high energies. For example, the existence of the integral $\int_{\alpha}^{\infty} \left| T(\omega)/\omega^2 \right|^2 d\omega$, where α is some positive energy, is sufficient for the validity of Eq. (1). Since, for forward scattering, $A(\omega)$ is related to the total cross section by the equation

$$A(\omega) = (k/4\pi)\sigma_T \quad ,$$

the above condition requires that the integral $\int_{\alpha}^{\infty} \left[\sigma(\omega)/\omega\right]^2 d\omega$ exist.

3. Symmetry Property with Respect to a Change in Sign of the Energy

In order to derive a simple dispersion relation, which involves only positive energies, one needs a simple symmetry property for T, either $T(-\omega) = T^*(\omega)$ or $T(-\omega) = -T^*(\omega)$. Equation (1) corresponds to the property $T(-\omega) = T^*(\omega)$. If $T(\omega)$ is Fourier analyzed,

$$T(\omega) = (2\pi)^{-3/2} \int_{-\infty}^{\infty} e^{i\omega t} T(t)dt \quad , \tag{2}$$

then the above two symmetry properties correspond to the cases in which $T(t)$ is either real or pure imagainary. In quantum field theory this corresponds to the requirement that the scattered particles transform into themselves under charge conjugation. Gamma rays and neutral pions do transform into themselves under charge conjugation, but π^+ and π^- mesons do not. In order to derive simple dispersion equations for charged pions, one must use the linear combinations $T^{(1)} = T_{\pi^+} + T_{\pi^-}$ and $T^{(2)} = T_{\pi^+} - T_{\pi^-}$, for these linear combinations do transform into themselves under charge conjugation. The amplitude $T^{(1)}$ satisfies Eq. (1).

4. Assumptions Concerning the Interaction

In ordinary quantum mechanics a dispersion formula may be derived for the scattering of any particle from a potential. In quantum field theory dispersion relations have been derived, so far, only for boson-fermion scattering, in which the interaction between fields is assumed to be of the contact type, that is, characterized by an interaction Hamiltonian of the type

$$H = \int d^3x \, \phi_F(x) \, \phi_B(x) \quad ,$$

where $\phi_F(x)$ is expressed in terms of the fermion field, and $\phi_B(x)$ is expressed in terms of the boson field. An example is the well-known proton interaction with the electromagnetic field,

$$H = e \sum_{i=1}^{4} \int d^3x \quad \psi^+(x) \alpha_i \, \psi(x) \, A_i(x)$$, where $\vec{A}(x)$ is the electro-magnetic field, ψ the Dirac field, and $\vec{\alpha}$ the well-known Dirac operator. It is not known whether or not such scattering amplitudes as the neutron-proton and proton-proton amplitudes satisfy dispersion formulas.

II. Relation of $D(\omega)$ and $A(\omega)$ to Physical Quantities.

In order to apply Eq. (1) to γ - N or π - N scattering, we must interpret $D(\omega)$ and $A(\omega)$ in terms of physical quantities. The essential meaning of the forward scattering amplitude is that it is related to the differential cross section in the forward direction by the equation $d\sigma/d\Omega = |T|^2$. However, there are two specific problems which must be discussed before Eq. (1') is applied. These problems are listed below.

1. The Nonphysical Energy Region

If the scattered particle is a pion, or other finite-mass particle, the energy region $0 < \omega < \mu$ does not correspond to a physical process, for in this region the pion momentum $k = (\omega^2 - \mu^2)^{\frac{1}{2}}$ is imaginary. The value of $M(\omega)$ in this region must be defined by analytic continuation. If there are no bound states, however it can be shown that the quantity $A(\omega)$ vanishes in the region $0 < \omega < \mu$. This may be shown by expanding $T(\omega)$ in partial waves and making use of the fact that the scattering phase shift $\delta(k)$ is an odd function of k. Then $\delta(k)$ is imaginary when k is imaginary, and $T(\omega) = e^{2i\delta - 1}/2ik$ is real.

If there exists a bound state of the system at some energy in the region $0 < \omega < \mu$, $M(\omega)$ will have a pole at this point, and this pole will give rise to an added term in Eq. (1). In pion-nucleon scattering, the state of the real nucleon acts like a bound state and leads to an additive term in the dispersion relation. This term may be estimated from meson theory.

Since this bound-state term is the only term entering into the nucleon-neutral pion dispersion relation in this energy region, we may rewrite Eq. (1), for this case,

$$D(\omega) - D(\mu) = \frac{2k^2}{\pi} P \int_{\mu}^{\infty} \frac{d\omega' \; \omega' \; A(\omega')}{k'^2(k'^2 - k^2)} + \Gamma(\omega) , \qquad (3)$$

where $\Gamma(\omega)$ represents the contribution from the real nucleon state.

2. Relation between $A(\omega)$ and the Total Cross Section

It was shown in Lecture 5 that, if the spins of the particles may be neglected, the imaginary part of the forward scattering amplitude is related to the total cross section by the relation

$$A(\omega) = (k/4\pi)\sigma_T . \qquad (4)$$

In pion-nucleon scattering there may be no spin flip for forward scattering, and so Eq. (4) may be used directly. However, for gamma ray-nucleon scattering, there may be spin flip for forward scattering, because of the spin of the photon. The γ - N forward scattering amplitude may be written

$$T(\omega) = f_1(\omega)\, \vec{e}_f \cdot \vec{e}_i + i f_2(\omega)\, \sigma \cdot \vec{e}_f \times \vec{e}_i ,$$

where $\vec{e}_i$ and $\vec{e}_f$ are the polarization vectors of the initial and final photons. The derivation of Eq. (4) makes use of the fact that the forward amplitude is a diagonal element of the entire T matrix, and thus implies no spin flip. Thus in this case it is $f_1(\omega)$ which is related by Eq. (4) to the total cross section σ_T, which here denotes the total cross section for unpolarized initial particles. Using Eq. (4) for the γ - N amplitude $f_1(\omega)$, we may write the dispersion relation in the form

$$\mathrm{Re}\left[f_1(\omega) - f_1(0)\right] = \frac{\omega^2}{2\pi^2} P \int_0^\infty \frac{d\omega' \; \sigma_T(\omega')}{(\omega'^2 - \omega^2)} . \qquad (5)$$

The amplitude $f_1(0)$ may be shown to be (e^2/Mc^2), where M is the mass of the scatterer. The dispersion relation and Eq. (4) allow the forward elastic scattering amplitude to be completely determined from a knowledge of the total cross section at all energies.

III. Difference between Cases of Strong and Weak Interaction

The principal use of the dispersion formula is quite different in the two cases, gamma-nucleon scattering, and pion-nucleon scattering, because the first interaction is weak and the second interaction is strong. We first consider the weak interaction case. At energies above the threshold for the photoproduction of pions from nucleons, the production cross section, which is proportional to $e^2/\hbar c$, is much larger than the elastic $\gamma - N$ scattering cross section, which is proportional to $(e^2/\hbar c)^2$. Therefore, if Eq. (5) is used to investigate $\gamma - N$ scattering in the energy region 50 to 200 Mev, the elastic scattering cross section may be neglected in computing the integral, and Eqs. (4) and (5) allow a determination of the elastic scattering amplitude from a knowledge of a different physical process, photomeson production.

If $T(\omega)$ is expanded in terms of spherical waves, the weak interaction corresponds to small phase shifts. Expanding $T(\omega)$ for a particular spherical wave in powers of the phase shift, and keeping only the lowest term, we obtain

$$T(\omega) = \frac{e^{2i\delta} - 1}{2ik} = \frac{\delta(\omega)}{k} . \tag{6}$$

Therefore, in this approximation, the real part of T corresponds to the real part of $\delta(\omega)$, and the imaginary part of $T(\omega)$ corresponds to the imaginary part of δ . Below the photoproduction threshold, only elastic scattering is possible, and $\delta(\omega)$ is real. Thus, in this weak interaction approximation, $T(\omega)$ is real also. Above photoproduction threshold, both $\delta(\omega)$ and $T(\omega)$ are complex. It may be seen also from Eq. (4) that the imaginary part of $T(\omega)$ vanishes below the photopion production threshold, if the elastic scattering total cross section is ignored.

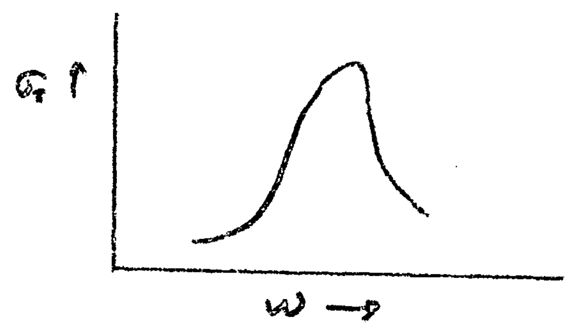
For the pion-nucleon problem the elastic scattering is strong, and cannot be neglected. The dispersion relations may be written in a simple form, however, if it is assumed that multiple meson production is neglected, and only the elastic process $\pi + N \rightarrow \pi + N$ is important. In a charge-independent theory, neglecting inelastic processes is equivalent to the statement that the phase shifts corresponding to the different values of orbital angular momentum, total angular momentum, and total isotopic spin are real. Then the real and imaginary parts of the scattering amplitude for a wave of given ℓ, j, and T may be written in terms of the real phase shifts $\delta_{\ell jT}$:

$$D_{\ell jT} = \frac{\cos\delta \sin\delta}{k} = \frac{\sin 2\delta}{2k}$$
$$A_{\ell jT} = \frac{\sin^2\delta}{k} \quad . \tag{7}$$

The dispersion relations now become nonlinear integral equations for the phase shifts. These equations do not have unique solutions, but they may be used to indicate certain features of the variation of phase shifts with energy. In order to write a very simple equation that still makes some sense, we consider an energy region in the neighborhood of the $\ell = 1$, $j = 3/2$, $T = 3/2$ pion-nucleon scattering resonance. We consider the dispersion relation for the amplitude $T^{(1)} = T^{(+)} + T^{(-)}$, expand in spherical waves, neglect all waves except the $\ell = 1$, $j = 3/2$, $T = 3/2$ wave, and neglect terms of order (μ/M), where M is the nucleon mass. The contribution from the real nucleon state is of order (μ/M) in this case, and therefore is neglected. The resulting equation is

$$\sin 2\delta_{33}(\omega) = \frac{4k^3}{\pi} P \int_0^\infty dk' \frac{1}{k'^2 - k^2} \frac{\sin^2 \delta_{33}(\omega')}{k'^2} \quad . \tag{8}$$

The right side may be written in terms of the total cross section by the relation $\sigma = 8\pi \sin^2\delta/k^2$, where the factor 8 is characteristic of the particular values of j and ℓ involved. Eq. (8) is an approximation, but it indicates correctly the essential features of the phase shift δ_{33} in the neighborhood of the resonance. Since the resonance is rather sharply peaked, i.e.,



thus, one can see from the energy denominator $(k'^2 - k^2)$ that $\sin 2\,\delta_{33}(\omega)$ is positive when ω is lower than the energy of the resonance, and $\sin 2\,\delta_{33}(\omega)$ is negative when ω is greater than the energy of the resonance. Thus $\delta_{33}(\omega)$ must go through 90° somewhere in the neighborhood of the resonance peak.